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Testing Optimal Rebalancing for Fully Invested Portfolios with Transaction Costs

An Empirical Analysis of Multiple Risky Assets Portfolios

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Abstract

The thesis will examine the performance of various rebalancing strategies in portfolios with two and four risky assets, where all capital is committed to risky assets. The primary objective is to demonstrate that portfolio rebalancing adds value for investors, even in the presence of transaction costs. We provide a comprehensive analysis identifying the optimal rebalancing strategy for a multi-asset portfolio. Within the mean-variance framework, the optimal strategy without considering transaction costs is to maintain constant weights in the asset allocation, irrespective of the performance of the risky assets. This necessitates rebalancing to restore the portfolio to its target weights, unlike a passive portfolio where weights remain unchanged throughout the investment period. The scarcity of analytical solutions for the optimal portfolio choice problem is largely due to the complexities introduced by transaction costs, which necessitate the use of intricate numerical methods within the domain of stochastic optimal control theory. Furthermore, the thesis draws inspiration from Dichtl, Drobetz, and Wambach (2014) and examines the performance of various rebalancing strategies for fully invested portfolios consisting of two risky assets, that incorporate transaction costs. We employ the methodology of Zakamulin (2024) to illustrate that the optimal portfolio choice problem can be solved using a practical and easily applicable method, derived from a single-period model. Our findings are consistent with those of Dichtl et al. (2014) and Zakamulin (2024), indicating that the Buy-and-Hold strategy is sub-optimal.

Keywords: Portfolio Optimization, Rebalancing strategies, US market, Multiple risky assets, Transaction costs, Fully invested portfolio

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Chapter 1

Introduction

In the domain of finance, portfolio optimization is a critical endeavor as investors seek to attain an optimal balance between risk and return. This imperative builds upon the seminal contributions of Markowitz (1952), who introduced the concept of the mean-variance portfolio. Further research by Merton (1969) and Merton (1973) extended the framework into a continuous-time model. Portfolio management involves determining an investor's risk preferences, followed by implementing the most suitable asset allocation strategy. The analytical solutions to the optimal portfolio choice problem, applicable in single-period and continuous-time scenarios, often recommend a constant-weight allocation strategy. However, unlike a passive strategy, where asset weights inevitably shift over time, a constant-weight strategy necessitates regular rebalancing to preserve the targeted asset weights. Rebalancing incurs transaction costs, which can substantially diminish the capital gains from realigning the portfolio to its target allocations.

The continuous-time model with transaction costs for portfolios of one risky-asset and one risk-free asset was initially studied by Magill and Constantinides (1976), Davis and Norman (1990) and Shreve and Soner (1994). Further contribution by Leland (2000) advanced the study by considering a portfolio of two risky assets, also accounting for transaction costs. Advancements by Lynch and Tan (2010), Gârleanu and Pedersen (2016) and Ma, Siu, and Zhu (2019), extended the study for portfolios encompassing multiple risky assets with transaction costs. A common limitation of the previous studies on portfolio optimization is the inclusion of a risk-free asset. However, it is unrealistic for institutional investors to allocate any portion to a risk-free asset, suggesting that these models are more relevant for individual investors. Despite this alignment, most individual investors typically do not engage in dynamic rebalancing.

The primary challenge is that when transaction costs are factored in, the optimization problem lacks analytical solution. Therefore, analytical solutions to portfolio optimization is limited to a one-period model with portfolios containing up to two risky assets. Thus necessitating numerical methods to solve the problem when considering portfolios of multiple risky assets. However, in the continuous-time context and discrete-time models, these numerical methods necessitate solving the dynamic optimal control problem. The complexities associated with the optimal control problem has led to solutions containing only portfolios of two risky assets, although they initially attempted to solve for multiple risky assets.

Numerical solutions to the optimal transaction policy with transaction costs reveal the existence of a no-trade region in the case of one or two risky assets. The no-trade region is a concept in portfolio management that defines a range within which the portfolio's asset weights can fluctuate without triggering rebalancing. This region exists because the transaction costs associated with buying and selling assets can sometimes exceed the benefits of realigning the portfolio to its exact target allocation. Within this region, it is optimal not to trade. Rebalancing is only triggered when the portfolio's weights drift outside the no-trade region, where the benefits of realigning the portfolio outweigh the transaction costs. However, the optimal rebalancing strategy for portfolios containing more than two risky assets remains largely unexplored and inadequately understood.

The empirical study by Dichtl et al. (2014) incorporates transaction cost and investigates whether rebalancing adds any value to investors, and which rebalancing strategy is optimal. They test three different rebalancing strategies against the Buy-and-Hold strategy to evaluate the effectiveness of rebalancing in a portfolio composed of two risky assets. The Naïve rebalancing strategy rebalances to the target asset weights each period, whereas the Threshold rebalancing strategy is a simple ad-hoc approach where the action of rebalancing depends on a no-trade band and transaction costs. Thus, when the deviation from target asset weights is small, it is not optimal to rebalance back to the target weights. Similarly, the Range strategy depends on a no-trade band and triggers a rebalancing to the nearest specified boundary, adjusting to either an upper or lower limit based on the direction of the deviation. The findings of Dichtl et al. (2014) show that the Buy-and-Hold strategy is sub-optimal, and further conclude that the choice of which rebalancing strategy to employ is of minor economic importance.

In the case of a portfolio with more than two risky assets, there is no analytical solution to the optimization problem. The challenge arises from the fact that, when considering portfolios with more than two assets, the no-trade region is not well-defined. To illustrate this, consider this example: the target portfolio weights are 30%, 30%, 20%, and 20%, with a No-Trade Band (NTB) of $\pm 3\%$. The current weights are 36%, 28%, 18%, and 18%. Only the first asset, at 36%, is outside the no-trade band, so its weight is reduced to 33%. Although the other assets are within the no-trade band, adjustments are still required to maintain the overall portfolio balance.

It is important to acknowledge that previous studies have extensively addressed the theoretical treatment of the single-period mean-variance optimal choice problem with transaction costs, evidently by the works of Scherer (2011) & Fabozzi and Markowitz (2011). However, these earlier studies do not consider portfolios fully invested in risky assets and struggle to efficiently address practical solutions for managing multiple risky assets with transaction costs.

Furthermore, D. Liu presents a theoretical paper that explores a single-period model addressing multiple assets. Despite this, the study relies on arbitrary parameters that are insufficiently explained, rendering it impractical for real-world application. Overall, the literature lacks a practical model that addresses portfolios fully invested in risky assets and effectively handles scenarios involving multiple risky assets with transaction costs.

While it was previously believed that the solution to the single-period problem could not be applied to multi-period contexts, more recent research by Zakamulin (2024) has demonstrated that it is indeed possible. Zakamulin (2024) proposes a practical approach for the optimal portfolio choice problem, specifically focusing on portfolios fully invested in multiple risky assets. The author proposes a practical approach that balances theoretical depth with numerical simplicity, where the main focus is on a multi-period model that approximates a policy derived from a single-period model. This method sidesteps the complex and time-intensive optimal stochastic control problem by offering a numerical method for quadratic optimization.

Motivated by the significant challenges identified in existing literature, this thesis aims to further the discussion on portfolio optimization considering a fully invested portfolios with proportional transaction costs and target asset weights. Furthermore, we seek to address the practical application challenges associated with managing multiple risky assets, opting for an efficient methodology to solve this problem. While previous studies provide extensive theoretical treatments and methodologies, they often lack empirical findings, particularly as the number of risky assets increases. Therefore we replicate the studies of Dichtl et al. (2014) and Zakamulin (2024) to provide a thorough empirical study that covers two portfolios: a fully invested portfolio of two and four risky assets. It contributes to the existing literature by offerering empirical findings for two and four risky assets scenarios and by stress-testing each case through variations in the model's underlying parameters.

The first portfolio is a fully-invested portfolio composed of two risky assets and draws inspiration from the study of Dichtl et al. (2014). Similarly to Dichtl et al. (2014) we employ an arbitrary no-trade band of $\pm 3\%$ and use the Sharpe ratio as a performance metric to evaluate the optimal rebalancing strategy for a two risky asset portfolio. The performance of Threshold, Range, and Naïve rebalancing strategies is evaluated in comparison to the Buy-and-Hold strategy. The Buy-and-Hold strategy maintains the initial asset allocation constant over the entire investment period, thereby foregoing any portfolio rebalancing. As a result, this strategy incurs no transaction costs. However, a passive approach allows the portfolio composition to change naturally with market fluctuations, often leading to a greater proportion of the portfolio being invested in stocks.

The second portfolio is composed of four risky assets, and we employ the methodology presented by Zakamulin (2024). In scenarios involving more than two risky assets, application within a predefined no-trade region is simply not possible, therefore the author propose a simple method to optimally rebalance a fully invested portfolio of multiple risky assets when transaction costs are present. We employ a numerical method designed to ensure that, in the absence of transaction costs, the optimal portfolio weights converge to those of the tangency portfolio. However, when transaction costs are incurred, the optimal weights deviate from the tangency portfolio weights, resulting in a portfolio that maximizes mean-variance utility after rebalancing with adjusted weights.

The methodology varies in complexity in the case of the two risky asset and four risky asset portfolios. For the portfolio consisting of two risky assets, we use an arbitrary parameter to determine the no-trade band. The computation is straightforward and easy to implement, allowing for precise portfolio management based on explicit formulas. The optimal portfolio choice relies on an objective function that is quadratic, with linear constraints, which inherently precludes the possibility of an analytical solution as the number of risky assets increases beyond two. Therefore, in order to derive a solution to the optimal portfolio choice problem, numerical solution becomes necessary. By applying numerical algorithms, we effectively tackle the quadratic optimization problem. The methodology employs certain assumptions, including proportional transaction costs, constant risk aversion, and a constant variance-covariance matrix. Under the optimal rebalancing strategy, the band size is calibrated to maximize the Sharpe ratio and is dependent on the model parameters.

However, it is important to note that the Optimal strategy converge to the Range strategy when the number of risky assets reduces to two, as both trigger rebalancing when the portfolio moves outside the no-trade region.

To validate the empirical findings and demonstrate the practical applicability of the models, we conduct historical simulations using real-world data from the US market spanning the years 1960 to 2022. The thesis aims to enhance the understanding of portfolio optimization for portfolio managers that are dealing with fully invested portfolios that incur transaction costs. The empirical study offers valuable insights for both academic researchers as the model relies on assumptions that are in line with reality, such as prohibiting short-sales and excluding a risk-free alternative in the portfolio. Additionally, through rigorous empirical testing, we illustrate the practical effectiveness of these strategies, particularly the Optimal strategy in more complex scenarios.

In light of the empirical validations, the findings of this thesis demonstrate the effectiveness of rebalancing strategies in portfolio optimization with varying numbers of risky assets. Our empirical investigation reveal that the traditional Buy-and-Hold and Naïve strategies is inferior to rebalancing strategies. Notably, the more sophisticated Threshold and Range strategies, which are specifically designed for portfolios with two risky assets, consistently achieve superior Sharpe ratios. Furthermore, when extending our analysis to include portfolios of four risky assets, the implementation of an Optimal strategy demonstrate superior performance compared to both the Naïve strategy and passive Buy-and-Hold approach in terms the Sharpe ratio.

The remaining thesis is organized as follows: Section 2 reviews the literature on portfolio management, emphasizing the predominance of studies on continuous-time models compared to the relatively fewer discussions on multi-period (discrete-time) models. Section 3 gives a description of the methodology and statistical model used, while Section 4 presents our data and its descriptive statistics. Section 5 reports the empirical results, where the 6th Section will be a discussion of the findings. Section 7 concludes the thesis.

Chapter 2

Literature Review

Modern portfolio theory, a cornerstone of portfolio management, was first presented by Markowitz (1952) in his seminal paper Portofolio Selection. He introduced a framework that balances risk and return. The model received criticism for its lack of realism, particularly due to its single-period approach, which fails to adequately capture the dynamic nature of real markets.

Consequently, Merton (1969) developed a more realistic model, although it assumes a perfect capital market. Merton significantly advanced portfolio optimization by extending the one-period mean-variance portfolio model into a continuous-time setting. The model is predicated on several assumptions: (1) the market functions continuously and competitively, (2) securities can be fully diversified, (3) all pertinent information is accessible and (4) investors earn an income over their lifetimes. Merton's model demonstrate that investment strategies are influenced by the investor's risk aversion, expected returns, and asset volatility.

Furthermore, Merton (1973) exploration of the inter-temporal Capital Asset Pricing Model (CAPM) provides a critical expansion of CAPM into a continuous-time framework, delving deeper into asset risk valuation under conditions such as, perfect capital markets, all investors are rationale and that there is continuous trading. These studies have been criticized for their lack of real-world application, particularly due to the underlying assumptions. For instance, the models assume perfect market conditions, where transaction costs and taxes are absent, and the availability of complete information, which do not hold in actual financial markets. However, in both single-period and continuous-time contexts, the optimal portfolio choice problem yields analytical solutions that often suggest a constant-weight asset allocation strategy.

Magill and Constantinides (1976) advanced the research on portfolio optimization by examining a continuous-time model designed for the optimal management of a portfolio containing both a risky asset and a risk-free asset, incorporating transaction costs. The findings in this paper demonstrate that the optimal rebalancing policy includes a "no-trade" region around the target proportion of the risky asset. When the proportion stays within this interval, no trading is necessary because the costs of rebalancing would outweigh the benefits. However, if the proportion drifts outside the no-trade interval, it should be adjusted back to the nearest edge of the interval, rather than directly to the target proportion. Furthermore, the study highlights that these no-trade regions are centered around the optimal weights that would be maintained in a frictionless market without transaction costs. This implies that, in the absence of transaction costs, the optimal strategy entails continuously rebalancing the portfolio to the optimal weights. Similar investigation have been pursued; see, among others, Davis and Norman (1990) & Shreve and Soner (1994). These studies further validated the presence of a no-trade region, emphasizing its importance in optimal portfolio management with transaction costs.

The model introduced by Leland (2000) serves as a foundational framework for understanding two risky asset portfolio optimization in the presence of transaction costs and capital gains taxes, for a fully invested portfolio. By using numerical methods to solve the optimization problem, Leland (2000) suggest that the optimal rebalancing strategy is dependent on a two-dimensional no-trade region. Subsequent study of Lynch and Tan (2010) utilizes a numerical method to determine the optimal portfolio choice for an investor who has access to two correlated risky assets, considering both proportional transaction costs and return predictability. The study reveals that return predictability significantly impacts the optimal portfolio strategy, altering the boundaries of the no-trade region. However, in the presence of return predictability and transaction costs, implementation of more than two-risky assets becomes extremely challenging.

To enhance the understanding of dynamic portfolio management, Gârleanu and Pedersen (2016) explore the portfolio choice and demonstrate how it can be modeled in continuous time for portfolios of multiple risky assets and one risk-free asset. They argue that their model is more traceable than previous models on portfolio optimization, that rely on complex numerical solutions. Instead, the framework allows for a closed form optimal portfolio choice, where costs are quadratic in the number of risky assets being traded. Moreover, the authors demonstrate how a continuous-time model can be approximated by discrete-time models with increasingly shorter time periods, provided the model parameters are scaled appropriately. Their findings highlight that the optimal portfolio strategy involves gradual trading towards an aim portfolio. The aim portfolio is a weighted sum of the expected Markowitz portfolios across all future dates, with weights adjusted to account for return predictability and transaction costs. The model implies continuous trading, which is consistent with observable institutional investor behavior.

Building on this foundation, Ma et al.(2019), consider a portfolio consisting of multiple assets including a risk-free asset. This approach incorporates a time-dependent effect and is based on an investor with constant-relative-risk-aversion (CRRA). The method includes the volatility of return-predicting signals and examines its impact on trading strategies. Their findings indicate that return predictability leads to aggressive portfolio rebalancing, moderated by transaction costs that prevent large trades. Ma et al. (2019) provide numerical analysis offering insights into portfolios of multiple risky assets by demonstrating the time-dependent effect. As the terminal date approaches, the aim portfolio converges towards the Merton portfolio. The analysis shows that assets with low volatility, slow mean-reversion rates, and higher persistence are preferable.

While rebalancing can offer risk management benefits and potentially higher returns, these advantages must be weighed against the incurred transaction costs. The empirical study by Dichtl et al. (2014) compare the performance of rebalancing strategies in a fully invested portfolio of two risky assets with transaction costs, against a Buy-and-Hold portfolio. Their findings indicate that, by analyzing data from the US, UK, and Germany, the rebalancing strategies tested outperforms the buy-and-hold strategy in terms of the Sharpe ratio, Sortino ratio, and Omega measure. They note that while the Buy-and-Hold strategy may yield higher returns, the primary purpose of rebalancing is to maintain target asset allocations, not necessarily to maximize return-to-risk. Supporting this view, Zilbering, Jaconetti, and Jr. (2015) emphasize that investors adopt rebalancing strategies to balance their risk tolerance against expected net returns from rebalancing. They argue, however, that there is no significant evidence of an optimal rebalancing threshold or frequency.

In practice, rebalancing has been a key component of investment strategies for institutional investors. For instance, the Norwegian pension fund has effectively utilized rebalancing as a key component of its investment strategy. Between 2001 and 2012, they adhered to a rebalancing rule whereby if the allocation deviated by more than $\pm 3\%$ from the target weights of 60% stocks and 40% bonds, they would initiate a rebalancing process, selling stocks and purchasing bonds to realign with the target allocations.[1] The report from Norges Bank Investment Management (2012)

[1] Since 2012, the norwegian pension fund has changed their strategy and increased the target weight for stocks to 70% and decreased the no-trade band to 2% Ministry of Finance, Norway (2021). This implies that if the value of one asset class deviates by more than 2%, a rebalancing of the fund occurs to bring it back to target weights of 70% stocks

looks at no-trade band rebalancing strategies for portfolios and reveals that wider bands can lower transaction costs by reducing trading frequency. However, they also increase average equity share and its variability. The effectiveness of no-trade bands in exploiting time-varying expected returns is limited and uncertain. As noted by both Dichtl et al. and Zilbering et al., the challenge of determining an optimal rebalancing strategy becomes significantly more complex when managing portfolios with more than two risky assets. The increased number of assets introduces additional variables and interactions that complicate the rebalancing decision.

Mei, DeMiguel, and Nogales (2016) introduces a discrete-time framework designed to address the optimal multi-period portfolio choice problem, in the presence of transaction costs. Using a discrete-time approach has benefits over continuous-time models, especially when managing numerous risky assets, as it occasionally permits analytical solutions. However, an interesting discovery from this study was that in a multi-period model, the optimal strategy involves executing transactions only in the first period. As a result, when trading costs are taken into account, the optimal transaction policy effectively reduces to a buy-and-hold strategy. The results contradict those of Dichtl et al., who demonstrated the inferiority of the Buy-and-Hold approach.

Previous models typically include a risk-free asset and aim to maximize the investor's utility. D. Liu (2019) deviates from this trend by examining a single-period model that includes multiple risky assets, both with and without a risk-free asset. D. Liu presents analytical results to a fully-invested portfolio with fixed target weights and proportional transaction costs by deriving the equation for the no-trade region. Moreover, the model is based on simplified assumptions of the asset covariance matrix. These assumptions relies on a one-factor model, uncorrelated returns and same non-zero pairwise correlation. The author examines a single-period model that optimizes the tradeoff between tracking error and trading costs. However, this tradeoff relies on an arbitrary parameter, which presents a significant challenge for portfolio managers due to the lack of a clear method for selecting this parameter. Consequently, the practical implementation of this approach reveals several complications.

On the contrary, Zakamulin (2024) presents an approach that balances theoretical and numerical simplicity with practical relevance for a fully invested portfolio with transaction costs encompassing mulitple risky assets. The study demonstrates a practical method, suggesting that the optimal

and 30% bonds.. The size of the fund complicates the rebalancing process, as too frequent rebalancing may impact the market prices Norges Bank Investment Management (2018).

rebalancing strategy for a portfolio containing multiple risky assets can be approximated by a policy derived from a single-period model. This approach simplifies the complex problem of portfolio rebalancing with transaction costs. The numerical experiment is affected by numerous model parameter: (1) the initial distribution of asset weights (2) transaction costs, (3) length of investment, (4) asset volatility, and (5) correlation between assets. The findings of Zakamulin (2024) indicate that the Optimal strategy is superior compared to the Buy-and-hold and Naïve rebalancing strategy as evidenced by the Sharpe ratio.

In summary, several papers have addressed the problem of portfolio optimization, focusing on one risky asset, two risky assets, and multiple risky assets. However, a common feature in previous research is the inclusion of a risk-free asset; see, among others, Magill and Constantinides, (1976); Davis and Norman, (1990); Shreve and Soner, (1994). The inclusion of a risk-free asset is impractical, and does not reflect the behaviour of portfolio managers as they typically do not allocate any proportion of wealth to risk-free alternatives. Additionally, while some studies have begun to address portfolios with multiple risky assets, their practical implementation remains problematic; see, among others, Gârleanu and Pedersen, (2016); Ma et al., (2019). The main issues with these studies are: (1) the reliance on extremely complex numerical solutions that may not be feasible in real-world applications, (2) the continued inclusion of a risk-free asset, which does not reflect the actual investment strategies of institutional investors and (3) the presence of transaction costs present difficulties, as is precludes analytical solution when the number of risky assets increase.

Furthermore, the literature underscores the necessity for a practical model that addresses the inherent complexities of excluding a risk-free asset with multiple risky assets in the presence of transaction costs. This was partially addressed by Mei et al. (2016), suggesting that it is optimal to rebalance a portfolio only the first period, and thereafter the portfolio converge to a buy-and-hold. However, these findings contrasts with those of Dichtl et al. (2014) and Zakamulin (2024), who identified buy-and-hold as suboptimal. Additionally, study by Zakamulin (2024) identifies a method for optimally rebalancing a portfolio fully invested in multiple risky assets, considering transaction costs. According to their findings, the optimal rebalancing policy in a complex, multi-period setting can be derived through a simplified single-period model; effectively addressing the main issues identified in previous research. Furthermore, to the best of our knowledge, Zakamulin (2024) is the only study that produces significant results for a fully invested portfolios comprising of multiple risky assets with transaction costs, excluding the risk-free asset. The motivation for this thesis is therefore to clarify the complex field of portfolio optimization by implementing a model for multiple risky assets with target weights and transaction costs.

Chapter 3

Methodology

This section outlines the practical applications and computations of the chosen methods, including the software and algorithms that support the analysis. By explaining the intricacies of our approach, we aim to provide transparency and provide a deeper understanding of the methodology and the reliability of our conclusions.

3.1 Transaction Costs in Capital Markets

In financial modeling, transaction costs involve the integration of costs associated with purchasing and selling securities. Typically, these models assume that the capital market is friction-free and that a stock's shares are infinitely divisible. The latter means that an investor can invest by buying or selling $\triangle$ shares of a stock (see, for example (Zakamouline, 2008)). Trading costs encompasses the following: (1) bid-ask spread, (2) brokerage fee (comission) and (3) impact costs (Sun, Fan, Chen, Schouwenaars, & Albota, 2006). For simplicity, we distinguish between two classes of investors to model realistic transaction costs: large and small.

Large investors usually executes large trading volumes in blocks through block trading or the brokerage house. Typically, these investors encounter a transaction cost schedule that does not specify a minimum fee. Moreover, large investors often establish agreements with brokerage houses in advance, securing a commission rate that applies uniformly to any given trade size. Conversely, small investors typically engage in 100-share round lot transactions through retail brokerage firms,[1] who impose a minimum fee on trades. For small investors, the total transaction costs are structured to offer lower rates up to a specified trade level.

[1] 100-share round lots refer to a standard trading unit on stock exchanges. Historically, round lots have been the basis for standard trading practices, and buying and selling shares in round lots often results in more favorable pricing.

Impact cost is associated with the influence that large orders exert on market prices. This becomes problematic when an investor places an order that is substantial relative to the average daily trading volume of the shares. It is important to note that impact cost is closely related to liquidity: the extent to which a stock can be influenced correlates with how easy it can be bought or sold. Therefore, the impact cost decreases as the liquidity of the stock increases.

Although transaction costs are multifaceted in capital markets, we make a simplified assumption that transaction costs are proportional to the volume traded[2]. The impact cost is therefore neglected and we mainly focus on the bid-ask spread. Throughout the thesis, we have distinguished transaction costs associated with stocks and bonds. The cost impact on trading bonds are lower than for stocks, which is realistic to real-world application,

Assuming transaction costs are proportional to the value traded, where N_i represents the number of units and P_i represents the price per unit for asset i, we have that $V_i = N_i \times P_i$. Thus, you pay:

$$\text{Transaction Costs} = \sum_{i=1}^{n} \lambda_i V_i,$$

where λ_i denotes the transaction costs incurred for asset i.

3.1.1 Multi-Period Portfolio: Multiple Risky-Assets

In this section, we present a methodology for a general portfolio comprising of n-risky assets. Furtermore, we assume a fully invested portfolio, invested exclusively in risky assets.

3.1.2 Time 0: Initial Value of Portfolio

The weight of each asset is represented by $w_{i,0}$, the sum of these weights should equal 1 (i.e., 100% of the portfolio value is allocated across all assets). Hence,

$$\sum_{i=1}^{n} w_{i,0} = 1, \tag{1}$$

where, w_{i0} is the initial weight for asset i. Thus, the general formula that represents the initial value of the portfolio V_0 as the sum of the values allocated to each of the n risky assets is:

[2]Proportional trading costs refer to the costs that are expressed as a percentage of the volume traded during rebalancing.

$$V_0 = \sum_{i=1}^{n} w_{i,0} V_0, \tag{2}$$

It is reasonable to assume that in this period, we start with:

$$w_{i,0} = \hat{w}_i, \text{for} \quad i = 1, 2, \ldots, n$$

where $\hat{w}_i$ represents the target asset weight for asset i.

This assumption holds, because, at time 0 we start with the target asset weights.

3.1.3 Time 1: Portfolio Value before Rebalancing

Holding the portfolio over one period, until time 1. At period one the returns of the risky assets are $r_{i,1}$ for $i = 1, 2, \ldots, n$. The portfolio value before rebalancing is expressed as:

$$V_{1-} = \sum_{i=1}^{n} w_{i,0} V_0 (1 + r_{i,1}), \tag{3}$$

where the portfolio gross return before the rebalancing, by definition:

$$V_{1-} = V_0 (1 + r_{p1-}) \tag{4}$$

hence, we can derive this equation:

$$1 + r_{p1-} = \sum_{i=1}^{n} w_{i,0} (1 + r_{i,1}) \tag{5}$$

the new weights of the risky assets, before rebalancing, are calculated as follows:

$$w_{i,1} = \frac{w_{i,0}(1 + r_{i,1})}{\sum_{k=1}^{n} w_{k,0}(1 + r_{k,1})}, \quad \text{for} \quad i = 1, 2, \ldots, n \tag{6}$$

where k is an index running from 1 to n, representing all the risky assets in the portfolio.

The new portfolio value, before the rebalancing, can be written as:

$$V_{1-} = \sum_{i=1}^{n} w_{i,1} V_{1-} \tag{7}$$

3.1.4 Time 1: Portfolio Value after Rebalancing

We must decide whether to rebalance the portfolio, which depends on which rebalancing strategy is being employed. When rebalancing a portfolio of n-risky assets occurs the portfolio rebalances to the target weights $\hat{w}_i$. The formulas are valid even when $w_{i,1} = \hat{w}_i$. If rebalancing is chosen:

$$V_{1+} = \underbrace{\hat{w}_1 V_{1+}}_{\text{Value of asset 1}} + \underbrace{\hat{w}_2 V_{1+}}_{\text{Value of asset 2}} + \cdots + \underbrace{\hat{w}_n V_{1+}}_{\text{Value of asset } n}$$

$$V_{1+} = \sum_{i=1}^{n} \hat{w}_i V_{1+}, \tag{8}$$

where V_{1+} is the value of the portfolio after rebalancing, and $\hat{w}_i V_{1+}$ represents the value allocated to the i-th asset based on the target weight $\hat{w}_i$.

The new portfolio value after rebalancing, accounting for transaction costs, can be written as:

$$V_{1+} = V_{1-} - \sum_{i=1}^{n} |w_{i,1} V_{1-} - \hat{w}_i V_{1+}| \, \lambda_i \tag{9}$$

Before trading, the value of the i-th asset is $w_{i,1} V_{1-}$. After trading, the new value of the i-th asset is $\hat{w}_i V_{1+}$. The absolute difference $|w_{i,1} V_{1-} - \hat{w}_i V_{1+} V_{1+}|$ is then multiplied by the transaction cost rate of the i-th asset λ_i.

Thus, the transaction cost for the i-th asset is:

$$|w_{i,1} V_{1-} - \hat{w}_i V_{1+}| \, \lambda_i$$

To find V_{1+}, we must solve the equation using a numerical method or an analytical solution, such as by considering the sale and purchase of assets, where the value of certain assets decreases and others increase:

$$V_{1+} = V_{1-} - \sum_{i=1}^{n} (w_{i,1} V_{1-} - \hat{w}_i V_{1+}) \, \lambda_i \tag{10}$$

which gives:

$$V_{1+} = V_{1-} \left(\frac{1 - \sum_{i=1}^{n} w_{i,1} \lambda_i}{1 - \sum_{i=1}^{n} \hat{w}_i \lambda_i} \right) \tag{11}$$

To compute the return after rebalancing, we start with the following condition:

$$V_{1+} = V_0 (1 + r_{p1+}), \tag{12}$$

rearrange to solve for r_{p1+}, we get:

$$r_{p1+} = \frac{V_{1+}}{V_0} - 1, \tag{13}$$

3.1.5 Threshold & Range Rebalancing

To study threshold rebalancing in an investment portfolio, we developed an algorithm that computes the current weights of assets based on their returns, incorporating transaction costs. The algorithm uses a predefined no-trade threshold to check whether the asset weights deviate significantly from the target weights. If the deviation exceeds this threshold, the algorithm automatically adjusts the weights back to their targets.

The concept of a range rebalancing strategy entails initiating rebalance actions when the asset weights fall outside a defined range of tolerance. On such an occasion, the portfolio is adjusted back within a narrower band to maintain the intended asset allocation. Thus, after the first rebalancing is triggered, instead of adjusting the weights back to target weights, it is adjusted to target weigts $\pm$ the predefined range.

3.2 Single-Period Optimal Portfolio

In our investment model, we include n risky assets and a singular risk-free asset. The portfolio weights are represented by vector $\boldsymbol{w}$, which corresponds to the proportion of the portfolio's capital allocated to each risky asset. This model mandates a fully invested, long-only stance, implying that 100% of the available capital is apportioned across the risky assets, and the allocation strategy precludes short selling, hence the weights cannot be negative. The weight constraints, stated as:

$$\mathbf{1}_{\boldsymbol{n}}'\boldsymbol{w} = 1, \quad \mathbf{0}_{\boldsymbol{n}} \leq \boldsymbol{w} \leq \mathbf{1}_{\boldsymbol{n}},$$

where the prime symbol (') denotes the transpose of a vector. The vectors $\mathbf{1}_n$ and $\mathbf{0}_n$ are n-dimensional with all elements being 1 and 0, respectively.

Define $\boldsymbol{\mu}^e$ as the $n \times 1$ vector representing the average excess returns of the assets, and let $\boldsymbol{\Sigma}$ be the $n \times n$ matrix characterizing the variance and covariances. For a portfolio that is fully invested, its mean excess return, $\boldsymbol{\mu}^e$, and its variance, σ_p^2, are given by:

$$\mu_p^e = \boldsymbol{w}'\boldsymbol{\mu}^e, \quad \sigma_p^2 = \boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}, \tag{14}$$

It is assumed that the utility of the investor can be captured by a function of mean-variance, expressed as:

$$U(\boldsymbol{w}) = \boldsymbol{w}'\boldsymbol{\mu}^e - \frac{1}{2}\gamma\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}, \tag{15}$$

where in γ symbolizes the degree of the investor's aversion to risk.

In the presence of a risk-free asset, the ideal risky portfolio for an investor who adheres to mean-variance optimization is known as the tangency portfolio. Constructing the tangency portfolio, particularly under a long-only constraint which prohibits short-selling, entails a two-step process. Initially, one seeks to maximize the mean-variance utility of a portfolio comprising solely long positions in assets alongside the risk-free asset:

$$\max_{\boldsymbol{w}} \boldsymbol{w}'\boldsymbol{\mu}^e + r - \frac{1}{2}\gamma\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}, \tag{16}$$

where r represents the risk-free rate of return. The optimal weights, confined to long positions, are derived from the solution:

$$\boldsymbol{w} = \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^e. \tag{17}$$

In addition, it is necessary for the sum of the weights of the risky assets, which are restricted to be non-negative to reflect the long-only policy, to equal one. Under this constraint, the investor's risk aversion coefficient, tailored for investing solely in long positions of the risky assets, can be calculated as:

$$\gamma = \boldsymbol{1}_n'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^e, \tag{18}$$

accordingly, the weights of the tangency portfolio are determined by:

$$\boldsymbol{w}_{\text{tan}} = \frac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^e}{\boldsymbol{1}_n'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^e} \tag{19}$$

It is presupposed that the tangency portfolio is constructed exclusively with long positions in the assets.

Consider that the initial portfolio of the investor is characterized by a vector of weights $\boldsymbol{w}_0$ in $n \times 1$ dimensions, which is distinct from the weights of the tangency portfolio, $\boldsymbol{w}_{\text{tan}}$. These initial weights are bound by the following conditions:

$$\mathbf{1}_n' \boldsymbol{w}_0 = 1, \quad \mathbf{0}_n \leq \boldsymbol{w}_0 \leq \mathbf{1}_n. \tag{20}$$

Without considering transaction costs, aligning the portfolio with the tangency portfolio weights would be the straightforward objective. However, the introduction of transaction costs complicates this approach, rendering a direct alignment to $\boldsymbol{w}_{\text{tan}}$ potentially suboptimal. Under these circumstances, it is advantageous to determine a rebalancing strategy that maximizes the portfolio's mean-variance utility after accounting for transaction costs.

3.2.1 Optimal Rebalancing in the Presence of Transaction costs

Proportional transaction costs are incorporated and expressed as a percentage of the traded asset values during rebalancing. $\boldsymbol{\lambda}_i$ represents the transaction cost associated with the i-th asset. Hence, the expense incurred from adjusting the weight of asset i is denoted by (Zakamulin, 2024) :

$$|w_{i,0} - w_{i,1}| \, \lambda_i,$$

which in turn reduces the expected return of the portfolio.

The optimization challenge to enhance the mean-variance utility of the portfolio, inclusive of transaction costs, is structured as:

$$\max_{\boldsymbol{w}} \boldsymbol{w}' \boldsymbol{\mu}^e - |\boldsymbol{w}_0 - \boldsymbol{w}| \boldsymbol{\lambda} - \frac{1}{2} \gamma \boldsymbol{w}' \boldsymbol{\Sigma} \boldsymbol{w}, \tag{21}$$

$$\text{subject to} \quad \begin{cases} \mathbf{1}_n' \boldsymbol{w} = 1, \\ \mathbf{0}_n \leq \boldsymbol{w} \leq \mathbf{1}_n, \end{cases}$$

where $\boldsymbol{w}_0$ is the initial weights and $\boldsymbol{w}$ represents the optimal weights, $|\boldsymbol{w}_0 - \boldsymbol{w}|$ symbolizes the element-wise absolute difference between $\boldsymbol{w}_0$ and $\boldsymbol{w}$, and $\boldsymbol{\lambda}$ is the vector of transaction costs per asset. The objective is to navigate from the initial weights $\boldsymbol{w}_0$ to the optimal weights $\boldsymbol{w}$, maximizing utility whilst factoring in transaction costs.

Typically, problem (21) does not yield to an analytical solution and is not inherently suited to conventional solver techniques due to its quadratic programming (QP) nature. The coefficient of risk aversion γ tends to be an extraneous variable in this scenario. Nevertheless, the judicious choice of γ should be guided by the requirement that the solution aligns with the tangency portfolio weights in the limit as transaction costs diminish. This condition is fulfilled when γ is ascertained from equation (18).

The ideal rebalancing approach is the one that amplifies the Sharpe ratio of the portfolio subsequent to rebalancing. Hence, the initial optimization problem is posited as:

$$\max_{\boldsymbol{w}} \frac{\boldsymbol{w}'\boldsymbol{\mu}^e - |\boldsymbol{w}_0 - \boldsymbol{w}|\boldsymbol{\lambda}}{\sqrt{\boldsymbol{w}'\Sigma\boldsymbol{w}}}. \tag{22}$$

Similar to problem (21), this optimization scenario also trends towards the tangency portfolio weights when transaction costs are not factored in. Yet, when transaction costs come into play, it necessitates the use of numerical optimization methods. The complexity here stems from the fact that optimizing for the Sharpe ratio does not naturally fit into the framework of quadratic programming, as it often presents as a non-convex issue with the possibility of numerous local maxima.

Target asset weights need to match the weights of the tangency portfolio because the process of portfolio optimization typically starts by establishing the ideal asset allocation, such as a 60% investment in stocks and a 40% investment in bonds. In scenarios devoid of transaction costs, solving problem (21) is expected to result in these predefined target weights. Hence, it is essential for the target asset weights to be in line with the tangency portfolio's weights. This concept echoes the assumptions made by Black and Litterman (1992), who posited that the intended portfolio allocation should reflect the optimal solution to the portfolio selection challenge. From the viewpoint of a portfolio manager, achieving these target weights is tantamount to realizing an optimal investment strategy. Should these target weights diverge from the optimal, the portfolio manager would rebalance the portfolio to better fit the optimal portfolio model.

3.3 Analytical Solution: Optimal Portfolio with Two Risky Assets

Considering a portfolio consisting of two risky assets, where starting weights are $\boldsymbol{w}_{1,0}$ and $\boldsymbol{w}_{2,0}$. In order to maximize the mean-variance utility we must rebalance to new weights, represented by $\boldsymbol{w}_{1,0}$ and $\boldsymbol{w}_{2,0}$, under the assumption that our initial weights are not optimal. As previously discussed, we have a long-only portfolio. The equation below shows the excess mean return when considering transaction costs, and is after a rebalancing:

$$\mu_p^e = w_1\mu_1^e + w_2\mu_2^e - |w_{1,0} - w_1|\lambda_1 - |w_{2,0} - w_2|\lambda_2, \tag{23}$$

where, μ_1^e is excess mean return for asset 1 and μ_2^e is excess mean return for asset 2. The calculated standard deviation can be shown as:

$$\sigma_p^2 = w_1^2\sigma_1^2 + 2w_1w_2\sigma_{12} + w_2^2\sigma_2^2, \tag{24}$$

where, σ_p^1 and σ_p^2 represent standard deviations for asset 1 and asset 2, respectively. σ_{12} represents the covariance between returns.

The following optimization problem must be solved to find the optimal portfolio weights, after rebalancing:

$$\max_{w_1,w_2} w_1\mu_1^e + w_2\mu_2^e - |w_{1,0} - w_1|\lambda_1 - |w_{2,0} - w_2|\lambda_2 - \frac{1}{2}\gamma(w_1^2\sigma_1^2 + 2w_1w_2\sigma_{12} + w_2^2\sigma_2^2), \tag{25}$$

$$\text{subject to} \quad w_{1,0} + w_{2,0} = 1 \quad \text{and} \quad w_1 + w_2 = 1.$$

The maximization problem, originally with constraints, can be simplified to an unconstrained optimization problem with a single-variable, given our focus on a long-only portfolio. In order to do so, we substitute w_2 with $1 - w_1$. The change in the weight of one asset results in a change in the weight of the other asset, maintaining a fully invested portfolio. Thus, we arrive at an unconstrained optimization problem:

$$\max_{w_1} w_1\mu_1^e + (1 - w_1)\mu_2^e - |w_{1,0} - w_1|(\lambda_1 + \lambda_2) - \frac{1}{2}\gamma\left(w_1^2\sigma_1^2 + 2w_1(1 - w_1)\sigma_{12} + (1 - w_1)^2\sigma_2^2\right), \tag{26}$$

The first-order condition for maximization yields the following equation:

$$(\mu_{e1} - \mu_{e2}) + (\lambda_1 + \lambda_2) + \gamma w_1(\sigma_1^2 - 2\sigma_{12} + \sigma_2^2) - \gamma(\sigma_{12} - \sigma_2^2) = 0. \tag{27}$$

The solution with respect to w_1 yields:

$$w_1^{\text{sell}} = \frac{(\mu_e^1 - \mu_e^2) + \gamma(\sigma_2^2 - \sigma_{12})}{\gamma(\sigma_1^2 - 2\sigma_{12} + \sigma_2^2)} + \frac{\lambda_1 + \lambda_2}{\gamma(\sigma_1^2 - 2\sigma_{12} + \sigma_2^2)}, \tag{28}$$

In the case where it is optimal to buy the risky asset, our optimization problem becomes:

$$(\mu_{e1} - \mu_{e2}) - (\lambda_1 + \lambda_2) + \gamma w_1(\sigma_1^2 - 2\sigma_{12} + \sigma_2^2) - \gamma(\sigma_{12} - \sigma_2^2) = 0, \tag{29}$$

the optimal weights are:

$$w_1^{\text{buy}} = \frac{(\mu_e^1 - \mu_e^2) + \gamma(\sigma_2^2 - \sigma_{12})}{\gamma(\sigma_1^2 - 2\sigma_{12} + \sigma_2^2)} - \frac{\lambda_1 + \lambda_2}{\gamma(\sigma_1^2 - 2\sigma_{12} + \sigma_2^2)}. \tag{30}$$

When we do not face any transaction costs, equation (28) and (30) of the above solutions reduce to the weight's of the tangency portfolio. Thus, we conclude that:

$$w_1^{\text{tan}} = \frac{(\mu_{e1} - \mu_{e2}) + \gamma(\sigma_2^2 - \sigma_{12})}{\gamma(\sigma_1^2 - 2\sigma_{12} + \sigma_2^2)}, \tag{31}$$

where w_1^{sell} and w_1^{buy} determine the upper and lower bounds of the no-trade region, respectively. The method stipulates that transactions occur only when the new asset weights exceed the defined boundaries of the no-trade region. Zakamulin (2024) suggest that the dimensions of this region are influenced by several factors, with transaction costs being the most significant. As transaction costs rise, the size of the no-trade region typically decreases, since frequent rebalancing becomes financially prohibitive, thereby diminishing the portfolio's overall returns. Additionally, the no-trade region is symmetrically aligned with the weights of the tangency portfolio.

Investor risk tolerance also plays a crucial role in defining the size of the no-trade region; greater risk aversion leads to a smaller no-trade region. Similarly, during periods of high asset volatility, investors often rebalance more frequently to achieve optimal portfolio diversification, which also results in a smaller no-trade region. The correlation among assets influences the region's size as well: positive correlations lead to an expansion, while negative correlations result in contraction.

This occurs because of the substitution effect that positively correlated assets exhibit. Consequently, an investor is less likely to rebalance if one asset is underweight and another is overweight relative to the target portfolio. Lastly, the investment duration affects the no-trade region, with longer investment horizons typically allowing for a broader no-trade zone. This adjustment accommodates the greater potential for value fluctuations over extended periods, reducing the need for frequent rebalancing.

3.4 Numerical Solution: Optimal Portfolio with Multiple Risky Assets

To transform the non-linear optimization problem (21) into a form amenable to quadratic programming (QP), Scherer (2011) and Richard and Roncalli (2015) recommend the addition of extra variables. For each asset i, they define increments Δw_i^+ and Δw_i^-, which represent the positive and negative changes in weights, respectively, such that the new weight w_i is the initial weight $w_{i,0}$ adjusted by these increments:

$$w_i = w_{i,0} + \Delta w_i^+ - \Delta w_i^-, \tag{32}$$

In this notation, $\Delta w_i^- = \max(w_{i,0} - w_i, 0)$ captures a decrease in the asset weight, while $\Delta w_i^+ = \max(w_i - w_{i,0}, 0)$ captures an increase.

Incorporating these adjustments, problem (21) can be reformulated as follows:

$$\max_{\boldsymbol{w},\boldsymbol{\Delta w}^-,\boldsymbol{\Delta w}^+} \boldsymbol{w}'\boldsymbol{\mu}^e - (\boldsymbol{\Delta w}^- + \boldsymbol{\Delta w}^+)'\boldsymbol{\lambda} - \frac{1}{2}\gamma\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} \tag{33}$$

$$\text{subject to} \begin{cases} \mathbf{1}_n'\boldsymbol{w} = 1, \\ -\boldsymbol{c_n} \leq \boldsymbol{w} \leq \mathbf{1}_n + \boldsymbol{c_n}, \\ \boldsymbol{w} + \boldsymbol{\Delta w}^- - \boldsymbol{\Delta w}^+ = \boldsymbol{w_0}, \\ \mathbf{0}_n \leq \boldsymbol{\Delta w}^- \leq \boldsymbol{w_0}, \\ \mathbf{0}_n \leq \boldsymbol{\Delta w}^+ \leq \mathbf{1}_n - \boldsymbol{w_0}, \end{cases}$$

The first constraints of the optimization problem ensure that the portfolio remains fully invested, with no cash holdings at any time. The second constraint prohibits short-sales. Where the c_n represent a constant that is non-negative, meaning that the weight of an asset cannot be, for example, -10% or 110%. The third constraint ensures that after any purchases $\boldsymbol{\Delta w}^+$ or sales $\boldsymbol{\Delta w}^-$

of assets, the resulting weights will align with the initial weights w_0. The third constraint aims to prevent short selling by ensuring all asset weights are non-negative. The fourth set of constraints guarantees that the weights of assets after selling $\boldsymbol{\Delta w}^-$ do not fall below their initial values $\boldsymbol{w_0}$, while the fifth set ensures that the weights after buying $\boldsymbol{\Delta w}^+$ do not exceed the maximum allowable proportion, which is one minus the initial weights. This maintains the predetermined balance of the portfolio within its original bounds.

This results in the non-linear optimization problem (21) being expanded into an augmented QP problem of dimension $3n$, using the variable

$$\boldsymbol{x} = \begin{pmatrix} \boldsymbol{w} \\ \boldsymbol{\Delta w}^- \\ \boldsymbol{\Delta w}^+. \end{pmatrix}. \tag{34}$$

The augmented QP problem is then expressed as[3]:

$$\min_{\boldsymbol{x}} \frac{1}{2}\boldsymbol{x}'\boldsymbol{D}\boldsymbol{x} - \boldsymbol{d}'\boldsymbol{x}, \tag{35}$$

subject to the linear constraints

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \quad \boldsymbol{x}^- \leq \boldsymbol{x} \leq \boldsymbol{x}^+, \tag{36}$$

where

$$\boldsymbol{d} = \frac{1}{\gamma}\begin{pmatrix} \boldsymbol{\mu}^e \\ -\boldsymbol{\lambda} \\ -\boldsymbol{\lambda} \end{pmatrix}, \quad \boldsymbol{D} = \begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{0}_{n,n} & \mathbf{0}_{n,n} \\ \mathbf{0}_{n,n} & \mathbf{0}_{n,n} & \mathbf{0}_{n,n} \\ \mathbf{0}_{n,n} & \mathbf{0}_{n,n} & \mathbf{0}_{n,n} \end{pmatrix},$$

where $\mathbf{0}_{n,n}$ is an $n \times n$ matrix with all elements equal to zero. Also, the equality constraints are specified by matrix A and vector b:

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1}_n' & \mathbf{0}_n & \mathbf{0}_n \\ \boldsymbol{I}_{n,n} & \boldsymbol{I}_{n,n} & -\boldsymbol{I}_{n,n} \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 1 \\ \boldsymbol{w}_0 \end{pmatrix},$$

[3]To solve the augmented QP problem, we use the R package Quadprog

whereas the inequality constraints are:

$$\boldsymbol{x}^- = \begin{pmatrix} -\boldsymbol{c}_n \\ \boldsymbol{0}_n \\ \boldsymbol{0}_n \end{pmatrix}, \quad \boldsymbol{x}^+ = \begin{pmatrix} \boldsymbol{1}_n + \boldsymbol{c}_n \\ \boldsymbol{w}_0 \\ \boldsymbol{1}_n - \boldsymbol{w}_0 \end{pmatrix},$$

In total, the extended QP problem is structured with $1+n$ equality constraints and $6n$ inequality constraints. The post-rebalancing optimal portfolio weights are delineated by the first n components of the vector $\boldsymbol{x}$.

With only two risky assets, symmetry gives rise to two non-trading zones, each linked to one of the assets. Yet, with more than two risky assets, the distinction of non-trading zones for each becomes less defined. There are circumstances where rebalancing the portfolio is unnecessary, and others where it is deemed essential. Scenarios may also arise where engaging in transactions across all risky assets is advantageous, while some situations may call for selective trading within the asset pool.

The numerical analysis suggests that the optimal transaction strategy is affected by both the initial distribution of asset weights and a spectrum of model parameters Zakamulin (2024). These factors include transaction costs, the volatility of assets, asset correlations, and the investment horizon's length. For instance, if model parameters are held constant, the optimal transaction policy is influenced by the initial asset weights. A greater divergence from their target values tends to prompt a stronger rebalancing action towards those targets. Efforts to maintain asset weights close to their target benchmarks are influenced by variables such as reducing transaction costs, increasing asset volatilities, lowering correlation coefficients towards negative values, and extending the investment period.

3.5 Statistical Estimations & Tests

3.5.1 Performance Measurement

The Sharpe ratio is essential in portfolio optimization for evaluating the performance of portfolios. It serves as an investment tool that enables investors to make informed decisions. The Sharpe ratio is calculated by:

$$SR = \frac{r_p - r_f}{\sigma_p}, \tag{37}$$

where r_p is the return of the portfolio,r_f is the risk-free return and the σ_p is the standard deviation of the portfolio. The null hypothesis for the Sharpe ratio test is:

$$H_0 : SR_R = SR_{BH}$$
$$H_A : SR_R > SR_{BH}$$

where SR_R represents the different rebalancing strategies and SR_{BH} represents the benchmark, Buy-and-hold. The null and alternative hypothesis is tested at a 5% significance level against the alternative hypothesis. The test statistics is given by the following formula:

$$z = \frac{SR_R - SR_{BH}}{\sqrt{\frac{1}{N}\left[2(1-\rho) + \frac{1}{2}(SR_R^2 + SR_{BH}^2) - 2\rho^2 SR_R SR_{BH}\right]}}, \tag{38}$$

where SR_R and SR_{BH} represent estimated Sharpe ratios for the portfolio being analysed and the Sharpe ratio for the buy and hold strategy, respectively. ρ is the correlation coefficient between returns. Under the null hypothesis, the distribution of z converges to a standard normal distribution as the sample size approaches infinity.

3.5.2 Sample Variance-Covariance Matrix

In our optimization problem, we employ a variance-covariance matrix, denoted as $\mathbf{\Sigma}$, derived from a specified historical period spanning January 1960 to December 1979. This matrix, once computed, remains constant and is applied throughout our analysis to assess data from 1980 to

2022. The variance covariance matrix is represented as:

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{pmatrix} \tag{39}$$

The matrix serves to quantify the variance of returns for each of the n assets under consideration and to elucidate the covariances, which represent how asset returns move in relation to one another. The diagonal elements, σ_{ii}, represent the variance of the returns of asset i. Variance is a measure of the spread of the returns around the mean, calculated as:

$$\sigma_{ii}^2 = \frac{1}{T-1}\sum_{t=1}^{T}(r_{it} - \bar{r}_i)^2, \tag{40}$$

where r_{it} is the return of asset i at time t, and $\bar{r}_i$ is the average return of asset i over T periods. The off-diagonal elements, σ_{ij} (where $i \neq j$), represent the covariance between the returns of asset i and asset j. Covariance measures how the returns of two assets move together, calculated as:

$$\sigma_{ij}^2 = \frac{1}{T-1}\sum_{t=1}^{T}(r_{it} - \bar{r}_i)(r_{jt} - \bar{r}_j), \tag{41}$$

3.5.3 Shrinkage Estimation

In the context of covariance matrix estimation, the shrinkage methodology offers a robust alternative to the direct use of the sample covariance matrix $\boldsymbol{S}$. The shrinkage method can be represented as:

$$\hat{\boldsymbol{\Sigma}}_{\text{Shrink}} = \delta^* F + (1 - \delta^*)\boldsymbol{S} \tag{42}$$

where, F typically represents a more stable but less flexible estimator, such as a diagonal matrix or the identity matrix scaled by the average variance across features, and $\boldsymbol{S}$ is the covariance matrix derived from the data. The coefficient δ^* is optimally chosen to minimize the mean squared error between the shrinkage estimator and the true covariance matrix. This methodology not only reduces the estimation error but also improves the conditioning of the covariance matrix, which is crucial for subsequent analyses such as portfolio optimization (Ledoit & Wolf, 2003).

3.5.4 Minimum Covariance Method

The minimum covariance method (MCD) is an estimator that provides a robust multivariate location and scatter. This robust method, introduced by Rousseeuw and Driessen (1999), is designed to address outliers and handle data that violates the assumption of normality. The MCD method aims to find the subset of the data whose covariance matrix has the lowest possible determinant. The calcuation of the MCD method can be shown (Hubert & Debruyne, 2010):

$$\hat{\boldsymbol{\mu}}_{\text{MCD}} = \frac{\sum_{i=1}^{n} W(d_i^2)\mathbf{x}_i}{\sum_{i=1}^{n} W(d_i^2)} \tag{43}$$

where, $\hat{\boldsymbol{\mu}}_{\text{MCD}}$ represents the mean vector estimated by the MCD method. $\mathbf{x}_i$, is the i-th observation vector in the dataset. $W(d_i^2)$, is a weight function applied to the squared Mahalanobis distances d_i^2. $\sum_{i=1}^{n}$ denotes the summation over all n observations.

Hence the MCD is shown as:

$$\hat{\Sigma}_{\text{MCD}} = c_1 \frac{1}{n} \sum_{i=1}^{n} W(d_i^2)(\mathbf{x}_i - \hat{\boldsymbol{\mu}}_{\text{MCD}})(\mathbf{x}_i - \hat{\boldsymbol{\mu}}_{\text{MCD}})', \tag{44}$$

where, $\hat{\Sigma}_{\text{MCD}}$ represents the covariance matrix estimated by the MCD method. c_1 is a consistency factor used to make the covariance matrix estimator consistent with the assumed multivariate normal distribution. n is the number of observations. $(\mathbf{x}_i - \hat{\boldsymbol{\mu}}_{\text{MCD}})$ represents the deviation of the i-th observation from the robust mean vector.

3.5.5 Maximum Likelihood Estimation

The maximum likelihood estimation (MLE) is also a robust method to estimate the covariance matrix. The method assumes that the data comes from a multivariate t-distribution. This approach is particularly useful for handling outliers and heavy-tailed distributions, which are common in financial data and other real-world datasets. The MLE for the multivariate t-distribution can be expressed as follows. Given a sample of n observations, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, the log-likelihood function for the multivariate t-distribution is (C. Liu, 1997):

$$\begin{aligned} L(\theta) = &-\frac{n(\nu+p)}{2}\ln(a(\theta)) - \frac{n}{2}\ln|\Psi| \\ &- \frac{1}{2a(\theta)}\text{trace}\left[\Psi^{-1}\sum_{i=1}^{n} w_i(Y_i-\mu)(Y_i-\mu)'\right] \\ &- \frac{\nu}{2a(\theta)}\sum_{i=1}^{n} w_i, \end{aligned} \tag{45}$$

where, $\boldsymbol{\mu}$ is the mean vector, $\boldsymbol{\Psi}$ is the scale matrix, ν is the degrees of freedom, $a(\theta)$ is a function of the parameters, w_i are weights, Y_i are the observations.

The robust MLE method iteratively estimates the parameters $\boldsymbol{\mu}$, $\boldsymbol{\Psi}$, and ν by maximizing the log-likelihood function $L(\theta)$. This iterative process ensures that the estimates are less influenced by outliers and more representative of the underlying data distribution. Additionally, the robustness of this method is enhanced by the t-distribution's ability to model heavier tails compared to the normal distribution. This feature allows the robust MLE to provide more accurate and reliable estimates of the covariance matrix in the presence of extreme values or deviations from normality. For more details on the robust MLE method and its applications, refer to (Kent, Tyler, & Vard, 1994).

3.5.6 Portfolio Turnover & Tracking Error

Turnover in portfolio management measures the frequency of trading activities, specifically the buying and selling of assets within a portfolio. This metric is crucial as it serves as an indicator of the degree of rebalancing that occurs within a portfolio. High turnover is asscociated with higher transaction costs affecting the overall return of the portfolio. The formula provided calculates the average turnover at time t. The equation is defined as (Kirby & Ostdiek, 2012):

$$\text{Turnover} = \frac{1}{N}\sum_{i=1}^{N}\left(|w_{i,t} - \hat{w}_{i,t}| + \left|\sum_{i=1}^{N}(w_{i,t} - \hat{w}_{i,t})\right|\right), \tag{46}$$

where N is the total number of observations in the period. $\sum_{i=1}^{N}$ this summation symbol indicates that the calculation is performed over all periods. $w_{i,t}$ is the weight of asset i in the portfolio at time t. $\hat{w}_{i,t}$ is the target weight of asset i in the portfolio at time t. $|w_{i,t} - \hat{w}_{i,t}|$ is the absolute difference between the actual weight and the target weight of asset i at time t. This term measures how much the weight of asset i deviates from its target weight.

Tracking error is calculated as the absolute difference between the return of the portfolio, the formula can be shown:

$$\text{Tracking Error} = \frac{1}{N}\sum_{i=1}^{N}|r_{\text{p},t} - r_{\text{b},t}|, \tag{47}$$

where $r_{p,t}$ represents the portfolio value for the specified rebalancing strategy and $r_{b,t}$ denotes the portfolio value of the benchmark. A low tracking error indicates that the portfolio closely matches the benchmark's returns, which is typically desired in passive management strategies.

3.5.7 Mean Excess Return

Ensuring that the target asset weights are in line with the tangency portfolio's weights can be achieved through estimation or specification of the variance-covariance matrix $\boldsymbol{\Sigma}$. Subsequently, the asset excess mean returns vector $\boldsymbol{\mu}^e$ is approximated using the investor's risk aversion coefficient. This vector is often denoted as the "implied" excess mean returns or implied risk premia, influenced by both the target asset weights and the variance-covariance matrix Black and Litterman (1992).

Let γ denote risk aversion coefficient, and $\boldsymbol{\Sigma}$ stand for the $n \times n$ variance-covariance matrix. The excess mean return, μ_p^e of a fully invested portfolio is calculated as:

$$\boldsymbol{\mu}^e = \gamma \boldsymbol{\Sigma} \hat{\boldsymbol{w}}, \tag{48}$$

where $\hat{\boldsymbol{w}}$ is the $n \times 1$ vector of portfolio target asset weights. We have the target asset weights and the variance-covariance matrix, but we need to derive an estimate of the risk aversion coefficient.

3.5.8 The Risk Aversion

The degree of risk aversion plays a pivotal role in determining the sensitivity of portfolio rebalancing to fluctuations in market conditions and transaction costs. Given that most investors exhibit risk-averse behavior, an increase in expected returns accompanied by a corresponding rise in portfolio risk complicates the decision-making process for portfolio selection. Thus, the portfolio choice is not straightforward, as it necessitates a careful evaluation of the trade-off between higher potential returns and the increased risk involved.

The investor's capital allocation between the risky and the risk-free assets consists in investing proportion y of his wealth in the risky asset such that the mean return and the standard deviation of returns are:

$$E[r_c] = y(\mu - r_f) + r_f, \quad \sigma_c = y\sigma, \tag{49}$$

where the $\sigma_c = y\sigma$ is denoted as standard deviation of returns. The expected return of the portfolio, $E[rc]$, is a function of the proportion invested in the risky asset y, the expected return of the risky asset μ, and the risk-free rate r_f.

We solve the following utility maximization problem to find the optimal proportion y:

$$\max_{y} U(r_c) = E[r_c] - \frac{1}{2}\gamma\sigma_c^2 = y(\mu - r_f) + r_f - \frac{1}{2}\gamma y^2\sigma^2. \tag{50}$$

The first-order condition for optimality of y:

$$\frac{dU(r_c)}{dy} = (\mu - r_f) - \gamma y^* \sigma^2 = 0. \tag{51}$$

The solution with respect to the optimal y^* yields

$$y^* = \frac{1}{\gamma}\frac{\mu - r_f}{\sigma^2}. \tag{52}$$

Since we allocate all wealth to risky assets, we know that $y = 1$, therefore, we rearrange and solve for γ:

$$\gamma = \frac{E[r] - r_f}{\sigma^2}, \tag{53}$$

where $E[r]$ and σ^2 denote the target portfolio mean return and variance, respectively. γ denotes the risk aversion coefficient.

A risk-neutral investor is according to Bodie, Kane, and Marcus (2014) one where the risk aversion coefficient γ = 0, whereas for a risk lover γ <0. Under the assumption that investors are risk averse, the chosen risk aversion coefficient used in the optimization function is 5. Which is an estimate ensuring that the investor's optimal portfolio matches the tangency portfolio, when transaction costs are absent, as further elaborated in the methodology section.

3.5.9 Out-of-sample Portfolio Simulation

To evaluate the robustness and practical applicability of our portfolio optimization strategies, we conduct out-of-sample testing using data from a period distinct from that used to derive our initial estimations. Specifically, we use the variance-covariance matrix derived from historical data spanning January 1960 to December 1979 for our in-sample analysis. The matrix is then applied to a subsequent period from 1980 to 2022 to assess the performance of our strategies out-of-sample. Using the constant $\mathbf{\Sigma}$, we evaluate the performance of various portfolio rebalancing strategies over the out-of-sample period.

For the purposes of this research, the variance-covariance matrix is held constant to avoid "look-ahead bias" and to simplify the modeling process. This approach assumes that market dynamics remain stable over time, which allows for a clearer evaluation of the historical performance of various investment strategies without the retrospective use of information that would not have been available at the time. This bias can lead to estimations that are not applicable in real-world practices. The decision to use a constant variance-covariance matrix, however, comes with the acknowledgment that it does not account for dynamic market conditions. Dynamic market forces and structural economic shifts are outside the purview of this model, and their exclusion is a trade-off for maintaining analytical simplicity and consistency.

Chapter 4

Data

Our research utilizes monthly return data sourced from a two stock and two bond indices, along with the risk-free rate. The data is extracted from the Ibbotson SBBI 2023 Yearbook, spanning from January 1960 through December 2022, consisting of 756 monthly observations [1]. Specifically, the stock indices encompass portfolios of large-cap and small-cap stocks, whereas the bond indices correspond to long-term and intermediate-term government bond portfolios. We employ the average one-month Treasury Bills as a proxy for the risk-free rate of return.

4.1 Descriptive Statistics

Table 4.1 reports the descriptive statistics, revealing that small stocks offer the greatest potential reward, reflected in their highest mean returns. However, they also carry the highest risk, as suggested by the standard deviation. Contrastingly, Treasury bills and long-term bonds exhibit lower standard deviation, with their standard deviations being the lowest among the assets analyzed, underscoring their stability in comparison to the highly volatile nature of small stocks.

Skewness and kurtosis further describe the distribution characteristics of these assets. Large and small stocks exhibit negative skewness, which indicates a distribution with a tail skewed towards lower returns, and positive excess kurtosis, suggesting a leptokurtic distribution characterized by pronounced heavy tails compared to the normal distribution. Treasury bills, with their positive skewness and lower kurtosis, suggest a distribution leaning towards higher returns and less propensity for extreme outcomes. The minimum and maximum values captures the range of returns,

[1]The Ibbotson SBBI 2023 Yearbook provides data covering the period from 1926 to 2022. The decision to limit the duration of the analysis is primarily driven by practical considerations. As the time horizon extends, the Buy-and-Hold strategy would progressively lead to a portfolio dominated exclusively by stocks.

	Lareg Stocks	Small Stocks	Long-term Bonds	Inter-term Bonds	TBills
Monthly					
Mean	1.0	1.1	0.7	0.5	0.3
St.dev	4.4	5.9	3.4	1.6	0.3
Skewness	-0.578	-0.487	0.285	0.687	0.913
Kurtosis	4.820	5.694	4.676	8.855	3.650
Min	-21.5	-29.2	-11.2	-6.4	0.0
Max	13.5	23.6	15.2	12.0	1.3
Quarterly					
Mean	3.1	3.6	2.1	1.6	1.0
St.dev	8.2	11.8	6.6	3.3	0.9
Skewness	-0.612	-0.177	0.576	0.900	0.836
Kurtosis	3.776	3.529	4.142	5.715	3.323
Min	-22.5	-31.7	-14.5	-6.4	0.0
Max	21.6	34.1	24.4	16.6	3.8
Yearly					
Mean	12.9	13.9	8.6	6.6	4.0
St.dev	16.9	20.5	13.7	7.5	3.6
Skewness	-0.840	-0.246	-0.130	0.280	0.839
Kurtosis	3.393	2.798	3.361	3.890	3.255
Min	-37.0	-36.7	-29.6	-12.4	0.0
Max	37.6	60.7	40.4	29.1	14.7

Table 4.1: Summary of descriptive statistics for four risky assets and the risk-free asset. Segmented into monthly, quarterly and yearly.

with small stocks again demonstrating the broadest range, hence the highest volatility. This range narrows considerably for Treasury bills, which maintain returns closer to their mean.

Figure 4.1 displays the cumulative log-returns of four risky assets alongside the Risk-free. This time-series plot reveals that initially, the returns of all assets are relatively close; however, as time progresses, the disparities in their performance become clear.

Upon closer analysis, it is evident that small-cap stocks yield the highest returns, followed closely by large-cap stocks, which is consistent with historical risk-return profiles of these asset classes. On the other hand, Treasury bills exhibit the lowest returns. Intermediate-term bonds generally outperform long-term bonds.

Table 4.2 displays the correlation matrix for both risky assets and a risk-free, revealing significant patterns in asset movements. Notably, there is a strong positive correlation between large-cap and small-cap stocks as well as between intermediate-term and long-term bonds. This positive correlation suggests that these assets typically move together, likely influenced by overlapping

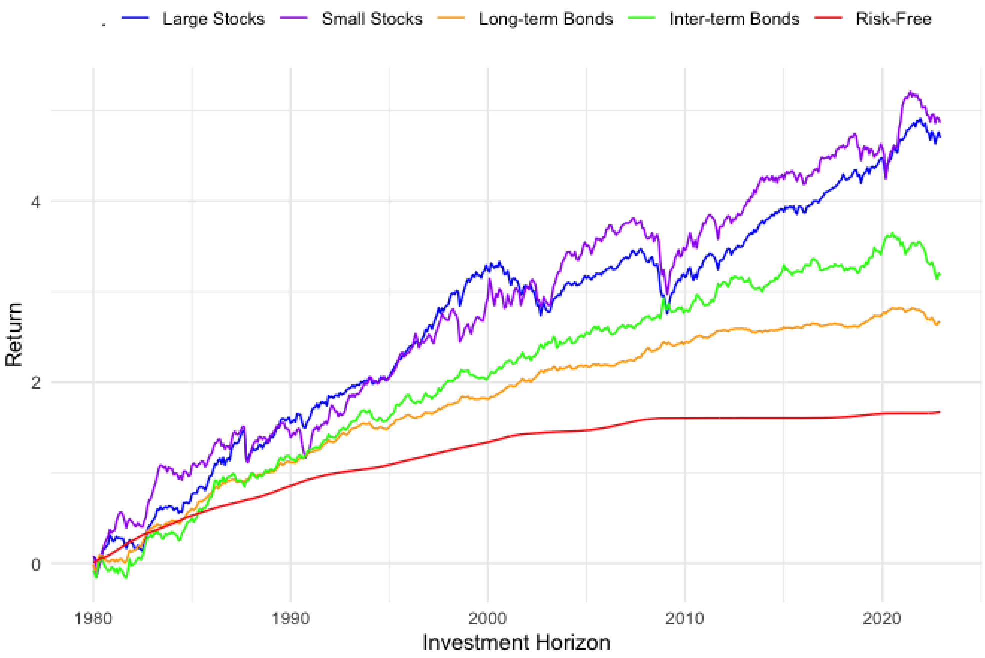

Figure 4.1: Cumulative logarithmic returns of $1 Invested in four risky assets and a risk-free asset, 1980–2022. The use of log returns facilitates the comparison of return dynamics over time.

	Large Stocks	Small Stocks	Long-term Bonds	Inter-term Bonds	TBills
Large Stocks	1.0000	0.7481	0.0899	0.1047	0.1102
Small Stocks	0.7481	1.0000	-0.0506	0.0197	0.0441
Long-term Bonds	0.0899	-0.0506	1.0000	0.8813	0.1438
Inter-term Bonds	0.1047	0.0197	0.8813	1.0000	0.2138
TBills	0.1102	0.0441	0.1438	0.2138	1.0000

Table 4.2: The correlation coefficients displaying correlations for four risky asset and the Risk-Free asset from 1980 - 2022.

market dynamics or similar investor behaviors. Conversely, there is almost zero correlation between both large stocks and small stocks with long-term bonds and intermediate-term bonds. This suggests no co-movement, such a relationship reflect common economic factors that influence both stocks and bonds. The negligible correlation could signify that in times of market stress, equities and bonds do not necessarily react in unison, possibly reflecting a flight-to-quality effect where investors pivot towards safer assets like bonds when stock prices are volatile.

Chapter 5

Empirical Analysis

In this section the empirical results are presented. Our analysis covers two portfolios, the first portfolio allocating 60% to large stocks and 40% to long-term bonds. The second portfolio maintains the same allocation to stocks and bonds, but diversifies further by including small stocks and intermediate-term bonds. Consequently, the allocation for the second portfolio is 30% to large stocks, 30% to small stocks, 20% to intermediate-term bonds, and 20% to long-term bonds[1]. We utilize four distinct investment strategies for managing the portfolio that contains two risky assets: Buy-and-Hold, Naïve, Threshold, Range. Among these, Buy-and-Hold and the Naïve strategies will serve as benchmarks. In contrast, for the portfolio comprising four risky assets, three strategies is utilized: Buy-and-Hold, Naïve, and Optimal.

The Buy-and-Hold benchmark portfolio follows a passive strategy, where investments are made initially and left unchanged for the given periods. The second benchmark portfolio adopts a "Naïve" rebalancing strategy, which regularly adjusts to maintain target asset weights. The Threshold strategy rebalances when the assets weight deviates beyond a predefined threshold, ensuring that the portfolio maintains its target allocation. The Range strategy triggers rebalancing when an assets weight crosses specified boundaries, aiming for adjustments back to either an upper or lower limit, depending on the direction of the deviation.

Furthermore, it is neccessary to emphasize that when managing a portfolio with two risky assets, the Range strategy employs an arbitrary no-trade band. This band represents a predetermined threshold that dictates when no rebalancing should occur. In principle, the Range strategy is built

[1]Since 2017, the Norwegian Pension Fund has adopted a 70/30 asset allocation strategy (Norges Bank Investment Management, 2018). As Norges Bank is positioned as a long-term investor, it allocates a larger portion to stocks. In contrast, our decision to utilize the traditional 60/40 portfolio is driven by our more conservative risk tolerance and a shorter investment horizon, which typically warrants a higher proportion of bonds.

upon the same logic as the optimal strategy in the case of two risky assets. In the context of managing a portfolio comprising four risky assets, the optimal rebalancing strategy is not based on arbitrary parameters. Instead, it systematically adjusts asset weights to enhance expected utility, accounting for the investors risk-aversion (γ), variance-covariance matrix, excess return, and transaction costs. The optimal strategy involves solving a quadratic programming problem, which efficiently optimize the weights based on the parameters of the model.

To determine the optimal portfolio rebalancing strategy, the methodology necessitates an estimation of the risk averison coefficient (γ). In this study, (γ) is calculated to be 5.20 based on the optimal fraction invested in the risky portfolio, which is 1 in our case. The derivation of the estimate is based upon the logic that in the absence of transaction costs, the optimal portfolio and the tangency portfolio aligns. For simplicity, (γ) is set to a constant value of 5. Using this (γ) value, we then employ equation (18) to find the vector of excess mean returns. Rebalancing is executed when the optimal weights, as determined by this model, differ sufficiently from the current weights, considering the trade-off between potential performance gains and the costs of rebalancing.

To configure the portfolio in the case of four risky assets, we utilize the long-term historical data spanning from 1960 to 1979 to estimate the variance-covariance matrix of portfolio returns. Subsequently, the estimated variance-covariance matrix is utilized to numerically compute the implied asset mean excess returns to achieve the desired asset weight for the portfolio. Each portfolio is managed under the assumption of strategic asset allocation, where target asset weights are established based on long-term forecasts for each asset class.

Furthermore, the simulation of the rebalancing strategies spans from the historical period of January 1980 to December 2022. The analysis incorporates realistic transaction cost of 25 bps and 10 bps for stocks and bonds, respectively[2]. To demonstrate the effectiveness of the rebalancing strategies, we test them against the benchmark using the Sharpe ratio as the performance measure. Our analysis has chosen the $\pm 3\%$ no-trade band width which is has been commonly applied by large institutions such as, the Norwegian Government Pension Fund Global has implemented a no-trade band of $\pm 3\%$ as indicated by Ministry of Finance, Norway (2021).

Table 5.1 presents the results from the historical simulation in the case of two risky asset.

[2]The chosen transaction costs are based on the work of Zakamulin (2024), who applied the same transaction costs in his baseline study. However, Dichtl et al. (2014), employed transaction costs of 10 basis points for stocks and 5 basis points for bonds, which, in our assessment, were insufficiently low.

Portfolio of 2 risky assets	BAH	Naïve	Threshold	Range
Mean	10.80	10.49	10.57	10.70
Std. deviation	11.87	10.53	10.53	10.54
Sharpe ratio	0.580	0.626	0.633	0.639
P-value		0.059	0.043	0.039
Avg turnover (%)	0.000	1.911	0.493	0.219

Table 5.1: Descriptive statistics for portfolio of two risky assets. The mean and standard deviation are annualized in percentages. The p-value is gathered by testing the null hypothesis against that the rebalancing strategies is equal to the Buy-and-Hold (BAH) strategy. Threshold and Range uses the $\pm 3\%$ No Trade-band.

Illustrating that the Buy-and-Hold strategy, when assessed using the Sharpe ratio, consistently exhibits statistically significant lower performance. However, the Buy-and-Hold reveal higher mean returns. This phenomenon can be attributed to the Buy-and-Hold strategy's tendency to shift towards a higher allocation of stocks over time. Nevertheless, it is important to note that the primary objective of a rebalancing strategy is to maximize mean-variance utility to asset allocation (Zilbering et al., 2015). In this context, the findings underscore the effectiveness of the strategy without presenting any contradictions to its intended purpose. Another key finding is that the Range strategy is the superior strategy in the case of two risky asset, attaining the highest Sharpe ratio[3]. Interestingly, it is also noted that the Range strategy exhibits the lowest average turnover compared to other rebalancing strategies, suggesting a more stable, cost-efficient approach that aligns well with long-term investment objectives.

NT Band	Rebalancing	Sharpe	P-value	Avg. TO, %	Avg. TE, %
1%	Threshold	0.626	0.069	0.230	0.010
1%	Range	0.630	0.053	0.395	0.029
3%	Threshold	0.633	0.042	0.493	0.040
3%	Range	0.633	0.039	0.219	0.077
5%	Threshold	0.629	0.058	0.355	0.080
5%	Range	0.629	0.042	0.115	0.114
7%	Threshold	0.639	0.044	0.361	0.095
7%	Range	0.627	0.039	0.082	0.161

Table 5.2: Impact of no-trade band width on Threshold and Range rebalancing strategies for portfolio of two risky assets: A comparison of Sharpe ratios, p-values, and annualized metrics for average turnover (Avg TO) and tracking error (Avg TE) from 1% to 7% Bands for the traditional 60/40 portfolio. $\lambda_S = 0.25\%$ $\lambda_B = 0.10\%$.

[3]It is essential to emphasize that the Range strategy = Optimal strategy in the case of two risky assets. The Range strategy utilizes an arbitrary no-trade region, allowing for asset weights to fluctuate within a predefined range before rebalancing is triggered for two portfolio of two risky assets. In contrast, for portfolios with four risky assets, the Optimal strategy is derived through numerical solutions that determine the "no-trade region" under which rebalancing should occur.

Table 5.2 presents the findings related to changes in the no-trade region, and how it affects the Sharpe ratio, turnover, and tracking error. The motivation for varying the width of the no-trade band is to understand its impact on the performance and efficiency of different rebalancing strategies. By expanding the NT band, we aim to explore whether reducing the frequency of trades can enhance overall portfolio performance without significantly compromising the alignment with targeted benchmarks. According to Table 5.2, expanding the width of the no-trade band causes an increase in the average tracking error for both Threshold and Range rebalancing strategies, while also leading to a decrease in the p-value. This suggests that broader no-trade bands may reduce the frequency of trades but at the cost of the utility in tracking the targeted benchmarks.

Additionally, varying the no-trade band minimally affects the Sharpe ratio. The Sharpe ratio for the Range strategy maintains a consistent level, whereas the Threshold strategy shows a slight increase in Sharpe ratio as the NT bandwidths widen. For instance, the Sharpe ratio for the Threshold strategy varies within a narrow range from 0.626 to 0.639, while for the Range strategy, it varies from 0.627 to 0.630. These findings imply that enlarging the NT band could be more advantageous for Threshold strategy, as opposed to the Range strategy. Furthermore, figure 5.1 illustrates the impact of widening the NT band on rebalancing strategies, demonstrating a decrease in average turnover for Range rebalancing and an increase for Threshold rebalancing as the NT band widens.

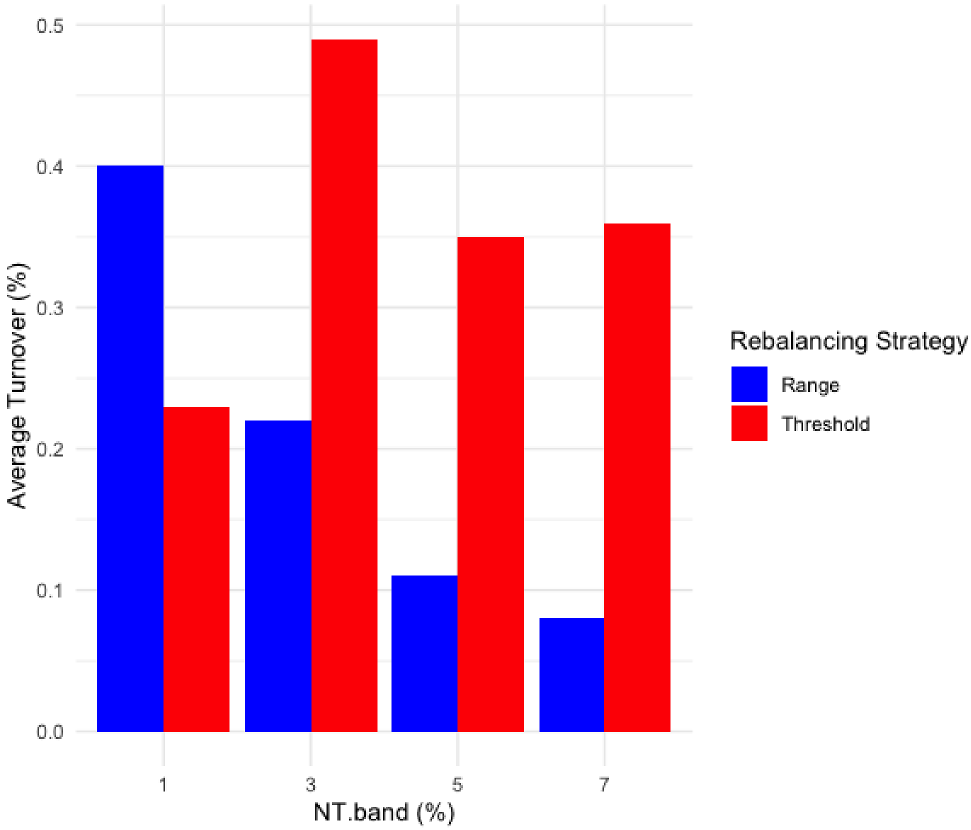

Figure 5.1: Average turnover rates for Range and Threshold rebalancing strategies across different no-trade band widths for a 60/40 portfolio with two risky assets, with $\lambda_S = 0.25\%$ $\lambda_B = 0.10\%$.

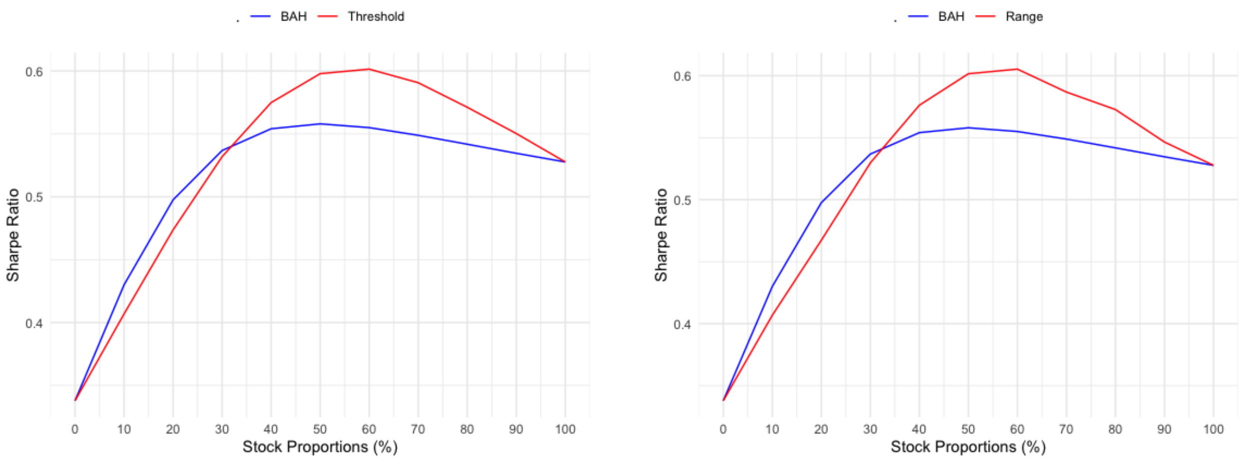

Figure 5.2: Variation of the annualized Sharpe ratio with stock allocation percentages for Buy-and-Hold (BAH) and Threshold-Based (Left panel) versus BAH and Range-Based (Right panel) in the case of two risky assets. Accounting for $\lambda_S = 0.25\%$ $\lambda_B = 0.10\%$.

Figure 5.2 demonstrates that increasing the proportion of stocks enhances the Sharpe ratio up to a certain threshold. Understanding the impact of different stock allocation percentages on the Sharpe ratio is crucial for determining the optimal asset mix that maximizes risk-adjusted returns. Figure 5.2 aims to identify the allocation that offers the best performance for both Range and Threshold rebalancing strategies, compared to the traditional Buy-and-Hold strategy. Notably, the Sharpe ratio reaches its peak with a 60/40 stock-to-bond allocation, indicating that this mix provides the optimal risk-adjusted returns for both Range and Threshold rebalancing strategies.

However, it is essential to note that the optimal portfolio mix can vary based on transaction costs or the no-trade band, as suggested by table 5.2. Beyond this optimal point, any further increase in stock allocation results in a slight decline in the Sharpe ratio, suggesting that higher stock proportions do not necessarily equate to improved risk-adjusted returns. Additionally, the Buy-and-Hold strategy exhibits higher Sharpe ratios for lower stock allocations up to 30%. However, beyond this level, rebalancing strategies consistently outperform Buy-and-Hold across various portfolio compositions.

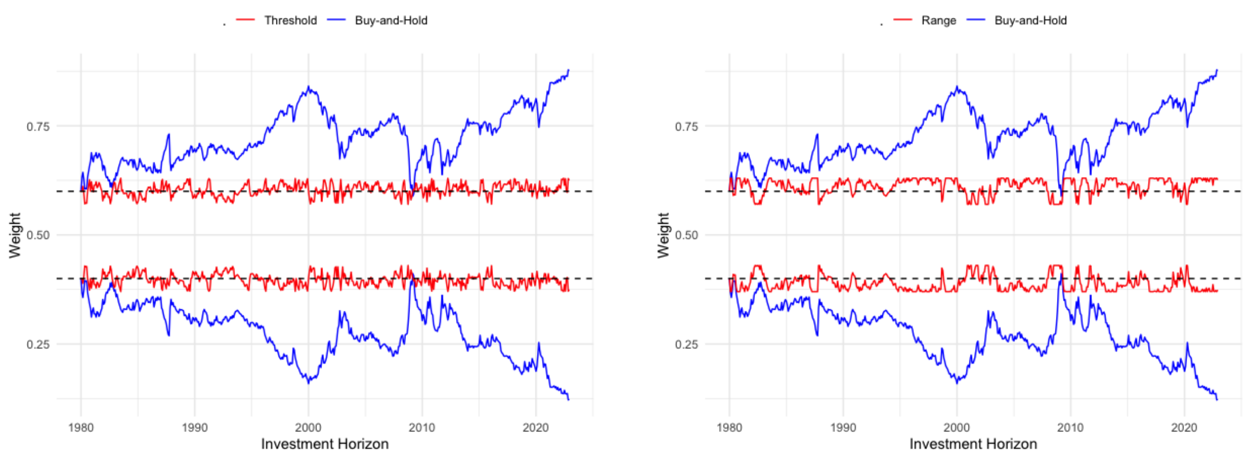

Figure 5.3: Comparative change in asset weight from 1980 to 2022 in the portfolio of two risky assets using threshold (Left) and Range (Right) rebalancing strategies. The dotted line showcases the target asset weight.

Figure 5.3 graphically illustrates the change in asset weight for two risky asset portfolio for the Threshold and Range. It is interesting to observe for the respective rebalancing strategies the asset weight fall within the predefined Range around the target asset weight. The computed no-trade region is approximately $\pm 3\%$ around the target asset weights, aligning well with asset management practices.

For instance, the Norwegian Government Pension Fund used the $\pm 3\%$ no-trade region; however, in 2017, reduced the no-trade region to $\pm 2\%$ (Norges Bank Investment Management, 2018). In contrast, the Buy-and-Hold strategy exhibits a trend towards a greater stock proportion over time. The deviation from the target asset weights underscores the passive nature of the Buy-and-Hold approach, which does not actively seek to revert back to an initial allocation. The development of the Buy-and-Hold strategy sheds light on significant declines in asset weights corresponding to periods of economic downturn, such as the pronounced plunge during the 2008 financial crisis and the significant drop in 2020 pandemic. These moments serve as an indicator of how vulnerable the Buy-and-Hold portfolio is to market downturns.

Portfolio of 4 risky assets	BAH	Naïve	Optimal
Mean return	11.19	10.62	10.86
Std. deviation	13.05	10.55	10.58
Sharpe ratio	0.557	0.635	0.656
P-value		0.006	0.001
Avg turnover (%)	0.000	2.261	0.477

Table 5.3: Comparative descriptive statistics for portfolio of four risky assets. The mean and standard deviation are annualized in percentages. The p-value is gathered by testing the null hypothesis against that the rebalancing strategies is equal to the Buy-and-Hold (BAH) strategy. γ = 5.

Table 5.3 displays the results of the historical simulation for a portfolio consisting of four risky assets. In line with the findings from table 5.1, the optimal strategy consistently outperforms the benchmark strategies, namely Buy-and-Hold and the Naïve, similarly to the Range strategy. However, compared to the portfolio of two risky assets, there is an observed increase in the Sharpe ratio for both the Optimal and Naïve strategies, while the Buy-and-Hold strategy exhibits a decrease in its Sharpe ratio in the case of four risky assets.

Figure 5.4 presents the asset weights for the optimal portfolio in the case of four risky assets. The left panel displays the individual weight of each asset over time, whereas the right panel combines these weights for the period from 1980 to 2022. The left panel reveals a more significant deviation from the target asset weight than the right panel. As per table 4.1, large stocks and small stocks, which are more volatile, deviate considerably from their target asset weights by approximately $\pm 6\%$ and $\pm 4\%$, respectively. In comparison, long-term and intermediate-bonds, characterized by lower volatility, illustrate smaller deviations from their target asset weights, about $\pm 2.5\%$. Further, it is observed that large stock and small stock tend to move in opposite directions, a behavior likely explained by their high correlation.

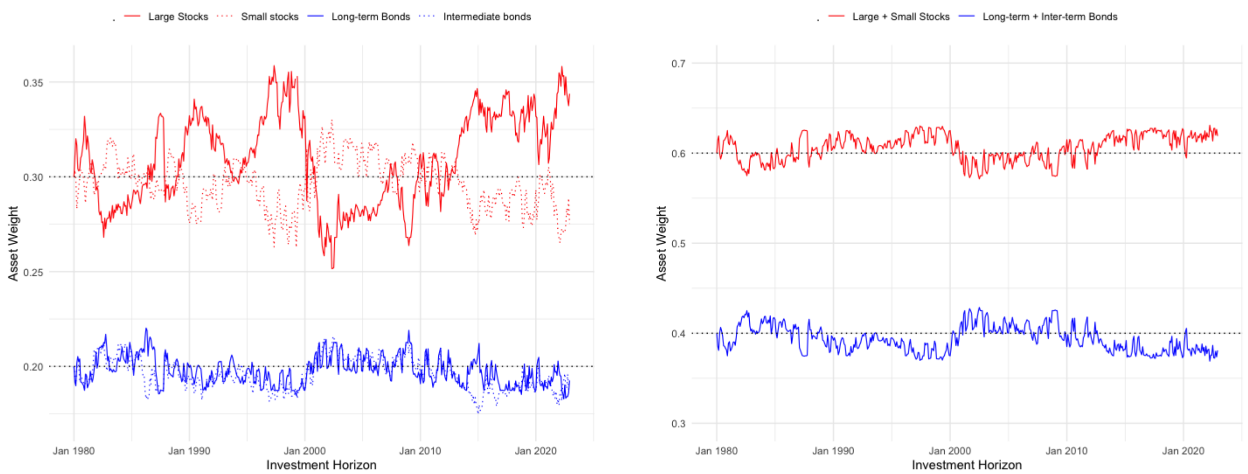

Figure 5.4: Left panel showcasing the change in weight for the four risky asset. Right panel displays the combined weight change over the period 1980 - 2022. The dotted lines showcases the target asset weights.

The optimal transaction policy is influenced by the level of trading costs, and by the inherent volatility of the assets and the correlation coefficients between them. Thus, it is pertinent to note that there exists a significant degree of correlation between stock indices and bond indices, causing these assets to act as substitutes for one another. This is visually presented in the left panel, where increases in large stock indices coincide with decrease in small stocks, and a similar relationship is observed between long-term bonds and intermediate-term bonds. The increase (decrease) of weight in large stock will reduce (increase) the weight in small stock.

The right panel of figure 5.4 demonstrates the reduced fluctuations around the target asset weights compared to individual asset classes. This stability is attributed to the counterbalancing effect achieved when the assets are aggregated. For stocks, the variations from the target asset weight hover around $\pm 2.5\%$, whereas for bonds, the deviation is somewhat tighter, at approximately $\pm 2\%$. Such an observation underscores the benefits of diversification, as the combined portfolio exhibits more consistency in adhering to target allocations.

5.1 Robustness Tests

The baseline analysis utilize the period from January 1960 to December 1979 for the in-sample. Whilst for the out-of-sample the period from January 1980 to December 2022 is utilized. Additionally, we incorporate proportional transaction costs of $\lambda_S = 0.25\%$ $\lambda_B = 0.10\%$, along with a risk aversion parameter of $\gamma = 5$. To validate our findings, we employ robustness tests to evaluate the stability and reliability of the results in the presence of estimation errors in input parameters, as well as variations in key model parameters and assumptions. The tables with the results are located in the appendix.

Our thesis employ a sample variance-covariance matrix, estimated from historical return covariances from January 1960 to December 1979. To evaluate the reliability of the variance-covariance matrix, we employ three distinct methods for its estimation: (1) the shrinkage method (Ledoit & Wolf, 2003), (2) the robust method of estimating the minimum covariance determinant (Rousseeuw & Driessen, 1999), and (3) the robust maximum likelihood estimation (Kent et al., 1994). It is observed that there were slight variations in the realization of the Sharpe ratio, but differ minimally between the methods. For instant, the optimal strategy varies from 0.651 to 0.660 in terms Sharpe ratio (see, table A.1 in appendix).

We further evaluated our empirical study by varying the simulation periods for the out-of-sample analysis. The baseline study utilize the historical simulation spanning from 1980 to 2022. To assess how the rebalancing strategies perform under different time frames, we conduct additional out-of-sample simulations for the periods 1990 to 2022 and 2000 to 2022. Despite the variation in time periods, our observations consistently indicate that the optimal strategy is superior in scenarios involving both two risky asset and four risky asset portfolios (see, table A.2 & A.5 in appendix). Additionally, the period from 2000 to 2022 reveal a significant decrease in the mean return, likely due to the dot-com crash and the 2008 financial crisis. Despite this, the optimal strategy consistently exhibit the highest Sharpe ratio, even in underperforming markets. The p-values for the period of 2000 to 2022 are generally not statistical significant, except for the optimal strategy with four risky assets.

Furthermore, to test the model's robustness, we conduct a similar test for a portfolio with three risky assets by varying the out-of-sample time period. The allocation to stocks and bonds are the same, but distributes 30% to large stocks, 30% to small stocks, and 40% to long-term bonds. Consistent with our previous findings, the optimal strategy demonstrate superiority. Interestingly, for the three-risky assets portfolio, the Sharpe ratio is higher compared to the portfolios with two and four risky assets across all scenarios (see, table A.7 in appendix).

As previously stated, our study adopts a constant risk-aversion parameter $\gamma = 5$. To assess the sensitivity of our models to changes in risk aversion, we perform a robustness check by varying the risk aversion parameter to $\gamma = 3$ and $\gamma = 7$. As shown in appendix A.8, we observe marginal changes in the Sharpe ratio when varying (γ). Despite these variations, the Optimal strategy consistently demonstrate significant superiority over the Buy-and-Hold and Naïve strategy. The Sharpe ratios for the Optimal strategy remain robust, highlighting its effectiveness across different levels of risk-aversion.

To further ensure the robustness of our findings, we conduct a robustness test by varying the portfolio mix to include portfolios of 50/50, 60/40, and 70/30 allocations in the case of two risky assets. Additionally, we adjust for transaction costs under two scenarios: (1) 25 basis points (bps) for stocks and 10 bps for bonds, (2) 50 bps for stocks and 20 bps for bonds for the three portfolio mix. Our findings indicate that for the portfolio with two risky assets, the Range strategy seems to be superior in almost all scenarios, except for the 70/30 portfolio mix, where the Threshold strategy marginally outperforms the Range strategy (see, table A.3 in appendix).

Furthermore, we extend the robustness test to a portfolio of four risky assets. For this robustness test, we applied the same approach as used for the two risky assets but introduced an additional parameter: zero transaction costs. The optimal strategy consistently presents the highest Sharpe ratio (see, table A.6 in appendix). Interestingly, all strategies seem to perform worse with the 70/30 portfolio mix in terms of the Sharpe ratio. This observation is particularly intriguing given that the Norwegian Pension Fund has adopted a 70/30 portfolio mix since 2017 (Norges Bank Investment Management, 2018). This choice may be justified by the fund's indefinite investment horizon, along with the principle that longer market engagement tends to reduce volatility.

Interestingly, in the case of four risky assets; when the transaction cost is zero the Optimal strategy result in a lower Sharpe ratio for the 70/30 portfolio mix compared to the scenarios with transaction cost of 35 bps and 70 bps (see, table A.6 in appendix). This result may appear counterintuitive at first glance. However, this observation can be rationalized through several factors. First, the inherent stochasticity in financial markets plays a crucial role; random fluctuations in asset returns can lead to variations in performance metrics such as the Sharpe ratio across different time periods. Second, for the sake of simplifying our model, the variance-covariance matrix of asset returns is held constant. While this approach streamlines our analysis, it omits the dynamic nature of market conditions—specifically, the time-varying volatility and correlations among financial assets. This model simplification may not fully reflect the more complex behaviors exhibited in financial markets.

Lastly, we test the impact of varying NT bands varying from $\pm 1\%$, $\pm 3\%$ and $\pm 5\%$, similarly as table 5.2. However, this robustness test further analyze the effect of these varying NT bands has on different portfolio mixes: 50/50, 60/40, and 70/30. The results indicate that the highest Sharpe ratio is achieved by the Range strategy for the 50/50 portfolio mix. However, this result is not statistically significant, as the p-value exceeds the 5% significance level. In the 60/40 portfolio, we see significant Sharpe ratios for both Threshold and Range are 0.633 with a no-trade band of $\pm 3\%$. Interestingly, the Threshold strategy marginally outperforms the Range strategy for NT bands of $\pm 1\%$ and $\pm 3\%$ for the 70/30 portfolio mix. Overall, the Range strategy performs better than the Threshold strategy in the majority of the cases, although they differ by increments (see, table A.9 in appendix).

Chapter 6

Discussion

The primary objective of this thesis is to evaluate various rebalancing strategies for portfolios containing two and four risky assets. The study assumes a fully invested portfolio, that incorporates proportional transaction costs and prohibits short-sales. In the case of two risky assets, our analysis covers four strategies: Buy-and-Hold, Naïve, Threshold, Range. The inspiration for the two risky asset case is based on the empirical work of Dichtl et al. (2014). Two-asset portfolios employing rebalancing strategies such as the Threshold and Range strategy reveal that the these active strategies outperform the Naive strategy and the Buy-and-Hold, based on the Sharpe ratio. These findings are consistent with the results reported by Dichtl et al. (2014).

Further expansion of the thesis draws inspiration from Zakamulin (2024). Zakamulin developed an optimal rebalancing strategy applicable to n-dimensional portfolios of risky assets using quadratic programming. Furthermore, Zakamulin's central idea is that the multi-period strategy can be effectively approximated by a policy derived from the single-period model. By implementing Zakamulin's methodology, we approximate the optimal portfolio for four risky assets, focusing on three strategies: Buy-and-Hold, Naïve and Optimal. Utilizing this methodology, we confirm the superiority of the Optimal strategy.

However, our findings in the multi-period context diverge from those of Mei et al. (2016), who concluded that rebalancing is optimal only after the first period, subsequently simplifying to a Buy-and-Hold strategy. Our results, along with those of Dichtl et al. (2014) and Zakamulin (2024) identifies the Buy-and-Hold strategy as suboptimal.

6.1 Impact of NT bandwidths & Transaction Costs

Our study confirms that the width of the no-trade bands is a significant determinant of rebalancing frequency and transaction costs. By testing various no-trade bandwidths, we provide empirical evidence that wider bands could mitigate transaction costs but potentially at the expense of portfolio performance. This aligns with the strategies employed by large institutional investors like the Norwegian Government Pension Fund Global, reinforcing the practical relevance of our findings Management (2018). Furthermore, appendix A.9 displays the NT band for threshold and range rebalancing strategies for different stock proportions. It shows that the turnover in the threshold strategy tends to increase as the NT bandwidth widens for almost all stock proportions. In contrast, the range strategy appears less affected by changes in NT bandwidth, maintaining a consistent turnover rate across different stock proportions and NT bandwidts.

Earlier research, such as the study by Norges Bank Investment Management (2012) exploring the effects of no-trade bandwidths on rebalancing strategies. Dichtl et al. (2014)'s assessment of these strategies using the Sharpe ratio as a performance measure, has provided important insights. Notably, implementing a no-trade band—where rebalancing occurs only if asset allocations deviate beyond specific thresholds—can substantially reduce transaction costs. However, their analysis is limited due to insufficient time periods, reliance on historical data, and simplified assumptions such as constant volatility and returns.

It's crucial to acknowledge that the baseline of the thesis has proportional transaction costs. This approach involves setting a constant transaction cost for both stocks and bonds—25 basis points (bps) for stocks and 10 bps for bonds—primarily to simplify the model. This simplification may not accurately reflect the real-world scenario where transaction costs can fluctuate. However, this simplification allows for a clearer focus on other variables but may not capture the full complexity of market dynamics. The complexities of identifying the optimal portfolio are exacerbated when incorporating transaction costs, particularly due to their quadratic nature. This feature significantly complicates the computational and analytical efforts required to optimize portfolio choices, as it introduces a nonlinear element into the cost-benefit analysis of different investment strategies.

6.2 Limitations & Further Studies

The body of the research in this particular area of financial study exhibits limitations. There are numerous examples of previous studies that explore the effect of transaction costs on portfolio optimization. However, these studies fail to provide examples of fully invested portfolio composed of risky assets beyond two, and often include a risk-free alternative. This has resulted in a deficiency of a practical model that can be effective for portfolio managers to employ. Besides the study of Zakamulin (2024), we were not able to find any previous research that present a practical model for a fully invested multi-asset portfolios that incorporates transaction costs. A notable limitation of our thesis is the adoption of proportional transaction costs and a constant risk aversion. Although simplifying the model, the proportional transaction cost does not accurately capture real-world trading conditions. Transaction costs may vary depending on market conditions, trade size, and asset liquidity, among other factors. Additionally, employing a constant risk aversion overlooks the potential for investor preferences to fluctuate in response to changing economic conditions.

Furthermore, the optimization function used in this study has only been tested within the US market and considers only four risky assets: large stocks, small stocks, inter-term bonds and long-term bonds. Further research should evaluate the optimization function in markets outside the US and incorporate a broader range of risky assets to assess its robustness. The reliance on a constant variance-covariance matrix, derived from our dataset, provides a robust estimate for our model. However, applying the same model to different datasets could potentially reveal limitations, as the assumptions of constant variance covariance might not hold. Thus, testing the model on differently sized datasets could offer deeper insights into its scalability and reliability across different contexts.

Further studies on this topic could include a more comprehensive analysis of transaction costs and taxes. While the current study has focused on proportional transaction costs, future research should explore a broader range of transaction costs and tax implications to better understand their impact on the results. Moreover, the model could be extended to handle uncertainty in inputs by employing stochastic models or robust optimization techniques.

Throughout this thesis, we have used ChatGPT (OpenAI, 2024) for the purpose of refining the language and clarity of the text. It was utilized to correct grammar and improve the overall readability of the thesis.

Chapter 7

Conclusion

This thesis explores various rebalancing strategies for fully invested portfolios consisting of two and four risky assets. Our analysis incorporates transaction costs, which are proportional to the amount traded. The study employs monthly data for an out-of-sample period spanning from January 1980 to December 2022. The in-sample period, from January 1960 to December 1979, is utilized to estimate the covariance matrix, which is essential for implementing the Optimal strategy.

To further the discussion on portfolio optimization it is crucial to address the limitations of earlier studies, particularly their lack of practical application. Firstly, previous research struggle to provide significant empirical results for optimal rebalancing strategies in portfolios fully invested in risky assets. Secondly, earlier studies encounter substantial challenges when dealing with multiple risky assets. Consequently, treatments of portfolios with multiple risky assets tend to allocate a significant portion to a risk-free asset, evident by the paper of Gârleanu and Pedersen (2016). A later study by D. Liu (2019) included multiple risky assets without a risk-free asset but encountered challenges in practical application, due to the complexities of portfolio optimization.

For the optimal portfolio choice problem involving up to two risky assets, analytical solutions are applicable and straightforward to implement within a single-period framework. Previous studies emphasise that when transaction costs are absent, it is optimal to rebalance to the target asset weights. However, as transaction costs must be accounted for, it is optimal to rebalance only when the benefits of rebalancing exceed the costs. The historical simulations provide evidence that in the case of two risky assets, the Buy-and-Hold strategy is sub-optimal, which aligns with the findings of Dichtl et al. (2014). Although Dichtl et al. (2014) argues that it is of minor economic importance which rebalancing strategy to employ, we show that the Range strategy builds upon the same logic as the Optimal strategy and our findings support the optimality of Range in the two-asset case.

For the case of multiple risky assets, we wanted to address the existing challenges, by opting for an approach with a portfolio fully invested in risky assets with transaction costs. Zakamulin (2024) argues that there is a solution to the problem that does not rely on solving the stochastic control problem. The methodology developed by Zakamulin (2024) provides a numerical solution within the multi-period framework, derived from a single-period model. By employing the methodology our results are consistent with those of Zakamulin (2024), the optimal strategy produces the highest Sharpe ratio and exhibits the most significant results. The method is highly applicable to portfolio managers due to its simplicity and practical relevance.

The significance of this thesis lies in its contribution to the literature on portfolio optimization and rebalancing strategies. By addressing the limitations of earlier models and incorporating realistic assumptions about transaction costs, our research provides valuable insights for both academic researchers and financial practitioners. The methodologies and findings presented here enhance the understanding of how to effectively manage fully invested portfolios, particularly in the presence of multiple risky assets and transaction costs.

References

Black, F., & Litterman, R. (1992). Global portfolio optimization. *Financial Analysts Journal*, *48*(5), 28–43.

Bodie, Z., Kane, A., & Marcus, A. J. (2014). *Investments* (10th ed.). McGraw Hill.

Constantinides, G. M. (1979). Multiperiod consumption and investment behavior with convex transactions costs. *Management Science*, *25*(11), 1127–1137. Retrieved from https://www.jstor.org/stable/2630413

Constantinides, G. M. (1986). Capital market equilibrium with transaction costs. *The Journal of Political Economy*, *94*(4), 842–862. Retrieved from https://doi.org/10.1086/261410

Davis, M. H., & Norman, A. R. (1990). Portfolio selection with transaction costs. *Mathematics of Operations Research*, *15*(4), 676–713. Retrieved from https://doi.org/10.1287/moor.15.4.676

Dichtl, H., Drobetz, W., & Wambach, M. (2014). Where is the value added of rebalancing? a systematic comparison of alternative rebalancing strategies. *Financial Markets and Portfolio Management*, *28*, 209–231. Retrieved from https://doi.org/10.1007/s11408-014-0231-3

Fabozzi, F. J., & Markowitz, H. M. (Eds.). (2011). *Equity valuation and portfolio management*. Wiley.

Gârleanu, N., & Pedersen, L. H. (2016). Dynamic portfolio choice with frictions. *Journal of Economic Theory*, *165*, 487–516. Retrieved from https://doi.org/10.1016/j.jet.2016.06.001

Hubert, M., & Debruyne, M. (2010). Minimum covariance determinant. *WIREs Computational Statistics*, *2*(1), 36-43.

Kent, J. T., Tyler, D. E., & Vard, Y. (1994). A curious likelihood identity for the multivariate t-distribution. *Communications in Statistics - Theory and Methods*, *23*(11), 2943–2960. Retrieved from https://www.oa.mg/work/10.1080/03610919408813180

Kirby, C., & Ostdiek, B. (2012). It's all in the timing: Simple active portfolio strategies that outperform naïve diversification. *Journal of Financial and Quantitative Analysis*, *47*(2), 437–467. Retrieved from https://doi.org/10.1017/S0022109012000117

Ledoit, O., & Wolf, M. (2003). Honey, i shrunk the sample covariance matrix. *Journal of Portfolio Management*, *30*(4), 110–119.

Leland, H. E. (2000). *Optimal portfolio implementation with transactions costs and capital gains taxes* (Technical Report). Haas School of Business. Retrieved from https://doi.org/10.2139/ssrn.206871

Liu, C. (1997). Ml estimation of the multivariate t distribution and the em algorithm. *Journal of Multivariate Analysis*, *63*(2), 296-312.

Liu, D. (2019). Analytical solutions of optimal portfolio rebalancing. *Quantitative Finance*, *19*(4), 683–697. Retrieved from https://doi.org/10.1080/14697688.2018.1520394

Lynch, A. W., & Tan, S. (2010). Multiple risky assets, transaction costs, and return predictability: Allocation rules and implications for us investors. *Journal of Financial and Quantitative Analysis*, *45*(4), 1015. Retrieved from https://doi.org/10.1017/S0022109010000360

Ma, G., Siu, C. C., & Zhu, S.-P. (2019). Dynamic portfolio choice with return predictability and transaction costs. *European Journal of Operational Research*, *278*(3), 976–988.

Magill, M. J., & Constantinides, G. M. (1976). Portfolio selection with transaction costs. *Journal of Economic Theory*, *13*(2), 245–263. Retrieved from https://doi.org/10.1016/0022-0531(76)90018-1

Management, N. B. I. (2018, August). No-trade band rebalancing rules: Expected returns and transaction costs [Discussion Note]. (1). (ISSN 1893-966X)

Markowitz, H. (1952). Portfolio selection*. *The Journal of Finance*, *7*(1), 77-91.

Mei, X., DeMiguel, V., & Nogales, F. J. (2016). Multiperiod portfolio optimization with multiple risky assets and general transaction costs. *Journal of Banking & Finance*, *69*, 108–120. Retrieved from https://doi.org/10.1016/j.jbankfin.2016.04.002

Merton, R. C. (1969). Lifetime portfolio selection under uncertainty: The continuous-time case. *Review of Economics and Statistics*, *51*(3), 247–257. Retrieved from https://doi.org/10.2307/1926560

Merton, R. C. (1973). An intertemporal capital asset pricing model. *Econometrica*, *41*(5), 867–887. Retrieved from https://doi.org/10.2307/1913811

Ministry of Finance, Norway. (2021). *Government pension fund global: Mandate.* Online. Retrieved from https://www.regjeringen.no/contentassets/9d68c55c272c41e99f0bf45d24397d8c/gpfg_mandate_2021.08.24.pdf

Norges Bank Investment Management. (2012, March). *The history of rebalancing of the fund.* Norges Bank Investment Management (NBIM). Retrieved from https://www.nbim.no/en/publications/discussion-notes/2012/the-history-of-rebalancing-of-the-fund/ (NBIM Discussion Note #4)

Norges Bank Investment Management. (2018). *The rule for rebalancing the equity share in the government pension fund global.* Online. Retrieved from https://www.nbim.no/en/publications/submissions-to-ministry/2018/the-rule-for-rebalancing-the-equity-share-in-the-government-pension-fund-global/ (Available: https://www.nbim.no/en/publications/submissions-to-ministry/2018/the-rule-for-rebalancing-the-equity-share-in-the-government-pension-fund-global/ [Accessed: Insert the date here])

OpenAI. (2024). *Chatgpt: An ai language model.* https://www.openai.com. (Version GPT-4)

Perold, A. F. (1984). Large-scale portfolio optimization. *Management Science*, *30*, 1143–1160. doi: 10.1287/mnsc.30.10.1143

Richard, J.-C., & Roncalli, T. (2015, March). Smart beta: Managing diversification of minimum variance portfolios. *Lyxor Asset Management, Paris*. Retrieved from https://doi.org/10.1016/B978-1-78548-008-9.50002-2

Roncalli, T. (2014). *Introduction to risk parity and budgeting*. CRC Press.

Rousseeuw, P. J., & Driessen, K. V. (1999). A fast algorithm for the minimum covariance determinant estimator. *Universitaire Instelling Antwerpen, Universiteitsplein 1, B-2610 Wilrijk, Belgium*.

Scherer, B. (2011). A note on the returns from minimum variance investing. *Journal of Empirical Finance*, *18*(4), 652-660. Retrieved from https://doi.org/10.1016/j.jempfin.2011.06.001

Sharpe, W. F. (1966). Mutual fund performance. *The Journal of Business*, *39*(1), 119–138.

Sharpe, W. F. (1994). The sharpe ratio. *The Journal of Portfolio Management*, *21*(1), 49-58. Retrieved from https://doi.org/10.3905/jpm.1994.409501

Shreve, S. E., & Soner, H. M. (1994). Optimal investment and consumption with transaction costs. *Annals of Applied Probability*, *4*(3), 609–692. Retrieved from https://doi.org/10.1214/aoap/1177004966

Sun, W., Fan, A., Chen, L.-W., Schouwenaars, T., & Albota, M. A. (2006). Optimal rebalancing for institutional portfolios: Minimizing costs using dynamic programming. *Journal Name*. (MIT Laboratory for Information and Decision Systems, Cambridge, MA)

Zakamouline, V. (2008). Option pricing and hedging in the presence of transaction costs and nonlinear partial differential equations. Retrieved from http://dx.doi.org/10.2139/ssrn.938933

Zakamulin, V. (2017). Superiority of optimized portfolios to naive diversification: Fact or fiction? *Finance Research Letters*, *22*, 122-128. Retrieved from https://www.sciencedirect.com/science/article/pii/S1544612316303555 doi: https://doi.org/10.1016/j.frl.2016.12.007

Zakamulin, V. (2024, April 2). Transaction cost optimization in a fully invested portfolio with target weights. Retrieved from https://ssrn.com/abstract=4781258

Zilbering, Y., Jaconetti, C. M., & Jr., F. M. K. (2015, November). *Best practices for portfolio rebalancing* (Research Report). Vanguard Research.

7.1 Appendix

The following tables A.1 - A.10, will present a robustness check to validate the findings of our empirical study. The appendix will present different variance-covariance matrixes, varying the simulation horizon on the 60/40 portfolio mix, a descriptive statistics for three-risky assets and varying the portfolio mix.

Table A.1 provides descriptive statistics for four risky assets, utilizing four different estimations of the variance-covariance matrix. The Buy-and-Hold and Naive strategies are identical to the different estimation methods, as these rebalancing strategies do not depend on the method used for estimating the covariance matrix. Conversely, the optimal strategy exhibits slight variations depending on the specific estimation method applied.

Metric	**BH**	**Naive**	**Optimal**
Sample variance-covariance matrix			
Mean	11.20	10.62	10.86
Standard deviation	13.05	10.55	10.58
Sharpe ratio	0.558	0.635	0.656
P-value		0.006	0.001
Avg. Turnover, %	0.000	2.261	0.477
Shrinkage method			
Mean	11.20	10.62	10.88
Standard deviation	13.05	10.55	10.59
Sharpe ratio	0.558	0.635	0.660
P-value		0.006	0.000
Avg. Turnover, %	0.000	2.261	0.474
Minimum covariance method			
Mean	11.20	10.62	10.84
Standard deviation	13.05	10.55	10.59
Sharpe ratio	0.558	0.635	0.653
P-value		0.006	0.001
Avg. Turnover, %	0.000	2.261	0.484
MLE			
Mean	11.20	10.62	10.82
Standard deviation	13.05	10.55	10.59
Sharpe ratio	0.558	0.635	0.651
P-value		0.006	0.002
Avg. Turnover, %	0.000	2.261	0.491

Table A.1: The descriptive statistics obtained by using different variance-covariance estimations. $\lambda_S = 0.25\%$ $\lambda_B = 0.10\%$ and γ = 5 with a 60/40 portfolio mix.

Table A.2 presents descriptive statistics of a 60/40 portfolio mix in the case of two risky assets for three out-of-sample periods: 1980-2022, 1990-2022 and 2000-2022. The results indicate that the Range strategy achieves the highest Sharpe ratio across all three periods, although marginally.

Metric	**BAH**	**Naive**	**Threshold**	**Range**
1980-2022				
Mean	10.80	10.49	10.57	10.58
Standard deviation	11.87	10.53	10.53	10.54
Sharpe ratio	0.580	0.626	0.633	0.634
P-value	0	0.025	0.039	0.039
Avg. turnover (%)	0.000	1.911	0.493	0.219
1990-2022				
Mean	8.97	8.91	9.03	9.24
Standard deviation	10.40	9.71	9.70	9.83
Sharpe ratio	0.621	0.660	0.670	0.673
P-value		0.006	0.019	0.019
Avg. turnover (%)	0.000	1.944	0.222	0.402
2000-2022				
Mean	6.11	6.64	6.75	6.81
Standard deviation	8.91	9.65	9.62	9.57
Sharpe ratio	0.516	0.531	0.544	0.552
P-value		0.208	0.321	0.321
Avg. turnover (%)	0.000	2.189	0.582	0.245

Table A.2: Comparative performance metrics over three periods for two-risky assets. The mean, standard deviation and Sharpe ratios are annualized in percentages. Threshold and Range uses the $\pm 3\%$ No Trade-band. $\lambda_S = 0.25\%$ $\lambda_B = 0.10\%$ and γ = 5.

	Sharpe (λ_S= 0.25 λ_B = 0.10)	Sharpe (λ_S= 0.50 λ_B = 0.20)
50/50 portfolio mix		
Buy-and-hold	0.5889	0.5889
	(-)	(-)
Naive	0.6265	0.6222
	(0.1768)	(0.2054)
Threshold	0.6329	0.6306
	(0.1379)	(0.1504)
Range	0.6354	0.6342
	(0.1165)	(0.1224)
60/40 portfolio mix		
Buy-and-hold	0.5800	0.5800
	(-)	(-)
Naive	0.6261	0.6217
	(0.0591)	(0.0935)
Threshold	0.6332	0.6309
	(0.0435)	(0.0493)
Range	0.6335	0.6321
	(0.0436)	(0.0419)
70/30 portfolio mix		
Buy-and-hold	0.5608	0.5608
	(-)	(-)
Naive	0.6103	0.6072
	(0.0320)	(0.0433)
Threshold	0.6196	0.6179
	(0.0095)	(0.0115)
Range	0.6156	0.6148
	(0.0127)	(0.0139)

Table A.3: Sharpe ratios and p-values for two risky asset portfolios by varying the transaction costs on different portfolio mix. Note: P-values are shown in parentheses.

Table A.3 presents the Sharpe ratio and corresponing p-value when varying the proportional transaction cost: $\lambda_S = 0.25\%$ $\lambda_B = 0.10\%$ and $\lambda_S = 0.50\%$ $\lambda_B = 0.20\%$. The portfolio mixes analyzed include 50/50, 60/40, and 70/30 allocations.

Metric	BH	Naive	Thresh	Range
1980-2022				
Mean	11.68	10.58	10.71	10.73
Standard deviation	16.28	12.32	12.34	12.39
Sharpe ratio	0.476	0.540	0.550	0.551
P-value		0.003	0.000	0.000
Avg. Turnover, %	0.000	2.222	0.563	0.241
1990-2022				
Mean	10.34	9.35	9.51	9.57
Standard deviation	15.86	12.30	12.29	12.34
Sharpe ratio	0.493	0.556	0.569	0.571
P-value		0.006	0.001	0.000
Avg. Turnover, %	0.000	2.307	0.563	0.248
2000-2022				
Mean	8.34	8.12	8.27	8.32
Standard deviation	14.50	12.99	12.97	12.99
Sharpe ratio	0.471	0.509	0.521	0.524
P-value		0.055	0.035	0.035
Avg. Turnover, %	0.000	2.484	0.591	0.285

Table A.4: Comparative descriptive statistics for portfolio of two risky asstes. The mean, standard deviation and Sharpe ratios are annualized in percentages. Threshold and Range uses the $\pm 3\%$ No Trade-band with a 60/40 portfolio mix.

Following table presents the descriptive statistics for the scenario involving two risky assets. In this case, the original assets of large stocks and long-term bonds are substituted with small stocks and Intermediate Bonds. This substitution aims to illustrate and validate our findings in the context of more volatile assets. The table validates our findings that the Optimal strategy is superior in all scenarios.

Metric	BAH	Naive	Optimal
1980-2022			
Mean return	11.19	10.62	10.86
Std. deviation	13.05	10.55	10.58
Sharpe ratio	0.557	0.635	0.656
P-value		0.001	0.001
Avg turnover (%)	0.000	2.261	0.477
1990-2022			
Mean return	9.65	9.24	9.51
Std. deviation	12.11	10.04	10.02
Sharpe ratio	0.589	0.669	0.698
P-value		0.0003	0.0003
Avg turnover (%)	0.000	2.330	0.473
2000-2022			
Mean return	7.32	7.53	7.85
Std. deviation	10.49	10.34	10.27
Sharpe ratio	0.552	0.582	0.616
P-value		0.053	0.053
Avg turnover (%)	0.000	2.530	0.484

Table A.5: Comparative performance metrics across different time periods in the case of four risky assets. $\lambda_S = 0.25\%$ $\lambda_B = 0.10\%$ and γ = 5 with the 60/40 portfolio mix.

Table A.5 presents descriptive statistics of a 60/40 portfolio mix in the case of four risky assets for three out-of-sample periods: 1980-2022, 1990-2022 and 2000-2022. The results indicate that the Optimal strategy achieves the highest significant Sharpe ratio across all three periods. Interestingly, the Sharpe ratio is the highest for all three strategies for out-of-sample period of 1990-2022.

	Sharpe (λ= 0)	Sharpe (λ_S= 0.25 λ_B = 0.10)	Sharpe (λ_S= 0.50 λ_B = 0.20)
	50/50 portfolio mix		
Buy-and-hold	0.5724	0.5724	0.5724
	(-)	(-)	(-)
Naive	0.6659	0.6606	0.6552
	(0.0132)	(0.0181)	(0.0245)
Optimal	0.6659	0.6786	0.6814
	(0.0132)	(0.0056)	(0.0048)
	60/40 portfolio mix		
Buy-and-hold	0.5576	0.5576	0.5576
	(-)	(-)	(-)
Naive	0.6402	0.6352	0.6307
	(0.0040)	(0.0061)	(0.0094)
Optimal	0.6532	0.6521	0.6541
	(0.0041)	(0.0012)	(0.0011)
	70/30 portfolio mix		
Buy-and-hold	0.5458	0.5458	0.5458
	(-)	(-)	(-)
Naive	0.6116	0.6075	0.6035
	(0.0015)	(0.0027)	(0.0056)
Optimal	0.6134	0.6222	0.6215
	(0.0007)	(0.0003)	(0.0004)

Table A.6: Sharpe ratios and p-values for four risky asset portfolios by varying the transaction costs on different portfolio mix. Note: P-value are shown in paranthesis.

Table A.6 presents the Sharpe ratio and corresponing p-value when varying the proportional transaction cost: $\lambda_S = 0.00\%$ $\lambda_B = 0.00\%$ and $\lambda_S = 0.25\%$ $\lambda_B = 0.10\%$ and $\lambda_S = 0.50\%$ $\lambda_B = 0.20\%$. The portfolio mixes analyzed include 50/50, 60/40, and 70/30 allocations in the case of four risky assets.

Metric	BH	Naive	Optimal
1980-2022			
Mean	11.23	10.97	11.19
Standard deviation	12.85	10.97	10.99
Sharpe ratio	0.568	0.643	0.662
P-value		0.020	0.005
Avg. Turnover, %	0.000	2.369	0.512
1990-2022			
Mean	9.70	9.62	9.87
Standard deviation	11.71	10.31	10.27
Sharpe ratio	0.614	0.689	0.716
P-value		0.196	0.019
Avg. Turnover, %	0.000	2.452	0.509
2000-2022			
Mean	7.47	7.94	8.24
Standard deviation	10.34	10.55	10.48
Sharpe ratio	0.576	0.609	0.641
P-value		0.239	0.056
Avg. Turnover, %	0.000	2.704	0.539

Table A.7: Descriptive statistics for a portfolio of three-risky assets. $\lambda_S = 0.25\%$ $\lambda_B = 0.10\%$ and γ = 5 with the 60/40 portfolio mix.

Table A.7 presents descriptive statistics for three risky assets, large stocks, small stocks and long-term bonds. This table is to further validate our empirical findings, where the weighting is 0.3%, 0.3% and 0.4% for large stocks, small stocks and long-term bonds, respectfully. Consistently with the previous funds, the Optimal strategy attains the highest Sharpe ratio.

Metric	BAH	Naïve	Optimal
γ **= 3**			
Mean	11.19	10.62	10.78
Standard deviation	13.05	10.55	10.59
Sharpe ratio	0.558	0.635	0.648
P-value		0.006	0.002
Avg. turnover	0.000	2.261	0.495
γ **= 5**			
Mean	11.19	10.62	10.86
Standard deviation	13.05	10.55	10.58
Sharpe ratio	0.558	0.635	0.656
P-value		0.006	0.001
Avg. turnover	0.000	2.261	0.477
γ **= 7**			
Mean	11.19	10.62	10.82
Standard deviation	13.05	10.55	10.57
Sharpe ratio	0.558	0.635	0.653
P-value		0.006	0.001
Avg. turnover	0.000	2.261	0.468

Table A.8: Comparison of BAH, Naïve, and Optimal strategies for different values of Gamma. Descriptive statistics for a portfolio of three-risky assets. $\lambda_S = 0.25\%$ $\lambda_B = 0.10\%$ with the 60/40 portfolio mix for the period of 1980-2022.

The following table presents the sensitivity analysis of the Optimal strategy when varying the risk aversion parameter (γ). . It is observed that when γ = 5. result in the highest Sharpe ratio. However, by varying (γ) the Sharpe ratio of the Optimal strategy, vary minimally, from 0.648 to 0.656.

50/50 Portfolio Mix					
NT band	Strategies	Sharpe ratio	P-value	Tracking Error ,%	Turnover ,%
1%	Thres	0.628	0.164	0.010	0.247
1%	Range	0.632	0.124	0.029	0.397
3%	Thres	0.632	0.137	0.037	0.448
3%	Range	0.635	0.116	0.076	0.245
5%	Thres	0.626	0.174	0.082	0.368
5%	Range	0.632	0.118	0.113	0.111
60/40 Portfolio Mix					
NT band	Strategies	Sharpe ratio	P-value	TE %	Turnover %
1%	Threshold	0.626	0.069	0.010	0.230
1%	Range	0.630	0.053	0.029	0.395
3%	Threshold	0.633	0.042	0.040	0.493
3%	Range	0.633	0.039	0.077	0.219
5%	Threshold	0.629	0.058	0.080	0.355
5%	Range	0.629	0.042	0.114	0.115
70/30 Portfolio Mix					
NT band	Strategies	Sharpe ratio	P-value	TE %	Turnover %
1%	Threshold	0.612	0.025	0.010	0.232
1%	Range	0.614	0.018	0.029	0.350
3%	Threshold	0.619	0.009	0.370	0.450
3%	Range	0.615	0.012	0.076	0.144
5%	Threshold	0.620	0.009	0.071	0.356
5%	Range	0.611	0.015	0.115	0.083

Table A.9: Performance Metrics for Large Stocks and Long-term Bonds (1980-2022).

Table A.9 presents the Sharpe ratio, p-value, tracking error, and turnover for three portfolio mixes—50/50, 60/40, and 70/30—across varying no-trade bands of 1%, 3%, and 5%. For the 50/50 and 60/40 portfolio mixes, the range strategy proves to be superior. However, in the case of the 70/30 mix, the Threshold strategy demonstrates a marginal advantage.

	Stock Proportions										
Sharpe ratios	0%	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%
BAH											
Yearly	0.3379	0.4300	0.4976	0.5368	0.5541	0.5580	0.5550	0.5489	0.5418	0.5346	0.5277
Quarterly	0.3355	0.4394	0.5205	0.5652	0.5793	0.5758	0.5649	0.5521	0.5397	0.5285	0.5188
Monthly	0.3604	0.4647	0.5389	0.5781	0.5909	0.5888	0.5801	0.5689	0.5576	0.5470	0.5347
Periodic Rebalancing											
Yearly	0.3379	0.4020	0.4677	0.5274	0.5723	0.5969	0.6015	0.5913	0.5726	0.5504	0.5277
Quarterly	0.3355	0.4038	0.4770	0.5458	0.5971	0.6219	0.6207	0.6020	0.5751	0.5463	0.5188
Monthly	0.3604	0.4255	0.4916	0.5504	0.5922	0.6124	0.6129	0.6000	0.5804	0.5586	0.5347
Threshold Rebalancing											
Yearly	0.3379	0.4069	0.4737	0.5319	0.5749	0.5979	0.6015	0.5907	0.5712	0.5503	0.5277
Quarterly	0.3355	0.4095	0.4836	0.5551	0.6068	0.6304	0.6276	0.6067	0.5768	0.5448	0.5188
Monthly	0.3604	0.4361	0.5058	0.5692	0.6124	0.6316	0.6298	0.6132	0.5881	0.5611	0.5347
Range Rebalancing											
Yearly	0.3379	0.4068	0.4673	0.5297	0.5762	0.6016	0.6053	0.5868	0.5727	0.5465	0.5277
Quarterly	0.3355	0.4089	0.4833	0.5535	0.6061	0.6287	0.6299	0.6041	0.5799	0.5469	0.5188
Monthly	0.3604	0.4292	0.5016	0.5665	0.6110	0.6255	0.6261	0.6142	0.5843	0.5645	0.5347

Table A.10: Annualized Sharpe ratios where transaction costs is $\lambda_S = 0.10\%$ $\lambda_B = 0.05\%$ with a portfolio mix of 0% to 100% with a 10% intervall. Segmented into Yearly, Quarterly and monthly for the period of 1980-2022 in the case two risky assets.

R code

```
rm(list=ls(all=TRUE))
library(zoo);library(moments);library(xts);library(xtable);library(quadprog);library(Ma
library(xts)
library(MTS)
library(xtable)
library(quadprog)
library(corpcor)
library(Rsolnp)
library(MASS)

# Working directory
source("markowitz.r")
#ret.aggregate <- function(x) prod(1+x) - 1

SR <- function(er) {
  # computes the Sharpe ratio
  return(mean(er)/sd(er)*sqrt(12))
}

SharpeTest <- function(ex.b, ex.a) {
  # test for equality of two Sharpe ratios
  # ex.b - excess returns to the BENCHMARK portfolio
  # ex.a - excess returns to the ACTIVE portfolio
  # returns the p-value of the test
  # the null hypothesis is rejected when p-value is small
  if (length(ex.a) != length(ex.b))
    stop("Different lengths of two returns!")
  SR.b <- mean(ex.b)/sd(ex.b)
  SR.a <- mean(ex.a)/sd(ex.a)
  ro <- cor(ex.b,ex.a)
  n <- length(ex.b)
  z <- (SR.a-SR.b)/sqrt( (2*(1-ro)+0.5*(SR.b^2+SR.a^2-2*SR.b*SR.a*ro^2))/n )
  pval <- pnorm(-sign(z)*abs(z))
  return(pval)
}

# Define the function to find optimal transactions
quadoptim <- function(w0, gamma, covmat, exret, tc) {
```

```
# Calculate matrices for quadratic programming
n <- length(w0)  # number of assets
#parameters
# Create the identity matrix
Inn <- diag(n) #identitiy matrix nxn
# Create the 3n x 3n identity matrix I_3n,3n
I.3nn <- diag(3*n)
In <- rep(0,n)
ones <- rep(1,n)
# Creating a column vector of zeros
On <- rep(0, n)
# Creating a column vector of ones
one.n <- rep(1, n)
w.minus <- w0 - one.n

# make Dmat
Onn <- diag(0,n)  # n*n matrix with all zeros
Dmat <- cbind(covmat, Onn, Onn)
Dmat <- rbind(Dmat, cbind(Onn, Onn, Onn), cbind(Onn, Onn, Onn))
Dmat <- make.positive.definite(Dmat)



dvec <- (1 / gamma) * c(exret,-tc, -tc)
#dvec<- -dvec  # NO NEED TO INVERT SIGN!
b <- c(1, w0, On, On, On, -one.n, -w0, w.minus)
bvec <-b
zeros <- rep(0,n)
Amat <- c(ones, zeros, zeros)
Amat <-rbind(Amat, cbind(Inn,Inn,-Inn))
Amat <- rbind (Amat,I.3nn,-I.3nn)
Amat <- t(Amat)

meq <- 1+n  # Number of equality constraints

# Solve the quadratic programming problem
solution <- solve.QP(Dmat = Dmat, dvec = dvec, Amat = Amat, bvec = bvec, ...
    meq = meq)

# Extract the new weights from the solution
opt.w <- solution$solution[1:n]
```

```
  return(opt.w)
}
#===========================================
# Program begins

# read the data table as a zoo-object
dataset <- read.zoo("newusdata.txt", header = TRUE, FUN = as.yearmon)

start.date <- as.yearmon(as.Date("1980-01-01", format="%Y-%m-%d"))
end.date <- as.yearmon(as.Date("2022-12-31", format="%Y-%m-%d"))
data <- window(dataset, start=start.date, end=end.date)

start.date.1 <- as.yearmon(as.Date("1960-01-01", format="%Y-%m-%d"))
end.date.1 <- as.yearmon(as.Date("1979-12-31", format="%Y-%m-%d"))
dataset <- window(dataset, start=start.date.1, end=end.date.1)


# select the variables
dates <- index(data) # retrieve dates from zoo-object
ret <- as.matrix(coredata(data)) # retrieve return data from zoo-object
ret.rf <- ret[,"TBills"]
ret.stocks <- ret[,"LgStocks"]
ret.bonds <- ret[,"LtBonds"]
ret.smstocks <-  ret[,"SmStocks"]
ret.ItBonds <-  ret[,"ItBonds"]


data.df <- fortify.zoo(data)

# Rename the columns appropriately if needed
colnames(data.df) <- c("Time", "ExcessReturn")
data.df$TBills <- coredata(data)[,"TBills"]
data.df$LgStocks <- coredata(data)[,"LgStocks"]
data.df$SmStocks <- coredata(data)[,"SmStocks"]
data.df$ItBonds <- coredata(data)[,"ItBonds"]
data.df$LtBonds <- coredata(data)[,"LtBonds"]
ggplot(data.df, aes(x = Time, y = TBills)) +
  geom_line() +
  theme_minimal() +
  labs(title = "Excess Returns for Risk-free",
       x = "Time",
```

```
      y = "Excess Return") +
  theme(plot.title = element_text(hjust = 0.5))
ggplot(data.df, aes(x = Time, y = TBills)) +
  geom_line() +
  theme_minimal() +
  labs(title = "Excess Returns for Large Stocks",
      x = "Time",
      y = "Excess Return") +
  theme(plot.title = element_text(hjust = 0.5))
ggplot(data.df, aes(x = Time, y = SmStocks)) +
  geom_line() +
  theme_minimal() +
  labs(title = "Excess Returns for Small Stocks",
      x = "Time",
      y = "Excess Return") +
  theme(plot.title = element_text(hjust = 0.5))
ggplot(data.df, aes(x = Time, y = ItBonds)) +
  geom_line() +
  theme_minimal() +
  labs(title = "Excess Returns for Intermediate-Term Bonds",
      x = "Time",
      y = "Excess Return") +
  theme(plot.title = element_text(hjust = 0.5))
ggplot(data.df, aes(x = Time, y = LtBonds)) +
  geom_line() +
  theme_minimal() +
  labs(title = "Excess Returns for Long-Term Bonds",
      x = "Time",
      y = "Excess Return") +
  theme(plot.title = element_text(hjust = 0.5))


# TARGET PORTFOLIO WEIGHTS
# 60% in Large Stocks and 40% in Long-term bonds
wt.stocks <- 0.6  # target weight in stocks
wt.bonds <- 1- wt.stocks # targe weight in bonds


# --- 60/40 Benchmark Strategy - Rebalance to Target Weights Assuming No ...
    Transaction Costs ---
```

```
ret.bench <- as.vector(cbind(ret.stocks, ret.bonds) %*% c(wt.stocks, ...
    wt.bonds))
cum.bench <- cumprod(1 + ret.bench)

# --- 60/40 Buy-and-Hold Strategy ---
nobs <- length(ret.stocks) # number of observations in the sample
ret.bh <- rep(0,nobs)
# we need to keep the track of previous portfolio weights
w.stocks.bh <- rep(wt.stocks, nobs)
w.bonds.bh <- rep(wt.bonds, nobs)


# --- 60/40 Always rebalancing strategy ---
w.stocks <- rep(0,nobs)
w.bonds <- rep(0,nobs)
ret.always <- rep(0,nobs)

# Turnover for the Always rebalancing strategy
turnover.always <- 0

# --- 60/40 Threshhold portfolio weights ---
ret.thres <- rep(0,nobs)
wtt.stocks <- rep(wt.stocks, nobs +1)
wtt.bonds <- rep(wt.bonds, nobs +1)


# Turnover for the threshhold rebalancing strategy
turnover.thres <- 0


# Define transaction cost rates
tc_rate_stocks <- 0.0025
tc_rate_bonds <- 0.001
nb <-3/100

# --- 60/40 range Strategy ---
ret.range <- rep(0,nobs)
w.stocks.range <- rep(wt.stocks, nobs + 1)
w.bonds.range <- rep(wt.bonds, nobs + 1)
new.w.stocks.range <- rep(0,nobs)
new.w.bonds.range <- rep (0,nobs)
```

```
w.stocks.up  <- 0
w.bonds.up <- 0
# Turnover for the range rebalancing strategy
turnover.range <- 0


#2 assets optimal strategy
n <- 2  # assets
w.target <- c(0.6, 0.4)
tc=c(0.0025,0.001)
cov.start.date <- as.yearmon(as.Date("1960-01-01", format="%Y-%m-%d"))
cov.end.date <- as.yearmon(as.Date("1979-12-31", format="%Y-%m-%d"))
cov.data <- window(dataset, start=cov.start.date, end=cov.end.date)

cov.dates <- index(cov.data) # retrieve dates from zoo-object
cov.ret <- as.matrix(coredata(cov.data)[, c("LgStocks", "LtBonds")])
covmat <- cov(cov.ret) * 12
ret<-cbind(ret.stocks,ret.bonds)


gamma <- 5
w0 <- c(0.6, 0.4)
exret <- gamma * covmat %*% w.target
# estimate parameters
r <- mean(ret.rf)

w.optimal <- matrix(NA, nrow=nobs, ncol=n)
w.optimal[1,] <- w.target #
ret.optimal  <- rep(0,nobs)
turnover.optimal <- 0

# main simulation loop
for (i in 1:nobs) {
  # return to the buy-and-hold strategy
  ret.bh[i] <- w.stocks.bh[i]*ret.stocks[i] + w.bonds.bh[i]*ret.bonds[i]
  if (i != nobs) {
    # compute the new weights for buy-and-hold
    top <- w.stocks.bh[i]*(1+ret.stocks[i])
    bot <- w.stocks.bh[i]*(1+ret.stocks[i]) + w.bonds.bh[i]*(1+ret.bonds[i])
    w.stocks.bh[i+1] = top/bot
    w.bonds.bh[i+1] = 1 - w.stocks.bh[i+1]
```

```
    turnover.bh = abs(w.stocks.bh[i+1] - w.stocks.bh[i]) + ...
        abs(w.bonds.bh[i+1] - w.bonds.bh[i])
  }



  # return to the always rebalance strategy
  ret.always[i] <- wt.stocks*ret.stocks[i] + wt.bonds*ret.bonds[i]
  turnover.now <- 0
  if (i != nobs) {
    # compute the new weights for always rebalance
    top <- wt.stocks*(1+ret.stocks[i])
    bot <- wt.stocks*(1+ret.stocks[i]) + wt.bonds*(1+ret.bonds[i])
    w.stocks = top/bot
    w.bonds = 1 - w.stocks
    ret.always[i] <- (1+ret.always[i])*(1 - ...
        abs(wt.stocks-w.stocks)*tc_rate_stocks -
                                              abs(wt.bonds-w.bonds)*tc_rate_bonds)-1
    turnover.now = abs(wt.stocks - w.stocks) + abs(wt.bonds - w.bonds)
    turnover.always = turnover.always + turnover.now
  }

  turnover.t <- 0
  ret.thres[i] <- wtt.stocks[i]*ret.stocks[i] + wtt.bonds[i]*ret.bonds[i]
  if (i != nobs) {
    # compute the new weights for threshold rebalancing
    top <- wtt.stocks[i]*(1+ret.stocks[i])
    bot <- wtt.stocks[i]*(1+ret.stocks[i]) + wtt.bonds[i]*(1+ret.bonds[i])
    wtt.stocks[i+1] = top/bot
    wtt.bonds[i+1] = 1 - wtt.stocks[i+1]
    # Check if rebalancing is needed
    dev.stocks <- abs(wtt.stocks[i+1] - wt.stocks)
    if (dev.stocks > nb) {

      ret.thres[i] <- (1+ret.thres[i])*(1 - ...
          abs(wt.stocks-wtt.stocks[i+1])*tc_rate_stocks
                                            - ...
                                              abs(wt.bonds-wtt.bonds[i+1])*tc_rate_bonds)-

      wtt.stocks [i+1] <- wt.stocks
      wtt.bonds [i+1] <- wt.bonds
```

```
      turnover.t = abs(wtt.stocks[i+1] - wtt.stocks[i])*tc_rate_stocks + ...
         abs(wtt.bonds[i+1] - wtt.bonds[i])
      turnover.thres <- turnover.thres + turnover.t
    }
  } else {
    wtt.stocks [i+1] <- wtt.stocks[i]
    wtt.bonds [i+1] <- wtt.bonds[i]
  }
  # RANGE REBALANCING
  turnover.this.period <- 0
  ret.range[i] <- w.stocks.range[i]*ret.stocks[i] + ...
     w.bonds.range[i]*ret.bonds[i]
  if (i != nobs) {
    # Compute new weights for range rebalancing
    top <- w.stocks.range[i] * (1 + ret.stocks[i])
    bot <- w.stocks.range[i] * (1 + ret.stocks[i]) + w.bonds.range[i] * (1 ...
       + ret.bonds[i])
    w.stocks.range[i+1] <- top / bot
    w.bonds.range[i+1] <- 1 - w.stocks.range[i+1]
    # Check if rebalancing is needed
    dev.stocks.r <- abs(w.stocks.range[i+1] - wt.stocks)
    if (dev.stocks.r > nb){
      # Rebalance back to target weights
      w.stocks.up <- wt.stocks + nb
      w.stocks.down <- wt.stocks - nb
      w.bonds.up<- wt.bonds+nb
      w.bonds.down<- wt.bonds-nb
      if(w.stocks.range[i+1] > w.stocks.up) {
        new.w.stocks <- w.stocks.up
      } else {
        new.w.stocks <- w.stocks.down
      }
      new.w.bonds <- 1- new.w.stocks

      ret.range[i] <- (1+ret.range[i])*(1 - abs( w.stocks.range[i+1] - ...
         new.w.stocks)*tc_rate_stocks
                                        - abs(w.bonds.range[i+1]- ...
                                          new.w.bonds)*tc_rate_bonds)-1
      w.stocks.range[i+1] <- new.w.stocks
      w.bonds.range[i+1] <- new.w.bonds
```

```
      turnover.this.period <- abs(w.stocks.range[i+1] - w.stocks.range[i]) ...
        + abs(w.bonds.range[i+1] - w.bonds.range[i])
      turnover.range <- turnover.range + turnover.this.period
    }
    #optimal strateyg
    turnover.opt <- 0
    ret.optimal[i] <- sum(w.optimal[i, ] * ret[i, ])
    if (i != nobs) {
      weights.optimal <- (w.optimal[i,] * (1 + ret[i,])) / ...
        sum(w.optimal[i,] * (1 + ret[i,]))
      opt.w <- quadoptim(w0 = weights.optimal, gamma = gamma, covmat = ...
        covmat, exret = exret, tc = tc)
      abs.diff <- abs(opt.w - weights.optimal)
      w.optimal[i + 1,] <- opt.w
      tc.impact = sum(abs.diff * tc)
      ret.optimal[i] <- (1 + ret.optimal[i]) * (1 - tc.impact) - 1
      turnover.opt <- abs(w.optimal[i+1] - w.optimal[i])
      turnover.optimal <- turnover.optimal + turnover.opt
    }

  }

}
# compute and report Sharpe ratios
SR.bh <- SR(ret.bh-ret.rf)
SR.bench <- SR(ret.bench-ret.rf)
SR.always <- SR(ret.always-ret.rf)
SR.thres <- SR(ret.thres-ret.rf)
SR.range <- SR(ret.range-ret.rf)
SR.optim <- SR(ret.optimal-ret.rf)



SR.bh
SR.bench
SR.always
SR.thres
SR.range
SR.optim

# Compute the p-values
pval.bench <- SharpeTest(ret.bh-ret.rf, ret.bench-ret.rf)
```

```
pval.bench

pval.always <- SharpeTest(ret.bh-ret.rf, ret.always-ret.rf)
pval.always

pval.thres <- SharpeTest(ret.bh-ret.rf, ret.thres-ret.rf)
pval.thres

pval.range <- SharpeTest(ret.bh-ret.rf, ret.range-ret.rf)
pval.range

pval.optimal <- SharpeTest(ret.bh-ret.rf, ret.optimal-ret.rf)
pval.optimal


# Compute the turnover
avg.always <- turnover.always/nobs
print(paste("Total Turnover alwahys: ", avg.always))
avg.thresh <- turnover.thres /nobs
print(paste("Average Turnover: ", avg.thresh))
average.turnover <- turnover.range / nobs
print(paste("Average Turnover: ", average.turnover))
average.opt <- turnover.optimal / nobs
print(paste("Average Turnover: ", average.opt))

# ---- compute average tracking error for all strategies -----------
#ret.bench as benckmark

#-----------------Naive--------------------
tracking.error.always <- abs(ret.always - ret.bench)
average.te.always <- mean(tracking.error.always)
print(paste("Average Tracking Error for Always Strategy: ", ...
    average.te.always))

#-----------------Thresshold--------------------
tracking.error.thres <- abs(ret.thres - ret.bench)
average.te.thres <- mean(tracking.error.thres)
print(paste("Average Tracking Error for Threshold Strategy: ", ...
    average.te.thres))

##-----------------Range--------------------
```

```
tracking.error.range <- abs(ret.range - ret.bench)
average.te.range <- mean(tracking.error.range)
print(paste("Average Tracking Error for Range Strategy: ", average.te.range))

#-----------------Optimal--------------------
tracking.error.optimal <- abs(ret.optimal - ret.bench)
average.te.optimal <- mean(tracking.error.optimal)
print(paste("Average Tracking Error for Optimal Strategy: ", ...
   average.te.optimal))


#-----------------PLOTS--------------------

z.w <- zoo(cbind(wtt.stocks, wtt.bonds), order.by=dates)
plot(z.w, ylim=c(0,1), col=c("red","blue"), plot.type="single")
abline(h=wt.stocks+nb, col="red", lty=3)
abline(h=wt.stocks-nb, col="red", lty=3)
abline(h=wt.bonds+nb, col="blue", lty=3)
abline(h=wt.bonds-nb, col="blue", lty=3)

z.wa <- zoo(cbind(w.stocks.range, w.bonds.range), order.by=dates)

plot(z.wa, ylim=c(0,1), col=c("red","blue"), plot.type="single")
abline(h=w.stocks.up, col="red", lty=3)
abline(h=w.stocks.down, col="red", lty=3)
abline(h=w.bonds.up, col="blue", lty=3)
abline(h=w.bonds.down, col="blue", lty=3)

# Descriptive statistics

# For stocks
stocks.stddev <- sqrt(var(ret.stocks))   # Standard deviation
stocks.mean <- mean(ret.stocks)
stocks.skewness <- skewness(ret.stocks)
stocks.kurtosis <- kurtosis(ret.stocks)
stocks.min <- min(ret.stocks)
stocks.max <- max(ret.stocks)

# For bonds
bonds.stddev <- sqrt(var(ret.bonds))  # Standard deviation
bonds.mean <- mean(ret.bonds)
```

```
bonds.skewness <- skewness(ret.bonds)
bonds.kurtosis <- kurtosis(ret.bonds)
bonds.min <- min(ret.bonds)
bonds.max <- max(ret.bonds)

# For risk-free rate
rf.stddev <- sqrt(var(ret.rf))   # Standard deviation
rf.mean <- mean(ret.rf)
rf.skewness <- skewness(ret.rf)
rf.kurtosis <- kurtosis(ret.rf)
rf.min <- min(ret.rf)
rf.max <- max(ret.rf)

# Combine vectors into a data frame
descriptives <- data.frame(ret.stocks, ret.bonds, ret.rf)
summary.stats <- summary(descriptives)
xtable(summary.stats)

descriptive_stats <- data.frame(
  mean = apply(data, 2, mean, na.rm = TRUE),
  sd = apply(data, 2, sd, na.rm = TRUE),
  skewness = apply(data, 2, skewness, na.rm = TRUE),
  kurtosis = apply(data, 2, kurtosis, na.rm = TRUE),
  min = apply(data, 2, min, na.rm = TRUE),
  max = apply(data, 2, max, na.rm = TRUE)
)

# Transpose the data frame so that the statistics are in columns
descriptive.stats.t <- t(descriptive_stats)
latex.table <- xtable(descriptive.stats.t, caption = "Descriptive ...
    Statistics", digits = c(3, 3, 3, 3, 3, 3))
latex.table

#-----------------------------------------------
# correlation matrix
correlation.matrix<-cor(ret)
latex <- xtable(correlation.matrix, caption="Correlation Matrix", digits=4)

print(latex, type = "latex", floating = FALSE, include.rownames = TRUE, ...
    include.colnames = TRUE)
ret.sub <- cbind(ret.stocks, ret.bonds,ret.rf)
```

```
cor(ret.sub)
#---------------------------------------------

# annual mean and std for the four rebalancing strategies
mean(ret.bh)*12*100
mean(ret.always)*12*100
mean(ret.thres)*12*100
mean(ret.range)*12*100
mean(ret.optimal)*12*100

sd(ret.bh)*sqrt(12)
sd(ret.always)*sqrt(12)
sd(ret.thres)*sqrt(12)
sd(ret.range)*sqrt(12)
sd(ret.optimal)*sqrt(12)

# frequency returns for stocks and bonds
hist(ret.stocks, breaks=50, col="grey", main="Histogram of Stock Returns",
     xlab="Returns", ylab="Frequency")
hist(ret.bonds, breaks=50, col="grey", main="Histogram of Bond Returns",
     xlab="Returns", ylab="Frequency")

hist(ret.thres, breaks=50, col="grey", main="Histogram of Bond Returns",
     xlab="Returns", ylab="Frequency")
hist(ret.range, breaks=50, col="grey", main="Histogram of Bond Returns",
     xlab="Returns", ylab="Frequency")

# plot the indices
cum.bh <- cumprod(1+ret.bh)
cum.always <- cumprod(1+ret.always)
cum.thres <- cumprod(1+ret.thres)
cum.range <- cumprod(1+ret.range)

z.ind <- zoo(cbind(cum.bh, cum.bench, cum.always, cum.thres, cum.range), ...
   order.by=dates)
plot(log(z.ind), plot.type="single", col=c("red","blue","green", "brown", ...
   "pink"))



# Asset weights thresh vs BAh
z.wtt.stocks <- zoo(wtt.stocks, order.by=index(wtt.stocks))
```

```r
z.w.stocks.bh <- zoo(w.stocks.bh, order.by=index(w.stocks.bh))
z.w.bonds.bh <- zoo(w.bonds.bh, order.by=index(w.bonds.bh))
z.wtt.bonds <- zoo(wtt.bonds, order.by=index(wtt.bonds))
thresh.w <- zoo(cbind(z.wtt.stocks, z.w.stocks.bh,z.wtt.bonds ...
    ,z.w.bonds.bh), order.by=dates)
thresh.w.df <- fortify.zoo(thresh.w)
names(thresh.w.df) <- c("dates", "Threshold", "Buy-and-Hold", ...
    "ThresholdBonds", "BondsBH")
thresh.w.df$dates <- as.Date(thresh.w.df$dates)
thresh.w.l <- pivot_longer(thresh.w.df,
                              cols = -dates,
                              names_to = ".",
                              values_to = "Weight")

ggplot(thresh.w.l, aes(x = dates, y = Weight, color = .)) +
  geom_line(size = 0.5) +
  scale_color_manual(values = c("Threshold" = "red", "Buy-and-Hold" = "blue",
                                "ThresholdBonds" = "red", "BondsBH" = "blue"),
                      breaks = c("Threshold", "Buy-and-Hold")) +
  labs(x = "Investment Horizon", y = "Weight") +
  geom_hline(yintercept = wt.stocks, linetype = "dashed", color = "black") +
  geom_hline(yintercept = wt.bonds, linetype = "dashed", color = "black") +
  theme_minimal() +
  theme(legend.position = "top")




# Asset weight range vs BAH
z.range.stocks <- zoo(w.stocks.range, order.by=index(w.stocks.range))
z.w.stocks.bh <- zoo(w.stocks.bh, order.by=index(w.stocks.bh))
z.w.bonds.bh <- zoo(w.bonds.bh, order.by=index(w.bonds.bh))
z.range.bonds <- zoo(w.bonds.range, order.by=index(w.bonds.range))
range.w <- zoo(cbind(z.range.stocks, z.w.stocks.bh,z.range.bonds, ...
    z.w.bonds.bh), order.by=dates)
range.w.df <- fortify.zoo(range.w)
names(range.w.df) <- c("dates", "Range", "Buy-and-Hold", "range.b", "bah.b")
range.w.df$dates <- as.Date(range.w.df$dates)
range.w.l <- pivot_longer(range.w.df,
                             cols = -dates,
                             names_to = ".",
```

```
                            values_to = "Weight")
ggplot(range.w.l, aes(x = dates, y = Weight, color = .)) +
  geom_line(size = 0.5) +
  scale_color_manual(values = c("Range" = "red", "Buy-and-Hold" = "blue",
                                "range.b" = "red", "bah.b" = "blue"),
                     breaks = c("Range", "Buy-and-Hold")) +
  labs(x = "Investment Horizon", y = "Weight") +
  geom_hline(yintercept = wt.stocks, linetype = "dashed", color = "black") +
  geom_hline(yintercept = wt.bonds, linetype = "dashed", color = "black") +
  theme_minimal() +
  theme(legend.position = "top")

# Asset weight Optimal vs BAH

z.optimal.stock <- zoo(w.optimal[, 1], order.by = index(w.optimal))
z.optimal.bond <- zoo(w.optimal[, 2], order.by = index(w.optimal))
optimal.w <- zoo(cbind(z.optimal.stock, z.w.stocks.bh,z.optimal.bond, ...
    z.w.bonds.bh), order.by=dates)
optimal.w.df <- fortify.zoo(optimal.w)
names(optimal.w.df) <- c("dates", "Optimal", "Buy-and-Hold", "optimal.b", ...
    "bah.b")
optimal.w.df$dates <- as.Date(optimal.w.df$dates)
optimal.w.l <- pivot_longer(optimal.w.df,
                            cols = -dates,
                            names_to = ".",
                            values_to = "Weight")
ggplot(optimal.w.l, aes(x = dates, y = Weight, color = .)) +
  geom_line(size = 0.5) +
  scale_color_manual(values = c("Optimal" = "red", "Buy-and-Hold" = "blue",
                                "optimal.b" = "red", "bah.b" = "blue"),
                     breaks = c("Optimal", "Buy-and-Hold")) +
  labs(x = "Investment Horizon", y = "Weight") +
  geom_hline(yintercept = wt.stocks, linetype = "dashed", color = "black") +
  geom_hline(yintercept = wt.bonds, linetype = "dashed", color = "black") +
  theme_minimal() +
  theme(legend.position = "top")



# threshhold vs bh 1$ invested
thresh.ind <- zoo(cbind(cum.thres, cum.bh), order.by=dates)
thresh.df <- fortify.zoo(thresh.ind)
```

```
names(thresh.df) <- c("dates", "Threshold", "Buy-and-Hold")
thresh.df$dates <- as.Date(thresh.df$dates)
thresh.l <- pivot_longer(thresh.df,
                         cols = -dates,
                         names_to = ".",
                         values_to = "Return")
ggplot(thresh.l, aes(x = dates, y = log(Return), color = .)) +
  geom_line(size = 0.5) +
  scale_color_manual(values = c("Threshold" = "red", "Buy-and-Hold" = ...
     "blue")) +
  labs(x = "Investment Horizon", y = "Log Return") +
  theme_minimal() +
  theme(legend.position = "top")


# Cumulative return range vs bah
range.ind <- zoo(cbind(cum.range, cum.bh), order.by=dates)
range.df <- fortify.zoo(range.ind)
names(range.df) <- c("dates", "Range", "Buy-and-Hold")
range.df$dates <- as.Date(range.df$dates)
range.l <- pivot_longer(range.df,
                        cols = -dates,
                        names_to = ".",
                        values_to = "Return")
ggplot(range.l, aes(x = dates, y = Return, color = .)) +
  geom_line(size = 0.5) +
  scale_color_manual(values = c("Range" = "red", "Buy-and-Hold" = "blue")) +
  labs(x = "Investment Horizon", y = "Return") +
  theme_minimal() +
  theme(legend.position = "top")

# Wealt indices for RF, large stocks and bonds

cum.rf <- cumprod(1+ret.rf)
cum.stocks <- cumprod(1+ret.stocks)
cum.bonds <- cumprod(1+ret.bonds)
cum.smstocks <- cumprod(1 + ret.smstocks)
cum.itbonds <- cumprod(1 + ret.ItBonds)

z.ind <- zoo(cbind(cum.rf, cum.stocks, cum.bonds, cum.smstocks, ...
    cum.itbonds), order.by=dates)
```

```r
data <- fortify.zoo(z.ind)
names(data) <- c("dates", "Risk-Free", "Large Stocks", "Inter-term Bonds", ...
    "Small Stocks", "Long-term Bonds")
data$dates <- as.Date(data$dates)
data.l <- pivot_longer(data,
                       cols = -dates,
                       names_to = ".",
                       values_to = "Return")
data.l$. <- factor(data.l$., levels = c("Large Stocks", "Small Stocks", ...
    "Long-term Bonds", "Inter-term Bonds", "Risk-Free"))

ggplot(data.l, aes(x = dates, y = log(Return), color = .)) +
  geom_line(size = 0.5) +
  scale_color_manual(values = c("Risk-Free" = "red",
                                "Large Stocks" = "blue",
                                "Inter-term Bonds" = "green",
                                "Small Stocks" = "purple",
                                "Long-term Bonds" = "orange"),) +
  labs(x = "Investment Horizon", y = "Return") +
  theme_minimal() +
  theme(legend.position = "top")




#Tables LaTex using csv

#fulltable of turnover for nt-band 1,3,5,7 - monthly, quarterly and yearly ...
    (10%-100%)
tableturn <- read.csv("fulltable.csv", header = TRUE, sep = ";")
head(tableturn)
print(tableturn)
table.xtable <- xtable(tableturn)
print.xtable(table.xtable, include.rownames = FALSE, hline.after = NULL, ...
    comment = FALSE)


#------- SCIRPT FOR FOUR RISKY ASSETS ------------
```

```r
#------- Parameteres ------------
n <- 4  # assets
w.target <- c(0.3, 0.3, 0.2, 0.2)
nobs <- nrow(ret)
mu <- er
er <- matrix(er, nrow = n, ncol = 1)
gamma <- 5
tc <- c(0.0050, 0.005, 0.002, 0.002)
#w0 <- c(0.3, 0.3, 0.2, 0.2)
exret <- gamma * covmat %*% w.target



#-------------Benchmark----------------------------

ret.bench <- as.vector(cbind(ret.lgstocks, ...
    ret.ltbonds,ret.smstocks,ret.itbonds) %*% w.target)
ret.bench

#-------------Buy and hold ----------------------------
nobs <- nrow(ret)
w.bh <- matrix(NA, nrow=nobs, ncol=n)
w.bh[1,] <- w.target
weight.bh <- rep(0,nobs)
ret.bh <- rep(0, nobs)
turnover.bh <- numeric(nobs)

#------------Always strategy----------------------------
w.always <- matrix(NA, nrow=nobs, ncol=n)
w.always[1,] <-w.target
weight.always <- rep(0,nobs)
ret.always <- rep(0, nobs)
turnover.always <- 0



#---------Optimal strategy---------------------------
w.optimal <- matrix(NA, nrow=nobs, ncol=n)
w.optimal[1,] <- w.target
ret.optimal  <- rep(0,nobs)
length(ret.optimal)
turnover.optimal <- 0
```

```
weight.always
w.always
turnover.always
for (i in 1:(nobs - 1)) {
  ret.bh [i] <- sum(w.bh[i,] * ret [i,])
  if (i != nobs){
    weight.bh <- w.bh[i,] * (1 + ret[i,])
   w.bh[i + 1,] <- weight.bh / sum(weight.bh)
  }
  turnover.now <- 0
   ret.always[i] <- sum(w.always[i,] * ret [i,])
  if (i != nobs) {

   weight.always <- (w.always[i,] * (1 + ret[i,])) / sum(w.always[i,] * (1 ...
      + ret[i,]))

     w.always[i + 1,] <- w.target

   abs.always <- abs(weight.always-w.target)



   tc.always <- sum(abs.always*tc)

   ret.always[i] <- (1 + ret.always[i]) * (1 - tc.always) -1
   turnover.now = abs(w.target - weight.always)

   turnover.always = turnover.always + turnover.now
  }
  turnover.opt <- 0
  ret.optimal[i] <- sum(w.optimal[i, ] * ret[i, ])
  if (i != nobs) {
  # Compute weights based on returns
  weights.optimal <- (w.optimal[i,] * (1 + ret[i,])) / sum(w.optimal[i,] * ...
      (1 + ret[i,]))
    opt.w <- quadoptim(w0 = weights.optimal, gamma = gamma, covmat = ...
        covmat, exret = exret, tc = tc)
    abs.diff <- abs(opt.w - weights.optimal)
   w.optimal[i + 1,] <- opt.w
  tc.impact = sum(abs.diff * tc)
  ret.optimal[i] <- (1 + ret.optimal[i]) * (1 - tc.impact) - 1
  turnover.opt <- abs(w.optimal[i+1] - w.optimal[i])
```

```
   turnover.optimal <- turnover.optimal + turnover.opt
  }
}


#tunrover optimal
average.alw <- sum(turnover.always) / nobs
print(paste("Average Turnover: ", average.alw))


average.opt <- sum(turnover.optimal) / nobs
print(paste("Average Turnover: ", average.opt))

SR.bh <- SR(ret.bh-ret.rf)
SR.bench <- SR(ret.bench-ret.rf)
SR.always <- SR(ret.always-ret.rf)
SR.optimal <- SR(ret.optimal-ret.rf)


SR.bh
SR.bench
SR.always
SR.optimal
# Compute the p-values
pval.bench <- SharpeTest(ret.bh-ret.rf, ret.bench-ret.rf)
pval.bench

pval.always <- SharpeTest(ret.bh-ret.rf, ret.always-ret.rf)
pval.always

pval.optimal <- SharpeTest(ret.bh-ret.rf, ret.optimal-ret.rf)
pval.optimal

#-------Descriptive------------
mean(ret.bh)*12*100
mean(ret.always)*12*100.
mean(ret.optimal)*12*100

sd(ret.bh)*sqrt(12)
sd(ret.always)*sqrt(12)
sd(ret.optimal)*sqrt(12)
```

```
#-------Asset weight optimal portoflio showing all 4 risky asset-----------
z.w.optimal <- zoo(w.optimal, order.by=dates)
optimal.weights.df <- fortify.zoo(z.w.optimal)
names(optimal.weights.df) <- c("dates", "Large Stocks", "Small stocks", ...
    "Long-term Bonds", "Intermediate bonds")
optimal.weights.l <- pivot_longer(optimal.weights.df,
                                  cols = -dates,
                                  names_to = ".",
                                  values_to = "Weight")

optimal.weights.l <- optimal.weights.l %>%
  mutate(LineType = case_when(
    . %in% c("Large Stocks", "Long-term Bonds") ~ "solid",
    . %in% c("Small stocks", "Intermediate bonds") ~ "dotted"
  ))
optimal.weights.l$. <- factor(optimal.weights.l$., levels = c("Large ...
    Stocks", "Small stocks", "Long-term Bonds", "Intermediate bonds"))
plot.opt <- ggplot(optimal.weights.l, aes(x = dates, y = Weight, color = ., ...
    linetype = LineType)) +
  geom_line(size = 0.4) +
  labs(x = "Investment Horizon", y = "Asset Weight") +
  scale_color_manual(values = c(
    "Large Stocks" = "red",
    "Small stocks" = "red",
    "Intermediate bonds" = "blue",
    "Long-term Bonds" = "blue")) +
  geom_hline(yintercept = 0.3, linetype = "dotted", color = "black") +
  geom_hline(yintercept = 0.2, linetype = "dotted", color = "black") +
  scale_linetype_manual(values = c("solid" = "solid", "dotted" = "dotted")) +
  theme_minimal() +
  theme(legend.position = "top") +
  guides(
    color = guide_legend(override.aes = list(linetype = c("solid", ...
        "dotted", "solid", "dotted"))),
    linetype = FALSE
  )
plot.opt <- plot.opt +
 scale_y_continuous(limits = c(0.17, 0.36))
print(plot.opt)
```

```r
#-------Asset weight optimal portoflio showing all 4 asset combined into ...
    "Stocsk and Bonds"------------
optimal.weights.df$Stocks <- optimal.weights.df$`Large Stocks` + ...
    optimal.weights.df$`Small stocks`
optimal.weights.df$Bonds <- optimal.weights.df$`Long-term Bonds` + ...
    optimal.weights.df$`Intermediate bonds`
optimal.weights.df <- optimal.weights.df %>%
  select(dates, Stocks, Bonds)

optimal.weights.l <- pivot_longer(optimal.weights.df,
                                  cols = -dates,
                                  names_to = ".",
                                  values_to = "Weight")
optimal.weights.l$. <- factor(optimal.weights.l$., levels = c("Stocks", ...
    "Bonds"))

plot.optt <- ggplot(optimal.weights.l, aes(x = dates, y = Weight, color = ...
    .)) +
  geom_line(size = 0.4) +
  labs(x = "Investment Horizon", y = "Asset Weight") +
  scale_color_manual(values = c(
    "Stocks" = "red",
    "Bonds" = "blue"
  ),
  labels = c("Large + Small Stocks", "Long-term + Inter-term Bonds")
  ) +
  geom_hline(yintercept = 0.6, linetype = "dotted", color = "black") +
  geom_hline(yintercept = 0.4, linetype = "dotted", color = "black") +
  theme_minimal() +
  theme(legend.position = "top")

plot.optt <- plot.optt +
  scale_y_continuous(limits = c(0.3, 0.7))
print(plot.optt)
```

Discussion Note - Johan Nguyen: International

To mark the end of my five years at the University of Agder, this discussion paper serves as the concluding remarks for my educational journey. Throughout my time in school, I have gathered knowledge in economics and analytical finance. This thesis is the final piece of my master's degree in business administration at the University of Agder, with a specialization in Analytical Finance.

I believe the process of writing this master thesis has taught me three key components; (1) patient, (2) consistency and (3) teamwork. Crafting a comprehensive thesis requires considerable time and effort, demanding a high level of patience. Additionally, it has taught me the importance of consistency; through steady and persistent work, significant progress can be achieved over time. Lastly, teamwork has been crucial, as understanding and effectively communicating with collaborators is vital for successful joint efforts. This experience has not only enhanced my ability to work well with others but also improved my skills in efficient and productive communication.

The master's thesis provides an empirical analysis of portfolio optimization, specifically focusing on fully invested portfolios with transaction costs and excluding the risk-free asset. The primary motivation for this study is to bridge the gap in existing research regarding portfolios with multiple risky assets, as previous studies have not extensively explored this area. This research aims to enhance our understanding of this complex field and contribute to ongoing discussions within the academic community. Our thesis demonstrates the ability to achieve significant results in portfolios comprising multiple risky assets, an area previously underserved in the literature due to its complexity. Notably, there is no analytical solution for optimizing portfolios with more than two risky assets when transaction costs are considered. Consequently, earlier studies have relied on numerical methods to address this challenge, which is the approach we also adopt. Overall, the motivation behind this thesis is to advance research in optimal portfolio choice, providing valuable insights and methodologies that can be applied to real-world investment scenarios involving multiple risky assets and transaction costs.

Global financial markets are increasingly interconnected, and international trends and forces significantly impact portfolio management strategies. In relation to the topic "international" I will discuss how my thesis is influenced by international trends and forces, such as globalization, technological advancements, regulatory changes, and economic cycles. Our master thesis addresses optimal rebalancing strategies for a fully invested portfolio with transaction costs. This area is deeply affected by international financial trends and forces, and are applicable for whoever that want to implement our approach. The research question revolves around determining the most effective rebalancing strategies for portfolio with varying numbers of risky assets with transaction costs, under different scenarios. International conditions, such as trends and forces shape questions by influencing the baseline assumptions like asset returns, volatility and transaction costs.

For our research we employed a 60/40 portfolio with transaction costs and we employ five different strategies for the case of two risky assets (large stocks and long-term bonds) and 3 strategies for the case of four risky assets (large stocks, small stocks, long-term bonds and intermediate term bonds). Our findings indicate that rebalancing strategies generally outperform the benchmark Buy-and-Hold strategy in terms of the Sharpe ratio. However, our thesis is limited to the US market. The performance of these strategies may differ based on the market; for example, results may vary in emerging markets due to different market dynamics, regulatory environments, and levels of market efficiency. The term "international" in relation to our thesis is highly relevant as it underscores the necessity of adapting rebalancing strategies to diverse global contexts. Further research could expand on this by testing the strategies in various international markets, taking into account the unique characteristics and conditions that may influence the effectiveness of portfolio rebalancing. This broader perspective would provide a more comprehensive understanding of how global financial trends and forces impact the optimal rebalancing strategies for fully invested portfolios.

Furthermore, our thesis focuses on the US market and consider Large stocks, Small stocks and Long-term and intermediate term bonds. We observe that there were several big economic decline for the periods of 2000 (dot.com crisis) and 2008 (housing crisis). By employing a rebalancing strategy for these events, will make the investor less vulnerable to do downside, as it is in fact stocks that decreases the most during these crisis compared to bonds. A key point of rebalancing strategies is the ability to get a portfolio which is considering the risk. If there is no rebalancing done, a portfolio over time, would eventually only consist of stocks. For such a case, the risk will increase as the inherent risk in stocks versus bonds is higher. Therefore, the thesis aims to provide valuable information on whether rebalancing is outperforming a strategy where you don't rebalance (Buy-and-Hold strategy). The rebalancing strategies were tested against this Buy-and-Hold strategy and we were able to confirm that the rebalancing strategies are able to outperform the Buy-and-Hold in terms of the Sharpe ratio.

By understanding and incorporating these international trends and forces, investors can enhance their portfolio management practices and achieve better outcomes. This discussion paper underscores the relevance of international considerations in financial research and provides a broader perspective on the impact of global financial trends on portfolio rebalancing strategies. Market efficiency varies significantly across different regions and countries, impacting the effectiveness of rebalancing strategies. In highly efficient markets, where information is quickly reflected in asset prices, the opportunities for gaining excess returns through rebalancing might be limited. Conversely, in less efficient markets, where information asymmetry and slower price adjustments are more common, rebalancing strategies can potentially exploit mispricings and yield higher returns.

In our thesis, we primarily focused on the US market, which is generally considered to be highly efficient. However, when applying our findings to international contexts, it is crucial to consider the varying degrees of market efficiency. For instance, emerging markets often exhibit lower levels of efficiency compared to developed markets. This presents unique opportunities and challenges for rebalancing strategies. Investors in these markets may benefit from more frequent rebalancing to capitalize on volatility and arbitrage opportunities. However, investors must also be mindful of higher transaction costs and potential liquidity issues that can arise in less efficient markets. Our thesis focuses on the impact of transaction costs on rebalancing strategies, emphasizing how these costs influence the overall effectiveness of rebalancing. Each time a portfolio is rebalanced, transaction costs are incurred, which in turn reduce the portfolio's return. This critical factor must be carefully considered when developing and evaluating rebalancing strategies, as the frequency and method of rebalancing can significantly affect the net performance of the portfolio. By extending our research to include a variety of international markets, we can gain a deeper understanding of how market efficiency influences the optimal frequency and methodology of rebalancing. Such insights would be of great value for global investors seeking to optimize their portfolios in different economic and regulatory environments. This expanded research could also help identify specific market conditions under which rebalancing strategies are most effective, providing a more comprehensive framework for portfolio management across diverse global markets.

Our thesis is focused on the US market, as previously mentioned. In terms of term "international" in relation to our master thesis, we know that cultural and behavioral factors play a significant role in how investor think. Investors are influnced by its cultural attitudes towards risk and investment. Our thesis, while grounded in the context of the US market, acknowledges the importance of these behavioral factors. Internationally, these factors can cause significant variations in market reactions to economic events, regulatory changes, and technological advancements. Understanding these cultural nuances is essential for adapting rebalancing strategies to align with the behavior of local investors and market participants.

Furhtermore the impact of geopolitcal events on rebalancing strategies. This can be in relation to events such as, political elections, trade disputes. Additionally, it can affect the market volatility and uncertainty, therefore impacting the prices in the market. It would be interesting to see if there was any significant differences in the different markets in such an area. Exploring the rebalancing strategies we have employed in different markets would be highly interestingly, and to assess if our approach is significant internationally. In our thesis, we primarily examined rebalancing strategies within the context of a 60/40 portfolio of large stocks and bonds. However, expanding this analysis to include a broader range of asset classes, such as real estate, commodities, and alternative investments, can provide a more holistic view of rebalancing strategies. Additionally, geographic diversification across developed and emerging markets can further reduce risk and improve portfolio resilience.

Technological innovations, will in the optimization field, be extremely relevant. Technology can advance and create new opportunities, such as artificial intelligence, machine learning, which transforms how data is analyzed and processed, how transactions are executed and also how portfolios are managed. Our thesis employs traditional analytical methods to evaluate rebalancing strategies, but integrating artificial intelligence and machine learning can significantly enhance the precision and adaptability of these strategies. For example, artificial intelligence algorithms can analyze vast amounts of data in real-time, identifying patterns and trends that human analysts might miss. This can lead to more informed and timely rebalancing decisions, improving portfolio performance. In a topic such as portfolio optimization, technology is highly relevant, as processing big data requires time. And if there was a possibility for something or an algorhitm that can track data and markets as it happens would potentially disrupt the financial markets. The optimal strategy we have utilized, uses an algorithm, although simplier and based on assumption that might not hold in the real world. I would believe that as time goes, technological advancement could implement our method without any simplification and assumption, making it highly reliabe for real-world market and investors.

In summary, our thesis on optimal rebalancing strategies for fully invested portfolios with transaction costs is deeply influenced by international trends and forces. Globalization, technological advancements, regulatory changes, and economic cycles all play a critical role in shaping portfolio management strategies. Our research demonstrates the importance of adapting rebalancing strategies to diverse global contexts and highlights the potential benefits of rebalancing in terms of risk management and portfolio performance.

Our research underscores the importance of adapting rebalancing strategies to diverse international contexts, providing valuable insights for global investors. This broader perspective not only enhances our understanding of the impact of global financial trends on portfolio management but also highlights the dynamic and interconnected nature of modern financial markets. As such, our thesis contributes to the ongoing discourse on international finance and offers practical guidance for optimizing portfolio performance in a rapidly evolving global landscape.

References

Van Oort, C. M. (2018). Market efficiency in U.S. stock markets: A study of the Dow 30 and the S&P 30 (Master's thesis, University of Vermont). Retrieved from: https://scholarworks.uvm.edu/graddis/917

OpenAI. (2024). ChatGPT: An AI language model. https://www.openai.com. Version GPT-4.

Discussion Paper - Frida Hindersland: Responsible

After three years as a student at the University of Technology, Sydney, I chose to move back to Norway to do my Master´s degree. I am satisfied with the knowledge I have attained over the past five years as a student. The University of Agder has provided me with a variety of courses through the specialisation within Analytical Finance, and I will highlight how it has given me insight about responsibility in relation to the financial industry.

Our thesis is about the optimal portfolio choice. We provide an empirical analysis of two portfolios. The first portfolio consists of two risky assets. The second portfolio consist of four risky assets. Both portfolios are fully invested, they incur proportional transaction costs and short-sales are prohibited. The motivation behind the thesis is that previous research has either too complex methods, or they fail to produce results when the number of assets increases beyond two. We reveal how the optimal portfolio looks when the computation is based on an arbitrary no-trade band for the two-asset case, whereas in the four-asset case we show results that are based on the numerical method provided by Zakamulin (2024).

As discussed in our thesis, most people are risk averse by nature. Therefore, we need to find the optimal balance between risk and return when considering how to allocate wealth Markowitz (1952).

Assume that you are a portfolio manager, and you help a client that has never invested in the financial market before. First, you are responsible of going through the risk associated with each asset class and how it relates to market risk. You are the professional party, and together with the client, you determine how much wealth should go in stocks, bonds, real estate and other asset classes. In the case where the client chooses 60% in the stocks and 40% in bonds, these target asset weights will change with time. If the stock market performs better than the bond market, then the predefined risk profile changes and the client does no longer have a portfolio that matches his risk-return appetite. Therefore, the portfolio need adjustment so that the investor does not have more risk than he is prepared for.

This introduces rebalancing, which involves a realignment of the portfolio. Whether rebalancing is triggered depends on the rebalancing strategy in action. There are numerous studies on the topic and most early studies use portfolios of one or two assets in addition to a risk-free alterative.

Periodic rebalancing is a strategy where the portfolio is being rebalanced at a pre-determined time, regardless of the deviation from the original asset allocation. Threshold is a strategy where the rebalancing occurs when there is a deviation beyond a predefined threshold. Similarly, with a Range strategy, rebalancing happens based on a predefined threshold, but it rebalances back to a boundary of the no-trade region. If no rebalancing action is taken, then the portfolio is essentially a Buy-and-Hold portfolio (Dichtl, Drobetz, & Wambach, 2014). Historically, the stock market has generally outperformed the bond market over the long term. Therefore, if you start with a 60/40 allocation (60% stocks and 40% bonds) and do not make any changes for the next 20 years, the proportion of stocks in your portfolio would likely increase significantly due to their higher returns. Consequently, stocks would dominate the portfolio, potentially shifting the risk profile and exposing the investor to more risk than initially intended.

Rebalancing incurs transaction costs; therefore, we must consider the cost associated with realigning the portfolio to establish whether or not it provides more benefit to us. The early studies on the topic assumes a perfect capital market, where transaction costs and taxes do not exist. However, if a portfolio manager would tell his clients that they should rebalance their portfolio and not to worry about the costs, that would not be a good recommendation. This is because, the costs can diminish the portfolio returns to an extent where it is simply not worth it.

Which rebalancing strategy maximizes returns and minimize risk? Well, that depends as the rebalancing strategies mentioned are based on arbitrary numbers in our thesis. Indicating that we need a realistic model that can be practical for portfolio managers, so that these managers can give a nuanced and more accurate answer to their clients. The main objective of the portfolio manager is to reduce risk at a given level of return. However, there are many factors to consider when meausring risk.

To provide an overview of risk, I will highlight critical aspects and examine past events that offer valuable insights. The Global Financial Crisis (GFC) of 2008 serves as a stark reminder of the catastrophic consequences that can arise from irresponsible financial practices. This crisis, precipitated by a combination of high-risk lending practices, insufficient regulatory oversight, and a pervasive culture of short-term profit maximization, underscores the imperative for a more responsible approach in the financial industry. Jennifer McGillivray and Hung-Gay Fung (2013) point out what happened during the GFC and what measures that have been made ever since.

Financial institutions engaged in aggressive lending to subprime borrowers, who were often not equipped to meet their mortgage obligations. This was driven by the desire to expand market share and generate immediate profits. Mortgages were bundled into complex financial products, such as mortgage-backed securities (MBS) and collateralized debt obligations (CDOs), which were sold to investors worldwide. These products obscured the underlying risks and were often rated highly by credit rating agencies, despite their inherent vulnerabilities.

Regulatory bodies failed to adequately oversee and manage the risks associated with these complex financial instruments. The lack of transparency and the rapid innovation in financial products outpaced the ability of regulators to monitor and mitigate systemic risks. Furthermore, financial institutions prioritized short-term profits over long-term stability. Compensation structures often incentives risky behavior, as executives and traders were rewarded for short-term gains without sufficient consideration of the long-term consequences. Credit rating agencies had conflicts of interest as they were paid by the issuers of the securities they rated. This compromised the objectivity of their ratings and contributed to the widespread mispricing of risk.

Financial institutions failed to act in the best interests of their clients. This breach of fiduciary duty was evident in the sale of high-risk financial products to investors without full disclosure of the associated risks. The complexity of financial products, combined with misleading credit ratings, meant that many investors were unaware of the true nature of their investments. This lack of transparency eroded trust in the financial system.

The Global Financial Crisis (GFC) shed light on significant flaws within the financial industry. One critical aspect is the need for transparency in financial products. Investors must have access to clear and comprehensive information about the risks and rewards associated with their investments. This level of transparency is crucial for maintaining trust in the financial system. Additionally, compensation structures need to be aligned with long-term performance and risk management. Bonuses and other incentives should be tied to sustainable financial health rather than short-term profits.

Another vital area is the ethical behavior within financial institutions. These institutions must prioritize ethical standards and embed them into their corporate culture. This involves training employees on ethical practices, encouraging whistleblowing, and holding individuals accountable for unethical behavior. Moreover, regulatory oversight is essential. Regulators must be empowered with the authority and resources to effectively oversee financial institutions and enforce compliance with rigorous standards. This includes regular stress testing, improved transparency requirements, and stricter controls on financial innovations.

The financial industry faces numerous ethical concerns, spanning both work ethics and investment practices. One significant issue is the gender imbalance; the proportion of men working in finance is significantly larger than that of women. This male-dominated culture can lead to challenges for women, including barriers to career advancement, unequal pay, and a lack of representation in leadership roles. Some companies have experienced issues related to this imbalance, resulting in an environment that can be unwelcoming to women (Irene van Staveren, 2010).

Monetary incentives and long working hours are also common in the financial industry. Portfolio managers and other finance professionals often deal with investments across multiple stock exchanges, each with different opening hours, affecting their daily schedules. In some firms, an unhealthy work ethic prevails, with excessive expectations placed on employees. This can lead to burnout, strained family relationships, and health-related issues due to chronic sleep deprivation. The intense pressure to perform and meet targets can also foster unethical behavior, such as misrepresenting financial products or engaging in high-risk trading without proper disclosure to clients. Such practices not only harm individual investors but can also lead to broader financial instability. Addressing these ethical concerns requires a concerted effort from the industry. Promoting diversity and inclusion, ensuring fair labor practices, and fostering a culture of integrity and transparency are essential steps. Companies must prioritize the well-being of their employees and clients, recognizing that sustainable success is built on ethical foundations.

Another factor to consider when mitigating risk is sustainability. Which is an increasing concern in most industries, and with that, portfolio managers face regulatory challenges. The push towards sustainable investing is driven by a growing recognition of the long-term risks associated with environmental, social, and governance (ESG) factors. This shift is influencing investment strategies and decision-making processes.

It is important to mention that sustainability involves more than just reducing emissions. It encompasses a broad spectrum of environmental, social, and governance (ESG) factors that contribute to the overall health and well-being of our planet and society. Social factors, such as child labor, working conditions, fair wages, diversity, equity, and inclusion, play a crucial role in defining sustainable practices. Ensuring ethical labor practices, promoting safe and healthy work environments, and providing fair compensation are fundamental to sustainable development.

Moreover, governance factors, which include corporate governance, transparency, business ethics, and anti-corruption measures, are essential in fostering a culture of accountability and integrity within organizations. Good governance practices ensure that companies are managed responsibly, with a focus on long-term value creation and the interests of all stakeholders, including shareholders, employees, customers, and the broader community.

In addition to addressing these social and governance aspects, sustainability also involves the efficient use of resources, waste reduction, and the adoption of circular economy principles. This means designing products and processes that minimize waste, encourage recycling and reuse, and reduce the overall environmental footprint. The integration of ESG factors into business strategies and investment decisions not only mitigates risks but also opens up opportunities for innovation and competitive advantage. Companies that prioritize sustainability are better positioned to respond to regulatory changes, attract and retain talent, enhance their brand reputation, and meet the growing demands of socially conscious consumers and investors.

Ultimately, sustainability is about creating a balanced and resilient economy that supports environmental stewardship, social well-being, and ethical governance. It requires a collaborative effort from businesses, governments, and civil society to drive meaningful change and build a sustainable future for generations to come.

For portfolio managers who choose not to consider sustainability in their investment practices, the stakes of higher risk are significant. Numerous examples of large enterprises involved in scandals related to human rights violations or environmental breaches highlight these risks. One notable example is the Dieselgate scandal, where Volkswagen manipulated their emissions data to appear more environmentally friendly than they actually were. When the news about their deception broke out, it caused the company's share price to plummet, resulting in substantial financial losses and reputational damage (Mujkic and Klingner, 2018). .

Ignoring sustainability factors can lead to exposure to regulatory penalties, legal liabilities, and a loss of investor confidence. Moreover, companies that fail to adhere to ethical practices may face boycotts, protests, and divestment campaigns, further impacting their financial performance and market position. In contrast, integrating ESG criteria into investment decisions helps mitigate these risks and can enhance long-term returns by identifying companies that are well-managed and resilient to social and environmental challenges.

The consequences of not adopting a sustainable approach can be significant, underscoring the critical responsibility of portfolio managers in making investment decisions for their clients. By considering ethical concerns and sustainability factors, portfolio managers can make informed decisions that align with their clients' risk-return profiles and broader societal goals. Ultimately, the pursuit of responsible investment practices will foster a more sustainable and equitable financial industry, benefiting both investors and the global community.

The text in the discussion note has been refined and enhanced by using ChatGPT (OpenAI, 2024) for grammar and readability of the text.

References

Dichtl, H., Drobetz, W., & Wambach, M. (2014). Where is the value added of rebalancing? A systematic comparison of alternative rebalancing strategies. Financial Markets and Portfolio Management, 28, 209–231. https://doi.org/10.1007/s11408-014-0231-3

van Staveren, I. (2010). Gender biases in finance. International Review of Accounting, Banking and Finance, 9–17. https://doi.org/10.1080/13552070127734

Markowitz, H. (1952). Portfolio selection. The Journal of Finance, 7(1), 77-91. https://doi.org/10.1111/j.15 6261.1952.tb01525.x

McGillivray, J., & Fung, H.-G. (2013). The need for ethical reform in the US financial industry. International Review of Accounting, Banking and Finance, 5(1), 17–41.

Mujkic, E., & Klingner, D. (2018). Dieselgate: How hubris and bad leadership caused the biggest scandal in automotive history. Public Integrity, 21(4), 365–377. https://doi.org/10.1080/10999922.2018

OpenAI. (2024). ChatGPT: An AI language model. https://www.openai.com. Version GPT-4.

Zakamulin, V. (2024). Transaction cost optimization in a fully invested portfolio with target weights. SSRN. https://ssrn.com/abstract=4781258