Master

Argumentation and Proof in Japanese and Norwegian Mathematical Textbooks Grades 5-9.

A Comparative Study.

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Preface

In my younger years, school didn't hold much appeal for me, so I pursued a career as a carpenter. After several years honing my craft in construction, I found myself drawn to the role of mentorship and supervision while overseeing apprentices. The gratification of guiding others through their learning process sparked a newfound passion within me, leading me to pursue the career of a teacher.

My journey towards this goal commenced five years ago, marked by numerous hurdles along the way. I vividly recall the skepticism of my initial teacher upon returning to formal education, doubting my teaching capabilities. However, I remained steadfast, determined to prove her wrong. This sense of accomplishment when overcoming such obstacles has been truly awarding.

Throughout my studies I've been fortunate to encounter exceptional classmates, particularly in the mathematics classes, whose support and fellowship has enriched my learning experience. The support and fun with this community have played a pivotal role in my enjoyment and success as a student.

Reflecting on my time at the University of Agder, I am truly grateful for some of the dedicated educators and professors, I would also like to thank Yusuke Shinno for all the help regarding Japanese textbooks and questions regarding my study. My biggest thanks to David Alexander Reid, his guidance and supervision throughout this master's program has been invaluable. David's support and willingness to address any concerns and questions have made this master's program a good learning experience.

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Summary

In recent years, the field of mathematics education has witnessed a significant emphasis on argumentation and proof (Valenta & Enge, 2020), with educational textbooks emerging as indispensable resources for both teachers and students alike (Ahl et al., 2015). The implementation of Norway's new curriculum, Kunnskapsløftet 2020, underscores the importance of argumentation, reasoning, and proving as core elements within mathematics education (Kunnskapsdepartementet, 2019). Similarly, the inclusion of argumentation and proof in the Japanese curriculum (Isoda, 2010) renders the comparison of these two countries' approaches particularly intriguing.

This master's study delves into the presentation and approach of argumentation and proof-related topics, such as the sum of three consecutive numbers, the Pythagorean theorem, the sum of two even/odd numbers, and the sum of three angles in a triangle, within mathematics textbooks used in grades 5-9 in Japan and Norway. Through a comparative analysis, the study aims to discern the differences and similarities in the strategies adopted by these two countries' mathematical textbooks.

The contents and design of mathematical textbooks play a vital role for students and teachers in classroom practice (Ahl et al., 2015), and ongoing discussions among researchers regarding the meaning of proof (Stylianides, 2007; Ball et al., 2002; Jeannotte & Kieran, 2017) highlights the importance of the approach taken within the textbooks, influencing both teachers and students' perspectives and comprehension of proof and argumentation. The presentation and structure of tasks related to argumentation and proof, as well as the language (representations) used, may play a role in shaping these perceptions.

In conducting this study, a document analysis methodology developed by Bowen (2009), supported by a framework established by Miyakawa and Shinno (2021), was employed. A comprehensive examination was undertaken, encompassing a total of 38 mathematical textbooks sourced from three publishers in each country. The findings reveal notable disparities in the treatment of proof between Japanese and Norwegian textbooks, with Japanese textbooks having more opportunities for proving. However, the Japanese textbooks exhibit a more uniform structure, particularly in grades 8-9, Norwegian textbooks display greater variability. Despite these differences, both countries' textbooks initially adopt an empirical approach before transitioning to a more deductive approach, albeit the Japanese more heavily than the Norwegian textbooks. Additionally, differences in the function of tasks were observed, with Norway emphasizing a more discovery-oriented function compared to the more illuminated function evident in Japanese mathematical textbooks. Both countries shared the use of verification, albeit with some differences.

This comparative analysis offers valuable insight into the diverse approaches to argumentation and proof within mathematics textbooks in Japan and Norway. By identifying both differences and similarities, educators can gain valuable insights, which ultimately may enhance student learning.

Sammendrag

I løpet av de siste årene har feltet for matematikkutdanning vært vitne til en betydelig vektlegging av argumentasjon og bevis (Valenta & Enge, 2020), med lærebøker som uunnværlige ressurser for både lærere og elever (Ahl et al., 2015). Implementeringen av Norges nye læreplan, Kunnskapsløftet 2020, understreker viktigheten av argumentasjon, resonnement og bevis som kjerneelementer innen matematikkutdanning (Kunnskapsdepartementet, 2019). På samme måte gjør inkluderingen av argumentasjon og bevis i det japanske læreplanen (Isoda, 2010) sammenligningen av disse to landenes tilnærminger særlig interessant.

Denne masterstudien dykker ned i presentasjonen og tilnærmingen til argumentasjon og bevisrelaterte emner, som summen av tre påfølgende tall, Pythagoras' teorem, summen av to partall/oddetall, og summen av tre vinkler i en trekant, innen matematikkbøker brukt i trinn 5-9 i Japan og Norge. Gjennom en sammenlignende analyse søker studien å skille ut forskjellene og likhetene i strategiene som er tatt i bruk av disse to landenes matematikkbøker.

Innholdet og utformingen av matematikkbøker spiller en avgjørende rolle for elever og lærere i klasserommet (Ahl et al., 2015), og pågående diskusjoner blant forskere om betydningen av bevis (Stylianides, 2007; Ball et al., 2002; Jeannotte & Kieran, 2017) understreker viktigheten av tilnærmingen som er tatt i bruk innen bøkene, noe som påvirker både læreres og elevers perspektiver og forståelse av bevis og argumentasjon. Presentasjonen og strukturen av oppgaver knyttet til argumentasjon og bevis, samt språket (representasjonene) som brukes, kan spille en rolle i å forme disse oppfatningene.

I gjennomføringen av denne studien ble en dokumentanalysemetodologi utviklet av Bowen (2009), støttet av et rammeverk etablert av Miyakawa og Shinno (2021) benyttet. En grundig undersøkelse ble gjennomført, og inkluderte totalt 38 matematikkbøker fra tre utgivere i hvert land. Funnene avslører betydelige forskjeller i behandlingen av bevis mellom japanske og norske lærebøker, med japanske lærebøker som gir flere muligheter for bevis. Imidlertid viser de japanske lærebøkene en mer uniform struktur, særlig på trinn 8-9, mens norske lærebøker viser mer variasjon. Til tross for disse forskjellene, tar både Japanske og Norske lærebøker først i bruk en empirisk tilnærming før de går over til en mer deduktiv tilnærming, selv om de japanske lærebøkene gjør dette tyngre enn de norske lærebøkene. I tillegg ble det observert forskjeller i oppgavenes funksjon, med Norge som legger vekt på en mer oppdagelsesorientert funksjon sammenlignet med den mer opplyste funksjonen som er tydelig i japanske matematikkbøker. Begge landene delte bruken av verifikasjon, selv om det var noen forskjeller.

Denne sammenlignende analysen gir verdifull innsikt i de varierte tilnærmingene til argumentasjon og bevis innen matematikkbøker i Japan og Norge. Ved å identifisere både forskjeller og likheter, kan lærere oppnå verdifull innsikt, som til syvende og sist kan forbedre elevens læring.

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1.0 Introduction

In the realm of mathematics education, the design and content of textbooks play a pivotal role in shaping the way students and teachers engage with mathematical classroom practice (Ahl et al., 2015, p. 179). This master's study delves into a comparative analysis of Norwegian and Japanese mathematics textbooks, specifically for the grades of 5 through 9. Its primary focus is to unravel the nuances in the presentation and approach to argumentation and proof between the two countries and cultures' textbooks.

The decision to explore mathematical textbooks in the Japanese and Norwegian education system stems from a combination of personal experiences, observations, and professional aspirations. During my practicum as a teacher, I encountered a prevailing sentiment of distrust toward the use of mathematical textbooks among supervising educators. Their consistent advice to avoid relying on textbooks puzzled me, especially considering the fundamental role mathematical textbooks traditionally play in supporting classroom practice.

This observation sparked my curiosity and led me to question the underlying reasons for this skepticism towards mathematical textbooks. I recognized the opportunity to delve deeper into mathematical textbooks with this study when I got the chance to work with the international research group, *Linguistic and Cultural Approaches to Classroom Argumentation.* The opportunity to compare Japanese and Norwegian mathematical textbooks would not only address the research gap in knowledge, but also benefit my own professional development as someone who pursue a career in education. As such I have made a problem statement (problemstilling) which aims to answer some of these issues.

> How does the approach to mathematical concepts, especially within the theme of argumentation and proof, vary in Norwegian and Japanese mathematics textbooks, as indicated in argumentation and proof related tasks and presentations in the textbooks?

Argumentation and proof have become a central part of mathematical work. This can be seen in multiple countries' curriculums (Valenta & Enge, 2020, p. 2). One such example is from the new Norwegian curriculum that was implemented on August 1st, 2020. The Norwegian curriculum says that in mathematics, argumentation entails pupils providing rationale for their methods, explanations, and solutions, demonstrating their validity through proof (Kunnskapsdepartementet, 2019, p. 3). The Japanese curriculum also mentions proof but in a more specific way, that the students shall learn the necessities, meanings, and methods of proof (Isoda, 2010).

Prior research has demonstrated that the positioning of proof and proving varies across the curricula, textbooks, and classroom practices of different countries (Miyakawa & Shinno, 2021, p. 242). This can also be apparent in the textbooks between countries and cultures, which leads to the following research questions that I aim to explore in this study:

1. How do the approach and presentation of argumentation and proof differ between Norwegian and Japanese mathematics textbooks in grades 5-9?
2. How are argumentation and proof presented in tasks related to proofs, such as:
    - The sum of two even/odd numbers
    - The sum of three consecutive numbers
    - The sum of three angles in a triangle
    - Pythagorean theorem

Chapter 2 will begin by outlining relevant definitions, terms, and previous research concerning argumentation and proof, as well as relevant prior research on textbooks used in school practices. Following this, Chapter 3 will detail the methodology utilized in this study, followed by the presentation of the results and findings of the analysis in Chapter 4. Chapter 5 will be dedicated to the discussion, before concluding and giving my insights on further research in Chapter 6.

2.0 Literature review

The goal of this study is to reveal and understand nuances for argumentation and proof between Japanese and Norwegian mathematics textbooks for grades 5-9. As such this chapter will be an introduction to relevant literature for this study. It will begin with Chapter 2.1, a conceptional framework to define relevant terms and concepts regarding argumentation and proof used in this study, before mentioning some prior research on argumentation and proof in Chapter 2.2. Lastly Chapter 2.3 will presenting prior research on textbooks.

2.1 Conceptional framework

2.1.1 Definition of proof in school

Researchers concur that placing a greater emphasis on mathematical proof in school, not only enhances students reasoning abilities but also fosters a deeper understanding of mathematical concepts and connections (Valenta & Enge, 2020, p. 1). This acknowledgment has sparked a notable shift in educational paradigms towards recognizing the value of proof in mathematics education. Notably, the recent Norwegian curriculum for school mathematics has integrated reasoning and argumentation as a core element, encompassing approaches to proving, reasoning, and solution strategies (Kunnskapsdepartementet, 2019). Similarly, the Japanese curriculum addresses proof explicitly, stipulating the understanding of the necessity, meaning and methods of proof in grade 2 of junior high school (Isoda, 2010).

However, the definition of proof does not have a unanimous acceptance among mathematicians or educators (Ball et al., 2002, p. 907). Therefore, I will be using Stylianides's (2007) definition of proof and argumentation. Stylianides (2007) definition of proof, which says that within the context of school mathematics, proof holds a significant importance as it involves a structured and logical argumentation process grounded in accepted statements. According to Stylianides (2007), proof is outlined as a mathematical argument, a connected sequence of assertions for or against a mathematical claim, characterized by the following components:

> **Set of accepted statements**: Proof relies on statements that are accepted by the classroom community as true and do not require further justification. These statements can include definitions, axioms, theorems, and other established mathematical facts.
> **Modes of argumentation**: The process of proof involves employing valid forms of reasoning known to, or within the conceptual reach of the classroom community. This includes logical rules of inference, systematic enumeration of cases, construction of counterexamples, and other methods of logical reasoning.
> **Modes of argument**: In presenting a proof, appropriate forms of expression are used that are familiar to the classroom community. This can include linguistic, diagrammatic, symbolic, or other forms of representation that effectively communicate the logical structure of the argument (Stylianides, 2007, p. 291)

Stylianides (2007) provides these examples of the three components of a mathematical argument mentioned in his definition of proof:

> **Set of accepted statements**: Definitions, axioms, theorems, etc.
> **Modes of argumentation**: Application of logical rules of inference (such as modus ponens, and modus tollens), use of definitions to derive general statements, systematic enumeration of all cases to which a statement is reduced (given that their number is finite), construction of counterexamples, development of a reasoning that shows that acceptance of a statement leads to a contradiction, etc.
> **Modes of argument**: Linguistic (e.g., oral language), physical, diagrammatic/pictorial, tabular, symbolic/algebraic, etc. (Stylianides, 2007, p. 292)

The term "classroom community" is considered to consist mainly of pupils, where the teacher functions as the representative of the discipline of mathematics. The mention about community is however not explicit to pupils but also to the community of professional mathematicians (Stylianides, 2007, p. 292).

Jeannotte and Kieran (2017) mention that proving in mathematics literature can be divided into groups, proving itself and its more structured counterpart, formal proving. Proving, akin to justification, is a social endeavor where explanations are validated, removing doubts about the truth. Unlike justification, proving leans on deductive reasoning guiding a narrative's credibility from probable to certain (Jeannotte & Kieran, 2017, p. 12). On the other hand, formal proving comes from a need for validation within mathematics. It uses strict structures and meta-rules, demanding explicit integration intro established mathematical theories (Jeannotte & Kieran, 2017, p. 13). While both proving and formal proving share the same goal of enhancing narrative credibility, proving embraces a degree of flexibility in narrative acceptance, while formal proving demands formalism and structured mathematical frameworks.

In conclusion, integrating proof as a foundational element in mathematics education is essential for nurturing a deeper understanding of mathematical thinking, particularly within school mathematics, which also include the use of educational textbooks. The inclusion of proof in both the Norwegian and Japanese educational frameworks highlights its role in fostering mathematical reasoning and understanding on a global scale as mentioned above.
By combining Stylianides (2007) argumentation process and Jeannotte and Kieran's (2017) definition of "proving", the term "proof" will be used in this study as a method where arguments and statements are used to try to validate a mathematical claim.

2.1.2 Reasoning and Argumentation in school

Reasoning in mathematics refers to the cognitive skill of comprehending, evaluating, and following mathematical chains of thought. It entails the ability to discern the logical connections between mathematical rules, concepts, and results, recognizing that they are founded on clear and systematic principles rather than arbitrary notions (Kunnskapsdepartementet, 2019). Argumentation in school mathematics involves students providing justifications for their approaches, reasonings, and solutions, thereby demonstrating the validity of their mathematical thinking. This entails not only arriving at correct answers but also explaining the rationale behind them in a coherent and logically manner, proving the validity of their mathematical claims (Kunnskapsdepartementet, 2019).

2.1.3 Deductive reasoning

Deductive reasoning in the context of an argument refers to the mode of argumentation where the logical inferences drawn from a given set of premises are necessarily valid. It entails constructing a sequence of assertions that logically follow accepted definitions to support a conclusion (Stylianides & Stylianides, 2008, p. 108). Deductive reasoning does not prescribe a specific mode of representation but focuses on the logical coherence of the argument. It involves drawing logically necessary inferences from a set of premises or givens (klaczynski & Narasimham, 1998, cited in Stylianides & Stylianides, 2008, p. 107). Deductive reasoning thus plays a role in mathematical argumentation and proof, prompting questions about its accessibility and development in the student.

2.1.4 Empirical reasoning

Empirical reasoning, as applied in mathematics and science, entails utilizing evidence obtained through observation, experimentation, or real-world data to substantiate or justify a mathematical claim or hypothesis (Weber, 2013, p. 101). In contrast to deductive reasoning, which derives conclusions from established premises through logical deduction, empirical reasoning relies on empirical evidence to validate mathematical concepts. This involves verifying a general claim about an infinite set of objects by examining a subset of those objects and confirming that they exhibit a specific property, thereby supporting the assertion with concrete examples within the claim's scope (Weber, 2013, p. 101).

2.2 Prior research within argumentation and proof

2.2.1 Translation of proof-related words in the case of Japanese textbooks

In the context of mathematics education, the translation of proof-related words cross different languages and cultures can pose significant challenges for researchers and educators, especially for non-native English speakers. Shinno (2023) highlights the complexity of translating terms like "proof" and "reasoning" into languages like Japanese, where direct translations may not exist. This lack of direct equivalence can hinder the understanding and communication of fundamental mathematical concepts, impacting both research and classroom instruction (Shinno, 2023, p. 23).

One key issue discussed by Shinno's (2023) research is the translation of the term "proof" into Japanese. The target term "証明すること (syōmei-surukoto)" is considered too lengthy for practical study, leading to the adoption of the term "証明活動 (syōmei-katsudo)" meaning "proof-activity" in Japanese mathematics education research. This presents a unidirectional equivalence translation from English to Japanese. The term "reasoning" has similar issues when translating from English to Japanese (Shinno, 2023, p. 24). As such when comparing Japanese and Norwegian mathematical textbooks, several challenges may arise due to the nuances of translation and cultural difference in mathematical terminology.

2.2.2 Relation between argumentation and proof

In mathematics education, the relationship between argumentation and proof has been a subject of significant discourse and investigation. Hemmi et al., (2013) delve into this relationship within the context of developing proof-related competences in the curricula of Estonia, Finland, and Sweden. Their research sheds light on how argumentation can be seen as essential or distinct from the proving process and the implications of these two different viewpoints.

Argumentation, as discussed by researchers are often perceived either as distinct from proving or as an integral part of the proving process. (Reid & Knipping, 2010, p. 218) conceptualize argumentation as involving non-deductive reasoning, drawing upon methods such as induction, abduction, examples, or visual models, an empirical approach. Conversely, others regard proving as a specialized form of argumentation, emphasizing on the logical organization of arguments to construct valid proofs (Hemmi et al, 2013, p. 358), a deductive approach.

These differing viewpoints have profound implications for mathematics education. Teachers who perceive proof as separate from argumentation typically prioritize teaching the logical structure of proofs, aiming to impart a conceptual framework independent of problem-solving. Conversely, those who view proof as a form of argumentation highlights the importance of producing arguments within problem-solving contexts, expecting those arguments to be logically organized into valid proofs (Hanna & Villers, 2008, cited in Hemmi et al, 2013, p. 358). Understanding these perspectives is crucial for analyzing statements in mathematics related to argumentation and proof. Hemmi et al., (2013) advocate for the integration of problem-solving and proving, emphasizing its significance in fostering students understanding of mathematical concepts. By acknowledging the interplay between problem-solving, understanding, and proving, teachers can guide students towards constructing valid mathematical arguments and proofs (hemmi et al., 2013, p. 358).

2.2.3 Empirical reasoning in proof related mathematics.

Weber (2013) writes that discussion of empirical reasoning in proof related mathematics is centered around the concept of naïve empiricism and its role in mathematical persuasion. Naïve empiricism refers to the use of empirical evidence, particularly in the form of concrete examples or observations, to support mathematical claims or hypotheses (Weber, 2013).
Weber (2013) highlights that mathematicians, contrary to common belief, may be influenced by naïve empirical evidence in forming convictions about mathematical conjectures even before formal proofs are established (Weber, 2013, p. 102). This suggests that empirical reasoning plays a significant role in shaping mathematicians' perspectives and confidence in mathematical truths. This is especially more convincing in certain domains, such as number theory, where claims can be verified through proof by induction, compared to other branches of mathematics where empirical evidence may be less reliable (Weber, 2013, p. 104).
However, Weber (2013) emphasizes that while empirical reasoning can be a valuable tool for generating mathematical conjectures, guiding intuition, and providing informal support for formal proofs, it is essential to recognize its limitations. Empirical evidence, while persuasive in certain context, does not constitute a substitute for rigorous mathematical proof (Weber, 2013, p. 104).

2.3 Prior research on textbook comparison

Pepin et al., (2013) did a research study where they investigated mathematics curriculum documents, textbooks, and teacher curricular practice in Norway and France. The aim of the study was to deepen the understanding of how these traditions permeate the education system, from policy documents to the classroom implementation (Pepin et al., 2013, p. 685).
They found that cultural and educational traditions linked to egalitarianism, shape the curricula and teaching approaches. They compared the emphasis on theoretical properties and mathematical reasoning in French textbooks with the practical and inquiry-based activities in Norwegian textbooks, highlighting the different interpretations of educational values. An example of this was how the mathematical "training exercises" were presented in the textbooks. The French did not distinguish between the difficulty of the exercises, thus not distinguishing between the level of understanding between students, a view highlighted by the egalitarian values. The Norwegian books split the "training exercises" into three difficulty levels, distinguishing between the level of mathematical understanding. However, this was also seen as an egalitarian view. Norway interpreted this as an adopted form of teaching so it would fit all students (Pepin et al., 2013, p. 695). Pepin et al., (2013) concluded that even if the educational traditions were similar, as in both countries used egalitarian values, Norway and France "lived" the views differently, and educational traditions, policy, and curricular practice permeate the system. Notably, they also concluded that these findings come together to highlight that the mathematical textbooks are an important resource for teachers (Pepin et al., 2013, p. 696).

A comparative study of textbooks in Ethiopia, South Sudan, and Norway was conducted by Tesfamicael et al., (2022). It mainly focused on problem posing activities. The study emphasized the importance of problem posing in relation to problem solving within mathematics education. The findings revealed the lack of comprehensive and varied problem posing activities, as well as a heavy reliance on textbooks, specifically in Norway compared to Ethiopia, and South Sudan. The findings of the research showed a sparse amount of problem posing activities in tasks related to algebra, and those found were restricted in form (Tesfamicael et al., 2022, p. 7), which was most in the form of semi-structured problem posing, a total of 54 of the 62 problem posing activities was of this form (Tesfamicael et al., 2022, p. 5). They concluded that teachers nonetheless heavily depend on textbooks (Tesfamicael et al., 2022, p. 7).

Cabassut and Paris (2005) conducted a study comparing the approaches to teaching mathematics in French and German secondary school, particularly focusing on the textbooks used. The study shed light on the encouragement of two distinct types of arguments in mathematical instruction, arguments of plausibility and arguments of necessity. A notable finding was some differing emphases between the German and French textbooks regarding methods of validation. In the German textbooks, there was a more notable emphasis on the student's explanation of the topic, while the French had more emphasis on a more visual approach, even if both textbooks used visual arguments. This could be seen in examples provided by Cabassut and Paris (2005), specifically in proofs regarding the Pythagorean theorem, and the sum of angles in a triangle (Cabassut & Paris, 2005, p. 6). There was also a highlighted pedagogical significance of employing visual arguments, especially to facilitate the verification and explanation of proof in cases where a proof may not be fully accomplished (Cabassut & Paris, 2005, p. 8). The visual arguments served as tools for conveying arguments of necessity, with the integration of both mathematical and non-mathematical arguments, particularly visual representations. This synthesis highlights the importance of employing diverse forms of argumentation to enhance students' comprehension of mathematical concepts (Cabassut & Paris, 2005, p. 9).

Pepin et al., (2001) wrote a study about the use of mathematics textbooks in lower secondary classrooms in England, France, and Germany. Methodologically, the research examined the similarities and differences among mathematics textbooks at the lower secondary level in England, France, and Germany. Furthermore, it delved into the manner in which the teachers utilized these textbooks in classroom practices (Pepin et al., 2001, p. 167). The findings highlighted that the textbooks used in mathematical education have a significant influence on shaping the classroom cultures, with the teachers acting as mediators of the curriculum through their utilization of the textbooks used (Pepin et al., 2001, p. 169). This finding shows the importance of mathematical textbooks in an educational practice, and how it is used as a mediator between the classroom and the curriculum in mathematics.

In the discussion part of my study, I will revisit the research of Pepin et al., (2001), Pepin et al., (2013), Tesfamicael et al., (2022), and Cabassut and Paris (2005) to draw comparisons with their research and my own research of Japanese and Norwegian mathematics textbooks. Emphasizing the similarities, differences, and potential intersections between the insights provided by prior research and the findings of my findings.

3.0 Methodology

This chapter will serve as a methodological guide for the research conducted in this study. The primary objective is to clarify the process through which the textbooks are examined and analyzed. As such, it will begin with Chapter 3.1 where the selection of the textbooks is discussed, before delving into the Bowen's (2009) document analysis in Chapter 3.2 – Chapter 3.4, which is supported by Miyakawa and Shinno's (2021) framework for examining tasks related to argumentation and proof. Then in Chapter 3.5 the method of the comparative analysis will be presented, before Chapter 3.6 which discusses the validity of the research. Ending the chapter will be Chapter 3.7 which discusses some ethical reflections.

3.1 Selection of Textbooks

To establish a robust foundation for the research, a total of 38 textbooks, 7929 pages, spanning from grades 5-9 underwent examination (see Table 1). The Norwegian data comprised of 21 textbooks from three publishers: Multi from Gyldendal, Matematikk from Cappelen Damm, and Matemagisk from Aschehoug. In the Japanese context, data was also collected from three publishers: Keirinkan, Gakko Tosho, and Tokyo Shoseki, totaling in 17 Japanese mathematical textbooks. Notably, all selected textbooks are primary instructional materials employed within current school curricula. Opting against the utilization of exercise books and teacher guides was a conscious decision driven by the acknowledgment that primary textbooks assume a pivotal role in the introduction of new knowledge, allowing for an examination of how they present and teach proofs and argumentation.

Country	**Publisher**	**Title**	**pages**
Japan	Keirinkan	Math 5A for elementary school	154
Japan	Keirinkan	Math 5B for elementary school	122
Japan	Keirinkan	Math 6A for elementary school	164
Japan	Keirinkan	Math 6B for elementary school	128
Japan	Keirinkan	Math 1 for junior high school	287
Japan	Keirinkan	Math 2 for junior high school	216
Japan	Keirinkan	Math 3 for junior high school	275
Japan	Gakko Tosho	Mathematics for elementary school 5th volume 1	170
Japan	Gakko Tosho	Mathematics for elementary school 5th volume 2	173
Japan	Gakko Tosho	Mathematics for elementary school 6th volume 1	251
Japan	Gakko Tosho	Mathematics for elementary school 6th volume 2	56
Japan	Gakko Tosho	Mathematics 1 for junior high school	310
Japan	Gakko Tosho	Mathematics 2 for junior high school	254
Japan	Gakko Tosho	Mathematics 3 for junior high school	310
Japan	Tokyo Shoseki	Mathematics 7	273
Japan	Tokyo Shoseki	Mathematics 8	223
Japan	Tokyo Shoseki	Mathematics 9	265
Norway	Gyldendal	Multi 5A	135
Norway	Gyldendal	Multi 5B	135
Norway	Gyldendal	Multi 6A	144

Norway	Gyldendal	Multi 6B	136
Norway	Gyldendal	Multi 7A	144
Norway	Gyldendal	Multi 7B	136
Norway	Gyldendal	Maximum 8	290
Norway	Gyldendal	Maximum 9	297
Norway	Cappelen Damm	Matematikk 5 fra Cappelen Damm	221
Norway	Cappelen Damm	Matematikk 6 fra Cappelen Damm	243
Norway	Cappelen Damm	Matematikk 7 fra Cappelen Damm	223
Norway	Cappelen Damm	Matematikk 8 fra Cappelen Damm	332
Norway	Cappelen Damm	Matematikk 9 fra Cappelen Damm	327
Norway	Aschehoug	Matemagisk 5A	141
Norway	Aschehoug	Matemagisk 5B	167
Norway	Aschehoug	Matemagisk 6A	121
Norway	Aschehoug	Matemagisk 6B	205
Norway	Aschehoug	Matemagisk 7A	155
Norway	Aschehoug	Matemagisk 7B	163
Norway	Aschehoug	Matemagisk 8	304
Norway	Aschehoug	Matemagisk 9	279
Total pages Japan			3631
Total pages Norway			4298
Total pages Japan and Norway			7929

Table 1: Overview of textbooks examined.

The data collection process used both physical and digital versions of the textbooks. While the content remains identical across formats, the digital versions offer the added advantage of search functions, making retrieval of specific information easier. It is noteworthy that the Norwegian textbooks retained their original language, whereas the Japanese textbooks were translated into English, and acknowledging that certain Japanese terms might not undergo a seamless translation into English, which might be a weakness in the integrity of this study.

3.2 Document analysis.

This study uses Bowen's (2009) definition of a document analysis. Bowen (2009) explains a document analysis as qualitative research that have a systematic way to evaluate or examine documents, and like other analytical approaches in qualitative research, document analysis requires the examination and interpretation of data to extract meaning, enhance understanding and construct empirical knowledge (Bowen, 2009, p. 27). Bowen (2009) also says document analysis are composed of three main parts. Superficial examination (skimming), Thorough examination (reading), and interpretation (Bowen, 2009, p. 32).

In the initial phase of the analysis, a superficial examination involves skimming through the data and organizing the information systematically into categories relevant to thoe research focus (Bowen, 2009, p. 32). To guide this process, a predefined set of tasks and problems related to argumentation and proofs was established. Drawing inspiration from the research of Bieda et al., (2014), a list of keywords was compiled to facilitate the identification of tasks and problems. Aiding in the organization and retrieval of relevant data for the study (Bieda et al., 2014, p. 75).

Transitioning to the second phase, in the thorough examination I revisited the identified pages and tasks found in the superficial examination. A detailed re-reading and closer examination were undertaken to assess how the textbooks presented opportunities for argumentation and proofs, as well as incorporating Miyakawa and Shinno's (2021) framework developed for international research. The aspects of this framework are the triplet: structure, language, and function (Miyakawa & Shinno, 2021, p. 244).

The third and final phase of the document analysis is interpretation, where the findings from the superficial and thorough examination are synthesized to facilitate a comparative analysis between the Japanese and Norwegian textbooks. Using the information collected with Miyakawa and Shinno's (2021) framework to discern patterns, variations and pedagogical approaches in presenting argumentation and proofs in the tasks selected.

3.3 The superficial examination

The initial phase of the document analysis, known as the superficial examination, plays a foundational role in setting the stage for the examination of the Japanese and Norwegian textbooks on argumentation and proof related tasks and problems. Drawing from Bowen's (2009) document analysis. This phase involves a preliminary and systematic skimming through the textbooks with the aim of organizing relevant data into pertinent categories (Bowen, 2009, p. 32). To guide this process, I made a list of predefined set of tasks involving argumentation and proofs. These tasks include: *the sum of odd/even numbers*, the sum of *three consecutive integers*, *the sum of three angles in a triangle*, and *the Pythagorean theorem*. Initially I planned to also include the inscribed angle theorem but in the superficial examination I observed that the Norwegian textbooks did not include the inscribed angle theorem, as such I decided to omit any further research on this topic.

The terminology of "task" and "topic" plays a crucial role in this study, organizing and detailing the analysis of mathematical content within the Japanese and Norwegian textbooks. The term "task" is employed to refer to exercises or problems presented within the textbooks, serving as a unit of analysis. Each task represents a specific mathematical problem or a group of problems on a specific page which will be referred to in the text. The other term "topic" is utilized for broader mathematical concepts or themes examined in this study. Examples of topics studied in this study include *the sum of two even/odd numbers*, *the sum of three angles in a triangle*, *the sum of three consecutive integers*, and *the Pythagorean theorem*. It is important to be able to distinguish between these two terms to understand the contents of this study.

To accompany this, I made a list of keywords inspired by the research of Bieda et al., (2014). These include *triangle*, *consecutive*, *angle*, *Pythagorean*/*Pythagoras*, *proof*/*prove* and *odd*/*even*. This was translated into the Norwegian equivalent: *trekant*, *etterfølgende*, *vinkel*, *Pythagoras*, *bevis* and *par/odd* for examination in the Norwegian textbooks. This strategic approach helps with the structure of the analysis and the identification process but also contributes to a more efficient organization and retrieval of relevant data collected from the textbooks. As a result, a table (see Table 2) was created containing relevant information and pages, facilitating easy retrieval for further examination in the subsequent thorough analysis.

Grade	Title	The sum of two even/odd numbers	The sum of three consecutive integers	The sum of three angles in a triangle	The pythagorean theorem

Table 2: Table for organizing information found in the superficial examination.

3.4 Thorough examination

The next part of the document analysis involves a thorough examination, where the information organized from the superficial examination are revisited and subjected to a deeper analysis using the framework developed by Miyakawa and Shinno (2021). This phase is integral to unraveling the presentation of argumentation and proofs within the selected Japanese and Norwegian textbooks.

3.4.1 Miyakawa & Shinno's Framework

Miyakawa and Shinno (2021) proposed a new perspective on identifying and characterizing cultural specificities of proving and proof in different country's classrooms. To this end they made a framework composed of a triplet of actions; structure, language, and function (Miyakawa & Shinno, 2021, p. 242). Even if the framework was made for the use in examining proofs in the classroom, it gives opportunities for examining aspects of argumentation and proofs in textbooks as well. To help with organizing each task I made a table (see table 3) for gathering my findings.

Textbook	**Topic**	**Structure**	**Language/Representation**	**Function**	**Notes:**

Table 3: Table for organizing findings from the thorough examination.

The structure refers to the systematic arrangement of reasoning or arguments, illustrating the connections between distinct statements within a proof. This may take the form of a step-by-step guide, providing instructions on problem-solving or outlining the process of proving a theorem (Miyakawa & Shinno, 2021, p. 244). I used this to examine the structure the textbooks used in the predetermined set of tasks, mainly if the structure was deductive, or empirical.

Miyakawa and Shinno's (2021) results showed when determining if a statement has been proven, the focus lies on establishing a structure of deductive reasoning from the hypothesis to the conclusion, which often uses formal logic and established principles when presenting mathematical proofs or demonstrating theorems (Miyakawa & Shinno, 2021, p. 248). On the other hand, Empirical structure rely on observation, experimentation, and evidence to form a conclusion.

The aspect of language is the representation used to express the arguments and structure of reasoning (Miyakawa & Shinno, 2021, p. 244). Representations used in textbooks can be verbal, symbolic, graphic etc. Proof can be presented differently in textbooks. Some might have a graphic illustration of how to prove the theorem, while another textbook focus on discussion or algebraic proof in their methods.

The function captures the purpose or objective behind the arguments or instructions (Miyakawa & Shinno, 2021, p. 246). Within this study the predefined set of tasks may serve a distinct educational goal. These could include enhancing problem-solving skills, acquiring new knowledge, and showing a truth of a concept through a proof. I have narrowed these down to the terms: illumination, discovery, and verification.

Illumination is a form of a "AHA!" experience, you are confused for a long time until everything suddenly makes sense. Liljedahl (2012) have this citation which describes illumination:

> Perhaps I could best describe my experience of doing mathematics in terms of entering a dark mansion. One goes into the first room, and it's dark, completely dark. One stumbles

> around bumping into the furniture, and gradually, you learn where each piece of furniture is, and finally, after six months or so, you find the light switch. You turn it on, and suddenly, it's all illuminated. (Wiles, 1993, cited in Liljedahl, 2012, p. 253).

As such, illumination is the sensation that arises when one comprehends a concept. It's akin to a light suddenly appearing, illuminating everything that came before, and making sense of it all (Liljedahl, 2012, p. 253).

Verification in the context of proof, refers to the use of deductive reasoning to confirm the truth of a mathematical statement. It involves providing logical and rigorous arguments to demonstrate the validity of a claim (Shongwe, 2021, p. 513).

Discovery involves the creation of new mathematical knowledge or insights thorough the process of constructing proofs. It provides learners with opportunities to explore and uncover new mathematical relationships, properties, or theorems (Shongwe, 2021, p. 513).

3.5 Comparative analysis

The comparative analysis stands as a pivotal phase of this study, where the methodologies and educational approaches within argumentation and proof of the Japanese and Norwegian textbooks are examined side by side. The comparison focuses on the findings from the thorough examination of the structure, language, and function to discern differences and similarities. Structurally, the analysis scrutinizes how the textbooks organize argumentation and proof-related tasks, exploring whether they favor deductive of empirical approaches. Linguistically, the focus shifts to the representation used within the mathematical textbooks, investigating the interplay between symbolic, graphical, and verbal representations. This aspect sheds some light on how the language differs between Norwegian and Japanese mathematical textbooks within the topic of argumentation and proof.

Functionally, the comparison probes the objectives underlying argumentation and proof-related tasks in the mathematical textbooks in Japan and Norway, whether they prioritize discovery, verification, or illumination. This offers an insight into the difference and similar aims of the Japanese and Norwegian mathematical textbooks.

3.6 Validity and reliability

Postholm and Jacobsen (2018) states that there are two aspects the researcher need to reflect on to ensure the quality of the research. These include what limitations are associated with one's own research, and how he or she, through their way of conducting the research, may influence the results of the research (Postholm & Jacobsen, 2018, p. 222).

This study is subject to several limitations. One such limitation pertains to the examination of textbooks and the list of keywords used for identifying relevant content. Despite efforts to compile a comprehensive list of keywords thorough manual and digital examination of the textbooks, there is a possibility that some relevant keywords or tasks related to argumentation and proof may have been overlooked. Additionally, the sheer number of textbooks included in the examination (38 textbooks totaling in 7929 pages) increases the likelihood of human error. Despite the use of manual and digital examination procedures, it is possible that errors or omissions may have occurred. Consequently, another researcher conducting a similar examination of the textbooks may identify additional relevant content that was inadvertently missed in this study. Another limitation of this study is that the collected data only originated from the main mathematical textbooks used by the students. Often, there exists a teacher guidebook designed to complement these textbooks. The absence of the analysis of these teacher guidebooks may result in certain tasks related to argumentation and proof appearing lackluster or unclear. Combining the analysis of relevant pages in the textbooks with corresponding sections in the teacher guidebooks could have yielded different results. Therefore, the exclusion of teacher guidebooks from this study may have impacted the comprehensiveness of my findings.

In addition to considering the limitations mentioned above, it's crucial to reflect on how the researcher's methods may influence the results of the research, as highlighted by Postholm and Jacobsen (2018). This consideration is essential for assessing the reliability of the study. In qualitative research, researchers are encouraged to critically reflect on their own influence on the research process. This involves ensuring transparency and openness in the research method, allowing others to scrutinize and evaluate the methodology employed (Postholm & Jacobsen, 2018, p. 224). In this study, my method encompasses the examination of textbooks from various publishers in both Japan and Norway. Furthermore, I provide a detailed description of relevant content, and the analytical process. By transparently documenting and elucidating the research methodology, this research aims to enhance its credibility and facilitate critical appraisal by other researchers.

3.7 Ethical reflections

In writing this research, ethical considerations have been considered to ensure the integrity, transparency, and respect for all involved parties by following the National Committee for Research Ethics (NESH). To ensure methodical norms such as objectivity, accountability, verifiability, clarity, and that research and scientific methods are respected, and following the truth norm which speaks of the search and understanding of the truth as well as honesty and integrity. Together these norms are fundamental to ensure scientific methods are followed in a proper way (Den nasjonale forskningsetiske komité for samfunnsvitenskap og humaniora, 2022, p. 5).

My study is not based on human behavior, it is based entirely on documents and books. Its goal is to compare Japanese and Norwegian textbooks. This renders some of the typical ethical considerations irrelevant. Nevertheless, there are still ethical reflections to consider. One such reflection is the use of previous research, this needs to be properly acknowledged. Every source of information of previous work of researchers needs to be correctly referred to, both in the text and the bibliography. A second reflection is that this study involves two different countries and cultures. Therefore, it is essential to follow the methodical norms to ensure objectivity, so that the researcher try to not involve personal opinions as much as possible. The data collected, and the comparisons and findings from the analysis are for educational and research purposes only, they are not to favor one country, culture, or textbook over the other.

4.0 Results from the analysis and research

The findings of this study reveal both subtle and significant differences between Norwegian and Japanese mathematical textbooks designed for grades 5-9. This chapter serves to interpret the findings and results derived from the analysis of the 38 textbooks. It begins with presenting the findings of the superficial examination in Chapter 4.1, followed by the thorough examination in Chapter 4.2. Then ending with the findings of the comparative analysis in Chapter 4.3, which compare the findings in the prior chapters.

4.1 Superficial examination

The initial phase of this study included a systematic process of superficially examining a selection of textbooks sourced from Norway and Japan. The primary objective was to extract and organize pertinent data and insights from a diverse array of sources. This served as the foundational step, laying the groundwork for subsequent, more thorough examination in later stages of the analysis process.

The initial plan included the examination of five topics: the sum of two even/odd numbers, the sum of three consecutive integers, the sum of three angles in a triangle, the Pythagorean theorem, and the inscribed angle theorem. However, during the superficial examination, the inscribed angle theorem was not found in the Norwegian textbooks. Consequently, it was then decided to exclude any further analysis pertaining to the inscribed angle theorem from the scope of this study.

	Norwegian pages	Japanese pages	Total
The sum of two even/odd numbers	3	9	12
The sum of three consecutive integers	1	9	10
The sum of three angles in a triangle	13	14	27
Inscribed angle theorem	0	21	21
Pythagorean theorem	30	31	61
Pages about proof/arg	22	115	137
Total relevant pages	69	199	268
Total pages in all textbooks	4298	3608	7929

Table 4: Overview of findings from the superficial examination.

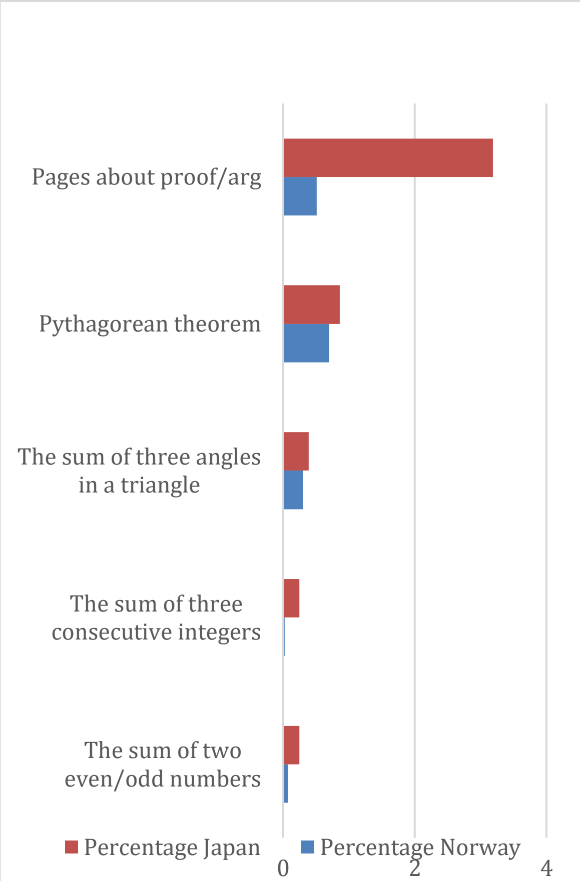


Table 5: Percentage of pages for each topic examined.

Table 4 shows the total number of pages as well as the relevant pages where I found information about the predefined topics, and Table 5 shows the percentage of pages for each topic, for both Norwegian and Japanese textbooks. In Table 5, it is evident that the Japanese textbooks contained a significantly greater number of pages relevant to this study compared to the Norwegian textbooks. Across all topics analyzed in this study, Japanese textbooks consistently have a higher page percentage. This discrepancy highlights the comparatively broader focus on the subject argumentation and proof in the Japanese textbooks than in the Norwegian textbooks. This also can be observed in the thorough examination.

4.2 Through examination

In this chapter, a deeper exploration of selected Japanese and Norwegian mathematical textbooks was undertaken, with a focus on specific topics and tasks aimed at addressing the research questions. The examination was conducted within the framework proposed by Miyakawa and Shinno (2021), as discussed in Chapters 3.2-3.4.

Given the amount of data material uncovered during the superficial examination, a decision was made to utilize the introduction pages for topics were possible, such as for the Pythagorean theorem as every book in 9th grade had a chapter dedicated to this topic. This approach was adopted to facilitate a comparative analysis of the textbooks from both countries, recognizing that both share the common objective of introducing and teaching the topic outlined in the textbook. The tasks selected for analysis can be found in Appendix B.

Table 6 provides a condensed summary of most of the tasks outlined in Appendix B, shedding light on the observation that Japanese textbooks feature 11 more tasks related to these topics. This discrepancy highlights potential differences in the depth and/or breadth of coverage between Japanese and Norwegian textbooks regarding the selected mathematical topics involved in argumentation and proof.

	The sum of angles in a triangle	**Pythagorea n Theorem**	**The sum of two even/odd numbers**	**The sum of three consecutive integers**	**Total**
Japan	5	3	7	5	20
Norway	3	3	2	1	9
Total	8	6	9	6	29

Table 6: Overview of relevant tasks chosen for further examination.

Appendix C presents the findings resulting from the examination of selected tasks (see appendix B), employing Miyakawa and Shinno's (2021) framework, which considers structure, language, and function. Table 7 provides a summary of the data seen in Appendix C. Note, that some of the percentages add to more than 100, reason being one task can have multiple languages or functions. Upon analysis, it becomes evident that the structure of the Japanese textbooks predominantly emphasizes a deductive approach, in contrast with the Norwegian textbooks which prioritize empirical tasks to a much higher degree. This distinction is reflected in the higher percentage of deductive tasks found in the Japanese textbooks (65% in the Japanese textbooks versus 22% in the Norwegian textbooks), while the Norwegian textbooks exhibit a noticeable higher percentage of empirical related tasks (78% in Norway compared to the 35% in Japan). Furthermore, differences in function are apparent between the two sets of textbooks. Japanese textbooks place a greater emphasis on illumination (60% of tasks examined) while Norwegian textbooks lean towards discovery (56% of tasks examined). Despite both sets featuring five tasks in the discovery function, there is a notable disparity in the percentage distribution (56% in Norway versus 25% in Japan). These observations underscore substantial variations in the pedagogical approaches adopted by the Japanese and Norwegian textbooks.

Grades 5-9	Structure		Language			Function			Total tasks
	Deductive	Empirical	Symbolic	Verbal	Graphic	Illumination	Verifi-cation	Disc-overy	
Norway	2	7	5	0	6	1	5	5	9
	22%	78%	56%	0%	67%	11%	56%	56%	100%
Japan	13	7	19	3	9	12	9	5	20
	65%	35%	95%	15%	45%	60%	45%	25%	100%

Table 7: Table for organizing findings from the analysis of Miyakawa and Shinno's (2021) framework.

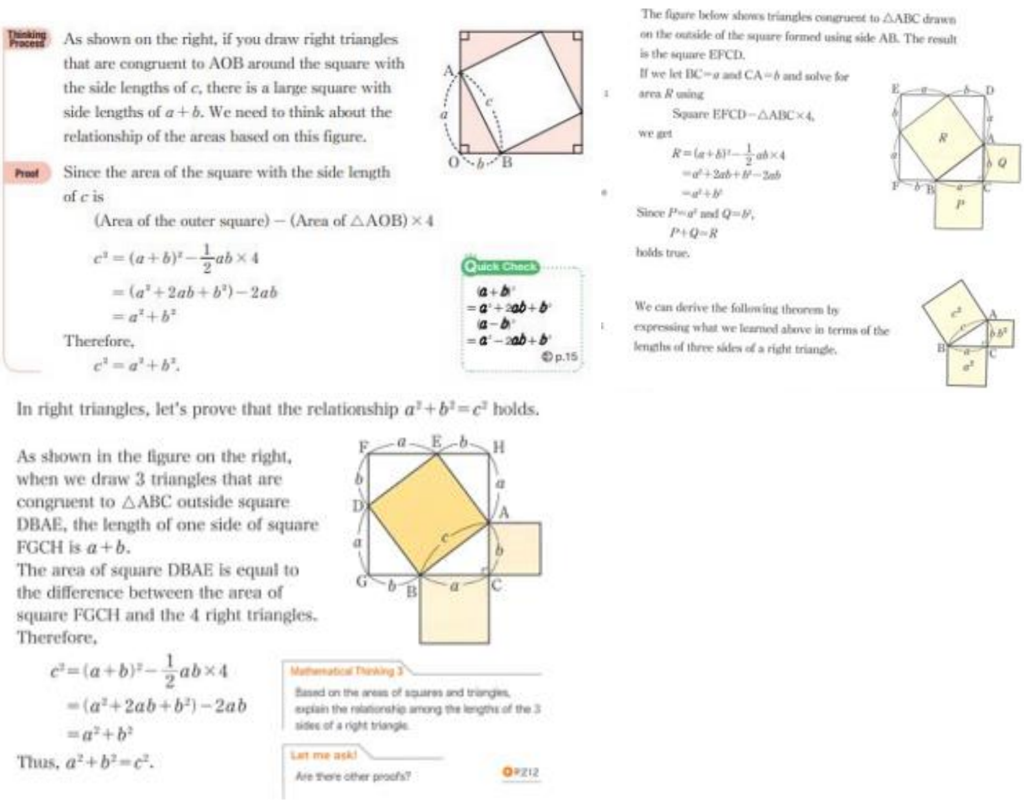
4.3 Comparison

In this chapter, the Japanese and Norwegian mathematical textbooks will be thoroughly compared using Miyakawa and Shinno's (2021) framework, which examines structure, language, and function. The comparison aims to uncover variations in how the argumentation and proof topics chosen are approached, ranging from surface-level differences to more nuanced differences. Through side-by-side examples of some of the selected tasks depicted in Appendix B, this chapter seeks to discern differences and similarities between the Japanese and Norwegian textbooks.

4.3.1 The structure

Upon delving into the structure of the predefined set of tasks in the Japanese and Norwegian textbooks, notable differences emerge in their presentation of the predefined topics and tasks. This contrast became evident during the thorough examination, where the majority of Japanese tasks were structured deductively compared to the empirical structure prevalent in the Norwegian textbooks (see Table 7). This discrepancy shows a fundamental difference in teaching approach between the two countries.

A clear illustration of this difference is evident in the treatment of the proof of the Pythagorean theorem within the textbooks. The Japanese textbooks consistently employ a standardized algebraic approach, which I will call the *Japanese Pythagorean proof* (see Figure 1) with the help of some visuals to demonstrate the proof. This remains consistent through all the 9th grade Japanese textbooks examined in this study. Following a uniform format, each textbook provides an explanation of the proof's rationale. After which, students are encouraged to undertake their own proofs after they have gone thorough standardized algebraic proofs. This is consistent in all 9th grade Japanese textbooks. In contrast, the Norwegian textbooks present the theorem for the topic first, in this case the Pythagorean theorem, and later towards the end of the chapter present a proof, if the textbooks have proofs in them.



Thinking Process: As shown on the right, if you draw right triangles that are congruent to AOB around the square with the side lengths of c, there is a large square with side lengths of $a+b$. We need to think about the relationship of the areas based on this figure.

Proof: Since the area of the square with the side length of c is

(Area of the outer square) − (Area of △AOB) × 4

$$c^2=(a+b)^2-\frac{1}{2}ab\times 4$$
$$=(a^2+2ab+b^2)-2ab$$
$$=a^2+b^2$$

Therefore,

$$c^2=a^2+b^2.$$

Quick Check: $(a+b)^2=a^2+2ab+b^2$, $(a-b)^2=a^2-2ab+b^2$ ©p.15

The figure below shows triangles congruent to △ABC drawn on the outside of the square formed using side AB. The result is the square EFCD.

If we let BC = a and CA = b and solve for area R using

Square EFCD − △ABC × 4,

we get

$$R=(a+b)^2-\frac{1}{2}ab\times 4$$
$$=a^2+2ab+b^2-2ab$$
$$=a^2+b^2$$

Since $P=a^2$ and $Q=b^2$,

$$P+Q=R$$

holds true.

We can derive the following theorem by expressing what we learned above in terms of the lengths of three sides of a right triangle.

In right triangles, let's prove that the relationship $a^2+b^2=c^2$ holds.

As shown in the figure on the right, when we draw 3 triangles that are congruent to △ABC outside square DBAE, the length of one side of square FGCH is $a+b$.

The area of square DBAE is equal to the difference between the area of square FGCH and the 4 right triangles.

Therefore,

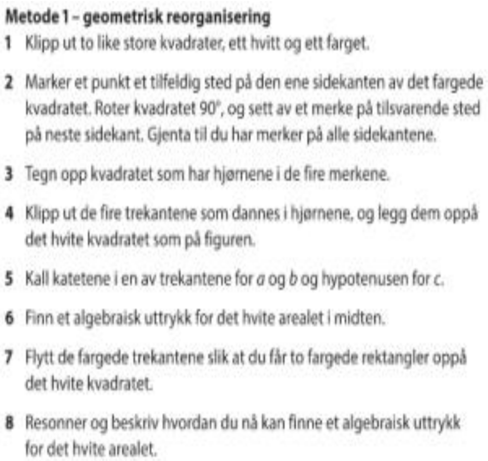
$$c^2=(a+b)^2-\frac{1}{2}ab\times 4$$
$$=(a^2+2ab+b^2)-2ab$$
$$=a^2+b^2$$

Thus, $a^2+b^2=c^2$.

Mathematical Thinking 3: Based on the areas of squares and triangles, explain the relationship among the lengths of the 3 sides of a right triangle.

Let me ask! Are there other proofs? ©P.212

Figure 1: Japanese Pythagorean proofs. Retrieved from Mathematics 9, Tokyo Shoseki, p.150., Junior High School Mathematics 3, Gakko Tosho, p.196. & Math 3 for Junior High School, Keirinkan, p.161.



Metode 1 – geometrisk reorganisering

1 Klipp ut to like store kvadrater, ett hvitt og ett farget.

2 Marker et punkt et tilfeldig sted på den ene sidekanten av det fargede kvadratet. Roter kvadratet 90°, og sett av et merke på tilsvarende sted på neste sidekant. Gjenta til du har merker på alle sidekantene.

3 Tegn opp kvadratet som har hjørnene i de fire merkene.

4 Klipp ut de fire trekantene som dannes i hjørnene, og legg dem oppå det hvite kvadratet som på figuren.

5 Kall katetene i en av trekantene for a og b og hypotenusen for c.

6 Finn et algebraisk uttrykk for det hvite arealet i midten.

7 Flytt de fargede trekantene slik at du får to fargede rektangler oppå det hvite kvadratet.

8 Resonner og beskriv hvordan du nå kan finne et algebraisk uttrykk for det hvite arealet.

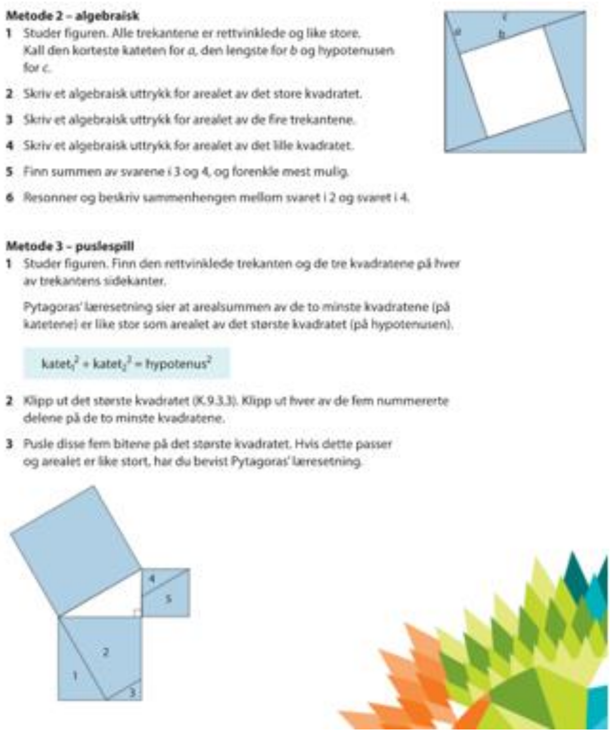
Figure 2: Method 1 - Geometric Reorganizing. Retrieved from Maximum 9, Gyldendal, p.190.

The Norwegian textbooks exhibit a more diverse range of approaches to the proof of the Pythagorean theorem, with some textbooks omitting proofs altogether. In the Norwegian textbooks where proofs are included, they vary in approach, often employing different forms of guided step-by-step methods devoid of a valid explanation at the end. For instance, *Geometric Reorganizing* (see Figure 2) showcases an 8-step process where students are prompted to reason and derive an algebraic expression for the white square. Translated into English the *Geometric Reorganizing* proof are as follows:

Method 1 - Geometric Reorganizing (see Figure 2)

1. Cut out two equal squares, one white and one colored.
2. Mark a random point somewhere on one of the sides of the colored square. Rotate the square 90 degrees and mark a point on an equivalent place on the next side. Repeat until you have marks on each side.
3. Draw the square that has a marked point at each corner.
4. Cut out the four triangles made in the corners and put them on top of the white square.
5. Call the legs in one corner a and b, and the hypotenuse c.
6. Find an algebraic expression for the white square.
7. Move the colored triangles in a way so you get two colored rectangles on top of the white square.
8. Reason and describe how you now can find an algebraic expression for the white square.

While this approach may encourage student-led discovery, it lacks the final explanation and validation found in the Japanese textbooks, leaving students without a valid conclusion to the proof without input from the teacher.



Metode 2 – algebraisk

1 Studer figuren. Alle trekantene er rettvinklede og like store. Kall den korteste kateten for *a*, den lengste for *b* og hypotenusen for *c*.

2 Skriv et algebraisk uttrykk for arealet av det store kvadratet.

3 Skriv et algebraisk uttrykk for arealet av de fire trekantene.

4 Skriv et algebraisk uttrykk for arealet av det lille kvadratet.

5 Finn summen av svarene i 3 og 4, og forenkle mest mulig.

6 Resonner og beskriv sammenhengen mellom svaret i 2 og svaret i 4.

Metode 3 – puslespill

1 Studer figuren. Finn den rettvinklede trekanten og de tre kvadratene på hver av trekantens sidekanter.

Pytagoras' læresetning sier at arealsummen av de to minste kvadratene (på katetene) er like stor som arealet av det største kvadratet (på hypotenusen).

$katet_1^2 + katet_2^2 = hypotenus^2$

2 Klipp ut det største kvadratet (K.9.3.3). Klipp ut hver av de fem nummererte delene på de to minste kvadratene.

3 Pusle disse fem bitene på det største kvadratet. Hvis dette passer og arealet er like stort, har du bevist Pytagoras' læresetning.

Figure 3: Method 3 - Puzzle Proof. Retrieved from Maximum 9, Gyldendal, p.191.

An example of a lackluster proof (see Figure 3) is method 3 - *Puzzle Proof* found in Maximum 9. Translated into English the *Puzzle Proof* are as follows:

Method 3 – Puzzle Proof (see Figure 3)

1. Study the Figure, Identify the "right triangle" and the three squares from each side of the triangle.
   The Pythagorean theorem says that the sum of the area of the two smaller squares (on the sides) equals the biggest square (on the hypotenuse). "$Katet^2 + Katet^2 = Hypotenuse^2$"
2. Cut out the biggest square (K.9.3.3). Cur out each of the five numbered pieces in both smaller squares.
3. Puzzle together all five pieces on top of the bigger square. If the five pieces fit the big square, you have proven the Pythagorean theorem.

The *Puzzle Proof* does not explain the reason this might be a proof. The whole method is based around the (K.9.3.3) task appendix. This method might only work with (K.9.3.3) and nothing else, which make this method lackluster and confusing.

The divergent approaches to the Pythagorean theorem exemplify differences in structural methodologies between the Japanese and the Norwegian textbooks. Japanese textbooks adhere a more standardized, traditional approach characterized by the consistency across all the Japanese textbooks. Conversely, the Norwegian textbooks adopt a more varied approach, by using a more empirical structure to see different methods to proving the theorem, characterized by the differences among the Norwegian textbooks. This disparity also extends beyond the Pythagorean theorem topic. An example of this is the proof involving the sum of three angles in a triangle. In the Japanese 8th grade textbooks, the proof is nearly identical in each of the three mathematics textbooks (see Figure 4), mirroring the uniformity observed with the Pythagorean theorem.

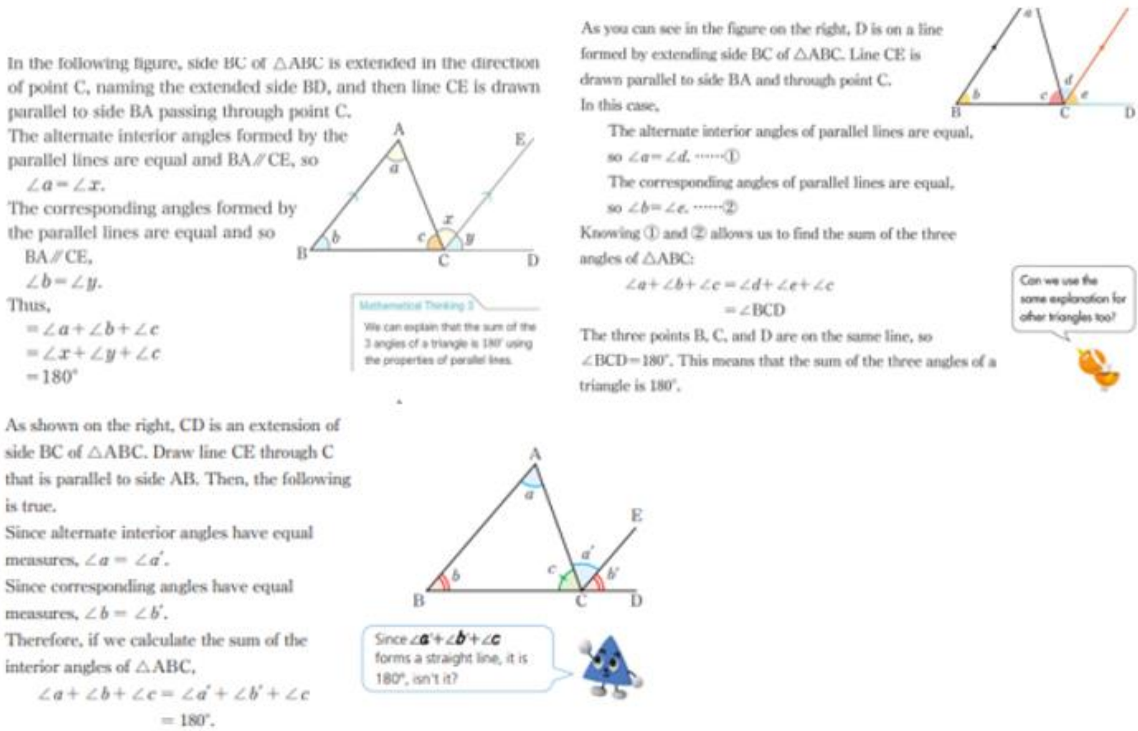
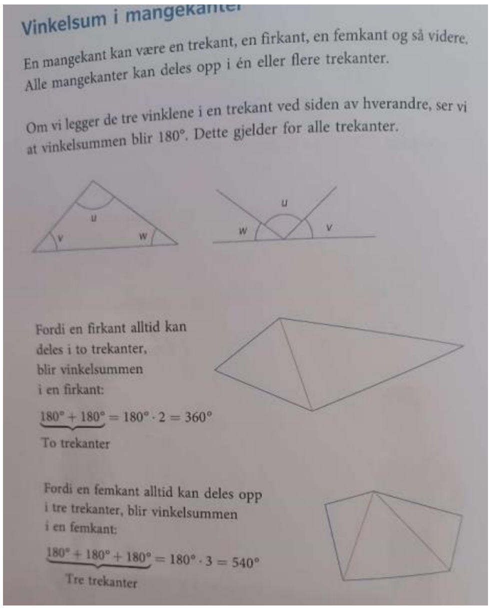


Figure 4: The Japanese sum of three angles in a triangle. Retrieved from Junior High School Mathematics 2, Gakko Tosho, p. 111., Math 2 for Junior High School, Keirinkan, p.88. & Mathematics 9, Tokyo Shoseki, p.98.

However, it is difficult to compare with the Norwegian textbooks for the later grades, as they do not present any proof related to this topic. Instead, Norwegian textbooks only state that the sum of three angles in a triangle equals 180 degrees in context of finding the sum of the interior angles in a polygon (see Figure 5).


Vinkelsum i mangekanter

En mangekant kan være en trekant, en firkant, en femkant og så videre. Alle mangekanter kan deles opp i én eller flere trekanter.

Om vi legger de tre vinklene i en trekant ved siden av hverandre, ser vi at vinkelsummen blir 180°. Dette gjelder for alle trekanter.

Fordi en firkant alltid kan deles i to trekanter, blir vinkelsummen i en firkant:

$\underbrace{180° + 180°}_{\text{To trekanter}} = 180° \cdot 2 = 360°$

Fordi en femkant alltid kan deles opp i tre trekanter, blir vinkelsummen i en femkant:

$\underbrace{180° + 180° + 180°}_{\text{Tre trekanter}} = 180° \cdot 3 = 540°$

Figure 5: The sum of the angle in a polygon. Retrieved from Matematikk 9 fra Cappelen Damm, Cappelen Damm, p.74.

However, in grades 5-6, there are notable similarities in the treatment of the sum of three angles in a triangle. Both the Norwegian and the Japanese textbooks employ an empirical approach to teaching that the sum of three angles in a triangle equals 180 degrees. They also utilize similar methodologies, prompting students to explore different triangles and add the sum of all the angles together to discover the sum of three angles in a triangle equals 180 degrees, albeit with slight variations in visual presentation. Both Japanese and Norwegian examples (see Figure 6 & 7) involve students recording angle measurements in a table and adding them together to observe that all triangles measured sums up to 180 degrees.

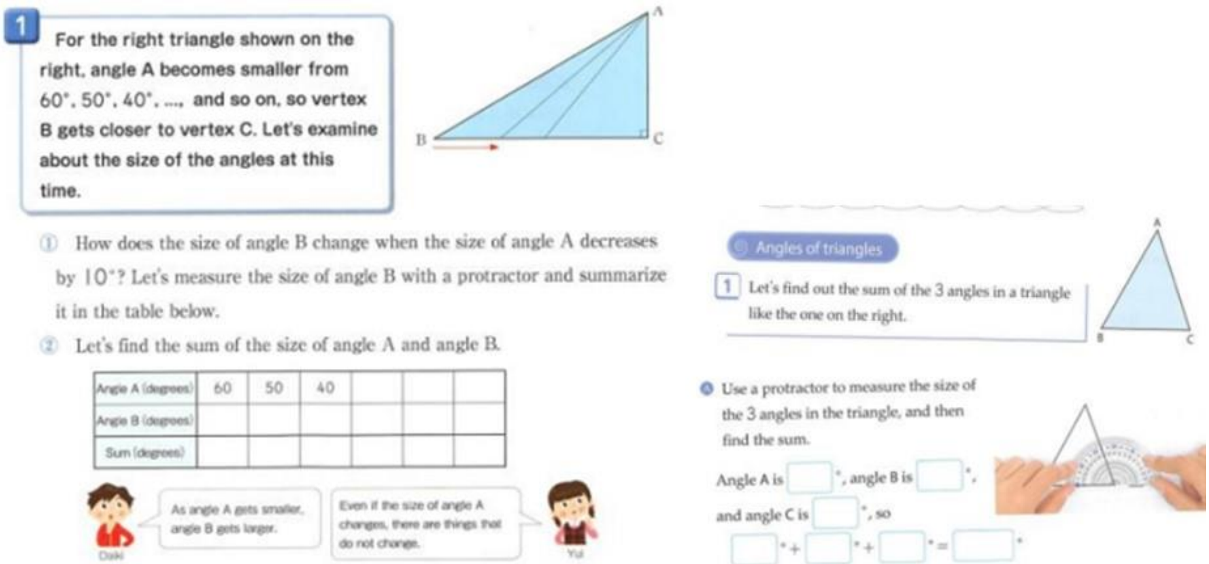


Figure 6: Japanese sum of three angles in a triangle. Retrieved from Mathematics 5.1 for Elementary School, Gakko Tosho, p.113. & Math 5A for Elementary School, Keirinkan, p.73-74.

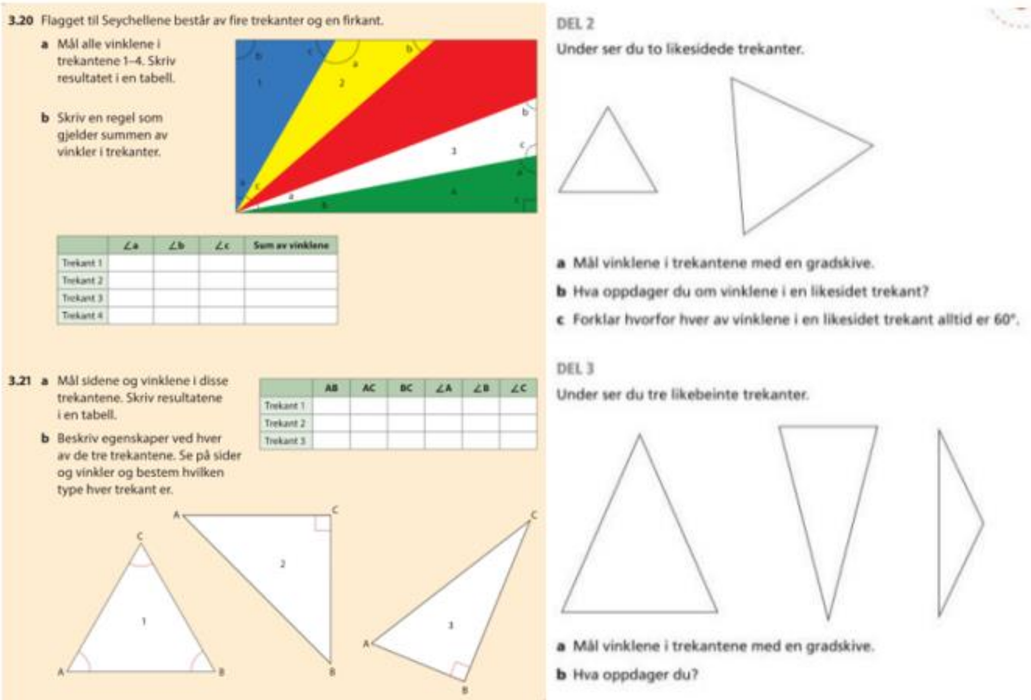


Figure 7: The sum of three angles in a triangle. Retrieved from Multi 6a, Gyldendal, p.88. & Matemagisk 6a, Aschehoug, p.41.

In summary, the analysis of the Norwegian and Japanese mathematical textbooks reveals a mix of commonalities and distinctions in their structure. Early grades exhibit significant resemblances between the two countries, with later grades showcasing more pronounced differences. Norwegian textbooks maintain a consistent empirical approach across grades 5-9, while the Japanese textbooks undergo a methodology shift in higher grades. This transition is marked by a shift towards a consistent deductive approach, contrasting with the more empirical methodology employed in the earlier grades, akin to that found in the Norwegian textbooks.

4.3.2 The Language

In Miyakawa and Shinno's (2021) framework, language (representations like: Visuals, symbolic, verbal, etc.) holds synonymous significance with mathematical representations (Miyakawa & Shinno, 2021, p. 244). The presentation of language within argumentation and proof-related tasks in Japanese and Norwegian textbooks reveals a notable degree of similarity, albeit with discernible differences in linguistic representation.

Japanese textbooks predominantly employ symbolic language (see Table 6) in argumentation and proof-related tasks found in the relevant topics. Although graphical language is also utilized, it frequently is used together with symbolic language (see Appendix C). resulting in a higher prevalence of tasks utilizing symbolic language. For instance, the topic of the Pythagorean theorem demonstrates the simultaneous use of both symbolic and graphic languages, whereas another topic, like the sum of two even numbers, rely mostly on symbolic language. This integration contributes to the predominance of symbolic language in Japanese mathematical textbooks.

Conversely, the Norwegian textbooks tend to lean towards either symbolic or graphical language, with fewer instances of tasks within the relevant topics utilizing multiple languages

simultaneously. While some tasks may incorporate multiple languages, such occurrences are comparatively infrequent in Norwegian textbooks compared to the Japanese textbooks.
In summary, while the Japanese and Norwegian textbooks exhibit similarities in their use of language, disparities emerge in the extent of integration between symbolic and graphical language.

4.3.3 The Function
The function of argumentation and proof related tasks within Japanese and Norwegian textbooks reflects distinct differences between the two countries. Highlighting differences in pedagogical methodology between the two countries. The deductive approach prevalent in Japanese textbooks contrasts with the empirical approach favored in Norwegian textbooks. Japanese textbooks prioritize the function of illumination and verification, supplemented by a hint of discovery. Through deductive reasoning and systematic proof, Japanese textbooks aim to illuminate mathematical concepts and principles, fostering a deeper understanding. Furthermore, argumentation and proofs serve as tools for verifying mathematical truths. While not as prominently featured as in the Norwegian textbooks, the function of discovery is integrated to stimulate critical thinking and inquiry. For instance, at the beginning of the Pythagorean theorem chapter, each textbook initiates with an exploratory task, encouraging students to uncover connections between the squares of the sides of a triangle (see Figure 8). This initial exploration serves to engage the students with the topic, leading them towards a comprehensive understanding through the application of deductive reasoning in subsequent

proofs. This approach emphasizes the illuminative function, progressively guiding students towards a nuanced comprehension of the mathematical concept.

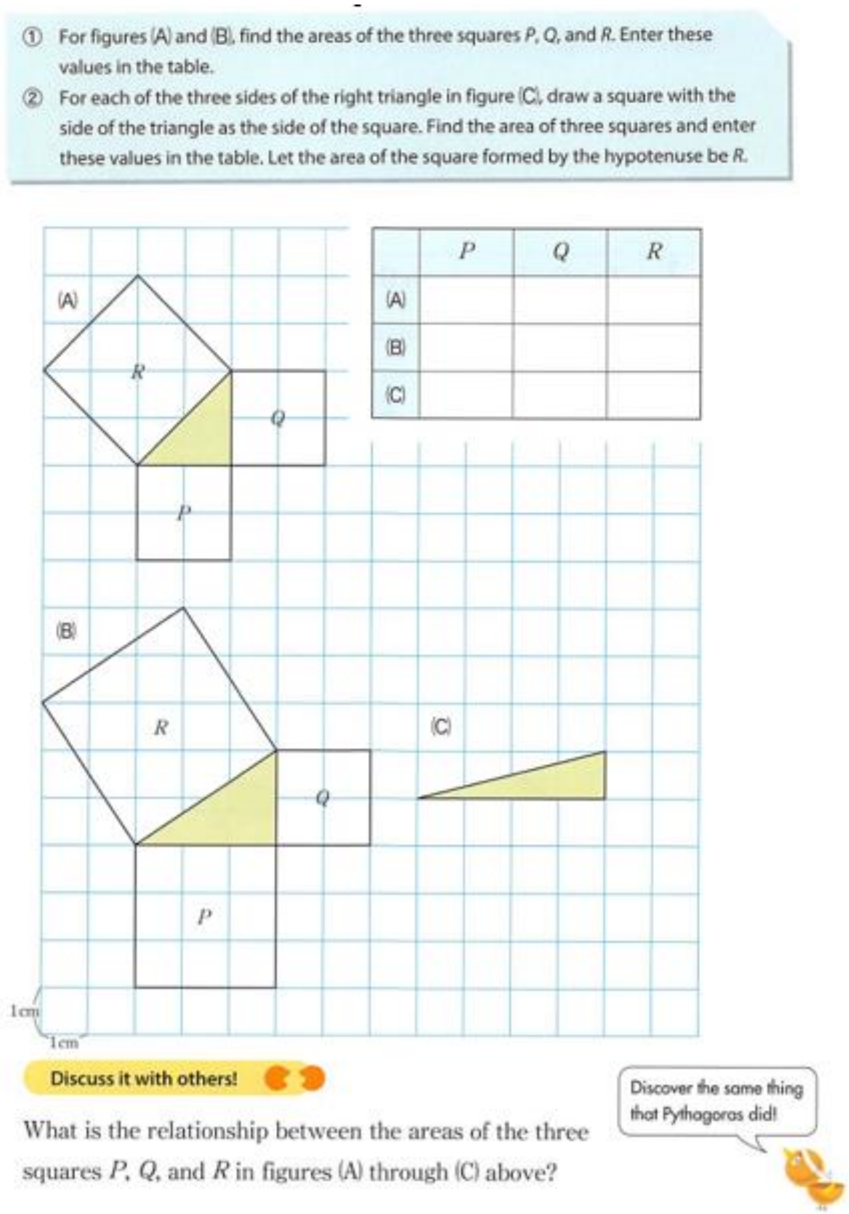


Figure 8: Explore the sides of a triangle. Retrieved from Math 3 for Junior High School, Keirinkan, p.160.

Utforsk sammen

Partall og oddetall.
Bruk tallene nedenfor og lag mange addisjonsstykker av to og to tall.
Se på svarene. Lag en regel som sier noe om svarene når dere
- adderer to partall
- adderer to oddetall
- adderer et partall og et oddetall

2 8 10 4 12
14 18 6 16 20
3 7 13 15 5
17 11 1 9 19

Figure 9: Explore even and odd numbers. Retrieved from Matematikk 7 fra Cappelen Damm, Cappelen Damm, p.23.

Conversely, the Norwegian textbooks prioritize the functions of verification and discovery. While this approach may not offer the same level of depth as the illuminative function in the Japanese textbooks, it equips Norwegian students with essential thinking skills to verify mathematical concepts and explore new ideas through empirical observation and experimentation. For instance, *Explore even and odd numbers* (see Figure 9), translated from Norwegian to English:

Explore even and odd numbers (see Figure 9).

Even numbers and odd numbers.
Use the numbers below and make many addition tasks of two and two numbers. Look at the answers. Make a rule that says something about the answers when you:

1. Add together two even numbers.
2. Add together two odd numbers.
3. Add one even and one odd number together.

This exercise makes students engage in discovering patterns that they then can verify later, employing logical and critical thinking skills to grasp the task's complexities.
In summary, while both the Japanese and the Norwegian textbooks aim to cultivate mathematical understanding and problem-solving skills, differences in functions reflect the different methodologies and priorities between the Japanese and the Norwegians.

5.0 Discussion

The aim of this study was to compare Japanese and Norwegian 5-9 grade mathematical textbooks within topics related to argumentation and proof. I have now presented prior research related to this in Chapter 2, as well as presented results of my own analysis and research in Chapter 4, and described how I conducted the analysis in Chapter 3. This chapter will try synthesize everything presented in the study so far to try to answer the research questions, which are as follows:

1. How do the approach and presentation of argumentation and proof differ between Norwegian and Japanese mathematics textbooks in grades 5-9?
2. How are argumentation and proof presented in tasks related to proofs, such as:
    - The sum of two even/odd numbers
    - The sum of three consecutive numbers
    - The sum of three angles in a triangle
    - Pythagorean theorem

In Chapter 5.1, I will give a summary and some thoughts of my findings in this study, followed by Chapter 5.2 which is a look at my own findings compared with prior research which I went through in the literature review.

5.1 Summary

I have in the present study skimmed thorough 7929 pages, from 38 books divided on Japanese and Norwegian textbooks. Among those I handpicked 29 tasks related to the topics of the *sum of two odd/even numbers, the sum of three angles in a triangle, the sum of three consecutive numbers, and the Pythagorean theorem* for a deeper comparable analysis. I have found notable differences, approaches, and similarities to argumentation and proof within this analysis. Especially when it comes to the methodology used by both countries, as they seem to have a different but at the same time similar goals to their education.

5.1.1 Findings for grades 5-7

The examination of tasks within the Japanese and Norwegian mathematical textbooks for grades 5-7 revealed notable similarities and differences in the approach and presentation of argumentation and proof. Despite the limited sample size (see Table 6) of six tasks, three for Japan and three for Norway, which comprised of two tasks on the topic of the sum of three angles in a triangle and one task on the topic of the sum of two odd/even numbers. Both the Japanese and Norwegian textbooks demonstrated a high degree of methodological similarity, with an empirical structure being predominant in all six tasks. This consistency suggests a shared emphasis on providing students with opportunities to engage in mathematical reasoning grounded in concrete examples and observations.

Additionally, both countries shared a similar utilization of the symbolic and graphic language, 66% of the tasks used both symbolic and graphic languages, indicating a balanced approach to conveying mathematical concepts through multiple representations for both countries. However, a notable difference was observed in the function of the tasks examined. While the Japanese textbooks focused on the function of verification with all three tasks using the function of verification, with two of the three tasks also using the function of discovery, the Norwegian textbooks displayed a reverse pattern, with having all three tasks using the function of discovery, with two of the three tasks also using the function of verification. This discrepancy shows some variation between the Japanese and Norwegian textbooks for grades 5-7, but since the sample size is quite small it's hard to say that these similarities and differences in the approach to topics related to argumentation and proof in the lower grades of 5-7.

5.1.2 Findings in grades 8-9

The findings of grades 8-9 reveals significant differences between the Japanese and Norwegian mathematical textbooks in terms of argumentation and proof. Unlike the previous grades, there were more tasks looked at for grades 8-9, encompassing in a total of 23 tasks across all topics, the sum of three angles in a triangle, the Pythagorean theorem, the sum of three consecutive integers, and the sum of two odd/even numbers. A vast difference between the Japanese and the Norwegian textbooks are the different emphasis on proof-related tasks, with the Japanese having 17 out of the 23 examined tasks for grades 8-9, where the Norwegian textbooks only had 6 tasks examined. This difference highlights the stronger emphasis on proof related content in the Japanese textbooks compared to the Norwegian textbooks.

The examination of the structure of the grades 8-9 revealed notable differences in methodology between the Japanese and the Norwegian textbooks. The Japanese textbooks employed a predominantly deductive structure, with 77% of tasks exhibiting deductive reasoning, while the Norwegian counterpart also shifted towards a more deductive structure but retained a significant empirical approach, with 67% of the task being of empirical structure and 33% deductive. While the Japanese shifted to a much more deductive structure than the Norwegian textbooks, both countries are shifting toward a more deductive approach compared to the earlier grades of 5-7. The reason why the Japanese textbooks had such a sharp change in methodology might be because of the strict curriculum used by the Japanese compared to the Norwegians. As the Japanese curriculum states that students shall learn about proofs, while the Norwegian curriculum are much more vague in how you can interpret the curriculum, which might result in a more careful shift to a deductive structure compared to the Japanese.

The language remained somewhat consistent with the previous grades of 5-7 for both the Japanese and Norwegian textbooks. However, Japanese textbooks exhibited a notable shift towards heavier reliance on symbolic language, with all 17 tasks featuring some sort of symbolic language, and only 7 tasks containing graphical language. This shift in language suggests a highlighted focus on symbolic reasoning for constructing and providing proofs in Japanese textbooks. The Norwegian counterparts continued with the even spread of symbolic and graphical language in their tasks that were examined. The reason why the Norwegians didn't head towards a heavier focus on symbolic language might be because of less focus on constructing and providing proof, the Norwegian textbooks focused more on empirical reasoning to make proofs valid compared to the Japanese use of deductive reasoning.

The function of tasks underwent notable changes in both the Japanese and the Norwegian mathematical textbooks. Norwegian textbooks transitioned towards a methodology akin to the Japanese textbooks in grades 5-7, with greater emphasis on the function of verification. 67% of the Norwegian tasks examined used the function of verification, 33% used the function of discovery, and 17% used the function of illumination. Conversely, the Japanese textbooks shifted toward a more illuminative approach, with 71% of the tasks examined used the function of illumination, 35% used the function of verification, and 18% used the function of discovery. This reveals a methodology which indicates a move to fostering deeper understanding and mathematical reasoning related to argumentation and proofs in the Japanese textbooks for the grades 8-9.

5.2 Comparison with prior research

The findings of this study align with and extend existing research on the role of mathematical textbooks in shaping pedagogical approaches and discourse for argumentation and proof. Pepin et al., (2013) conducted a comparative study of mathematics curriculum documents and textbooks in Norway and France, highlighting the influence of cultural and educational traditions on curricula design and teaching approaches. Similarly, Pepin et al., (2001) explored the utilization of mathematics in lower secondary classrooms in England, France, and Germany, emphasizing the significant influence of textbooks on classroom cultures and instructional practices.

One key finding from Pepin et al., (2013) is the distinct interpretation of egalitarian values in Norway and France, reflected in the presentation of mathematical exercises in textbooks. While the French textbooks emphasized theoretical properties and mathematical reasoning without distinguishing between the difficulty levels of exercises, Norwegian textbooks adopted a more differentiated approach, aligning with egalitarian principles by catering to students' diverse levels of mathematical understanding. This finding resonates with the observed differences in the function of tasks between the Japanese and Norwegian textbooks, where Norwegian textbooks exhibited a greater emphasis on discovery tasks, potentially reflecting on a pedagogical emphasis on inquiry-based learning and varied instruction in the classrooms.

Furthermore, Cabassut and Paris (2005) highlighted the pedagogical significance of employing visual arguments in mathematics instruction, particularly in facilitating the verification and explanation of proofs. The emphasis on visual and symbolic representations in both the German and French textbooks aligns with the use of symbolic and visual languages in the Norwegian and Japanese textbooks I have examined. This synthesis suggests a common pedagogical strategy aimed at enhancing students' comprehension of mathematical concepts through multiple forms of argumentation.

Additionally, the study by Tesfamicael et al., 2022 underscores the prevalence of textbook reliance in mathematics education, particularly in Norway compared to Ethiopia and South Sudan. As well as a sparse amount of problem posing activities in the Norwegian textbooks compared to the other countries. This aligns with the 9 tasks examined in the Norwegian textbooks, which were much less than the 20 tasks examined in the Japanese textbooks. An observation that highlights the methodological differences between Norway and Japan. Japanese textbooks focus more on specific topics to get a deeper understanding which aims to use the function of illumination compared to the Norwegian function of verification, which aims more for the practical use of mathematic concepts instead of a deep understanding of mathematic concepts.

6.0 Conclusion

The examination of Norwegian and Japanese mathematics textbooks across grades 5-9 offers valuable insights into the approach and presentation of argumentation and proof-related topics. By analyzing specific topics and tasks related to argumentation and proof, this study aimed to compare the presentation and approach to argumentation and proofs, shedding light on potential differences and similarities between the Norwegian and Japanese mathematical textbooks.

In the grades 5-7, both Norwegian and Japanese textbooks demonstrated the shared emphasis on empirical reasoning, providing students with concrete examples to engage students in mathematical reasoning. However, variations emerged in the function of the tasks examined, where Japanese textbooks favored verification over discovery, the Norwegian was the opposite. This suggests that the Japanese textbooks have a more narrated methodology, while the Norwegian textbooks have a more open methodology on how topics in the lower grades of 5-7 should be learned, although by a small margin.

There are more significant differences in the approach to argumentation and proofs in the higher grades of 8-9. Japanese textbooks have a stronger emphasis on proof than compared to Norwegian textbooks, the Japanese utilizing a more deductive structure compared to the Norwegian textbooks which utilized a more empirical structure similar to the lower grades of 5-7. This suggests that the Japanese textbooks aims for their students to have a deep understanding and logical thinking process of argumentation and proof related topics, while the Norwegian textbooks focuses on students' ability to practically use the mathematical concepts while still maintaining the critical thinking process needed to understand mathematical concepts.

This observation is further evident in the presentation of the related topics (see Table 4), such as the sum of two odd/even numbers, the sum of three angles in a triangle, and the sum of three consecutive integers. While both Japanese and Norwegian textbooks offer tasks covering these topics, the emphasis may vary, with Japanese textbooks providing a more extensive deductive reasoning to verify and prove proofs compared to the Norwegian textbooks.

6.1 Further research

While this study has contributed to the understanding of the approach and presentation of argumentation and proof in Japanese and Norwegian mathematical textbooks for grades 5-9, there are numerous opportunities for further research to deepen the knowledge in this area. One notable finding in this study was that the Japanese had a significant higher number of related tasks involving argumentation and proofs compared to the Norwegian textbooks. What is the reason for this? As such it would be interesting to study why this is the case. I mentioned that in my conclusion I thought that the Norwegian textbooks aimed more for a practical use for the mathematical concepts instead of a very deep understanding of the concept presented. With this in mind the Norwegian textbooks might have a wider area of topics presented in the textbooks, something that further suggests this is the number of pages in the Norwegian textbooks 4298 pages compared to the Japanese textbooks 3608 pages.

The use of teacher guides was not studied at all in my research. A study on the synthesis of teacher guides and textbooks could identify various missing concepts in the main student textbooks. For example, the lack of proofs in the Norwegian textbooks could be explained in the teacher guides.

While I did mention that there could be some translation issues when translating from Japanese to English, I did not do much research in this area. As such there could be a difference in the amount of pages identified with argumentation and proof in the Japanese textbooks analyzed.

This offers an opportunity for individuals proficient in both Japanese and English to study the translation of educational mathematics textbooks, focusing on argumentation and proof in the translated texts. This would make further research in this area more accurate in the future.

While this study had made some contributions to our understanding of argumentation and proof in Japanese and Norwegian mathematical textbooks for grades 5-9, there are multiple opportunities for further research to address unanswered questions. My study is but a small rock in a big pond.

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8.0 Appendix

8.1 Appendix A: Results of superficial examination, overview of relevant pages in the textbooks.

8.1.1 Norwegian textbooks.

Title	The sum of two even/odd numbers	The sum of three consecutive integers	The sum of three angles in a triangle	Pythagorean Theorem
Multi 5A				
Multi 5B				
Matematikk 5 fra Cappelen Damm				
Matemagisk 5A				
Matemagisk 5B				
Multi 6A			**p. 89** (The sum of angles in triangles)	
Multi 6B				
Matematikk 6 fra Cappelen Damm				
Matemagisk 6A				
Matemagisk 6B			**p. 29** (Why the total degree on a line is 180 degrees) **p. 40-43** (Total sum of angles in a triangle) **p. 66-68** (Practice problems involving triangles and angles) **p. 76** (Summary triangles)	
Multi 7A				
Multi 7B				
Matematikk 7 fra Cappelen Damm	**p. 23** (explore addition and subtraction of odd/even numbers)			
Matemagisk 7A				
Matemagisk 7B				

Matemagisk 8				
Maximum 8, Grunnbok				
Matematikk 8 fra Cappelen Damm				
Maximum 9, grunnbok			**p. 193** (If two triangles have the same angles then they are congruent)	**p. 184** (Pythagorean theorem) **p. 188** (3, 4, 5 triangles "special cases") **p. 190-191** (Proof of Pythagorean theorem)
Matematikk 9 fra Cappelen Damm			**p. 74** (The angle sum of polygons)	**p. 126 ->** (Pythagorean theorem chapter) **p. 136** (Explore the Chinese proof, but don't say where to find it.)
Matemagisk 9	**p.34** (Explain with the help of drawings that the sum of two consecutive numbers always becomes an odd number, algebra tiles) **p. 37** (Show with calculation that the sum of an odd and an even number equals an even number)	**p. 38** (Show with calculation that three consecutive numbers can always be divided by three)	p. 140	p. 184-205 (Chapter 16 about Pythagorean theorem) p. 227 (Euclid's proof for Pythagorean theorem)

Table 7: Results of superficial examination, Norwegian Textbooks.

8.1.2 Japanese textbooks.

Title	The sum of two even/odd numbers	The sum of three consecutive integers	The sum of three angles in a triangle	Pythagorean Theorem
Kerin grade 5A			**p. 74** (The 3 angles make a straight line, so the sum is 180 degrees)	
Kerin grade 5B				

Gakuto grade 5, 1			**p. 112** (The sum of 3 angles in a triangle is 180 degrees) **p. 114** (Summary: for any triangle the sum of the three angles is 180 degrees)	
Gakuto grade 5, 2				
Kerin grade 6A			**p. 3** (We know that the sum of 3 angles in a triangle is 180 degrees)	
Kerin grade 6B			**p. 69** (The sum of 3 angles in a triangle is 180 degrees) **p. 103** (Proof that the sum of 3 angles in a triangle is 180 degrees)	
Gakuto grade 6, 1				
Gakuto grade 6, 2				
Kerin grade 7				
Gakuto Grade 7			**p. 8** (Known and given: Sum of three interior angles of a triangle is 180 degrees)	
Math 7, Tokyo Shoseki	**p. 73** (Sum of two consecutive numbers?)			
Kerin grade 8	**p. 25** (The sum of two odd number equals an even number) **p. 27** (sum of two even numbers equals an even number)	**p. 160** (The sum of 10 consecutive natural numbers)	**p. 88-89** (Properties of interior and exterior angles of triangles)	
Gakuto grade 8	**p. 32** (Two consecutive odd numbers)	**p. 26** (Find the sum of three consecutive integers) **p. 223** (Sum of 3 consecutive even numbers)	**p. 111** (Interior and exterior angles of triangles) **p. 111-112** (Proof of the sum of angles are 180	

			degrees in a triangle)	
Math 8, Tokyo Shoseki	**P. 28** (Let's try)	**P. 20** (sum of 5 numbers) **P. 27** (Sum of any 3 consecutive numbers)	**p. 90** (Sum of multiple angles triangles) **p. 98-99** (proof of sum of triangles = 180 degrees)	
Kerin grade 9	**p. 31-32** (the product of two consecutive even numbers plus 1 is the square of an odd number, and proof on p. 32)			**Chapter 7:** Pythagorean Theorem **p. 161** (Pythagorean theorem) **p. 163-164** (Converse of the pythagorean theorem) **p. 166** (using the pythagorean theorem) **p. 170** (3-4-5 triangles) **p.171** (Use in space figures) **p. 175**-> (Practice problems) **p. 210-211** (Proving the pythagorean theorem) **p. 212** (Proving the converse pythagorean theorem) **p. 213** (Pythagorean theorem and area, working with area around triangle)

Gakuto grade 9	**p. 36** (Add 1 to the product of two consecutive even numbers, and proof on p. 37)	**p. 38** (Tasks involving consecutive numbers) **p. 41** (task 6 is relevant)		**Chapter 7:** Pythagorean Theorem **p. 197** (The Pythagorean Theorem) **p. 199** (The converse of pythagorean theorem) **p. 201** (3, 4, 5 triangles) **p. 202->** (Using the pythagorean theorem) **p. 212** (Diverse proofs of pythagorean theorem) **p. 278** (Relevant practice tasks)
Math 9, Tokyo Shoseki	**p. 31** (prove that the result of adding 1 to the product of 2 consecutive even numbers is a square of an odd number)	**p. 28** (sum of n consecutive integers) **p. 80** (sum of 3 consecutive integers = 302. Find the 3 consecutive integers)		**p. 148-150** (introduction to the pythagorean theorem) **p. 150-151** (The pythagorean theorem) **p. 153->** (practice tasks) and (tasks involving the converse pythagorean theorem) **p.154** (Proving that a angle is 90 degrees with the pythagorean theorem) and (converse pythagorean

				theorem) **p. 157-162** (uses of the pythagorean theorem) **p. 160->** (determine the length of different figures using the pythagorean theorem) **p. 219** (various proofs of the pythagorean theorem) **p. 236, 243** (extra problems)

Table 8: Overview of superficial examination, Japanese Textbooks.

8.2 Appendix B: Chosen relevant tasks for examination.

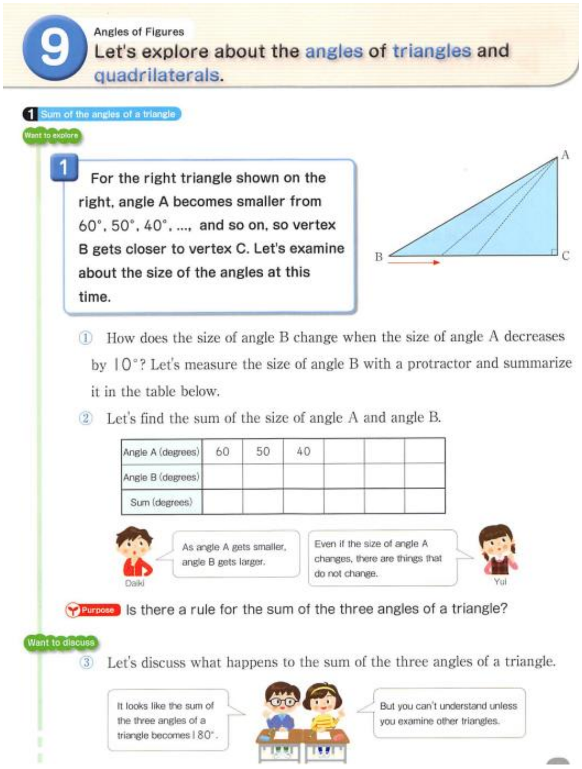


Figure 10: Explore angles of triangles. Retrieved from Mathematics 5.1 for Elementary School, Gakko Tosho, p.113.

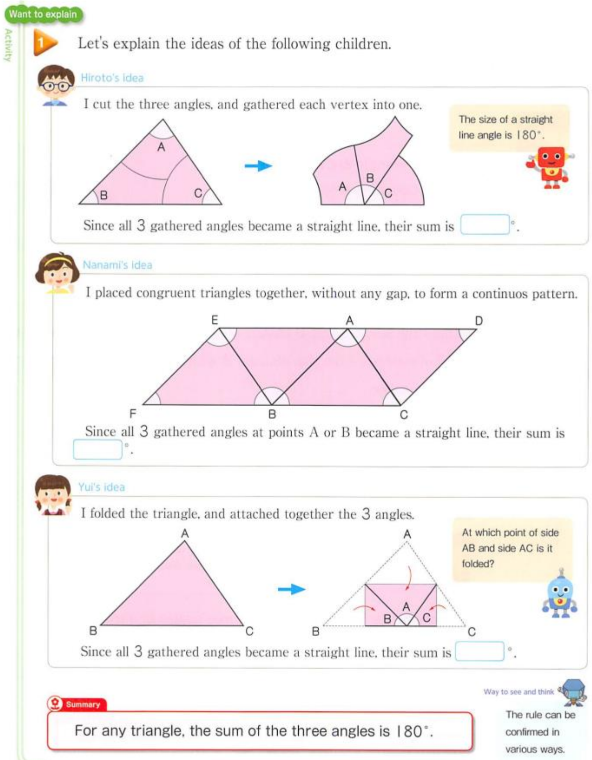


Figure 11: Explore angles of triangles. Retrieved from Mathematics 5.1 for Elementary School, Gakko Tosho, p.114.

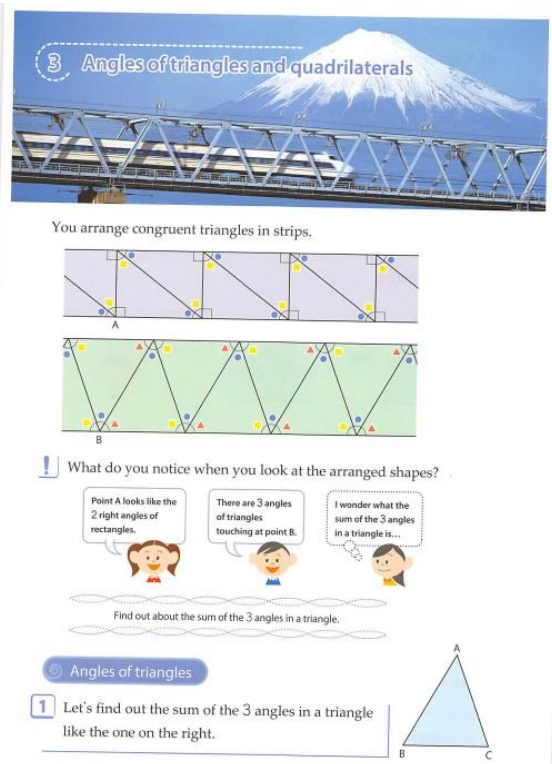
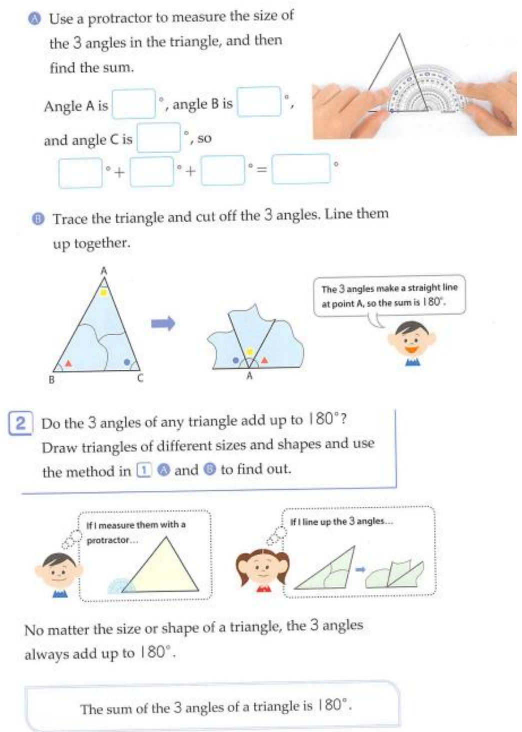


Figure 12: Angles of triangles. Retrieved from Math 5A for Elementary School, Keirinkan, p.73.



Ⓐ Use a protractor to measure the size of the 3 angles in the triangle, and then find the sum.

Angle A is ☐°, angle B is ☐°, and angle C is ☐°, so

☐° + ☐° + ☐° = ☐°

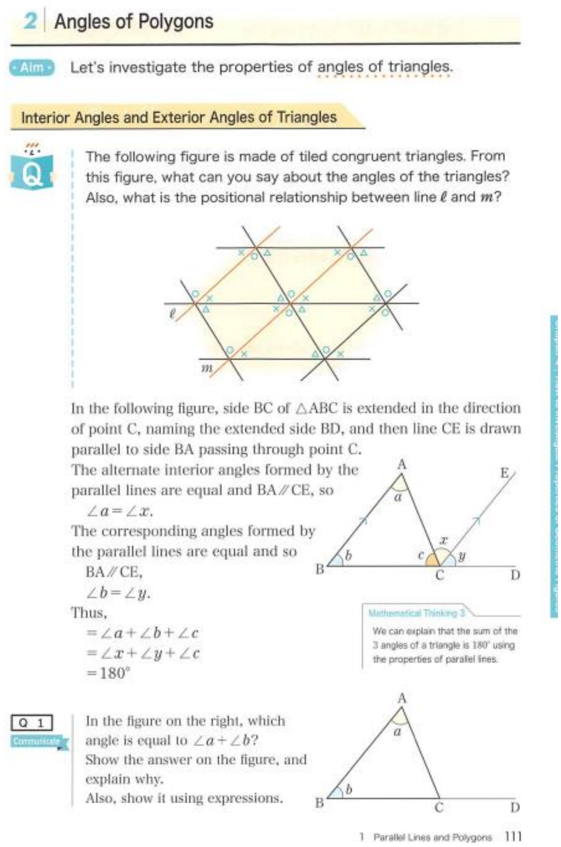
Ⓑ Trace the triangle and cut off the 3 angles. Line them up together.

2 Do the 3 angles of any triangle add up to 180°? Draw triangles of different sizes and shapes and use the method in 1 Ⓐ and Ⓑ to find out.

No matter the size or shape of a triangle, the 3 angles always add up to 180°.

The sum of the 3 angles of a triangle is 180°.

Figure 13: Angles of triangles. Retrieved from Math 5A for Elementary School, Keirinkan, p.74.



2 Angles of Polygons

Aim Let's investigate the properties of angles of triangles.

Interior Angles and Exterior Angles of Triangles

Q The following figure is made of tiled congruent triangles. From this figure, what can you say about the angles of the triangles? Also, what is the positional relationship between line ℓ and m?

In the following figure, side BC of △ABC is extended in the direction of point C, naming the extended side BD, and then line CE is drawn parallel to side BA passing through point C.

The alternate interior angles formed by the parallel lines are equal and BA∥CE, so

$\angle a = \angle x$.

The corresponding angles formed by the parallel lines are equal and so

BA∥CE,

$\angle b = \angle y$.

Thus,

$= \angle a + \angle b + \angle c$

$= \angle x + \angle y + \angle c$

$= 180°$

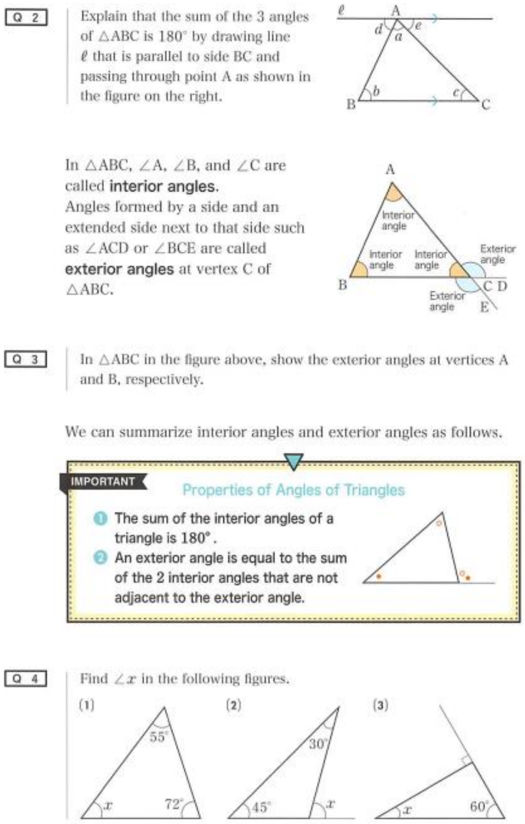
Mathematical Thinking 3

We can explain that the sum of the 3 angles of a triangle is 180° using the properties of parallel lines.

Q 1 Communicate In the figure on the right, which angle is equal to $\angle a + \angle b$? Show the answer on the figure, and explain why. Also, show it using expressions.

1 Parallel Lines and Polygons 111

Figure 14: Angles of Polygons. Retrieved from Junior High School Mathematics 2, Gakko Tosho, p.111.



Q 2 Explain that the sum of the 3 angles of △ABC is 180° by drawing line ℓ that is parallel to side BC and passing through point A as shown in the figure on the right.

In △ABC, ∠A, ∠B, and ∠C are called **interior angles**. Angles formed by a side and an extended side next to that side such as ∠ACD or ∠BCE are called **exterior angles** at vertex C of △ABC.

Q 3 In △ABC in the figure above, show the exterior angles at vertices A and B, respectively.

We can summarize interior angles and exterior angles as follows.

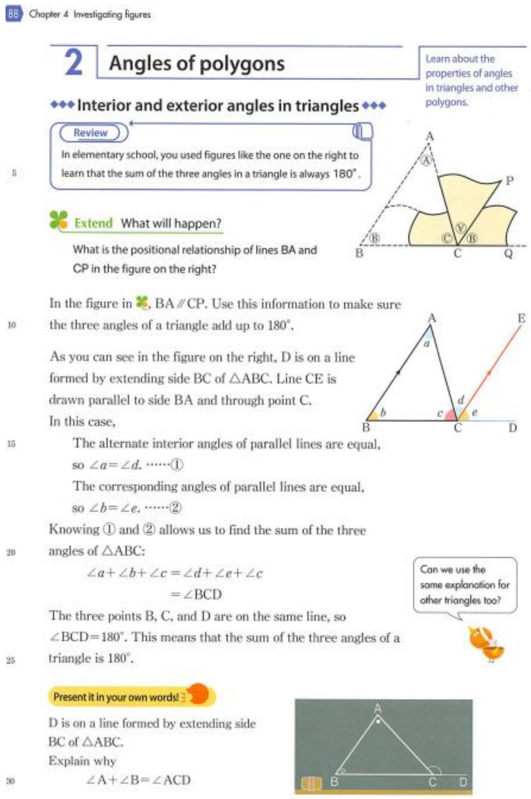
IMPORTANT Properties of Angles of Triangles

1. The sum of the interior angles of a triangle is 180°.
2. An exterior angle is equal to the sum of the 2 interior angles that are not adjacent to the exterior angle.

Q 4 Find $\angle x$ in the following figures.

(1) (2) (3)

Figure 15: Angles of Polygons. Retrieved from Junior High School Mathematics 2, Gakko Tosho, p.112.



88 Chapter 4 Investigating figures

2 Angles of polygons

Learn about the properties of angles in triangles and other polygons.

Interior and exterior angles in triangles

Review In elementary school, you used figures like the one on the right to learn that the sum of the three angles in a triangle is always 180°.

Extend What will happen?

What is the positional relationship of lines BA and CP in the figure on the right?

In the figure in Extend, BA // CP. Use this information to make sure the three angles of a triangle add up to 180°.

As you can see in the figure on the right, D is on a line formed by extending side BC of △ABC. Line CE is drawn parallel to side BA and through point C.

In this case,

The alternate interior angles of parallel lines are equal, so $\angle a = \angle d$. ······①

The corresponding angles of parallel lines are equal, so $\angle b = \angle e$. ······②

Knowing ① and ② allows us to find the sum of the three angles of △ABC:

$$\angle a + \angle b + \angle c = \angle d + \angle e + \angle c = \angle BCD$$

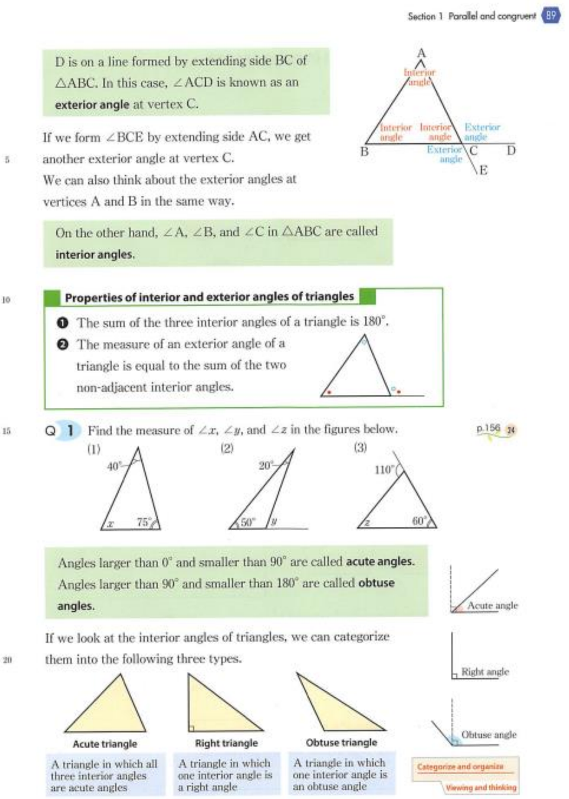
The three points B, C, and D are on the same line, so ∠BCD = 180°. This means that the sum of the three angles of a triangle is 180°.

Present it in your own words!

D is on a line formed by extending side BC of △ABC. Explain why

$$\angle A + \angle B = \angle ACD$$

Figure 16: Angles of polygons. Retrieved from Math 2 for Junior High School, Keirinkan, p.88.



Section 1 Parallel and congruent 89

D is on a line formed by extending side BC of △ABC. In this case, ∠ACD is known as an **exterior angle** at vertex C.

If we form ∠BCE by extending side AC, we get another exterior angle at vertex C.
We can also think about the exterior angles at vertices A and B in the same way.

On the other hand, ∠A, ∠B, and ∠C in △ABC are called **interior angles.**

Properties of interior and exterior angles of triangles

1. The sum of the three interior angles of a triangle is 180°.
2. The measure of an exterior angle of a triangle is equal to the sum of the two non-adjacent interior angles.

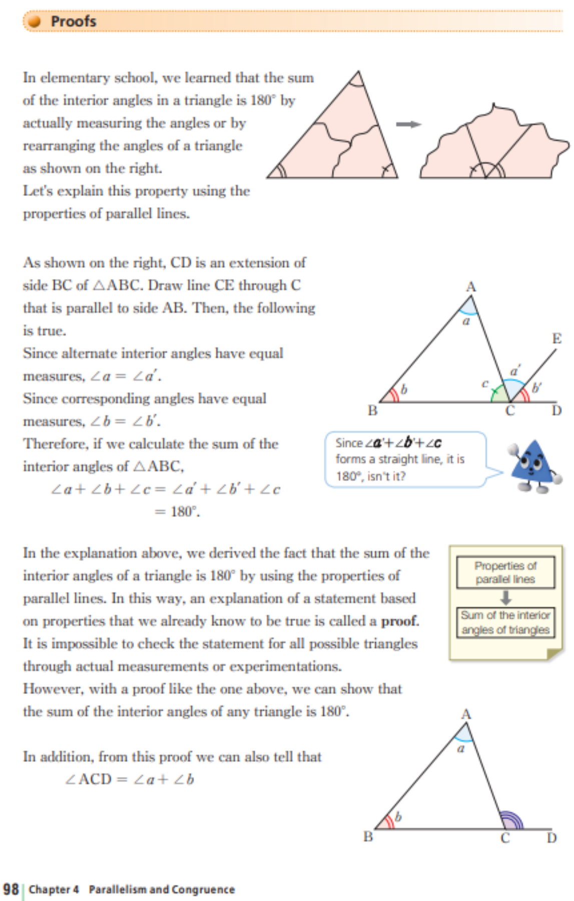
Q 1 Find the measure of ∠x, ∠y, and ∠z in the figures below. p.156

Angles larger than 0° and smaller than 90° are called **acute angles.** Angles larger than 90° and smaller than 180° are called **obtuse angles.**

If we look at the interior angles of triangles, we can categorize them into the following three types.

Acute triangle: A triangle in which all three interior angles are acute angles
Right triangle: A triangle in which one interior angle is a right angle
Obtuse triangle: A triangle in which one interior angle is an obtuse angle

Figure 17: Angles of polygons. Retrieved from Math 2 for Junior High School, Keirinkan, p.89.



Proofs

In elementary school, we learned that the sum of the interior angles in a triangle is 180° by actually measuring the angles or by rearranging the angles of a triangle as shown on the right.
Let's explain this property using the properties of parallel lines.

As shown on the right, CD is an extension of side BC of △ABC. Draw line CE through C that is parallel to side AB. Then, the following is true.
Since alternate interior angles have equal measures, $\angle a = \angle a'$.
Since corresponding angles have equal measures, $\angle b = \angle b'$.
Therefore, if we calculate the sum of the interior angles of △ABC,

$$\angle a + \angle b + \angle c = \angle a' + \angle b' + \angle c = 180°.$$

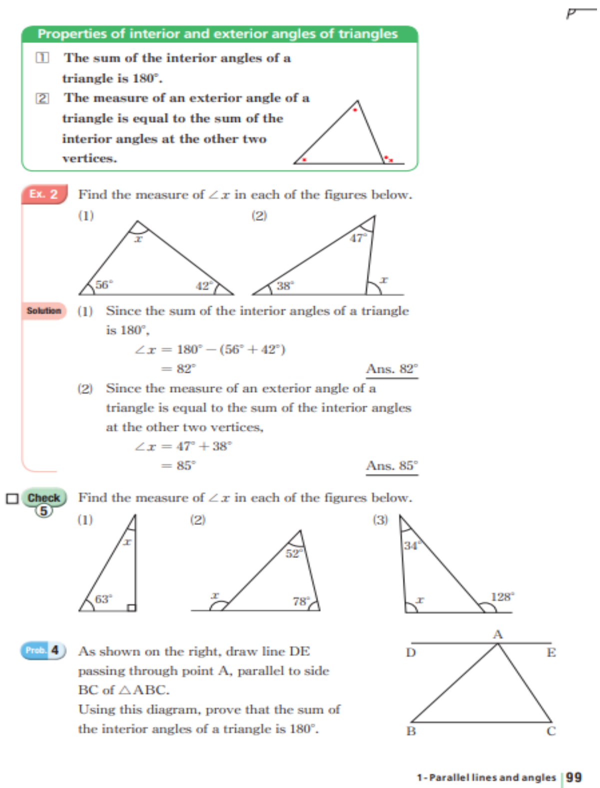
In the explanation above, we derived the fact that the sum of the interior angles of a triangle is 180° by using the properties of parallel lines. In this way, an explanation of a statement based on properties that we already know to be true is called a **proof**.
It is impossible to check the statement for all possible triangles through actual measurements or experimentations.
However, with a proof like the one above, we can show that the sum of the interior angles of any triangle is 180°.

In addition, from this proof we can also tell that

$$\angle ACD = \angle a + \angle b$$

98 Chapter 4 Parallelism and Congruence

Figure 18: Proofs the sum of three angles in a triangle. Retrieved from Mathematics 8, Tokyo Shoseki, p.98.



Properties of interior and exterior angles of triangles

[1] The sum of the interior angles of a triangle is 180°.

[2] The measure of an exterior angle of a triangle is equal to the sum of the interior angles at the other two vertices.

Ex. 2 Find the measure of $\angle x$ in each of the figures below.

(1) (2)

Solution (1) Since the sum of the interior angles of a triangle is 180°,

$\angle x = 180° - (56° + 42°)$

$= 82°$ Ans. 82°

(2) Since the measure of an exterior angle of a triangle is equal to the sum of the interior angles at the other two vertices,

$\angle x = 47° + 38°$

$= 85°$ Ans. 85°

Check 5 Find the measure of $\angle x$ in each of the figures below.

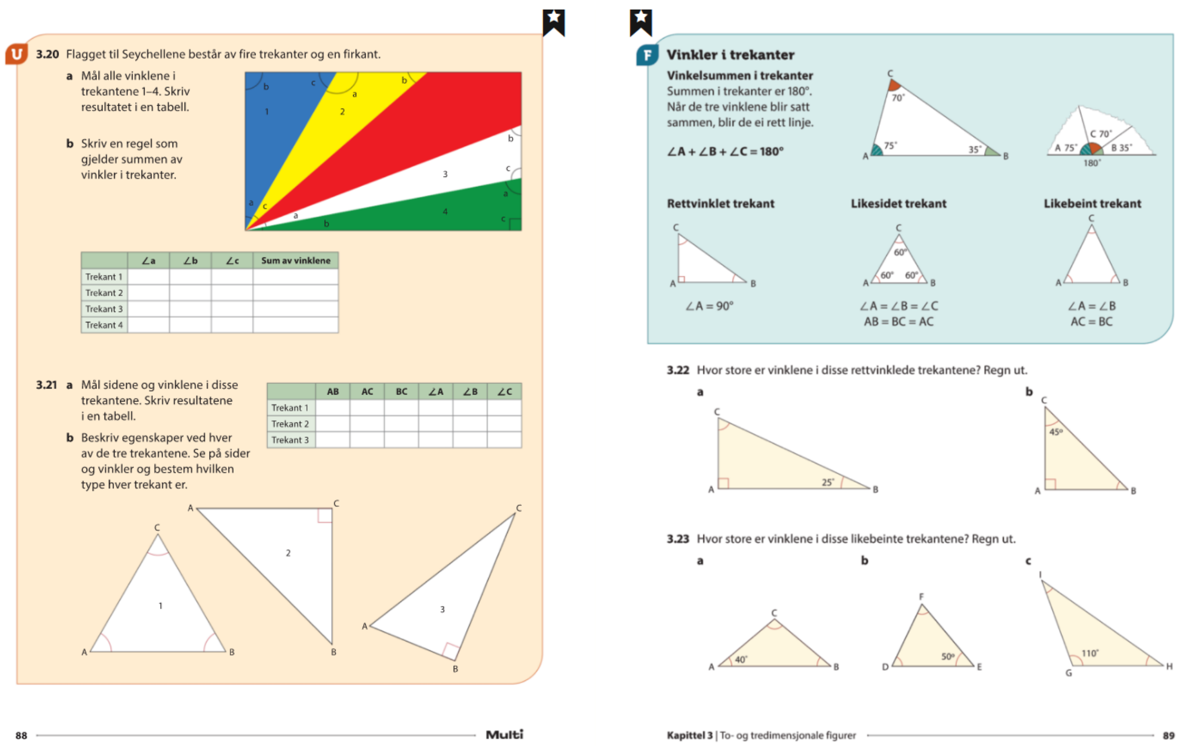
(1) (2) (3)

Prob. 4 As shown on the right, draw line DE passing through point A, parallel to side BC of △ABC.

Using this diagram, prove that the sum of the interior angles of a triangle is 180°.

1 - Parallel lines and angles | 99

Figure 19: Proofs the sum of three angles in a triangle. Retrieved from Mathematics 8, Tokyo Shoseki, p.99.



U **3.20** Flagget til Seychellene består av fire trekanter og en firkant.

a Mål alle vinklene i trekantene 1–4. Skriv resultatet i en tabell.

b Skriv en regel som gjelder summen av vinkler i trekanter.

	∠a	∠b	∠c	Sum av vinklene
Trekant 1				
Trekant 2				
Trekant 3				
Trekant 4				

3.21 **a** Mål sidene og vinklene i disse trekantene. Skriv resultatene i en tabell.

b Beskriv egenskaper ved hver av de tre trekantene. Se på sider og vinkler og bestem hvilken type hver trekant er.

	AB	AC	BC	∠A	∠B	∠C
Trekant 1						
Trekant 2						
Trekant 3						

F **Vinkler i trekanter**

Vinkelsummen i trekanter

Summen i trekanter er 180°. Når de tre vinklene blir satt sammen, blir de ei rett linje.

$\angle A + \angle B + \angle C = 180°$

Rettvinklet trekant $\angle A = 90°$

Likesidet trekant $\angle A = \angle B = \angle C$, AB = BC = AC

Likebeint trekant $\angle A = \angle B$, AC = BC

3.22 Hvor store er vinklene i disse rettvinklede trekantene? Regn ut.

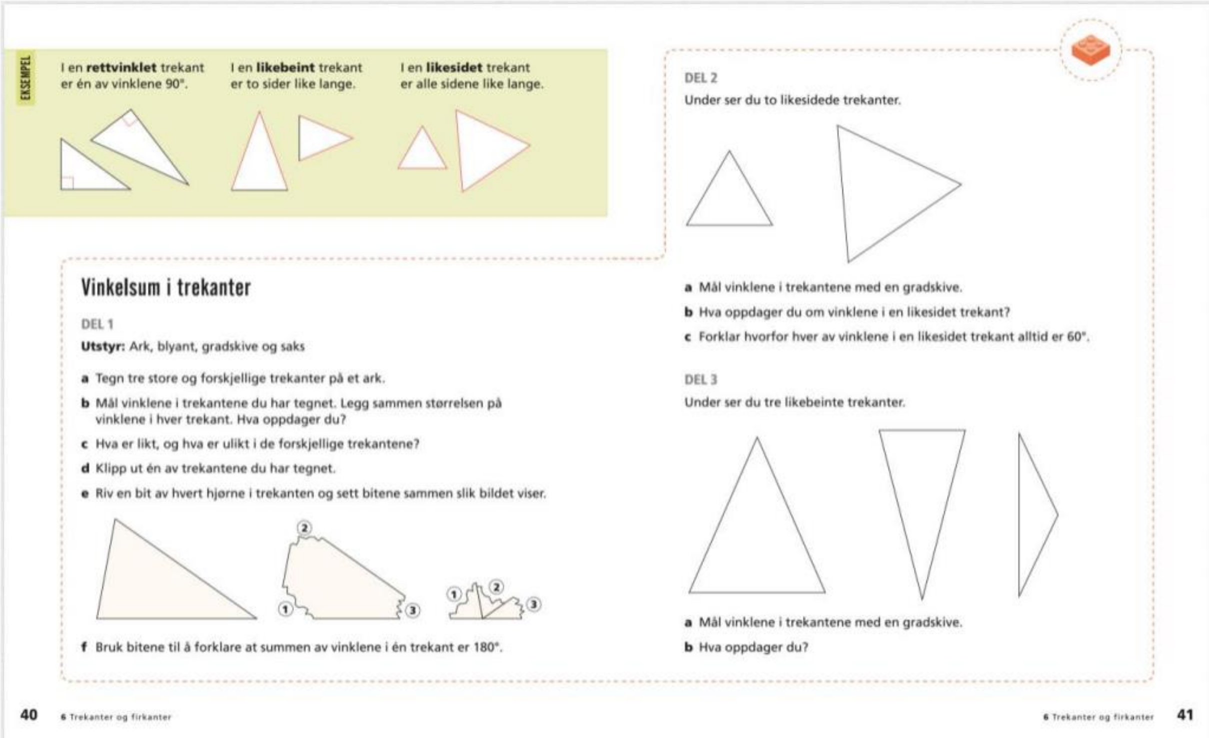
a b

3.23 Hvor store er vinklene i disse likebeinte trekantene? Regn ut.

a b c

88 Multi Kapittel 3 | To- og tredimensjonale figurer 89

Figure 20: The sum of three angles in a triangle. Retrieved from Multi 6a, Gyldendal, p.88-89.



EKSEMPEL

I en **rettvinklet** trekant er én av vinklene 90°.

I en **likebeint** trekant er to sider like lange.

I en **likesidet** trekant er alle sidene like lange.

Vinkelsum i trekanter

DEL 1

Utstyr: Ark, blyant, gradskive og saks

a Tegn tre store og forskjellige trekanter på et ark.

b Mål vinklene i trekantene du har tegnet. Legg sammen størrelsen på vinklene i hver trekant. Hva oppdager du?

c Hva er likt, og hva er ulikt i de forskjellige trekantene?

d Klipp ut én av trekantene du har tegnet.

e Riv en bit av hvert hjørne i trekanten og sett bitene sammen slik bildet viser.

f Bruk bitene til å forklare at summen av vinklene i én trekant er 180°.

DEL 2

Under ser du to likesidede trekanter.

a Mål vinklene i trekantene med en gradskive.

b Hva oppdager du om vinklene i en likesidet trekant?

c Forklar hvorfor hver av vinklene i en likesidet trekant alltid er 60°.

DEL 3

Under ser du tre likebeinte trekanter.

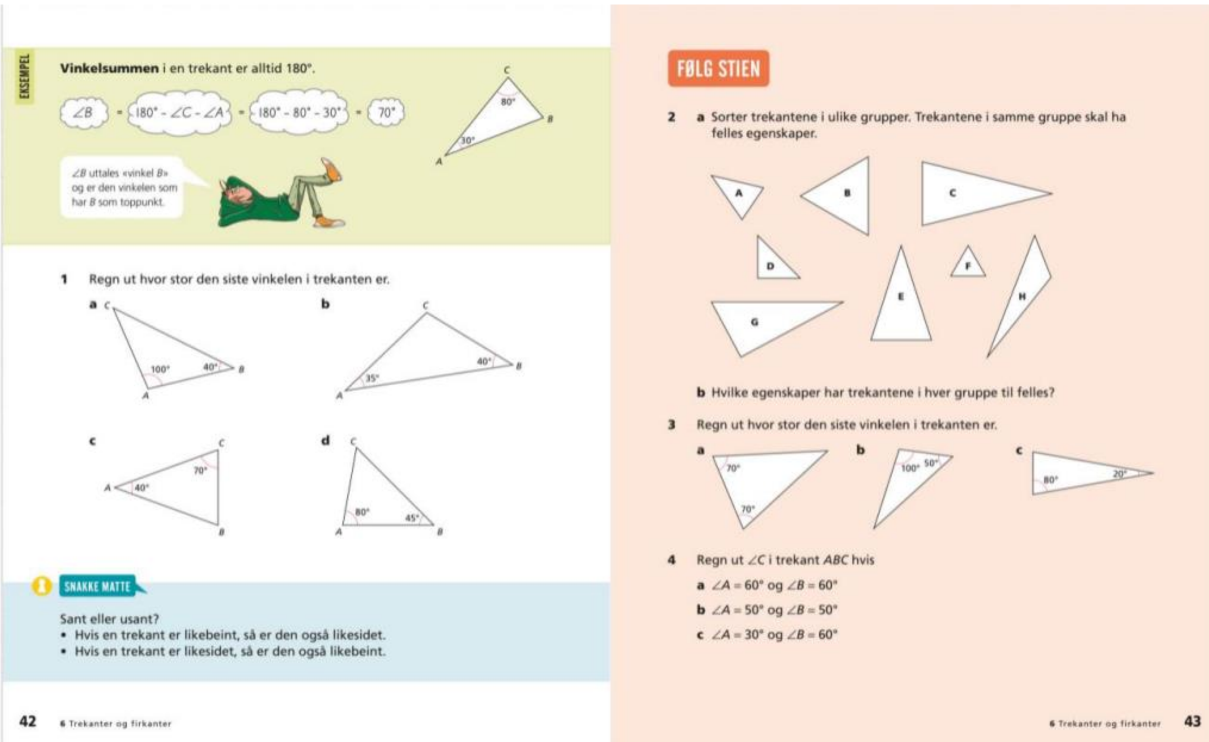
a Mål vinklene i trekantene med en gradskive.

b Hva oppdager du?

40 6 Trekanter og firkanter

6 Trekanter og firkanter 41

Figure 21: The sum of angles in triangles. Retrieved from Matemagisk 6a, Aschehoug, p.40-41.



EKSEMPEL

Vinkelsummen i en trekant er alltid 180°.

$\angle B = 180° - \angle C - \angle A = 180° - 80° - 30° = 70°$

$\angle B$ uttales «vinkel *B*» og er den vinkelen som har *B* som toppunkt.

1 Regn ut hvor stor den siste vinkelen i trekanten er.

SNAKKE MATTE

Sant eller usant?

- Hvis en trekant er likebeint, så er den også likesidet.
- Hvis en trekant er likesidet, så er den også likebeint.

FØLG STIEN

2 **a** Sorter trekantene i ulike grupper. Trekantene i samme gruppe skal ha felles egenskaper.

b Hvilke egenskaper har trekantene i hver gruppe til felles?

3 Regn ut hvor stor den siste vinkelen i trekanten er.

4 Regn ut $\angle C$ i trekant *ABC* hvis

a $\angle A = 60°$ og $\angle B = 60°$

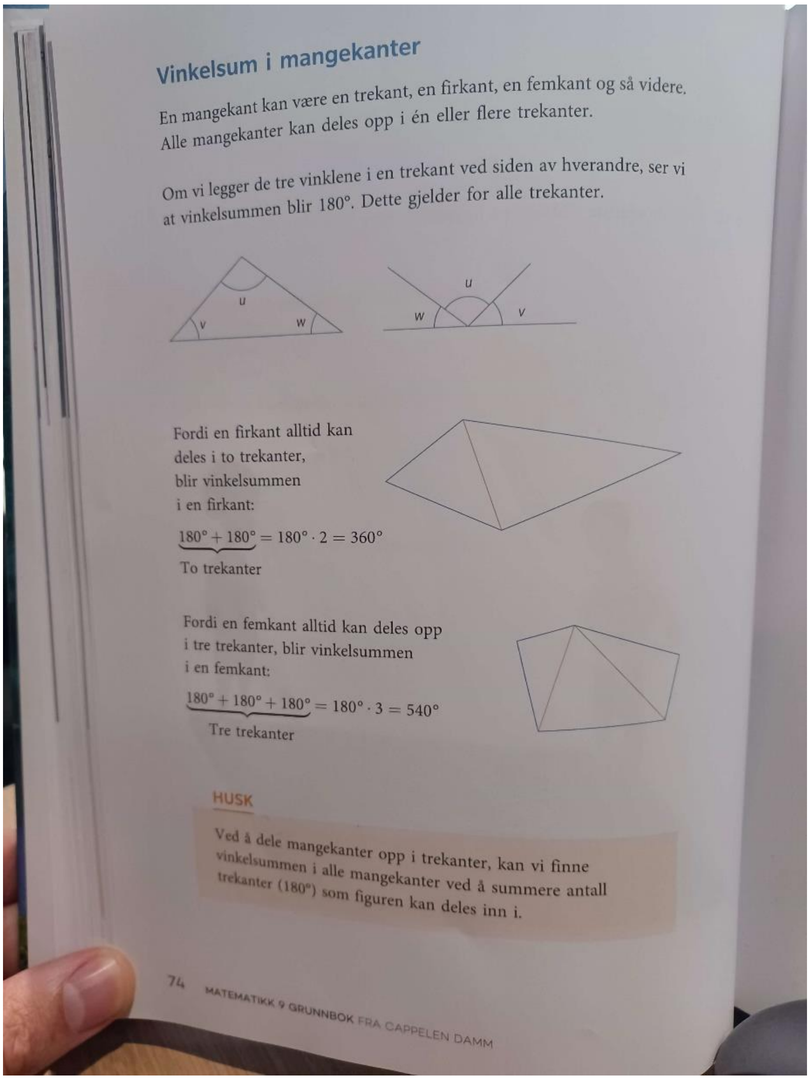
b $\angle A = 50°$ og $\angle B = 50°$

c $\angle A = 30°$ og $\angle B = 60°$

42 6 Trekanter og firkanter

6 Trekanter og firkanter 43

Figure 22: The angle sum in a triangle equals 180 degrees. Retrieved from Matemagisk 6a, Aschehoug, p.42-43.



Vinkelsum i mangekanter

En mangekant kan være en trekant, en firkant, en femkant og så videre. Alle mangekanter kan deles opp i én eller flere trekanter.

Om vi legger de tre vinklene i en trekant ved siden av hverandre, ser vi at vinkelsummen blir 180°. Dette gjelder for alle trekanter.

Fordi en firkant alltid kan deles i to trekanter, blir vinkelsummen i en firkant:

$\underbrace{180° + 180°}_{\text{To trekanter}} = 180° \cdot 2 = 360°$

Fordi en femkant alltid kan deles opp i tre trekanter, blir vinkelsummen i en femkant:

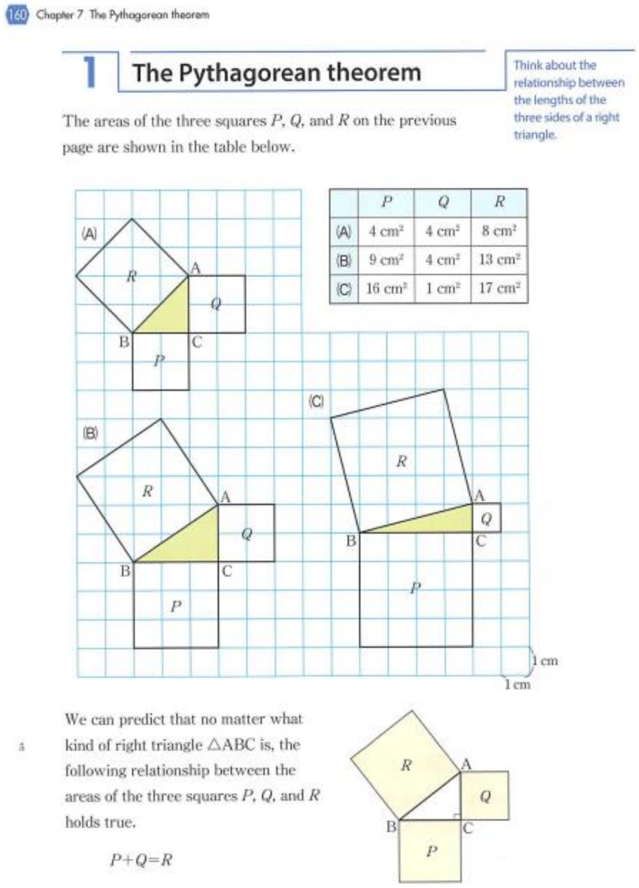
$\underbrace{180° + 180° + 180°}_{\text{Tre trekanter}} = 180° \cdot 3 = 540°$

HUSK

Ved å dele mangekanter opp i trekanter, kan vi finne vinkelsummen i alle mangekanter ved å summere antall trekanter (180°) som figuren kan deles inn i.

74 MATEMATIKK 9 GRUNNBOK FRA CAPPELEN DAMM

Figure 23: The angle sum in polygons. Retrieved from Matematikk 9 fra Cappelen Damm, Cappelen Damm, p.74.


160 Chapter 7 The Pythagorean theorem

1 The Pythagorean theorem

Think about the relationship between the lengths of the three sides of a right triangle.

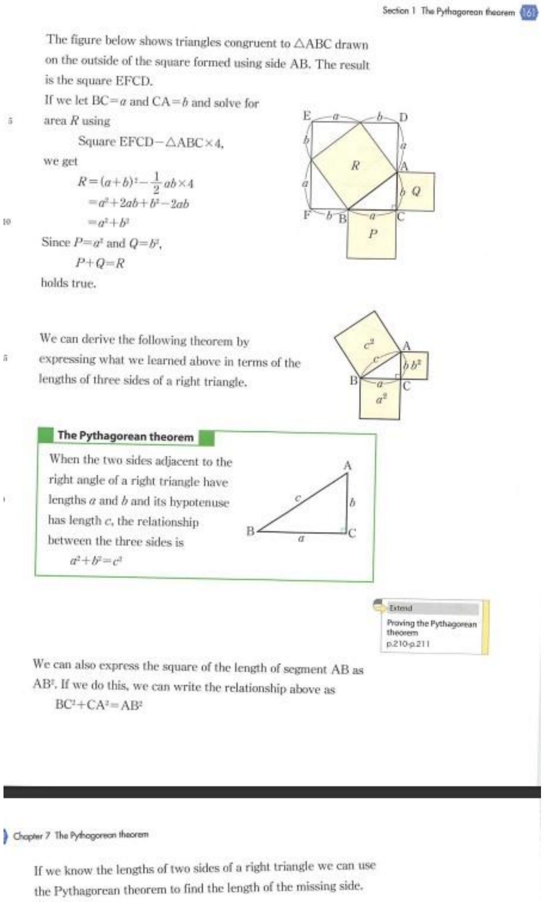
The areas of the three squares P, Q, and R on the previous page are shown in the table below.

	P	Q	R
(A)	4 cm²	4 cm²	8 cm²
(B)	9 cm²	4 cm²	13 cm²
(C)	16 cm²	1 cm²	17 cm²

We can predict that no matter what kind of right triangle △ABC is, the following relationship between the areas of the three squares P, Q, and R holds true.

$P+Q=R$

Figure 24: Explore the sides of a triangle. Retrieved from Math 3 for Junior High School, Keirinkan, p.160.


Section 1 The Pythagorean theorem 161

The figure below shows triangles congruent to △ABC drawn on the outside of the square formed using side AB. The result is the square EFCD.

If we let BC $=a$ and CA $=b$ and solve for area R using

Square EFCD $-$ △ABC $\times 4$,

we get

$$R=(a+b)^2-\frac{1}{2}ab\times 4$$
$$=a^2+2ab+b^2-2ab$$
$$=a^2+b^2$$

Since $P=a^2$ and $Q=b^2$,

$$P+Q=R$$

holds true.

We can derive the following theorem by expressing what we learned above in terms of the lengths of three sides of a right triangle.

The Pythagorean theorem

When the two sides adjacent to the right angle of a right triangle have lengths a and b and its hypotenuse has length c, the relationship between the three sides is

$$a^2+b^2=c^2$$

Extend: Proving the Pythagorean theorem p.210-p.211

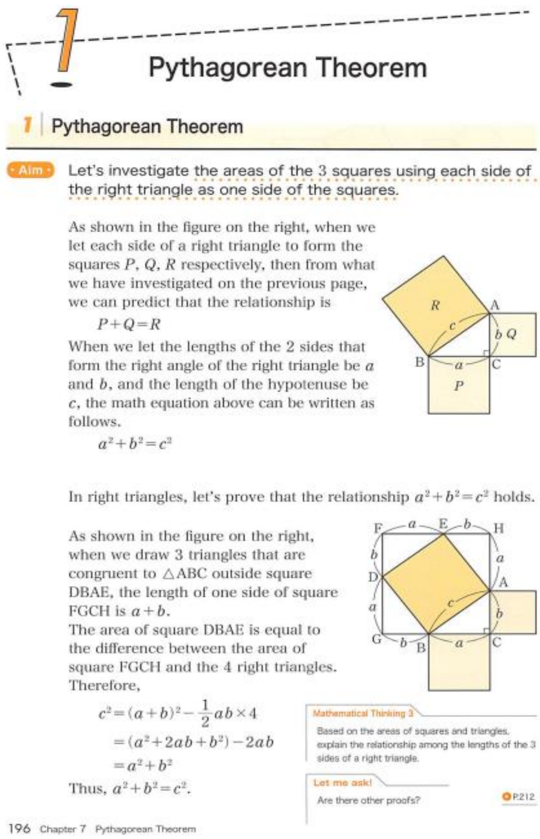
We can also express the square of the length of segment AB as AB^2. If we do this, we can write the relationship above as

$$BC^2+CA^2=AB^2$$

Chapter 7 The Pythagorean theorem

If we know the lengths of two sides of a right triangle we can use the Pythagorean theorem to find the length of the missing side.

Figure 25: Explore the sides of a triangle. Retrieved from Math 3 for Junior High School, Keirinkan, p.161.


7 Pythagorean Theorem

1 Pythagorean Theorem

Aim: Let's investigate the areas of the 3 squares using each side of the right triangle as one side of the squares.

As shown in the figure on the right, when we let each side of a right triangle to form the squares P, Q, R respectively, then from what we have investigated on the previous page, we can predict that the relationship is

$$P+Q=R$$

When we let the lengths of the 2 sides that form the right angle of the right triangle be a and b, and the length of the hypotenuse be c, the math equation above can be written as follows.

$$a^2+b^2=c^2$$

In right triangles, let's prove that the relationship $a^2+b^2=c^2$ holds.

As shown in the figure on the right, when we draw 3 triangles that are congruent to △ABC outside square DBAE, the length of one side of square FGCH is $a+b$.

The area of square DBAE is equal to the difference between the area of square FGCH and the 4 right triangles.

Therefore,

$$c^2=(a+b)^2-\frac{1}{2}ab\times 4$$
$$=(a^2+2ab+b^2)-2ab$$
$$=a^2+b^2$$

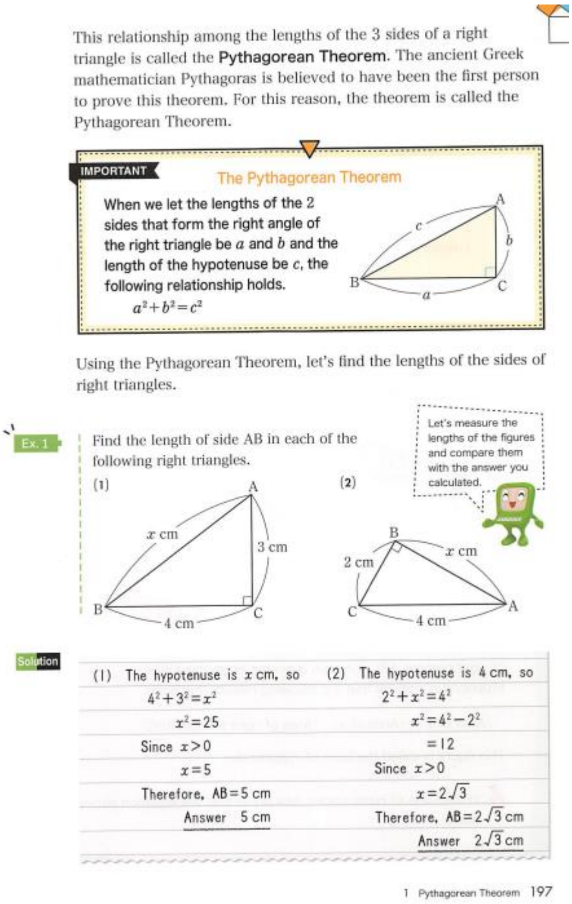
Thus, $a^2+b^2=c^2$.

Mathematical Thinking 3: Based on the areas of squares and triangles, explain the relationship among the lengths of the 3 sides of a right triangle.

Let me ask! Are there other proofs? P.212

196 Chapter 7 Pythagorean Theorem

Figure 26: The Pythagorean theorem. Retrieved from Junior High School Mathematics 1, Gakko Tosho, p.196.



This relationship among the lengths of the 3 sides of a right triangle is called the **Pythagorean Theorem**. The ancient Greek mathematician Pythagoras is believed to have been the first person to prove this theorem. For this reason, the theorem is called the Pythagorean Theorem.

IMPORTANT **The Pythagorean Theorem**

When we let the lengths of the 2 sides that form the right angle of the right triangle be a and b and the length of the hypotenuse be c, the following relationship holds.

$a^2+b^2=c^2$

Using the Pythagorean Theorem, let's find the lengths of the sides of right triangles.

Ex. 1 Find the length of side AB in each of the following right triangles.

(1) (2)

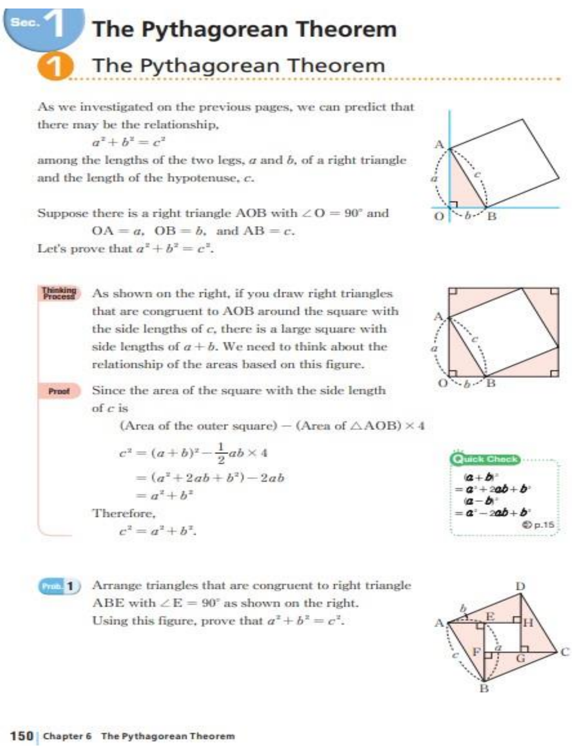
Let's measure the lengths of the figures and compare them with the answer you calculated.

Solution

(1) The hypotenuse is x cm, so	(2) The hypotenuse is 4 cm, so
$4^2+3^2=x^2$	$2^2+x^2=4^2$
$x^2=25$	$x^2=4^2-2^2$
Since $x>0$	$=12$
$x=5$	Since $x>0$
Therefore, AB = 5 cm	$x=2\sqrt{3}$
Answer 5 cm	Therefore, AB $=2\sqrt{3}$ cm
	Answer $2\sqrt{3}$ cm

1 Pythagorean Theorem 197

Figure 27: The Pythagorean theorem. Retrieved from Junior High School Mathematics 1, Gakko Tosho, p.197.



Sec. 1 **The Pythagorean Theorem**

1 **The Pythagorean Theorem**

As we investigated on the previous pages, we can predict that there may be the relationship,

$a^2+b^2=c^2$

among the lengths of the two legs, a and b, of a right triangle and the length of the hypotenuse, c.

Suppose there is a right triangle AOB with $\angle O = 90°$ and

$OA = a$, $OB = b$, and $AB = c$.

Let's prove that $a^2+b^2=c^2$.

Thinking Process As shown on the right, if you draw right triangles that are congruent to AOB around the square with the side lengths of c, there is a large square with side lengths of $a+b$. We need to think about the relationship of the areas based on this figure.

Proof Since the area of the square with the side length of c is

(Area of the outer square) − (Area of △AOB) × 4

$$c^2 = (a+b)^2 - \frac{1}{2}ab \times 4$$
$$= (a^2+2ab+b^2) - 2ab$$
$$= a^2+b^2$$

Therefore,

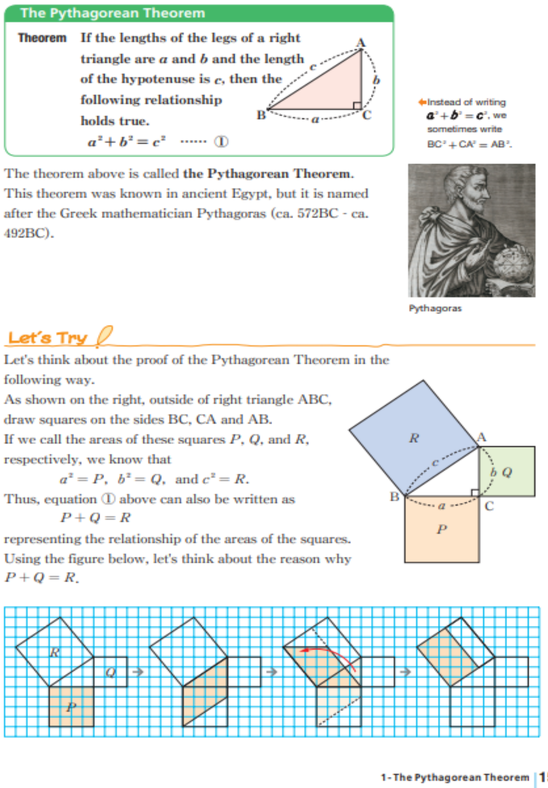
$c^2=a^2+b^2$.

Quick Check
$(a+b)^2 = a^2+2ab+b^2$
$(a-b)^2 = a^2-2ab+b^2$
p.15

Prob. 1 Arrange triangles that are congruent to right triangle ABE with $\angle E = 90°$ as shown on the right. Using this figure, prove that $a^2+b^2=c^2$.

150 Chapter 6 The Pythagorean Theorem

Figure 28: The Pythagorean theorem. Retrieved from Mathematics 9, Tokyo Shoseki, p.150.



The Pythagorean Theorem

Theorem If the lengths of the legs of a right triangle are a and b and the length of the hypotenuse is c, then the following relationship holds true.

$a^2 + b^2 = c^2$ …… ①

Instead of writing $a^2 + b^2 = c^2$, we sometimes write $BC^2 + CA^2 = AB^2$.

The theorem above is called **the Pythagorean Theorem**. This theorem was known in ancient Egypt, but it is named after the Greek mathematician Pythagoras (ca. 572BC - ca. 492BC).

Pythagoras

Let's Try

Let's think about the proof of the Pythagorean Theorem in the following way.

As shown on the right, outside of right triangle ABC, draw squares on the sides BC, CA and AB.

If we call the areas of these squares P, Q, and R, respectively, we know that

$a^2 = P$, $b^2 = Q$, and $c^2 = R$.

Thus, equation ① above can also be written as

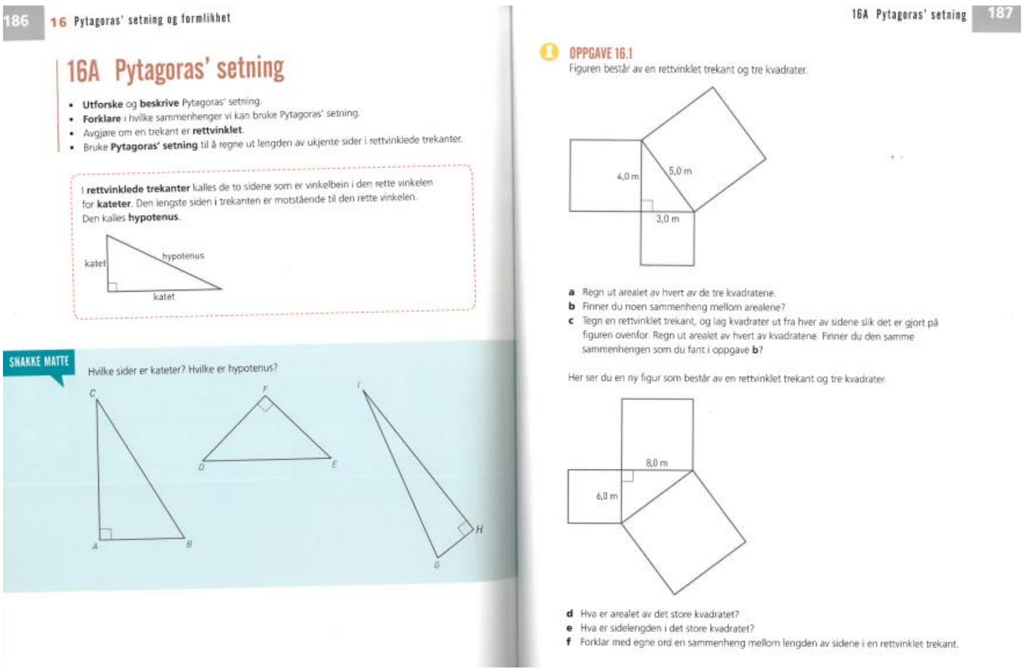
$P + Q = R$

representing the relationship of the areas of the squares.

Using the figure below, let's think about the reason why $P + Q = R$.

1 - The Pythagorean Theorem | 151

Figure 29: The Pythagorean theorem. Retrieved from Mathematics 9, Tokyo Shoseki, p.151.



186 16 Pytagoras' setning og formlikhet

16A Pytagoras' setning

- **Utforske** og **beskrive** Pytagoras' setning.
- **Forklare** i hvilke sammenhenger vi kan bruke Pytagoras' setning.
- Avgjøre om en trekant er **rettvinklet**.
- Bruke **Pytagoras' setning** til å regne ut lengden av ukjente sider i rettvinklede trekanter.

I **rettvinklede trekanter** kalles de to sidene som er vinkelbein i den rette vinkelen for **kateter**. Den lengste siden i trekanten er motstående til den rette vinkelen. Den kalles **hypotenus**.

SNAKKE MATTE

Hvilke sider er kateter? Hvilke er hypotenus?

16A Pytagoras' setning 187

OPPGAVE 16.1

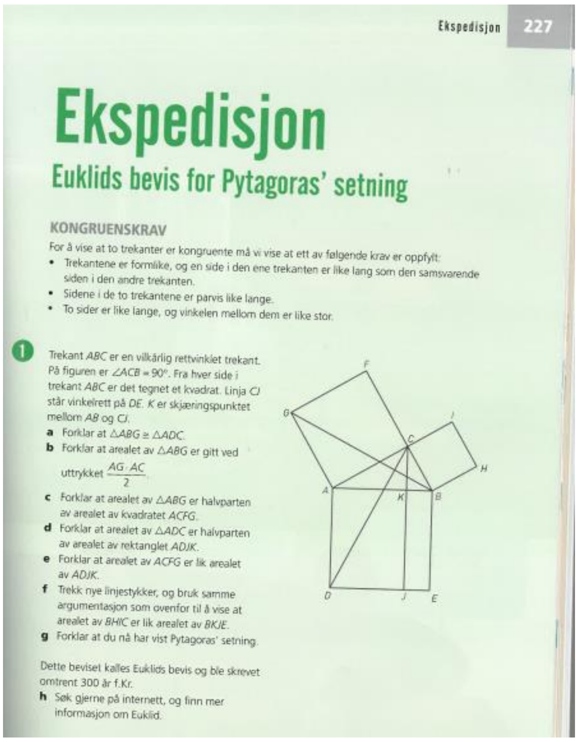
Figuren består av en rettvinklet trekant og tre kvadrater.

a Regn ut arealet av hvert av de tre kvadratene.
b Finner du noen sammenheng mellom arealene?
c Tegn en rettvinklet trekant, og lag kvadrater ut fra hver av sidene slik det er gjort på figuren ovenfor. Regn ut arealet av hvert av kvadratene. Finner du den samme sammenhengen som du fant i oppgave **b**?

Her ser du en ny figur som består av en rettvinklet trekant og tre kvadrater.

d Hva er arealet av det store kvadratet?
e Hva er sidelengden i det store kvadratet?
f Forklar med egne ord en sammenheng mellom lengden av sidene i en rettvinklet trekant.

Figure 30: The Pythagorean theorem. Retrieved from Matemagisk 9, Aschehoug, p.186-187.



Ekspedisjon 227

Ekspedisjon

Euklids bevis for Pytagoras' setning

KONGRUENSKRAV

For å vise at to trekanter er kongruente må vi vise at ett av følgende krav er oppfylt:

- Trekantene er formlike, og en side i den ene trekanten er like lang som den samsvarende siden i den andre trekanten.
- Sidene i de to trekantene er parvis like lange.
- To sider er like lange, og vinkelen mellom dem er like stor.

1 Trekant *ABC* er en vilkårlig rettvinklet trekant. På figuren er $\angle ACB = 90°$. Fra hver side i trekant *ABC* er det tegnet et kvadrat. Linja *CJ* står vinkelrett på *DE*. *K* er skjæringspunktet mellom *AB* og *CJ*.

a Forklar at $\triangle ABG \cong \triangle ADC$.

b Forklar at arealet av $\triangle ABG$ er gitt ved uttrykket $\frac{AG \cdot AC}{2}$.

c Forklar at arealet av $\triangle ABG$ er halvparten av arealet av kvadratet *ACFG*.

d Forklar at arealet av $\triangle ADC$ er halvparten av arealet av rektanglet *ADJK*.

e Forklar at arealet av *ACFG* er lik arealet av *ADJK*.

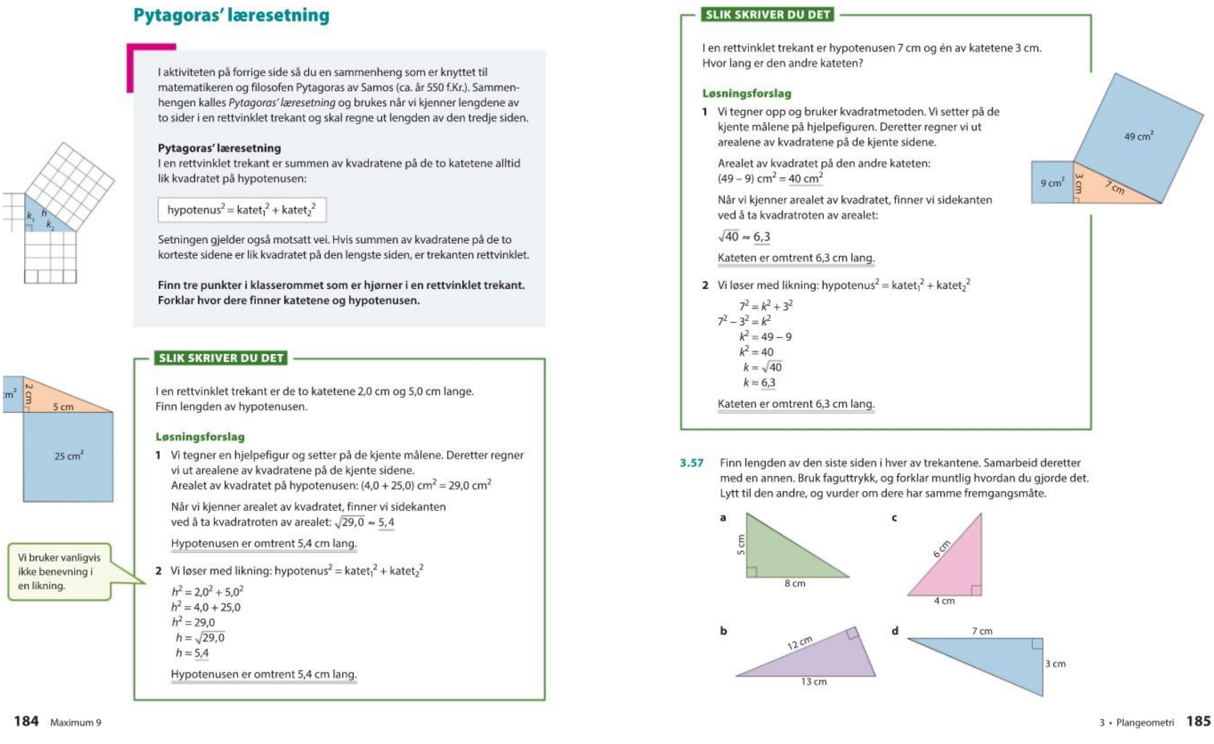
f Trekk nye linjestykker, og bruk samme argumentasjon som ovenfor til å vise at arealet av *BHIC* er lik arealet av *BKJE*.

g Forklar at du nå har vist Pytagoras' setning.

Dette beviset kalles Euklids bevis og ble skrevet omtrent 300 år f.Kr.

h Søk gjerne på internett, og finn mer informasjon om Euklid.

Figure 31: Euclid's proof for the Pythagorean theorem. Retrieved from Matemagisk 9, Aschehoug, p.227.



Pytagoras' læresetning

I aktiviteten på forrige side så du en sammenheng som er knyttet til matematikeren og filosofen Pytagoras av Samos (ca. år 550 f.Kr.). Sammenhengen kalles *Pytagoras' læresetning* og brukes når vi kjenner lengdene av to sider i en rettvinklet trekant og skal regne ut lengden av den tredje siden.

Pytagoras' læresetning
I en rettvinklet trekant er summen av kvadratene på de to katetene alltid lik kvadratet på hypotenusen:

$\text{hypotenus}^2 = \text{katet}_1^2 + \text{katet}_2^2$

Setningen gjelder også motsatt vei. Hvis summen av kvadratene på de to korteste sidene er lik kvadratet på den lengste siden, er trekanten rettvinklet.

Finn tre punkter i klasserommet som er hjørner i en rettvinklet trekant. Forklar hvor dere finner katetene og hypotenusen.

SLIK SKRIVER DU DET

I en rettvinklet trekant er de to katetene 2,0 cm og 5,0 cm lange. Finn lengden av hypotenusen.

Løsningsforslag

1 Vi tegner en hjelpefigur og setter på de kjente målene. Deretter regner vi ut arealene av kvadratene på de kjente sidene.
Arealet av kvadratet på hypotenusen: $(4{,}0 + 25{,}0)\ \text{cm}^2 = 29{,}0\ \text{cm}^2$

Når vi kjenner arealet av kvadratet, finner vi sidekanten ved å ta kvadratroten av arealet: $\sqrt{29{,}0} \approx 5{,}4$

Hypotenusen er omtrent 5,4 cm lang.

2 Vi løser med likning: $\text{hypotenus}^2 = \text{katet}_1^2 + \text{katet}_2^2$

$h^2 = 2{,}0^2 + 5{,}0^2$
$h^2 = 4{,}0 + 25{,}0$
$h^2 = 29{,}0$
$h = \sqrt{29{,}0}$
$h \approx 5{,}4$

Hypotenusen er omtrent 5,4 cm lang.

Vi bruker vanligvis ikke benevning i en likning.

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SLIK SKRIVER DU DET

I en rettvinklet trekant er hypotenusen 7 cm og én av katetene 3 cm. Hvor lang er den andre kateten?

Løsningsforslag

1 Vi tegner opp og bruker kvadratmetoden. Vi setter på de kjente målene på hjelpefiguren. Deretter regner vi ut arealene av kvadratene på de kjente sidene.

Arealet av kvadratet på den andre kateten:
$(49 - 9)\ \text{cm}^2 = 40\ \text{cm}^2$

Når vi kjenner arealet av kvadratet, finner vi sidekanten ved å ta kvadratroten av arealet:

$\sqrt{40} \approx 6{,}3$

Kateten er omtrent 6,3 cm lang.

2 Vi løser med likning: $\text{hypotenus}^2 = \text{katet}_1^2 + \text{katet}_2^2$

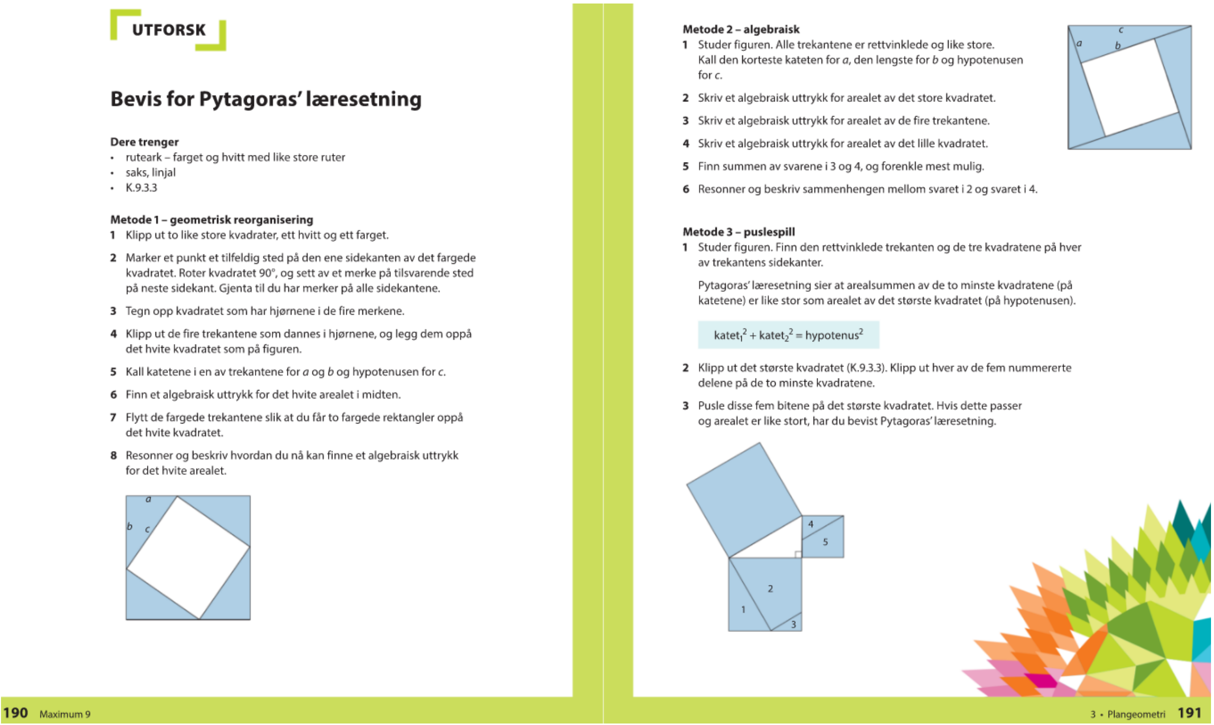
$7^2 = k^2 + 3^2$
$7^2 - 3^2 = k^2$
$k^2 = 49 - 9$
$k^2 = 40$
$k = \sqrt{40}$
$k \approx 6{,}3$

Kateten er omtrent 6,3 cm lang.

3.57 Finn lengden av den siste siden i hver av trekantene. Samarbeid deretter med en annen. Bruk faguttrykk, og forklar muntlig hvordan du gjorde det. Lytt til den andre, og vurder om dere har samme fremgangsmåte.

3 • Plangeometri 185

Figure 32: The Pythagorean theorem. Retrieved from Maximum 9, Gyldendal, p.184-185.



UTFORSK

Bevis for Pytagoras' læresetning

Dere trenger
- ruteark – farget og hvitt med like store ruter
- saks, linjal
- K.9.3.3

Metode 1 – geometrisk reorganisering
1 Klipp ut to like store kvadrater, ett hvitt og ett farget.
2 Marker et punkt et tilfeldig sted på den ene sidekanten av det fargede kvadratet. Roter kvadratet 90°, og sett av et merke på tilsvarende sted på neste sidekant. Gjenta til du har merker på alle sidekantene.
3 Tegn opp kvadratet som har hjørnene i de fire merkene.
4 Klipp ut de fire trekantene som dannes i hjørnene, og legg dem oppå det hvite kvadratet som på figuren.
5 Kall katetene i en av trekantene for *a* og *b* og hypotenusen for *c*.
6 Finn et algebraisk uttrykk for det hvite arealet i midten.
7 Flytt de fargede trekantene slik at du får to fargede rektangler oppå det hvite kvadratet.
8 Resonner og beskriv hvordan du nå kan finne et algebraisk uttrykk for det hvite arealet.

Metode 2 – algebraisk
1 Studer figuren. Alle trekantene er rettvinklede og like store. Kall den korteste kateten for *a*, den lengste for *b* og hypotenusen for *c*.
2 Skriv et algebraisk uttrykk for arealet av det store kvadratet.
3 Skriv et algebraisk uttrykk for arealet av de fire trekantene.
4 Skriv et algebraisk uttrykk for arealet av det lille kvadratet.
5 Finn summen av svarene i 3 og 4, og forenkle mest mulig.
6 Resonner og beskriv sammenhengen mellom svaret i 2 og svaret i 4.

Metode 3 – puslespill
1 Studer figuren. Finn den rettvinklede trekanten og de tre kvadratene på hver av trekantens sidekanter.

Pytagoras' læresetning sier at arealsummen av de to minste kvadratene (på katetene) er like stor som arealet av det største kvadratet (på hypotenusen).

$\text{katet}_1^2 + \text{katet}_2^2 = \text{hypotenus}^2$

2 Klipp ut det største kvadratet (K.9.3.3). Klipp ut hver av de fem nummererte delene på de to minste kvadratene.
3 Pusle disse fem bitene på det største kvadratet. Hvis dette passer og arealet er like stort, har du bevist Pytagoras' læresetning.

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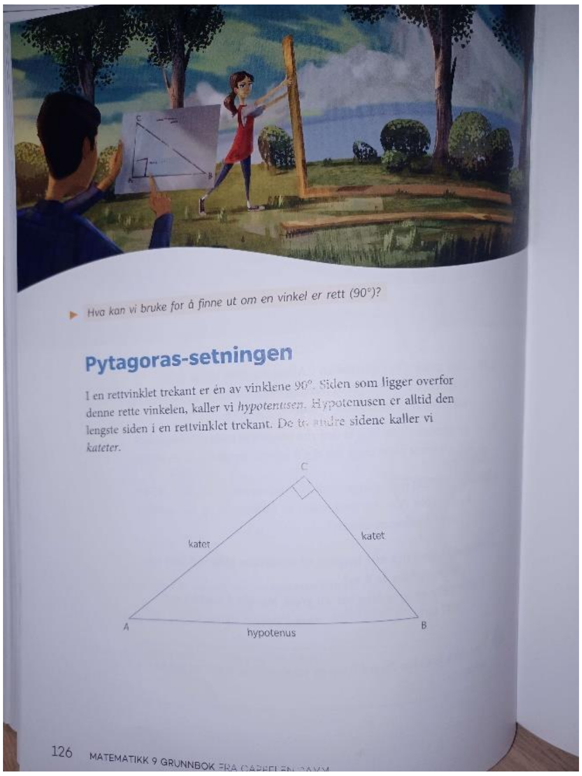
Figure 33: Proof for the Pythagorean theorem. Retrieved from Maximum 9, Gyldendal, p.190-191.



▶ *Hva kan vi bruke for å finne ut om en vinkel er rett (90°)?*

Pytagoras-setningen

I en rettvinklet trekant er én av vinklene 90°. Siden som ligger overfor denne rette vinkelen, kaller vi *hypotenusen*. Hypotenusen er alltid den lengste siden i en rettvinklet trekant. De to andre sidene kaller vi *kateter*.

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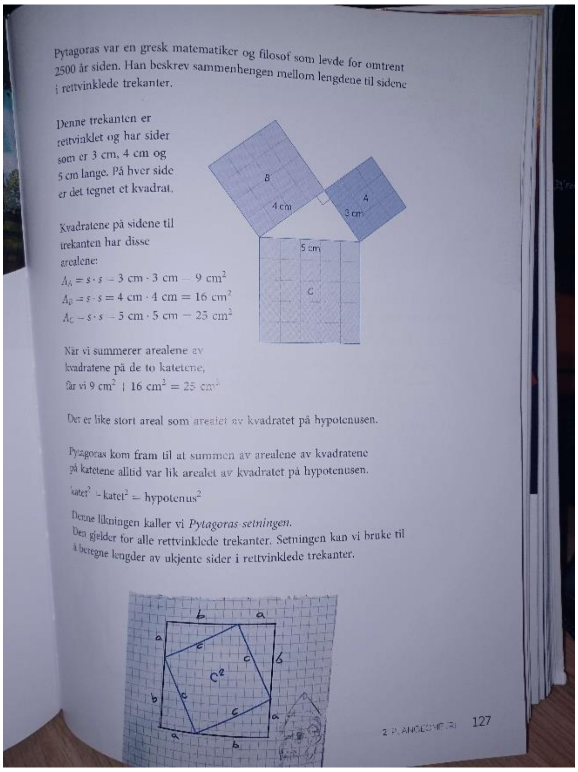
Figure 34: The Pythagorean theorem. Retrieved from Matematikk 9 fra Cappelen Damm, Cappelen Damm, p.126.



Pytagoras var en gresk matematiker og filosof som levde for omtrent 2500 år siden. Han beskrev sammenhengen mellom lengdene til sidene i rettvinklede trekanter.

Denne trekanten er rettvinklet og har sider som er 3 cm, 4 cm og 5 cm lange. På hver side er det tegnet et kvadrat.

Kvadratene på sidene til trekanten har disse arealene:

$A_A = s \cdot s = 3\text{ cm} \cdot 3\text{ cm} = 9\text{ cm}^2$

$A_B = s \cdot s = 4\text{ cm} \cdot 4\text{ cm} = 16\text{ cm}^2$

$A_C = s \cdot s = 5\text{ cm} \cdot 5\text{ cm} = 25\text{ cm}^2$

Når vi summerer arealene av kvadratene på de to katetene, får vi $9\text{ cm}^2 + 16\text{ cm}^2 = 25\text{ cm}^2$

Det er like stort areal som arealet av kvadratet på hypotenusen.

Pytagoras kom fram til at summen av arealene av kvadratene på katetene alltid var lik arealet av kvadratet på hypotenusen.

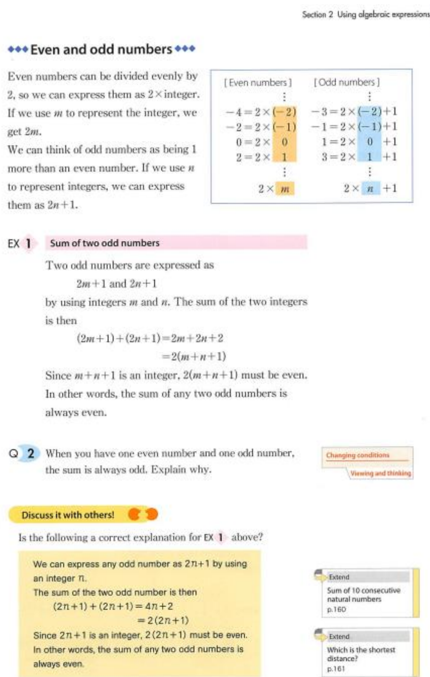
$\text{katet}^2 + \text{katet}^2 = \text{hypotenus}^2$

Denne likningen kaller vi *Pytagoras-setningen*.
Den gjelder for alle rettvinklede trekanter. Setningen kan vi bruke til å beregne lengder av ukjente sider i rettvinklede trekanter.

Figure 35: The Pythagorean theorem. Matematikk 9 fra Cappelen Damm, Cappelen Damm, p.127.

Using n as an integer, 2 consecutive integers can be represented as n, $n+1$.
What kind of number will the sum of 2 consecutive integers be?

Figure 36: Two consecutive integers. Retrieved from Mathematics 7, Tokyo Shoseki, p.73.



Section 2 Using algebraic expressions 25

Even and odd numbers

Even numbers can be divided evenly by 2, so we can express them as 2×integer. If we use m to represent the integer, we get $2m$.
We can think of odd numbers as being 1 more than an even number. If we use n to represent integers, we can express them as $2n+1$.

[Even numbers]	[Odd numbers]
⋮	⋮
$-4=2\times(-2)$	$-3=2\times(-2)+1$
$-2=2\times(-1)$	$-1=2\times(-1)+1$
$0=2\times 0$	$1=2\times 0+1$
$2=2\times 1$	$3=2\times 1+1$
⋮	⋮
$2\times m$	$2\times n+1$

EX 1 **Sum of two odd numbers**

Two odd numbers are expressed as
$2m+1$ and $2n+1$
by using integers m and n. The sum of the two integers is then

$$(2m+1)+(2n+1)=2m+2n+2 \\ =2(m+n+1)$$

Since $m+n+1$ is an integer, $2(m+n+1)$ must be even.
In other words, the sum of any two odd numbers is always even.

Q 2 When you have one even number and one odd number, the sum is always odd. Explain why.

Changing conditions
Viewing and thinking

Discuss it with others!

Is the following a correct explanation for EX 1 above?

We can express any odd number as $2n+1$ by using an integer n.
The sum of the two odd number is then
$(2n+1)+(2n+1)=4n+2$
$=2(2n+1)$
Since $2n+1$ is an integer, $2(2n+1)$ must be even.
In other words, the sum of any two odd numbers is always even.

Extend
Sum of 10 consecutive natural numbers
p.160

Extend
Which is the shortest distance?
p.161

Figure 37: Even and odd numbers. Retrieved from Math 2 for junior High School, Keirinkan, p.25.

6 The following explains why the sum of two even numbers is an even number.
Fill in the □.

Using algebraic expressions
↓
p.23-p.25

> We can express any two even integers as □ and □ by using integers m and n.
>
> □ + □ = □ $(m+n)$
>
> Since $m+n$ is an integer, it must be even.
> In other words, the sum of any two even numbers is always even.

Figure 38: Even and odd numbers. Retrieved from Math 2 for Junior High School, Keirinkan, p.27.

Let's Try

Student A tried to explain "the sum of 2 odd numbers is an even number" as shown below. However, he realized he made a mistake. What was his mistake?

✕ Example of a Mistake

> If n is an integer, two odd numbers can be expressed as $2n+1$ and $2n+3$.
>
> Therefore, the sum of 2 odd numbers is
>
> $(2n+1)+(2n+3)$
>
> $= 4n+4$
>
> $= 2(2n+2)$
>
> Since $2n+2$ is an integer, the sum of 2 odd numbers will be an even number.

Figure 39: Two odd numbers. Retrieved from Mathematics 8, Tokyo Shoseki, p.28.

7
Explaining with Algebraic Expressions
[P.26] Ex.1
[P.29] 3

Answer the following regarding two consecutive odd numbers, such as 5 and 7.

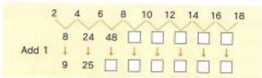
(1) Letting n be an integer, if we let the smaller odd number be $2n+1$, how can we express the larger odd number?

(2) Explain why the sum of these two consecutive odd numbers is a multiple of 4.

Figure 40: Two consecutive odd numbers. Retrieved from Junior High School Mathematics 2, Gakko Tosho, p.32.

Extend What will happen?

List even numbers in order. Multiply adjacent numbers and then add 1 to the product. What kind of numbers did you get?



We can predict the following based on ✤ above.

> The product of two consecutive even numbers plus 1 is the square of an odd number.

32 Chapter 1 Expanding and factoring expressions

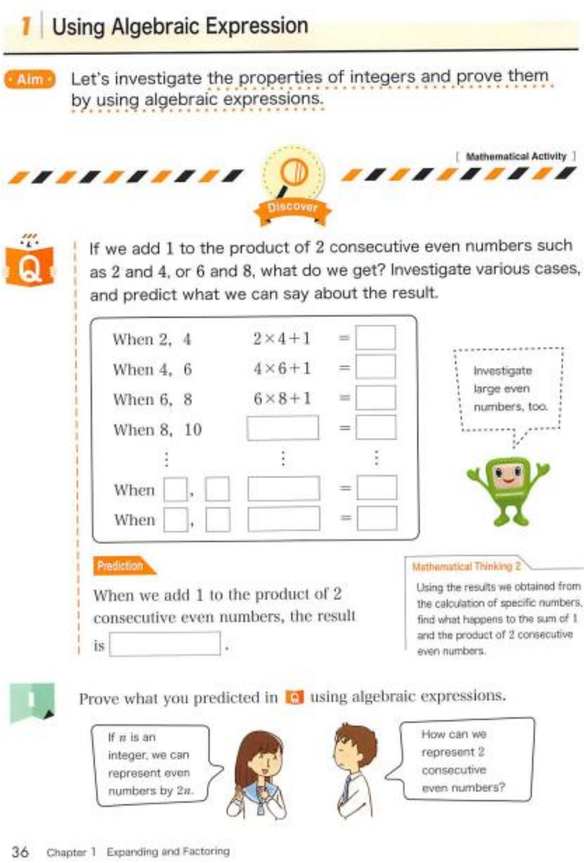
We can prove that our prediction on the previous page is true by using expression calculation.

Proof

> We can represent two consecutive even numbers as
>
> $2n$ and $2n+2$,
>
> where n is an integer.
> If we add 1 to the product of those two numbers, we get
>
> $2n(2n+2)+1=4n^2+4n+1$
>
> $=(2n+1)^2$
>
> which is the square of the odd number $2n+1$.

This proof demonstrates that one more than the product of two consecutive even numbers is the square of the odd number between them.

Figure 41: Two consecutive even numbers. Retrieved from Math 3 for Junior High School, Keirinkan, p.31-32.


1 Using Algebraic Expression

Aim: Let's investigate the properties of integers and prove them by using algebraic expressions.

[Mathematical Activity] Discover

Q If we add 1 to the product of 2 consecutive even numbers such as 2 and 4, or 6 and 8, what do we get? Investigate various cases, and predict what we can say about the result.

When 2, 4	$2\times4+1$	$=\square$
When 4, 6	$4\times6+1$	$=\square$
When 6, 8	$6\times8+1$	$=\square$
When 8, 10	$\square$	$=\square$
⋮	⋮	⋮
When $\square$, $\square$	$\square$	$=\square$
When $\square$, $\square$	$\square$	$=\square$

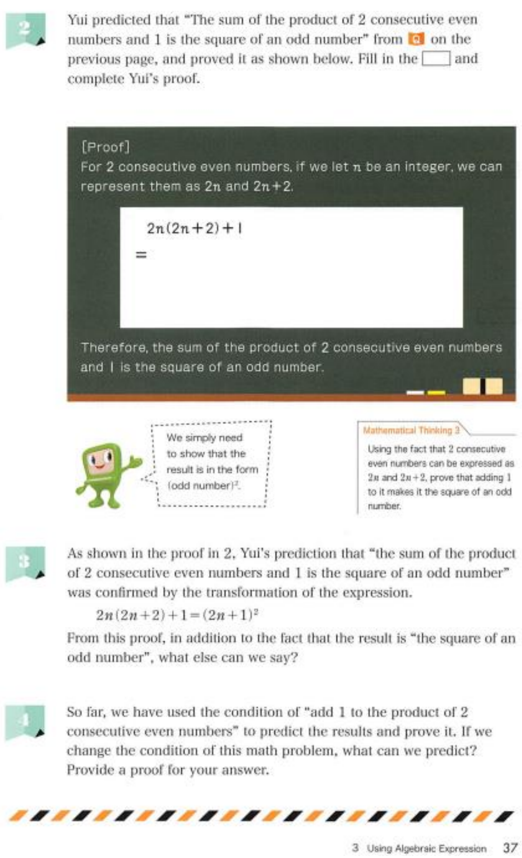
Prediction
When we add 1 to the product of 2 consecutive even numbers, the result is $\square$.

Mathematical Thinking 2
Using the results we obtained from the calculation of specific numbers, find what happens to the sum of 1 and the product of 2 consecutive even numbers.

1 Prove what you predicted in Q using algebraic expressions.

36 Chapter 1 Expanding and Factoring

Figure 42: Two consecutive even numbers. Retrieved from Junior High School Mathematics 3, Gakko Tosho, p.36.


2 Yui predicted that "The sum of the product of 2 consecutive even numbers and 1 is the square of an odd number" from Q on the previous page, and proved it as shown below. Fill in the $\square$ and complete Yui's proof.

[Proof]
For 2 consecutive even numbers, if we let n be an integer, we can represent them as $2n$ and $2n+2$.

$2n(2n+2)+1$
$=$

Therefore, the sum of the product of 2 consecutive even numbers and 1 is the square of an odd number.

Mathematical Thinking 3
Using the fact that 2 consecutive even numbers can be expressed as $2n$ and $2n+2$, prove that adding 1 to it makes it the square of an odd number.

3 As shown in the proof in 2, Yui's prediction that "the sum of the product of 2 consecutive even numbers and 1 is the square of an odd number" was confirmed by the transformation of the expression.

$2n(2n+2)+1=(2n+1)^2$

From this proof, in addition to the fact that the result is "the square of an odd number", what else can we say?

4 So far, we have used the condition of "add 1 to the product of 2 consecutive even numbers" to predict the results and prove it. If we change the condition of this math problem, what can we predict? Provide a proof for your answer.

3 Using Algebraic Expression 37

Figure 43: Adding one to two consecutive even numbers. Retrieved from Junior High School Mathematics 3, Gakko Tosho, p.36.

The result of adding 1 to the product of 2 consecutive even numbers is a square of an odd number. Prove this. **Think about it**

Figure 44: Adding one to the product of two consecutive even numbers. Retrieved from Mathematics 9, Tokyo Shoseki, p.31.

Utforsk sammen

Partall og oddetall.

Bruk tallene nedenfor og lag mange addisjonsstykker av to og to tall.
Se på svarene. Lag en regel som sier noe om svarene når dere

- adderer to partall
- adderer to oddetall
- adderer et partall og et oddetall

2 8 10 4 12

14 18 6 16 20

3 7 13 15 5

17 11 1 9 19

Figure 45: Explore even and odd numbers. Retrieved from Matematikk 7 fra Cappelen Damm, Cappelen Damm, p.23.

160 Math Dojo

Extend **Sum of 10 consecutive natural numbers**

p.23-p.25 Using algebraic expressions

Find the sum of the following sets of 10 consecutive numbers.

$1+2+3+4+5+6+7+8+9+10=\square$

$8+9+10+11+12+13+14+15+16+17=\square$

$72+73+74+75+76+77+78+79+80+81=\square$

What can you say based on results?

Here are Hayato's and Akane's ideas

Hayato's idea

$1+2+3+4+5+6+7+8+9+10=55$

The sum is always 11

It's 5 times the sum of the first and last number.

Akane's idea

$1+2+3+4+5+6+7+8+9+10=55$

$8+9+10+11+12+13+14+15+16+17=125$

↑ $120+5$

It's the fifth number from the beginning with a 5 added to the end.

Use letters to explain why Hayato's idea holds true for any 10 consecutive natural numbers.

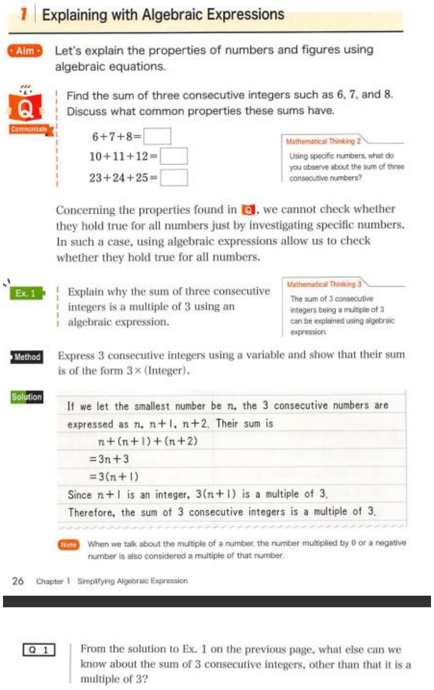
If we let the first number be a, the sum of the next 10 consecutive natural numbers is

$$a+(a+1)+(a+2)+(a+3)+(a+4)+(a+5)+(a+6)+(a+7)+(a+8)+(a+9)$$
$$=10a+45$$
$$=5(2a+9)$$
$$=5(a+a+9)$$

We can see that this number is five times the sum of the first number a and the last number $a+9$.

1 Use letters to explain why Akane's idea holds true for any 10 consecutive natural numbers.

Figure 46: Sum of ten consecutive natural numbers. Retrieved from Math 2 for Junior High School, Keirinkan, p.160.


1 Explaining with Algebraic Expressions

Aim: Let's explain the properties of numbers and figures using algebraic equations.

Q (Communicate): Find the sum of three consecutive integers such as 6, 7, and 8. Discuss what common properties these sums have.

6+7+8=□
10+11+12=□
23+24+25=□

Mathematical Thinking 2: Using specific numbers, what do you observe about the sum of three consecutive numbers?

Concerning the properties found in Q, we cannot check whether they hold true for all numbers just by investigating specific numbers. In such a case, using algebraic expressions allow us to check whether they hold true for all numbers.

Ex. 1: Explain why the sum of three consecutive integers is a multiple of 3 using an algebraic expression.

Mathematical Thinking 3: The sum of 3 consecutive integers being a multiple of 3 can be explained using algebraic expression.

Method: Express 3 consecutive integers using a variable and show that their sum is of the form 3 × (Integer).

Solution: If we let the smallest number be n, the 3 consecutive numbers are expressed as n, $n+1$, $n+2$. Their sum is

$n+(n+1)+(n+2)$
$=3n+3$
$=3(n+1)$

Since $n+1$ is an integer, $3(n+1)$ is a multiple of 3.
Therefore, the sum of 3 consecutive integers is a multiple of 3.

Note: When we talk about the multiple of a number, the number multiplied by 0 or a negative number is also considered a multiple of that number.

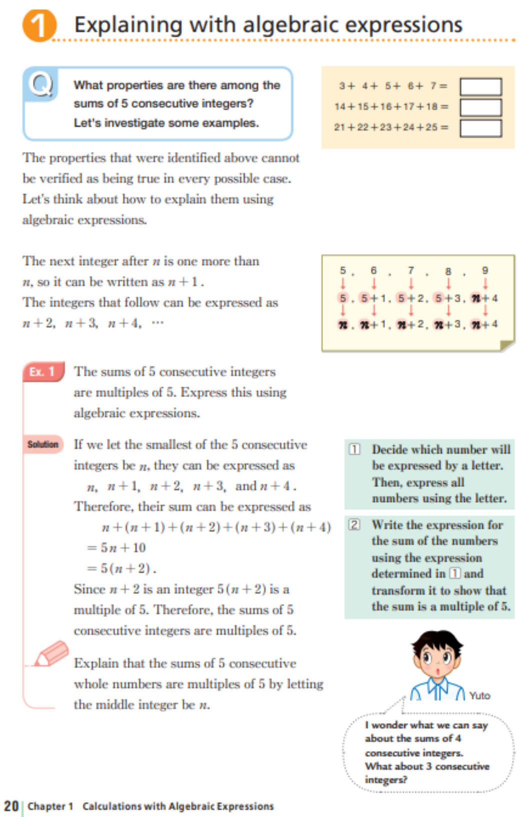
26 Chapter 1 Simplifying Algebraic Expression

Q 1: From the solution to Ex. 1 on the previous page, what else can we know about the sum of 3 consecutive integers, other than that it is a multiple of 3?

Figure 47: Sum of three consecutive integers. Retrieved from Junior High School Mathematics 2, Gakko Tosho, p.26-27.


5 Explain why the sum of 3 consecutive even numbers such as 2, 4 and 6 is a multiple of 6 using algebraic expression.

Figure 48: Sum of three consecutive even numbers. Retrieved from Junior High School Mathematics 2, Gakko Tosho, p.223.


1 Explaining with algebraic expressions

Q: What properties are there among the sums of 5 consecutive integers? Let's investigate some examples.

3+ 4+ 5+ 6+ 7 = □
14+15+16+17+18 = □
21+22+23+24+25 = □

The properties that were identified above cannot be verified as being true in every possible case. Let's think about how to explain them using algebraic expressions.

The next integer after n is one more than n, so it can be written as $n+1$.
The integers that follow can be expressed as $n+2$, $n+3$, $n+4$, …

5, 6, 7, 8, 9
5, 5+1, 5+2, 5+3, 5+4
n, $n+1$, $n+2$, $n+3$, $n+4$

Ex. 1: The sums of 5 consecutive integers are multiples of 5. Express this using algebraic expressions.

Solution: If we let the smallest of the 5 consecutive integers be n, they can be expressed as
n, $n+1$, $n+2$, $n+3$, and $n+4$.
Therefore, their sum can be expressed as
$n+(n+1)+(n+2)+(n+3)+(n+4)$
$=5n+10$
$=5(n+2)$.
Since $n+2$ is an integer $5(n+2)$ is a multiple of 5. Therefore, the sums of 5 consecutive integers are multiples of 5.

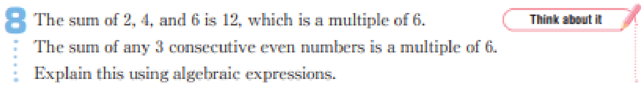
1 Decide which number will be expressed by a letter. Then, express all numbers using the letter.

2 Write the expression for the sum of the numbers using the expression determined in 1 and transform it to show that the sum is a multiple of 5.

Explain that the sums of 5 consecutive whole numbers are multiples of 5 by letting the middle integer be n.

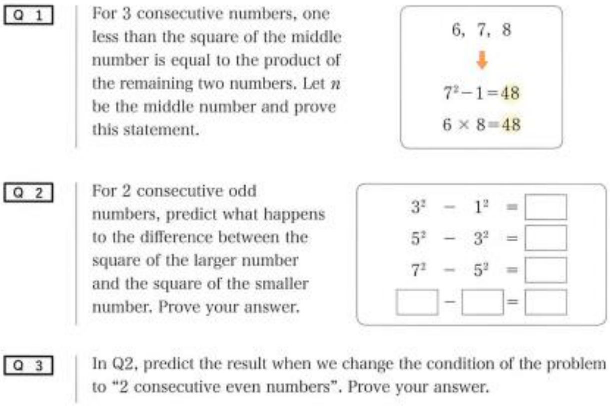
20 Chapter 1 Calculations with Algebraic Expressions

Figure 49: The sum of five consecutive numbers. Retrieved from Mathematics 8, Tokyo Shoseki, p.20.


8 The sum of 2, 4, and 6 is 12, which is a multiple of 6.
The sum of any 3 consecutive even numbers is a multiple of 6.
Explain this using algebraic expressions.

Think about it

Figure 50: The sum of three consecutive numbers. Retrieved from Mathematics 8, Tokyo Shoseki, p.27.



Q 1 For 3 consecutive numbers, one less than the square of the middle number is equal to the product of the remaining two numbers. Let n be the middle number and prove this statement.

6, 7, 8

$7^2-1=48$

$6\times8=48$

Q 2 For 2 consecutive odd numbers, predict what happens to the difference between the square of the larger number and the square of the smaller number. Prove your answer.

$3^2 - 1^2 = \square$

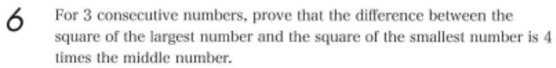
$5^2 - 3^2 = \square$

$7^2 - 5^2 = \square$

$\square - \square = \square$

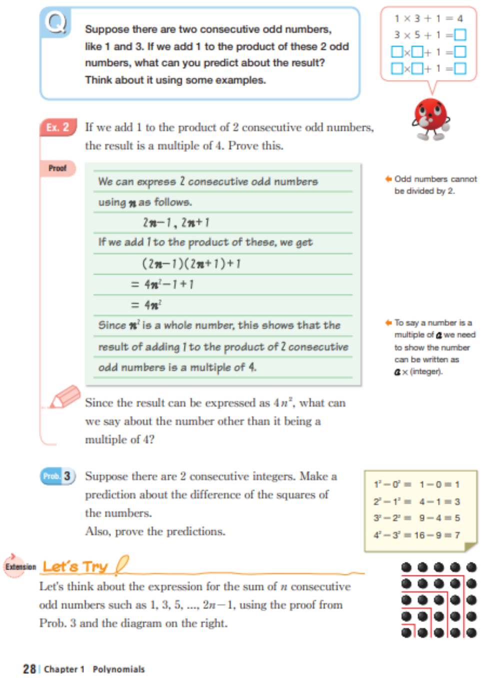
Q 3 In Q2, predict the result when we change the condition of the problem to "2 consecutive even numbers". Prove your answer.

Figure 51: The sum of three consecutive numbers. Retrieved from Junior High School Mathematics 3, Gakko Tosho, p.38.



6 For 3 consecutive numbers, prove that the difference between the square of the largest number and the square of the smallest number is 4 times the middle number.

Figure 52: Three consecutive numbers. Retrieved from Junior High School Mathematics 3, Gakko Tosho, p.41.



Q Suppose there are two consecutive odd numbers, like 1 and 3. If we add 1 to the product of these 2 odd numbers, what can you predict about the result? Think about it using some examples.

$1\times3+1=4$

$3\times5+1=\square$

$\square\times\square+1=\square$

$\square\times\square+1=\square$

Ex. 2 If we add 1 to the product of 2 consecutive odd numbers, the result is a multiple of 4. Prove this.

Proof

We can express 2 consecutive odd numbers using n as follows.

$2n-1,\ 2n+1$

If we add 1 to the product of these, we get

$(2n-1)(2n+1)+1$

$=4n^2-1+1$

$=4n^2$

Since n^2 is a whole number, this shows that the result of adding 1 to the product of 2 consecutive odd numbers is a multiple of 4.

← Odd numbers cannot be divided by 2.

← To say a number is a multiple of a we need to show the number can be written as $a\times$ (integer).

Since the result can be expressed as $4n^2$, what can we say about the number other than it being a multiple of 4?

Prob. 3 Suppose there are 2 consecutive integers. Make a prediction about the difference of the squares of the numbers.
Also, prove the predictions.

$1^2-0^2=1-0=1$

$2^2-1^2=4-1=3$

$3^2-2^2=9-4=5$

$4^2-3^2=16-9=7$

Extension Let's Try

Let's think about the expression for the sum of n consecutive odd numbers such as 1, 3, 5, ..., $2n-1$, using the proof from Prob. 3 and the diagram on the right.

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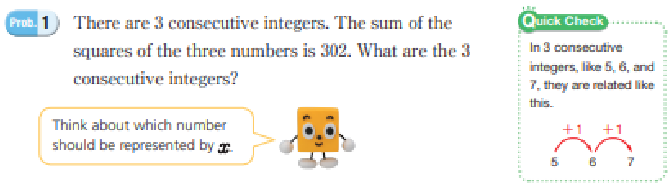
Figure 53: Add one to the product of two consecutive odd numbers. Retrieved from Mathematics 9, Tokyo Shoseki, p.28.



Prob. 1 There are 3 consecutive integers. The sum of the squares of the three numbers is 302. What are the 3 consecutive integers?

Think about which number should be represented by x.

Quick Check

In 3 consecutive integers, like 5, 6, and 7, they are related like this.

+1 +1

5 6 7

Figure 54: What are the three consecutive integers?. Retrieved from Mathematics 9, Tokyo Shoseki, p.80.

8.3 Appendix C: Overview of Thorough examination.

Textbook	Topic	Structure	Language/representation	Function	Notes:

Gakko 5.1	The sum of angles in a triangle.	Empirical	Graphic and verbal	Verification	
Keirinkan 5a	The sum of angles in a triangle.	Empirical	Symbolic and graphic	Verification and Discovery	Testing with multiple triangles.
Gakko 8	The sum of angles in a triangle.	Deductive	Symbolic and graphic	Illumination	
Keirinkan 8	The sum of angles in a triangle.	Deductive	Symbolic and graphic	Illumination	
Tokyosyoseki 8	The sum of angles in a triangle.	Deductive	Symbolic and graphic	Illumination and Verification	Showed and explained what a proof is.
Multi 6a	The sum of angles in a triangle.	Empirical	Symbolic and Graphic	Verification and Discovery	Testing with multiple triangles.
Matemagisk 6a	The sum of angles in a triangle.	Empirical	Graphic	Verification and Discovery	Also needs some deductive thinking
Matematikk 9	The sum of angles in a triangle.	Empirical	Graphic	“Statement”	It just states why it is 180 degrees in a triangle.
Keirinkan 9	Pythagorean Theorem	Deductive	Symbolic and Graphic	Illumination and Discovery	Same proof in all Japanese textbooks
Gakko 9	Pythagorean Theorem	Deductive	Symbolic and Graphic	Illumination and Discovery	
Tokyosyoseki 9	Pythagorean Theorem	Deductive	Symbolic and Graphic	Illumination and Discovery	

Matemagisk 9	Pythagorean Theorem	Empirical	Graphic	Discovery	Shows Euklids proof in a “advanced” problem later in the textbook.
Maximum 9	Pythagorean Theorem	Empirical	Symbolic and Graphic	Verification, illumination and discovery	Three examples of how to “prove” the Pythagorean theorem.
Matematikk 9	Pythagorean Theorem	Empirical	Graphic	Verification	
Tokyosyoseki 7	The sum of two even/odd numbers.	Empirical	Symbolic	Verification and discovery	
Keirinkan 8	The sum of two even/odd numbers.	Deductive	Symbolic and verbal	Illumination	
Tokyosyoseki 8	The sum of two even/odd numbers.	Empirical	Symbolic	Verification	“Find the mistake”
Gakko 8	The sum of two even/odd numbers.	Empirical	Symbolic	Verification	
Keirinkan 9	The sum of two even/odd numbers.	Deductive	Symbolic	Illumination	
Gakko 9	The sum of two even/odd numbers.	Empirical	Symbolic	Illumination	
Tokyosyoseki 9	The sum of two even/odd numbers.	Deductive	Symbolic	Verification	

Matematikk 7	The sum of two even/odd numbers.	Empirical	Symbolic	Discovery	
Matemagisk 9	The sum of two even/odd numbers.	Deductive	Symbolic	Verification	Lots of problem solving
Keirinkan 8	The sum of three consecutive integers.	Empirical	Symbolic	Verification	One deductive problem rest empirical.
Gakko 8	The sum of three consecutive integers.	Deductive	Symbolic and verbal	Illumination	
Tokyosyoseki 8	The sum of three consecutive integers.	Deductive	Symbolic	Illumination	
Gakko 9	The sum of three consecutive integers.	Deductive	Symbolic and graphic	Verification	
Tokyosyoseki 9	The sum of three consecutive integers.	Deductive	Symbolic	Illumination	
Matemagisk 9	The sum of three consecutive integers.	Deductive	Symbolic	Verification	Also uses programming

Table 9: Summary of the thorough examination.