Adaptive Backstepping based Consensus Tracking of Uncertain Nonlinear Systems with Event-Triggered Communication ⋆

Wei Wang [a], Jiang Long [a], Jing Zhou [b], Jiangshuai Huang [c], Changyun Wen [d]

[a] *School of Automation Science and Electrical Engineering, Beihang University, Beijing 100191, China*

[b] *Department of Engineering Sciences, University of Agder, Grimstad, Norway 4898, Norway*

[c] *School of Automation, Chongqing University, Chongqing 400044, China*

[d] *School of Electrical and Electronic Engineering, Nanyang Technological University, Nanyang Avenue, Singapore 639798, Singapore*

Abstract

This paper investigates the consensus tracking problem for a class of uncertain high-order nonlinear systems with parametric uncertainties and event-triggered communication. Under a directed communication condition, a totally distributed adaptive backstepping based control scheme is presented. Specifically, a decentralized triggering condition is adopted in this paper such that continuous monitoring of neighboring states, as required in some existing results, can be avoided. Besides, to handle the non-differentiability problem of virtual controllers, which arises from the utilization of neighboring states collected only at the triggering instants, the virtual controllers in each recursive step are firstly designed with continuous communication. Then, the partial derivatives of these designed virtual controllers are adopted to construct distributed adaptive consensus controllers for the event based communication case. It is shown that with the presented distributed adaptive consensus control scheme and even-triggered communication mechanism, all the closed-loop signals are uniformly bounded and the output consensus tracking errors will converge to a compact set. Besides, the tracking performance in the mean square sense can be improved by appropriately adjusting design parameters.

Key words: Distributed adaptive control; consensus; event-triggered communication; uncertain high-order nonlinear systems.

1 Introduction

Distributed consensus control of multi-agent systems has received huge attention in resent years, due to its wide potential applications in various fields such as mobile robot networks, intelligent transportation management, surveillance and monitoring. However, currently available control algorithms are mostly developed based on continuous communication among connected subsystems. Unavoidably, such communication mechanism will consume considerable communication resources. To solve this issue, event-based consensus becomes a significant and hot research topic in recent years. A plenty of representative works in this area have been reported; see Dimarogonas, Frazzoli & Johansson (2012); Seyboth, Dimarogonas & Johansson (2013); Xing, Wen, Guo, Liu & Su (2017); Zhu, Jiang & Feng (2014) for instance.

Note that the aforementioned results are mainly established for linear multi-agent systems. However, physical systems are usually nonlinear with system uncertainties in practice. As we know, adaptive control has been proven as an effective approach to handle the system uncertainties (Krstic, Kanellakopoulos & Kokotovic, 1995). Recently, several adaptive event-triggered consensus control schemes have been proposed for first-order nonlinear systems (Wang, Wen, Huang & Zhou, 2020; Zhan, Hu & Li, 2019) and second-order nonlinear systems (Li, Yan, Yu & Qiu, 2020; Yang, Li, Yue & Yue, 2020). For uncertain high-order nonlinear multi-agent

⋆ This paper was not presented at any IFAC meeting. Corresponding author: Jiangshuai Huang.

Email addresses: `w.wang@buaa.edu.cn` (Wei Wang), `jlong@buaa.edu.cn` (Jiang Long), `jing.zhou@uia.no` (Jing Zhou), `jshuang@cqu.edu.cn` (Jiangshuai Huang), `ecywen@ntu.edu.cn` (Changyun Wen).

systems, various adaptive backstepping based consensus control algorithms are developed based on continuous communication. Interested readers may refer to Chen, Wen, Liu & Liu (2015); Long, Wang, Huang, Zhou & Liu (2019); Shen & Shi (2015); Wang, Wen & Huang (2017) and the references therein. However, to the best of our knowledge, the event-triggered results are still limited. Under an undirected communication graph, a fuzzy adaptive event-triggered leader-following consensus control algorithm is presented in Li, Yang & Tong (2018), where continuous monitoring of neighboring states is required to implement the designed triggering condition. In Wang & Li (2020), an observer-based event-triggered adaptive fuzzy control scheme is proposed from the output feedback viewpoint. A triggering condition is elaborately designed to update each subsystem' controller. However, the communication among connected subsystems is still continuous.

Motivated by the above limitations, we shall investigate the distributed adaptive consensus tracking problem for a class of uncertain high-order nonlinear systems with directed communication topology and event-triggered communication. For each subsystem, a group of event triggering conditions to broadcast its state information are designed, which are only dependent on its local state changing rate. Hence each subsystem needs no longer monitor its neighbors' states continuously as required in some existing results. Then a totally distributed consensus tracking control scheme based on backstepping technique (Krstic, Kanellakopoulos & Kokotovic, 1995) is proposed. The main challenge is that the virtual controllers designed in each subsystem will contain piecewise continuous state signal received from its neighbors, since event based communication mechanism is adopted. And the non-differentiable virtual control signals make the recursive design steps of backstepping difficult to proceed. To overcome this obstacle, distributed adaptive backtepping based consensus controllers are firstly designed with continuous communication among the subsystems. For the event based communication case, a constructive method is adopted to design the distributed adaptive controllers, where the partial derivatives in previously designed virtual controllers and the neighboring states collected at the triggering instants are utilized. It is shown that uniform boundedness of all the closed-loop signalscan be ensured, while Zeno behavior is ruled out. Besides, it is worth emphasizing that the consensus tracking performance in the mean square sense can be improved by properly adjusting the design parameters.

The rest of this paper is organized as follows. In Section 2, the considered multi-agent system model, communication topology condition and event-based broadcast mechanism are introduced. Distributed adaptive consensus controllers for continuous and event-triggered communication cases are designed in Section 3 and 4, respectively. The closed-loop system stability is analyzed in Section 5 followed by simulation results in Section 6. Finally, a conclusion is drawn in Section 7.

2 Problem formulation

2.1 System model

In this paper, we consider a group of nth-order nonlinear subsystems modeled as follows.

$$\begin{aligned} \dot{x}_{i,q} &= x_{i,q+1}, \quad q = 1,2,\ldots,n-1 \\ \dot{x}_{i,n} &= u_i + \psi_i(x_i) + \varphi_i(x_i)^T\theta_i \\ y_i &= x_{i,1}, \quad i = 1,2,\ldots,N \end{aligned} \tag{1}$$

where $x_i = [x_{i,1}, x_{i,2}, \ldots, x_{i,n}]^T \in \Re^n$, $y_i \in \Re$ and $u_i \in \Re$ are the state vector, output and input of the ith subsystem, respectively. $\theta_i \in \Re^{p_i}$ is a vector of unknown constants. $\psi_i(x_i) \in \Re$ and $\varphi_i(x_i) : \Re^n \to \Re^{p_i}$ are column vectors of known continuous nonlinear functions.

2.2 Communication condition among the N subsystems

Suppose that the communication among the N subsystems can be represented by a fixed directed graph $\mathcal{G} \triangleq (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{1, \ldots, N\}$ denotes the set of indexes (or vertices) corresponding to each subsystem, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges between two distinct subsystems. An edge $(i,j) \in \mathcal{E}$ indicates that subsystem j can obtain information from subsystem i, but not necessarily vice versa (Ren & Cao, 2010). In this case, subsystem i is called a in-neighbor of subsystem j and in turn subsystem j is a out-neighbor of subsystem i. We denote the set of neighbors for subsystem i as $\mathcal{N}_i$. Self edges (i,i) is not allowed in this paper, thus $(i,i) \notin \mathcal{E}$ and $i \notin \mathcal{N}_i$. The connectivity matrix $A = [a_{ij}] \in \Re^{N\times N}$ is defined such that $a_{ij} = 1$ if $(j,i) \in \mathcal{E}$ and $a_{ij} = 0$ if $(j,i) \notin \mathcal{E}$. Clearly, the diagonal elements $a_{ii} = 0$. We introduce an in-degree matrix $\triangle$ such that $\triangle = \mathrm{diag}(\triangle_i) \in \Re^{N\times N}$ with $\triangle_i = \sum_{j\in\mathcal{N}_i} a_{ij}$ being the ith row sum of A. Then, the Laplacian matrix of $\mathcal{G}$ is defined as $\mathcal{L} = \triangle - A$.

2.3 Control objective

The desired trajectory for all subsystem outputs is characterized by a bounded time varying function $y_0(t)$. We now use $\mu_i = 1$ to indicate the case that $y_0(t)$ is accessible directly to subsystem i; otherwise, $\mu_i = 0$.

The control objective in this paper is to determine appropriate triggering condition and effective distributed adaptive controllers $u_i(t)$ for each subsystem by utilizing continuous local states $(x_i(t))$ and the discrete-time neighboring states $(x_{j,q}(t^j_{q,k})$, if $a_{ij} = 1)$ such that
$i)$ all closed-loop signals are uniformly bounded;
$ii)$ all subsystem outputs can track the desired trajectory $y_0(t)$ as closely as possible.

To achieve the objective, the following assumptions are imposed.

Assumption 1 *The directed graph $\mathcal{G}$ is balanced and weakly connected. The full knowledge of $y_0(t)$ is directly accessible by at least one subsystem, i.e.* $\sum_{i=1}^{N} \mu_i > 0$.

Assumption 2 *The first nth-order derivatives $y_0(t)^{(n)}$ of $y_0(t)$ are bounded, piecewise continuous, and directly known by subsystem i with $\mu_i = 1$, that is, $|y_0(t)^{(n)}| < F_i$ where F_i is an unknown positive constant.*

The following lemmas are introduced, which will be useful in our design and analysis of distributed adaptive controllers.

Lemma 1 *(Ren & Cao, 2010) Let $\mathcal{B}$ be a diagonal matrix $\mathcal{B} = diag\{\mu_1, \ldots, \mu_N\}$ and define $Q = (\mathcal{L} + \mathcal{B}) + (\mathcal{L} + \mathcal{B})^T$. Based on Assumption 1, the matrix Q is symmetric positive definite.*

3 Preliminary design of distributed adaptive controllers with continuous communications

In this section, a backstepping (Krstic et al., 1995) based distributed adaptive consensus control scheme will be presented. In each subsystem with $\mu_i = 0$, we introduce $\hat{y}_{i,0}(t) \in \Re$ to estimate the unknown reference function $y_0(t)$. The following error variables are defined.

$$\begin{aligned} z_{i,1} &= y_i - \mu_i y_0 - (1-\mu_i)\hat{y}_{i,0} \\ &= \delta_i + (1-\mu_i)\tilde{y}_{i,0} \end{aligned} \tag{2}$$

$$z_{i,q} = x_{i,q} - \alpha_{i,q-1}, \quad q = 2, \ldots, n \tag{3}$$

$$e_i = \sum_{j=1}^{N} a_{ij}(y_i - y_j) + \mu_i(y_i - y_0) \tag{4}$$

where $\delta_i = y_i - y_0$ is the tracking error for each subsystem i. $\tilde{y}_{i,0} = y_0 - \hat{y}_{i,0}$ is the estimation error for $\mu_i = 0$.

The virtual control signals $\alpha_{i,q}$ for $q = 1, \ldots, n$ and the actual controller u_i are designed as follows.

$$\alpha_{i,1} = -c_{i,1}z_{i,1} - ke_i + \mu_i\dot{y}_0 + (1-\mu_i)\dot{\hat{y}}_{i,0} \tag{5}$$

$$\begin{aligned} \alpha_{i,q} &= -c_{i,q}z_{i,q} - z_{i,q-1} + \sum_{k=1}^{q-1}\frac{\partial \alpha_{i,q-1}}{\partial x_{i,k}}x_{i,k+1} \\ &\quad + \sum_{j=1}^{N} a_{ij}\sum_{k=1}^{q-1}\frac{\partial \alpha_{i,q-1}}{\partial x_{j,k}}x_{j,k+1} + \mu_i\sum_{k=1}^{q}\frac{\partial \alpha_{i,q-1}}{\partial y_0^{(k-1)}}y_0^{(k)} \\ &\quad + (1-\mu_i)\frac{\partial \alpha_{i,q-1}}{\partial \hat{y}_{i,0}}\dot{\hat{y}}_{i,0}, \quad q = 2, \ldots, n \end{aligned} \tag{6}$$

$$u_i = \alpha_{i,n} - \psi_i - \varphi_i^T\hat{\theta}_i \tag{7}$$

where $c_{i,1}$, $c_{i,q}$ and k are positive design parameters, $\hat{\theta}_i$ is the estimate of unknown system parameter θ_i.

The parameter update laws $\hat{y}_{i,0}$ and $\hat{\theta}_i$ are designed as

$$\dot{\hat{y}}_{i,0} = -\gamma_{y_{i0}}e_i - \gamma_{y_{i0}}\kappa_{y_{i0}}\left(\hat{y}_{i,0} - y_{i,0}\right) \tag{8}$$

$$\dot{\hat{\theta}}_i = \Gamma_{\theta_i}\varphi_i z_{i,n} - \Gamma_{\theta_i}\kappa_{\theta_i}\left(\hat{\theta}_i - \theta_{i,0}\right) \tag{9}$$

where $\gamma_{y_{i0}}$, Γ_{θ_i}, $\kappa_{y_{i0}}$, κ_{θ_i}, $y_{i,0}$ and $\theta_{i,0}$ are positive constants with suitable dimension.

The main results in this section can be formally stated in the following theorem.

Theorem 1 *Consider a group of N uncertain subsystems as modeled in (1) with a desired trajectory $y_0(t)$ under Assumptions 1-2. By designing the distributed adaptive controllers as (7) with parameter update laws (8) and (9), the following results can be guaranteed.*
1) All closed-loop signals are uniformly bounded.
2) The tracking error signals $\delta = [\delta_1, \delta_2, ..., \delta_N]^T$ will converge to a compact set.
3) The upper bound of $\|\delta(t)\|^2_{[0,T]} = \frac{1}{T}\int_0^T \|\delta(t)\|^2 dt$ can be decreased by choosing suitable design parameters.

Proof. Define a Lyapunov function candidate V_1 as $V_1 = \sum_{i=1}^{N}\left[\frac{1}{2}z_{i,1}^2 + \frac{k(1-\mu_i)}{2\gamma_{y_{i0}}}\tilde{y}_{i,0}^2\right]$, where $\tilde{y}_{i,0} = y_0 - \hat{y}_{i,0}$. Let $\delta = [\delta_1, \ldots, \delta_N]^T$. From (2), (5) and (8), the derivative of V_1 is computed as

$$\begin{aligned} \dot{V}_1 &\le -\frac{k}{2}\delta^T Q\delta + \sum_{i=1}^{N}\left(-c_{i,1}z_{i,1}^2 + z_{i,1}z_{i,2}\right) \\ &\quad - \sum_{i=1}^{N}\frac{k(1-\mu_i)\kappa_{y_{i0}}}{2}\tilde{y}_{i,0}^2 + \sum_{i=1}^{N}\frac{k(1-\mu_i)}{\gamma_{y_{i0}}}|\tilde{y}_{i,0}|F_1 \\ &\quad + \sum_{i=1}^{N}\frac{k(1-\mu_i)\kappa_{y_{i0}}}{2}(y_0 - y_{i,0})^2 \\ &\le -\frac{k}{2}\lambda_{\min}(Q)\|\delta\|^2 + \sum_{i=1}^{N}\left(-\frac{1}{2}c_{i,1}z_{i,1}^2 + z_{i,1}z_{i,2}\right) \\ &\quad - \sum_{i=1}^{N}\frac{k(1-\mu_i)\kappa_{y_{i0}}}{4}\tilde{y}_{i,0}^2 + M_1 \end{aligned} \tag{10}$$

where $M_1 = \sum_{i=1}^{N}\frac{k(1-\mu_i)}{\gamma_{y_{i0}}^2\kappa_{y_{i0}}}F_1^2 + \sum_{i=1}^{N}\frac{k(1-\mu_i)\kappa_{y_{i0}}}{2}(y_0 - y_{i,0})^2$.

The Lyapunov function candidate V_n for the overall system is defined as $V_n = V_1 + \sum_{i=1}^{N}\sum_{q=2}^{n-1}\frac{1}{2}z_{i,q}^2 + \sum_{i=1}^{N}\frac{1}{2}\tilde{\theta}_i^T\Gamma_{\theta_i}^{-1}\tilde{\theta}_i$. From (2)-(9) and (10), the derivative of V_n can be computed as

$$\dot{V}_n \le -\frac{k}{2}\lambda_{\min}(Q)\|\delta\|^2 + \sum_{i=1}^{N}\left(-\frac{1}{2}c_{i,1}z_{i,1}^2 - \sum_{q=2}^{n}c_{i,q}z_{i,q}^2\right)$$

$$
\begin{aligned}
& -\sum_{i=1}^{N}\frac{k(1-\mu_i)\kappa_{y_{i0}}}{4}\tilde{y}_{i,0}^2-\sum_{i=1}^{N}\frac{\kappa_{\theta_i}\|\tilde{\theta}_i\|^2}{2}+M_n \\
& \leq -\frac{k}{2}\lambda_{\min}(Q)\|\delta\|^2-\sigma V_n+M_n \qquad (11)
\end{aligned}
$$

where $\sigma=\min\{c_{i,1},2c_{i,2},\dots,2c_{i,n},\frac{\gamma_{y_{i0}}\kappa_{y_{i0}}}{2},\frac{\kappa_{\theta_i}}{\lambda_{\max}(\Gamma_{\theta_i}^{-1})}\}$, $M_n=\sum_{i=1}^{N}\frac{k(1-\mu_i)}{\gamma_{y_{i0}}^2\kappa_{y_{i0}}}F_1^2+\sum_{i=1}^{N}\frac{k(1-\mu_i)\kappa_{y_{i0}}}{2}(y_0-y_{i,0})^2+\sum_{i=1}^{N}\frac{\kappa_{\theta_i}\|\theta_i-\theta_{i,0}\|^2}{2}$.

We now establish the results in Theorem 1 one by one.

1) From (11), it yields $\dot{V}_n\leq-\sigma V_n+M_n$. By direct integrations of this inequality, we have

$$
V_n(t)\leq V_n(0)e^{-\sigma t}+\frac{M_n}{\sigma}\left(1-e^{-\sigma t}\right)\leq V_n(0)+\frac{M_n}{\sigma} \qquad (12)
$$

which shows that V is uniformly bounded. Thus the error signals $z_{i,q}$ for $1\leq q\leq n$, $\tilde{\theta}_i$ and $\tilde{y}_{i,0}$ are bounded. From (2), δ_i is bounded. Since $\tilde{y}_{i,0}=y_0-\hat{y}_{i,0}$ and y_0 is bounded, thus $\hat{y}_{i,0}$ is bounded. From (2), (5) and (6), $x_{i,q}$ and $\alpha_{i,q}$ are bounded. From (7), the boundedness of u_i is also ensured. Therefore all the closed-loop signals are uniformly bounded.

2) From (2), the definitions of V_1 and V_n, we have $\|\delta\|^2\leq\sum_{i=1}^{N}\left[\frac{1}{2}z_{i,1}^2+\frac{1-\mu_i}{2}\tilde{y}_{i,0}^2\right]\leq\xi V_n$, where $\xi=\max\{1,\frac{\gamma_{y10}}{k},\dots,\frac{\gamma_{yN0}}{k}\}$. With (12), it further follows that $\|\delta(t)\|^2\leq\xi\left[V_n(0)e^{-\sigma t}+\frac{M_n}{\sigma}(1-e^{-\sigma t})\right]$. This implies that the tracking errors in Euclidean norm will converge to a compact set $E_r=\{\delta|\|\delta\|^2\leq\xi(M_n+\varsigma)/\sigma\}$ for $t\geq(1/\sigma)\ln(|V_n(0)\sigma-M_n|/\varsigma)$ with ς an arbitrarily small positive constant.

3) From (11), we have $\dot{V}_n\leq-\frac{k}{2}\lambda_{\min}(Q)\|\delta\|^2+M_n$. Integrating both sides of this inequality yields that

$$
\begin{aligned}
\|\delta(t)\|_{[0,T]}^2 &= \frac{1}{T}\int_0^T\|\delta(t)\|^2dt \\
&\leq \frac{2}{k\lambda_{\min}(Q)}\left[\frac{V_n(0)-V_n(T)}{T}+M_n\right] \\
&\leq \frac{2}{k\lambda_{\min}(Q)}\left[\frac{V_n(0)}{T}+M_n\right] \qquad (13)
\end{aligned}
$$

From the definition of V_1, V_n, M_1, M_n, it follows that the upper bound of the overall tracking errors in the mean square sense can be decreased by decreasing κ_{θ_i}, $\kappa_{y_{i0}}$ and increasing k, $c_{i,q}$ for $q=1,\dots,n$, Γ_{θ_i}, $\gamma_{y_{i0}}$. □

4 Design of distributed adaptive controllers with event-triggered communications

In this section, an event-based distributed adaptive control scheme will be presented to achieve the control objective. In addition to Assumptions 1 and 2, the following assumptions are also imposed.

Assumption 3 *$\|\varphi_i(x_i)\|\leq L_{i,1}\|x_i\|+L_{i,0}$, where $\|\cdot\|$ denotes Euclidean norm, $L_{i,1}$ and $L_{i,0}$ are unknown positive constants.*

Assumption 4 *The unknown parameter vector $\theta_i\in\Re^{p_i}$ is within a compact convex set $\mathcal{C}_{\theta_i}$ with $\|\theta_i\|\leq L_{\theta_i}$. The value of L_{θ_i} is only known by subsystem i.*

4.1 Design of event triggering condition

Notations $t_{q,0}^j,t_{q,1}^j,\dots,t_{q,k}^j,\dots$ with $0=t_{q,0}^j<t_{q,1}^j<t_{q,2}^j<\dots<t_{q,k}^j<t_{q,k+1}^j<\dots<\infty$, $k\in Z^+$, $j\in\mathcal{V}$, $q=1,\dots,n$ are adopted to denote the sequence of event times for subsystem j to broadcast the information of its qth state to subsystem i, if $a_{ij}=1$. $t_{q,0}^j$ is the initial time instant when agent j starts up. For each subsystem i, the instantaneous information of the qth state for its neighboring subsystems is updated only at the time instants $t_{q,k}^j$ for $j\in\mathcal{N}_i$. This indicates that for time $t\in[t_{q,k}^j,t_{q,k+1}^j)$, the neighbour's qth states available for subsystem i are kept unchanged as $\bar{x}_{j,q}(t)=x_{j,q}(t_{q,k}^j)$, $j\in\mathcal{N}_i$.

The triggering condition is chosen as

$$
t_{q,k+1}^j=\inf\{t>t_{q,k}^j,|x_{j,q}(t)-\bar{x}_{j,q}(t)|>m_q^j\} \qquad (14)
$$

where $j\in\{0,\mathcal{V}\}$, $\bar{x}_{j,q}(t)=x_{j,q}(t_{q,k}^j)$ and m_q^j is a positive constant to be designed. It is noted from (14) that the designed triggering condition for each subsystem is dependent only on its local state changing rates. Hence, continuous monitoring of neighbors' states as required in Dimarogonas et al. (2012),Zhu et al. (2014), You et al. (2017), Li et al. (2018) and Yang et al. (2020) can be avoided.

4.2 Design of distributed adaptive controllers

The following error variables are defined

$$
\begin{aligned}
\bar{z}_{i,1} &= y_i-\mu_iy_0-(1-\mu_i)\hat{\bar{y}}_{i,0} \qquad (15)\\
&= \delta_i+(1-\mu_i)\tilde{\bar{y}}_{i,0} \\
\bar{z}_{i,q} &= x_{i,q}-\bar{\alpha}_{i,q-1},\quad q=2,\dots,n \qquad (16)\\
\epsilon_i &= \sum_{j=1}^{N}a_{ij}(y_i-\bar{y}_j)+\mu_i(y_i-y_0) \qquad (17)
\end{aligned}
$$

where $\delta_i=y_i-y_0$ is the tracking error for each subsystem i and $\tilde{\bar{y}}_{i,0}=y_0-\hat{\bar{y}}_{i,0}$ is the estimation error. $\hat{\bar{y}}_{i,0}$ is the estimate of y_0 introduced in subsystem i with $\mu_i=0$. Note that the estimator $\hat{y}_{i,0}$ in (8) cannot be implemented for the case with event-based communications, as the error variable e_i in (4) is unavailable. Defining that $\bar{x}_{j,q}(t)=x_{j,q}(t_{q,k}^j)$, $j\in\{0,\mathcal{V}\}$, $t\in[t_{q,k}^j,t_{q,k+1}^j)$.

$\bar{y}_j(t) = \bar{x}_{j,1}(t)$.

In this case, the virtual control inputs and the actual control input are designed as

$$\bar{\alpha}_{i,1} = -c_{i,1}\bar{z}_{i,1} - k\epsilon_i + \mu_i\dot{y}_0 + (1-\mu_i)\dot{\hat{\bar{y}}}_{i,0} \tag{18}$$

$$\begin{aligned}\bar{\alpha}_{i,q} =& -c_{i,q}\bar{z}_{i,q} - \bar{z}_{i,q-1} + \sum_{k=1}^{q-1}\frac{\partial\alpha_{i,q-1}}{\partial x_{i,k}}x_{i,k+1} \\ &+\sum_{j=1}^{N}a_{ij}\sum_{k=1}^{q-1}\frac{\partial\alpha_{i,q-1}}{\partial x_{j,k}}\bar{x}_{j,k+1} + \mu_i\sum_{k=1}^{q}\frac{\partial\alpha_{i,q-1}}{\partial y_0^{(k-1)}}y_0^{(k)} \\ &+(1-\mu_i)\frac{\partial\alpha_{i,q-1}}{\partial\hat{y}_{i,0}}\dot{\hat{\bar{y}}}_{i,0}, \quad q = 2,\ldots,n\end{aligned} \tag{19}$$

$$u_i = \bar{\alpha}_{i,n} - \psi_i - \varphi_i^T\hat{\bar{\theta}}_i \tag{20}$$

where $c_{i,1}$, k and $c_{i,q}$ are positive design parameters. $\hat{\bar{\theta}}_i$ is the estimate of unknown system parameter θ_i. $\frac{\partial\alpha_{i,q-1}}{\partial x_{i,k}}$, $\frac{\partial\alpha_{i,q-1}}{\partial x_{j,k}}$, $\frac{\partial\alpha_{i,q-1}}{\partial y_0^{(k-1)}}$ and $\frac{\partial\alpha_{i,q-1}}{\partial\hat{y}_{i,0}}$ are the partial derivatives adopted in previously designed $\alpha_{i,q}$ in (6).

The estimator $\hat{\bar{y}}_{i,0}$ and parameter update law for $\hat{\bar{\theta}}_i$ are designed as

$$\dot{\hat{\bar{y}}}_{i,0} = -\gamma_{y_{i0}}\epsilon_i - \gamma_{y_{i0}}\kappa_{y_{i0}}(\hat{\bar{y}}_{i,0} - y_{i,0}) \tag{21}$$

$$\dot{\hat{\bar{\theta}}}_i = Proj\{\tau_i\} \tag{22}$$

where $\tau_i = \Gamma_{\bar{\theta}_i}\varphi_i\bar{z}_{i,n}$, $Proj\{\cdot\}$ is the projector operator originated from Krstic et al. (1995). $\gamma_{y_{i0}}$, $\kappa_{y_{i0}}$, $y_{i,0}$ and $\Gamma_{\bar{\theta}_i}$ are positive constants with appropriate dimension.

The following lemma is useful in the system stability analysis.

Lemma 2 *By applying the projector operator $Proj\{\cdot\}$ in Krstic et al. (1995), the property $-\tilde{\bar{\theta}}_i^T\Gamma_{\bar{\theta}_i}^{-1}Proj\{\tau_i\} \leq -\tilde{\bar{\theta}}_i^T\Gamma_{\bar{\theta}_i}^{-1}\tau_i, \forall\hat{\bar{\theta}}_i \in \mathcal{C}_{\theta_i}, \theta_i \in \mathcal{C}_{\theta_i}$ exists, where $\tilde{\bar{\theta}}_i = \theta_i - \hat{\bar{\theta}}_i$ and $\hat{\bar{\theta}}_i$ is the estimate of unknown parameter θ_i. $\Gamma_{\bar{\theta}_i}$ is a positive constant matrix with appropriate dimension.*

Remark 1 *By comparing (18)-(19) and (5)-(6), it can be seen that $\bar{\alpha}_{i,q}$ is designed in a similar form as $\alpha_{i,q}$. The only difference is that all the continuous neighboring states $x_{j,q}(t)$ involved in $\alpha_{i,q}$ are replaced with piecewise continuous states $\bar{x}_{j,q}(t)$. The effects due to such replacement will be rigorously analyzed in subsequent section. On the other hand, the partial derivatives terms adopted in $\alpha_{i,q}$ (i.e. $\frac{\partial\alpha_{i,q-1}}{\partial x_{i,k}}$, $\frac{\partial\alpha_{i,q-1}}{\partial x_{j,k}}$, $\frac{\partial\alpha_{i,q-1}}{\partial y_0^{(k-1)}}$ and $\frac{\partial\alpha_{i,q-1}}{\partial\hat{y}_{i,0}}$) are kept unchanged to construct $\bar{\alpha}_{i,q}$. More detailed discussions will be presented in Remark 3 and Remark 5.*

5 System stability and consensus analysis

Lemma 3 *The errors between $z_{i,q}$ in (3) and $\bar{z}_{i,q}$ in (16), $\alpha_{i,q}$ in (6) and $\bar{\alpha}_{i,q}$ in (19) are bounded. Thus*

$$|z_{i,q} - \bar{z}_{i,q}| \leq \Delta_{z_{i,q}} \tag{23}$$

$$|\alpha_{i,q} - \bar{\alpha}_{i,q}| \leq \Delta_{\alpha_{i,q}} \tag{24}$$

where $\Delta_{z_{i,q}}$ and $\Delta_{\alpha_{i,q}}$ are positive constants related to topology parameters Δ_i, μ_i, individual design parameters k, $c_{i,q}$, $\gamma_{y_{i0}}$, $\kappa_{y_{i0}}$, m_q^i and neighboring design parameters $c_{j,q}$, $\gamma_{y_{j0}}$, $\kappa_{y_{j0}}$, m_q^j for $a_{ij} = 1$ and $q = 1,\ldots,n$.

Proof. The proof is provided in Appendix A. □

Remark 2 *It is observed from the Proof of Lemma 3 that the partial derivative terms in (6) are all constants depending on topology parameters Δ_i, μ_i, individual design parameters k, $c_{i,q}$, $\gamma_{y_{i0}}$, $\kappa_{y_{i0}}$, m_q^i and neighboring design parameters $c_{j,q}$, $\gamma_{y_{j0}}$, $\kappa_{y_{j0}}$, m_q^j for $a_{ij} = 1$. This important property enables the utilization of the partial derivatives in designing $\bar{\alpha}_{i,q}$ in (19).*

Lemma 4 *Define $\bar{z}_1 = [\bar{z}_{1,1}, \bar{z}_{2,1}, \ldots, \bar{z}_{N,1}]^T$, $z_q = [z_{1,q}, z_{2,q}, \ldots, z_{N,q}]^T$ for $2 \leq q \leq n$, $\tilde{\bar{\theta}} = [\tilde{\bar{\theta}}_1^T, \ldots, \tilde{\bar{\theta}}_N^T]^T$ and $\tilde{\bar{y}}_0$ as the vector of $\tilde{\bar{y}}_{i,0} = y_0 - \hat{\bar{y}}_{i,0}$ for all subsystems with $\mu_i = 0$ and $\tilde{\bar{\theta}}_i = \theta_i - \hat{\bar{\theta}}_i$. The states x_i for $1 \leq i \leq N$ satisfy the following inequality*

$$\|x_i\| \leq L_{x_i}\left\|(\bar{z}_1^T, z_2^T, \ldots, z_n^T, \tilde{\bar{y}}_0^T, \tilde{\bar{\theta}}^T)^T\right\| + B_{x_i}, \tag{25}$$

where L_{x_i} and B_{x_i} are positive constants related to topology parameters and design parameters as stated in Lemma 3.

Proof. The proof is provided in Appendix B. □

The main results in this section are formally stated in the following theorem.

Theorem 2 *Consider a group of N uncertain subsystems as modeled in (1) with a desired trajectory $y_0(t)$ under Assumptions 1-4. By designing the event-triggering communication rules as (14) and the distributed adaptive controllers as (20), distributed estimators (21), parameter update laws (22), the following results can be guaranteed.*
1) All closed-loop signals are uniformly bounded.
2) The tracking error signals $\delta = [\delta_1, \delta_2, ..., \delta_N]^T$ will converge to a compact set.
3) The upper bound of $\|\delta(t)\|_{[0,T]}^2 = \frac{1}{T}\int_0^T\|\delta(t)\|^2dt$ can be decreased by choosing suitable design parameters.
4) Zeno behavior is excluded.

Proof. Define a Lyapunov function as $V_1 = \sum_{i=1}^{N} \left[\frac{1}{2}\bar{z}_{i,1}^2 + \frac{k(1-\mu_i)}{2\gamma_{y_{i0}}}\tilde{\bar{y}}_{i,0}^2\right]$. From (15), (18) and (21), the derivative of V_1 is computed as

$$\begin{aligned}
\dot{V}_1 =& \sum_{i=1}^{N} \left[-c_{i,1}\bar{z}_{i,1}^2 - k\bar{z}_{i,1}(e_i + \epsilon_i - e_i) + \bar{z}_{i,1}\bar{z}_{i,2}\right] \\
&+ \sum_{i=1}^{N} \frac{k(1-\mu_i)}{\gamma_{y_{i0}}}\tilde{\bar{y}}_{i,0}\left(\dot{y}_0 - \dot{\hat{\bar{y}}}_{i,0}\right) \\
\leq& \sum_{i=1}^{N} -c_{i,1}\bar{z}_{i,1}^2 - \sum_{i=1}^{N} k\left[\delta_i + (1-\mu_i)\tilde{\bar{y}}_{i,0}\right]e_i \\
&+ \sum_{i=1}^{N}\left[k|\bar{z}_{i,1}|\Delta_i m + \bar{z}_{i,1}\bar{z}_{i,2} + \frac{k(1-\mu_i)}{\gamma_{y_{i0}}}|\tilde{\bar{y}}_{i,0}|F_1\right] \\
&- \sum_{i=1}^{N} \frac{k(1-\mu_i)}{\gamma_{y_{i0}}}\tilde{\bar{y}}_{i,0}\dot{\hat{\bar{y}}}_{i,0} \\
\leq& -\frac{k}{2}\lambda_{\min}(Q)\|\delta\|^2 + \sum_{i=1}^{N}\left(-\frac{1}{2}c_{i,1}\bar{z}_{i,1}^2 + \bar{z}_{i,1}\bar{z}_{i,2}\right) \\
&- \sum_{i=1}^{N}\frac{k(1-\mu_i)\kappa_{y_{i0}}}{4}\tilde{\bar{y}}_{i,0}^2 + \bar{M}_1 \qquad (26)
\end{aligned}$$

where $m = \max\{m_q^1, \ldots, m_q^N\}$ and $\bar{M}_1 = \sum_{i=1}^{N}\frac{k^2\Delta_i^2 m^2}{2c_{i,1}} + \sum_{i=1}^{N}\frac{k(1-\mu_i)}{\gamma_{y_{i0}}^2\kappa_{y_{i0}}}\left(\gamma_{y_{i0}}\Delta_i m + F_1\right)^2 + \sum_{i=1}^{N}\frac{k(1-\mu_i)\kappa_{y_{i0}}}{2}(y_0 - y_{i,0})^2$.

Define a Lyapunov function candidate V_2 as $V_2 = V_1 + \sum_{i=1}^{N}\frac{1}{2}z_{i,2}^2$. From (3) and (6), the derivative of V_2 is computed as

$$\begin{aligned}
\dot{V}_2 \leq& -\frac{k}{2}\lambda_{\min}(Q)\|\delta\|^2 + \sum_{i=1}^{N}\left(-\frac{1}{2}c_{i,1}\bar{z}_{i,1}^2 + \bar{z}_{i,1}\bar{z}_{i,2}\right) \\
&+ \sum_{i=1}^{N} -\frac{k(1-\mu_i)\kappa_{y_{i0}}}{4}\tilde{\bar{y}}_{i,0}^2 + \bar{M}_1 \\
&+ \sum_{i=1}^{N}\left(-c_{i,2}z_{i,2}^2 - z_{i,1}z_{i,2} + z_{i,2}z_{i,3}\right) \qquad (27)
\end{aligned}$$

Since $\bar{z}_{i,1}\bar{z}_{i,2} - z_{i,1}z_{i,2} = (\bar{z}_{i,1} - z_{i,1})z_{i,2} + \bar{z}_{i,1}(\bar{z}_{i,2} - z_{i,2}) \leq |z_{i,2}|\Delta_{z_{i,1}} + |\bar{z}_{i,1}|\Delta_{z_{i,2}}$, $\dot{V}_2$ can be further derived as

$$\begin{aligned}
\dot{V}_2 \leq& -\frac{k}{2}\lambda_{\min}(Q)\|\delta\|^2 + \sum_{i=1}^{N}\left(-\frac{1}{4}c_{i,1}\bar{z}_{i,1}^2 - \frac{1}{2}c_{i,2}z_{i,2}^2\right. \\
&\left.-\frac{k(1-\mu_i)\kappa_{y_{i0}}}{4}\tilde{\bar{y}}_{i,0}^2 + z_{i,2}z_{i,3}\right) + \bar{M}_2 \qquad (28)
\end{aligned}$$

where $\bar{M}_2 = \bar{M}_1 + \sum_{i=1}^{N}\left(\frac{1}{c_{i,1}}\Delta_{z_{i,2}}^2 + \frac{1}{2c_{i,2}}\Delta_{z_{i,1}}^2\right)$.

The Lyapunov function for the entire closed-loop system is chosen as $V_n = V_2 + \sum_{i=1}^{N}\left(\sum_{i=3}^{n}\frac{1}{2}z_{i,q}^2 + \frac{1}{2}\tilde{\bar{\theta}}_i^T\Gamma_{\theta_i}^{-1}\tilde{\bar{\theta}}_i\right)$ From (3), (6), (20), (22) and Lemma 2, the derivative of V_n is computed as

$$\begin{aligned}
\dot{V}_n \leq& -\frac{k}{2}\lambda_{\min}(Q)\|\delta\|^2 + \sum_{i=1}^{N}\left(-\frac{1}{4}c_{i,1}\bar{z}_{i,1}^2 - \frac{1}{2}c_{i,2}z_{i,2}^2\right. \\
&\left.-\sum_{q=3}^{n-1}c_{i,q}z_{i,q}^2 - \frac{1}{2}c_{i,n}z_{i,n}^2 - \frac{k(1-\mu_i)\kappa_{y_{i0}}}{4}\tilde{\bar{y}}_{i,0}^2\right) \\
&-\sum_{i=1}^{N}\frac{\|\tilde{\bar{\theta}}_i\|^2}{2} + \sum_{i=1}^{N}\|\tilde{\bar{\theta}}_i\|\|\varphi_i\|\Delta_{z_{i,n}} \\
&+\bar{M}_2 + \sum_{i=1}^{N}\frac{\Delta_{\alpha_{i,n}}^2}{2c_{i,n}} + \sum_{i=1}^{N}\frac{\|\tilde{\bar{\theta}}_i\|^2}{2} \qquad (29)
\end{aligned}$$

According to Assumption 3, Assumption 4 and Lemma 4, the term $\|\tilde{\bar{\theta}}_i\|\|\varphi_i\|\Delta_{z_{i,n}}$ can be directly derived as

$$\begin{aligned}
&\|\tilde{\bar{\theta}}_i\|\|\varphi_i\|\Delta_{z_{i,n}} \\
&\leq L_{\theta_i}(L_{i,1}\|x_i\| + L_{i,0})\Delta_{z_{i,n}} \\
&\leq L_{\theta_i}L_{i,1}\Delta_{z_{i,n}}\left(L_{x_i}\left\|(\bar{z}_1^T, z_2^T, \ldots, z_n^T, \tilde{\bar{y}}_0^T, \tilde{\bar{\theta}}^T)^T\right\| + B_{x_i}\right) \\
&\quad +L_{\theta_i}L_{i,0}\Delta_{z_{i,n}} \\
&\triangleq \frac{c}{2N}\left\|(\bar{z}_1^T, z_2^T, \ldots, z_n^T, \tilde{\bar{y}}_0^T, \tilde{\bar{\theta}}^T)^T\right\|^2 + \bar{M}_{i3} \qquad (30)
\end{aligned}$$

where $c = \min\left\{\frac{1}{4}c_{i,1}, \frac{1}{2}c_{i,2}, c_{i,q}, \frac{k(1-\mu_i)\kappa_{y_{i0}}}{2}, \frac{1}{2}\right\}$ for $q = 3, \ldots, n-1$ and $\mu_i = 0$. $\bar{M}_{i3} = \frac{N(L_{\theta_i}L_{i,1}\Delta_{z_{i,n}}L_{x_i})^2}{2c} + L_{\theta_i}L_{i,1}\Delta_{z_{i,n}}B_{x_i} + L_{\theta_i}L_{i,0}\Delta_{z_{i,n}}$.

Substituting (30) into (29) yields that

$$\begin{aligned}
\dot{V}_n \leq& -\frac{k}{2}\lambda_{\min}(Q)\|\delta\|^2 - c\left\|(\bar{z}_1^T, z_2^T, \ldots, z_n^T, \tilde{\bar{y}}_0^T, \tilde{\bar{\theta}}^T)^T\right\|^2 \\
&+\frac{c}{2}\left\|(\bar{z}_1^T, z_2^T, \ldots, z_n^T, \tilde{\bar{y}}_0^T, \tilde{\bar{\theta}}^T)^T\right\|^2 \\
&+\bar{M}_2 + \sum_{i=1}^{N}\left(\frac{\Delta_{\alpha_{i,n}}^2}{2c_{i,n}} + \bar{M}_{i3}\right) + \sum_{i=1}^{N}\frac{\|\tilde{\bar{\theta}}_i\|^2}{2} \\
\leq& -\frac{k}{2}\lambda_{\min}(Q)\|\delta\|^2 - \frac{c}{2}\left\|(\bar{z}_1^T, z_2^T, \ldots, z_n^T, \tilde{\bar{y}}_0^T, \tilde{\bar{\theta}}^T)^T\right\|^2 \\
&+M^* \\
\leq& -\frac{k}{2}\lambda_{\min}(Q)\|\delta\|^2 - \sigma V_n + M^* \qquad (31)
\end{aligned}$$

where $M^* = \bar{M}_2 + \sum_{i=1}^{N}\left(\frac{\Delta_{\alpha_{i,n}}^2}{2c_{i,n}} + M_{i3}\right) + \sum_{i=1}^{N}\frac{\|\tilde{\bar{\theta}}_i\|^2}{2}$ and $\sigma = \min\left\{c, \frac{c\kappa_{y_{i0}}}{k(1-\mu_i)}, \frac{c}{\lambda_{\max}(\Gamma_{\bar{\theta}_i}^{-1})}\right\}$.

By following the similar analysis in the proof of Theorem 1, the conclusions 1), 2) and 3) can be drawn. To avoid repetition, the details are omitted here.

To show the exclusion of Zeno behavior, we shall show that the inter-execution intervals $(t^j_{q,k+1} - t^j_{q,k})$ for $j \in \mathcal{V}$, $\forall k \in Z^+$ are lower-bounded by a positive constant. Define $\eta^j_{q,k}(t) = x_{j,q}(t) - \bar{x}_{j,q}(t)$ for $t \in [t^j_{q,k}, t^j_{q,k+1})$, whose derivative is computed as $\frac{d|\eta^j_{q,k}|}{dt} = \frac{d(\eta^j_{q,k} \times \eta^j_{q,k})^{\frac{1}{2}}}{dt} = \mathrm{sgn}(\eta^j_{q,k})\dot{\eta}^j_{q,k} \leq \left|\dot{\eta}^j_{q,k}\right|$. Since $\bar{x}_{j,q}(t)$ keeps unchanged for $t \in [t^j_{q,k}, t^j_{q,k+1})$, we have

$$\left|\dot{\eta}^j_{q,k}(t)\right| = |x_{j,q+1}|, \quad q = 2, \ldots, n-1 \tag{32}$$

$$\left|\dot{\eta}^j_{n,k}(t)\right| = \left|u_j + \psi_j + \varphi_j^T \theta_j\right|, \quad \text{for } j \in \mathcal{V} \tag{33}$$

From the boundedness of $x_{j,q}, u_j, \varphi_j$, it is concluded that there exist a positive constant $\iota_{q,j}$ such that $\left|\dot{\eta}^j_{q,k}(t)\right| \leq \iota^j_q$ for $j \in \bar{\mathcal{V}}$. Then the inter-execution intervals must satisfy that $t^j_{q,k+1} - t^j_{q,k} \geq m^j_q/\iota^j_q$, i.e. Zeno behavior is excluded. □

Remark 3 *Different from Theorem 1, the coupling term $\|\tilde{\bar{\theta}}_i\|\|\varphi_i\|\Delta_{z_{i,n}}$ in (29) arises in the derivative of V_n, which is the Lyapunov function defined for the entire closed-loop system in event-based communication case. To effectively handle this term, two additional assumptions, i.e. Assumption 3 and Assumption 4 are imposed in this section. Besides, the projector operator $Proj\{\cdot\}$ in Krstic et al. (1995) is used to ensure the boundedness of parameter estimate $\hat{\bar{\theta}}_i$ and the corresponding estimation error $\tilde{\bar{\theta}}_i$, i.e. $\|\tilde{\bar{\theta}}_i\| \leq L_{\theta_i}$.*

Remark 4 *As observed from (11), (13) and (31), the tracking performance of all subsystem outputs in the mean square error sense is mainly determined by state initials, design parameters and the size of M^*. Besides, it can be seen from $\bar{M}_1$, $\bar{M}_2$ and M^* that event triggering thresholds m^j_q have huge impact on the size of M^*. Clearly, if m^j_q is increased, the triggering times for communication among subsystems can effectively be reduced. However, the upper bound of $\|\delta\|^2_{[0,T]}$ will be increased with a larger m^j_q. Therefore, determining the values of m^j_q is a tradeoff between the cost of communication resources and consensus tracking performance.*

Remark 5 *The main challenge to design backstepping based distributed adaptive consensus controllers for high-order nonlinear systems with event-based communications lies in the fact that traditional backstepping technique (Krstic et al., 1995) requires differentiating virtual control inputs recursively. If the virtual control input designed in one step involves piecewise continuous signals, computing its derivatives is impossible, thus the subsequent design step is difficult to proceed. To overcome this difficulty, backstepping technique is not adopted directly in designing the virtual controllers and final control law for event-based communication case. Instead, we design the backstepping based distributed adaptive controllers for all subsystems with continuous communications firstly. Thus all the chosen virtual controllers (i.e. $\alpha_{i,q}$) are differentiable. For the case with event triggered communications, the partial derivative terms $\frac{\partial \alpha_{i,q-1}}{\partial x_{i,k}}$, $\frac{\partial \alpha_{i,q-1}}{\partial x_{j,k}}$, $\frac{\partial \alpha_{i,q-1}}{\partial y_0^{(k-1)}}$ and $\frac{\partial \alpha_{i,q-1}}{\partial \hat{y}_{i,0}}$, which are shown to be constants, are utilized to construct the virtual control $\bar{\alpha}_{i,q}$ in (19).*

Remark 6 *Different from some existing distributed adaptive backstepping based results with continuous communication among connected subsystems as in Chen et al. (2015); Long et al. (2019); Shen & Shi (2015); Wang & Li (2020); Wang et al. (2017), the communication in our results is decided by a predesigned triggering condition. Compared with Li et al. (2018) with undirected graph, the considered directed communication topology in this paper is more general and continuous monitoring of neighbors' states is not required. Besides, the obtained results formulate a backstepping based controller design and stability analysis framework for the chained nonlinear systems to solve the non-differentiability problem of virtual controllers. It can be applied to solve other related issues, like quantized communication and denial-of-services attacks (DoS) on the communication channels.*

6 Simulation studies

In this section, we consider a group of 4 pendulum systems (Zhou et al., 2019) modeled with the following dynamics

$$m_i l_i \ddot{\vartheta}_i + m_i g \sin(\vartheta_i) + k_i l_i \dot{\vartheta}_i = u_i \tag{34}$$

where ϑ_i denotes the angle of pendulum, m_i and l_i are the mass $[kg]$ and length of the robe $[m]$, g denotes the acceleration due to the gravity, k_i is an unknown friction coefficient, u_i represents an input torque provided by a DC motor. System parameters are chosen the same as these in Zhou et al. (2019) i.e. $m_i = 1kg$, $l_i = 1m$ and $g = 9.8m/s^2$. The friction coefficients are set as $k_1 = 0.2$, $k_2 = 0.2$, $k_3 = 0.1$ and $k_4 = 0.1$, respectively. The objective is to design distributed adaptive controllers u_i for all subsystems such that ϑ_i can track a common desired trajectory $\vartheta_0(t) = 2\cos(0.1t)$ for $1 \leq i \leq 4$. The communication condition among the 4 subsystems and $\vartheta_0(t)$ is represented by the directed graph in Fig. 1.

Defining the state variables $x_{i,1} = \vartheta_i$ and $x_{i,2} = \dot{\vartheta}_i$ for $i = 1, \ldots, 4$, then (34) can be rewritten as the same form as in (1).

$$\begin{aligned} \dot{x}_{i,1} &= x_{i,2}; \\ \dot{x}_{i,2} &= \frac{1}{m_i l_i} u_i - \frac{g}{l_i} \sin(x_{i,1}) - \frac{k_i}{m_i} x_{i,2} \end{aligned} \tag{35}$$

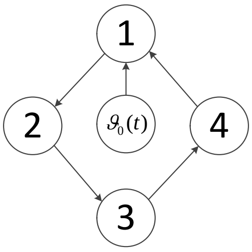

Fig. 1. Information transmission graph for the 4 subsystems.

Clearly, the parameter k_i/m_i is unknown. The triggering condition for inter-subsystem communication, distributed adaptive controllers and parameter estimators are designed as in (14), (20)-(22).

In simulation, the state initials including $\vartheta_i(0)$, $\dot{\vartheta}_i(0)$, $\hat{\bar{\theta}}_i(0)$, $\hat{\bar{y}}_{i,0}(0)$ are set as zeros for $i \in \{1,2,3,4\}$. The design parameters are chosen as follows. $k = 0.1$, $c_{i,1} = c_{i,2} = 3$, $\gamma_{\vartheta_i} = 20$, $\Gamma_{\bar{\theta}_i} = 0.3$, $\kappa_{\vartheta_i} = 0.005$, $\vartheta_{i0} = 0.01$, $m_1^i = 0.05$, $m_2^i = 0.05$ for $i = 1,\ldots,4$. The tracking performance of all subsystems' outputs ϑ_i with comparison to ϑ_0, the tracking errors δ_i and states $\dot{\vartheta}_i$ for $i = 1,2,3,4$ are shown in Fig. 2-Fig. 5 respectively. Fig. 5-Fig. 6 exhibit the control inputs and the triggering time of all subsystems, respectively. The triggering count and minimum inter-event times are provided in Table 1. It can be seen that desired tracking performance for all subsystems' outputs can be achieved, while all the observed signals are bounded. Moreover, Zeno behavior in each pendulum does not exist. In order to show the effects of triggering thresholds on the tracking performance, m_1^i is changed to $m_1^i = 0.2$ while keeping all the remaining design parameters unchanged. From Fig. 7-Fig. 8, it can be observed that the tracking performance can be improved by reducing the triggering threshold m_1^i at the cost of increasing the triggering frequency.

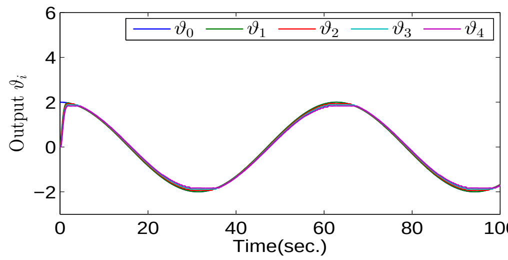

Fig. 2. The outputs ϑ_i, $i = 1,\ldots,4$.

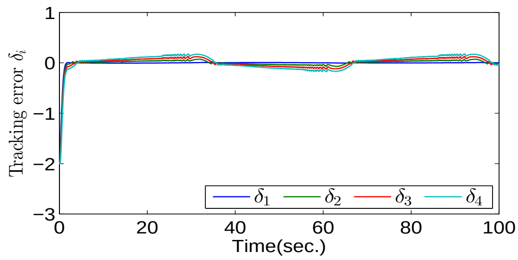

Fig. 3. Tracking errors $\delta_i = \vartheta_i - \vartheta_0$, $i = 1,\ldots,4$.

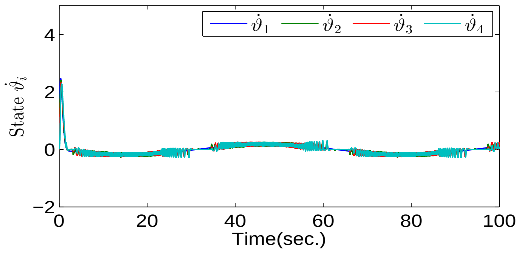

Fig. 4. The states $\dot{\vartheta}_i$, $i = 1,\ldots,4$.

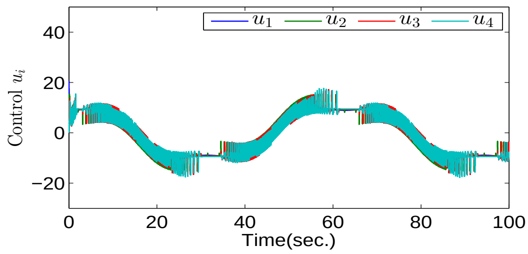

Fig. 5. Control inputs u_i, $i = 1,\ldots,4$.

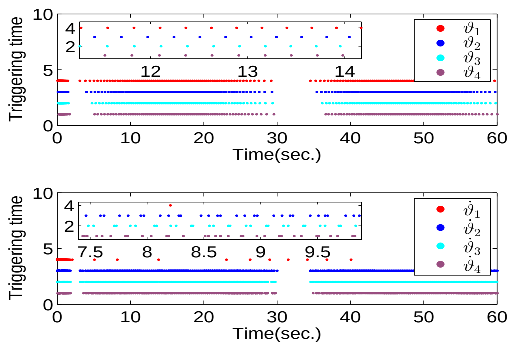

Fig. 6. Triggering times of ϑ_i and $\dot{\vartheta}_i$, $i = 1,\ldots,4$.

7 Conclusion

In this paper, the consensus tracking control problem for uncertain high-order nonlinear systems is investigated. Under directed communication condition, two totally distributed adaptive control schemes are proposed with or without event-triggered communication. For event-triggered case, the continuous monitoring problem of neighboring states has been removed. The boundedness of all closed-loop signals and the tracking performance have been analysed. Finally, the simulation results show the effectiveness of our proposed control scheme.

Table 1
Event count and minimum inter-event times for the states in each agent.

		Count	Inter-event times (s)
Pendulum 1	ϑ_1	278	0.0203
	$\dot{\vartheta}_1$	122	0.0025
Pendulum 2	ϑ_2	270	0.0209
	$\dot{\vartheta}_2$	981	0.0037
Pendulum 3	ϑ_3	261	0.0215
	$\dot{\vartheta}_3$	1084	0.0042
Pendulum 4	ϑ_4	257	0.0224
	$\dot{\vartheta}_4$	1038	0.0049

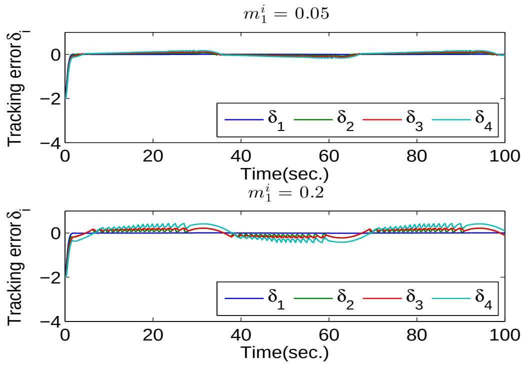

Fig. 7. Errors $\delta_i, i = 1, 2, 3, 4$ for subsystems with different m_1^i.

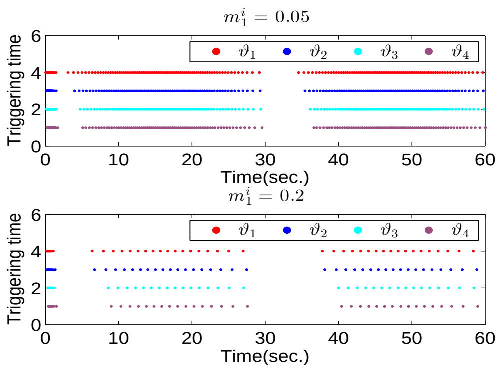

Fig. 8. Triggering times of $\vartheta_i, i = 1, 2, 3, 4$ with different m_1^i.

Acknowledgements

This work was supported by National Key Research and Development Program of China under grants no. 2018AAA0101100, 2019YFB1312002, and National Natural Science Foundation of China under grants 61973017, 61673035, 61703061.

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Appendix

A Proof of Lemma 3

Proof. Define $s_{y_{i0}} = \hat{\tilde{y}}_{i,0} - \hat{y}_{i,0}$. From (8) and (21), there is $\dot{s}_{y_{i0}} = -\gamma_{y_{i0}}\kappa_{y_{i0}}s_{y_{i0}} + \gamma_{y_{i0}}(e_i - \epsilon_i) = -\gamma_{y_{i0}}\kappa_{y_{i0}}s_{y_{i0}} + \gamma_{y_{i0}}\left[\sum_{j=1}^N a_{ij}\left(\bar{y}_j - y_j\right)\right]$. The solution of this differential equation is computed to satisfy that $|s_{y_{i0}}| \le |s_{y_{i0}}(0)|e^{-\gamma_{y_{i0}}\kappa_{y_{i0}}t} + \frac{\Delta_i m}{\kappa_{y_{i0}}}(1 - e^{-\gamma_{y_{i0}}\kappa_{y_{i0}}t}) \le |s_{y_{i0}}(0)| + \frac{\Delta_i m}{\kappa_{y_{i0}}} \triangleq \Delta_{y_{i0}}$, where $m = \max\{m_1^1, \ldots, m_1^N, \ldots, m_n^1, \ldots, m_n^N\}$. From (2) and (15), it further results in $|z_{i,1} - \bar{z}_{i,1}| = (1-\mu_i)\left|\hat{\tilde{y}}_{i,0} - \hat{y}_{i,0}\right| \le (1-\mu_i)|s_{y_{i0}}| \le (1-\mu_i)\Delta_{y_{i0}} \triangleq \Delta_{z_{i,1}}$. With (4) and (17), we have $\left|\dot{\hat{y}}_{i,0} - \dot{\hat{\tilde{y}}}_{i,0}\right| \le \gamma_{y_{i0}}|\epsilon_i - e_i| + \gamma_{y_{i0}}\kappa_{y_{i0}}|s_{y_{i0}}| \le \gamma_{y_{i0}}(\Delta_i m + \kappa_{y_{i0}}\Delta_{y_{i0}})$. From (5) and (18), there is $|\alpha_{i,1} - \bar{\alpha}_{i,1} = \left|-c_{i,1}(z_{i,1} - \bar{z}_{i,1}) - k(e_i - \epsilon_i) + (1-\mu_i)\left(\dot{\hat{y}}_{i,0} - \dot{\hat{\tilde{y}}}_{i,0}\right)\right| \le c_{i,1}\Delta_{z_{i,1}} + k\Delta_i m + (1-\mu_i)\gamma_{y_{i0}}(\Delta_i m + \kappa_{y_{i0}}\Delta_{y_{i0}}) \triangleq \Delta_{\alpha_{i,1}}$.

Note that $\alpha_{i,1}$ is the function of $x_{i,1}$, $x_{j,1}$ if $a_{ij} = 1$, y_0, $\dot{y}_0$ if $\mu_i = 1$, $\hat{y}_{i,0}$ if $\mu_i = 0$. It will be shown that all the partial derivatives of $\alpha_{i,1}$ are constants which are associated with the design parameters and the triggering threshold.

$$\frac{\partial e_i}{\partial x_{i,1}} = \Delta_i + \mu_i,\ \frac{\partial e_i}{\partial x_{j,1}} = -a_{ij},\ \frac{\partial e_i}{\partial y_0} = -\mu_i \tag{A.1}$$

$$\begin{aligned}\frac{\partial \alpha_{i,1}}{\partial x_{i,1}} &= -c_{i,1} - k\frac{\partial e_i}{\partial x_{i,1}} + (1-\mu_i)(-\gamma_{y_{i0}})\frac{\partial e_i}{\partial x_{i,1}} \\ &= -c_{i,1} - (\Delta_i + \mu_i)[k + \gamma_{y_{i0}}(1-\mu_i)]\end{aligned} \tag{A.2}$$

$$\begin{aligned}\frac{\partial \alpha_{i,1}}{\partial x_{j,1}} &= -k\frac{\partial e_i}{\partial x_{j,1}} + (1-\mu_i)(-\gamma_{y_{i0}})\frac{\partial e_i}{\partial x_{j,1}} \\ &= a_{ij}[k + \gamma_{y_{i0}}(1-\mu_i)]\end{aligned} \tag{A.3}$$

$$\begin{aligned}\frac{\partial \alpha_{i,1}}{\partial y_0} &= -c_{i,1}(-\mu_i) - k\frac{\partial e_i}{\partial y_0} + (1-\mu_i)(\gamma_{y_{i0}})\frac{\partial e_i}{\partial y_0} \\ &= (c_{i,1} + k)\mu_i - (1-\mu_i)\gamma_{y_{i0}}\mu_i\end{aligned} \tag{A.4}$$

$$\frac{\partial \alpha_{i,1}}{\partial \dot{y}_0} = \mu_i \tag{A.5}$$

$$\begin{aligned}\frac{\partial \alpha_{i,1}}{\partial \hat{y}_{i,0}} &= c_{i,1}(1-\mu_i) + (1-\mu_i)(-\gamma_{y_{i0}}\kappa_{y_{i0}}) \\ &= (1-\mu_i)(c_{i,1} - \gamma_{y_{i0}}\kappa_{y_{i0}})\end{aligned} \tag{A.6}$$

Therefore, by some straightforward manipulation, we can directly get $|z_{i,2} - \bar{z}_{i,2}| \le |\bar{\alpha}_{i,1} - \alpha_{i,1}| \le \Delta_{\alpha_{i,1}} \triangleq \Delta_{z_{i,2}}$ and $|\alpha_{i,2} - \bar{\alpha}_{i,2}| \le c_{i,2}\Delta_{z_{i,2}} + \Delta_{z_{i,1}} + \sum\limits_{j=1}^N a_{ij}\left|\frac{\partial \alpha_{i,1}}{\partial x_{j,1}}\right| m + (1-\mu_i)\left|\frac{\partial \alpha_{i,1}}{\partial \hat{y}_{i,0}}\right|\gamma_{y_{i0}}(\Delta_i m + \kappa_{y_{i0}}\Delta_{y_{i0}}) \triangleq \Delta_{\alpha_{i,2}}$.

Following the same procedure based on $z_{i,q}$ in (3), $\alpha_{i,q}$ in (6), $\bar{z}_{i,q}$ in (16), $\bar{\alpha}_{i,q}$ in (19), we have $|z_{i,q} - \bar{z}_{i,q}| \le |\bar{\alpha}_{i,q-1} - \alpha_{i,q-1}| \le \Delta_{\alpha_{i,q-1}} \triangleq \Delta_{z_{i,q}}$ and $|\alpha_{i,q} - \bar{\alpha}_{i,q}| \le c_{i,q}\Delta_{z_{i,q}} + \Delta_{z_{i,q-1}} + \sum\limits_{j=1}^N a_{ij}\sum\limits_{k=1}^{q-1}\left|\frac{\partial \alpha_{i,q-1}}{\partial x_{j,k}}\right| m + (1-\mu_i)\left|\frac{\alpha_{i,q-1}}{\partial \hat{y}_{i,0}}\right|\gamma_{y_{i0}}\left(\Delta_i m + \kappa_{y_{i0}}\Delta_{y_{i,0}}\right) \triangleq \Delta_{\alpha_{i,q}}$ □

B Proof of Lemma 4

Proof. Note that for the subsystems with $\mu_i = 0$, we have $|\hat{y}_{i0}| = |\hat{\tilde{y}}_{i,0} - s_{y_{i0}}| \le |y_0| + |\tilde{\tilde{y}}_{i,0}| + |s_{y_{i0}}| \le |\tilde{\tilde{y}}_{i,0}| + B_{y_{i0}}$, where $B_{y_{i0}} = F_0 + \Delta_{y_{i0}}$ and F_0 is the upper bound of $|y_0|$.

From (2), we have $|x_{i,1}| \le \sqrt{2}\|(\bar{z}_{i,1}, \tilde{\tilde{y}}_{i,0})^T\| + B_{y_{i0}} + \Delta_{z_{i,1}} \triangleq L_{x_{i1}}\|(\bar{z}_{i,1}, \tilde{\tilde{y}}_{i,0})^T\| + B_{x_{i1}}$, where $L_{x_{i1}} = \sqrt{2}$ and $B_{x_{i,1}} = B_{y_{i0}} + \Delta_{z_{i,1}}$. From (5) and (8), there is $|\alpha_{i,1}| \le \sqrt{3}\max\{c_{i,1}, k + \gamma_{y_{i0}}, \gamma_{y_{i0}}\kappa_{y_{i0}}\}\|(\bar{z}_{i,1}, e_i, \tilde{\tilde{y}}_{i,0})^T\| + c_{i,1}\Delta_{z_{i,1}} + \mu_i F_1 + (1-\mu_i)\gamma_{y_{i0}}\kappa_{y_{i0}}(y_{i,0} + B_{y_{i0}}) \triangleq L_{\alpha_{i1}}\|(\bar{z}_{i,1}, e_i, \tilde{\tilde{y}}_{i,0})^T\| + B_{\alpha_{i1}}$, where $L_{\alpha_{i1}} = \sqrt{3}\max\{c_{i,1}, k + \gamma_{y_{i0}}, \gamma_{y_{i0}}\kappa_{y_{i0}}\}$ and $B_{\alpha_{i1}} = c_{i,1}\Delta_{z_{i,1}} + \mu_i F_1 + (1-\mu_i)\gamma_{y_{i0}}\kappa_{y_{i0}}(y_{i,0} + B_{y_{i0}})$.

From (3), there is $|x_{i,2}| \le (1 + L_{\alpha_{i1}})\|(\bar{z}_{i,1}, z_{i,2}, e_i, \tilde{\tilde{y}}_{i,0})^T\| + B_{\alpha_{i1}} \triangleq L_{x_{i2}}\|(\bar{z}_{i,1}, z_{i,2}, e_i, \tilde{\tilde{y}}_{i,0})^T\| + B_{x_{i2}}$, where $L_{x_{i2}} = 1 + L_{\alpha_{i1}}$ and $B_{x_{i2}} = B_{\alpha_{i1}}$. From (6), we have $|\alpha_{i,2}| \le c_{i,2}|z_{i,2}| + |z_{i,1}| + \left|\frac{\partial \alpha_{i,1}}{\partial x_{i,1}}\right||x_{i,2}| + \sum_{j=1}^N a_{ij}\left|\frac{\partial \alpha_{i,1}}{\partial x_{j,1}}\right||x_{j,2}| + \mu_i\sum_{k=1}^2\left|\frac{\partial \alpha_{i,1}}{\partial y_0^{(k-1)}}\right|\left|y_0^{(k)}\right| + (1-\mu_i)\left|\frac{\partial \alpha_{i,1}}{\partial \hat{y}_{i,0}}\right|[\gamma_{y_{i0}}|e_i| + \gamma_{y_{i0}}\kappa_{y_{i0}}(|\hat{y}_{i,0}| + y_{i,0})] \le L_{\alpha_{i2}}\|(\bar{z}_{i,1}, z_{i,2}, e_i, \tilde{\tilde{y}}_{i,0})^T\| + B_{\alpha_{i2}} + \sum_{j=1}^N a_{ij}L_{\alpha_{ij2}}\|(\bar{z}_{j,1}, z_{j,2}, e_j, \tilde{\tilde{y}}_{j,0})^T\|$, where $L_{\alpha_{i2}} \triangleq c_{i,2} + 1 + \left|\frac{\partial \alpha_{i,1}}{\partial x_{i,1}}\right|L_{xi2} + (1-\mu_i)\left|\frac{\partial \alpha_{i,1}}{\partial \hat{y}_{i,0}}\right|\gamma_{y_{i0}} + \left|\frac{\partial \alpha_{i,1}}{\partial \hat{y}_{i,0}}\right|\gamma_{y_{i0}}\kappa_{y_{i0}}$, $B_{\alpha_{i2}} \triangleq \Delta_{z_{i,1}} + \left|\frac{\partial \alpha_{i,1}}{\partial x_{i,1}}\right|B_{x_{i2}} + \mu_i\sum\limits_{k=1}^2\left|\frac{\partial \alpha_{i,1}}{\partial y_0^{(k-1)}}\right|F_k + (1-\mu_i)\left|\frac{\partial \alpha_{i,1}}{\partial \hat{y}_{i,0}}\right|\gamma_{y_{i0}}\kappa_{y_{i0}}(B_{y_{i0}} + y_{i,0}) + \sum\limits_{j=1}^N a_{ij}\left|\frac{\partial \alpha_{i,1}}{\partial x_{j,1}}\right|B_{x_{j2}}$, $L_{\alpha_{ij2}} \triangleq \left|\frac{\partial \alpha_{i,1}}{\partial x_{j,1}}\right|L_{xj2}$.

By following similar analysis, it can be shown that $|x_{i,q}| \le \sum\limits_{i=1}^N L_{x_{iq}}\|(\bar{z}_{i,1}, z_{i,2}, \ldots, z_{i,q}, e_i, \tilde{\tilde{y}}_{i,0})^T\| + B_{x_{iq}} \le \left(\sum\limits_{i=1}^N L_{x_{iq}}\right)\|(\bar{z}_1^T, z_2^T, \ldots, z_q^T, e^T, \tilde{\tilde{y}}_0^T)^T\| + B_{x_{iq}}$, where $e = [e_1, \ldots, e_N]^T$. From (4), we have $|e_i| \leq \sum\limits_{j=1}^{N} a_{ij}(|x_{i,1}| + |x_{j,1}|) + \mu_i(|x_{i,1}| + |y_0|) \leq (2\sqrt{2}N + \sqrt{2})\|(\bar{z}_1^T, \tilde{\bar{y}}_0^T)^T\| + \Delta_{e_i} \triangleq L_{e_i}\|(\bar{z}_1^T, \tilde{\bar{y}}_0^T)^T\| + \Delta_{e_i}$, where $L_{e_i} = 2\sqrt{2}N + \sqrt{2}$ and $\Delta_{e_i} = B_{x_{i,1}} + \sum_{j=1}^N a_{ij}(B_{x_{i,1}} + B_{x_{j,1}})$. Thus, $\|e\| \leq |e_1| + |e_2| + \ldots + |e_n| \leq NL_{e_i}\|(\bar{z}_1^T, \tilde{\bar{y}}_0^T)^T\| + N\Delta_{e_i} \leq NL_{e_i}\|(\bar{z}_1^T, z_2^T, \ldots, z_n^T, \tilde{\bar{y}}_0^T)^T\| + N\Delta_{e_i}$. Using $\left(\sum\limits_{i=1}^{N} L_{x_{iq}}\right)\|(\bar{z}_1^T, z_2^T, \ldots, z_q^T, e^T, \tilde{\bar{y}}_0^T)^T\| + B_{x_{iq}}$, we can further get $|x_{i,q}| \leq \left(\sum\limits_{i=1}^{N} L_{x_{iq}}\right) \times[\|(\bar{z}_1^T, z_2^T, \ldots, z_q^T, \tilde{\bar{y}}_0^T)^T\| + \|e\|] + B_{x_{iq}} \leq \left(\sum\limits_{i=1}^{N} L_{x_{iq}}\right) \times[(1+NL_{e_i})\|(\bar{z}_1^T, z_2^T, \ldots, z_n^T, \tilde{\bar{y}}_0^T)^T\|] + \left(\sum\limits_{i=1}^{N} L_{x_{iq}}\right) N\Delta_{e_i} + B_{x_{iq}}$. Thus,

$$
\begin{aligned}
\|x_i\| &\leq |x_{i,1}| + |x_{i,2}| + \ldots + |x_{i,n}| \\
&\leq \sum_{q=1}^{n}\sum_{i=1}^{N} L_{x_{iq}}[(1 + NL_{e_i})\|(\bar{z}_1^T, z_2^T, \ldots, z_n^T, \tilde{\bar{y}}_0^T)^T\|] \\
&\quad + \sum_{q=1}^{n}\left(\sum_{i=1}^{N} L_{x_{iq}} N\Delta_{e_i} + B_{x_{iq}}\right) \\
&\triangleq L_{x_i}\|(\bar{z}_1^T, z_2^T, \ldots, z_n^T, \tilde{\bar{y}}_0^T)^T\| + B_{x_i} \\
&\leq L_{x_i}\|(\bar{z}_1^T, z_2^T, \ldots, z_n^T, \tilde{\bar{y}}_0^T, \tilde{\bar{\theta}}^T)^T\| + B_{x_i} \qquad \text{(B.1)}
\end{aligned}
$$

where $L_{x_i} = \sum_{q=1}^n \sum_{i=1}^N L_{x_{iq}}(1 + NL_{e_i})$ and $B_{x_i} = \sum_{q=1}^n \left(\sum_{i=1}^N L_{x_{iq}} N\Delta_{e_i} + B_{x_{iq}}\right)$. □