TOULMIN ANALYSIS OF META-MATHEMATICAL ARGUMENTATION IN A JAPANESE GRADE 8 CLASSROOM

David Reid[1], Yusuke Shinno[2], Kotaro Komatsu[3] and Yosuke Tsujiyama[4]

[1]Unversity of Agder, Norway

[2]Hiroshima University, Japan

[3]Shinshu University, Japan

[4]Chiba University, Japan

In this research report we aim to analyse argumentation at two levels, using the so-called Toulmin model. We examine the structure of the mathematical argumentation, as well as the nature of the meta-mathematical argumentation justifying the validity of some proofs and the rejection of others in a Japanese grade 8 classroom. The results show that the analysis of a meta-mathematical argument allows us to gain a deeper insight into the proving process, although the role of the statements is more difficult to determine.

INTRODUCTION

Proof and proving are considered to be essential but challenging in the mathematics classroom. How argumentation may develop through proving processes in classroom interactions is one of the main research foci in recent publications (e.g., Mariotti et al., 2018; Stylianides et al., 2016). Argumentation analysis based on the work of Toulmin (1958) has been used extensively in mathematics education to investigate students' mathematical arguments in different ways. For instance, Pedemonte (2007) has analysed the relationship between argumentation and proof in students' working in pairs, while Krummheuer (2007) has adopted the model to analyse students' argumentation and participation in classroom processes. Although the Toulmin model was developed to be applicable to different rational arguments in different fields, including mathematical and non-mathematical argument, in mathematics education research it is less common to use it to investigate non-mathematical arguments. One exception is Potari and Psycharis (2018), who analysed pre-service mathematics teachers' argumentations while interpreting classroom incidents. They report that "different argumentation structures and types of warrants, backings and rebuttals [occur] in the … interpretations of students' mathematical activity." (p. 169). The lack of research on non-mathematical argumentation implies that a broader perspective is needed to take into account both mathematical and non-mathematical argumentation and how they can be related to proof and proving in classroom teaching and learning. In this research report we aim to analyse argumentation in a Japanese classroom at two levels.

2021. In Inprasitha, M., Changsri, N., Boonsena (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 251-259. Khon Kaen, Thailand: PME.

We examine the structure of the mathematical argumentation, and also the nature of the meta-mathematical argumentation justifying the validity of some proofs and the rejection of others. To attain this aim, we first consider how Toulmin model can be adopted to analyse the mathematical and meta-mathematical argumentation, and then reconstruct the classroom process in terms of the two-levels of argumentation.

THEORETICAL AND METHODOLOGICAL CONSIDERATIONS

Argumentation Analysis

We focus on argumentative structures which can be identified in collective processes in classroom interaction. For this purpose, the Toulmin model is adopted, following the methods of *argumentation analysis* described by Knipping and Reid (2015, 2019). At the heart of this method is a reduced Toulmin scheme describing arguments in terms of Claims/Conclusions, Data, Warrants, and Backings (see Figure 1). Briefly, an argument aims to establish a claim, based on specific data and general warrants (and backings). Toulmin's full scheme includes other elements (Qualifiers, Rebuttals) and Knipping and Reid include also Refutations in their analysis.

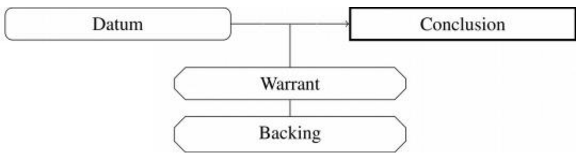


Figure 1: Toulmin model

Mathematical and Meta-mathematical Argumentation

Knipping and Reid (2015, 2019) analysed argumentation over time and reconstructed local (detailed) and the global (gross) argumentative structures. In this study, in order to reveal the nature of mathematical argumentation in-depth, we consider both mathematical argumentation and meta-mathematical argumentation. Our analysis shows that a mathematical argument can be supported (or rejected) by meta-mathematical argumentation which is talking about the argument. Since the meta-mathematical argumentation sometimes involves non-mathematical argumentation which represented by ordinary language, it is more complex and more difficult to analyse. The research question in this study is as follows: *How can the method of argumentation analysis be used to analyse both mathematical and meta-mathematical argumentation?*

Method of Analysis

Argumentation analysis (Knipping & Reid, 2015, 2019) consists of three stages for reconstructing arguments in classroom: first identifying 'episodes' of mathematical activity, then assigning roles (Conclusions, Data, Warrants, Backings) to statements, and grouping these into 'steps', assembling these steps into larger 'streams' in which a conclusion of one step is used as a datum (or occasionally a warrant) for the next step. These streams are in turn joined together into a 'structure' of the entire argumentation. Here we are interested in comparing the argumentation streams at two levels that occur in episodes of the lesson.

Data and Context

Our analysis is based on a transcript from a mathematics lesson in a mathematics class with 37 eighth-graders (age 13-14) at a junior high school in Japan. Before the lesson, the students learnt the terms 'proof' and 'definition', with several definitions of geometrical objects and fundamental assumptions (e.g., conditions for congruent triangles, SSS, SAS, ASA, and properties of parallel lines and angles) in Euclidean geometry. Based on these definitions and assumptions, they learned how to prove several statements related to triangles; in order to prove that two segments are equal in length, they learnt to focus on 'finding' a pair of triangles to which the two segments belong respectively as their sides (see Tsujiyama & Yui, 2018, for details). This experience is related to the main problem in the lesson (see below).

DATA ANALYSIS

Identifying Episodes

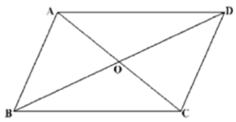
The lesson can be divided into episodes, listed in Table 1. In Episode 1 the teacher asked the students about their prior knowledge about parallelograms; the students identified five properties of parallelograms. The teacher then (Episode 2) posed the problem: Prove that in a quadrilateral ABCD, if $AB \parallel DC$ and $AD \parallel BC$, then $AB = DC$. After all the students had individually written down a plan and proof in their own way (Episode 3), the teacher asked two students to present their results. One showed $\triangle ABC \equiv \triangle CDA$ (Episode 4). Miya then presented her proof, based on showing $\triangle ABD \equiv \triangle CDB$ (Episode 5). The teacher then asked students who had done the first proof, why they had drawn the segment AC (Episode 6) and those who had taken Miya's approach why they had constructed the segment BD (Episode 7). At the end of the lesson, the teacher discussed the faulty proof produced by a fictive student Mikio (Episode 9, see Figure 2).

Episode	Description	Transcript Lines
1	Recalling Prior Knowledge	Not included
2	Posing Problem	Not included
3	Individual Work	Not included
4	First Argument: △ABC ≡ △CDA	Not included
5	Miya's Argument	1–4
6	Why AC? Jiro's argument	4–9
7	Why BD? Etsu's argument	10–24
8	Omitted section	25
9	Mikio's 'proof'	26–72

Table 1: Episodes of the lesson

Here we analyse mathematical argumentations from Episodes 5 and 9, and meta-mathematical arguments from Episodes 6, 7 and 9. Although we have identified two types of meta-mathematical arguments (Episode 6 & 7) concerning the argument from Episode 5, we omit one of the meta-arguments due to limited space.

To derive AB=CD, I will show the congruence of △AOB and △COD to which AB and DC respectively belong.



Mikio's proof

I draw AC and BD and make the triangles AOB and COD to which AB and CD respectively belong. Then I show △AOB ≡ △COD to derive the conclusion AB=CD.

In △AOB and △COD, since AB || DC and alternate angles of parallel lines have equal measures, ∠BAO = ∠DCO (1), ∠ABO = ∠CDO (2)

Figure 2: Mikio's 'proof'

Miya's Mathematical Argument

Miya's argument is as follows:

2 Miya: I drew diagonal BD to show AB = CD, and proved that △ABD and △CDB are congruent. AB and BC are parallel from the assumption. And, since alternate interior angles of parallel lines are equal, ∠ABD and ∠CDB are equal, also ∠ADB and ∠CBD are equal. And, since they are common sides, BD = DB. Thus, ASA holds, and therefore △ABD and △CDB are congruent. Since corresponding sides of congruent figures are equal, well, I thought that AB = CD should be correct.

The argumentation stream for this argument is shown in Figure 3.

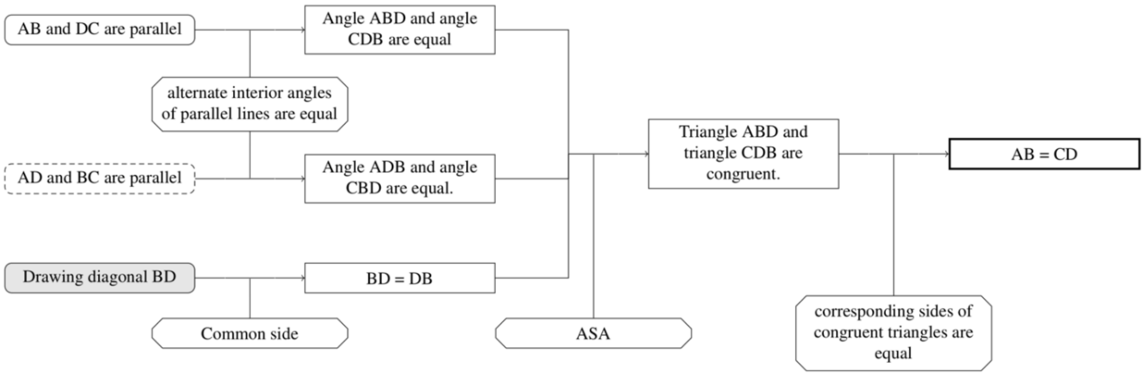


Figure 3: Miya's argument

Miya's argument is unusual only in that there are fewer implicit data or warrants, which are indicated by boxes with dashed lines, than often occurs in classrooms. The only implicit statement is AD || BC, which is not uttered but written on the blackboard. She asserts the existence of BD by drawing it into the diagram.

Why AC? Jiro's Meta-Mathematical Argument

In Episode 6, Jiro offers the following argument to justify the construction of the diagonal AC.

4	T:	Why was it necessary to draw AC? Jiro.
5	Jiro:	Well, what we want to show now, we want to show that two sides are congruent.
6	T:	Two sides are congruent?
7	Jiro:	Two sides are equal, we want to show that two sides are equal in length. And, to show it, now I have a quadrilateral, but I do not know how to show congruency of quadrilaterals. I only have one quadrilateral and do not know how to show, so I tried to transform it to the one that I already knew. Then I drew the line [AC] and, well, made two triangles in that way, and I thought of showing the congruency [of two triangles]. I drew the line in this way.

The diagram for his argumentation is shown in Figure 4. It begins from the fact that they want to show that two sides are congruent. From this fact, via the implicit warrant that corresponding sides of congruent figures are congruent, Jiro implicitly concludes that he wants to show that two figures are congruent. The only figure he can see is a quadrilateral and he does not know how to show quadrilaterals are congruent, so he concludes he should show the congruency of two triangles. To produce the two triangles it was necessary to draw the segment AC.

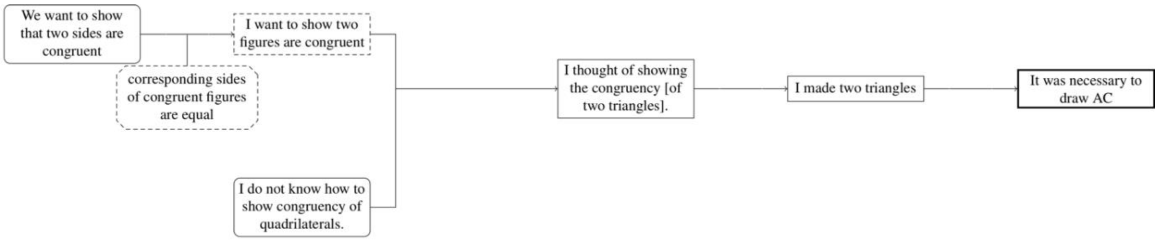


Figure 4: Our analysis of Jiro's meta-mathematical argument

The analysis of a meta-mathematical argument is similar to the analysis of a mathematical argument, although the role of the statements is more difficult to determine. It is useful to begin from the conclusion, the necessity of drawing AC, and to find the data and warrants supporting it. This can be expressed as 'It was necessary to draw AC, because I wanted to make two triangles.' In turn, 'I wanted to make two triangles, because I wanted to show congruency of two triangles' and 'I wanted to show congruency of two triangles because I wanted to show congruency of two figures, and I do not know how to show the congruency of quadrilaterals.' More elements are implicit than in Miya's argument.

Mikio's Mathematical Argument

In Episode 9, the teacher showed Mikio's 'proof', which is the incomplete proof 'attempt' shown in Figure 2. Our analysis of it is shown in Figure 5.

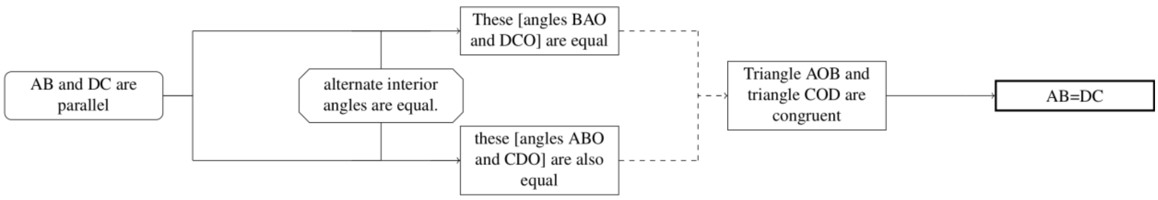


Figure 5: Our analysis of the mathematical argumentation in Mikio's 'proof'

The diagram of Mikio's 'proof' reveals that is leaves two warrants implicit. One of these implict warrants conceals the flaw in this 'proof'. To assert that $\triangle AOB \equiv \triangle COD$ he needs a congruent side so that he can use either AAS or ASA. But without asserting either the property he is trying to prove (opposite sides are congrent) or one of the other properties (that diagonals bisect each other) he cannot establish the congruent side.

The Meta-Mathematical Argument Concerning Mikio's 'Proof'

Related to the meta-mathematical argument concerning Mikio's proof, some parts of the transcripts are omitted here, since space is limited. Nevertheless, the transcript below shows how the teacher and students arrived at the reason why the given proof is impossible.

51	T:	You could not make it?
52	Ken:	Since there was no congruent side, well, no equal side.

53	T:	Then, Mikio tries to prove, in this way. To create triangles that include the conclusion, AB and CD, he connects points A and C, B and D respectively. See? [These triangles] include the conclusion[i]. Fine. Now he makes [triangles] AOB and COD. And then, if he shows congruency [of △AOB and △COD], he can deduce the equality [of AB and CD] since they include [AB and CD]. So he tries to prove that. But here, he stopped at (1) (2). Why [did Mikio] get stuck? What do you think? Ken.
54	Ken:	Well, under the assumption, uh, sides... We cannot show equality of [any pair of] sides, so we cannot prove this.
…		
59	Ss	[…] We do not know properties of parallelogram.
60	T	Yes. We cannot use other properties [of parallelogram].
61	Shin	Then, uh, sides
62	T	These are parallel. [with marking AB and DC in Mikio's diagram]
63	Shin	Length of other sides, sides are not necessarily equal in length.
…		
67	Mizu:	Well, we can only use those above two [properties of parallelogram: (1) opposite sides are parallel, (2) opposite sides are equal in length] and cannot use the three below [properties: (3) the diagonals intersect at their midpoint, (4) opposite angles have equal measures and so on].
…		
70	T:	Mizu, you wanted to use this [referring to the property (3)], didn't you? You thought that you could succeed if you used this, didn't you? Then, bad teacher came and told you that you were not allowed to use this, so you were in trouble, didn't you? Then, this and this are like this [marking angles in Mikio's diagram], alternate interior angles are equal. So, it comes to this way. But how does it work? These [angles] are equal since they are alternate interior angles of parallel lines, these are also equal. So it is fine if we use AB = DC, isn't it.
71	Ss:	The conclusion. / We cannot use the conclusion.

The meta-argument around Mikio's 'proof' has rather complex structure as shown in Figure 6. To justify the rejection of the proof, many statements involve negations and both warrants and implicit backings can be considered as meta-level reasoning or proving; such as 'a statement cannot be used for proving unless it is proven', 'circular reasoning', and 'it is necessary that a pair of sides at least is equal to use congruence conditions of triangles'. In Figure 6, the warrant 'other properties of parallelograms have not been established" and its backing are implicit, but it can be interpreted as such (Lines 59-50, 67, 70). The 'circular reasoning' as an implicit backing is also concerned with what the teacher and students uttered (Lines 70-71). Thus, it is interpreted that their discussions about Mikio's 'proof' are not only about how to prove the statement but also about what constitutes a proof.

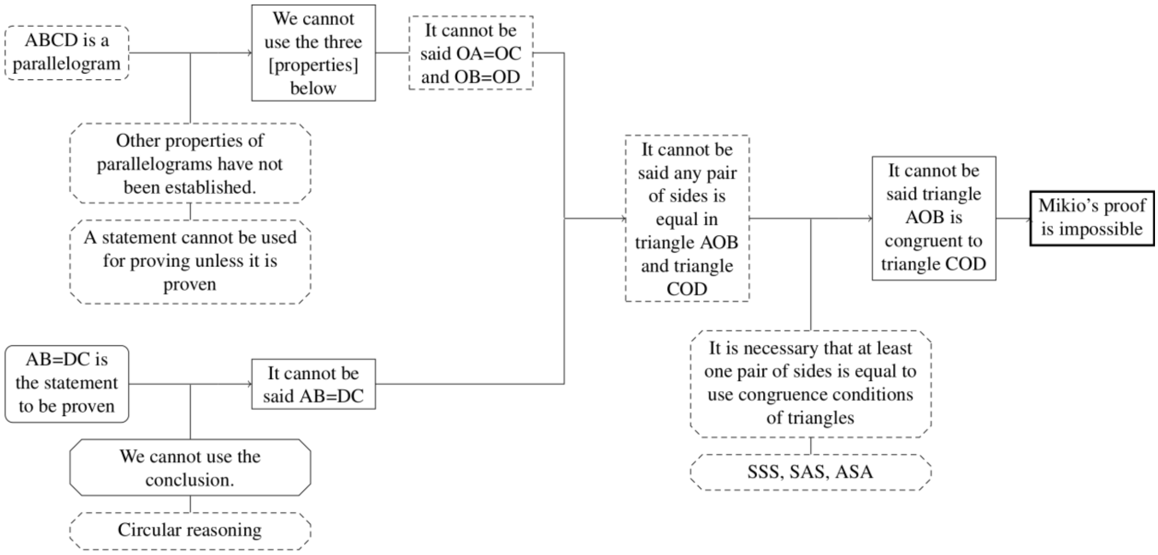


Figure 6: Our analysis of the meta-mathematical argumentation around Mikio's 'proof'

DISCUSSIONS AND CONCLUSIONS

One of the characteristics of our paper is the use of Toulmin's model for the analysis of meta-mathematical argumentation, which allows us to gain a deeper insight into classroom processes involving proving activity. In this study, we considered the meta-mathematical argument as the justification of the validity and the rejection of proofs and analysed their structures. For example, Jiro's meta-mathematical arguments were for justifying the validify of Miya's proof. It seems that the teacher's question "Why was it necessary to draw AC?" (line 4) facilitated their discussions. Our analysis (Figure 4) showed that some statements include ordinary sentences, such as "we want to show…" and "I do not know…", rather than purely mathematical sentences. Another meta-mathematical argument was for reasoning about the rejection of Mikio's 'proof'. This meta-level discussion allowed them to reflect "what has (not) been proven" and "the conclusion is a statement to be proven (it cannot be used to prove)" and to understand what constitutes a proof in the class (Figure 6). Since our analysis is limited, further research is needed to investigate different structures and functions of meta-mathematical argument.

Acknowledgements

The analysis in this study is based on the workshop on argumentation analysis which was held in Tokyo in March 2019. We are grateful to all participants of the workshop for their work.

References

Knipping, C., & Reid, D.A. (2015). Reconstructing argumentation structures: A perspective on proving processes in secondary mathematics classroom interactions. In A. Bikner-Ahsbahs, C. Knipping & N. Presmeg (Eds.) *Approaches to qualitative research in mathematics education.* (pp. 75–101). Cham, Switzerland: Springer.

Knipping, C., & Reid, D.A. (2019). Argumentation analysis for early career researchers. In Kaiser, G., & Presmeg, N. (Eds.) *Compendium for early career researchers in mathematics education.* (pp. 3–31). Cham, Switzerland: Springer. https://doi.org/10.1007/978-3-030-15636-7_1

Krummheuer, G. (2007). Argumentation and participation in the primary mathematics classroom. Two episodes and related theoretical abductions. *Journal of Mathematical Behavior*, *26*, 60–82.

Mariotti, M. A., Durand-Guerrier, V., & Stylianides, G. J. (2018): Argumentation and proof. In T. Dreyfus et al. (Eds.), *Developing research in mathematics education: twenty years of communication, cooperation, and collaboration in Europe* (pp. 75–89). New York, NY: Routledge.

Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics*, *66*, 23–41.

Potari, D., & Psycharis, G. (2018). Prospective mathematics teacher argumentation while interpreting classroom incidents. In Strutchens, M.E., Huang, R., Potari, D., Losano, L. (Eds.) *Educating prospective secondary mathematics teachers* (pp. 169–187). Cham, Switzerland: Springer.

Stylianides, A., Bieda, K. N., & Morselli, F. (2016). Proof and argumentation in mathematics education research. In Gutiérrez, Á. et al. (Eds.). *The second handbook of research on the psychology of mathematics education* (pp. 315–351). Rotterdam, The Netherlands: Sense Publishers.

Toulmin, S. E. (1958). *The uses of argument.* Cambridge, UK: Cambridge University Press.

Tsujiyama, Y., & Yui, K. (2018). Using examples of unsuccessful arguments to facilitate students' reflection on their processes of proving. In A. J. Stylianides & G. H. Harel (Eds.), *Advances in mathematics education research on proof and proving: An international perspective* (pp. 269–281). Cham, Switzerland: Springer.

PME44 VIRTUAL

THE 44th CONFERENCE OF THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

July 19-22, 2021

Hosted by Khon Kaen, Thailand

Virtually hosted by Technion, Israel Institute of Technology

AUTHOR INDEX