'Reasoning' in national curricula and standards

David A Reid

‘Reasoning’ in national curricula and standards

David A Reid

University of Agder, Norway; david.reid@uia.no

I examine the use of the word ‘reasoning’ in the 2020 Norwegian national mathematics curriculum, in the 2000 National Council of Teachers of Mathematics (NCTM) Standards and in the 2003 Education Standards of the German Kultusminister Konferenz (KMK). I identify differences in usage, make comparisons to the classification of aspects of reasoning proposed by Jeannotte and Kieran (2017), and suggest expanding their framework by addressing the distinction between the activity of reasoning and the process of reasoning, and also addressing the goal of reasoning. Specifically, the use of reasoning to explain is neglected in their framework.

Keywords: Reasoning, Argumentation, Proof, Mathematics curriculum, Language

Introduction

The differing meanings and usages of terms related to argumentation and proof have been discussed in the literature and related to differences in language (e.g., Sekiguchi & Miyazaki, 2000), professional context (e.g., Godino & Recio, 1997), and epistemological perspectives (e.g., Balacheff, 2008). Here I will contribute to this literature by examining the use of the word ‘reasoning’ in the 2020 Norwegian national mathematics curriculum, in the 2000 National Council of Teachers of Mathematics (NCTM) Standards and in the 2003 Education Standards of the German Kultusminister Konferenz (KMK). I will identify differences in usage, make connections to related terms such as ‘argumentation’ and ‘proof’, and suggest a framework for further discussion of these differences.

Related literature

Jeannotte and Kieran (2017) conducted a thorough survey of the ways mathematical reasoning is described in the mathematics education literature and they propose a conceptual model, based in a within a commognitive theoretical framework, to describe mathematical reasoning. In their model they distinguish between structural and process aspects. The structural aspect refers to the form of the reasoning: deductive, inductive or abductive. The process aspect is more complex, and is divided into three processes related to the search for similarities and differences, validating, and exemplifying. The first two processes are further divided into subprocesses.

The search for similarities and differences includes generalizing, conjecturing, identifying a pattern, comparing, and classifying. By ‘generalizing’ they mean inferring something from a given set that applies to a larger set containing it. Conjecturing is characterized by the epistemic value it assigns to an inference: probable or likely. These two processes can (and perhaps often) occur together, but they are distinct. One can generalize with making a claim that the generalization is probable, and one can make a claim that an inference is probable that does not involve generalizing from a subset to a set. Jeannotte and Kieran use ‘identifying a pattern’ to refer to the process of identifying a relation among a set of objects, but this relation need not be extended to a larger set (as in generalizing) nor given a probable epistemic value (as in conjecturing). Comparing can occur along with the other process, and is necessary for identifying a pattern. Comparing, however, refers only to the observation of similarities and differences, without identification of a relation between the objects. Classifying involves making a class of objects based on shared properties.

Processes related to validating include justifying, proving and formal proving. Validating refers to any process that is directed towards changing epistemic value, towards higher or lower likelihood. Justifying is validating that includes a search for data, warrants and backing to modify epistemic value. Proving is validating that specifically modifies epistemic value to truth. It is linked to a set of accepted truths, the use of deductive reasoning (at least in its final stages), and particular, socially accepted, forms of expression. Formal proving is proving that meets stricter criteria for the accepted truths used and the final forms of expression. As Jeannotte and Kieran put it, "formal proving relies on mathematical theory built a priori and on formalized realizations (axioms and theorems)." (2017, p. 13)

Jeannotte and Kieran list a final process, exemplifying, which they say supports the other processes. It consists of producing examples which can then allow patterns to be identified, conjectures and generalizations to be made, and which can be used to justify claims.

It should be noted that Jeannotte and Kieran considered other aspects of mathematical reasoning that were not directly included in their final model, as they felt that the distinctions that they had made captured these other distinctions. One distinction they encountered in the literature but did not explicitly include is the activity/product dichotomy, which separates the product of reasoning from the inaccessible mental activity that gives rise to it. They also considered the inferential nature of mathematical reasoning, that is, the origin of novel ideas through reasoning, and the goal and functions of mathematical reasoning, which refers to the purpose of reasoning which is often verification but might also be exploration or explanation.

Jeannotte and Kieran found that there is no universal agreement on the meaning of 'mathematical reasoning' in the research literature, but they were able to, by considering carefully the ways this term is used, to identify key aspects of it. They note that policy documents like curricula and standards around the world emphasize mathematical reasoning as a goal, but that the description of it in such policy documents "tends to be vague, unsystematic, and even contradictory from one document to the other" (p. 2). Nonetheless, such policy documents seem likely to have a stronger influence on what teachers think mathematical reasoning is, and hence what goes on in classrooms with the goal of fostering mathematical reasoning, than the research literature. Hence, I have chosen to examine one such curriculum document and two national standards documents to explore the use, and hence the meaning, of 'reasoning' in them. I have been informed by, but have not strictly applied, Jeannotte and Kieran's framework, in order to allow for the possibility that distinctions are made in policy documents that differ significantly from those included in Jeannotte and Kieran's framework.

The Norwegian mathematics curriculum

The current Norwegian curriculum (Kunnskapsdepartementet 2019a, b) lists *Reasoning and argumentation* as one of six "core elements", each of which is described in a paragraph. The other core elements are: *Exploration and problem solving, Modelling and applications, Representation and communication, Abstraction and generalization,* and *Mathematical fields of knowledge* (which includes number, algebra, functions, geometry, statistics and probability).

The official English translation of the *Reasoning and argumentation* core element is:

> **Reasoning** in mathematics means the ability to follow, assess and understand mathematical chains of thought. It means that the pupils shall understand that mathematical rules and results are not random, but have clear and logical grounds. The pupils shall formulate their own **reasoning** to understand and to solve problems. Argumentation in mathematics means that the pupils give grounds for their methods, **reasoning** and solutions, and prove that these are valid. (2019b, p. 3, bold added)

An important distinction in the original is missing from this translation. The original text is headed "Resonnering og argumentasjon" and reads:

> **Resonnering** i matematikk handlar om å kunne følgje, vurdere og forstå matematiske tankerekkjer. Det inneber at elevane skal forstå at matematiske reglar og resultat ikkje er tilfeldige, men har klare grunngivingar. Elevane skal utforme eigne **resonnement** både for å forstå og for å løyse problem. Argumentasjon i matematikk handlar om at elevane grunngir framgangsmåtar, **resonnement** og løysingar og beviser at dei er gyldige. (2019a, p. 2, bold added)

Notice that the word 'reasoning' in the English translation is used for two Norwegian words, 'resonnering' and 'resonnement'. The word 'resonnering' does not have a dictionary entry of its own. It is formed from the verb "resonnere" (to reason) with the suffix "-ing" to make it a noun. The word 'resonnement' is a noun means 'thinking, way of thinking, concluding'. Both are nouns, but 'resonnering' is closer to the verb form and is used to refer to the process of reasoning, while 'resonnement' refers to the product of reasoning. Recall that the activity/product dichotomy is a distinction that Jeannotte and Kieran chose not to specifically include in their model. However, in this case, where the language allows this distinction to be explicitly marked, the authors of the Norwegian curriculum have chosen to do so. This suggests that the dichotomy is an important one to them, and that teachers being guided by the curriculum might make a similar distinction. It also reminds us that this distinction is harder to observe in languages like English that have only a single word for reasoning.

Sources

One of the influences on the Norwegian curriculum is the work of Mogens Niss and his colleagues (e.g., Niss & Jensen, 2002; Niss & Højgaard, 2019). The idea of 'chains of thought' originates from Niss's work. For example, Niss and Jensen (2002) write:

> [The reasoning] competence consists, on the one hand, in being able to follow and assess a mathematical reasoning, i.e. a chain of arguments put forward by others in writing or in speech in support of a statement (p. 54, my translation)

However, Niss and Jensen also say that the competence involves:

> understanding what a mathematical proof is and how it differs from other forms of mathematical reasoning, e.g., heuristic reasoning resting on intuition or on consideration of specific cases, and to be able to determine when a mathematical reasoning actually constitutes a proof and when not. (p. 54, my translation)

and

> consists of being able to devise and implement informal and formal reasoning (on the basis of intuition), including transforming heuristic reasoning into actual (valid) proofs. (p. 54, my translation)

The 'reasoning' in the Norwegian curriculum does not mention intuition, consideration of specific cases, or heuristic reasoning, and its emphasis on "clear and logical grounds" suggests that such reasoning is not included.

Summary

The description of reasoning in the Norwegian curriculum touches on several characteristics: the process of reasoning involves following, assessing and understanding mathematical chains of thought; it is related to the grounds of mathematics; the product of reasoning can be formulated; and giving grounds for that product is central to argumentation. It excludes heuristic reasoning and reasoning based on specific cases and intuition, which Niss and his colleagues include as mathematical 'reasoning'.

The NCTM Standards

The 2000 NCTM *Principles and Standards for School Mathematics* have influenced curricula both in the United States and internationally. The structuring of the Norwegian curriculum into core elements may have been influenced by the structure of the NCTM *Standards.* Three of the core elements (*Exploration and problem solving, Reasoning and argumentation,* and *Representation and communication*), have, at least in part, the same names as four of the NCTM 'Process Standards' (*Problem solving, Reasoning and Proof, Communication,* and *Representation*). A fourth (*Modelling and applications*) is very similar in content to the NCTM's *Connections* standard.

The NCTM standard *Reasoning and Proof* is described in a four page long general section, as well as in separate sections for the Pre-K–2, 3–5, 6–8 and 9–12 grade bands. The general section begins with a listing of four goals:

> Instructional programs from prekindergarten through grade 12 should enable all students to—
>
> - recognize reasoning and proof as fundamental aspects of mathematics;
> - make and investigate mathematical conjectures;
> - develop and evaluate mathematical arguments and proofs;
> - select and use various types of reasoning and methods of proof. (p. 56)

The goal "develop and evaluate mathematical arguments and proofs" is also the central focus of the Norwegian core element *Reasoning and argumentation.* However, the NCTM standard is broader as it includes making and investigating mathematical conjectures. Conjecturing is not mentioned in the Norwegian curriculum but the core element *Exploration and problem solving* includes "searching for patterns, finding relationships" (p. 2). This parallels the NCTM's "note patterns, structure, or regularities in both real-world situations and symbolic objects" (p. 56) in the *Reasoning and Proof* standard.

The NCTM Standards also includes the ability to "select and use various types of reasoning and methods of proof." (p. 56). This suggests that, like Niss, the NCTM sees mathematical reasoning as including several kinds of reasoning. One distinction the NCTM makes may be similar to Niss's distinction between heuristic reasoning and proofs:

> At all levels, students will reason inductively from patterns and specific cases. Increasingly over the grades, they should also learn to make effective deductive arguments based on the mathematical truths they are establishing in class. (NCTM, 2000, p. 59)

Several characteristics of reasoning are evident in the description given.

> People who reason and think analytically tend to note patterns, structure, or regularities in both real-world situations and symbolic objects; they ask if those patterns are accidental or if they occur for a reason; and they conjecture and prove. Ultimately, a mathematical proof is a formal way of expressing particular kinds of reasoning and justification. (p. 56)

The asking if patterns are accidental or if they occur for a reason parallels the Norwegian curriculum's "pupils shall understand that mathematical rules and results are not random, but have clear and logical grounds" (Kunnskapsdepartementet 2019b, p. 3) but for the NCTM reasoning includes deciding if a pattern occurs for a reason, while the Norwegian curriculum is more narrowly focused on rules and results that have reasons.

The NCTM associates reasoning with analytic thinking, though what the distinction is between them is not clear. Later the *Reasoning and Proof* standard states "Classrooms in which students are encouraged to present their thinking and in which everyone contributes by evaluating one another's thinking provide rich environments for learning mathematical reasoning" (NCTM, 2000, p. 57). This further suggests a link to a mental activity.

The NCTM is also explicit about what 'proof' means; the Norwegian curriculum does not mention 'proof'. For the NCTM there is an emphasis on proofs being formulated and expressing a particular kind of reasoning, which the passage quoted earlier (from p. 59) suggests is deductive reasoning.

For the NCTM, reasoning is related to understanding.

> Being able to reason is essential to understanding mathematics. By developing ideas, exploring phenomena, justifying results, and using mathematical conjectures in all content areas and—with different expectations of sophistication—at all grade levels, students should see and expect that mathematics makes sense. (NCTM 2000, p. 56)

Similarly, the Norwegian curriculum states "The pupils shall formulate their own reasoning to understand" (Kunnskapsdepartementet 2019b, p. 3).

Summary

The word 'reasoning' in the NCTM Standards is used more broadly than in the Norwegian curriculum. It includes not only giving reasons, but also making conjectures. These two activities are associated with deductive and inductive reasoning, respectively. Proof is formulated deductive reasoning. As in the Norwegian curriculum, in the NCTM Standards, 'reasoning' is connected to a mental activity, to understanding, and to finding the reasons underlying a pattern, rule or result.

The KMK Standards

In 2003 and 2004 The Standing Conference of the Ministers of Education and Cultural Affairs of the Länder in the Federal Republic of Germany (KMK) issued educational standards for mathematics for the different German school forms. The first one, issued in 2003 for middle schools ending at Grade 10, is my focus here.

The KMK Standards are structured into competences, analogous to the NCTM's Standards and the Norwegian curriculum's core elements. They are *Argumentation, Problem solving, Modelling, Using representations, Dealing with symbolic, formal and technical elements,* and *Communication*. The names of these strongly parallel the names of the NCTM Standards with the exception of *Modelling* (which the NCTM calls *Connections*) and *Argumentation* (which the NCTM calls *Reasoning and Proof*).

The naming of the *Argumentation* competence reveals an interesting linguistic difference between German, English and Norwegian. German has not adopted a word based on the French *raisonner,* and it seems to lack a direct equivalent. Possible translations for the verb 'to reason' include *schlussfolgern* (to conclude) and *begründen* (to give reasons for, to justify). The noun form 'reasoning' can be translated as *logisches Denken* (logical thinking), or *Argumentation* (argumentation). It is interesting that the question "How is the word 'reasoning' used in the KMK Standards?" can be answered briefly "It isn't", and also that the question cannot even be asked in German.

However, we can compare the use of *Argumentation* in the KMK Standards to the use of 'reasoning' elsewhere. The *Argumentation* competence states that:

> Mathematical argumentation … includes:
>
> - Posing questions that are characteristic of mathematics ("Does there exist ...?", "What changes if...?", "Is that always so?") and expressing justified conjectures,
> - Developing mathematical arguments (such as explanations, justifications and proofs)
> - Describing and justifying solution methods. (p. 8, my translation, original in Appendix)

Conjecturing, or at least expressing and justifying conjectures, is included under *Argumentation*, similarly to the NCTM's *Reasoning and Proof*. Developing mathematical arguments and describing and justifying solution methods are both found in both the Norwegian curriculum and the NCTM Standards. Those documents, however, include references to "chains of thought" (Kunnskapsdepartementet, 2019b, p.3) and students presenting "their thinking" (NCTM, 2000, p. 58). These refer to the mental process of reasoning. The KMK Standards, however, do not mention this process and instead focuses on the observable social product of reasoning.

The three kinds of argument named: explanations, justifications and proofs ("Erläuterungen, Begründungen, Beweise" in the original, KMK, 2003, p. 8), suggest different goals for an argument: explaining, justifying, and doing so in a way acceptable to the mathematical community. However, these terms are themselves not defined, and so this connection to goals might not have been what the authors had in mind. If this is what is intended, then these goals overlap with those expressed in the

Norwegian curriculum. which refers to “understanding”, ”give grounds” and “prove” (Kunnskapsdepartementet, 2019b, p. 3). Similarly, the NCTM Standards (2000) state that “being able to reason is essential to understanding mathematics” and that “a mathematical proof is a formal way of expressing particular kinds of reasoning and justification” (p. 56).

Conclusion

Examining the Norwegian curriculum and the NCTM and KMK Standards shows that one distinction Jeannotte and Kieran made, between processes related to the search for similarities and differences and processes related to validating, is also useful for describing a key difference between the Norwegian curriculum and the two Standards documents. In the Norwegian curriculum only processes related to validating are included under the core element *Reasoning and argumentation.* Processes related to the search for similarities and differences, such as conjecturing, are instead included under *Exploration and problem solving*.

Jeannotte and Kieran’s wider distinction, between processes and structures is also interesting, as only the NCTM Standards mentions different structures of reasoning, specifically inductive and deductive reasoning. This may be simply because the other two documents are much shorter.

One difference that Jeannotte and Kieran’s framework does not capture is that between the mental activity of reasoning and the social product of reasoning. This distinction is important in the Norwegian curriculum and reflected in the words used. Though the NCTM Standards uses ‘reasoning’ to refer to both the activity and the product, both are included in the descriptions. The KMK Standards, however, refer only to the product, as ‘Argumentation’. This activity/process distinction reflects most strongly the possible influence of language on the mathematics curriculum in this area. As Jeannotte and Kieran were working within a commognitive theoretical framework, which denies the existence of a split between mental activity and social discourse, it is not surprising that this distinction is not captured in their framework. It does, however, seem to be an aspect of the use of the word ‘reasoning’ in policy documents, and therefore worth attending to.

A final distinction not captured in Jeannotte and Kieran’s framework, but important in the policy documents, is the distinction between different goals of reasoning. The explicit listing of explanations, justifications and proofs as three kinds of arguments in the KMK Standards is also reflected in the other two documents, but seems impossible to capture in the categories listed by Jeannotte and Kieran. The goal of explanation, which is perhaps the most important in educational settings (Hanna, 1989) seems not to be included at all in Jeannotte and Kieran concept of ‘reasoning’.

To capture the way the word ‘reasoning’ is used in policy documents, it would be useful to add a fourth process, *explaining*, to Jeannotte and Kieran’s search for similarities and differences, validating, and exemplifying. Furthermore, some way to make a distinction between mental activities and social discourse seems to be needed. It is not evident how this could be added to Jeannotte and Kieran’s framework, except perhaps as a third aspect.

References

Balacheff, N. (2008). The role of the researcher's epistemology in mathematics education: An essay on the case of proof. *ZDM The International Journal on Mathematics Education 40*, 501–512. https://doi.org/10.1007/s11858-008-0103-2.

Godino, J., & Recio, A. (1997). Meaning of proofs in mathematics education. In Pekhonen, Erkki (Ed.), *Proceedings of the Twenty-first Conference of the International Group for the Psychology of Mathematics Education.* (Vol. 2, pp. 313–320). University of Helsinki.

Hanna, G. (1989). Proofs that prove and proofs that explain. In G. Vergnaud, J. Rogalski, & M. Artigue (Eds.), *Proceedings of the Thirteenth International Conference on the Psychology of Mathematics Education*, (Vol. 2, pp. 45–51). PME.

Jeannotte, D., & Kieran, C. (2017). A conceptual model of mathematical reasoning for school mathematics. *Educational Studies in Mathematics, 96*(1), 1–16. https://doi.org/10.1007/s10649-017-9761-8

Kultusministerkonferenz (KMK) (2003). Bildungsstandards im Fach Mathematik für den Mittleren Schulabschluss [Educational Standards in Mathematics for the completion of middle school] Luchterhand.

Kunnskapsdepartementet (2019a). Læreplan i matematikk 1.–10. trinn http://www.udir.no/kl06/MAT01-05

Kunnskapsdepartementet (2019b). Curriculum for Mathematics Year 1–10 (Official translation of Kunnskapsdepartementet, 2019a) https://www.udir.no/lk20/mat01-05?lang=eng

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics.* NCTM.

Niss, M, & Jensen, T. H. (Eds.) (2002). *Kompetancer og matematiklæring. Ideer og inspiration til udvikling af matematikundervisning i Danmark. Utdannelsesstyrelsens temahefter nr. 18 – 2002;* Undervisningsministeriet.

Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics, 102*(1), 9–28. https://doi.org/10.1007/s10649-019-09903-9

Appendix

The original text of the Argumentation competence in the KMK Standards is as follows:

Mathematisch argumentieren

Dazu gehört:

- Fragen stellen, die für die Mathematik charakteristisch sind („Gibt es ...?“, „Wie verändert sich...?“, „Ist das immer so ...?“) und Vermutungen begründet äußern,
- mathematische Argumentationen entwickeln (wie Erläuterungen, Begründungen, Beweise),
- Lösungswege beschreiben und begründen. (KMK, 2003, p. 8)