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CHAPTER 8

Mathematical Modelling and Inquiry-Based Mathematics Education

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8.1. Introduction

Mathematical modelling (MM) is a powerful tool used by scientists and engineers to solve important problems for humankind. MM opens many possibilities for inquiry and has been included in the PLATINUM project as one of the Intellectual Outputs, IO5 (see Chapter 5 for the complete list). We consider MM an important part of the teaching and learning process. We believe that helping to develop students' modelling competencies we equip them with a valuable understanding of practical and theoretical concepts, prepare them for a life-long learning, and form them as critical citizens.

This chapter is organised as follows. In Section 8.2, we share our views on why we teach MM, noting that our teaching practices are adapted differently to suit local educational contexts (types of students, programs of study, institutional traditions, constraints, etc.). We proceed with the discussion of what are the most important to us characteristics of Inquiry-Based Mathematics Education (IBME) as a teaching approach. To explain authors' understanding of how the MM relates to IBME, the concept of 'active knowledge' is introduced in Section 8.3. The key idea of this concept is that in response to the use of IBME in the classroom students' engagement with MM activates previously acquired knowledge and facilitates its efficient use. In Section 8.4, partners present examples of the use of MM within IBME practices and comment on how students activate their mathematical knowledge. Each example shows multifaceted connections between MM and IBME. We reflect on the lessons learned from our contributions to the Intellectual Output IO5 in Section 8.5.

8.2. Mathematical Modelling and Inquiry-Based Mathematics Education in Our Teaching

Theoretical foundations of Inquiry-Based Mathematics Education (IBME) presented in Chapter 2 emphasise the importance of improving the balance between procedural and conceptual learning of mathematics through an inquiry approach that offers students opportunities for deeper engagement with the subject. On the one hand, inquiry-based tasks motivate and encourage students to get involved with the subject more actively by posing questions and trying to answer them, and by exploring processes and concepts. On the other hand, mathematical modelling (MM) tasks motivate students' engagement into activities that contribute to the development of their creativity and exploratory skills that are characteristic of professional mathematicians, with the aim of developing students' mathematical literacy and prepare them for professional life. Therefore, the main ideas of the IBME are well suited for the use in mathematical modelling and can be employed to motivate students to learn actively in and outside the classroom through individual and collaborative problem solving and project work. This is often achieved when students work with authentic tasks and are provided with a timely strategic support.

8.2.1. Why Do We Use Mathematical Modelling in Our Teaching? In their work on mathematical modelling and applications, Blum and Niss (1991) defined a mathematical model as a triple (S, M, R) consisting of some real problem situation S, some collection M of mathematical entities and some relation R by which objects and relations of S are related to objects and relations of M. Then, MM is the entire process leading from the real problem situation to a mathematical model. Whilst this definition seems easy to understand, in practice MM activity is very complex and there is a certain disagreement in the mathematics education community as to what exactly counts as a model/modelling, what the aims of MM are, and how it can be taught best (cf., Kaiser & Sriraman, 2006; Hernandez-Martinez et al., 2021).

It is therefore not surprising that the authors of this chapter have similarities and differences in their views of MM. In order to elucidate where these similarities and differences lie, we look at the perspectives on MM that Kaiser and Sriraman (2006) elaborated, and the questionnaire that Treffert-Thomas et al. (2017) developed based on those perspectives plus an additional one called "Enjoyment perspective." The five categories connected to goals of teaching modelling are:

(1) Realistic (or applied) perspective, which describes the aims of MM as pragmatic or utilitarian, that is, to solve practical problems in the way that applied mathematicians would do in their professional practice;
(2) Epistemological (or theoretical) perspective, which sees the aims of MM as theory-oriented, that is, to develop theory without paying too much attention to the realistic aspects of a problem;
(3) Socio-critical (or emancipatory) perspective, which characterises MM as aiming to develop critical understandings of the world and the role that mathematics plays in making important societal decisions;
(4) Contextual perspective, which characterises MM as a tool for psychological development, that is, MM activity should elicit the invention, extension, and refinement of mathematical (psychological) constructs;
(5) Educational perspective, which sees the aims of MM as pedagogical, that is, MM should foster the understanding of mathematical concepts and structure the learning processes.

Treffert-Thomas et al. (2017) complemented this categorisation with a sixth one (see also, Rogovchenko et al., 2020):

(6) Enjoyment (or affective) perspective, in which the aim of MM is the intrinsic satisfaction derived from engaging in MM activity.

The authors completed the questionnaire on MM and IBME, where part D is based on the items used by Treffert-Thomas et al. (2017),[1] and we next discuss the results from those categories where the majority of the partners agreed or disagreed. We all strongly agreed or agreed that MM should aim to develop skills in solving authentic

[1] `https://bit.ly/32h3cy1`

problems and that MM with real data leads to significant insights, both characteristics related to a realistic perspective. MM team members also strongly agreed or agreed that the aims of MM are to develop general problem-solving skills and develop students' critical thinking skills, which would correspond to the educational and socio-critical perspectives, respectively. Finally, MM team members strongly agreed or agreed that MM should be based on the theoretical understanding of the phenomenon to be modelled, which corresponds to an epistemological perspective.

These similarities are reflected in how each of the authors views MM and how s/he operationalises her/his views in their teaching as a local community of inquiry. For example, partners at Brno University of Technology (BUT) believe that MM activities should be based on or motivated by applications of mathematics to real life problems (realistic perspective), but this does not exclude pure mathematics activities (epistemological perspective). Partners at Borys Grinchenko Kyiv University (BGKU) believe that the main purpose of using MM in the educational process is for students to acquire the knowledge of new mathematical facts, master new mathematical methods useful for studying different phenomena and processes, improve conceptual understanding of mathematics and advanced mathematical thinking, gain some experience in applying mathematical knowledge, develop collaboration and communication skills, deepen motivation for lifelong learning, and so on. That is, MM for partners at BGKU is a tool for forming students' mathematical competence, which mainly corresponds to an educational perspective. Partners at the University of Agder (UiA) believe that MM should allow students to reach a certain mathematical maturity, to have knowledge of different areas of mathematics, to develop critical thinking and the ability to collaborate and communicate efficiently, with all these components embedded in traditional mathematics courses. They also believe that MM should stimulate joy and excitement of MM in students, and bring about creativity and inspiration, particularly in the advanced educational levels. These views correspond to the realistic, educational and enjoyment perspectives.

We should be aware that these views are mediated by a myriad of factors that affect practice. For example, partners at UiA need to deal with the fact that there are no dedicated courses where MM is taught, so they include this important aspect of mathematics in mathematics courses they teach. Partners at BGKU are responsible for teaching of students in mathematics and ICT undertaking a larger variety of courses, including those aimed at real-life applications, and hence MM plays a more prominent role in their pedagogical strategy. Partners at BUT are new to the use of MM in teaching but have plenty of experience in using MM in professional settings, where theory is valued. All these contexts and circumstances shape what we value in MM and what role we ascribe to MM in teaching (Hernandez-Martinez et al., 2021). Therefore, while we agree on several basic features of MM and the ways it should be taught, these features might look quite different in each of the partners' practices.

Finally, we discuss questionnaire items in part D where most of the partners disagreed. Partners were split in their opinion that every student should learn modelling, if the aim of creating a model is to obtain a solution or if solving word problems constitutes MM. We hypothesise that these disagreements stem from the realities and situations in which each partner operates. For example, for partners at BUT, word problems would not constitute MM because they are inauthentic while for partners at BGKU these types of problems might provide important tools to achieve their teaching goals. For partners at UiA and BUT, the process would be more important than the solution to a problem because it is through the process of MM that students become interested, engaged or even creative. On the other hand, for partners at BGKU, correct and complete solutions to problems carry a great deal of value because for it indicates if students have achieved the learning that the teachers desire them to do. Hence, we see disagreements as part and parcel of the different circumstances in which partners find themselves but not as a barrier for discussing MM.

The partners agree that a simplified four-step cycle (Understanding task—Establishing model—Using mathematics—Explaining result) described by Blum and Borromeo Ferri (2009) represents a convenient format for the work with modelling tasks with students. We also agree that successful students' work on modelling tasks requires certain mathematical maturity, knowledge in different areas of mathematics, critical thinking and ability to collaborate and communicate efficiently; MM at a higher professional level brings also creativity and inspiration which turn it into an "art of MM" rather than a process. One important goal in teaching MM is to share with students the excitement and joy of mathematical modelling (Rogovchenko et al., 2020) which can be experienced in a mathematics classroom even with an entry or intermediate level modelling projects and problems.

8.2.2. How Do We Use Inquiry in Our Teaching? The ideas of inquiry-based mathematics Education (IBME) can be traced back at least to the work of the American philosopher and educator John Dewey (1859-1952) who published two cornerstone monographs "How we think: A restatement of reflective thinking to the educative process" (1933) and "Logic: The theory of inquiry" (1938). In the latter book, Dewey defined inquiry as "the controlled or directed transformation of an indeterminate situation into one that is as determinate in its constituent distinctions and relations as to convert the elements of the original situation into a unified whole" (p. 108). Nowadays, student inquiry is characterised as "an educational activity in which students are placed in the position of scientists gathering knowledge about the world. Students direct their own investigative activity, completing all the stages of scientific investigation such as formulating hypotheses, designing experiments to test them, collecting information, and drawing conclusions" (Keselman, 2003, p. 898). This definition emphasises active participation and learner's responsibility for constructing this knowledge (de Jong & van Joolingen, 1998). Modern views on inquiry-based education develop further Dewey's educational philosophy promoting learning through reflective inquiry which combines inductive and deductive methods and emphasises pragmatic efficiency of knowledge and connections to real-life situations and professional practice (Artigue & Blomhøj, 2013). In line with Dewey's educational philosophy, inquiry in the mathematics classroom often starts with the discussion of realistically looking situations which may naturally lead to modelling.

> The starting points for mathematical inquiry are the multiple live issues that students possess; mathematics becomes the set of tools from which they can choose to help carry out their inquiries. In this type of mathematics class, the teacher becomes a skilled guide who can help shape student inquiries, aiding in the construction of mathematical models and the selection of appropriate mathematical tools of inquiry and in supervising the evaluation of such activities. (Stemhagen & Smith, 2008, p. 34)

Pedaste et al. (2015) conducted a systematic literature review identifying the core phases of IBME and their involvement in learning distinguishing five distinct general inquiry phases, some also split into subphases. The list of phases includes *Orientation*, understood as the process of stimulating curiosity and addressing a learning challenge through a problem; *Conceptualisation*, the process of stating hypotheses or stating theory-based questions, which splits into subphases of *Questioning* and *Hypothesis Generation*; *Investigation*, the process of exploring, experimenting, planning, and collecting and analysing data with the two subphases of (i) *Exploration, Experimentation* *and Data Interpretation* and (ii) *Conclusion and Discussion*, understood, respectively, as the process of drawing conclusions from the data analysis and answering the hypothesis or research questions and the process of presenting findings, communicating with others and engaging in reflection, with the subphases *Communication* and *Reflection*. Inquiry activities are organised in cycles, which combine different phases. This fits especially well the process of learning mathematical modelling because recent research links modelling competency with the ability to successfully perform all steps in a modelling cycle (Blomhøj & Højgaard Jensen, 2003; Blum & Borromeo Ferri, 2009; Blum & Leiß, 2007; Blum & Niss, 1991). Different models of a modelling cycle are used in mathematics education ranging from a seven-step model for research and teaching (Blum & Leiß, 2007) to a simpler four step schema (Understanding task—Establishing model—Using mathematics—Explaining result) deemed to be more appropriate for students' work (Blum & Borromeo Ferri, 2009).

We explored partners' views on IBME by asking them to respond to parts A-C of the four-part questionnaire "Relevance of inquiry for mathematical modelling."[2]

In part A, the respondents were asked to indicate their preferences to 34 key inquiry activities listed by Pedaste et al. (2015) with the understanding that not all activities may be equally well suited for the inquiry in a mathematics classroom; this is clearly reflected in the answers. It turns out that partners did not indicate particular interest in the three activities related to the first phase, Orientation. Partners viewed the following five activities as very relevant or relevant for inquiry-based learning;[3] phase names are written in italics in the parentheses; the items are ordered from the highest ranked on the top of our list to the lowest ranked at the bottom):

(5) Determining what needs to be known, Define problem, Identifying the problem, Identification of question or questions (*Conceptualisation*);

(11) Investigate, Observe, Observation, Collect my evidence, Conduct observation, Explore, Exploration, Initial observation (*Investigation*);

(24) Construction, Reasoning with models, Problem solving and developing a course/ experiment (*Conclusion*);

(27) Evaluating success, Evaluate, Evaluation, Evaluate action, Evaluate inquiry, Comparing new knowledge to prior knowledge, Test the explanations (Falling between *Conclusion* and *Discussion*);

(28) Discuss, Debate, Share and discuss my inquiry, Discussing with others, Communicating new understandings, Elaborate, Communicating results, Argument, Discussion and presentation of new content, Communication, Learner communicates and justifies explanation, Present inquiry (*Discussion*).

On the list of inquiry activities where the partners' views on relevance diverged the most, we find three items: (14) Sign system exploration and (21) Transmediation (both from the phase Investigation) and (23) Celebration (from the phase Conclusion).

Part B regards Essential Ingredients in inquiry-based mathematics education (Artigue & Blomhøj, 2013). Summarising the results, we observe that the partners valued most highly two items: Pose questions and Inquire—the 5 E's: Engage, Explore, Explain, Extend, Evaluate—both grouped into What Students Do, and that the partners expressed differing views on the two items from the group Classroom culture: Shared sense of purpose/justification, and Shared ownership. The views on the ingredients of IBME collected under the umbrellas of Valued outcomes, Teacher guidance and Type of questions were much less pronounced. This suggests that all

[2]The questionnaire and the summary of the answers of six team members can be found at the following link: `https://bit.ly/32h3cy1`

[3]Activity numbers are from Pedaste et al. (2015).

partners indicate clear interest in using inquiry as the learning strategy and value multifaceted experiences that inquiry offers students. Not surprisingly, classroom culture in the Czech Republic, Norway and Ukraine differ significantly which is reflected in the answers.

Part C of our questionnaire is based on the list of Components of Inquiry Process (Artigue & Blomhøj, 2013), indicate three favourites: New experience/question, Plan and conduct investigation, and Interpret data. Much less agreement between the partners was observed with regard to the remaining components: Possible explanation, Existing idea, Alternative ideas, Bigger idea, Prediction, and Conclusion. This again points towards partners' prioritisation of the organisation of students learning through inquiry emphasising questioning, analysis, and validation of results, all three components critical for the proper structuring of the modelling activities. The final part D of the questionnaire was based on the paper by Treffert-Thomas et al. (2017); our answers to this part were already discussed in Section 8.2.1.

The sources of mathematical inquiry in IBME emerge not only from mathematical objects themselves, but also from daily life problems, industrial and technical problems, processes, and phenomena in nature, and even from art and human artefacts.

> In relation to IBME, the concept of modelling offers a systematic way of understanding and working with the relationship between mathematics and problem situations or phenomena in other disciplines and in extra-mathematical contexts in general. From a learning perspective, modelling can thus be a bridge between the mathematical concepts and ideas and real-life experiences. Through modelling activities, the learner can make sense of the concepts as well as gain new insights into the problem situations modelled. (Artigue & Blomhøj, 2013, p. 805)

Inquiry cycles and mathematical modelling cycles discussed in the research literature exhibit striking similarity; therefore, the work with mathematical models leads to "valuable understanding of inquiry as a more general process with different particular realisations in different disciplines and contexts" (Artigue & Blomhøj, 2013, p. 805).

8.3. Active Knowledge: Connecting IBME and MM

Introducing the notion of mathematical modelling competence, Blomhøj and Højgaard Jensen (2003) highlighted the following steps in the modelling process:

(a) Formulation of the task and identifying the characteristics of perceived reality.
(b) Selection of the relevant objects and relations, use of idealisation.
(c) Translation to mathematics.
(d) Use of mathematical methods.
(e) Interpretation of results.
(f) Evaluation and validation of the model by comparison with data.

We believe that these steps require the following abilities from the students:

(a) Curiosity, motivation, exploring, engagement
(b) Exploring, engagement
(c) Engagement
(d) Engagement
(e) Evaluation
(f) Evaluation

Drawing on these ideas, we introduce the concept of active knowledge to explain how the modelling process is related to inquiry-based learning and how this relationship enhances the active knowledge formation. We want to distinguish active knowledge from passive knowledge in a following way: active knowledge is used on the regular basis; passive knowledge is what we recognise when we encounter it but do not use often. This can be compared with the use of languages: active and passive vocabulary. Transformation from passive to active knowledge in MM can be seen as an activation of the knowledge previously acquired by students during the modelling process in response to the use of IBL in the classroom. On the other hand, active learning describes the process of gaining knowledge based on learner's activity and agency; students acquire an important role of co-creators of new knowledge. We see the process as follows: students use active learning to obtain knowledge; activate it during MM (learn how to use it); and use active knowledge for theoretical and professional tasks on the regular basis. The connection between the ways of gaining knowledge and different areas of applying it within the active knowledge framework is shown in Figure 8.1; we use the concept of active knowledge to explain the passage from "how to gain knowledge" to "when to use it." Students use active knowledge both during the study process (solving realistic tasks with help of modelling, using modelling to develop mathematical concepts) and as professional tool (creation of mathematical theory, innovation activities, solving professional tasks).

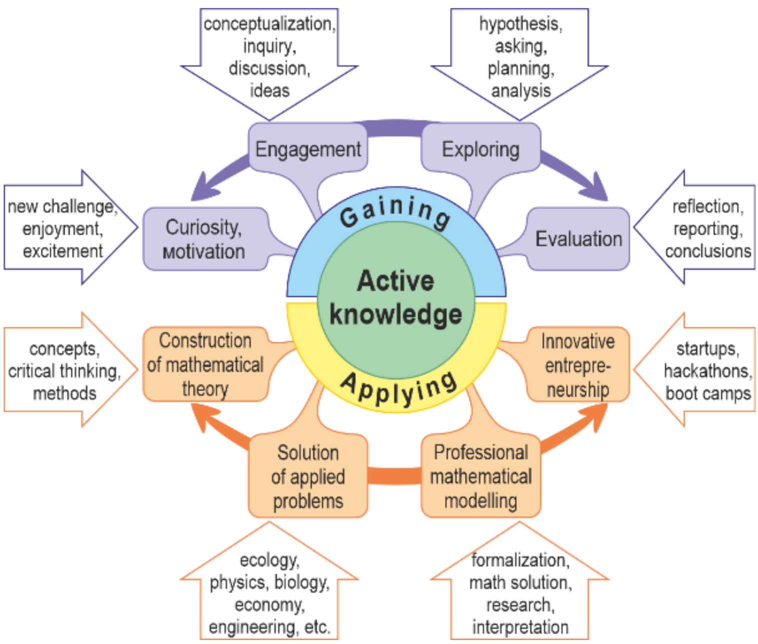

FIGURE 8.1. Main components of *active knowledge*.

We believe that teachers can promote inquiry within mathematical modelling as the form of active knowledge; this brings students closer to the atmosphere of mathematical discovery which is usually associated with the work of professional mathematicians. Students are encouraged to reflect about new material in an explorative manner, by asking and answering questions which help to understand mathematical concepts, the reasoning in the proofs, logical chains of arguments in the solutions of problems. When students understand that passiveness in the class does not help to learn mathematics efficiently and accept the challenge of being challenged, the routine work in the classroom turns into an exciting adventure into the Universe of mathematics. Ambiguity and confusion experienced on times by students should not discourage them as they lead to small but important discoveries and victories; uncertainty is the feeling often experienced by research mathematicians and mathematical modellers creating and applying new mathematics in their work. As Byers (2007, p. 78) pointed out, “ambiguity can be the doorway to understanding, the doorway to creativity.” By designing inquiry tasks, especially modelling problems, and including them in our teaching, we provide opportunities for students’ learning and engagement in critical reflection. On the other hand, mathematics teachers also engage in critical reflection during the design, testing, and refining tasks; critical reflection is even more requested when teaching-learning interactions occur in a context of genuine inquiry. Furthermore, existing students’ knowledge can be activated when teachers design tasks that link students’ learning to an authentic inquiry, as it is done in academic and industrial research. Finally, teachers also engage in critical reflection when they adjust their thinking by discussing their practices with other teachers and researchers, as was the case in PLATINUM. The three cycles of inquiry related to the active knowledge are illustrated in Figure 8.2.

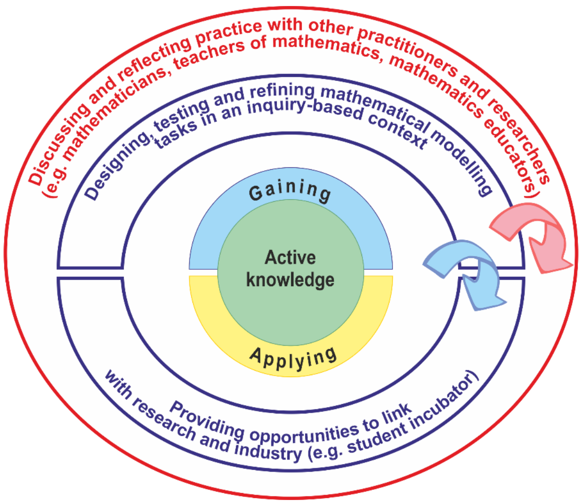

FIGURE 8.2. Active knowledge and mathematical modelling within the three-layer model of inquiry.

The idea of active knowledge is reflected in the examples from the three partner universities discussed in Section 8.4. First, we observe how Ukrainian mathematics undergraduates create models for practical problems which lead to a rigorous formal definition of a definite integral by using their previous experience of calculating areas of elementary geometric shapes and the knowledge of the additive property of area. Then we follow the process of the knowledge activation of Czech undergraduate students who relate calculations of the volume and surface area for a sphere and a cylinder in the final years of the high school, definition of the derivative of a function, its basic properties and optimisation applications from a calculus course, and real-life experience with the shapes of different beverage cans to design their own can that is optimal from both mathematical and practical standpoints. Finally, Norwegian master’s students activate the knowledge of physics laws, basic calculus, differential equations, and computational skills to verify the applicability of the existence and uniqueness theorem for a differential equation constructed to model a leaking bucket. In all three examples, active knowledge featured multifaceted connections between inquiry with mathematical modelling.

8.4. Examples From Three PLATINUM Partners

In this section, we describe partners' experience with the use of mathematical modelling in their teaching and explore how the inquiry was organized by the lecturers and perceived by students.

8.4.1. Borys Grinchenko Kyiv University. An important characteristic of mathematical competence and, at the same time, a necessary condition for the effective application of mathematics for solving applied problems is the proper mastery of mathematical concepts. Despite a large number of theoretical and empirical studies on modelling and related mathematical activities, not many examples promote the use of modelling as a teaching philosophy aimed, in the first place, at the formation of students' conceptual understanding. We agree with Gravemeijer (1999), who argued that formal mathematics should be created by students themselves, and believe that ideas of emergent modelling that encourage and stimulate the process of discovery (construction) of mathematical theory by students themselves are useful. This is especially true for advanced mathematics because many key mathematical notions and structures, like the concept of a limit of a function and related notions of derivative, integral or the method of mathematical induction can be presented initially to students as imperfect models introducing intuitive, non-rigorous ideas about mathematical concepts and structures, or even as metacognitive models for the process of thinking about them. Departing from authentic real or real-like situations originating usually in an extra-mathematical domain, students construct the initial, naive understanding of a new concept. The teacher's task is to offer such stimulating problems. Then, through the abstraction from the subject content of the specific problem, a mathematical model of a rigorous mathematical concept is created on the basis of the preliminary intuitive non-rigorous concept. During the shift from the real world back to abstraction, a rigorous, formal concept gains a new quality, it carries with it an imprint of reality becoming an efficient tool for its study, a "building block" for mathematical modelling. Therefore, mathematical models and mathematical concepts develop simultaneously, mutually stimulating each other's development. Thus, we can view the formation of the rigorous mathematical concepts and their consequent application as a cyclic triad presented in Figure 8.3 where our views on MM and concept formation align with the French tradition of Chevallard (1999), Brousseau (2002), and other authors (García & Ruiz, 2006; Dorier, 2006) who consider all mathematical activity including problem solving in pure mathematics as modelling.

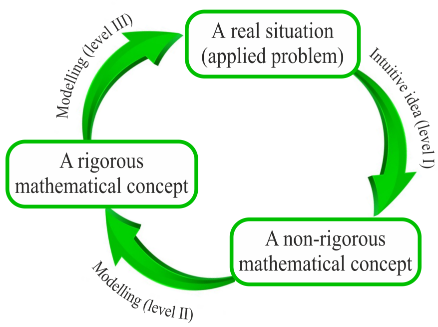

FIGURE 8.3. The triad of formation and application of a mathematical concept.

The cyclic triad presented in Figure 8.3 is implemented at BGKU throughout the entire study period. Depending on the year of study, discipline, and specialisation, teaching accents are shifted in accordance with the changes in learning objectives and the course content. The IBME approach can be used effectively at every level of this triad because its goal is not limited only to finding the right answer (which is often not available in modelling tasks), but to find a good approach to the solution. Modelling requires much more than a mere retrieval and use of the previously gained knowledge, it requires a deep understanding of concepts, facts, processes, methods, and an ability to apply them under new conditions working continuously on the border between the known and the unknown.

During the first years of study, students at BGKU go through fundamental mathematical disciplines where the modelling is efficiently used to assist in the formation of rigorous mathematical concepts and methods. Conceptual understanding of mathematics lays down good foundations for the study of advanced mathematical subjects, including those oriented at the applications and solution of practical problems coming from various branches of science, business, and engineering. At this stage, for educational purposes students are mainly trained on "toy" problems designed for the direct application of mathematical concepts, facts, and methods. The subject and complexity of applied problems, both real and pseudo-real, eventually increase following the developments in the study curriculum and relevant mathematics disciplines at the bachelor level. The list of applied and interdisciplinary disciplines includes such subjects as operations research and econometrics where the problems generally do not require the use of the full mathematical modelling cycle or interaction with specialists.

In accordance with the methodological model presented in Figure 8.3, teaching of "real" mathematical modelling for students majoring at BGKU in mathematics starts only at master's level within the programme "Mathematical Modelling." The purpose of the programme is to provide students with the solid training in mathematics emphasising the-state-of-the-art theories and methods that have wide applications in different fields of science and professional practice, including the basic methods of mathematical modelling. The study curriculum includes the in-depth study of the following disciplines: fundamentals of mathematical modelling, systems analysis, forecasting, applied functional analysis, dynamic systems, applied mathematical and computer modelling. Students must complete an undergraduate internship and write a master's thesis in which they develop a mathematical model for a particular field (economics, finance, computer science, physics).

One of the possibilities to apply theoretical knowledge in mathematics and mathematical modelling is a university business incubator (UBI) created to provide practice-oriented applied learning and to increase students' motivation for studying mathematics. The model of UBI uses existing successful practices in Poland, Israel, Norway, and Estonia; it reflects both the market needs and peculiarities of higher education. The UBI at BGKU provides a creative platform for the development of students' innovative projects in various fields of science including the development of mathematical models for the solution of practical problems in business and industry; its structural organisation is presented in Figure 8.4.

We present now an example illustrating the first level of the triad, namely the use of mathematical modelling for the formation of the concept of the Riemann integral in the Mathematical Analysis class for the first-year mathematics undergraduates at BGKU taught by Dr. Mariia Astafieva. The purpose of the lecture was to form the concept of a definite integral, make students understand what classes of applied problems can be solved by using definite integral and practice modelling skills. The lecture was conducted using an IBME approach, in particular, structured and guided inquiry. Interactive teaching methods used in the classroom included brainstorming, small

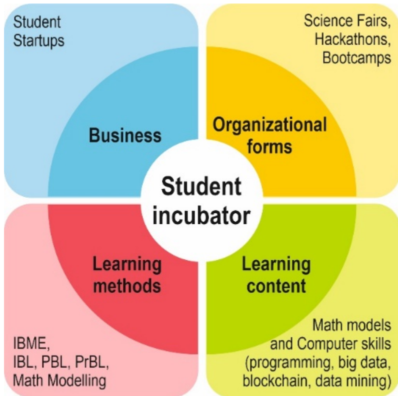

FIGURE 8.4. Organisation of the University Business Incubator at Borys Grinchenko Kyiv University.

group discussions of the problems followed by the presentation of the outcomes to the class, reflection, and the whole class discussion.

To involve students into cognitive and explorative activities, four applied tasks were given where students had to find (1) the area of a curvilinear trapezoid, (2) the mass of an inhomogeneous rod; (3) the distance travelled with a variable speed; and (4) the volume of the output at variable productivity. To stimulate the student activity, the work was initially organised in small groups (two groups of four students), each group received identical figures cut out of paper (Figure 8.5), scissors and rulers.

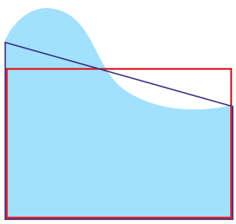

FIGURE 8.5. Replacing the figure: with a trapezoid or rectangle.

Both groups had to suggest how to find the area of the given figure and determine it in just one minute. After a one-minute discussion, the group's spokespersons presented the solution idea and suggested an approximate value for the area. Group N° 1 noticed that the figure looked like a rectangular trapezoid. Therefore, their proposal was to replace it with an ordinary trapezoid (i.e., to replace the curved "side" with a line segment, see Figure 8.5) and use the formula for the area of the trapezoid. Approximate value of the area: $(11 + 7) \div 2 \times 12 = 108\ \text{cm}^2$.

The idea of Group N° 2 was to cut off the "hump" with scissors and fill the "hole" with a piece that was cut off. That is, to replace the original figure with the rectangle (see Figure 8.5). This group had as many as four different answers ($108\ \text{cm}^2$, $102\ \text{cm}^2$, $99.6\ \text{cm}^2$, and $105.6\ \text{cm}^2$), because each student built "his/her own" rectangle.

To facilitate students' formation of the concept of a definite integral, the teacher encouraged them to think about the pros and cons of the proposed methods. In a teacher-led discussion, students also suggested how to improve the procedure for determining the approximate value of an area so that the calculation error is minimised. In particular, they easily concluded that the methods proposed by both teams were not very successful. Their only advantage is simplicity. The biggest drawback is that the obtained approximations are very rough. The productive idea of improving the procedure in order to achieve better accuracy of the result came up with more difficulty. However, the teacher was not in a hurry to readily suggest the idea using instead a sequence of questions that eventually prompted the required idea to cut the figure into vertical strips, which, like the whole figure, will be curvilinear trapezoids, and then to add the areas of all the strips found by the approximation method of any of the teams.

At the teacher's suggestion to analyse the solution, the student S1 stated that we did not solve the problem because our procedure gave an approximate but not the exact value of the area, required in the problem. The problem identified by the student is important in the context of constructing a definite integral. Thus, the student correctly recognised the mathematical essence of the problem.

The identified problem encouraged a new search for correct answer illustrated in the following excerpts (L=lecturer, S=student(s)).

Excerpt 1

L: Assume we cut the figure into one-centimetre-wide strips, calculated the approximate values of the area of each of them, added them up and got a value which (with a certain error!) approximates the area of the whole figure. Is it possible to reduce this error?
S: (chorus) Yes. It is necessary to cut the figure into narrower strips.
L: How wide? Half a centimetre? One millimetre?
S: (chorus) As narrow as possible.
S1,S2: The narrower the strips, the more accurately will the area be calculated.
L: Right. So, can we now draw a conclusion about the exact value of the area?
S1: This will be the limiting value! We already did that when we were looking for the value of the instantaneous velocity.

Describing in mathematical terms the sequence of actions required for the solution, that is, creating a mathematical model of the definite integral, students answered lecturer's questions by making reasonable assumptions and determining the required parameters, as illustrated in the following excerpt.

Excerpt 2

L: Is it important to divide the figure into the strips of equal width?
S2,S3: No, it is not. It is important that the strips are narrow. But when they are of the same width, it is easier to calculate the area.
L: Shall we choose trapezoidal strips, as suggested by group N^o 1, or rectangular, as suggested by group N^o 2?
S1,S4,S6: It also doesn't matter because the strips are very narrow. But it is better to choose rectangular form because the area of the rectangle is easier to calculate.
L: So, we will assume each narrow strip to be rectangular. And what is its height?
S5: Well, let's take approximately about half of the measurement between the points on a "hump" and in the "hole."
S1: We can actually measure this height anywhere.
L: I also believe that there is no need to "aim" at any specific point between the "hump" and the "hole". Since the width of the strips goes to zero, any perpendicular to the base of the strip can be considered the height of the rectangle. Do you agree?
(...)
L: Finally, when we write down the expression for the area of a "stepped" figure made of n rectangles, it is necessary to pass to the limit. What limit do we need?
S2,S7: When the number of strips n goes to infinity.
L: Does everyone agree?

Pause. Nobody replies.

L: Why do we want to increase the number of stripes indefinitely? What is the purpose?

S3,S7,S8: To make the strips narrower.

L: Okay. And by increasing their number will we achieve this goal?

S1: No! Look (*demonstrates, cutting a curved trapezoid into two strips of approximately the same width, and then continuing to cut in half only one of the two parts*), the number of strips increases, but one strip remains wide.

S7: Okay. But if you cut into strips of the same width, then with an unlimited increase in the number of strips, their width will approach zero. So maybe it's better to divide into strips of the same width?

L: No need. I think there is a way out of this situation. We will require that if the width of the widest strip approaches zero, so do the widths of all remaining strips.

The lecturer projected Figure 8.6 on the screen, introduced necessary mathematical notation ($\Delta x_k = x_k - x_{k-1}, \xi_k \in [x_{k-1}, x_k], k = 1, 2, \ldots, n$) and asked student $S8$ to recall all solution steps writing them down on the blackboard up to the final result:

$$\lim_{\lambda \to 0} \sum_{k=1}^{n} f(\xi_k)\Delta x_k, \quad \text{where } \lambda = \max_k\{\Delta x_k\}. \tag{8.1}$$

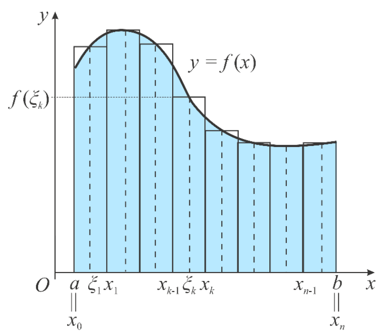

FIGURE 8.6. Replacement of a curvilinear trapezoid with a stepped figure.

After that students worked all together finding the mass of a heterogeneous rod and then, in small groups, they found the distance travelled by a particle at variable speed and production volume presenting group solutions to the class. Comparing solutions to all four problems, the students concluded that the mathematical model for these problems is based on the use of the limit as in Formula 8.1, where the function f defines, respectively, the equation of a curve that limits the curvilinear trapezoid, linear rod density, variable speed, and variable productivity; the same sequence of actions leads to this mathematical model. It is worth mentioning one important observation made by students: in all problems the required quantity A that they calculated (area, mass, distance, volume of manufactured products), possesses the additive property:

$$A(\phi_1 + \cdots + \phi_n) = A(\phi_1) + \cdots + A(\phi_n).$$

Students actively assisted the lecturer who introduced on the board the concept of a definite integral of a function $f(x)$ from a to b and explained the relevant notation. Then they revisited all four problems, wrote solutions as definite integrals, and discussed the context in which the definite integrals were used: geometric (area), physical (mass), mechanical (distance travelled), economic (production volume). In the end of the class, students were asked for the feedback about the lecture. The lecture and students' feedback were discussed by the community of inquiry at BGKU. In particular, the team members focused their attention on lecturer's strategies and actions taken to achieve the educational goals set for the lecture and reflected whether these goals were achieved, and if not, then why (see Table 8.1).

Goals set by the lecturer	*Strategy/Action to achieve the goals*
Creating positive motivation	Solution of applied problems in various fields
Involvement of students in mathematics exploration	Use of IBME approach, in particular, structured inquiry and guided inquiry. Students independently hypothesised, discussed ideas, conducted their own explorations, and drew conclusions. The lecturer encouraged students to be critical of these ideas, to focus on solving the problem. In fact, in students' work, all stages of the 5E-model of inquiry were implemented (Bybee et al., 2006). Interactive teaching methods: brainstorming was effectively used to mobilize students to find productive ideas, discussion of the problem in small groups, followed by presentation of the results, group discussions.
Intuitive understanding of the concept of a definite integral through the sequence of steps needed for its construction	Inductive approach and use of informal considerations. The lecturer did not suggest the action plan to students, they found it themselves. The lecturer set aside time for students working in small groups to discuss ideas and find solutions.
Conceptual understanding	The lecturer did not rush and allowed enough time for discussions (setting the time limit of only one minute in the beginning was an element of the game that created excitement and the atmosphere of competition). She followed closely discussions in the groups joining them unobtrusively when needed becoming a peer participant (see, for example, Excerpt 2). The lecturer supported students during the discussions by providing constructive feedback. She did not answer the questions directly encouraging students to look for answers on their own. The lecturer patiently led students from real situations to abstraction and construction of a formal, rigorous definition.
Mastering mathematical language	Using symbolic representation of the problems
Mathematical competence	Using the existing knowledge in a new situation (solving four different applied tasks)
Formation of active, independent, creative thinking	IBME supported the curiosity of students and their desire to be independent in the search for new knowledge; it stimulated students to ask questions and look for answers, which ultimately contributed to the improvement of students' knowledge, their ability to apply it in new situations, and an overall development of students' advanced mathematical thinking.

TABLE 8.1. The discussion of the lecture by the members of the BGKU community of inquiry.

The feedback from students signalled that the goals set by the lecturer were achieved. It should be noted that students worked actively and with visible pleasure throughout the lecture. This is what they were saying about the class:

> We studied the definite integral at school. And, perhaps, the teacher told something similar there. Maybe I forgot. But I only remember from school how to calculate, for example, $\int_1^3 x^2\,dx$ and in today's lecture, I could not even think of a definite integral as a limit. (...) I also do not know where the formula for the calculation of the integral came from (the one called Newton-Leibnitz formula). Most likely, the teacher just wrote it down on the board and showed how to use it, but I remember it. But now I am sure that in the following lectures we will find this out too. The teacher never tells us to just remember, but tries to make us understand, and we often deduce this or that formula ourselves, notice some fact, pattern. And I really like this kind of training.

> From school I only remember how the area of a curvilinear trapezoid is found with a definite integral. But only during today's lecture I understood why this is so and how do we come to this. (...) Now I understand that in the same way as we were looking for the area of a curved trapezoid, the mass of an inhomogeneous rod, the volume of production, the path travelled at a variable speed, we can find other quantities. For example, the length of a curve. It is necessary to break it into parts and replace each part with a line segment. And then add the lengths of all the segments and look for a limit. And it will be some kind of integral.

> It's hard for me to remember the definitions of concepts that are studied in mathematical analysis, such as limits. But the lecturer always introduces us to such concepts on concrete examples. If it weren't the specific examples, I would have never remembered those "epsilon-deltas." The same applies to the abstract and complex definition of a definite integral. But after I cut the figure into narrower and narrower strips and looked for the approximate value of the area, and then we attempted to find an exact one, we obviously needed to pass to the limit when the width of the strips goes to zero and then I understood how to construct a definite integral and what does it mean. Now I think I can formulate the definition correctly and remember it.

The lecturer also commented on the chosen strategy and the effectiveness of lecture:

> During the lecture a new concept was introduced. The most important task was to form its conceptual understanding because it creates the basis for active knowledge, lays the foundation for the conscious construction of its generalisations (multiple, curvilinear and surface integrals). The definite integral, like many other concepts of mathematical analysis, is not a model of certain objects but rather a model for the ways to study objects and understand the ideas that underlie their construction. That's why I chose the IBME strategy: I tried to organise the classes so that students acted like professional mathematicians building their understanding of mathematical concepts (in this case, the concept of a definite integral). In particular, they asked questions, hypothesised, discussed and substantiated their own thinking, evaluated the thinking of other students in a constructive and supportive environment. The creation of an adequate mathematical model organically combines all the key areas and forms of mathematical activity, launches all the necessary psychological mechanisms in their interaction. The four "real" situations proposed for modelling were presented as inquiry-tasks. Clearly, in a week or so none of mentioned tasks would have any inquiry potential for these students. The tasks would become simple routines for training the skills in application and calculation of the integral.
>
> As for the efficiency of the lecture itself, it is difficult to analyse it right away as well as the overall achievement of the goals. In particular, it takes time to assess students' conceptual understanding of the notion of a definite integral, how well have they formed this concept. Clearly, it takes much longer than just one lecture. Since the conceptual knowledge becomes an instrument for the subsequent cognitive and practical work, it should not only be acquired by own efforts and correspond to person's natural curiosity,

it must be well organised, reliably and optimally placed in the long-term memory, and be ready for further use. For this purpose, the initial perception is not sufficient, one needs a long practice of the knowledge application in different contexts.

There was an episode in the lecture when due to the nature of inquiry and questions posed it was possible to reveal a gap in students' understanding of the concept that they studied earlier prompting that this concept did not develop into a component of active knowledge. To find the area of a curvilinear trapezoid, Group N° 2 proposed to replace it with a rectangle (Figure 8.5). This prompted me to ask students impromptu an inquiry question: "Is there a rectangle whose area is exactly (and not approximately!) equal to the area of a curvilinear trapezoid?" It was suggested as a homework. And guess what turned out? The next day, 4 students (out of 8) said that they think (intuitively feel) that such a rectangle exists, but do not know how to prove it. Three students answered the question in negative because "it is impossible to know exactly what the height of the rectangle should be." And only one student gave the affirmative answer and substantiated it by "the property of a continuous function (in our case, it was the area) to take on all intermediate values (Bolzano-Cauchy Theorem)."

So, most of the students didn't notice that the area was a continuous function, or couldn't apply the property of a continuous function in different unexpected to them context. The reason, obviously, is that the continuity of the function was studied a long time ago, in the beginning of the first semester, and the students did not activate the relevant knowledge due to its insufficiently frequent use or this knowledge was not properly stored in the long-term memory and didn't become active.

This example confirms once again the following:

- An indicator for the correct formation of active knowledge is the ability to apply it for solving problems of practical and exploratory nature.
- The use of the IBME and its application in problems from various disciplines, including mathematical modelling, are expedient for the formation of conceptual understanding of mathematics and active knowledge.

8.4.2. Brno University of Technology. For partners at Brno University of Technology (BUT), MM means activities motivated by applications of mathematics to real life problems. We believe that MM should contain at least the formulation of a concrete problem, development of an abstract model, and mathematical work with the model (not only simulations, that is part of an engineer's job!). As mentioned in Section 8.2.1, we believe that the 'mathematical work with the model' part can also include pure mathematics activities (e.g., 'playing' with model components and studying consequences) that are not motivated by applications. Being mathematicians, we practise such 'playing with the model per se' that we see as a 'pure math' component of MM. Therefore, we do not see MM as a 'proper subset' of applied mathematics.

We acknowledge that teaching MM at a technical university like BUT, where the faculty in the Mathematics Department mostly offers service courses, represents certain challenge. Two major reasons for this are: (1) elevated number of topics 'packed' into a few semesters of mathematics leaves little space for MM activities; (2) modelling and work with the models are key topics in other specialised engineering subjects. We believe that including more mathematical activities (e.g., analysis) into engineering activities would provide students with a better insight into the modelled problem and consequently contribute to a more efficient learning.

Taking a closer look at the concepts of IBME and MM, we realise that these approaches are similar in many aspects and use similar means to achieve different goals. It is not easy for us to unravel and analyse where an individual aspect belongs and what it contributes to, IBME or MM. However, we feel that teaching MM without inquiry is just a dry transfer of knowledge regardless of the needs of students and learning MM without inquiry is unproductive and unnecessarily difficult. Taking into account the three-layer model of inquiry and MM described in Section 8.3 (see also Chapter 2) and the challenges we see in teaching MM introduced in the preceding paragraph, we realise that educational practices at BUT can, with some restrictions, accommodate inquiry in the first two layers (inquiry into students' learning mathematics and inquiry into teaching mathematics).

Motivated by

- the positive experience of our colleagues at BGKU with MM as an educational approach,
- our own experience with IBME within the PLATINUM project,
- the revealed proximity of the two approaches, and
- the possibility of achieving better long-term learning outcomes,

we decided to incorporate a MM activity into the dense teaching schedule. Here we exemplify our experience with a task that we believe to fall within the intersection of IBME and MM.

The task has been tested in three lessons with groups of 6–12 first-year engineering students, mostly male. Due to the COVID-19 outbreak, the teaching had to be realised in a virtual environment, a combination of an MS Teams online meeting and a shared virtual space substituting the whiteboard. The students were supposed to work in groups in the virtual environment. The formulation of the task was brief: *Given the volume of 0.5 litre of a liquid, minimise the material needed to make a can that would contain the liquid.*

This task has been given to the students in a Calculus I course after they learned the necessary prerequisites: derivatives, applications of the derivative to analyse a graph of a function and to search for local and global extrema. The students were familiar with the examples of applications leading to finding local extrema.

During the lessons, we were able to identify the following elements of the four-step MM cycle:

1. Understanding task. In this introductory phase, an informal formulation of the task has been made more precise. The students reflected about possible shapes of the can and investigated real samples of half-litre cans.

2. Establishing the model. All groups succeeded in turning the task formulation into a mathematical description. Students searched for expressions for the surface area and the volume of a cylinder and a sphere and expressed them as functions dependent on one unknown variable. They recognised that the function defining the surface area must be minimised.

3. Using mathematics. In each group, there were students who suggested to solve the optimisation task using derivatives. All students agreed to that and participated in the solution process. The fact that the minimum value of the function which defines the surface area is achieved at the stationary point has only been commented on, but not verified mathematically.

4. Explaining the model. In the final phase, the students compared results for cylindrical and spherical cans and observed that, as expected, the spherical shape has a smaller surface area. However, they also reasoned that making a spherical can would be technically more complicated and thus more expensive and, last but not least, a spherical can would be difficult to drink from. The relation between the optimal radius and height has been discussed, as well as possible modifications of the task.

We were also able to identify the following phases of the inquiry process and activities:

1. Orientation. In the beginning, students discussed possible can shapes and investigated real samples.

2. Conceptualisation. The students had to figure out that they need to know how to calculate the volume of a solid and its surface area. They identified the problem as an optimisation problem, and that the derivative might be a convenient tool.
3. Investigation. Students searched on the internet for formulas for the volume and the surface area of a cylinder and a sphere, and they measured the dimensions of real cans for comparison. They predicted that the sphere would have minimal surface area. They investigated functions that define the surface area for extrema and claimed that the extremum that has been found is the minimum.
4. Conclusion. The students identified that the sphere would have the minimal surface. However, they used reasoning to conclude that cylindrical shape is more practical for a real-life application.
5. Discussion. At the end of the lesson, students were asked for reflection and feedback on the lesson. They also discussed possible generalisations of the results they obtained to volumes other than 0.5 litre and extensions of the problem to more realistic conditions, e.g., when some free space inside the can is needed to open it without spilling the content.

All three groups came to a solution within the given time. However, in all lessons the student teams made some mistakes. The teacher did not warn the students about that and let them find out themselves. This approach has been valued positively by the students in the feedback. One of the students commented:

> Well, I take it positively. I liked the moment when we calculated a wrong derivative and you (the teacher) let us continue and we made further calculations with that, until we got to the point when it was clear that something is wrong. For example, when I work on a test and make such arithmetic mistake, I get to the point where we were not taught what to do because what we are doing is wrong. Nobody taught us how to do things wrongly, so it is difficult to get back and start again. Here (in the lesson) we could get back and that was good.

Some teacher's reflections after the activity:

- prior to the first lesson, there was some anxiety about the teamwork in a virtual environment;
- the choice of an appropriate shared virtual space (whiteboard) was a challenge, but the work in MICROSOFT ONENOTE was satisfactory;
- all three lessons went well and brought a new experience both to students and the teacher;
- the main disadvantage of the activity was the excessive time demand/consumption caused by the virtual environment;
- it was good to acknowledge that students use the knowledge from other subjects that may have a positive effect on the development of ideas and reasoning;
- it would be interesting to try the activity in a physical classroom setting and compare the outcomes.

In conclusion, note that we perceive the inclusion of the problem of mathematical modelling with elements supporting interest in the standard course of mathematics as fruitful and meaningful. According to our experience, the students were made to think out of the boxes, which helped them not only in understanding the topic but also in improving their problem-solving skills.

8.4.3. University of Agder. At the University of Agder (UiA), there are no dedicated courses in which mathematical modelling (MM) is taught; but the study curriculum assumes that mathematics and engineering students gain some knowledge of MM and methods of applied mathematics. Under these circumstances, efforts are made to include modelling tasks in traditional mathematics courses in the form of small group projects or individual assessments. Calculus, Linear Algebra and Differential Equations courses at UiA are particularly suited for this purpose. For illustration, we discuss an example of a modelling task introduced in a Differential Equations course for the fourth-year engineering students who neither had taken dedicated modelling courses nor had previous modelling experience, but all had some basic knowledge of Calculus and Linear Algebra. The complete analysis of this activity and students' work from the commognitive perspective (Sfard, 2008) has been reported in Rogovchenko (2021), where further details may be found.

The assessment involving multiple modelling tasks was suggested to students in line with the process of modification of the course traditionally taught to master's students in Mechatronics at UiA. The course upgrade was motivated by the lecturer's intention to connect better the knowledge students gained in physics and mathematics courses by introducing several MM assignments rooted in engineering or physical applications in the form of graded course projects. The lecturer set several pedagogical goals: to enrich students' mathematical narratives about the nature of differential equations, promote students' advanced mathematical thinking and the use of mathematical language, contribute to the development of general modelling routines, explain how known mathematical ideas and procedures can be combined and employed to generate new ones, and motivate an explorative approach to MM as a particular problem-solving strategy. In addition, organisation of students' work in small groups introduces important elements of collaborative learning to the classroom and enhances students' social skills.

Forty students in the course (38 males and 2 females, all in their twenties) worked in small groups of two to three students on different sets of modelling problems for one week, discussed their solutions to problems and produced individual written reports. The selection of MM tasks was primarily linked to the subject area of engineering studies, mechatronics, and the complexity level of the problems ranged from closed to open-end problems. Students were asked to employ mathematical methods for finding solutions and use mathematical software of their choice, MAPLE or MATLAB, to support their work. In addition, the analysis of the validity of the mathematical model regarding its correspondence to the real-world experience or data was required. Students audio-recorded group discussions themselves in the absence of the lecturer and provided recordings at her request for research purposes. Afterwards, group solutions to various problems were presented by each group in a whole-class session. Students' individual written reports were graded as a part of the course work; the mark counted towards the final grade.

A MM task presented below has been designed for the topic Existence and Uniqueness Theorems (EUTs) for initial-value problems for differential equations. Contrary to traditional teaching practices requiring merely formal verification of the conditions of the theorems, students engaged this time with EUTs through modelling problems. Another interesting experience with the use of nonstandard problems for testing students conceptual understanding of EUTs is described in the conference paper by Treffert-Thomas et al. (2018).

Problem. Consider a cylindrical bucket of constant cross-sectional area A with a hole of cross-sectional area a at the bottom of the bucket. The small hole is plugged, and the bucket is filled to height h_0. A clock is started as the plug is removed and the water begins to leak out of the hole. Construct a DE model to determine the height $h(t)$ (m) with respect to time t (s). Take $g = 10$ m/s^2. Choose your values for A and a so that the ratio $A/a = \sqrt{5}$.

(*a*) Explain all your steps while setting the model.
(*b*) Take $t_0 = 0$, $h_0 = 4$, set the IVP, explain its physical meaning.
(*c*) Solve the IVP and observe that the solution is defined for all t but after some time it is no longer a realistic description of the height. What physical event occurs at this moment?
(*d*) Build a realistic continuous solution to this problem and show that the solution is valid for all times t. Is this solution continuously differentiable?
(*e*) Do these results contradict the Existence and Uniqueness Theorem? Explain your reasoning in detail.
(*f*) Plot the solutions found in subsquestons (c) and (d), and analyse the graphs.

Students suggested several physical descriptions of the problem, some of which are illustrated below in Excerpt 3, and discussed corresponding mathematical models. Many students used diagrams to illustrate the translation from verbal description to mathematical formulation and combined in the process of solution several familiar routines. They set up and solved an IVP; this required several steps including the integration of a differential equation, identification of the general solution, and application of initial conditions for finding a particular solution. The analysis of the mathematical discourse presented in students' written solutions suggests that they have developed the ability "to express things in the language of mathematics" (Schoenfeld, 1992, p. 337). In Excerpt 3 from the discussion in Group 1 one can clearly witness the "repetitiveness, and thus patterns which is the source of communicational effectiveness" (Sfard, 2008, p. 195). It was important for this group to agree on the common solution method and to "indorse" the narrative, but not all groups came to an agreement in the end; in such cases, students presented their individual, different versions of solutions.

Excerpt 3

S11: I used the Bernoulli equation to set the differential equation.
S12: I did something similar, but I started from the Conservation of Energy Law to find the velocity out...
S13: I also used the energy law, and worked with some constants and found a nice equation...
S11: Yes, we can use different values for constants, but I chose to have the simplest...

Students in Group 2 used physics laws to derive the following initial value problem:

$$\frac{dh}{dt} = -2\sqrt{h}, \quad h(0) = 4.$$

Formal integration yields the exact solution $h(t) = (2 - t)^2$ to the IVP which is obviously valid for all times but due to the problem setting should be considered only on the interval $[0, 2]$ until the bucket empties completely. From the instant $t = 2$ and further on, the bucket is empty and the second 'piece' of the solution to the problem on the half-axis $[2, \infty)$, $h(t) = 0$, can be obtained only by the reasoning in context. Conditions of the Existence and Uniqueness Theorem are not satisfied when the height $h = 0$; this occurs at the instant $t = 2$ and yields multiplicity of solutions to the given initial value problem. During the discussion about 'realistic' solutions, students also pointed out the existence of a formal mathematical solution for 'negative times' obtained by reversing time and argued about the meaning of solution in physical terms ("the water level will approach infinity"). Students' explorations were facilitated by the use of a computer algebra system (CAS); the relevant discussion illustrated in the following Excerpt 4 demonstrates the explorative character of the mathematical discourse.

Excerpt 4

S21: The solution we got is a parabola. (...) After $t = 2$ the solution is no longer realistic. What physical event occurs at this moment? What occurs is that the tank becomes empty and then some sort of refilling starts to happen at the tank, which would not obviously happen with the real tank... It would be very practical for my car (laughter), but unfortunately this is not the case.

S22: We agree that the solution is not realistic after this point, like the tank starts to magically fill in again (laughter).

S21: The way I started to solve this is to make the solution a piecewise function and say that it follows the original solution up to the point when the tank is empty and the second part of the piecewise function is zero for all values of t larger or equal than 2.

S23: Yes, we can use different values for constants, but I chose to have the simplest... I also tried to fit the exponential function, like saying it is linearly independent, but it did not fit very well so I ended up splitting the function.

The attempt of student $S23$ to fit an exponential function can be interpreted as an explorative routine in the use of CAS. Other students in this group solved the differential equation and plotted graphs manually; surprisingly, students did not solve the differential equation analytically with CAS although they already knew how to do this from other assignments. In Figure 8.7(a), the student plotted the formal solution to the problem and then plotted a piecewise defined function corresponding to the 'realistic' solution arguing: "As our solution is a parabola, the reasonable thing to suspect is that after the level has decreased to its bottom value, it will start increasing again. As we plot the graph, we can see that at $t = 2$ the container is empty, and mathematically it starts filling again. So, the solution stops being a realistic description of the height after the container becomes empty." For the 'realistic' solution in Figure 8.7(b), the student defined a function $h(t)$ by two expressions, $(2-t)^2$ for $0 \leq t \leq 2$ and 0 for $t \geq 2$.

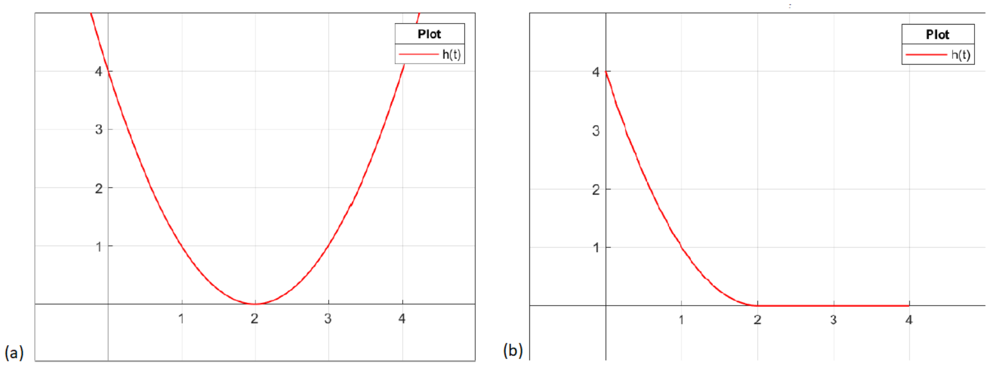

FIGURE 8.7. (a) Student's graph of a formal solution; (b) Student's graph of the 'realistic' solution.

Not surprisingly, in the development of a theoretical mathematical discourse, engineering students felt less confident with the use of rigorous mathematical concepts of continuity and differentiability and often relied on geometric arguments in explanations as illustrated, for instance, in Figure 8.8.

From out first part of the piece-wise function:

$$\frac{dh(t)}{dt} = -2 * (2 - t), \qquad t \in [0, 2\rangle$$

$$\lim_{t \to 2} \frac{dh(t)}{dt} = 0.$$

This gives a line that starts at (0, -4) and ends in (2, 0). Differentiating our second definition gives us 0 for all t and is also continuous. The differentiated functions meet at the point (2, 0). We can therefore say that the function is continuously differentiable and of class C^1. Which is enough for our problem.

FIGURE 8.8. Student's reasoning regarding continuity and differentiability of the solution.

The use of mathematical language in this fragment can be described as immature and the explanation provided by the students sounds rather intuitive, but computer simulation helped to develop the mathematical discourse and supported it. In fact, similar sets of graphs are present in the student's report twice: in the explanation of the solution as shown in Figure 8.8 and in the answer to subquestion (f) of the problem where the explanations to the graphs were explicitly requested.

The analysis of written reports shows that students relied on different representations (realisations) of the modelling task: mathematical description with the help of a diagram (visual), mathematisation using an appropriate differential equation (symbolic), graph plotting (visual), solution of the differential equation with the help of the CAS (symbolic). Students' written reports document striking differences in their ability to use CASs and demonstrate that technology was mainly used as a computational and verification tool and, to some extent, as a helpful visualization tool, but, similarly to the findings of Doerr and Zangor (2000), CASs did not become a transformational tool, nor a data collection and analysis tool.

8.5. Conclusions

Theoretical and empirical research indicates that students' success with MM tasks requires "a well-developed repertoire of cognitive and metacognitive strategies as well as a rich store of mathematical concepts, facts, procedures, and experiences; vicarious general encyclopedic knowledge of the world and word meanings; and truly experiential knowledge from personal experiences outside school or in more practical school subjects" (Stillman, 2015, p. 796). Much of this description is included in some form into the concept of active knowledge introduced in this chapter to relate MM to IBME. MM is not easy to teach and one of the main difficulties is the dependence of learning on the specific context; this requires that MM has to be learnt specifically (Blum & Borromeo Ferri, 2009). None of the three examples presented by PLATINUM partners were related to teaching MM per se but MM was embedded into different contexts. The first example of discovery of the definition of the definite integral with a modelling approach is close to the educational perspective of Realistic Mathematics Education (RME) whose fundamental principle is guided reinvention (Freudenthal, 1991). As pointed by Artigue and Blomhøj (2013), "in RME, modelling, and especially mathematisation, plays an essential role as a vehicle for the conceptual knowledge aimed at with no clear distinction being made between mathematisation of extra-mathematical situations and mathematisation within mathematics" (p. 805). This describes the approach used by the colleagues at BGKU to promote conceptual understanding of the definite integral through modelling tasks. On the other hand, the PLATINUM partners at BUT and UiA fostered students' understanding of applications of mathematics in Calculus and Differential Equations courses through modelling tasks which in both cases required active knowledge to relate mathematics to not very complex but realistic daily situations. All three examples confirmed recent research findings that "working with modelling in mathematics and in other subjects can thereby lead to valuable understanding of inquiry as a more general process with different particular realisations in different disciplines and contexts" (Artigue & Blomhøj, 2013, p. 805).

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