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Parameter Identification of Fractional-Order LTI Systems using Modulating Functions with Memory Reduction

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***Abstract*— The parameter estimation problem of a linear time-invariant fractional-order system is investigated by means of the modulating function method. Based on the assumption of known model structure and derivative orders, the modulating function method can be generalized to the fractional-order case in three different ways. We show that two approaches are identical for linear systems. This facilitates the computation of the fractional-order derivatives of modulating functions. In comparison to integer-order systems we have to include the initialization of the fractional-order system. We show that the spline type modulating function is capable of reducing the effect of the memory on the parameter estimation. However, it is not possible to compensate the memory initialization completely. In contrast to these tuning principles also the robustness against measurement noise must be considered. For this purpose we decouple the memory and noise compensation. The adjusted spline-type modulating functions reduce the initialization effect and the recursive least squares estimation provides the possibility to increase the numbers of equations such that the effect of the noise is reduced.**

I. Introduction

Recently the concept of non-integer order models, with so-called fractional-order derivatives, has gained increasing attention in fields like material science [1] or electro-chemistry [2]. As the fractional-order operators are non-local, these models include a long-term memory which can be exploited, for example, to model viscoelastic behavior [1], [3], [4].

Since the seventies well-established algorithms to estimate the parameters of integer-order systems have been extended to the fractional-order domain, e.g. the subspace method [5], [6], the instrumental variable method [7], frequency domain algorithms [8], [9] and the modulating function approach [10], [11], [12], [13], [14]. Some of these algorithms have already been applied to non-academic problems, like the parameter estimation of a lithium-ion battery [14].

Compared to integer-order systems, the identification of fractional-order systems is more challenging, as it shows at least two additional problems. First of all the orders of differentiation have to be considered as unknown model parameters, when compared to the integer-order case where only the model order (model structure) has to be determined. For integer-order systems this can be determined by applying the subspace method [15] for example. So far all generalizations of identification algorithms consider either the a priori knowledge of the differentiation orders or estimate it separately via nonlinear and iterative optimization techniques [16]. The second challenge is less grave, but has a huge impact in view of the online estimation of parameters in the time domain. The fractional-order operators contain memory leading to additional terms in the estimation equations. This also effects identification methods based in the frequency domain, as the frequency domain data is usually obtained from time domain data. Only if the systems is at rest at the beginning of an identification cycle these terms disappear. This is undesirable as it increases the time demand for the identification. Especially since the memory in fractional-order systems decays rather slowly, the system has to be at rest for hours to reduce its effect sufficiently.

Hence, the memory has to be handled by the identification algorithm. This has been investigated in detail in [13] and we are going to revisit parts of this contribution and extend it by including noise to the observation applying the ideas presented in [12]. The main focus of this paper lies on the estimation of the linear parameters of an initialized fractional-order system (assuming known orders of differentiation) from noisy input-output data.

The paper is organized as follows. The second section recalls the basic fractional-order derivatives of different types, including the left- and right-side approaches. In addition to that, the initialization functions are revisited to motivate the compensation of the memory within the estimation procedure. Section III revisits three different approaches of how the modulating function method can be extended to the fractional-order case. We show that two of these approaches are identical, which is one contribution of this paper. We derive the right-side Caputo's derivative for the spline-type modulating function and show that it can be used to reduce the memory effect in the estimation, although a complete compensation is not possible. We provide tuning approaches to find a suitable balance between the cancellation of the memory effect and robustness against measurement noise. Finally, we set up the parameter estimator with a single modulating function applied on an increasing time interval and use the recursive least squares algorithm to compute the system parameters. This approach enables us to split the identification task into two separate stages. The modulating function method is applied to reduce the memory effect, whereas the recursive least squares algorithm can reduce the measurement noise. These results are finally illustrated with a simulation example in Section IV.

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II. Preliminary Results

A. Operator Definitions

Within this work we will use a collection of fractional-order operators to reformulate the estimation problem by means of the modulating function method. The basis of all these operators is the fractional-order Riemann integral for the function $f(\cdot):[t_0,t_f]\to\mathbb{R}$ given by [17], [18]

$$ {}_{t_0}\mathcal{I}_t^{\alpha} f(t)=\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}f(\tau)\mathrm{d}\tau,\quad t\in[t_0,t_f], \tag{1}$$

where $\Gamma(\cdot)$ is Euler's gamma function. In this convolution integral, the time instant t is approached from the left side. The right-side counterpart is given by

$$ {}_{t}\mathcal{I}_{t_f}^{\alpha} f(t)=\frac{1}{\Gamma(\alpha)}\int_{t}^{t_f}(\tau-t)^{\alpha-1}f(\tau)\mathrm{d}\tau,\quad t\in[t_0,t_f]. \tag{2}$$

Using Fubini's theorem one can show how these operators change under integration [19]:

$$\int_{t_0}^{t_f}\varphi(\tau)\,{}_{t_0}\mathcal{I}_\tau^{\alpha}f(\tau)\mathrm{d}\tau=\int_{t_0}^{t_f}f(\tau)\,{}_{\tau}\mathcal{I}_{t_f}^{\alpha}\varphi(\tau)\mathrm{d}\tau. \tag{3}$$

Fractional-order derivatives are defined in combination with integer-order derivatives. First of all we have the left-sided Riemann-Liouville derivative

$$ {}_{t_0}^{\mathrm{R}}\mathcal{D}_t^{\alpha}f(t)=\frac{\mathrm{d}^m}{\mathrm{d}t^m}\left(\frac{1}{\Gamma(m-\alpha)}\int_{t_0}^{t}\frac{f(\tau)}{(t-\tau)^{\alpha-m+1}}\mathrm{d}\tau\right) \tag{4}$$

and the left-sided Caputo's fractional derivative [17]

$$ {}_{t_0}^{\mathrm{C}}\mathcal{D}_t^{\alpha}f(t)=\frac{1}{\Gamma(m-\alpha)}\int_{t_0}^{t}\frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}}\mathrm{d}\tau \tag{5}$$

with $m-1<\alpha<m$. Note that these operators are connected by the corresponding initial conditions [17] as per

$$ {}_{t_0}^{\mathrm{R}}\mathcal{D}_t^{\alpha}f(t)={}_{t_0}^{\mathrm{C}}\mathcal{D}_t^{\alpha}f(t)+\sum_{k=0}^{m-1}\frac{(t-t_0)^{k-\alpha}}{\Gamma(k-\alpha+1)}f^{(k)}(t_0^{+}). \tag{6}$$

This can be shown by differentiating the Riemann definition and applying integration by parts to Caputo's derivative. This connection can be exploited to adopt the system description to the identification procedure, as shown in [12].

The corresponding right-side derivatives are given by

$$ {}_{t}^{\mathrm{R}}\mathcal{D}_{t_f}^{\alpha}f(t)=\frac{(-1)^m}{\Gamma(m-\alpha)}\frac{\mathrm{d}^m}{\mathrm{d}t^m}\left(\int_{t}^{t_f}\frac{f(\tau)}{(\tau-t)^{\alpha-m+1}}\mathrm{d}\tau\right), \tag{7}$$

$$ {}_{t}^{\mathrm{C}}\mathcal{D}_{t_f}^{\alpha}f(t)=\frac{(-1)^m}{\Gamma(m-\alpha)}\int_{t}^{t_f}\frac{f^{(m)}(\tau)}{(\tau-t)^{\alpha-m+1}}\mathrm{d}\tau. \tag{8}$$

B. Initialized Fractional-Order Calculus

So far we have only given the basic definition of fractional-order operators. They are built on the fractional-order integration, hence, these operators are non-local. To take the past of a function into account, Lorenzo and Hartley [20] proposed the use the so-called initialization functions $\Psi(\cdot)$ to allow for history in the past, that is $f(t)\neq 0$ for $t\in[t_p,t_0]$

$$ {}_{t_0}^{\mathrm{R}}\mathcal{D}_t^{\alpha}f(t)={}_{a}^{\mathrm{R}}\mathcal{D}_t^{\alpha}f(t)-{}^{\mathrm{R}}\Psi(f,\alpha,t_0,a,t), \tag{9}$$

$$ {}_{t_0}^{\mathrm{C}}\mathcal{D}_t^{\alpha}f(t)={}_{a}^{\mathrm{C}}\mathcal{D}_t^{\alpha}f(t)-{}^{\mathrm{C}}\Psi(f,\alpha,t_0,a,t). \tag{10}$$

These initialization functions depend on the history of the function and can be derived, for example, by splitting the convolution integral in the definition of the operator, i.e.

$$\begin{aligned}{}^{\mathrm{C}}\Psi(f,\alpha,t_0,a,t)&={}_{a}^{\mathrm{C}}\mathcal{D}_t^{\alpha}f(t)-{}_{t_0}^{\mathrm{C}}\mathcal{D}_t^{\alpha}f(t)\\ &=\frac{1}{\Gamma(m-\alpha)}\int_{a}^{t_0}\frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}}\mathrm{d}\tau\,.\end{aligned} \tag{11}$$

As shown in [21] these initialization functions decay if the function respectively its derivative are bounded in the past interval $t\in[t_p,t_0]$ such that

$$|{}^{\mathrm{C}}\Psi(f,\alpha,t_0,a,t)|\leq\bar{M}_{\mathrm{C}}(t-t_0)^{m-\alpha-1}, \tag{12}$$

$$|{}^{\mathrm{R}}\Psi(f,\alpha,t_0,a,t)|\leq\bar{M}_{\mathrm{R}}(t-t_0)^{-\alpha-1} \tag{13}$$

with constants $\bar{M}_{\mathrm{C}}$ and $\bar{M}_{\mathrm{R}}$ depending on the function $f(\cdot)$ in the past interval. Note that the memory associated with the Riemann-Liouville derivative decays faster.

III. Modulating Functions Approach

The modulating function method is already well-established for estimating the parameters for integer-order linear [22], [23] and some nonlinear systems [22]. The measurements of the output or input are multiplied with the modulating function $\varphi(\cdot)$ and the following integration moves the derivative from the possible noisy measurement towards the known modulating function using integration by parts. To eliminate any initial conditions the modulating function $\varphi(\cdot)$ has to satisfy the following conditions [22].

Definition 1 (Modulating Function)**.** *The function* $\varphi(\cdot):[t_0,T]\to\mathbb{R}$ *is called a modulating function of order* m*, if*

(P1:) $\quad\varphi(t)\in\mathcal{C}^m,\qquad t\in[t_0,T], \quad (14)$

(P2:) $\quad\varphi^{(i)}(t_0)=\varphi^{(i)}(T)=0,\qquad i=0,1,\ldots,m-1. \quad (15)$

In this contribution, we consider fractional-order SISO LTI systems of the following structure

$$\sum_{i=0}^{n}a_i\,{}_{t_p}^{\mathrm{R}}\mathcal{D}_t^{\alpha_i}y(t)=\sum_{i=0}^{m}b_i\,{}_{t_p}^{\mathrm{R}}\mathcal{D}_t^{\beta_i}u(t), \tag{16}$$

where $u(\cdot)$ and $y(\cdot)$ are the input and output, respectively. We assume ordered differentiation orders, i.e. $\alpha_n>\cdots>\alpha_0$ and $\beta_m>\cdots>\beta_0$. For causality we require $\alpha_n\geq\beta_m$. The parameters a_i and b_i enter linearly and are to be determined. Without loss of generality we assume $a_n=1$. Note that we do not require the system to be commensurate, i.e. $\alpha_i=i\gamma$ and $\beta_i=i\gamma$ with $i\in\mathbb{N}$. However in order to generate well-conditioned estimation equations these pseudo polynomials

$$\lambda^{\alpha_n}+\cdots+a_1\lambda^{\alpha_1}+a_0\lambda^{\alpha_0}, \tag{17}$$

$$b_m\lambda^{\beta_m}+\cdots+b_1\lambda^{\beta_1}+b_0\lambda^{\beta_0} \tag{18}$$

need to be coprime with respect to the highest commensurable order γ. Note that in the uncommensurable case, we require a similar concept as coprimeness, to generate linear independent equations, otherwise the internal dynamics cannot be identified only from in- and output data.

Within this setup the system is not initialized, but the input and measurement are only available for time $t \in [t_0, t_f]$. We assume the measured output to be corrupted by noise

$$\tilde{y}(t) = y(t) + \epsilon(t), \tag{19}$$

where $\epsilon(t)$ is assumed to be Gaussian white noise.

In the literature various types of modulating functions have been applied to the fractional-order case, e.g. combined polynomials [10]

$$\tilde{\varphi}_n(t) = t^{\kappa_1+n}(t_f - t)^{\kappa_2-n}, \quad n, \kappa_1, \kappa_2 \in \mathbb{N}, \tag{20}$$

single sided polynomials [12]

$$\hat{\varphi}_n(t) = \sum_{i=1}^{n} c_i t^i \tag{21}$$

and spline-type modulating-functions [13]

$$\varphi_{k,l}(t) = \underbrace{\int_{t_0}^{t_f}\dots\int_{t_0}^{t_f}}_{o=k-l\text{-times}} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \delta\left(t - i\frac{t_f}{k} - t_h\right) \mathrm{d}t^o \tag{22}$$

which is a weighted sum of integrated Dirac impulses $\delta(\cdot)$.

For the implementation of these modulating functions, we use the integrated version

$$\varphi_{k,l}(t) = \sum_{i=0}^{k} \underbrace{\frac{(-1)^i}{(o-1)!}\binom{k}{i}}_{a_i} \underbrace{(t-d_i)^{o-1}\,\sigma\,(t-d_i)}_{\bar{\varphi}_i(t)} \tag{23}$$

with $d_i = i\frac{t_f}{k} + t_h$ and $\sigma(\cdot)$ representing the Heaviside function. This description will be used later to analytically compute the fractional-order derivatives.

Regarding fractional-order systems there are in general three different approaches to extend this method. The first approach [10] constructs a convolution integral, and with the property (15), $t_0 = 0$ and commutativity of the convolution we have

$$\begin{aligned}\int_0^{t_f} \varphi(t_f - \tau)\,{}_0^{\mathrm{R}}\mathcal{D}_\tau^\alpha y(\tau)\mathrm{d}\tau &= \int_0^{t_f} {}_0^{\mathrm{R}}\mathcal{D}_{t_f-\tau}^\alpha \varphi(t_f-\tau) y(\tau)\mathrm{d}\tau \\ &= \int_0^{t_f} {}_0^{\mathrm{R}}\mathcal{D}_\tau^\alpha \varphi(\tau) y(t_f - \tau)\mathrm{d}\tau.\end{aligned}$$

This approach is based on considerations in the Laplace domain, as the arising term s^α is moved towards the modulating function [10]. It can be extended to Caputo's operator by applying either (6) or including the estimation of the initial conditions separately. Note that in this case we can drop the right-side boundary condition on the modulating function.

The second method uses the so-called fractional integration by parts formula [24], arising from Fubini's theorem

$$\begin{aligned}\int_0^{t_f} \varphi(\tau)\,{}_0^{\mathrm{R}}\mathcal{D}_\tau^\alpha y(\tau)\mathrm{d}\tau &= \int_0^{t_f} {}_\tau^{\mathrm{C}}\mathcal{D}_{t_f}^\alpha \varphi(\tau) y(\tau)\mathrm{d}\tau - \\ &\sum_{j=0}^{m} \left[{}_\tau^{\mathrm{R}}\mathcal{D}_{t_f}^{m-1-j} \varphi(\tau)\,{}_\tau^{\mathrm{R}}\mathcal{D}_{t_f}^{\alpha+j-m} y(\tau)\right].\end{aligned} \tag{24}$$

Applying Condition (15) leads to

$$\int_0^{t_f} \varphi(\tau)\,{}_0^{\mathrm{R}}\mathcal{D}_\tau^\alpha y(\tau)\mathrm{d}\tau = \int_0^{t_f} {}_\tau^{\mathrm{C}}\mathcal{D}_{t_f}^\alpha \varphi(\tau) y(\tau)\mathrm{d}\tau. \tag{25}$$

Note that the left-side Riemann derivative changes to the right-side Caputo derivative. This approach allows to include nonlinear terms in the identification.

Remark 2. *We can show that this approach leads to similar results if the modulating functions are symmetric within the observation interval, i.e.* $\varphi(t) = \varphi(t_f - t)$ *or point symmetrical* $\varphi(t) = -\varphi(t_f - t)$*. Thus, we consider the left-side Riemann derivative of the flipped modulating function* $\varphi(t_f - t)$*. For zero initial conditions we can apply* (6) *and directly use Caputo's left-side definition*

$${}_0^{\mathrm{R}}\mathcal{D}_t^\alpha \varphi(t) = {}_0^{\mathrm{C}}\mathcal{D}_t^\alpha \varphi(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{\varphi^m(\tau)}{(t-\tau)^{\alpha-m+1}}\mathrm{d}\tau.$$

Flipping time $t \to t_f - t$ *and substituting* $\bar{\tau} = t_f - \tau$ *yields*

$$\begin{aligned}&\frac{1}{\Gamma(m-\alpha)} \int_0^{t_f-t} \frac{\varphi^m(\tau)}{(t_f-t-\tau)^{\alpha-m+1}}\mathrm{d}\tau = \\ &\frac{1}{\Gamma(m-\alpha)} \int_0^{t_f-t} \frac{\varphi^m(t_f-\bar{\tau})}{(\bar{\tau}-t)^{\alpha-m+1}}\mathrm{d}\bar{\tau}.\end{aligned} \tag{26}$$

Including the symmetry assumptions on the modulating functions leads to the right-side Caputo derivative (including a sign change). For $\varphi(t) = \varphi(t_f - t)$ *we have*

$${}_0^{\mathrm{R}}\mathcal{D}_{t_f-t}^\alpha \varphi(t_f - t) = (-1)^{-m}\ {}_t^{\mathrm{C}}\mathcal{D}_{t_f}^\alpha \varphi(t) \tag{27}$$

and for $\varphi(t) = -\varphi(t_f - t)$ *the sign changes*

$${}_0^{\mathrm{R}}\mathcal{D}_{t_f-t}^\alpha \varphi(t_f - t) = (-1)^{1-m}\ {}_t^{\mathrm{C}}\mathcal{D}_{t_f}^\alpha \varphi(t). \tag{28}$$

Hence, it is also possible to compare both approaches by investigating the corresponding flipped modulating function.

The third method applies the classical integer-order integration by parts [25], such that a fractional-order integration of the measurement signal remains within the integral, i.e. for $\alpha \in (0, 1)$ we have with $\varphi(0) = \varphi(T) = 0$ that

$$\int_0^T \varphi(\tau)\,{}_0^{\mathrm{R}}\mathcal{D}_\tau^\alpha y(\tau)\mathrm{d}\tau = -\int_0^T \dot{\varphi}(\tau)\,{}_0\mathcal{I}_\tau^{1-\alpha} y(\tau)\mathrm{d}\tau. \tag{29}$$

This method allows to use advanced modulating functions since only integer-order derivatives are required, which can be provided completely analytically. However, the fractional-order integration of the measurement signals might be computationally intense, especially if the differentiation order is unknown and has to be estimated by nonlinear optimization.

A. Memory Compensation

The compensation of the memory effects (unknown measurements of past input- and output data, respectively) has been discussed in [13]. Although the theoretical results are sound, its final conclusions have to be extended.

The basic idea is to move the lower limit of the integration from t_p to t_0. We consider that the input and output measurements are only available on the time interval $t \in [t_0, t_f]$.

Taking this into account, we assume that the modulating function $\varphi(\cdot)$ is zero within the past interval $t \in [t_p, t_0]$, leading to

$$\int_{t_p}^{t_f} \varphi(\tau)\,{}^{\mathrm{R}}_{t_p}\mathcal{D}^{\alpha_i}_{\tau} y(\tau)\mathrm{d}\tau = \int_{t_0}^{t_f} \varphi(\tau)\,{}^{\mathrm{R}}_{t_p}\mathcal{D}^{\alpha_i}_{\tau} y(\tau)\mathrm{d}\tau. \quad (30)$$

This equality, however, does not hold for the result of the fractional-order integration by parts and some error E occurs:

$$\begin{aligned}\int_{t_p}^{t_f}\varphi(\tau)\,{}^{\mathrm{R}}_{t_p}\mathcal{D}^{\alpha_i}_{\tau} y(\tau)\mathrm{d}\tau &= \int_{t_p}^{t_f} {}^{\mathrm{C}}_{\tau}\mathcal{D}^{\alpha_i}_{t_f}\varphi(\tau)y(\tau)\mathrm{d}\tau \\ &= \int_{t_0}^{t_f} {}^{\mathrm{C}}_{\tau}\mathcal{D}^{\alpha_i}_{t_f}\varphi(\tau)y(\tau)\mathrm{d}\tau + E(t_0).\end{aligned} \quad (31)$$

In order to compensate the memory, the modulating function has to fulfil the additional property [13]

$$(P3) \quad {}^{\mathrm{C}}_{t}\mathcal{D}^{\gamma}_{t_f}\varphi(t) = 0, \quad \forall t \in [t_p, t_0] \text{ and } \gamma \in \{\alpha_i, \beta_i\}. \quad (32)$$

The authors in [13] state that the spline-type modulating functions satisfy this condition. However, this is only true in the integer-order case. We compute the right-side Caputo derivative by inserting $\bar{\varphi}_i$ from (23) into Definition (8) with $m-1 < \alpha < m$:

$$\begin{aligned}&{}^{\mathrm{C}}_{t}\mathcal{D}^{\alpha}_{t_f}\bar{\varphi}_i(t) = \frac{1}{\Gamma(m-\alpha)}\int_t^{t_f} \frac{(-1)^m}{(\tau-t)^{\alpha-m+1}}\bar{\varphi}_i^{(m)}(\tau)\mathrm{d}\tau \\ &= \int_t^{t_f} \frac{\Gamma^{-1}(m-\alpha)}{(\tau-t)^{\alpha-m+1}} \underbrace{\frac{(-1)^m(o-1)!}{(o-m-1)!}}_{c} \frac{\sigma(\tau-d_i)}{(\tau-d_i)^{-o+1+m}}\mathrm{d}\tau \\ &= \tilde{c}\int_{\max(t,d_i)}^{t_f} (\tau-t)^{m-\alpha-1}(\tau-d_i)^{o-1-m}\,\mathrm{d}\tau\end{aligned}$$

with $\tilde{c} = \frac{c}{\Gamma(m-\alpha)}$. Substituting $\bar{\tau} = \tau - d_i$ and $\bar{t} = t - d_i$ in the integral leads to

$${}^{\mathrm{C}}_{t}\mathcal{D}^{\alpha}_{t_f}\bar{\varphi}_i(t) = \tilde{c}\int_{\max(t-d_i,0)}^{t_f-d_i} \left(\bar{\tau}-\bar{t}\right)^{m-\alpha-1}\bar{\tau}^{o-1-m}\mathrm{d}\bar{\tau}.$$

Applying the Binomial theorem as in [12] yields

$$\begin{aligned}&{}^{\mathrm{C}}_{t}\mathcal{D}^{\alpha}_{t_f}\bar{\varphi}_i(t) = \\ &\tilde{c}\int_{\max(t-d_i,0)}^{t_f-d_i} \left(\bar{\tau}-\bar{t}\right)^{m-\alpha-1}\sum_{q=0}^{o-1-m}\binom{o-1-m}{q}\frac{\left(\bar{\tau}-\bar{t}\right)^q}{\bar{t}^{-o+1+m+q}}\mathrm{d}\bar{\tau}.\end{aligned}$$

Going back to the original coordinates we have (33), see the next page. This fractional-order derivative is non-zero for $t \in [t_p, t_0]$ for the fractional-order case. An illustration is given in Figure 1. Only in the integer-order case the derivative vanishes. But we may notice that the magnitude decays for $t \to -\infty$. Hence we can reduce the resulting error E, see (31), by minimizing the right-side fractional-order derivative for $t < t_0$.

Note the difficulty here: If other candidates of modulating functions are to be considered, then the right-side Caputo derivative has to be zero on an entire interval and for the various differentiation orders α_i and β_i. As the spline type modulating function at least satisfies these conditions in the integer-order case, we may use (33) to tune the spline modulating function so as to reduce the effect of the unknown memory on the estimation.

Remark 3 (Tuning of the Modulating Functions)**.** *The basic tuning parameters to explore are the length t_f of the time interval, the starting time t_h of the impulses and the order difference $o = k - l$. With a higher integration order o the algebraic decay for $t \to -\infty$ is increased, e.g. in [13] the order $o = 20$ is used. However, such high orders reduce the numerical robustness and increase the sensitivity with respect to measurement noise. By increasing the starting time t_h we shift the modulating function (and its derivative) towards the end of the time interval. Hence, the information of the recent past may be weighted higher in the integral.*

B. Equation Generation for Parameter Estimation

In order to determine the $n_p = m+n+1$ parameters, we have to generate at least this number of equations. The modulating function approach offers two ways to construct these equations. First of all a set of $N > n_p$ linearly independent modulating functions on a specified time interval can be used (see e.g. [10], [13]). The parameters can be estimated with the pseudo inverse of the measurement matrix.

The problem with this approach is the lack of robustness against measurement noise. This can be improved by increasing the number of equations, but increases the numerical demands drastically.

The second approach uses a single modulating function with a changing time horizon $t_f = \kappa T$ as shown in [12]. This leaves us with

$$Y(\kappa) = M(\kappa)\theta(\kappa) \quad (34)$$

with $\hat{\theta} = \begin{pmatrix}\hat{a}_{n-1} & \dots & \hat{a}_0 & \hat{b}_m & \dots & \hat{b}_0\end{pmatrix}^\top$ and

$$Y(\kappa) = \int_{t_0}^{\kappa\bar{T}} {}^{\mathrm{C}}_{\tau}\mathcal{D}^{\alpha_n}_{\kappa\bar{T}}\varphi(\tau)y(\tau)\mathrm{d}\tau \quad (35)$$

$$M(\kappa) = \begin{pmatrix}-M_{y,n-1} & \dots & -M_{y,0} & M_{u,m} & \dots & M_{u,0}\end{pmatrix} \quad (36)$$

$$M_{y,i} = \int_{t_0}^{\kappa\bar{T}} {}^{\mathrm{C}}_{\tau}\mathcal{D}^{\alpha_i}_{\kappa\bar{T}}\varphi(\tau)y(\tau)\mathrm{d}\tau, \quad i = 0,\dots n-1 \quad (37)$$

$$M_{u,i} = \int_{t_0}^{\kappa\bar{T}} {}^{\mathrm{C}}_{\tau}\mathcal{D}^{\beta_i}_{\kappa\bar{T}}\varphi(\tau)u(\tau)\mathrm{d}\tau, \quad i = 0,\dots m. \quad (38)$$

For a fixed estimation horizon ΔT the lower limit in the upper integrals has to be changed to $t_0 \coloneqq \kappa\bar{T} - \Delta T$.

The resulting algebraic equation can be solved recursively without the necessity to compute inverses of large matrices. Advanced continuous-time algorithms can be used, see [26].

We applied the standard recursive least squares estimator [27] given by the following regression

$$\gamma(\kappa) = \frac{P(\kappa-1)M^\top(\kappa)}{1 + M(\kappa)P(\kappa-1)M^\top(\kappa)} \quad (39)$$

$$P(\kappa) = P(\kappa-1) - \gamma(\kappa)M(\kappa)P(\kappa-1) \quad (40)$$

$$\hat{\theta}(\kappa) = \hat{\theta}(\kappa-1) + \gamma(\kappa)(Y(\kappa) - M(\kappa)\hat{\theta}(\kappa-1)) \quad (41)$$

$$ {}_{t}^{\mathrm{C}}\mathcal{D}_{t_f}^{\alpha}\bar{\varphi}_i(t) = \begin{cases} \tilde{c}\sum_{q=0}^{o-1-m}\binom{o-1-m}{q}\frac{\big(t-d_i\big)^{o-1-m-q}}{m-\alpha+q}\big(t_f-t\big)^{m-\alpha+q}, & t>d_i \\ \tilde{c}\sum_{q=0}^{o-1-m}\binom{o-1-m}{q}\frac{\big(t-d_i\big)^{o-1-m-q}}{m-\alpha+q}\Big[\big(t_f-t\big)^{m-\alpha+q}-(d_i-t)^{m-\alpha+q}\Big], & t<d_i. \end{cases} \tag{33} $$

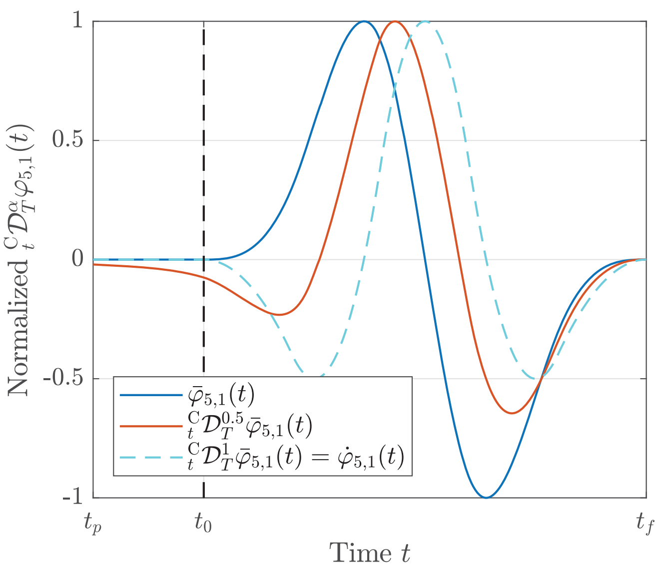

Fig. 1. Illustration of the normalized spline-type modulating function and its normalized integer- and fractional-order derivatives. Only the integer-order derivative is zero on the interval $t \in [t_p, t_0]$.

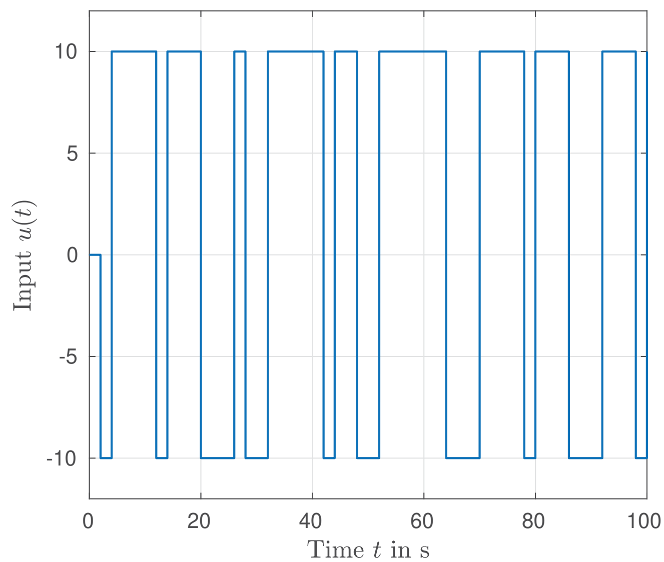

Fig. 2. Input signal.

with the initialization

$$ \hat{\theta}(0) = \Lambda^{-1}\begin{pmatrix}1 & 1 & \dots & 1\end{pmatrix}^{\top}, \quad P = \Lambda I, \tag{42} $$

where Λ is chosen relatively large.

IV. Example

We consider the academic example motivated by [12]

$$ {}_{0}^{\mathrm{R}}\mathcal{D}_t^{\alpha_2}y(t) + a_1\,{}_{0}^{\mathrm{R}}\mathcal{D}_t^{\alpha_1}y(t) + a_0 y(t) = b_0 u(t) \tag{43} $$

with $a_1 = \frac{2}{3}$, $a_0 = \frac{1}{3}$, $b_0 = \frac{4}{3}$ and the orders of differentiation $\alpha_1 = 0.4$ and $\alpha_2 = 0.8$. We use the input given in Figure 2 because the series of steps excites a broad range of frequencies. This is necessary as the fractional-order system is infinite dimensional in regard of the non-local derivative operators. For simulation we use the solver `fde12.m` presented in [28]. Since this solver is only designed for fractional-order differential equations with Caputo's definition, we set the initial conditions to zero, following (6). This solver is only suitable to simulate fractional-order differential equations without memory initialization (besides the integer-order initial conditions related to Caputo's operator), therefore the effect of the initialization is included by starting not at the initial time. The noisy observation is shown in Figure 3. The sampling time is set to $T_s = 10\,\mathrm{ms}$ and the estimation interval starts at $t_0 = 5\,\mathrm{s}$, $\bar{T} = 0.5\,\mathrm{s}$ and $t_f(\kappa = 0) = 20\,\mathrm{s}$. As the output data is corrupted by noise, different simulation techniques, like Oustaloup filters [17], will lead to comparable data sets.

We use the spline modulating function $\varphi_{8,1}(t)$ with $t_h = 15\,\mathrm{s}$ and normalize it such that the maximum derivative is one.

The estimation results for a growing estimation window $t_f \in [20\,\mathrm{s}, 100\,\mathrm{s}]$ are shown in Figure 5. The final parameter estimations for different noise levels are listed in Table I and II. For the lower noise levels both approaches perform similarly as the parameter error $e_\theta = \|\hat{\theta} - \theta\|_2/\|\theta\|_2$ is more or less comparable. In both cases the memory of the first 5 s has to be compensated. With the increasing window length of the first approach, this effect is reduced which results in a better performance, but to the price of a higher noise level.

To illustrate the effect of the memory reduction, the second approach uses a fixed estimation horizon. The results are shown in Figure 4. As given in [12] the recursive least squares algorithm leads to an offset in the estimation. Although a compensation algorithm is presented in [12] it is not applied here because the discrete error $\epsilon(\kappa)$ is not Gaussian white noise. Further investigations should focus on the reduction of the noise sensitivity.

V. Conclusions

In this work we revisit the modulating function method to identify the parameters of a linear time-invariant fractional-order system given by an input-output equation. In the literature there are three approaches to generalize this idea

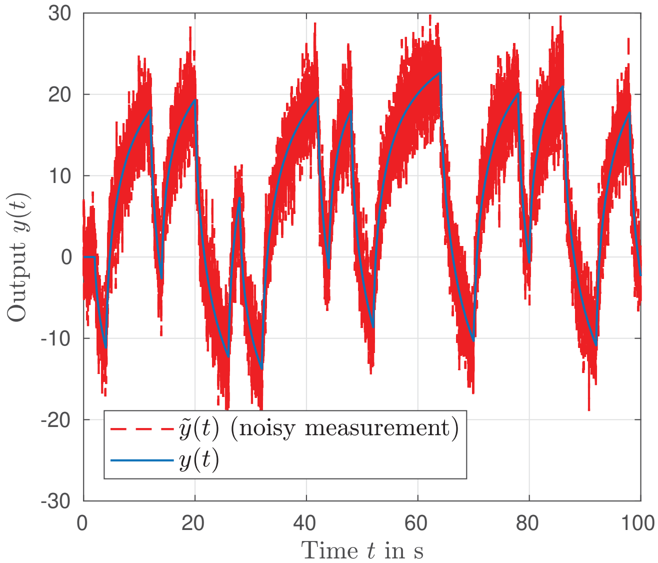

Fig. 3. Comparison of pure output $y(t)$ and noise-corrupted output $\tilde{y}(t)$, setting $\sigma^2 = 10$.

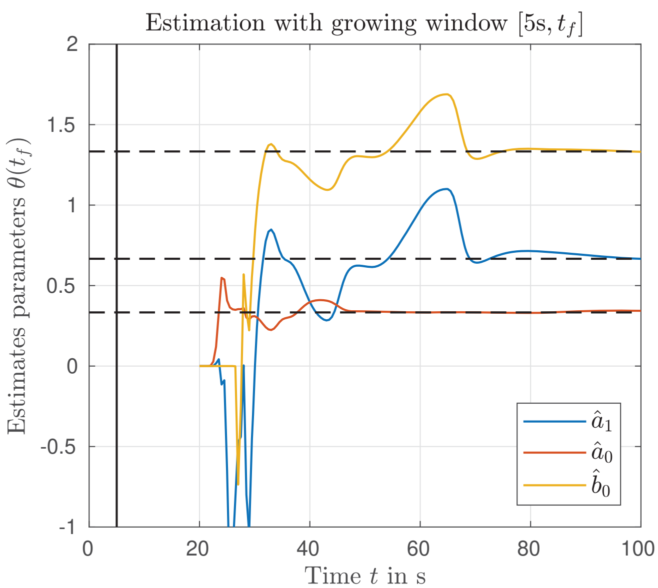

Fig. 4. Estimation results starting at $t = 5\,\mathrm{s}$ with an increasing estimation window for a moderate noise level $\sigma^2 = 10$. The estimation does not converge to an equilibrium within the time interval due to the integration error and the noise contribution.

TABLE I

BIASED ESTIMATIONS OF THE COEFFICIENTS WITH AN INCREASING ESTIMATION WINDOW WITH $t_f = 100\,\mathrm{s}$.

σ^2	SNR	$\hat{a}_1$	$\hat{a}_0$	$\hat{b}_0$	$e_\theta(\%)$
0	∞	0.6804	0.3301	1.3405	1.0
1	21.75	0.5997	0.3239	1.2695	6.1
10	12.02	0.6665	0.3434	1.3313	6.7
100	3.92	0.0168	0.4333	0.8999	51.6

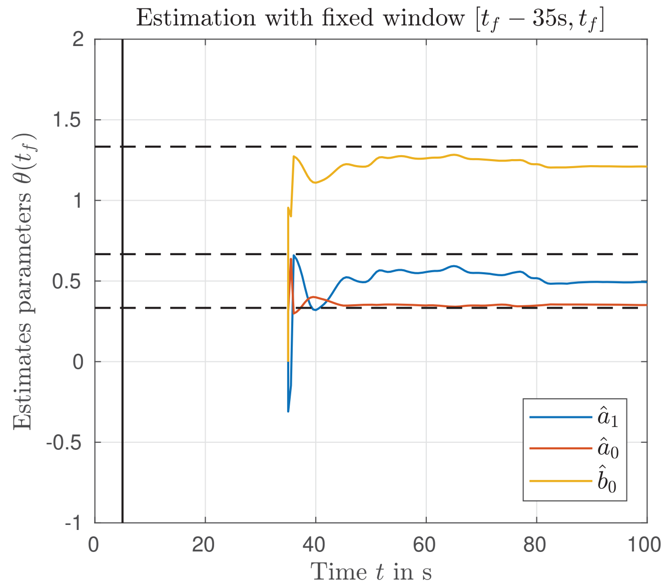

Fig. 5. Estimation results starting at $t = 5\,\mathrm{s}$ with a fixed estimation window of $35\,\mathrm{s}$ for a moderate noise level $\sigma^2 = 10$. The estimation converges to a biased parameter set.

TABLE II

BIASED ESTIMATIONS OF THE COEFFICIENTS AT $t_f = 100\,\mathrm{s}$ WITH A FIXED ESTIMATION WINDOW.

σ^2	SNR	$\hat{a}_1$	$\hat{a}_0$	$\hat{b}_0$	$e_\theta(\%)$
0	∞	0.6338	0.3321	1.3068	2.7
1	21.81	0.6199	0.3353	1.2992	3.8
10	12.14	0.4928	0.3511	1.2098	14.0
100	3.98	-0.0637	0.4282	0.8264	51.9

to the fractional-order case. We show that the convolution arrangement and the right-side Caputo derivative lead to the same results if we consider the flipped modulating function. This simplifies the computation of the required derivatives, as analytical derivatives can be applied.

The second part of this contribution focuses on the compensation of the memory effect. We show that the spline-type modulating functions are only able to reduce the effect of the memory on the estimation, however a complete compensation is not possible. From the simulation example we may draw the conclusion that the reduction of the initialization increases the sensitivity with respect to measurement noise. To overcome this problem we tune the modulating function carefully and increase the number of equations. This is possible by applying the recursive least squares estimation. In a simulation example the approach yields good results for low and medium noise levels.

Future work will focus on further reducing the error caused by measurement noise. Possible approaches may include the Kalman-like estimator structure in [12]. With an estimation independent of the memory, one may generate additional equations enhancing the estimation of possibly unknown derivative orders. The idea is that the (nonlinear) estimated derivative orders are constant and independent of some time interval (on which the memory effect is sufficiently reduced).

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