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Delta-points and their implications for the geometry of Banach spaces

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Abstract

We show that the Lipschitz-free space with the Radon–Nikodým property and a Daugavet point recently constructed by Veeorg is in fact a dual space isomorphic to ℓ_1. Furthermore, we answer an open problem from the literature by showing that there exists a superreflexive space, in the form of a renorming of ℓ_2, with a Δ-point. Building on these two results, we are able to renorm every infinite-dimensional Banach space to have a Δ-point. Next, we establish powerful relations between existence of Δ-points in Banach spaces and their duals. As an application, we obtain sharp results about the influence of Δ-points for the asymptotic geometry of Banach spaces. In addition, we prove that if X is a Banach space with a shrinking k-unconditional basis with $k < 2$, or if X is a Hahn–Banach smooth space with a dual satisfying the Kadets–Klee property, then X and its dual X^* fail to contain Δ-points. In particular, we get that no Lipschitz-free space with a Hahn–Banach smooth predual contains Δ-points. Finally, we present a purely metric characterization of the molecules in Lipschitz-free spaces that are Δ-points, and we solve an open problem about representation of finitely supported Δ-points in Lipschitz-free spaces.

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1 INTRODUCTION

Let X be a (real) Banach space with unit ball B_X, unit sphere S_X, and topological dual X^*. For $x \in S_X$, we will write

$$D(x) := \{x^* \in S_{X^*} : x^*(x) = 1\}.$$

A *slice* of a nonempty, bounded, and convex subset C of X is a nonempty intersection of C with an open half-space of X. Thus, a slice of B_X can be written

$$S(B_X, x^*, \varepsilon) = \{y \in B_X : x^*(y) > 1 - \varepsilon\},$$

where $x^* \in S_{X^*}$ and $\varepsilon > 0$. We usually omit B_X and write $S(x^*, \varepsilon)$ instead of $S(B_X, x^*, \varepsilon)$ when it is clear from the context what set is being sliced. If X is a dual space and the defining functional x^* is in the predual of X, then we call the slice a weak*-slice.

The main characters in our story are pointwise versions of the well-known Daugavet property, the slightly less known space with bad projections, and the more obscure property $\mathfrak{D}$. We start by recalling the definitions from [3] and [32].

Definition 1.1. Let X be a Banach space, and let $x \in S_X$. We say that

(i) x is a *Daugavet point* if $\sup_{y \in S} \|x - y\| = 2$ for every slice S of B_X.
(ii) x is a Δ*-point* if $\sup_{y \in S} \|x - y\| = 2$ for every slice S of B_X with $x \in S$.
(iii) x is a $\mathfrak{D}$*-point* if $\sup_{y \in S} \|x - y\| = 2$ for every slice $S = S(x^*, \varepsilon)$ of B_X with $x^* \in D(x)$ and $\varepsilon > 0$.
(iv) x is a *super* Δ*-point* if $\sup_{y \in W} \|x - y\| = 2$ for every relatively weakly open subset W of B_X with $x \in W$.

For dual spaces, we will also consider the natural weak* versions of Daugavet and Δ-points where we simply replace the phrase “every slice S” in the definition with “every weak*-slice S,” and replace “weakly open” by “weak* open” for super Δ-points.

Note that if X is a subspace of a Banach space Y and $x \in S_X$ is a Δ-point, $\mathfrak{D}$-point, or super Δ-point, then x is still such a point regarded as an element in Y. This is not the case for Daugavet points as can be seen by regarding $C[0,1]$ as a subspace of $C[0,1] \oplus_2 C[0,1]$ (this is Example 4.7 from [3]).

In Sections 2 through 4, we show that Daugavet points, super Δ-points, and Δ-points can exist in some well-behaved Banach spaces. The role played by $\mathfrak{D}$-points will be more negative, in the sense that spaces with certain properties do not even admit a $\mathfrak{D}$-point.

We start the paper with a study of the metric space $\mathcal{M}$ constructed by Veeorg in [41]. The Lipschitz-free space $\mathcal{F}(\mathcal{M})$ was the first example of a Banach space with the Radon–Nikodým property admitting a Daugavet point. We will show that in fact $\mathcal{F}(\mathcal{M})$ is isomorphic to ℓ_1, and is isometrically a dual space. Thus, there exits a separable dual space with a Daugavet point (see Theorem 2.1).

It was shown in [7, Corollary 6.10] that finite-dimensional Banach spaces do not admit Δ-points, and it was asked if the same holds for superreflexive spaces in [7, Question 6.1] and [32, Question 7.7]. In Section 3, we answer this question negatively. We modify a renorming of ℓ_2 from [18] to show that ℓ_2 can be renormed so that both ℓ_2 and its dual have a super Δ-point. These super Δ-points are not Daugavet points as there are strongly exposed bits of the original unit ball, which are still left in the new unit ball and which are at a distance strictly less than 2 to them. We also provide a positive answer to [32, Question 7.12] by proving that those super Δ-points actually belong to the closure of the set of strongly exposed-points for the new norm.

In Section 4, we combine the ideas and results from the previous two sections and show that any infinite-dimensional Banach space admits a renorming with a Δ-point. Spaces failing the Schur property (and in particular spaces that do not contain a copy of ℓ_1) always contain a normalized weakly null basic sequence, and using this as a starting point, we can adapt the (dual) renorming of ℓ_2 from Section 3 to get a renorming with a super Δ-point. For spaces containing ℓ_1, the ℓ_1 isomorphism from Section 2 together with a classic norm extension result will allow for a renorming with a Δ-point.

Having established that Daugavet points and Δ-points are very much isometric notions we go looking for conditions that will prevent the existence of such points in a Banach space. Let us mention that it is known that neither uniformly nonsquare [7, Corollary 2.4], asymptotically uniformly smooth Banach spaces [7, Theorem 3.7] nor real Gleit and McGuigan spaces [33, Corollary 3.8] contain Δ-points. It is obvious that no strongly exposed point can be a $\mathfrak{D}$-point and that no denting point can be a Δ-point and, in fact, neither can be quasi-denting points by [40, Corollary 2.2]. It was also recently proved in [27, Theorem 4.2] that no locally uniformly nonsquare point can be a Δ-point.

The main result of Section 5 is Theorem 5.6, which says that as soon as a space contains a $\mathfrak{D}$-point or its dual contains a weak* Δ-point, then the dual actually contains a weak* super Δ-point. This powerful result will have many implications so let us mention a few.

In Section 5.1, we prove Theorem 5.6 and use it to improve results from [7] and [40] and show that asymptotically uniformly smooth spaces cannot contain $\mathfrak{D}$-points and their duals cannot contain weak* Δ-points.

It is well known that a separable Banach space with the Daugavet property does not embed into a Banach space with an unconditional basis [28, Corollary 2.7]. However, for Daugavet points, there is no such obstruction. There exists a Banach space with a 1-unconditional basis such that the set of Daugavet points is weakly dense in the unit ball [8, Theorem 4.7]. A 1-unconditional basis does however prevent the existence of super Δ-points [8, Proposition 2.12]. The main result in Section 5.2 is Theorem 5.12. The proof of this theorem relies on Theorem 5.6 and the theorem is used to show that if a Banach space X has a shrinking k-unconditional basis for $k < 2$, then X contains no $\mathfrak{D}$-points and X^* contains no weak* Δ-points (see Corollary 5.15). This result answers [6, Question 5.6] affirmatively and can be used to strengthen [6, Proposition 4.6]. If X has a monotone boundedly complete k-unconditional basis for $k < 2$, we only get that X has no Δ-points. Furthermore Corollary 5.14 says that if X is a reflexive Banach space with a k-unconditional basis for $k < 2$, then X and X^* do not contain $\mathfrak{D}$-points. This strengthens previous results in this direction.

In Section 5.3, we study the implications of Theorem 5.6 for M-embedded spaces. A question that has not appeared in print, but has been in the back of the mind of several people studying Δ-points, is the following: Do (nonreflexive) M-embedded Banach spaces and their duals fail to contain Δ-points? Recall that X is *M-embedded* if X is an M-ideal in its bidual, which means that we can write $X^{***} = X^* \oplus_1 X^\perp$. Using the renorming of ℓ_2 from Section 3, we can answer the above question negatively by showing that there exists a nonreflexive M-embedded Banach space X such that both X and X^* have a super Δ-point.

However, all is not lost. M-embedded spaces are known to be Hahn–Banach smooth and Asplund (cf. [24, Chapter III]). We are able to show that if the dual unit ball is weak* sequentially compact, then we get a sequential version of Theorem 5.6 and that if X contains a $\mathfrak{D}$-point or if X^* contains weak* Δ-point, then X^* fails to be Kadets–Klee. For Lipschitz-free spaces, this implies that if M is a metric space such that $\mathcal{F}(M)$ is a dual space and Y is an M-embedded (or more generally, Hahn–Banach smooth) predual, then Y contains no $\mathfrak{D}$-points and $\mathcal{F}(M)$ contains no weak* Δ-points (see Corollary 5.27).

Finally, in Section 6, our main goal is to obtain a purely metric characterization of those molecules m_{xy} that are Δ-points of $\mathcal{F}(M)$, called simply Δ-*molecules* (see Theorem 6.7). In particular, we get that, when M is proper, m_{xy} is a Δ-point if and only if x and y are connected with a geodesic (see Corollary 6.8). These results improve Proposition 4.2 and Theorem 4.13 in [26], respectively. We also prove that every Δ-point of $\mathcal{F}(M)$ with finite support is a finite convex sum of Δ-molecules (see Theorem 6.9), solving [41, Problem 3] in the positive.

Notation

Let X be a Banach space and let $x \in S_X$. Recall that x is a *denting point* of B_X if for any $\delta > 0$, there exists a slice S of B_X such that $x \in S$ and $\operatorname{diam}(S) < \delta$. Furthermore, x is said to be *strongly exposed* if there exists $x^* \in D(x)$ such that $\operatorname{diam} S(x^*, \varepsilon) \to 0$ as $\varepsilon \to 0$. An element $x^* \in B_{X^*}$ is weak* strongly exposed if its strongly exposed by some $x \in D(x^*) \cap X$. Replacing the diameter of the slices with the Kuratowski measure of noncompactness α, we can define the notion of α-*strongly exposed point*. This is similar to how quasi-denting points were generalized from denting points by Giles and Moors [22].

We will follow standard Banach space notation as found in the books [4] and [20], but let us also say a few things about Lipschitz-free space notation since it is not yet completely standard.

For a metric space (M, d), we denote by $B(x, r)$ the closed ball centered at $x \in M$ with radius r, and we define the *metric segment* between points $x, y \in M$ by

$$[x, y] := \{z \in M : d(x, z) + d(z, y) = d(x, y)\}.$$

If M is *pointed*, that is, it is equipped with a distinguished base point usually denoted by 0, we let $\operatorname{Lip}_0(M)$ be the space of Lipschitz functions $f : M \to \mathbb{R}$ such that $f(0) = 0$ equipped with the

Lipschitz norm

$$\|f\|_L := \sup\left\{\frac{|f(x)-f(y)|}{d(x,y)} : x, y \in M, x \neq y\right\}.$$

Let $\delta : M \to \mathrm{Lip}_0(M)^*$ be the map that assigns each $x \in M$ to the corresponding point-evaluation $\delta(x)$ in $\mathrm{Lip}_0(M)^*$, that is, $f(x) = \langle \delta(x), f\rangle$ for $x \in M$ and $f \in \mathrm{Lip}_0(M)$. It is well known that δ is a nonlinear isometry and that $\mathcal{F}(M) = \overline{\mathrm{span}}(\delta(M))$ is a predual of $\mathrm{Lip}_0(M)$ called the *Lipschitz-free space* over M. The Lipschitz-free space is also known by the name Arens–Eells space and the notation Æ(M) is sometimes used, for example, in [42].

Recall that a function $f : M \to \mathbb{R}$, where M is a metric space, is *locally flat* if

$$\lim_{x,y\to z} \frac{f(x)-f(y)}{d(x,y)} = 0$$

for every $z \in M$. We will follow [2] and define

$$\mathrm{lip}_0(M) := \left\{f \in \mathrm{Lip}_0(M) : f \text{ is locally flat and } \lim_{r\to\infty} \left\| f|_{M\setminus B(0,r)}\right\|_L = 0\right\}.$$

For compact M, $\mathrm{lip}_0(M)$ is simply the set of locally flat $f : M \to \mathbb{R}$ such that $f(0) = 0$.

A *molecule* $m_{xy} \in \mathcal{F}(M)$, $x \neq y$, is an element of $S_{\mathcal{F}(M)}$ of the form

$$m_{xy} := \frac{\delta(x)-\delta(y)}{d(x,y)}.$$

For a subset K of M, the Lipschitz-free space $\mathcal{F}(K \cup \{0\})$ is identified with the subspace $\overline{\mathrm{span}}(\delta(K))$ of $\mathcal{F}(M)$. The *support* of $\mu \in \mathcal{F}(M)$, denoted $\mathrm{supp}(\mu)$, is the intersection of all closed $K \subseteq M$ such that $\mu \in \mathcal{F}(K \cup \{0\}) \subseteq \mathcal{F}(M)$. It holds that $\mu \in \mathcal{F}(\mathrm{supp}(\mu) \cup \{0\})$ (see [11, Section 2]).

2 | A SEPARABLE DUAL SPACE WITH A DAUGAVET POINT

This section is dedicated to studying the example given by Veeorg [41, Example 3.1] of a Banach space with the Radon–Nikodým property whose unit sphere contains a Daugavet point. We consider a metric space $\mathcal{M}$ constructed from a subset of $\mathbb{R}^2$ as follows. Let $p := (0,0)$, $q := (1,0)$, and for every $n \in \mathbb{N}$, let

$$S_n := \{(2^{-n}k, 2^{-n}) : k = 0, 1, \dots, 2^n\}$$

and finally $\mathcal{M} := \{p, q\} \cup \bigcup_{n=1}^{\infty} S_n$. (See Figure 1.) Endow $\mathcal{M}$ with the metric

$$d((x_1,y_1),(x_2,y_2)) := \begin{cases} |x_1 - x_2|, & \text{if } y_1 = y_2; \\ |y_1 - y_2| + \min\{x_1 + x_2, 2 - (x_1 + x_2)\}, & \text{if } y_1 \neq y_2; \end{cases}$$

and take p as its base point.

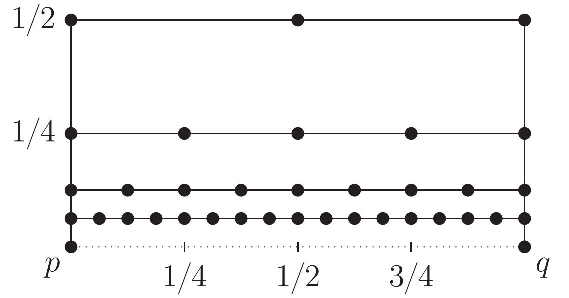

FIGURE 1 The sets $S_0, \ldots, S_4$.

Veeorg showed that $\mathcal{F}(\mathcal{M})$ has the Radon–Nikodým property and that the molecule m_{pq} is a Daugavet point. In this section, we will show that $\mathcal{F}(\mathcal{M})$ is isomorphic to ℓ_1 and that it actually has a predual. Let us start by introducing our candidate for a predual.

Denote

$$V := \{(x, y) \in \mathcal{M} : x = 0 \text{ or } x = 1\}$$

and let $h : \mathcal{M} \to \mathbb{R}$ be the function defined by

$$h(x, y) := x.$$

Observe that $h \in S_{\mathrm{Lip}_0(\mathcal{M})}$. Define

$$Y := \left\{ f \in \mathrm{Lip}_0(\mathcal{M}) : \lim_n \left\| (f - f(q) \cdot h)|_{S_n} \right\|_L \to 0 \text{ and } f|_V \text{ is locally flat} \right\}.$$

We can now state the main theorem of this section.

Theorem 2.1. *Let $\mathcal{M}$ and Y be as defined above. Then the following holds:*

(i) *The Banach space Y satisfies $Y^* = \mathcal{F}(\mathcal{M})$;*
(ii) *$\mathcal{F}(\mathcal{M})$ is isomorphic to ℓ_1.*

Thus, there is a separable dual space isomorphic to ℓ_1 that admits a Daugavet point.

Proof of Theorem 2.1(i). According to a theorem by Petunin and Plichko [35, Theorem 4], given a separable Banach space X, a subspace Y of X^* is an isometric predual of X when it satisfies the following conditions:

(a) Y is norm closed,
(b) Y separates points of X,
(c) all elements of Y attain their norm on S_X.

We will apply this result to $X := \mathcal{F}(\mathcal{M})$ and the subspace Y of $\mathrm{Lip}_0(\mathcal{M})$. Let us verify that Y satisfies all three Petunin–Plichko conditions.

We start with (a). Suppose that (f_n) is a sequence in Y that converges in norm to $f \in \mathrm{Lip}_0(\mathcal{M})$. Fix $\varepsilon > 0$, then there is k such that $\|f - f_k\|_L < \varepsilon$, and we get for every n,

$$\left\|(f-f(q)\cdot h)|_{S_n}\right\|_L \leqslant \left\|(f-f_k)|_{S_n}\right\|_L + \left\|(f_k-f_k(q)\cdot h)|_{S_n}\right\|_L + \left\|((f_k(q)-f(q))\cdot h)|_{S_n}\right\|_L$$
$$< 2\varepsilon + \left\|(f_k-f_k(q)\cdot h)|_{S_n}\right\|_L.$$

Since $f_k \in Y$, this will be less than 3ε for n large enough. Moreover, V is compact and so the space $\mathrm{lip}_0(V)$ is closed (see, e.g., [42, Corollary 4.5]). Since $f_k|_V \to f|_V$ in norm, we get $f|_V \in \mathrm{lip}_0(V)$ as well, so that $f \in Y$. Thus, Y is closed.

Let us now verify (b). Fix $\mu \in \mathcal{F}(\mathcal{M})$, $\mu \neq 0$. Suppose first that $\mathrm{supp}(\mu)$ contains no isolated point of $\mathcal{M}$. Then, $\mathrm{supp}(\mu) = \{q\}$ as, by definition, the base point cannot be an isolated point of $\mathrm{supp}(\mu)$. Thus, μ is a nonzero multiple of $\delta(q)$, so that $h(\mu) \neq 0$ with $h \in Y$. Now suppose that $\mathrm{supp}(\mu)$ contains some isolated point x of $\mathcal{M}$. Then, $\{x\}$ is an open neighborhood of x, and so by [11, Proposition 2.7], there exists $f \in \mathrm{Lip}_0(\mathcal{M})$, supported on $\{x\}$, such that $f(\mu) \neq 0$. In other words, $\chi_{\{x\}}(\mu) \neq 0$, where $\chi_{\{x\}}$ is the characteristic function of the set $\{x\}$. But $\chi_{\{x\}} \in Y$, so this finishes the proof of (b).

Finally we check (c). We will see that, in fact, every $f \in Y$ attains its Lipschitz constant between two points of $\mathcal{M}$, and thus it attains its norm as a functional at some molecule. Fix $f \in Y$ and assume $\|f\|_L = 1$. Suppose that f does not attain its Lipschitz constant. Then, we may still find a sequence of pairs of points (u_n, v_n) in $\mathcal{M}$ such that $f(m_{u_n v_n}) \to 1$. Note that $x \in [u, v]$ implies that m_{uv} is a convex combination of m_{ux} and m_{xv}, hence

$$\max\{f(m_{ux}), f(m_{xv})\} \geqslant f(m_{uv}).$$

Thus, by replacing each pair u_n, v_n with other points in $[u_n, v_n]$ and passing to a subsequence if necessary, we may assume that either $u_n, v_n \in V$ for all n or there is a sequence (k_n) in $\mathbb{N}$ such that $u_n, v_n \in S_{k_n}$ for all n.

In the first case, all u_n, v_n belong to the compact set V, so by passing to a subsequence, we get $u_n \to u$, $v_n \to v$ for some $u, v \in V$. Note that it is impossible to get $u = v$, as that would contradict the fact that $f|_V$ is locally flat. Thus, $u \neq v$ and so $f(m_{uv}) = 1$.

Now suppose $u_n, v_n \in S_{k_n}$ for all n. If (k_n) is bounded, say by N, then this implies that f restricted to the finite set $S_1 \cup \cdots \cup S_N$ has Lipschitz constant 1, and so it must attain its Lipschitz constant in that set. Otherwise, we may assume $k_n \to \infty$. Then we have $\left\|f|_{S_{k_n}}\right\|_L \geqslant f(m_{u_n v_n})$ and, given that $\left\|f|_{S_{k_n}}\right\|_L \to |f(q)|$, we obtain $|f(q)| = 1$. So f attains its Lipschitz constant between p and q.

Thus, all conditions in the Petunin–Plichko theorem are satisfied, and this finishes the proof that $Y^* = \mathcal{F}(\mathcal{M})$. □

Remark 2.2. The definition of Y is not equivalent if we ask that f is locally flat instead of $f|_V$. For instance, h is not locally flat while $h|_V$ is. It is conjectured that whenever a Lipschitz-free space is a separable dual, it must admit a predual that consists entirely of locally flat functions. Preduals of Lipschitz-free spaces are not unique in general, so there might be an alternative predual Y satisfying that condition.

Remark 2.3. The argument in the proof of Theorem 2.1(i) can be easily adapted to show that the Lipschitz-free space from [9, Example 4.2] is also a dual space. The corresponding predual has a simpler description, as the local flatness condition can be dropped. Note that the molecule m_{0q} in that example is a Δ-point (this can be shown, e.g., by using Theorem 6.7 below) but not a Daugavet point as there are denting points in the unit ball of this space at distance strictly less than 2 to it.

Before we engage with the proof of Theorem 2.1(ii), let us note that the proof below can be adapted to show that the Lipschitz-free space over other similar metric spaces, such as [9, Examples 4.2 and 4.3], is also isomorphic to ℓ_1. It is based on the next general lemma, which can be understood as a finite version of the approach followed in [1].

Lemma 2.4. *Let M be a complete pointed metric space, and $\varphi_1, \dots, \varphi_n$ be nonnegative Lipschitz functions on M with bounded support and such that $\varphi_1 + \dots + \varphi_n = 1$. Suppose that $A_1, \dots, A_n$ are subsets of M containing the base point, and* $\operatorname{supp}(\varphi_k) \subset A_k$ *for all k. Then, $\mathcal{F}(M)$ is isomorphic to a complemented subspace of $\mathcal{F}(A_1) \oplus \dots \oplus \mathcal{F}(A_n)$.*

Proof. For each $k = 1, \dots, n$ let $W_k : \mathcal{F}(M) \to \mathcal{F}(M)$ be the weighting operator defined by

$$\langle W_k \mu, f \rangle := \langle \mu, f \cdot \varphi_k \rangle$$

for $\mu \in \mathcal{F}(M)$ and $f \in \operatorname{Lip}_0(M)$. By the results in [11, section 2], this is a well-defined bounded operator. Moreover, its range is contained in $\mathcal{F}(A_k)$, which we identify with the corresponding subspace of $\mathcal{F}(M)$. Now define operators

$$\begin{aligned} T &: \mathcal{F}(M) \to \mathcal{F}(A_1) \oplus \dots \oplus \mathcal{F}(A_n), \\ S &: \mathcal{F}(A_1) \oplus \dots \oplus \mathcal{F}(A_n) \to \mathcal{F}(M), \end{aligned}$$

by $T\mu := (W_1\mu, \dots, W_n\mu)$ and $S(\mu_1, \dots, \mu_n) := \mu_1 + \dots + \mu_n$. Both of them are clearly bounded, and for every $\mu \in \mathcal{F}(M)$ and $f \in \operatorname{Lip}_0(M)$, we have

$$\begin{aligned} \langle ST\mu, f \rangle &= \langle W_1\mu, f \rangle + \dots + \langle W_n\mu, f \rangle \\ &= \langle \mu, f \cdot \varphi_1 \rangle + \dots + \langle \mu, f \cdot \varphi_n \rangle \\ &= \langle \mu, f \rangle \end{aligned}$$

by the choice of φ_k. Thus, ST is the identity on $\mathcal{F}(M)$. Therefore, $P := TS$ is a projection of $\mathcal{F}(A_1) \oplus \dots \oplus \mathcal{F}(A_n)$ onto its subspace $T(\mathcal{F}(M))$, which is isomorphic to $\mathcal{F}(M)$. □

The following provides a simple sufficient condition allowing Lemma 2.4 to be applied.

Lemma 2.5. *Let M be a bounded complete metric space, and let $U_1, \dots, U_n$ be an open cover of M. Suppose*

$$\inf_{x \in M} \sum_{k=1}^{n} d(x, M \setminus U_k) > 0.$$

Then, there exist $\varphi_1, \dots, \varphi_n$, nonnegative Lipschitz functions on M, such that $\varphi_1 + \dots + \varphi_n = 1$ and each φ_k vanishes outside of U_k.

Proof. It is enough to take

$$\varphi_k(x) := \frac{d(x, M \setminus U_k)}{\sum_{i=1}^n d(x, M \setminus U_i)}$$

for $x \in M$. The denominator is Lipschitz and bounded below by assumption, so each φ_k is Lipschitz (see, e.g., [42, Proposition 1.30]), and the other conditions are satisfied trivially. □

Proof of Theorem 2.1(ii). Fix real numbers α, β with $0 < \beta < \alpha < \frac{1}{2}$ and consider the sets

$$A := \{(x, y) \in \mathcal{M} \ : \ x < \alpha\},$$

$$B := \{(x, y) \in \mathcal{M} \ : \ \beta < x < 1 - \beta\},$$

$$C := \{(x, y) \in \mathcal{M} \ : \ x > 1 - \alpha\},$$

which form an open cover of $\mathcal{M}$ to which we wish to apply Lemma 2.5. Note that $A \cap C = \emptyset$ while B intersects both A and C. For $z = (x, y) \in \mathcal{M}$, denote

$$D(z) := d(z, \mathcal{M} \setminus A) + d(z, \mathcal{M} \setminus B) + d(z, \mathcal{M} \setminus C).$$

Let us see that $D(z) \geqslant \alpha - \beta$ for all z. By symmetry, it is enough to verify this when $x \leqslant \frac{1}{2}$. Then, we have three cases:

- If $z \in A \setminus B$, then $D(z) = d(z, \mathcal{M} \setminus A)$. So either $z = p$ and then $D(z) \geqslant \alpha$, or $x \leqslant \beta$ and $y > 0$, then the closest point to z in $\mathcal{M} \setminus A$ has the form (x_a, y) for $x_a \geqslant \alpha$, and so $D(z) = x_a - x \geqslant \alpha - \beta$.
- If $z \in A \cap B$, then $D(z) = d(z, \mathcal{M} \setminus A) + d(z, \mathcal{M} \setminus B)$, and we have $y > 0$ and $\beta < x < \alpha$. Thus, the closest points in $\mathcal{M} \setminus A$ and $\mathcal{M} \setminus B$ are (x_a, y) and (x_b, y), where $x_a \geqslant \alpha$ and $x_b \leqslant \beta$, respectively, so $D(z) = (x_a - x) + (x - x_b) \geqslant \alpha - \beta$.
- If $z \in B \setminus (A \cup C)$, then $D(z) = d(z, \mathcal{M} \setminus B)$, and we have $y > 0$ and $\alpha \leqslant x \leqslant \frac{1}{2}$. Thus, the closest point in $\mathcal{M} \setminus B$ is (x_b, y), where $x_b \leqslant \beta$, and $D(z) = x - x_b \geqslant \alpha - \beta$.

Thus, $\inf D(z) > 0$, and we may apply Lemma 2.5 followed by Lemma 2.4 to conclude that $\mathcal{F}(\mathcal{M})$ is isomorphic to a complemented subspace of

$$\mathcal{F}(\overline{A}) \oplus \mathcal{F}(\overline{B} \cup \{p\}) \oplus \mathcal{F}(\overline{C} \cup \{p\}).$$

We will prove that each of these three Lipschitz-free spaces is isomorphic to ℓ_1, and then the result will follow by Pełczyński's classical theorem that complemented infinite-dimensional subspaces of ℓ_1 are isomorphic to ℓ_1.

Note that the set $\overline{A}$ is an infinite weighted tree, that is, a connected graph with no cycle. In particular, it is isometric to a subset of an $\mathbb{R}$-tree that contains all of its branching points $(0, 2^{-n})$, $n \in \mathbb{N}$. Thus, $\mathcal{F}(\overline{A})$ is isometric to ℓ_1 by [23, Corollary 3.4].

Let $K_0 := \{p\}$ and $K_n := \left\{(x, y) \in \overline{B} \ : \ y = 2^{-n}\right\}$ for each $n \in \mathbb{N}$. Then, each $\mathcal{F}(K_n)$ is isometric to a finite-dimensional ℓ_1-space, and there is a bound above and below on the distance between elements in distinct K_n's. Using [23, Proposition 5.1], we get that $\mathcal{F}(\overline{B} \cup \{p\})$ is isomorphic to ℓ_1.

Finally, notice that $\mathcal{F}(\overline{C}) = \mathcal{F}(\overline{A})$ as $\overline{C}$ and $\overline{A}$ are isometric. On the other hand, $\mathcal{F}(\overline{C})$ and $\mathcal{F}(\overline{C} \cup \{p\})$ are isomorphic by [1, Lemma 2.8]. Thus, $\mathcal{F}(\overline{C} \cup \{p\})$ is also isomorphic to ℓ_1 and this ends the proof. $\square$

We end this section by showing a couple of geometric properties of the predual Y of $\mathcal{F}(\mathcal{M})$. Recall that a Banach space X is *almost square* if for every finite subset $x_1, \ldots, x_n \in S_X$ and $\varepsilon > 0$, there exists $y \in S_X$ such that $\|x_i \pm y\| \leqslant 1 + \varepsilon$ for $i = 1, \ldots, n$ [5]. The space c_0 is the prototypical almost square Banach space.

For a proper metric space M, we have that $\mathrm{lip}_0(M)$ is, for any $\varepsilon > 0$, $(1+\varepsilon)$-isometric to a subspace of c_0 by [16, Lemma 3.9]. Hence, $\mathrm{lip}_0(M)$ is almost square [5, Example 3.2]. Next we show that the predual of $\mathcal{F}(\mathcal{M})$ shares this property. At the end of Section 5.3, we will see that, unlike $\mathrm{lip}_0(M)$ for M proper and purely 1-unrectifiable, Y is not M-embedded.

Proposition 2.6. *The predual Y of the space $\mathcal{F}(\mathcal{M})$ is almost square.*

Proof. Let $(f_i)_{i=1}^N \subset S_Y$ and $\varepsilon > 0$. As $f_i|_V$ is locally flat, we can choose $\delta > 0$ such that for all $a, b \in B(q, \delta) \cap V$,

$$|\langle f_i, m_{ab} \rangle| < \varepsilon$$

for all $i = 1, \ldots, N$. Choose k such that $2^{-k} < \varepsilon$ and such that the points $(1, 2^{-k+1})$, $(1, 2^{-k})$, and $(1, 2^{-k-1})$ are all in $B(q, \delta)$. Define $g \in Y$ by

$$g(a) := \begin{cases} 2^{-(k+1)} h(a), & a \in S_k \\ 0, & \text{otherwise.} \end{cases}$$

Let $a_0 = (1, 2^{-k}) \in S_k$. It is clear that $\|g\|_L = 1$, since the closest point to a_0 is at distance $2^{-(k+1)}$ and $h(a_0) = 1$. Let us check that $\|f_i \pm g\|_L \leqslant 1 + \varepsilon$.

If $a, b \notin \mathrm{supp}(g) \subset S_k$, then

$$|\langle f_i \pm g, m_{ab} \rangle| = |\langle f_i, m_{ab} \rangle| \leqslant 1.$$

If $a, b \in S_k$, then

$$|\langle f_i \pm g, m_{ab} \rangle| \leqslant |\langle f_i, m_{ab} \rangle| + 2^{-(k+1)} |\langle h, m_{ab} \rangle| \leqslant 1 + \varepsilon.$$

If $a \in S_k$ and $b \notin S_k$, then we can find $a' \in V \cap S_k$ and $b' \in V$ (at the level of b) such that $a' \in [a, b]$ and $b' \in [a', b]$. Now m_{ab} is a convex combination of $m_{aa'}$ and $m_{a'b}$, and $m_{a'b}$ is a convex combination of $m_{a'b'}$ and $m_{b'b}$. Hence,

$$\begin{aligned} |\langle f_i \pm g, m_{ab} \rangle| &\leqslant \max\{|\langle f_i \pm g, m_{aa'} \rangle|, |\langle f_i \pm g, m_{a'b'} \rangle|, |\langle f_i \pm g, m_{b'b} \rangle|\} \\ &\leqslant \max\{1 + \varepsilon, |\langle f_i \pm g, m_{a'b'} \rangle|\}. \end{aligned}$$

If a' and b' have first coordinate 0, then we are done by the above. If a' and b' have first coordinate 1, then by the convex combination trick, we may assume $b' \in B(q, \delta)$ (e.g., $b' = (1, 2^{-k+1})$ or $b' =$

$(1, 2^{-k-1})$). But for $a', b' \in B(q, \delta)$, we have

$$|\langle f_i \pm g, m_{a'b'} \rangle| \leqslant |\langle f_i, m_{a'b'} \rangle| + |\langle g, m_{a'b'} \rangle| \leqslant \varepsilon + 1$$

as desired. □

So while our predual Y shares some properties with c_0, we will now give a short proof that, unlike c_0, Y is not polyhedral. Recall that a Banach space X is *polyhedral* is the unit ball of every finite-dimensional subspace of X is a polytope.

Proposition 2.7. *The predual Y of $\mathcal{F}(\mathcal{M})$ is not polyhedral.*

Proof. It suffices to prove that the set of E of weak* strongly exposed points of $B_{\mathcal{F}(\mathcal{M})}$ is not a boundary for Y (cf., e.g., Theorem 1.4 in [21]). Now every weak* strongly exposed point is also a preserved extreme point and so, by, for example, [42, Corollary 3.44] and [10, Theorem 1.1], for every $\mu \in E$, there are $a, b \in \mathcal{M}$, $a \neq b$, with $[a, b] = \{a, b\}$ such that $\mu = m_{ab}$. Now, define the function $f : \mathcal{M} \to \mathbb{R}$ by

$$f(x, y) := x(1 - y^2)$$

for $z = (x, y) \in \mathcal{M}$. It is easy to check that $f \in S_Y$.

By construction, $|\langle f, m_{ab} \rangle| < 1$ for all a, b with $[a, b] = \{a, b\}$, except for $|\left\langle f, m_{pq} \right\rangle| = 1$. But m_{pq} is a Daugavet point, so E is not a boundary, and thus Y is not polyhedral. □

3 | A SUPERREFLEXIVE BANACH SPACE WITH A Δ-POINT

In [18], a renorming of ℓ_2 was used to show that there exists a Banach space whose norm is asymptotically midpoint uniformly convex, but not asymptotically uniformly convex. We will use a slight variation of this norm to answer Question 6.1 from [7] negatively; there exists an equivalent renorming of ℓ_2 with a Δ-point. We will also show that this Δ-point fails to be a Daugavet point in a very strong way by providing a sequence of strongly exposed points of the new unit ball that converges in norm to it. It is still open whether there exists a reflexive or superreflexive space with a Daugavet point.

Let $(e_n)_{n\geqslant 1}$ be the usual basis in ℓ_2 and denote the biorthogonal elements in ℓ_2 by $(e_n^*)_{n\geqslant 1}$. We follow [18] and introduce an equivalent norm on ℓ_2 by defining the unit ball by

$$B(\ell_2, \|\cdot\|) := \overline{\mathrm{conv}}\left(B_{\ell_2} \cup \left\{\pm(e_1 + e_n)_{n\geqslant 2}\right\}\right),$$

where we take the closure of the convex hull in the topology of $\|\cdot\|_2$. From [18, Lemma 2.5], we have that $\|\cdot\| \leqslant \|\cdot\|_2 \leqslant \sqrt{2}\|\cdot\|$, that $(e_n)_{n\geqslant 1}$ is a normalized bimonotone basis for $(\ell_2, \|\cdot\|)$, and that $\|e_1 + e_n\| = 1$ for $n \geqslant 2$.

We trim down the $\|\cdot\|$ norm and define for $x = \sum_{n=1}^{\infty} x_n e_n \in \ell_2$

$$|||x||| := \max\{\|x\|, \sup_{n\geqslant 2} |x_1 - 2x_n|\},$$

and $Y := (\ell_2, |||\cdot|||)$. We have $\|y\| \leqslant |||y||| \leqslant 3\|y\|$ for all $y \in Y$. Figure 2 shows a picture in $\mathrm{span}(e_1, e_n)$ of $S_{(\ell_2,\|\cdot\|)}$ in red and S_Y in blue.

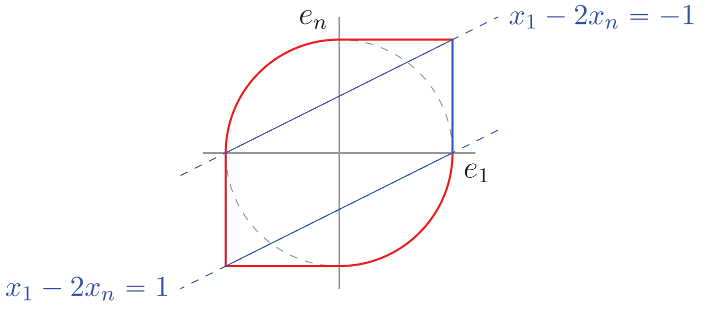

FIGURE 2 Geometric idea of the renorming.

It is clear that $|||e_1||| = 1$, $|||e_n||| = 2$, and $|||e_1 + e_n||| = 1$ for $n \geqslant 2$. These are all the ingredients needed to show that e_1 is a Δ-point.

To prove that e_1 is not a Daugavet point, we will find strongly exposed points at distance strictly less than 2 from e_1. As previously mentioned, we will actually prove something more, as we will show that e_1 belongs to the closure of the set of all strongly exposed points of B_Y. This will be done in the lemmas that follow the main theorem of this section that we will now state.

Theorem 3.1. *Let Y be the renorming of ℓ_2 above. We have that both $e_1 \in S_Y$ and $e_1^* \in S_{Y^*}$ are super Δ-points, but neither of them is a Daugavet point.*

The proof of the theorem uses a few simple lemmas. In [18], explicit expressions for how to calculate the $\|\cdot\|$-norm and its dual were given. These will turn out to be very useful even for the $|||\cdot|||$-norm, so let us provide right away some more detail.

Recall that in any given vector space V, we have

$$\operatorname{conv}(A \cup B) = \{\lambda a + (1-\lambda)b \;:\; a \in A,\; b \in B,\; \lambda \in [0,1]\}$$

for every convex set $A, B \subset V$. In particular, $B(\ell_2, \|\cdot\|)$ is equal to the closure of the set

$$C := \{\lambda y + (1-\lambda)u \;:\; y \in B_{\ell_2},\; u \in \operatorname{conv}\{\pm(e_1 + e_n),\; n \geqslant 2\},\; \lambda \in [0,1]\}. \tag{1}$$

The above description of the unit ball of $(\ell_2, \|\cdot\|)$ is used in [18, Lemma 2.5(b)] to show that the norm of $x^* = \sum_{n=1}^{\infty} a_n e_n^*$ in the dual is given by

$$\|x^*\| = \max\{\|x^*\|_2, \sup_{n \geqslant 2} |a_1 + a_n|\}.$$

A more or less identical argument shows that

$$B_{Y^*} = \overline{\operatorname{conv}}\left\{B_{(\ell_2,\|\cdot\|)^*} \cup \{\pm(e_1^* - 2e_n^*)_{n\geqslant 2}\}\right\},$$

hence B_{Y^*} is equal to the closure of the set

$$D := \{\lambda y^* + (1-\lambda)u^* \;:\; y^* \in B_{(\ell_2,\|\cdot\|)^*},\; u^* \in \operatorname{conv}\{\pm(e_1^* - 2e_n^*),\; n \geqslant 2\},\; \lambda \in [0,1]\}. \tag{2}$$

Next, let us identify some strongly exposed points of B_Y near e_1 and in B_{Y^*} near e_1^*.

Lemma 3.2. *For each $n \in \mathbb{N}$ define $x := x(n) \in Y$ and $x^* := x^*(n) \in Y^*$ by letting $k := 32n - 16$ and setting*

$$x := \left(1 - \frac{1}{n}\right)e_1 + \frac{1}{4n}\sum_{i=2}^{k+1} e_i \in Y$$

and

$$x^* := \left(1 - \frac{1}{n}\right)e_1^* + \frac{1}{4n}\sum_{i=2}^{k+1} e_i^* \in Y^*.$$

We have $|||x||| = \|x\| = \|x\|_2 = x^*(x) = 1$, and $|||x^*||| = \|x^*\| = \|x^*\|_2 = 1$ for each $n \in \mathbb{N}$.

Furthermore, $|||e_1 - x||| \underset{n}{\to} 0$ and $|||e_1^* - x^*||| \underset{n}{\to} 0$.

Proof. We have

$$x^*(x) = \|x\|_2^2 = \left(1 - \frac{1}{n}\right)^2 + k \cdot \frac{1}{16n^2} = 1 - \frac{2}{n} + \frac{1}{n^2} + \frac{2n-1}{n^2} = 1.$$

As for the dual, we always have $|||x^*||| \leqslant \|x^*\|$ and if we write $x^* = \sum_{i=1}^{\infty} a_i e_i^*$, then by Lemma 2.5(b) in [18],

$$\|x^*\| = \max\left\{\|x^*\|_2, \sup_{i \geqslant 2} |a_1 + a_i|\right\} = \max\left\{1, 1 - \frac{3}{4n}\right\} = 1.$$

Since $|(1 - \frac{1}{n}) - \frac{2}{4n}| < 1$, we also have $|||x||| = \|x\| \leqslant \|x\|_2 = 1$ (Lemma 2.5(a) in [18]). Hence, $|||x||| = |||x^*||| = 1$.

Finally,

$$\|e_1 - x\|_2 = \left\| \frac{1}{n}e_1 - \frac{1}{4n}\sum_{i=2}^{k+1} e_i \right\|_2 = \frac{\sqrt{16 + k}}{4n} = \sqrt{\frac{2}{n}}$$

and this expression tends to 0. Hence, $|||e_1 - x||| \to_n 0$. The calculation for $|||e_1^* - x^*||| \to_n 0$ is similar. $\square$

Lemma 3.3. *Let $n \in \mathbb{N}$. With $x := x(n) \in S_Y$ and $x^* := x^*(n) \in S_{Y^*}$ as in the previous lemma we have that x is strongly exposed by x^*.*

Proof. Recall that $\|x^*\| = |||x^*||| = 1$. The slice $S(B_{|||\cdot|||}, x^*, \delta)$ is contained in $S(B_{\|\cdot\|}, x^*, \delta)$, so it is enough to show $|||x - z||| \to 0$ uniformly on $S(B_{\|\cdot\|}, x^*, \delta) \cap C$ as $\delta \to 0$, where C is the set given by (1), and whose closure is equal to $B(\ell_2, \|\cdot\|)$.

Let $\varepsilon > 0$. By uniform convexity, there exists $\delta > 0$ such that $\|x - y\|_2 < \varepsilon$ whenever $\|x + y\|_2 > 2 - \delta$ and $\|x\|_2, \|y\|_2 \leqslant 1$. We may assume $\delta < \min\left\{\varepsilon, \frac{3}{4n}\right\}$. Let $z := \lambda y + (1 - \lambda)u$, where $0 \leqslant \lambda \leqslant 1$, $y \in B_{\ell_2}$, and $u \in \text{conv}\{\pm(e_1 + e_i)_{i \geqslant 2}\}$. We have

$$|x^*(u)| \leqslant \sup_{i \geqslant 2} |x^*(e_1 + e_i)| \leqslant 1 - \frac{3}{4n} < 1 - \delta.$$

So if $x^*(z) > 1-\delta$, this implies $x^*(y) > 1-\delta$. Hence

$$\|x+y\|_2 \geqslant x^*(x+y) > 2-\delta$$

and thus $\|x-y\|_2 < \varepsilon$. We also get

$$1-\delta < \lambda x^*(y) + (1-\lambda)x^*(u) \leqslant \lambda + \left(1-\frac{3}{4n}\right)(1-\lambda) = 1-\frac{3}{4n}(1-\lambda)$$

so that $1-\lambda < \frac{4n}{3}\delta$. Now

$$\begin{aligned}\frac{1}{3}|||x-z||| &\leqslant \|x-z\| \leqslant \|x-y\| + \|y-\lambda y\| + \|(1-\lambda)u\| \\ &\leqslant \sqrt{2}\|x-y\|_2 + (1-\lambda) + (1-\lambda) < 2\varepsilon + \frac{8n}{3}\delta \leqslant \left(2+\frac{8n}{3}\right)\varepsilon,\end{aligned}$$

and since n is a fixed constant, the conclusion follows. □

Lemma 3.4. *Let $n \in \mathbb{N}$. With $x := x(n) \in S_Y$ and $x^* := x^*(n) \in S_{Y^*}$ as in Lemma 3.2, we have that x^* is strongly exposed by x.*

Proof. Again, it is enough to show $|||x^*-z^*||| \to 0$ uniformly on $S(x,\delta) \cap D$ as $\delta \to 0$, where D is the set given by (2), and whose closure is equal to B_{Y^*}. Let $\varepsilon > 0$. By uniform convexity, there exists $\delta > 0$ such that $\|u-v\|_2 < \varepsilon$ whenever $\|u+v\|_2 > 2-\delta$ and $\|u\|_2, \|v\|_2 \leqslant 1$. We may assume $\delta < \min(\varepsilon, \frac{1}{n})$. Let $z^* := \lambda y^* + (1-\lambda)u^*$, where $0 \leqslant \lambda \leqslant 1$, $y^* \in B_{(\ell_2, \|\cdot\|)^*}$ and $u^* \in \operatorname{conv}\{\pm(e_1^* - 2e_i^*)_{i\geqslant 2}\}$. We have

$$|u^*(x)| \leqslant \sup_{i\geqslant 2} |(e_1^* - 2e_i^*)(x)| \leqslant 1-\frac{1}{n} < 1-\delta.$$

So if $z^*(x) > 1-\delta$, this implies $y^*(x) > 1-\delta$. Note that $\|y^*\|_2 \leqslant \|y^*\| \leqslant 1$ and $\|x^*\|_2 = \|x^*\| = 1$. Hence,

$$\|y^* + x^*\|_2 \geqslant (x^*+y^*)(x) > 2-\delta$$

and thus $\|y^*-x^*\|_2 < \varepsilon$. But then $\|y^*-x^*\| \leqslant \sqrt{2}\|y^*-x^*\|_2 \leqslant \sqrt{2}\varepsilon$. We also get

$$1-\delta < \lambda y^*(x) + (1-\lambda)u^*(x) \leqslant \lambda + \left(1-\frac{1}{n}\right)(1-\lambda) = 1-\frac{1}{n}(1-\lambda)$$

so that $1-\lambda < n\delta$. Finally,

$$\begin{aligned}|||x^*-z^*||| &\leqslant |||x^*-y^*||| + (1-\lambda)|||y^*||| + (1-\lambda)|||u^*||| \\ &\leqslant \|x^*-y^*\| + 2(1-\lambda) \\ &\leqslant \sqrt{2}\varepsilon + 2n\delta \leqslant \left(\sqrt{2}+2n\right)\varepsilon,\end{aligned}$$

and we are done. □

We are now ready to prove the main theorem of this section.

Proof of Theorem 3.1. By Lemmas 3.2 and 3.3, there exists a sequence $(x(n))_{n\geqslant 1}$ of strongly exposed points in S_Y converging to e_1 in norm, so e_1 is clearly not a Daugavet point. Similarly, by Lemmas 3.2 and 3.4, there exists a sequence $(x^*(n))_{n\geqslant 1}$ of strongly exposed points in S_{Y^*} converging to e_1^* in norm, so e_1^* is clearly not a Daugavet point.

It is quite obvious that $x \in S_X$ is a super Δ-point if and only if there is a net $(x_\alpha)_{\alpha\in\mathcal{A}} \subset S_X$ such that $x_\alpha \to x$ weakly and $\|x - x_\alpha\| \to 2$. This was observed in [32, Proposition 3.4]. Moreover, if X^* is separable, then we can clearly replace the net with a sequence in this characterization. Thus, taking $x = e_1$ and $x_n = e_1 + e_n$, $n \geqslant 2$, it follows that e_1 is a super Δ-point.

Finally, let us show that e_1^* is a super Δ-point. We have $|||x^*||| \leqslant \|x^*\|$ for all $x^* \in Y^*$. In particular, $|||e_n^*||| \leqslant \|e_n^*\| = 1$ for all $n \in \mathbb{N}$. For e_1^* we have $e_1^*(e_1) = 1$ and for $n \geqslant 2$ we have $e_n^*(e_1 + e_n) = 1$ so $e_n^* \in S_{Y^*}$ for all $n \in \mathbb{N}$. Next, we have for $x \in Y$ that

$$|(e_1^* - 2e_n^*)(x)| = |x_1 - 2x_n| \leqslant |||x|||$$

and since $(e_1^* - 2e_n^*)(e_1) = 1$, we get $|||e_1^* - 2e_n^*||| = 1$. As a conclusion, the sequence $(e_1^* - 2e_n^*)_{n\geqslant 1} \subset S_{Y^*}$ converges weakly to e_1^* and satisfies

$$|||e_1^* - (e_1^* - 2e_n^*)||| = 2|||e_n^*||| = 2,$$

for every $n \geqslant 2$, so e_1^* is a super Δ-point in B_{Y^*}. □

We can actually say a bit more about how the points e_1 and e_1^* sit in their respective unit balls—they are both extreme points.

Proposition 3.5. *We have that $e_1 \in \operatorname{ext} B_Y$ and $e_1 + e_n \in \operatorname{ext} B_Y$ for all $n \geqslant 2$. Similarly, $e_1^* \in \operatorname{ext} B_{Y^*}$ and $e_1^* - 2e_n^* \in \operatorname{ext} B_{Y^*}$ for all $n \geqslant 2$.*

Proof. Let $x := (x_n)_{n\geqslant 1} \in \ell_2$. We have the following easy facts.

(i) If $x_n > 1$ for some $n \in \mathbb{N}$, then $x \notin B(\ell_2, \|\cdot\|)$.
(ii) If $x_1 = 1$ and if $x_n < 0$ for some $n \geqslant 2$, then $x \notin B_Y$.

Let us see why (i) holds. We have

$$B_{\ell_2} \cup \{\pm(e_1 + e_n),\ n \geqslant 2\} \subset \{e_1^* \leqslant 1\} \cap \bigcap_{n\geqslant 2} \{e_n^* \leqslant 1\},$$

and since this set is clearly (weakly) closed and convex, we also have

$$B \subset \{e_1^* \leqslant 1\} \cap \bigcap_{n\geqslant 2} \{e_n^* \leqslant 1\},$$

and (i) follows. (ii) is clear by definition of $|||\cdot|||$. From this, it readily follows that e_1 and $e_1 + e_n$, $n \geqslant 2$, are extreme points in B_Y.

Next, let $x^* := (x_n)_{n\geqslant 1} \in \ell_2$. For the dual case, we have following.

(iii) If $x_1 > 1$ or $x_n > 2$ for some $n \in \mathbb{N}$, then $x^* \notin B_{Y^*}$.

(iv) If $x_1 = 1$, and if $x_n > 0$ for some $n \geqslant 2$, then $x^* \notin B_{Y^*}$.

Similarly to the above, we have

$$B_{(\ell_2,\|\cdot\|)^*} \cup \{\pm(e_1^* - 2e_n^*),\ n \geqslant 2\} \subset \{e_1 \leqslant 1\} \cap \bigcap_{n\geqslant 2}\{e_n \leqslant 2\},$$

and since this set is clearly (weakly) closed and convex, we also have

$$B_{Y^*} \subset \{e_1 \leqslant 1\} \cap \bigcap_{n\geqslant 2}\{e_n \leqslant 2\},$$

and (iii) follows. (iv) is clear since $|||e_1 + e_n||| = 1$ and $x^*(e_1 + e_n) = 1 + x_n$.

From (iii) and (iv), it readily follows that e_1^* and $e_1^* - 2e_n^*$ are extreme points in B_{Y^*}. □

Using notation from [32], we can say more about e_1 and e_1^*. If C is a convex set, then a given subset D of C is a *convex combination of relatively weakly open subsets* (*ccw* for short) of C, if D is of the form

$$D := \sum_{i=1}^{n} \lambda_i W_i,$$

where $n \in \mathbb{N}$, $\lambda_i \in (0, 1]$ for $i = 1, \dots, n$, $\sum_{i=1}^{n} \lambda_i = 1$ and W_i are relatively weakly open subsets of C.

Then, we have that e_1 and e_1^* are not only super Δ-points, but actually ccw Δ-points in the sense of [32].

Corollary 3.6. *If C is a ccw of B_Y such that $e_1 \in C$, then* $\sup_{y\in C} |||e_1 - y||| = 2$. *Similarly, if D is a ccw of B_{Y^*} such that $e_1^* \in D$, then* $\sup_{y^*\in D} |||e_1^* - y^*||| = 2$.

Proof. That super Δ-points that are extreme are ccw Δ-points is proved in [32, Proposition 3.13] (see also the remark following the proposition). □

To end the section, let us emphasize that the points e_1 and e_1^* satisfy the strongest possible Δ-property in Y and Y^* as they are both ccw Δ, but that they fail to be Daugavet points in an extreme way as they both belong to the closure of the set of all strongly exposed points in their respective unit balls. In particular, this example provides a positive answer to [32, Question 7.12].

4 | RENORMING ANY BANACH SPACE TO HAVE A Δ-POINT

From [7, Corollary 6.10], we know that no finite-dimensional Banach space admits a Δ-point. The following theorem, which is the main theorem in this section, highlights that the question of whether or not a Banach space contains a Δ-point is very much an isometric and not an isomorphic question. The proof combines ideas and results from the previous two sections.

Theorem 4.1. *Let X be an infinite-dimensional Banach space. Then the following holds:*

(i) *If X fails the Schur property, and in particular if X does not contain a copy of ℓ_1, then there exists an equivalent norm on X for which X contains a super Δ-point.*
(ii) *If X contains a copy of ℓ_1, then there exists an equivalent norm on X for which X contains a Δ-point.*

As we will show in Section 5 (see Corollary 5.8), the existence of a Δ-point in a Banach space automatically implies the existence of a weak* super Δ-point in its dual, so we will essentially focus here on the construction of Δ-points in our target spaces.

Our first step is showing that we can use similar ideas to the ones from Section 3 to renorm any infinite-dimensional Banach space that fails the Schur property with a super Δ-point.

Proof of Theorem 4.1(i). That spaces which do not contain a copy of ℓ_1 fail the Schur property follows from Rosenthal's ℓ_1 theorem. Now if X fails the Schur property, then using classic results on extraction of basic sequences (see, e.g., [4, Proposition 1.5.4]), we can construct a weakly null basic sequence $(e_n)_{n\geqslant 1} \subset S_X$. Let $(e_n^*)_{n\geqslant 1}$ be the sequence of biorthogonal functionals on the space $\overline{\operatorname{span}}(e_n)_{n\geqslant 1}$, and for every $n \geqslant 2$, take a Hahn–Banach extension $f_n \in X^*$ of the functional $e_1^* - 2e_n^*$. As $(e_n)_{n\geqslant 1}$ is basic, there exists a constant $K \geqslant 1$ such that $\sup_{n\geqslant 2} \|f_n\| \leqslant K$. We define an equivalent norm $|||\cdot|||$ on X by

$$|||x||| := \max\left\{\frac{1}{2}\|x\|, \sup_{n\geqslant 2} |f_n(x)|\right\}$$

for every $x \in X$. Then, $\frac{1}{2}\|\cdot\| \leqslant |||\cdot||| \leqslant K\|\cdot\|$, and by construction, we have, for every $n \geqslant 2$,

$$|||e_1||| = |||e_1 + e_n||| = 1 \text{ and } |||e_n||| = 2.$$

As $e_1 + e_n \to e_1$ weakly, it clearly follows that e_1 is a super Δ-point in $(X, |||\cdot|||)$. □

Before proving Theorem 4.1(ii), let us state the following lemma that is an easy consequence of a classical norm extension result.

Lemma 4.2. *Let X be a Banach space. If there exists a subspace Y of X that can be renormed to admit a Δ-point, then X can be renormed to admit a Δ-point.*

The same holds for super Δ-points.

Proof. Assume that there is a norm $|\cdot|$ on Y that admits a Δ- or a super Δ-point. By [17, Lemma II.8.1], we can extend $|\cdot|$ to an equivalent norm $|||\cdot|||$ on X, which coincides with $|\cdot|$ on Y. In other words, $(Y, |\cdot|)$ is isometrically isomorphic to a subspace of $(X, |||\cdot|||)$, and since both Δ- and super Δ-points pass to superspaces, the conclusion follows. □

Proof of Theorem 4.1(ii). If X contains a subspace Y, which is isomorphic to ℓ_1, then Y can be renormed to admit a Δ-point by Theorem 2.1. We finish by using Lemma 4.2. □

Let us end the present section with a few remarks.

Remark 4.3.

(a) Note that it is essential in the proof of Theorem 4.1(i) to go first through this process of extraction of a basic sequence in order to have complete control over the values of $f_n(e_m)$ for distinct m, n. It is thus unclear whether a similar construction could be done on a weakly null normalized net, and in particular whether it could be implemented in ℓ_1. Also, it is still unknown whether the Daugavet point in the space $\mathcal{F}(\mathcal{M})$ from [41, Example 3.1] studied in Section 2 is a super Δ-point (see Question 7.1 in [32] for further discussions). So we do not know whether ℓ_1 can be renormed with a super Δ-point.
(b) If X is a Banach space with a normalized weakly null Schauder basis $(e_n)_{n\geqslant 1}$, then the construction from Section 3 can be implemented in a natural way on this sequence in order to provide a renorming of X for which e_1 is a super Δ-point and e_1^* is a weak* super Δ-point. So using Lemma 4.2, we get an alternative geometric proof for Theorem 4.1(i).
(c) Let X be a Banach space with a normalized weakly null Schauder basis $(e_n)_{n\geqslant 1}$. Up to renorming, we may assume that $(e_n)_{n\geqslant 1}$ is bimonotone. Then, it is straightforward to check that for either the renorming from Theorem 4.1(i) or for the renorming copied from Section 3 that is discussed in item (b) above, the point e_1 is also an extreme point of the new ball, hence a ccw Δ-point in the sense of [32] (see Proposition 3.5 and Corollary 3.6). The same goes for e_1^* if the basis is moreover assumed to be shrinking, and in this case e_1^* becomes a weak* ccw Δ-point. However, it is unclear whether those points pass to superspaces in general, and thus we do not know whether Theorem 4.1(i) admits an analog for ccw Δ-points.

5 | DUALITY FOR Δ-POINTS AND APPLICATIONS

In this section, we provide a new powerful duality result for Δ-points, and collect a few striking consequences for the geometry of Banach spaces. The applications range from asymptotic geometry and unconditional bases to Hahn–Banach smooth spaces. The main theorem in this section is Theorem 5.6 where we prove that if a Banach space X contains a $\mathfrak{D}$-point, or if its dual X^* contains a weak* Δ-point, then in both cases, X^* actually contains a weak* super Δ-point. As a consequence, we show that $\mathfrak{D}$-points and weak* Δ-points are incompatible with some geometric properties of Banach spaces, such as asymptotic smoothness, shrinking, or monotone boundedly complete k-unconditional bases with $k < 2$, or Hahn–Banach smooth spaces that have a dual space with the Kadets–Klee property.

5.1 | Duality for Δ-points and asymptotic geometry

It was proved in [7] that Δ-points are incompatible with some asymptotic properties of smoothness and convexity of norms. The following can be obtained by combining results from [7] and [40].

Theorem 5.1. *Let X be a Banach space. If X is asymptotically uniformly smooth, then X contains no Δ-point, and X^* contains no* weak* *Δ-point.*

The ideas behind this result were related to the duality between asymptotic smoothness and weak* asymptotic convexity of the dual on one side, and to considerations on the Kuratowski measure of noncompactness α of weak* slices on the other. More precisely, the two following facts were obtained:

(i) If a point $x \in S_X$ is a Δ-point, then $\alpha(S(x,\delta)) = 2$ for every $\delta > 0$ [7, Theorem 3.5] or [40, Corollary 2.2]. In particular, no *asymptotically smooth point* can be a Δ-point [7, Proposition 3.6]. That is, x is not a Δ-point if $\lim_{t\to 0} \bar{\rho}(t,x)/t = 0$, where $\bar{\rho}(t,x)$ is the modulus of asymptotic smoothness at x (see, e.g., [4, Definition 14.6.1]).

(ii) If a point $x^* \in S_{X^*}$ is a weak* Δ-point, then every weak* slice S of B_{X^*} containing x^* has Kuratowski measure $\alpha(S) = 2$ [40, Corollary 2.4]. In particular, no *weak* quasi denting-point* can be a weak* Δ-point.

As every point in the unit sphere of a weak* asymptotically uniformly convex dual space is weak* quasi denting (see, e.g., the discussion following Corollary 4.7 in [7]), Theorem 5.1 immediately follows.

The relation between asymptotic properties and $\mathfrak{D}$-points was left aside in those papers, but let us point out that from the proof of [7, Theorem 4.2], we can collect the following lemma. Recall that a unit sphere element x in a Banach space X is an α-strongly exposed point if there exists $x^* \in D(x)$ such that $\lim_{\delta\to 0} \alpha(S(x^*,\delta)) = 0$.

Lemma 5.2. *Let X be a Banach space and let $x \in S_X$ be an α-strongly exposed point. Then x is not a $\mathfrak{D}$-point.*

Recall that a Banach space X has *Rolewicz' property* (α) if for every $x^* \in S_{X^*}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $\alpha(S(x^*,\delta)) \leqslant \varepsilon$. We say that X has *uniform property* (α) if the same δ works for all $x^* \in S_{X^*}$. These properties were introduced by Rolewicz in [36]. Implicit in Rolewicz [36, Theorem 3] is the result that X is asymptotically uniformly convex and reflexive if and only if X has uniform property (α); Rolewicz uses the term "X is Δ-uniformly convex" instead of X is asymptotically uniformly convex and reflexive.

As corollaries of Lemma 5.2, we get the two following results. The first corollary is a strengthening of [7, Theorem 4.4].

Corollary 5.3. *Let X be a Banach space. If X has Rolewicz' property (α), and in particular if X has finite dimension or if X is reflexive and asymptotically uniformly convex, then X contains no $\mathfrak{D}$-point.*

Proof. This immediately follows from Lemma 5.2 since from Rolewicz' property (α), every $x \in S_X$ is α-strongly exposed. □

Corollary 5.4. *Let X be a Banach space such that X^* is* weak* *asymptotically uniformly convex. Then, X^* contains no* weak* *Δ-points and no $x^* \in S_{X^*}$ that attains its norm on X is a $\mathfrak{D}$-point.*

Proof. That X^* has no weak* Δ-points is part of Theorem 5.1. Let $x^* \in S_{X^*}$ be such that there exists $x \in S_X$ with $x^*(x) = 1$. Then x^* is (weak*) α-strongly exposed by [7, Corollary 3.4], hence not a $\mathfrak{D}$-point by Lemma 5.2. □

Remark 5.5. Note that this result is sharp, because as the dual of c_0, we have that ℓ_1 is weak* asymptotically uniformly convex and that every $x \in S_{\ell_1}$ with infinite support is a $\mathfrak{D}$-point (see Proposition 2.3 in [3]).

It is unclear whether either [7, Theorem 3.5] or [40, Corollary 2.2] admit analogs for $\mathfrak{D}$-points. Yet, we can provide stronger duality results for those points that will in particular give new information about asymptotically smooth points and asymptotically uniformly smooth spaces.

Theorem 5.6. *Let X be a Banach space. If X contains a $\mathfrak{D}$-point, or if X^* contains a* weak* *Δ-point, then X^* contains a* weak* *super Δ-point.*

Proof. First, let us assume $x \in S_X$ is a $\mathfrak{D}$-point. We will distinguish between two cases. First assume that the set $D(x)$ contains exactly one element x^*. For every $n \in \mathbb{N}$, there exists $x_n \in S(x^*, 1/n)$ such that $\|x - x_n\| > 2 - 1/n$. Let $x_n^* \in S_{X^*}$ be such that $x_n^*(x) - x_n^*(x_n) > 2 - 1/n$. By weak* compactness of B_{X^*}, there exists a subnet $(y_\alpha^*)_{\alpha\in\mathcal{A}}$ of (x_n^*) that is weak*-convergent to some element $y^* \in B_{X^*}$. Since $x_n^*(x) > 1 - \frac{1}{n}$ for every $n \in \mathbb{N}$, we get $x_n^*(x) \to 1$ and thus also $y_\alpha^*(x) \to 1$. Hence, $y^* \in D(x)$, that is, $y^* = x^*$. Furthermore,

$$\|x^* - x_n^*\| \geqslant x^*(x_n) - x_n^*(x_n) > 2 - \frac{2}{n}$$

for every $n \in \mathbb{N}$. Therefore, $\|x^* - x_n^*\| \to 2$ and thus also $\|x^* - y_\alpha^*\| \to 2$, meaning x^* is a weak* super Δ-point.

Now assume that $D(x)$ has at least two distinct elements. Since $D(x)$ is convex, then $D(x)$ is infinite. Denote by $\mathcal{A}$ the directed set of all finite subsets of $D(x)$ ordered by inclusion.

For every $A \in \mathcal{A}$, we have $\frac{1}{|A|}\sum_{x^*\in A} x^*(x) = 1$, thus there exists $x_A \in S_X$ such that

$$\frac{1}{|A|}\sum_{x^*\in A} x^*(x_A) > 1 - \frac{1}{|A|^2}$$

and $\|x - x_A\| > 2 - \frac{1}{|A|}$. Therefore, there also exists $x_A^* \in S_{X^*}$ such that $x_A^*(x - x_A) > 2 - \frac{1}{|A|}$. There exists a subnet $(y_B^*)_{B\in\mathcal{B}}$ of $(x_A^*)_{A\in\mathcal{A}}$ that is weak*-convergent to some element $y^* \in B_{X^*}$. Since $x_A^*(x) > 1 - \frac{1}{|A|}$ for every $A \in \mathcal{A}$, we get $x_A^*(x) \to 1$ and thus also $y_B^*(x) \to 1$. Hence $y^* \in D(x)$.

Fix $\alpha > 0$. Let $A_0 \in \mathcal{A}$ be such that $y^* \in A_0$ and $\frac{2}{|A_0|} < \alpha$. Then for all $A \succeq A_0$, we have $y^* \in A$ and thus

$$\frac{1}{|A|}y^*(x_A) \geqslant \frac{1}{|A|}\sum_{x^*\in A} x^*(x_A) - 1 + \frac{1}{|A|} > \frac{1}{|A|} - \frac{1}{|A|^2},$$

which gives us

$$\|y^* - x_A^*\| \geqslant y^*(x_A) - x_A^*(x_A) > 1 - \frac{1}{|A|} + 1 - \frac{1}{|A|} \geqslant 2 - \frac{2}{|A_0|} > 2 - \alpha.$$

Therefore, $\|y^* - x_A^*\| \to 2$ and thus also $\|y^* - y_B^*\| \to 2$, meaning y^* is a weak* super Δ-point.

Second, let us assume $x_0^* \in S_{X^*}$ is a weak* Δ-point. Let $x_0 \in S_X$ be such that $x_0^*(x_0) > 1/2$. We construct recursively two sequences $(x_i)_{i\geqslant 1}$ and $(x_i^*)_{i\geqslant 1}$ in S_X and S_{X^*} in the following way. First choose $x_1^* \in S(x_0, 1/2)$ such that $\|x_0^* - x_1^*\| > 2 - 1/4$, and choose $x_1 \in S_X$ such that $x_0^*(x_1) > 1 - 1/4$ and $x_1^*(-x_1) > 1 - 1/4$.

Then, assume that we have found $x_1, \dots, x_{n-1} \in S_X$ and $x_1^*, \dots, x_{n-1}^* \in S_{X^*}$ such that

$$-x_k^*(x_k) > 1 - \frac{1}{2^{k+1}} \quad \text{and} \quad \sum_{i=0}^{k-1} x_k^*(x_i) > k-1 \quad \text{and} \quad x_0^*(x_k) > 1 - \frac{1}{2^{k+1}}$$

for every $k \in \{1, \dots, n-1\}$. Then,

$$x_0^*\left(\sum_{i=0}^{n-1} x_i\right) > \sum_{i=0}^{n-1}\left(1 - \frac{1}{2^{i+1}}\right) > n-1.$$

Since x_0^* is a weak* Δ-point, there exist $x_n^* \in S_{X^*}$ such that

$$x_n^*\left(\sum_{i=0}^{n-1} x_i\right) > n-1$$

and $\|x_0^* - x_n^*\| > 2 - 1/2^{n+1}$. Choose $x_n \in S_X$ such that $x_0^*(x_n) > 1 - 1/2^{n+1}$ and $-x_n^*(x_n) > 1 - 1/2^{n+1}$.

So we end up with two sequences $(x_i)_{i\geqslant 1}$ and $(x_i^*)_{i\geqslant 1}$ in S_X and S_{X^*} such that $x_i^*(x_i) \to -1$. Furthermore, we have

$$\sum_{i\in I} x_n^*(x_i) \geqslant \sum_{i=0}^{n-1} x_n^*(x_i) - (n - |I|) > n - 1 - (n - |I|) = |I| - 1$$

for every finite set $I \subseteq \{1, \dots, n-1\}$. As B_{X^*} is weak* compact, there exist subnets $(y_\alpha)_{\alpha\in\mathcal{A}}$ and $(y_\alpha^*)_{\alpha\in\mathcal{A}}$ of those sequences such that $(y_\alpha^*)_{\alpha\in\mathcal{A}}$ is weak*-convergent to some element $y^* \in B_{X^*}$. Since

$$x_n^*\left(\frac{1}{|I|}\sum_{i\in I} x_i\right) > 1 - \frac{1}{|I|}$$

whenever $n > \max I$, we get

$$y^*\left(\frac{1}{|I|}\sum_{i\in I} x_i\right) > 1 - \frac{1}{|I|}$$

for every finite set $I \subseteq \mathbb{N}$. We will finally show that there exists a subnet $(z_\beta)_{\beta\in\mathcal{B}}$ such that $y^*(z_\beta) \to 1$. Fix $\gamma > 0$ and $\alpha_0 \in \mathcal{A}$. Then, we can find a finite set $A \subseteq \{\alpha \in \mathcal{A} : \alpha \succeq \alpha_0\}$ such that $1/|A| < \gamma$ and all elements in A correspond to different elements in $\mathbb{N}$. Then,

$$y^*\left(\frac{1}{|A|}\sum_{\alpha\in A} y_\alpha\right) > 1 - \frac{1}{|A|}$$

and thus there exists $\alpha \in A$ such that $y^*(y_\alpha) > 1 - 1/|A| > 1 - \gamma$. It follows that 1 is a cluster point of the net $(y^*(y_\alpha))_{\alpha\in\mathcal{A}}$, and therefore there exist subnets $(z_\beta)_{\beta\in\mathcal{B}}$ and $(z_\beta^*)_{\beta\in\mathcal{B}}$ such that $y^*(z_\beta) \to$

1. As we are working with subnets, $z_\beta^*(z_\beta) \to -1$ by construction of the original sequences, so

$$\|y^* - z_\beta^*\| \geqslant y^*(z_\beta) - z_\beta^*(z_\beta) \to 2.$$

Therefore, y^* is a weak* super Δ-point. □

Remark 5.7. It is known that there are Δ-points that are not super Δ, and that there even exists a Banach space with a large subset of Daugavet points that contains no super Δ-point (combine Proposition 2.12 and Theorem 3.1 or Theorem 4.7 in [8]). So the previous result is completely specific to the weak* topology, and it is quite clear from the above proof that essentially all comes down to the weak* compactness of the dual unit ball.

Note that as a corollary of this result and Theorem 4.1, we get the following.

Corollary 5.8. *Let X be an infinite-dimensional Banach space. Then, X can be renormed so that X admits a Δ-point and X^* admits a* weak* *super Δ-point. Moreover, if X fails the Schur property, then X can be renormed so that X admits a super Δ-point and X^* admits a* weak* *super Δ-point.*

Recall that a (dual) Banach space X has the (weak*) *Kadets property* if the weak (respectively, the weak*) and norm topology coincide on the unit sphere of X. As pointed out in [32, section 3], it is clear that (weak*) super Δ-points are incompatible with being (weak*) Kadets. So observe that it immediately follows from Theorem 5.6 that if a Banach space X contains a $\mathfrak{D}$-point, or if its dual X^* contains a weak*-Δ-point, then X^* fails to be weak* Kadets. In particular, as weak* asymptotically uniformly convex duals satisfy a uniform weak* Kadets property in the sense of [30], we immediately get the following improved version of Theorem 5.1.

Corollary 5.9. *Let X be a Banach space. If X is asymptotically uniformly smooth, then X fails to contain $\mathfrak{D}$-points, and X^* fails to contain* weak* *Δ-points.*

Also observe that we can clearly get something more precise out of the proof of Theorem 5.6. Indeed, looking back there, we can see that if a point $x \in S_X$ is a $\mathfrak{D}$-point, then the set $D(x)$ contains a weak* super Δ-point. So as a corollary, we get the following pointwise result (see also the discussion following Theorem 5.1).

Corollary 5.10. *No asymptotically smooth point is a $\mathfrak{D}$-point.*

Proof. Let X be a Banach space. If $x \in S_X$ is a $\mathfrak{D}$ point, then as observed in the previous discussion, the set $D(x)$ contains a weak* super Δ-point y^*. As this point is also a weak* Δ-point, it follows from [40, Corollary 2.4] that $\alpha(S) = 2$ for every weak* slice S of B_{X^*} containing y^*. In particular, $\alpha(S(x, \delta)) = 2$ for every $\delta > 0$ since $y^*(x) = 1$. Then, by [7, Corollary 3.4], x is not an asymptotically smooth point (as the Kuratowski measure of the weak* slices of the dual unit ball defined by an asymptotically smooth point goes to 0 as δ goes to 0). □

Finally, note that we also get the following $\mathfrak{D}$-version of [7, Corollary 4.6].

Corollary 5.11. *If X is asymptotically uniformly convex and reflexive, then neither X nor X^* contain $\mathfrak{D}$-points.*

Proof. Since X is reflexive and asymptotically uniformly convex, X has no $\mathfrak{D}$-points by Corollary 5.3. Since X^* is asymptotically uniformly smooth X^* has no $\mathfrak{D}$-points by Corollary 5.9. □

5.2 | Shrinking and boundedly complete unconditional bases

We know from [8] that there exists a Banach space X with a 1-unconditional basis that admits a Daugavet point. This X contains copies of both c_0 and ℓ_1. In this section, we show that this is no coincidence, a consequence of our results is that if X has a shrinking or boundedly complete 1-unconditional basis, then X cannot contain Δ-points (see Corollary 5.15 below for the precise statement).

Instead of working with a basis, we can work with an approximating sequence of compact operators. A kind of reverse bounded approximation property will then prevent the existence of weak* super Δ-points in the dual in a similar way to [32, Proposition 3.20], and as a consequence of Theorem 5.6, it will also prevent the existence of $\mathfrak{D}$-point in the space and of weak* Δ-points in the dual. So here is the main theorem of this subsection.

Theorem 5.12. *Let X be a Banach space. Assume that there exists a family $(T_\lambda)_{\lambda\in\Lambda}$ of compact operators on X with $\sup_{\lambda\in\Lambda}\left\|I_X - T_\lambda\right\| < 2$, such that $T_\lambda^* \to I_{X^*}$ in the strong operator topology. Then, X contains no $\mathfrak{D}$-points, and X^* contains no* weak* *Δ-points.*

Proof of Theorem 5.12. Let $\varepsilon > 0$ be such that

$$\sup_{\lambda\in\Lambda}\left\|I - T_\lambda\right\| + 4\varepsilon \leqslant 2.$$

By Theorem 5.6, it is enough to show that if a net $(x_\alpha^*)_{\alpha\in\mathcal{A}} \subset B_{X^*}$ converges weak* to some $y^* \in B_{X^*}$, then for every $\alpha_0 \in \mathcal{A}$, there exists $\alpha \succeq \alpha_0$ such that

$$\|y^* - x_\alpha^*\| \leqslant 2 - 2\varepsilon.$$

Pick $\lambda \in \Lambda$ such that $\left\|y^* - T_\lambda^* y^*\right\| \leqslant \varepsilon$. By Schauder's theorem, T_λ^* is continuous as a map from (B_{X^*}, w^*) to $(X^*, \|\cdot\|)$ (see, e.g., [14, Theorem 6.26]) and it follows that $(T_\lambda^*(x_\alpha^*))_{\alpha\in\mathcal{A}}$ converges in norm to $T_\lambda^*(y^*)$. Thus, there exists $\alpha_0 \in \mathcal{A}$ such that $\left\|T_\lambda^*(y^* - x_\alpha^*)\right\| \leqslant \varepsilon$ for every $\alpha \succeq \alpha_0$.

Then, for every $\alpha \succeq \alpha_0$, we get

$$\begin{aligned}\|y^* - x_\alpha^*\| &\leqslant \left\|y^* - T_\lambda^* y^*\right\| + \left\|T_\lambda^*(y^* - x_\alpha^*)\right\| + \left\|x_\alpha^* - T_\lambda^* x_\alpha^*\right\| \\ &\leqslant \left\|I^* - T_\lambda^*\right\| + 2\varepsilon \\ &\leqslant 2 - 2\varepsilon,\end{aligned}$$

and the conclusion follows. □

Remark 5.13. In Theorem 5.12, we assume that X^* satisfies a version of the bounded compact approximation property with conjugate operators. It is clear from the above proof that actually all we need is a family $\mathcal{A} \subset \mathcal{K}(X)$ such that $\sup_{T\in\mathcal{A}} \|I_X - T\| < 2$ and such that given $\varepsilon > 0$ and $x^* \in X^*$, there exists $T \in \mathcal{A}$ such that $\|x^* - T^*x^*\| < \varepsilon$.

Let us now collect a few corollaries from Theorem 5.12.

Corollary 5.14. *Let X be a reflexive Banach space with a shrinking basis with partial sum projections (P_k). If $\sup_{k\in\mathbb{N}} \|I - P_k\| < 2$, then X and X^* contain no $\mathfrak{D}$-points.*

In particular, if X is a reflexive Banach space with k-unconditional basis for $k < 2$, then X and X^* contain no $\mathfrak{D}$-points.

In [8, Theorem 2.17], it was shown that Banach spaces with a subsymmetric basis have no Δ-points and this can be applied to show that the Schlumprecht space has no Δ-points. Unlike previous results in this direction, Corollary 5.14 applies also to the Tsirelson space and many of its relatives.

From Theorem 5.12 and the classic results from James on bases in Banach spaces, we also get the result announced at the start of this subsection.

Corollary 5.15. *Let X be a Banach space with shrinking k-unconditional basis for $k < 2$, then X contains no $\mathfrak{D}$-points and X^* contains no* weak* *Δ-points.*

Let X be a Banach space with a monotone boundedly complete k-unconditional basis for $k < 2$, then X contains no Δ-points.

In particular, a Banach space with 1-unconditional basis and a Δ-point contains a copy of c_0 and ℓ_1.

Proof. The first part follows directly from Theorem 5.12 using the partial sum projections.

If X has a monotone boundedly complete basis (e_n), then the biorthogonal functionals (e_n^*) is a shrinking basis for $Z = \overline{\operatorname{span}}(e_n^*)$ and X is isometric to Z^* (cf., e.g., [4, Theorem 3.2.15]). The unconditionality constants for (e_n) and (e_n^*) are the same and hence the second part follows from the first.

Finally, if (e_n) is an unconditional basis for X, then it fails to be shrinking if and only if X contains a subspace isomorphic to ℓ_1 (cf., e.g., [4, Theorem 3.3.1]) and it fails to be boundedly complete if and only if X contains a subspace isomorphic to c_0 (cf., e.g., [4, Theorem 3.3.2]). Since a 1-unconditional basis is monotone, the last part of the statement follows from the first two. □

Remark 5.16. Again, note that those results are sharp, as every norm-one element of ℓ_1 with infinite support is a $\mathfrak{D}$-point (see [3, Proposition 2.3]) and every norm-one sequence in c converging to 1 is a Daugavet point [3, Theorem 3.4].

Also, note that if we view ℓ_1 as the dual of c, then ℓ_1 contains a weak* super Δ-point by Theorem 5.6. However, ℓ_1 as the dual of c_0 contains no weak* Δ-points by Corollary 5.15.

We have one final corollary of this subsection. Recall that $\mathfrak{D}$-points can be lifted from subspaces.

Corollary 5.17. *If X is a subspace of a Banach space with a shrinking k-unconditional basis for $k < 2$, then X contains no $\mathfrak{D}$-points.*

Let us note that there is no quotient version of Corollary 5.17 since every separable Banach space is a quotient of ℓ_1. However, it is natural to ask the following.

Question 5.18. Can we conclude that X^* contains no weak* Δ-points if X is a subspace of a Banach space with a shrinking k-unconditional basis for $k < 2$?

According to Cowell and Kalton [15], a separable Banach space X has *property* (au^*) if $\lim_{n\to\infty} \|x^* + x_n^*\| = \lim_{n\to\infty} \|x^* - x_n^*\|$ whenever $x^* \in X^*$, $(x_n^*)_{n\geqslant 1}$ is a weak*-null sequence and both limits exist. In particular, spaces with Kalton's property (M*) and spaces with a 1-unconditional finite-dimensional decomposition satisfy this property.

It was shown in [15, Theorem 4.2] that a separable Banach space X has property (au^*) if and only if for every $\varepsilon > 0$, we have that X is $(1+\varepsilon)$-isomorphic to a subspace of a Banach space Y with a shrinking 1-unconditional basis. This is in turn equivalent to the fact that for every $\varepsilon > 0$, we have that X is isometric to a subspace of a Banach space Y with a shrinking $(1+\varepsilon)$-unconditional basis. Hence by Corollary 5.17, separable Banach spaces with property (au^*) have no $\mathfrak{D}$-points.

5.3 | Sequential super points and Hahn–Banach smooth spaces

Among the first nonreflexive examples of Banach spaces with no Δ-points were some M-embedded spaces and their duals, for example, c_0 and ℓ_1, $\mathcal{K}(\ell_2)$ and its dual, and the Schreier space and its dual. For the Schreier space and its dual, this was shown in [6], but it also follows from Corollary 5.15. The first two are covered by Corollary 5.9 and, in fact, both c_0 and $\mathcal{K}(\ell_2)$ have Kalton's property (M*). A not completely unnatural question to ask is the following:

Question 5.19. Let X be a nonreflexive M-embedded Banach space. Do X and its dual X^* fail to contain Δ-points?

In this section, we will show that the answer is positive when X^* is a Lipschitz-free space or more generally when X^* is Kadets–Klee. However, the following example shows that in general the answer is negative.

Example 5.20. There exists a nonreflexive M-embedded Banach space X such that both X and X^* have a super Δ-point.

Let $Y := (\ell_2, |||\cdot|||)$ be the renorming of ℓ_2 from Theorem 3.1. Then, both Y and Y^* admit a super Δ-point. Let $X := c_0(Y)$. Then, both X and $X^* = \ell_1(Y^*)$ isometrically contain a subspace with a super Δ-point. Since such points can be lifted to any superspace, both X and X^* admit a super Δ-point. Reflexive spaces are trivially M-embedded hence X is M-embedded by [24, Theorem III.1.6].

Before we move on to Hahn–Banach smooth spaces in more detail, let us point out that Corollary 5.9 already provides a positive answer to Question 5.19 for $X^* = \mathcal{F}(M)$ where M is a proper purely 1-unrectifiable metric space. Indeed, we have the following result.

Corollary 5.21. *If M is a proper metric space, then* $\mathrm{lip}_0(M)$ *is M-embedded and does not admit* $\mathfrak{D}$*-points.*

If, in addition, M is a proper purely 1-unrectifiable metric space, then $\mathcal{F}(M)$, the dual of $\mathrm{lip}_0(M)$, *has no* weak* Δ*-points.*

Proof. Let M be a proper metric space. Dalet showed that, for any $\varepsilon > 0$, the space $\mathrm{lip}_0(M)$ is $(1+\varepsilon)$-isomorphic to a subspace of c_0 [16, Lemma 3.9]. From the three-ball property [24, Theorem I.2.2], it follows that $\mathrm{lip}_0(M)$ is M-embedded. It also follows that $\mathrm{lip}_0(M)$ is asymptotically uniformly smooth (see, e.g., [34, Lemma 4.4.1]). By Corollary 5.9, we get that $\mathrm{lip}_0(M)$ does not admit $\mathfrak{D}$-points and its dual has no weak* Δ-points.

It is known that if M is a proper metric space, then $\mathcal{F}(M)$ is a dual space if and only if M is purely 1-unrectifiable and, in that case, $\mathrm{lip}_0(M)$ is a predual [2, Theorem 3.2]. Note that duals of M-embedded spaces are L-summands in their biduals and any Banach space has at most one predual that is M-embedded [24, Proposition IV.1.9]. □

It was proved by Fabian and Godefroy [19, Theorem 3] that every M-embedded space is weakly compactly generated and Asplund. In particular, the dual of any M-embedded space has the Radon–Nikodým property. In fact, every M-embedded space X is *Hahn–Banach smooth* (see, e.g., [24, Proposition I.1.12]), meaning that every $x^* \in X^*$ has a unique norm-preserving extension to X^{**}.

Sullivan proved in [39] that for a Hahn–Banach smooth space X, the relative weak and weak* topologies on B_{X^*} agree on S_{X^*} (also see [24, III.Lemma 2.14]). So a direct consequence of Theorem 5.6 is that if X is Hahn–Banach smooth, and X admits a $\mathfrak{D}$-point or X^* admits a weak* Δ-point, then X^* admits a super Δ-point. In particular, X^* fails the Kadets property.

Note that the sequential version of the Kadets property (the Kadets–Klee property, see below for the definition) is not incompatible with super Δ-points in general since there exists a Banach space with the Daugavet property and the Schur property [29]. However, it was pointed out in [32, Remark 3.5] that natural sequential versions of those points would automatically prevent the Schur property. So let us introduce the following.

Definition 5.22. Let X be a Banach space. We say that

(i) a point $x \in S_X$ is a *sequential super Δ-point* if there exists a sequence $(x_n)_{n\geqslant 1}$ in S_X that converges weakly to x and such that $\|x - x_n\| \to 2$;
(ii) a point $x^* \in S_{X^*}$ is a *weak* sequential super Δ-point* if there exists a sequence $(x_n^*)_{n\geqslant 1}$ in S_{X^*} that converges weak* to x^* and such that $\|x^* - x_n^*\| \to 2$.

Clearly, every (weak*) sequential super Δ-point is a (weak*) super Δ-point, and as we pointed out in the previous discussion, the converse does not hold in general.

However, it is clear that weak* sequential super Δ-points coincide with weak* super Δ-points in duals of separable spaces. With respect to the weak topology, recall that it is always possible by the results from Rosenthal to extract from any given bounded weakly converging net in a space that does not contain ℓ_1 a weakly converging sequence, so sequential super Δ-point and super Δ-points always coincide in this context. In particular, the two notions are equivalent in Asplund spaces.

As mentioned, sequential super Δ-points are incompatible with the Schur property. More precisely, they are incompatible with the Kadets–Klee property. Recall that a (dual) Banach space X is said to be (weak*) *Kadets–Klee* if every for every sequence $(x_n)_{n\geqslant 1}$ in S_X, and for every $x \in S_X$, we have that $x_n \to x$ weakly (respectively weak*) if and only if $\|x - x_n\| \to 0$.

Clearly, the existence of a (weak*) sequential super Δ-point is incompatible with the (weak*) Kadets–Klee property.

It turns out that Theorem 5.6 actually provides a weak* sequential super Δ-point under the assumption that the unit ball of the dual space is weak* sequentially compact.

Theorem 5.23. *Let X be a Banach space such that B_{X^*} is* weak* *sequentially compact. If X contains a $\mathfrak{D}$-point, or if X^* contains a* weak* *Δ-point, then X^* contains a sequential* weak* *super Δ-point.*

Proof. If X contains a $\mathfrak{D}$-point, then by Theorem 5.6, X^* admits a weak* super Δ-point, hence a weak* Δ-point, so it is sufficient to prove the second part of the statement.

So let us assume that X^* admits a weak* Δ-point x^*. Looking back at the second part of the proof of Theorem 5.6, it is clear that we can use weak* sequentially compactness to extract sequences instead of nets at each key step, so that we end up with a sequential weak* super Δ-point y^*. The conclusion follows. □

Stegall proved in [38, Theorem 3.5] that if a Banach space X is weak Asplund, then the unit ball of X^* is weak* sequentially compact. So every Asplund space satisfies this property, and the same goes for weakly compactly generated spaces by the results from Asplund's seminal paper [13].

Smith and Sullivan proved in [37] that Hahn–Banach smooth spaces are Asplund, so for Hahn–Banach smooth spaces, we get the following corollary.

Corollary 5.24. *Let X be a Hahn–Banach smooth space. If X admits a $\mathfrak{D}$-point or if X^* admits a* weak**Δ-point, then X^* contains a sequential super Δ-point. In particular, X^* fails to be Kadets–Klee.*

In particular, note that in the reflexive setting, we even get the following result.

Corollary 5.25. *Let X be a reflexive Banach space. If X contains a $\mathfrak{D}$-point, then X and X^* both contain a sequential super Δ-point. In particular, neither X nor X^* are Kadets–Klee.*

Proof. As reflexive spaces are Hahn–Banach smooth, the existence of a $\mathfrak{D}$-point in X implies the existence of a sequential super Δ-point in X^* by Corollary 5.24. But as super Δ-points are also $\mathfrak{D}$-points, this in turns implies the existence of a sequential super Δ-point in X. □

Remark 5.26. In particular, note that this result shows that it is no coincidence that the example from Section 3, which is the first example of a reflexive Banach space with a Δ-point, in fact provides an example of a reflexive space with a super Δ-point and a super Δ-point in its dual.

Also let us point out, in relation to the discussions from Section 5.1, that it was proved by Montesinos [31, Theorem 3] that a reflexive Banach space X has property (α) if and only if it is Kadets–Klee (the latter being referred to there as "property (H) of Radon-Riesz"). So Corollary 5.25 can be seen as an improved version of Corollary 5.11.

Finally, we get some general results for Lipschitz-free spaces with a Hahn–Banach smooth predual. For Lipschitz-free spaces, the Radon–Nikodým property and the Schur property are equivalent by [2, Theorem 4.6]. Hence, as noted above, since Hahn–Banach smooth spaces are Asplund, we get from Corollary 5.24 the following.

Corollary 5.27. *Let M be a metric space such that $\mathcal{F}(M)$ is a dual space with a Hahn–Banach smooth predual Y. Then, Y does not contain any $\mathfrak{D}$-point and $\mathcal{F}(M)$ does not contain any* weak* *Δ-point.*

Let $\mathcal{M}$ be the metric space defined by Veeorg that we studied in Section 2. Since $\mathcal{F}(\mathcal{M})$ contains a Daugavet point, we get from Corollary 5.27 that no predual of $\mathcal{F}(\mathcal{M})$ is Hahn–Banach smooth. Similarly one can show that for the metric space M in [9, Example 4.2] the molecule m_{0q} is a Δ-point (but not a Daugavet point), hence no predual of $\mathcal{F}(M)$ is Hahn–Banach smooth. In next section, we will study in more detail Δ-points in Lipschitz-free spaces. In particular, we will prove in Theorem 6.7 that for any two distinct points x and y in a metric space M, the molecule $m_{x,y}$ is a Δ-point if and only if x and y are discretely connectable in M (see Definition 6.1). So let us point out right away that together with this result, Corollary 5.27 also provides new information about the metric spaces M such that the associated Lipschitz-free space has a Hahn–Banach smooth predual.

Corollary 5.28. *Let M be a metric space such that $\mathcal{F}(M)$ is a dual space with a Hahn–Banach smooth predual Y. Then, no two distinct points in M are discretely connectable.*

We started this subsection with an example showing that nonreflexive M-embedded Banach spaces and their duals can contain Δ-points. We end with the corresponding question for Daugavet points.

Question 5.29. Does there exist a nonreflexive M-embedded space X such that X or X^* contains a Daugavet point?

6 | METRIC CHARACTERIZATION OF Δ-MOLECULES

In this last section, we will focus on Δ-points in Lipschitz-free spaces. Our main goal is to obtain a purely metric characterization of those molecules m_{xy} that are Δ-points of $\mathcal{F}(M)$ (see Theorem 6.7). In [26, Proposition 4.2], Jung and Rueda Zoca gave a sufficient metric condition: that x and y be *connectable*, that is, that they can be joined by Lipschitz paths in M whose length is arbitrarily close to $d(x, y)$. In particular, if x and y are connected by a geodesic, then m_{xy} is a Δ-point. Moreover, the converse is true when M is compact under some additional assumptions [26, Theorem 4.13]. We will show below that these extra hypotheses are, in fact, superfluous and the existence of a geodesic characterizes Δ-molecules for proper M (see Corollary 6.8). For general M, the notion of connectability is unnecessarily strong and it may be relaxed to allow for discrete paths as follows.

Definition 6.1. Let $x, y \in M$. Given $\varepsilon > 0$, we say that x and y are *ε-discretely connectable* (*in M*) if there exists a finite sequence of points $p_0, p_1, \dots, p_n, p_{n+1}$ in M, where $p_0 = x$ and $p_{n+1} = y$, with the following properties:

(1) $d(p_i, p_{i+1}) < \varepsilon$ for each $i = 0, \dots, n$, and
(2) $\sum_{i=0}^{n} d(p_i, p_{i+1}) < d(x, y) + \varepsilon$.

We say that x and y are *discretely connectable* if they are ε-discretely connectable for every $\varepsilon > 0$.

In the case where M is proper, this notion is still equivalent to the existence of geodesics.

Proposition 6.2. *If M is a proper metric space and $x \neq y \in M$, then x and y are discretely connectable if and only if they are connected by a geodesic.*

Proof. Denote $I = [0, d(x,y)] \subset \mathbb{R}$ and let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$.

For every $n \in \mathbb{N}$, we can find $p_0^n, p_1^n, \dots, p_{m(n)+1}^n$ in M with $p_0^n = x$ and $p_{m(n)+1}^n = y$ with $d(p_i^n, p_{i+1}^n) < 1/n$ for $0 \leqslant i \leqslant m(n)$ and

$$\sum_{i=0}^{m(n)} d(p_i^n, p_{i+1}^n) < d(x,y) + \frac{1}{n}.$$

Fix $n \in \mathbb{N}$. For every $a \in I$, there exists $k = k_n(a)$ with $1 \leqslant k \leqslant m(n)+1$ such that

$$a < \sum_{i=0}^{k-1} d(p_i^n, p_{i+1}^n) \leqslant a + \frac{1}{n}.$$

Define $p_a^n = p_k^n$. Then,

$$d(x, p_a^n) \leqslant \sum_{i=0}^{k-1} d(p_i^n, p_{i+1}^n) \leqslant a + \frac{1}{n}$$

and

$$d(p_a^n, y) \leqslant \sum_{i=k}^{m(n)} d(p_i^n, p_{i+1}^n) = \sum_{i=0}^{m(n)} d(p_i^n, p_{i+1}^n) - \sum_{i=0}^{k-1} d(p_i^n, p_{i+1}^n) < d(x,y) + \frac{1}{n} - a,$$

so that

$$p_a^n \in B\Big(x, a + \frac{1}{n}\Big) \cap B\Big(y, d(x,y) - a + \frac{1}{n}\Big).$$

Note that all points under consideration belong to the compact set $B(x, d(x,y)+1)$, so we can define $p_a = \lim_{\mathcal{U}} p_a^n$. In particular, we must have $x = p_0$ and $y = p_{d(x,y)}$.

Let us check that the map $a \mapsto p_a$ from I to M is an isometry. Let $a, b \in I$ with $a < b$. Given $\delta > 0$, choose $A \in \mathcal{U}$ such that $p_a^n \in B(p_a, \delta)$ for all $n \in A$ and $B \in \mathcal{U}$ such that $p_b^n \in B(p_b, \delta)$ for all $n \in B$. Let $C = A \cap B \in \mathcal{U}$. Since $\mathcal{U}$ is free, we know that C is infinite. Therefore, we can choose $N \in C$ such that $1/N < \delta$. If $k_N(a) = k_N(b)$, then $p_a^N = p_b^N \in B(p_a, \delta) \cap B(p_b, \delta)$ and $d(p_a, p_b) \leqslant 2\delta$. Otherwise $k_N(a) < k_N(b)$ and we have

$$\begin{aligned} d(p_a, p_b) &\leqslant \delta + d(p_a^N, p_b^N) + \delta \leqslant 2\delta + \sum_{i=k_N(a)}^{k_N(b)-1} d(p_i^N, p_{i+1}^N) \\ &= 2\delta + \sum_{i=0}^{k_N(b)-1} d(p_i^N, p_{i+1}^N) - \sum_{i=0}^{k_N(a)-1} d(p_i^N, p_{i+1}^N) \\ &< 2\delta + b + \frac{1}{N} - a < b - a + 3\delta. \end{aligned}$$

Since $\delta > 0$ was arbitrary, we get $d(p_a, p_b) \leqslant |b - a|$ for all $a, b \in I$. Now, we also have

$$d(x, y) \leqslant d(x, p_a) + d(p_a, p_b) + d(p_b, y) \leqslant (a - 0) + (b - a) + (d(x, y) - b) = d(x, y)$$

so all inequalities must be equalities and $d(p_a, p_b) = |b - a|$ for all $a, b \in I$. □

The argument used in [26, Proposition 4.2] can be discretized to show that m_{xy} is a Δ-point whenever x and y are discretely connectable. With a slight variation of that argument, we get the following, more general result.

Proposition 6.3. *Let $\mu \in S_{\mathcal{F}(M)}$. Suppose that, for every $\eta > 0$, μ can be expressed as a series of molecules*

$$\mu = \sum_{k=1}^{\infty} a_k m_{x_k y_k} \quad \textit{with} \quad \sum_{k=1}^{\infty} |a_k| < 1 + \eta \tag{3}$$

such that each pair (x_k, y_k) is discretely connectable in M. Then, μ is a Δ-point in $\mathcal{F}(M)$.

The hypothesis clearly holds if $\mu = m_{xy}$ where x and y are discretely connectable. More generally, it also holds if every pair of points in $\mathrm{supp}(\mu) \cup \{0\}$ is discretely connectable in M, as every $\mu \in S_{\mathcal{F}(M)}$ admits an expression of the form (3) (see, e.g., [10, Lemma 2.1]).

Proof of Proposition 6.3. Let $S = \{\nu \in B_{\mathcal{F}(M)} : f(\nu) > 1 - \alpha\}$ be a slice containing μ, for some $f \in S_{\mathrm{Lip}_0(M)}$ and $\alpha > 0$. Fix $\eta > 0$ such that $f(\mu) > (1 - \alpha)(1 + \eta)$, and choose a representation of μ of the form (3) where every pair (x_k, y_k) is discretely connectable. We may assume $a_k \geqslant 0$ for all k by swapping x_k with y_k if needed. Then, by convexity, we must have $f(m_{x_k y_k}) > 1 - \alpha$ for some k, that is, there are $x = x_k, y = y_k$ in M that are discretely connectable and such that $m_{xy} \in S$.

Now fix $\delta > 0$ such that $f(m_{xy}) > (1 - \alpha)(1 + \delta)$, and let $\varepsilon < \delta \cdot d(x, y)$ be arbitrary. Choose a sequence of points $p_1, \ldots, p_n \in M$ as in Definition 6.1, and denote $p_0 = x$, $p_{n+1} = y$. Then, we have

$$\begin{aligned}
\max\left\{f(m_{p_i, p_{i+1}}) : i = 0, \ldots, n\right\} &= \max\left\{\frac{f(p_i) - f(p_{i+1})}{d(p_i, p_{i+1})} : i = 0, \ldots, n\right\} \\
&\geqslant \frac{\sum_{i=0}^{n} (f(p_i) - f(p_{i+1}))}{\sum_{i=0}^{n} d(p_i, p_{i+1})} \\
&> \frac{f(x) - f(y)}{d(x, y) + \varepsilon} \\
&> \frac{f(m_{xy})}{1 + \delta} > 1 - \alpha.
\end{aligned}$$

Therefore, we may choose $u = p_k, v = p_{k+1}$ with $d(u, v) < \varepsilon$ and $f(m_{uv}) > 1 - \alpha$, that is, $m_{uv} \in S$. By [26, Theorem 2.6], it follows that S contains elements whose distance to μ is arbitrarily close to 2, hence μ is a Δ-point. □

Discrete connectability does, in fact, characterize Δ-molecules in Lipschitz-free spaces. In order to prove this, we will now construct a family of alternative metrics on M that provide information about how "well connected" (in the sense of Definition 6.1) a given pair of points is, by reducing their distance whenever there is a partial discrete path between them. We define them precisely as the shortest possible distance when giving a preference to discrete paths with sufficiently small step.

Fix $\alpha \in (0,1)$. For any $\varepsilon > 0$ and $x, y \in M$, we write

$$w_{\alpha,\varepsilon}(x,y) := \begin{cases} d(x,y), & \text{if } d(x,y) \geqslant \varepsilon \\ (1-\alpha)d(x,y), & \text{if } d(x,y) < \varepsilon \end{cases}$$

and

$$b_{\alpha,\varepsilon}(x,y) := \inf\left\{ \sum_{i=0}^{n} w_{\alpha,\varepsilon}(p_i, p_{i+1}) \ : \ p_0, p_1, \dots, p_{n+1} \in M, p_0 = x, p_{n+1} = y \right\}.$$

Note that $w_{\alpha,\varepsilon}$ and $b_{\alpha,\varepsilon}$ increase as ε decreases, so we can also define

$$b_\alpha(x,y) := \sup_{\varepsilon>0} b_{\alpha,\varepsilon}(x,y) = \lim_{\varepsilon\to 0} b_{\alpha,\varepsilon}(x,y).$$

Lemma 6.4. *Fix $\alpha \in (0,1)$ and $\varepsilon > 0$.*

(i) *b_α and $b_{\alpha,\varepsilon}$ are bi-Lipschitz equivalent metrics on M.*

(ii) *For any $x, y \in M$, we have*

$$(1-\alpha)d(x,y) \leqslant b_{\alpha,\varepsilon}(x,y) \leqslant b_\alpha(x,y) \leqslant d(x,y).$$

(iii) *If $x, y \in M$ are not ε-discretely connectable, then*

$$b_{\alpha,\varepsilon}(x,y) \geqslant (1-\alpha)d(x,y) + \varepsilon \cdot \min\{\alpha, 1-\alpha\}.$$

(iv) *Two points $x, y \in M$ are discretely connectable if and only if*

$$b_\alpha(x,y) = (1-\alpha)d(x,y).$$

Proof. For any $x, y \in M$, we have $b_{\alpha,\varepsilon}(x,y) \leqslant w_{\alpha,\varepsilon}(x,y) \leqslant d(x,y)$. Notice also that for any finite sequence $p_0 = x, p_1, \dots, p_n, p_{n+1} = y$ in M, we have

$$\sum_{i=0}^{n} w_{\alpha,\varepsilon}(p_i, p_{i+1}) \geqslant \sum_{i=0}^{n} (1-\alpha)d(p_i, p_{i+1}) \geqslant (1-\alpha)d(x,y),$$

and hence $b_{\alpha,\varepsilon}(x,y) \geqslant (1-\alpha)d(x,y)$. This proves (ii) for $b_{\alpha,\varepsilon}$ and thus also for b_α by taking limits.

It is clear that $w_{\alpha,\varepsilon}$ and $b_{\alpha,\varepsilon}$ are symmetric. For any $x, y, z \in M$ and $\delta > 0$, we may find two finite sequences $p_1, \dots, p_n$ and $p_1', \dots, p_m'$ of points in M such that

$$w_{\alpha,\varepsilon}(x, p_1) + w_{\alpha,\varepsilon}(p_1, p_2) + \cdots + w_{\alpha,\varepsilon}(p_n, z) < b_{\alpha,\varepsilon}(x, z) + \delta,$$
$$w_{\alpha,\varepsilon}(z, p_1') + w_{\alpha,\varepsilon}(p_1', p_2') + \cdots + w_{\alpha,\varepsilon}(p_m', y) < b_{\alpha,\varepsilon}(z, y) + \delta,$$

and therefore,

$$\begin{aligned} b_{\alpha,\varepsilon}(x, y) &\leqslant w_{\alpha,\varepsilon}(x, p_1) + \cdots + w_{\alpha,\varepsilon}(p_n, z) + w_{\alpha,\varepsilon}(z, p_1') + \cdots + w_{\alpha,\varepsilon}(p_m', y) \\ &< b_{\alpha,\varepsilon}(x, z) + b_{\alpha,\varepsilon}(z, y) + 2\delta. \end{aligned}$$

Letting $\delta \to 0$ yields the triangle inequality for $b_{\alpha,\varepsilon}$. Together with (ii) this shows that $b_{\alpha,\varepsilon}$ is an equivalent metric on M, and letting $\varepsilon \to 0$ we get (i).

For part (iii), the assumption is that for any finite sequence $p_0 = x, p_1, \dots, p_{n+1} = y$ in M, at least one of the two following statements holds:

(a) $\sum_{i=0}^{n} d(p_i, p_{i+1}) \geqslant d(x, y) + \varepsilon$.
(b) $d(p_k, p_{k+1}) \geqslant \varepsilon$ for some $k \in \{0, \dots, n\}$.

In case (a), we have

$$\sum_{i=0}^{n} w_{\alpha,\varepsilon}(p_i, p_{i+1}) \geqslant \sum_{i=0}^{n} (1-\alpha) d(p_i, p_{i+1}) \geqslant (1-\alpha)(d(x, y) + \varepsilon) \geqslant (1-\alpha) d(x, y) + (1-\alpha)\varepsilon.$$

In case (b), we have

$$\begin{aligned} \sum_{i=0}^{n} w_{\alpha,\varepsilon}(p_i, p_{i+1}) &\geqslant (1-\alpha) \sum_{i=0}^{k-1} d(p_i, p_{i+1}) + d(p_k, p_{k+1}) + (1-\alpha) \sum_{i=k+1}^{n} d(p_i, p_{i+1}) \\ &= (1-\alpha) \sum_{i=0}^{n} d(p_i, p_{i+1}) + \alpha d(p_k, p_{k+1}) \\ &\geqslant (1-\alpha) d(x, y) + \alpha\varepsilon. \end{aligned}$$

Taking the infimum over all choices of p_i yields (iii).

Finally, one of the implications in (iv) is given by (iii). For the converse, assume that x and y are discretely connectable. Let $\varepsilon' > 0$ and $\delta \in (0, \varepsilon')$. Then, one may find finitely many points $p_0 = x, p_1, \dots, p_n, p_{n+1} = y$ in M such that $d(p_i, p_{i+1}) < \delta$ and $\sum_{i=0}^{n} d(p_i, p_{i+1}) < d(x, y) + \delta$. Thus,

$$b_{\alpha,\varepsilon'}(x, y) \leqslant \sum_{i=0}^{n} w_{\alpha,\varepsilon'}(p_i, p_{i+1}) = (1-\alpha) \sum_{i=0}^{n} d(p_i, p_{i+1}) < (1-\alpha)(d(x, y) + \delta).$$

Letting $\delta \to 0$ followed by $\varepsilon' \to 0$ yields $b_\alpha(x, y) \leqslant (1-\alpha) d(x, y)$, and an appeal to (ii) ends the proof. □

Note that Lemma 6.4(ii) implies that $\mathcal{F}(M,d)$ and $\mathcal{F}(M,b_\alpha)$ are linearly isomorphic completions of span $\delta(M)$, and in particular $\|\cdot\|_{\mathcal{F}(M,b_\alpha)}$ is an equivalent norm on $\mathcal{F}(M)$. The same holds for $b_{\alpha,\varepsilon}$ in place of b_α. We have

$$(1-\alpha)\|\mu\|_{\mathcal{F}(M)} \leqslant \|\mu\|_{\mathcal{F}(M,b_{\alpha,\varepsilon})} \leqslant \|\mu\|_{\mathcal{F}(M,b_\alpha)} \leqslant \|\mu\|_{\mathcal{F}(M)}$$

for $\mu \in \mathcal{F}(M)$, and similarly

$$(1-\alpha)B_{\mathrm{Lip}_0(M,d)} \subseteq B_{\mathrm{Lip}_0(M,b_{\alpha,\varepsilon})} \subseteq B_{\mathrm{Lip}_0(M,b_\alpha)} \subseteq B_{\mathrm{Lip}_0(M,d)}.$$

We also have the following.

Lemma 6.5. *For any $\mu \in \mathcal{F}(M)$ and $\alpha \in (0,1)$, we have*

$$\lim_{\varepsilon\to 0} \|\mu\|_{\mathcal{F}(M,b_{\alpha,\varepsilon})} = \|\mu\|_{\mathcal{F}(M,b_\alpha)}.$$

Proof. Suppose first that μ has finite support and put $S = \mathrm{supp}(\mu) \cup \{0\}$. Let $\eta > 0$. Then, since S is finite, we can find $\varepsilon_0 > 0$ such that $b_\alpha(x,y) \leqslant (1+\eta)b_{\alpha,\varepsilon}(x,y)$ for all $x,y \in S$ and all $\varepsilon \in (0,\varepsilon_0)$. For any such ε, we can, by, for example, [42, Proposition 3.16], write μ as a finite sum of molecules in $\mathcal{F}(M,b_{\alpha,\varepsilon})$ in the form

$$\mu = \sum_{k=1}^{n} a_k \frac{\delta(x_k)-\delta(y_k)}{b_{\alpha,\varepsilon}(x_k,y_k)},$$

where $x_k \neq y_k \in S$ and $\sum_k |a_k| = \|\mu\|_{\mathcal{F}(M,b_{\alpha,\varepsilon})}$. This implies

$$\begin{aligned}
\|\mu\|_{\mathcal{F}(M,b_{\alpha,\varepsilon})} \leqslant \|\mu\|_{\mathcal{F}(M,b_\alpha)} &= \left\| \sum_{k=1}^{n} a_k \frac{b_\alpha(x_k,y_k)}{b_{\alpha,\varepsilon}(x_k,y_k)} \frac{\delta(x_k)-\delta(y_k)}{b_\alpha(x_k,y_k)} \right\|_{\mathcal{F}(M,b_\alpha)} \\
&\leqslant \sum_{k=1}^{n} |a_k| \frac{b_\alpha(x_k,y_k)}{b_{\alpha,\varepsilon}(x_k,y_k)} \leqslant (1+\eta)\|\mu\|_{\mathcal{F}(M,b_{\alpha,\varepsilon})}
\end{aligned}$$

for $\varepsilon < \varepsilon_0$. Thus, the lemma holds for this μ.

Now let $\mu \in \mathcal{F}(M)$ be arbitrary, and take $\eta > 0$. Find $\nu \in \mathcal{F}(M)$ with finite support and such that $\|\mu-\nu\|_{\mathcal{F}(M,b_\alpha)} \leqslant \eta$. Then, $\|\nu\|_{\mathcal{F}(M,b_\alpha)} \leqslant \|\nu\|_{\mathcal{F}(M,b_{\alpha,\varepsilon})} + \eta$ when ε is small enough, by the previous paragraph. For such ε, we have

$$\begin{aligned}
\|\mu\|_{\mathcal{F}(M,b_{\alpha,\varepsilon})} \leqslant \|\mu\|_{\mathcal{F}(M,b_\alpha)} &\leqslant \|\nu\|_{\mathcal{F}(M,b_\alpha)} + \eta \\
&\leqslant \|\nu\|_{\mathcal{F}(M,b_{\alpha,\varepsilon})} + 2\eta \\
&\leqslant \|\mu\|_{\mathcal{F}(M,b_{\alpha,\varepsilon})} + \|\mu-\nu\|_{\mathcal{F}(M,b_{\alpha,\varepsilon})} + 2\eta \\
&\leqslant \|\mu\|_{\mathcal{F}(M,b_{\alpha,\varepsilon})} + \|\mu-\nu\|_{\mathcal{F}(M,b_\alpha)} + 2\eta \\
&\leqslant \|\mu\|_{\mathcal{F}(M,b_{\alpha,\varepsilon})} + 3\eta.
\end{aligned}$$

So the lemma also holds for this μ. □

The relevance of the next lemma lies in the fact that condition (i) characterizes Δ-points when μ is a molecule (see [26, Theorem 4.7]) or, more generally, a finitely supported element of $S_{\mathcal{F}(M)}$ (see [41, Theorem 4.4]).

Lemma 6.6. *Let $\mu \in S_{\mathcal{F}(M)}$. Then the following are equivalent:*

(i) *Every slice of $B_{\mathcal{F}(M)}$ that contains μ also contains molecules m_{uv} for arbitrarily small $d(u,v)$.*
(ii) $\|\mu\|_{\mathcal{F}(M,b_{\alpha,\varepsilon})} = 1-\alpha$ *for all $\alpha \in (0,1)$ and all $\varepsilon > 0$.*
(iii) $\|\mu\|_{\mathcal{F}(M,b_{\alpha})} = 1-\alpha$ *for all $\alpha \in (0,1)$.*

Proof. (i) ⇒ (ii). Assume that (i) holds and suppose $\|\mu\|_{\mathcal{F}(M,b_{\alpha,\varepsilon})} > 1-\alpha$ for some $\varepsilon > 0$ and $\alpha \in (0,1)$. Then, there exists $h \in S_{\mathrm{Lip}_0(M,b_{\alpha,\varepsilon})} \subset B_{\mathrm{Lip}_0(M)}$ such that $h(\mu) > 1-\alpha$. By assumption, there exists a molecule m_{uv} in $\mathcal{F}(M)$ such that $h(m_{uv}) > 1-\alpha$ and $d(u,v) < \varepsilon$. Thus,

$$b_{\alpha,\varepsilon}(u,v) \geqslant h(u) - h(v) > (1-\alpha)d(u,v) = w_{\alpha,\varepsilon}(u,v) \geqslant b_{\alpha,\varepsilon}(u,v).$$

This contradiction proves (ii).

(ii) ⇒ (iii) follows from Lemma 6.5.

(iii) ⇒ (i). Assume that (iii) holds and fix $\varepsilon > 0$ and a slice $S(f,\alpha)$ such that $\mu \in S(f,\alpha)$, where $f \in S_{\mathrm{Lip}_0(M)}$ and $\alpha > 0$. By [25, Lemma 2.1], we may assume $\alpha \in (0,1)$. From (iii), we have

$$\|f\|_{\mathrm{Lip}_0(M,b_\alpha)} \geqslant \frac{f(\mu)}{\|\mu\|_{\mathcal{F}(M,b_\alpha)}} > \frac{1-\alpha}{1-\alpha} = 1.$$

Thus, there exist $x, y \in M$ such that

$$f(x) - f(y) > b_\alpha(x,y) \geqslant b_{\alpha,\varepsilon}(x,y).$$

By the definition of $b_{\alpha,\varepsilon}(x,y)$, we can find $p_0, p_1, \dots, p_{n+1} \in M$ such that $p_0 = x$, $p_{n+1} = y$ and

$$f(x) - f(y) > \sum_{i=0}^{n} w_{\alpha,\varepsilon}(p_i, p_{i+1}).$$

Let $I_1 = \big\{ i \in \{0,\dots,n\} : d(p_i,p_{i+1}) < \varepsilon \big\}$ and let $I_2 = \{0,\dots,n\} \setminus I_1$. Then,

$$\begin{aligned}(1-\alpha)\sum_{i\in I_1} d(p_i,p_{i+1}) + \sum_{i\in I_2} d(p_i,p_{i+1}) &= \sum_{i=0}^{n} w_{\alpha,\varepsilon}(p_i,p_{i+1}) \\ &< f(x) - f(y) \\ &= \sum_{i=0}^{n} \big(f(p_i) - f(p_{i+1})\big) \\ &\leqslant \sum_{i\in I_1} \big(f(p_i) - f(p_{i+1})\big) + \sum_{i\in I_2} d(p_i,p_{i+1})\end{aligned}$$

and therefore there exists $i \in I_1$ such that $f(p_i) - f(p_{i+1}) > (1-\alpha)d(p_i,p_{i+1})$. Since we have also $d(p_i,p_{i+1}) < \varepsilon$, we conclude that (i) holds with $u = p_i$, $v = p_{i+1}$. □

We are now in a position to prove our characterization of Δ-molecules.

Theorem 6.7. *Let $x \neq y \in M$. Then, m_{xy} is a Δ-point of $\mathcal{F}(M)$ if and only if x and y are discretely connectable in M.*

Proof. One implication follows immediately from Proposition 6.3. For the converse, suppose that m_{xy} is a Δ-point and fix $\alpha \in (0,1)$. Then, $\mu = m_{xy}$ satisfies condition (i) from Lemma 6.6 by [26, Theorem 4.7], so it also satisfies (iii) and

$$1-\alpha = \left\|m_{xy}\right\|_{\mathcal{F}(M,b_\alpha)} = \left\|\frac{\delta(x)-\delta(y)}{d(x,y)}\right\|_{\mathcal{F}(M,b_\alpha)} = \frac{b_\alpha(x,y)}{d(x,y)}.$$

That is, $b_\alpha(x,y) = (1-\alpha)d(x,y)$. Now Lemma 6.4(iv) shows that x and y are discretely connectable. □

Corollary 6.8. *Let M be a proper metric space and $x \neq y \in M$. Then, m_{xy} is a Δ-point of $\mathcal{F}(M)$ if and only if x and y are connected with a geodesic.*

By [41, Corollary 4.5], every convex sum (finite or infinite) of Δ-molecules of $\mathcal{F}(M)$ is again a Δ-point. It is then natural to ask whether the converse holds. Note that the question only makes sense for elements of $S_{\mathcal{F}(M)}$ that are actually convex sums of molecules, which is not all of them in general (see [12, Section 4]). This question was raised explicitly in [41, Problem 3] for those Δ-points $\mu \in S_{\mathcal{F}(M)}$ with finite support. In that case we have:

- μ can always be written as a convex sum of molecules (see, e.g., [42, Proposition 3.16]), and
- μ also satisfies property (i) from Lemma 6.6, by [41, Theorem 4.4].

The techniques developed in this section allow us to answer the question in the positive.

Theorem 6.9 (cf. [41, Problem 3]). *Suppose $\mu \in S_{\mathcal{F}(M)}$ is a Δ-point with finite support. Then, μ can be written as a finite convex combination of Δ-molecules in $S_{\mathcal{F}(M)}$.*

Proof. Fix $\alpha \in (0,1)$. By Lemma 6.6 and [41, Theorem 4.4], we have $\|\mu\|_{\mathcal{F}(M,b_\alpha)} = 1-\alpha$. Thus, since μ is a finitely supported element of $\mathcal{F}(M,b_\alpha)$, we may write $\mu / \|\mu\|_{\mathcal{F}(M,b_\alpha)}$ as a finite convex combination of b_α-molecules (e.g., by [42, Proposition 3.16]). That is,

$$\frac{\mu}{1-\alpha} = \sum_{i=1}^{n} \lambda_i \frac{\delta(x_i)-\delta(y_i)}{b_\alpha(x_i,y_i)}$$

for some $x_i \neq y_i \in M$ and $\lambda_i > 0$ such that $\sum_{i=1}^{n} \lambda_i = 1$. Then,

$$\begin{aligned} 1 = \|\mu\|_{\mathcal{F}(M,d)} &= (1-\alpha)\left\|\sum_{i=1}^{n} \lambda_i \frac{d(x_i,y_i)}{b_\alpha(x_i,y_i)} \frac{\delta(x_i)-\delta(y_i)}{d(x_i,y_i)}\right\|_{\mathcal{F}(M,d)} \\ &\leqslant \sum_{i=1}^{n} \lambda_i \cdot (1-\alpha)\frac{d(x_i,y_i)}{b_\alpha(x_i,y_i)} \leqslant \sum_{i=1}^{n} \lambda_i \cdot \frac{1-\alpha}{1-\alpha} = 1 \end{aligned}$$

and so all inequalities are actually equalities. In particular, $\mu = \sum_{i=1}^{n} \lambda_i m_{x_i y_i}$ is a finite convex combination of molecules such that $b_\alpha(x_i, y_i) = (1-\alpha)d(x_i, y_i)$ for all i. By Lemma 6.4(iv) and Theorem 6.7, this means that each $m_{x_i y_i}$ is a Δ-point. $\square$

We finish by remarking that the existence of Δ-points in $\mathcal{F}(M)$ does not necessarily imply the existence of Δ-molecules in general. For instance, if M is the Smith–Volterra–Cantor set, then $\mathcal{F}(M)$ is isometric to $L_1 \oplus_1 \ell_1$ by the proof of [23, Corollary 3.4], which admits Δ-points since L_1 does. However, M is compact and totally disconnected, so $\mathcal{F}(M)$ cannot contain Δ-molecules. In particular, the converse of Proposition 6.3 does not hold.

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