Rough linear PDE's with discontinuous coefficients - existence of solutions via regularization by fractional Brownian motion

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Abstract

We consider two related linear PDE's perturbed by a fractional Brownian motion. We allow the drift to be discontinuous, in which case the corresponding deterministic equation is ill-posed. However, the noise will be shown to have a regularizing effect on the equations in the sense that we can prove existence of solutions for almost all paths of the fractional Brownian motion.

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1   Introduction

In this paper we study examples of the so called regularization by noise phenomenon for a class of linear equations perturbed by fractional Brownian motion. In short, this is the name given to the phenomenon that occurs when ill-posed deterministic equations becomes well-posed by adding stochastic terms.

More specifically, assume $b \in L^1(\mathbb{R}^d; L^\infty([0,T]; \mathbb{R}^d)) \cap L^\infty([0,T] \times \mathbb{R}^d; \mathbb{R}^d)$ is a given function and let B^H be a d-dimensional fractional Brownian motion (fBm). In this paper we will study two different but related linear stochastic PDE's where b acts as a drift term. The stochastic transport equation reads

$$\partial_t u(t,x) + b(t,x) \cdot \nabla u(t,x) + c(t,x)u(t,x) + \nabla u(t,x) \cdot \dot{B}_t^H = 0, \qquad u(0,x) = u_0(x) \tag{1}$$

where $u_0 \in C_b^1(\mathbb{R})$, and we allow c to be a distribution in the sense that c is the distributional derivative of a bounded function. In the case that $c = \operatorname{div} b$ this is called the continuity equation which we also may define as the measure valued equation

$$\partial_t \mu_t + \operatorname{div}(b\mu_t) + \operatorname{div}(\mu_t \dot{B}_t^H) = 0, \qquad \mu|_{t=0} = \mu_0 \tag{2}$$

where μ_0 is a given measure. We see that $u(t,x)$ is equal to the Radon-Nykodim derivative of μ_t w.r.t. Lebesgue measure.

Both equations are related to the stochastic ordinary equation

$$\phi_t(x) = x + \int_0^t b(r, \phi_r(x))dr + B_t^H, \tag{3}$$

in the sense that the push-forward $\mu_t := (\phi_t)_\sharp \mu_0$ solves the continuity equation (2) and the composition $u(t,x) := u_0(\phi_t^{-1}(x)) \exp\{-\int_0^t c(r, \phi_r(\phi_t^{-1}(x)))dr\}$ solves the transport equation (1). This means that if we can show the regularization effect of the fBm on (3) there is hope to solve the corresponding stochastic PDE's (1) and (2).

Both equations involves terms of the form $Y_t \dot{B}_t^H$, but we know that the fBm is P-a.s. not differentiable so one should integrate the equations in time to produce terms on the form $\int_0^t Y_s dB_s^H$. But even at this stage there is ambiguity. Indeed, since for $H \neq \frac{1}{2}$ the fBm is not a semi-martingale there is no Itô-theory to make sense of this integral. Moreover, to enjoy the regularization effect of fBm on (3) we need to have $H < \frac{1}{2}$. Since the trajectories of B^H are P-a.s. Hölder continuous with exponent strictly smaller than H, and the solutions themselves cannot be expected to have higher regularity, also the integration theory by Young is out of reach for these equations.

As the title of the paper suggest, we shall interpret the integrals in the rough path setting, meaning we will use the iterated integrals of B^H and the theory of controlled paths to give meaning to these integrals.

We will discuss the equations separately. For notational simplicity we write B for the fBm.

1.1   The stochastic continuity equation

Integrating the continuity equation in time, and assuming we have the above mentioned integration theory, we get

$$\mu_t + \int_0^t \text{div}(b\mu_s)ds + \int_0^t \text{div}(\mu_s dB_s) = \mu_0 \tag{4}$$

regarded as a measure valued equation, namely for every $\eta \in C_c^\infty(\mathbb{R}^d)$

$$\mu_t(\eta) = \mu_0(\eta) + \int_0^t \mu_s(b(s,\cdot)\cdot\nabla\eta)ds + \int_0^t \mu_s(\nabla\eta \cdot dB_s)$$

where $\mu_t(\eta) := \int_{\mathbb{R}^d} \eta(x)d\mu_t(x)$, $\mu_t(\nabla\eta) = (\mu_t(\partial_{x_1}\eta), \ldots, \mu_t(\partial_{x_d}\eta))$ and $\cdot$ is the dot-product on $\mathbb{R}^d$.

We will show that the solution is on the form $\mu_t = (\phi_t)_\sharp \mu_0$. To see this, heuristically, take $\eta \in C_c^\infty(\mathbb{R}^d)$ and suppose we have some kind of Itô-Stratonovich-formula for the fractional Brownian motion in the rough path setting. We should have

$$\eta(\phi_t(x)) = \eta(x) + \int_0^t \nabla\eta(\phi_r(x))\cdot b(r,\phi_r(x))dr + \int_0^t \nabla\eta(\phi_r(x)) \cdot dB_r^H.$$

Integrating w.r.t. μ_0 produces the desired formula provided we can use integration by parts.

The authors in [2] show existence of a unique solution to (3) and the results will be included in Section 4.

1.2   The stochastic transport equation

Integrating the linear transport equation in time gives

$$u(t,x) + \int_0^t b(s,x)\cdot\nabla u(s,x)ds + \int_0^t c(s,x)u(s,x)ds + \int_0^t \nabla u(s,x) \cdot dB_s = u_0(x). \tag{5}$$

It is well known that the corresponding deterministic equation might develop discontinuities when b is irregular. Moreover, a weak formulation of the deterministic equation is not straightforward. Integrating against a test function $\eta \in C_c^\infty(\mathbb{R})$, we see that the term $\int_{\mathbb{R}} b(t,x)\cdot\nabla u(t,x)\eta(x)dx$ does not allow for integration by parts unless there is some regularity on b. We will choose the noise in such a way that the solution is weakly differentiable, thus circumventing integration by parts. Notice, however, that we will still use a (spatially) weak formulation of the equation.

1.3   Related literature and main contributions

The linear transport equation has been studied extensively. When the noise term is removed, Di Perna and Lions [5], showed that when $b \in L^1([0,T]; W_{loc}^{1,1}(\mathbb{R}^d))$ with linear growth and $\text{div } b \in L^1([0,T]\times\mathbb{R}^d)$, the equation is well posed. Notice that the regularity restrictions on b is needed in order to make a definition of a solution as indicated above. These results were later generalized to the setting of bounded variation vector fields by Ambrosio in [1].

The stochastic version driven by Brownian motion with Stratonovich formulation, i.e. $\int_0^t \nabla u(s,x) \circ dB_s$, has also received some attention. We mention the results in [9] and [17], developed simultaneously and independently, using two somewhat different techniques.

An approach of using rough paths for regularization by noise was used in [6], building on [7]. The techniques of [6] and [7] are similar in spirit to the present paper in the sense that they both use calculation on the occupation measures. The main advantage of [6] and [7] is that they offer a more defined separation between the probabilistic considerations and the analysis of the involved ODE and PDE's, thus making the approach suitable for different types of driving noise. In the present paper one needs to carefully keep track of P-null sets because many of the estimates are only shown to be true under expectation. On the other hand it gives some flexibility since some of the expressions are semi explicit via the local time.

The paper [6] consider drifts for which $\operatorname{div} b \in L^\infty([0,T] \times \mathbb{R}^d)$, and allow for linear growth. When $d = 1$ this is restricts to (locally) Lipschitz drift, but when $d > 1$ this condition is much weaker than Lipschitz. Another difference from the current paper is that [6] considers $H \in (\frac{1}{3}, \frac{1}{2})$. For the technique in the current paper to work, we *need* to have $H < \frac{1}{3}$ which makes the rough path theory a bit more involved.

The main advantage of the technique of the present paper is that the solution can easily be seen to be smoother in space, so that there is no need for integration by parts on the drift term, which is the reason for restricting to bounded divergence on b in [6].

In addition, we include a part where $d = 1$ where the proof is much simpler. The proof is based on a local-time technique that was introduced in [18] to study the Stochastic Heat Equation.

The main contribution of the present paper is to introduce the notion of "local time solutions", see Definition 6.3. This notion is introduced to compensate for the lack of integration by parts for the drift term in (5), and in fact allows to define a notion of solution when the multiplicative term c in (5) is allowed to be a distribution. Section 6.3 presents examples where one can check that the local time solutions actually gives rise to solutions that fits into the framework of the rough path integration theory.

1.4 Notation

For Banach spaces V, W we denote $\mathcal{L}(V; W)$ the set of all continuous linear mappings from V to W. For simplicity we denote $\mathcal{L}(V) := \mathcal{L}(V; \mathbb{R})$. If the spaces V and W are finite dimensional, and we can identify $\mathcal{L}(V \otimes W)$ with $\mathcal{L}(V; \mathcal{L}(W))$. In particular, for a sufficiently smooth function $f : V \to W$ the k'th derivative is considered as a map $\nabla^k f : V \to \mathcal{L}(V^{\otimes k}; W)$.

For $T > 0$ define the simplex $\Delta^{(n)}(s,t) := \{(r_1, \ldots, r_n) \in [0,T]^n : s < r_1 < \cdots < r_n < t\}$. For $\gamma > 0$ denote by $C_2^\gamma([0,T]; V)$ the space of all functions $f : \Delta^{(2)}(0,T) \to V$ such that $\|f\|_\gamma := \sup_{s<t} \frac{|f(s,t)|}{|t-s|^\gamma} < \infty$. Given a path $X : [0,T] \to V$ its increment is denoted $X_{st} := X_t - X_s$ and we denote by $C^\gamma([0,T]; V)$ the set of all path such that its increment belongs to $C_2^\gamma([0,T]; V)$.

For an integer p the p-step truncated tensor algebra

$$T^{(p)}(\mathbb{R}^d) := \bigoplus_{n=0}^{p} (\mathbb{R}^d)^{\otimes n}$$

is equipped with the product $(\mathbf{a} \otimes \mathbf{b})^{(n)} = \sum_{k=0}^n a^{(n-k)} \otimes b^{(k)}$.

We recall the following Taylor formula for a function $f : V \to W$ that is $m + 1$ times differentiable

$$f(x) - f(y) = \sum_{k=1}^{m} \frac{\nabla^k f(y)}{k!} (x-y)^{\otimes k} + R_m^f(x,y) \tag{6}$$

where $|R_m^f(x,y)| \lesssim |x-y|^{m+1}$. More specifically, we shall use the explicit formula

$$R_m^f(x,y) = \frac{1}{m!} \int_0^1 \nabla^{m+1} f(y + u(x-y))(1-u)^m du (x-y)^{\otimes(m+1)}. \tag{7}$$

We shall frequently use the space $L^1(\mathbb{R}^d; L^\infty([0,T]; \mathbb{R}^d))$ with norm denoted by

$$\|f\|_{\infty,1} := \int_{\mathbb{R}^d} \|f(\cdot, x)\|_{L^\infty([0,T]; \mathbb{R}^d)} dx.$$

For simplicity the norm in the space $L^\infty([0,T] \times \mathbb{R}^d; \mathbb{R}^d)$ will be denoted $\|\cdot\|_\infty$.

2 Elements of Controlled Rough Paths

The theory of rough paths was first introduced by Terry Lyons in the late 90's, see [16]. The insight of this work is that even though solutions to ODE's driven by rough signals are typically not continuous as a function of the signals themselves, by adding extra information, namely the iterated integrals of the driving signals, one obtains a topology for which there is continuity of the solutions. The theory was further developed by Gubinelli, [12] and [13], who introduced the notion of controlled paths which defines spaces that are well suited for constructing solutions of the rough ODE's.

In the present paper we shall use controlled paths as one of our main tools. See [10] for an introduction. It should be noted that the full theory of rough paths is not necessary for the present paper since there will be cancellations of the "area" of the rough paths due to the structure of the noise term in (1) and (2). Still, this machinery is convenient to understand the equations as expansions in the driving noise.

Throughout this section we fix some $\gamma \in (0, \frac{1}{2})$ and let p be the integer part of $\frac{1}{\gamma}$. A γ-rough path is a mapping

$$\mathbf{X} : \Delta^{(2)}(0,T) \to T^{(p)}(\mathbb{R}^d)$$
$$(s,t) \mapsto (1, X_{st}^{(1)}, \ldots, X_{st}^{(p)})$$

that satisfies an algebraic (Chen's) relation

$$\mathbf{X}_{st} = \mathbf{X}_{su} \otimes \mathbf{X}_{ut}, \tag{8}$$

and an analytic relation

$$|X_{st}^{(n)}| \le C|t-s|^{n\gamma} \qquad n = 1, \ldots, p, \quad \text{for some} \quad C > 0. \tag{9}$$

We denote by $\mathscr{C}^\gamma$ the set of all rough paths equipped with the metric

$$\varrho_\gamma(\mathbf{X}, \tilde{\mathbf{X}}) := \sum_{n=1}^{p} \sup_{t \ne s} \frac{|X_{st}^{(n)} - \tilde{X}_{st}^{(n)}|}{|t-s|^{n\gamma}}.$$

Given a function $X \in C^1([0,T]; \mathbb{R}^d)$ we can consider its canonical lift to a rough path

$$\mathbf{X}_{st} := (1, X_{st}, \int_s^t X_{sr} \otimes \dot{X}_r dr, \ldots, \int_{\Delta^{(p)}(s,t)} \dot{X}_{r_1} \otimes \cdots \otimes \dot{X}_{r_p} dr_1 \ldots dr_p). \tag{10}$$

We denote by $\mathscr{C}_g^\gamma$ the closure of the canonical lift of $C^1([0,T]; \mathbb{R}^d)$ in the rough path topology [1]. An element $\mathbf{X} \in \mathscr{C}_g^\gamma$ will be referred to as a geometric rough path and it satisfies the identity

$$\text{sym}(X_{st}^{(n)}) = \frac{1}{n!}\left(X_{st}^{(1)}\right)^{\otimes n}. \tag{11}$$

Given a rough path $\mathbf{X} \in \mathscr{C}^\gamma$, we shall say that a mapping

$$\mathbf{Y} : [0,T] \longrightarrow \bigoplus_{n=1}^{p} \mathcal{L}((\mathbb{R}^d)^{\otimes n})$$
$$t \longmapsto (Y_t^{(1)}, \ldots Y_t^{(p)})$$

is a controlled (by $\mathbf{X}$) path if the functions

$$Y_{st}^{(k)\sharp} := Y_t^{(k)} - \sum_{n=k}^{p} Y_s^{(n)} X_{st}^{(n-k)} \quad k = 1, \ldots, p$$

are such that $Y^{(k)\sharp} \in C_2^{(p+1-k)\gamma}([0,T]; \mathcal{L}((\mathbb{R}^d)^{\otimes k}))$, i.e.

$$|Y_{st}^{(k)\sharp}| \lesssim |t-s|^{(p+1-k)\gamma}. \tag{12}$$

We denote by $\mathscr{D}_{\mathbf{X}}^{p\gamma}$ the set of all paths controlled by $\mathbf{X}$, and we equip this linear space with the semi-norm

$$\|\mathbf{Y}\|_{\mathbf{X}} = \sum_{k=1}^{p} \|Y^{(k)\sharp}\|_{(p+1-k)\gamma}.$$

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[1] sometimes written $\mathscr{C}_g^{0,\gamma}$ in the literature, whereas $\mathscr{C}_g^\gamma$ is reserved for paths satisfying (11). While $\mathscr{C}_g^{0,\gamma}$ is strictly included in $\mathscr{C}_g^\gamma$ one can use "geodesic approximations" and interpolation to show $\mathscr{C}_g^{\gamma'} \subset \mathscr{C}_g^{0,\gamma} \subset \mathscr{C}_g^\gamma$ for $\gamma' < \gamma$, so that one can still approximate elements satisfying (11) at the expense of choosing a smaller γ.

Conditioned on $(Y_0^{(1)}, \ldots, Y_0^{(p)})$ we get a norm which controls the $\|\cdot\|_\infty$-norm of $\mathbf{Y}$ in the following way. We have $Y_t^{(k)} = Y_{0t}^{(k)\sharp} + \sum_{n=k}^p Y_0^{(n)} X_{0t}^{(n-k)}$ so that

$$\|Y^{(k)}\|_\infty \le T^{(p+1-k)\gamma} \|Y^{(k)\sharp}\|_{(p+1-k)\gamma} + \sum_{n=k}^p |Y_0^{(n)}| \|X\|^{(n-k)\gamma} T^{(n-k)\gamma}$$

$$\lesssim \|\mathbf{Y}\|_{\mathbf{X}} + \varrho_\gamma(0, \mathbf{X}) |\mathbf{Y}_0|. \tag{13}$$

If we consider two paths $\mathbf{Y}$ and $\tilde{\mathbf{Y}}$, controlled by $\mathbf{X}$ and $\tilde{\mathbf{X}}$ respectively, we introduce the "distance"

$$\|\mathbf{Y}; \tilde{\mathbf{Y}}\|_{\mathbf{X}, \tilde{\mathbf{X}}} := \sum_{k=1}^p \|Y^{(k)\sharp} - \tilde{Y}^{(k)\sharp}\|_{(p+1-k)\gamma}.$$

Similar as above have the following estimate

$$\max_{n=1,\ldots,p} \|Y^{(n)} - \tilde{Y}^{(n)}\|_\infty \le \|\mathbf{Y}; \tilde{\mathbf{Y}}\|_{\mathbf{X}, \tilde{\mathbf{X}}} + \varrho_\gamma(\mathbf{X}, 0) |\mathbf{Y}_0 - \tilde{\mathbf{Y}}_0| + \varrho_\gamma(\mathbf{X}, \tilde{\mathbf{X}}) |\tilde{\mathbf{Y}}_0|.$$

We define the total space

$$\mathscr{C}^\gamma \ltimes \mathscr{D}^{p\gamma} := \bigsqcup_{\mathbf{X} \in \mathscr{C}^\gamma} \{\mathbf{X}\} \times \mathscr{D}_{\mathbf{X}}^{p\gamma}$$

equipped with its natural topology, i.e. the weakest topology such that

$$\mathscr{C}^\gamma \ltimes \mathscr{D}^{p\gamma} \longrightarrow \mathscr{C}^\gamma \times \bigoplus_{k=1}^p C_2^{(p+1-k)\gamma}([0,T]; \mathcal{L}((\mathbb{R}^d)^{\otimes k}))$$

$$(\mathbf{X}, \mathbf{Y}) \longmapsto \left(\mathbf{X}, \oplus_{k=1}^p Y^{(k)\sharp}\right)$$

is continuous.

If f is a scalar valued function with higher Hölder regularity, i.e. $|f_{st}| \lesssim |t-s|^\beta$ for some $\beta \ge p\gamma$ and a controlled path $\mathbf{Y} \in \mathscr{D}_{\mathbf{X}}^{p\gamma}$ we can define the product as a controlled path $f\mathbf{Y}$.

Lemma 2.1. *The mapping*

$$C^\beta \times \mathscr{D}_{\mathbf{X}}^{p\gamma} \to \mathscr{D}_{\mathbf{X}}^{p\gamma}$$

$$(f, \mathbf{Y}) \mapsto (fY^{(1)}, \ldots, fY^{(p)})$$

is bilinear and continuous when $\beta \ge p\gamma$.

Proof. To see that the mapping is well defined is sufficies to notice that

$$(fY)_{st}^{(k)\sharp} = f_{st} Y_t^{(k)} + f_s Y_{st}^{(k)\sharp}$$

satisfies the required time-regularity when $\beta \ge p\gamma$. To see continuity of this map we can similarly write

$$|(fY)_{st}^{(k)\sharp} - (\tilde{f}\tilde{Y})_{st}^{(k)\sharp}| \le |t-s|^\beta \|f - \tilde{f}\|_\beta \|Y^{(k)}\|_\infty$$
$$+ |t-s|^{(p+1-k)\gamma} \|f - \tilde{f}\|_\infty \|Y^{(k)\sharp} - \tilde{Y}^{(k)\sharp}\|_{(p+1-k)\gamma}.$$

$\square$

2.1 Integration of Controlled Rough Paths

Following [10] we denote by $C_2^{\alpha,\beta}([0,T])$ the space of functions $\Xi : \Delta^{(2)}(0,T) \to \mathbb{R}$ such that

$$\|\Xi\|_\alpha := \sup_{s<t} \frac{|\Xi_{st}|}{|t-s|^\alpha} < \infty \quad \text{and} \quad \|\delta\Xi\|_\beta := \sup_{s<u<t} \frac{|\delta\Xi_{sut}|}{|t-s|^\beta} < \infty$$

where $\delta\Xi_{sut} := \Xi_{st} - \Xi_{su} - \Xi_{ut}$. We equip the space with the semi-norm $\|\Xi\|_{\alpha,\beta} := \|\Xi\|_\alpha + \|\delta\Xi\|_\beta$. The following result is sometimes referred to as the "sewing lemma":

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Lemma 2.2. *Assume $0 < \alpha \leq 1 < \beta$. Then there exists a unique continuous linear map*

$$\mathcal{I} : C_2^{\alpha,\beta}([0,T]) \to C^\alpha([0,T])$$

such that $(\mathcal{I}\Xi)_0 = 0$ and

$$|(\mathcal{I}\Xi)_{st} - \Xi_{st}| \lesssim |t-s|^\beta.$$

More specifically,

$$\mathcal{I}(\Xi)_{st} = \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} \Xi_{uv} \tag{14}$$

where $\mathcal{P}$ denotes a partition of $[s,t]$ and $|\mathcal{P}|$ its mesh. The limit can be taken along any sequence of partitions and is independent of this choice.

For a proof, see [10]. It is clear from (14) that $C_2^\theta([0,T]) \subset ker(\mathcal{I})$ for $\theta > 1$.
We are ready to define the integral of a controlled rough path. For $\mathbf{X} \in \mathscr{C}^\gamma$ and $\mathbf{Y} \in \mathscr{D}_{\mathbf{X}}^{p\gamma}$ let

$$\Xi_{st} := \sum_{n=1}^{p} Y_s^{(n)} X_{st}^{(n)}.$$

Chen's relation (8) gives $X_{st}^{(n)} = \sum_{k=0}^{n} X_{su}^{(n-k)} \otimes X_{ut}^{(k)}$, so that

$$\delta\Xi_{sut} = \sum_{n=1}^{p} Y_s^{(n)}(X_{st}^{(n)} - X_{su}^{(n)}) - \sum_{n=1}^{p} Y_u^{(n)} X_{ut}^{(n)} = \sum_{n=1}^{p} Y_s^{(n)} \sum_{k=1}^{n} X_{su}^{(n-k)} \otimes X_{ut}^{(k)} - \sum_{n=1}^{p} Y_u^{(n)} X_{ut}^{(n)}$$

$$= \sum_{k=1}^{p} \sum_{n=k}^{p} Y_s^{(n)} X_{su}^{(n-k)} \otimes X_{ut}^{(k)} - \sum_{k=1}^{p} Y_u^{(k)} X_{ut}^{(k)} = \sum_{k=1}^{p} \left(\sum_{n=k}^{p} Y_s^{(n)} X_{su}^{(n-k)} - Y_u^{(k)} \right) X_{ut}^{(k)}$$

$$= -\sum_{k=1}^{p} Y_{su}^{(k)\sharp} X_{ut}^{(k)}.$$

From (9) and (12) each term can be bounded by $C|t-s|^{(p+1)\gamma}$ for an appropriate constant C. Consequently $|\delta\Xi_{sut}| \lesssim |t-s|^{(p+1)\gamma}$. Since $(p+1)\gamma > 1$ we arrive at the following definition:

Definition 2.3. *Let $\mathbf{X} \in \mathscr{C}^\gamma$ and let $\mathbf{Y} \in \mathscr{D}_{\mathbf{X}}^{p\gamma}$. We define the rough path integral of $\mathbf{Y}$ w.r.t. $\mathbf{X}$ as*

$$\int_s^t \mathbf{Y}_r d\mathbf{X}_r := (\mathcal{I}\Xi)_{st} \tag{15}$$

with $\mathcal{I}$ and Ξ as above.

Remark 2.4. *For a smooth path X with its geometric lift (10) the rough path integral and the usual calculus coincide, i.e.*

$$\int_s^t Y_r d\mathbf{X}_r = \int_s^t Y_r \dot{X}_r dr,$$

for all $Y \in C^\gamma([0,T]; \mathcal{L}(\mathbb{R}^d))$. Indeed, we may define $Y^{(n)} = 0$ for $n = 2, \ldots p$. Even though in general (12) is not satisfied for $k = 1$, if we define

$$\Xi_{st} := Y_s X_{st}$$

we get $\delta\Xi_{sut} = -Y_{su} X_{ut}$ so that $\Xi \in C_2^{1,1+\gamma}([0,T])$.

The rest of this section is devoted to obtaining a "local Lipschitz"-type estimate when we regard the above as a mapping

$$\mathscr{C}^\gamma \ltimes \mathscr{D}^{p\gamma} \to C_2^{\gamma,(p+1)\gamma}([0,T]).$$

Indeed, let $\mathbf{X}, \tilde{\mathbf{X}} \in \mathscr{C}^\gamma$ and let $\mathbf{Y}$ and $\tilde{\mathbf{Y}}$ be controlled by $\mathbf{X}$ and $\tilde{\mathbf{X}}$ respectively. Define Ξ as before and

$$\tilde{\Xi}_{st} := \sum_{n=1}^{p} \tilde{Y}_s^{(n)} \tilde{X}_{st}^{(n)}.$$

Lemma 2.5. *Assume* $\varrho_\gamma(0, \mathbf{X}), \|\mathbf{Y}\|_{\mathbf{X}}, |\mathbf{Y}_0| \le M$ *for some constant* M, *and similarly for* $\tilde{\mathbf{X}}$ *and* $\tilde{\mathbf{Y}}$. *Then there exists a constant* C_M *such that*

$$\|\Xi - \tilde{\Xi}\|_{\gamma, (p+1)\gamma} \le C_M(|\mathbf{Y}_0 - \tilde{\mathbf{Y}}_0| + \|\mathbf{Y}; \tilde{\mathbf{Y}}\|_{\mathbf{X}; \tilde{\mathbf{X}}} + \varrho_\gamma(\mathbf{X}, \tilde{\mathbf{X}})).$$

Proof. We begin by decomposing

$$\Xi_{st} - \tilde{\Xi}_{st} = \sum_{n=1}^p Y_s^{(n)} X_{st}^{(n)} - \sum_{n=1}^p \tilde{Y}_s^{(n)} \tilde{X}_{st}^{(n)}$$

$$= \sum_{n=1}^p Y_s^{(n)} (X_{st}^{(n)} - \tilde{X}_{st}^{(n)}) + \sum_{n=1}^p (Y_s^{(n)} - \tilde{Y}_s^{(n)}) \tilde{X}_{st}^{(n)}$$

so that

$$|\Xi_{st} - \tilde{\Xi}_{st}| \le \sum_{n=1}^p \|Y^{(n)}\|_\infty \|X^{(n)} - \tilde{X}^{(n)}\|_{n\gamma} |t-s|^{n\gamma} + \sum_{n=1}^p \|Y^{(n)} - \tilde{Y}^{(n)}\|_\infty \|\tilde{X}^{(n)}\|_{n\gamma} |t-s|^{n\gamma}$$

$$\le |t-s|^\gamma \max_{n=1,\dots,p} \|Y^{(n)}\|_\infty \varrho_\gamma(\mathbf{X}, \tilde{\mathbf{X}}) + |t-s|^\gamma \varrho_\gamma(0, \tilde{\mathbf{X}}) \max_{n=1,\dots,p} \|Y^{(n)} - \tilde{Y}^{(n)}\|_\infty.$$

Using (13) we can find a constant $\tilde{C}_M$ such that

$$\|\Xi - \tilde{\Xi}\|_\gamma \le \tilde{C}_M(\|\mathbf{Y}; \tilde{\mathbf{Y}}\|_{\mathbf{X}, \tilde{\mathbf{X}}} + |\mathbf{Y}_0 - \tilde{\mathbf{Y}}_0| + \varrho_\gamma(\mathbf{X}, \tilde{\mathbf{X}})).$$

Similarly,

$$\delta\Xi_{sut} - \delta\tilde{\Xi}_{sut} = -\sum_{n=1}^p Y_{su}^{(n)\sharp} X_{ut}^{(n)} + \sum_{n=1}^p \tilde{Y}_{su}^{(n)\sharp} \tilde{X}_{ut}^{(n)}$$

$$= -\sum_{n=1}^p Y_{su}^{(n)\sharp} (X_{ut}^{(n)} - \tilde{X}_{ut}^{(n)}) + \sum_{n=1}^p (Y_{su}^{(n)\sharp} - \tilde{Y}_{su}^{(n)\sharp}) \tilde{X}_{ut}^{(n)}$$

so that

$$\|\delta(\Xi - \tilde{\Xi})\|_{(p+1)\gamma} \le \sum_{n=1}^p \|Y^{(n)\sharp}\|_{(p+1-n)\gamma} \|X^{(n)} - \tilde{X}^{(n)}\|_{n\gamma} + \sum_{n=1}^p \|Y^{(n)\sharp} - \tilde{Y}^{(n)\sharp}\|_{(p+1-n)\gamma} \|\tilde{X}^{(n)}\|_{n\gamma}$$

$$\le M(\varrho_\gamma(\mathbf{X}, \tilde{\mathbf{X}}) + \|\mathbf{Y}; \tilde{\mathbf{Y}}\|_{\mathbf{X}, \tilde{\mathbf{X}}}).$$

$\square$

2.2 Controlling solutions of ODE's

In this section we will show how to control solutions of ODE's perturbed by a rough path $\mathbf{X} \in \mathscr{C}^\gamma$. Fix a function $b \in C_b^1([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$ and denote by $\phi_\cdot(x)$ the solution of the perturbed ODE

$$\phi_t(x) = x + \int_0^t b(r, \phi_r(x))dr + X_t. \tag{16}$$

When there is no chance of confusion we shall denote the solution of (16) by ϕ_t for notational convenience. Notice that we shall later on be interested in ϕ_t as a function of x, but for this section we leave it fixed. We have

$$\phi_{st} = \int_s^t b(r, \phi_r)dr + X_{st} =: R_{st}^\phi + X_{st}$$

where $|R_{st}^\phi| \lesssim |t-s|$ by the boundedness of b. Let $f \in C_b^p(\mathbb{R}^d; \mathbb{R}^d)$, so that we can view $\nabla^k f : \mathbb{R}^d \to \mathcal{L}((\mathbb{R}^d)^{\otimes(k+1)})$. We shall lift the composition $f(\phi)$ to a controlled path in $\mathscr{D}_{\mathbf{X}}^{p\gamma}$.

Lemma 2.6. *Assume* $\mathbf{X}$ *is a geometric rough path. Then the mapping* $s \mapsto (f(\phi_s), \dots, \nabla^{p-1} f(\phi_s))$ *belongs to* $\mathscr{D}_{\mathbf{X}}^{p\gamma}$, *i.e. if we introduce the ad-hoc notation*

$$f(\phi)_{st}^{(k)\sharp} := \nabla^{k-1} f(\phi_t) - \sum_{n=k}^p \nabla^n f(\phi_s) X_{st}^{(n-k)}, \quad k = 1, \dots, p$$

we have $f(\phi)^{(k)\sharp} \in C_2^{(p+1-k)\gamma}([0, T]; \mathcal{L}((\mathbb{R}^d)^{\otimes(k+1)}))$.

Proof. Begin by writing

$$\phi_{st}^{\otimes n} = (R_{st}^{\phi} + X_{st})^{\otimes n} = \sum_{q=0}^{n} \binom{n}{q} \text{sym}((R_{st}^{\phi})^{\otimes(n-q)} \otimes X_{st}^{\otimes q}).$$

For a sufficiently smooth function $g : \mathbb{R}^d \to \mathcal{L}(V)$ where V is a finite-dimensional Banach space, we have from Taylor's formula

$$\begin{aligned}
g(\phi_t) - g(\phi_s) &= \sum_{n=1}^{m} \frac{\nabla^n g(\phi_s)}{n!} (\phi_{st})^{\otimes n} + R_m^g(\phi_s, \phi_t) \\
&= \sum_{n=1}^{m} \nabla^n g(\phi_s) X_{st}^{(n)} + R_m^g(\phi_s, \phi_t) \\
&\quad + \sum_{n=1}^{m} \sum_{q=1}^{n} \binom{n}{q} \frac{\nabla^n g(\phi_s)}{n!} ((R_{st}^{\phi})^{\otimes(n-q)} \otimes X_{st}^{\otimes q}).
\end{aligned} \tag{17}$$

In the above we have used that $\mathbf{X}$ satisfies (11) so that $\nabla^n g(\phi_s) \frac{X_{st}^{\otimes n}}{n!} = \nabla^n g(\phi_s) X_{st}^{(n)}$ since $\nabla^n g$ only acts on symmetric tensors. Furthermore, the second term $\lesssim |\phi_{st}|^{m+1} \lesssim |t-s|^{(m+1)\gamma}$, and the third term $\lesssim |t-s|$. With $g = \nabla^k f$ and $m = p - k - 1$ it follows that $f(\phi)^{(k)\sharp} \in C_2^{(p-k)\gamma}([0,T]; \mathcal{L}((\mathbb{R}^d)^{\otimes(k+1)})$, thus proving the lemma. $\square$

Remark 2.7. *We note that the symmetry of $\nabla^n g$ in the proof of Lemma 2.6 is the reason that the full generality of the theory of rough paths is not needed in this paper.*

Corollary 2.8. *For $f \in C_b^p(\mathbb{R}^d; \mathbb{R}^d)$ we may define $\int f(\phi_r) d\mathbf{X}_r$ as the rough path integral of $f(\phi)$ w.r.t. $\mathbf{X}$ as in (15).*

2.3 Stability w.r.t. the driving path

The purpose of this section is to prove local Lipschitz continuity of the mapping

$$\mathscr{C}^{\gamma} \to \mathscr{C}^{\gamma} \ltimes \mathscr{D}^{p\gamma}$$
$$\mathbf{X} \mapsto (\mathbf{X}, f(\phi))$$

where ϕ is the solution to (16), $f \in C_b^p(\mathbb{R}^d; \mathbb{R}^d)$ and $f(\phi)$ denotes the lift as described in the previous section. We begin with some trivial bounds, namely let $\tilde{\mathbf{X}} \in \mathscr{C}^{\gamma}$ and denote by $\tilde{\phi}$ the solution to (16) when we replace $\mathbf{X}$ by $\tilde{\mathbf{X}}$, i.e.

$$\tilde{\phi}_{st} = \int_s^t b(r, \tilde{\phi}_r) dr + \tilde{X}_{st} =: R_{st}^{\tilde{\phi}} + \tilde{X}_{st}.$$

One can check that (see [6], Lemma A.7)

$$\|\phi - \tilde{\phi}\|_{\gamma} \leq C(T, \nabla b) \|X - \tilde{X}\|_{\gamma}. \tag{18}$$

Clearly this implies $\|\phi - \tilde{\phi}\|_{\gamma} \lesssim \varrho_{\gamma}(\mathbf{X}, \tilde{\mathbf{X}})$ and also $\|R^{\phi} - R^{\tilde{\phi}}\|_{\gamma} \lesssim \varrho_{\gamma}(\mathbf{X}, \tilde{\mathbf{X}})$.

It follows that $\|\phi^{\otimes n} - \tilde{\phi}^{\otimes n}\|_{n\gamma} \lesssim \varrho_{\gamma}(\mathbf{X}, \tilde{\mathbf{X}})$ by induction: assume this holds for $n - 1$. Then

$$\begin{aligned}
|\phi_{st}^{\otimes n} - \tilde{\phi}_{st}^{\otimes n}| &\leq |\phi_{st}^{\otimes(n-1)}| |\phi_{st} - \tilde{\phi}_{st}| + |\phi_{st}^{\otimes(n-1)} - \tilde{\phi}_{st}^{\otimes(n-1)}| |\tilde{\phi}_{st}| \\
&\leq 2|t-s|^{n\gamma} \varrho_{\gamma}(\mathbf{X}, \tilde{\mathbf{X}})
\end{aligned}$$

by the induction hypothesis combined with (18).

The main result of this section is the following.

Lemma 2.9. *Assume $\varrho_{\gamma}(\mathbf{X}, 0), \varrho_{\gamma}(\tilde{\mathbf{X}}, 0) \leq M$ and $f \in C_b^p(\mathbb{R}^d; \mathbb{R}^d)$. Then there exists a constant C_M such that*

$$\|f(\phi); f(\tilde{\phi})\|_{\mathbf{X}, \tilde{\mathbf{X}}} \leq C_M \varrho_{\gamma}(\mathbf{X}, \tilde{\mathbf{X}}).$$

Proof. We shall use the formula (17) to show that $\|f(\phi)^{(k)\sharp} - f(\tilde{\phi})^{(k)\sharp}\|_{(p-k)\gamma} \leq C_M \varrho_\gamma(\mathbf{X}, \tilde{\mathbf{X}})$, which will prove the claim. Towards this goal, for a function g smooth enough, the remainder term of the Taylor expansion satisfies

$$R_m^g(\phi_s, \phi_t) - R_m^g(\tilde{\phi}_s, \tilde{\phi}_t) = \int_0^1 \frac{(1-r)^{m+1}}{m!} \nabla^{m+1} g(\phi_s + r\phi_{st}) dr \left(\phi_{st}^{\otimes(m+1)} - \tilde{\phi}_{st}^{\otimes(m+1)} \right)$$
$$+ \int_0^1 \frac{(1-r)^{m+1}}{m!} \left(\nabla^{m+1} g(\phi_s + r\phi_{st}) - \nabla^{m+1} g(\tilde{\phi}_s + r\tilde{\phi}_{st}) \right) dr \left(\tilde{\phi}_{st}^{\otimes(m+1)} \right).$$

For the first term above we have $\lesssim |t-s|^{(m+1)\gamma} \|\nabla^{m+1} g\|_\infty \varrho_\gamma(\mathbf{X}, \tilde{\mathbf{X}})$. For the second term we use, uniformly in $r \in [0,1]$

$$|\nabla^{m+1} g(\phi_s + r\phi_{st}) - \nabla^{m+1} g(\tilde{\phi}_s + r\tilde{\phi}_{st})| \leq \|\nabla^{m+2} g\|_\infty (|\phi_s - \tilde{\phi}_s| + r|\phi_{st} - \tilde{\phi}_{st}|)$$
$$\lesssim \|\nabla^{m+2} g\|_\infty \varrho_\gamma(\mathbf{X}, \tilde{\mathbf{X}}).$$

Together with the bound $|\tilde{\phi}_{st}^{\otimes(m+1)}| \lesssim |t-s|^{(m+1)\gamma}$ we see that

$$\|R_m^g(\phi_\cdot, \phi_\cdot) - R_m^g(\tilde{\phi}_\cdot, \tilde{\phi}_\cdot)\|_{(m+1)\gamma} \lesssim \varrho_\gamma(\mathbf{X}, \tilde{\mathbf{X}}).$$

Fix integers $q \geq 1$ and $n \geq 0$. Using the estimate $|a \otimes b - a' \otimes b'| \leq |a - a'||b| + |a'||b - b'|$ repeatedly, it is easy to check that

$$|\nabla g(\phi_s)(X_{st}^{\otimes n} \otimes (R_{st}^\phi)^{\otimes q}) - \nabla g(\tilde{\phi}_s)(\tilde{X}_{st}^{\otimes n} \otimes (R_{st}^{\tilde{\phi}})^{\otimes q})| \lesssim |t-s|\varrho_\gamma(\mathbf{X}, \tilde{\mathbf{X}}).$$

This combined with (17) gives

$$\|f(\phi)^{(k)\sharp} - f(\tilde{\phi})^{(k)\sharp}\|_{(p-k)\gamma} \lesssim \varrho_\gamma(\mathbf{X}, \tilde{\mathbf{X}})$$

which ends the proof of the lemma. $\qquad\square$

Combining the above Lemma, Lemma 2.5 and Remark 2.4 we get

Corollary 2.10. *Let $\mathbf{X} \in \mathscr{C}_g^\gamma$. Then there exists a family of smooth paths X^ϵ such that*

$$\int_0^\cdot f(\phi_r^\epsilon) \dot{X}_r^\epsilon dr \to \int_0^\cdot f(\phi_r) d\mathbf{X}_r \qquad in\ C^\gamma([0,T]),$$

as $\epsilon \to 0$.

2.4 Stability w.r.t. the drift

Let us fix $\mathbf{X} \in \mathscr{C}^\gamma$ and we consider the ODE (16). Assume we have a sequence of functions b_ϵ such that there exists a solution of for every $\epsilon > 0$ to

$$\phi_t^\epsilon = x + \int_0^t b_\epsilon(r, \phi_r^\epsilon) dr + X_t.$$

We will show stability in the sense of controlled rough paths when we *assume* that ϕ^ϵ converges in an appropriate topology. This convergence will be shown to hold in Proposition 4.15 for our particular case.

Lemma 2.11. *Assume ϕ^ϵ converges in C^γ to the solution of (16). Then for any $f \in C_b^p(\mathbb{R}^d; \mathbb{R}^d)$ we have that the lift of $f(\phi^\epsilon)$ converges in $\mathscr{D}_{\mathbf{X}}^{p\gamma}$ to $f(\phi)$, and as $\epsilon \to 0$*

$$\int_0^\cdot f(\phi_r^\epsilon) d\mathbf{X}_r \to \int_0^\cdot f(\phi_r) d\mathbf{X}_r$$

where the above convergence is in C^γ.

Proof. Note that the second claim follows from the first in connection with Remark 2.2.

To see the first claim, one has to show

$$\lim_{\epsilon \to 0} \|f(\phi)^{(k)\sharp} - f(\phi^\epsilon)^{(k)\sharp}\|_{(p-k)\gamma} = 0$$

for all $k = 0, 1, \ldots, p-1$. The proof follows the same lines as the proof of Lemma 2.9 with minor modifications, noting that $\mathbf{X} = \tilde{\mathbf{X}}$. $\qquad\square$

2.5 An Itô-Stratonovich formula

For the sake of being self-contained, we include a change-of-variable formula for our particular case. Let $\eta \in C_c^\infty(\mathbb{R}^d)$ and assume $\phi.$ solves (16). If X is a smooth path usual calculus yields,

$$\frac{d}{dt}\eta(\phi_t) = \nabla\eta(\phi_t) \cdot b(t, \phi_t) + \nabla\eta(\phi_t) \cdot \dot{X}_t.$$

We can generalize this to geometric rough paths.

Lemma 2.12. *Suppose $\eta \in C_c^\infty(\mathbb{R}^d)$ and $\mathbf{X}$ is a rough path above X. Then we have*

$$\eta(\phi_t) = \eta(x) + \int_0^t \eta(\phi_r) \cdot b(r, \phi_r)dr + \int_0^t \nabla\eta(\phi_r)d\mathbf{X}_r.$$

where the last term is the rough path integral.

Proof. Let $0 \le u \le v \le t$ and use Taylor's formula to write, as in (17)

$$\eta(\phi)_{uv} = \sum_{n=1}^p \frac{\nabla^n\eta(\phi_u)}{n!}(\phi_{uv})^{\otimes n} + R_p^\eta(\phi_u, \phi_v) = \nabla\eta(\phi_u)R_{uv}^\phi + \sum_{n=1}^p \nabla^n\eta(\phi_u)X_{uv}^{(n)} + \Xi_{uv}$$

where

$$\Xi_{uv} := R_p^\eta(\phi_u, \phi_v) + \sum_{n=2}^p \sum_{q=1}^{n-1} \binom{n}{q} \frac{\nabla^n\eta(\phi_u)}{n!}((R_{uv}^\phi)^{\otimes(n-q)} \otimes X_{uv}^{\otimes q})$$

and notice that $\Xi \in C_2^{1+\gamma}([0, T]) \subset ker(\mathcal{I})$. We have

$$\lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} \nabla\eta(\phi_u) \cdot \int_u^v b(r, \phi_r)dr = \lim_{|\mathcal{P}|\to 0} \int_0^t \sum_{[u,v]\in\mathcal{P}} \nabla\eta(\phi_u)1_{[u,v]}(r) \cdot b(r, \phi_r)dr$$

$$= \int_0^t \nabla\eta(\phi_r) \cdot b(r, \phi_r)dr$$

where we used continuity of $\nabla\eta$ and dominated convergence in the last step. Note that the above reasoning does not use any regularity requirements on b.

Finally, we have

$$\eta(\phi_t) - \eta(x) = \mathcal{I}(\eta(\phi.))_{0,t} = \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} \eta(\phi)_{uv}$$

$$= \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} \left(\nabla\eta(\phi_u)R_{uv}^\phi + \sum_{n=1}^p \nabla^n\eta(\phi_u)X_{uv}^{(n)} + \Xi_{uv}\right)$$

$$= \mathcal{I}(\nabla\eta(\phi.)R_{..}^\phi) + \mathcal{I}(\sum_{n=1}^p \nabla^n\eta(\phi.)X_{..}^{(n)}) + \mathcal{I}(\Xi)$$

$$= \int_0^t \nabla\eta(\phi_r) \cdot b(r, \phi_r)dr + \int_0^t \nabla\eta(\phi_r)d\mathbf{X}_t$$

by definition of the rough path integral. $\qquad\square$

2.6 Integrated ODE's

To emphasize that the solution of (16) depends on the initial value x, we denote its solution by $\phi.(x)$, i.e.

$$\phi_t(x) = x + \int_0^t b(r, \phi_r(x))dr + X_t.$$

Let ν be a finite signed measure on $\mathbb{R}^d$, and $f = (f^{(1)}, \ldots, f^{(d)}) \in C_b^p(\mathbb{R}^d; \mathbb{R}^d)$. In later chapters we shall be interested in expressions on the form

$$\nu(f(\phi.)) := \left(\int_\mathbb{R} f^{(1)}(\phi.(x))d\nu(x), \ldots, f^{(d)}(\phi.(x))d\nu(x)\right) \in \mathcal{L}(\mathbb{R}^d)$$

as a controlled path in order to define $\int_0^t \nu(f(\phi_r))d\mathbf{X}_r$ in the rough path sense. Similar results as the previous chapters holds, summarized below.

Proposition 2.13. *Retain the hypotheses and notations respectively from Corollary 2.8, Corollary 2.10, Lemma 2.11 and assume $f \in C_b^p(\mathbb{R}^d; \mathbb{R}^d)$. The following holds.*

1. The rough path integral $\int_0^t \nu(f(\phi_r)) d\mathbf{X}_r$ is well defined.

2. Let $\mathbf{X} \in \mathscr{C}_g^\gamma$. Then there exists a family of smooth paths X^ϵ such that

$$\int_0^\cdot \nu(f(\phi_r^\epsilon)) \dot{X}_r^\epsilon dr \to \int_0^\cdot \nu(f(\phi_r)) d\mathbf{X}_r \quad \text{in } C^\gamma([0,T]),$$

   as $\epsilon \to 0$, where ϕ^ϵ denotes the solution of (16) with X replaced by X^ϵ.

3. If $\nu(f(\phi_\cdot^\epsilon)) \to \nu(f(\phi_\cdot))$ in C^γ we have

$$\int_0^\cdot \nu(f(\phi_r^\epsilon)) d\mathbf{X}_r \to \int_0^\cdot \nu(f(\phi_r)) d\mathbf{X}_r \quad \text{in } C^\gamma([0,T]),$$

   as $\epsilon \to 0$.

Proof. Begin with the first assertion. Integrating (17) w.r.t. ν gives

$$\int_{\mathbb{R}^d} f(\phi(x))_{st}^{(k)\sharp} d\nu(x) = \sum_{n=1}^{p-k-1} \sum_{q=1}^{n} \int_{\mathbb{R}^d} \frac{\nabla f^{(k+n)}(\phi_s(x))}{q!} (R_{st}^{\phi(x)})^{\otimes q} \otimes X_{st}^{\otimes(n-q)} d\nu(x)$$
$$+ \int_{\mathbb{R}^d} R_{p-k-1}^{\nabla^k f}(\phi_s(x), \phi_t(x)) d\nu(x).$$

Since ν is finite and b is bounded we get for each k, n and q

$$\left| \int_{\mathbb{R}^d} \frac{\nabla^{k+n} f(\phi_s(x))}{q!} (R_{st}^{\phi(x)})^{\otimes q} d\nu(x) \right| \lesssim |t - s|.$$

Furthermore

$$\left| \int_{\mathbb{R}^d} R_{p-k-1}^{\nabla^k f}(\phi_s(x), \phi_t(x)) d\nu(x) \right| \lesssim \int_{\mathbb{R}^d} |\phi_{st}(x)|^{p-k} d\nu(x) \lesssim |t - s|^{(p-k)\gamma},$$

so that $\int_{\mathbb{R}^d} f(\phi_\cdot(x)) d\nu(x)$ is a controlled path and

$$\left(\int_{\mathbb{R}^d} f(\phi(x)) d\nu(x) \right)_{st}^{(k)\sharp} = \int_{\mathbb{R}^d} f(\phi(x))_{st}^{(k)\sharp} d\nu(x).$$

Using linearity, boundedness of b and dominated convergence the reader is invited to complete the remaining steps of the proof. $\qquad\square$

3 Fractional Brownian motion and Girsanov's theorem

In this section we introduce the fractional Brownian motion as well as the technical tools we shall need in the remainder of the paper. More specifically the representation in terms of a fractional integral operator allows us to formulate the appropriate version of Girsanov theorem. The notion of strong local non-determinism is then used to infer technical bounds that are useful for studying local time estimates later in the paper. Finally we mention how one can construct a rough path lift of the fractional Brownian motion.

A 1-dimensional centered Gaussian process, $B = \{B_t, t \in [0, T]\}$, is called a *fractional Brownian motion* (fBm) with Hurst parameter $H \in (0, \frac{1}{2})$ if the covariance is given by

$$R_H(t, s) := E[B_t B_s] = \frac{1}{2} \left(t^{2H} + s^{2H} - |t - s|^{2H} \right).$$

Observe that B has stationary increments and its trajectories are Hölder continuous of index $H - \varepsilon$ for all $\varepsilon > 0$.

Denote by $\mathcal{E}$ the set of step functions on $[0,T]$ and denote by $\mathcal{H}$ the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the inner product

$$\langle 1_{[0,t]}, 1_{[0,s]}\rangle_{\mathcal{H}} = R_H(t,s).$$

The mapping $1_{[0,t]} \mapsto B_t$ can be extended to an isometry between $\mathcal{H}$ and a Gaussian subspace of $L^2(\Omega)$.

For a function $f \in L^2([a,b])$, we define the *left fractional Riemann-Liouville integral* by

$$I_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-y)^{\alpha-1} f(y) dy$$

for $\alpha > 0$. Denote by $I_{0+}^{\alpha}(L^2([a,b]))$ the image of $L^2([a,b])$ under I_{0+}^{α} and by D_{a+}^{α} its inverse.

We define $K_H(t,s)$ as

$$K_H(t,s) = c_H \Gamma\left(H + \frac{1}{2}\right) s^{\frac{1}{2}-H} \left(D_{t-}^{\frac{1}{2}-H} u^{H-\frac{1}{2}}\right)(s),$$

for some constant c_H and write K_H for the operator from $L^2([0,T])$ onto $I_{0+}^{H+\frac{1}{2}}(L^2)$ associated with the kernel $K_H(t,s)$. It follows that

$$R_H(t,s) = \int_0^{t\wedge s} K_H(t,u) K_H(s,u) du.$$

Moreover, if $W = \{W_t : t \in [0,T]\}$ is a standard Brownian motion B can be represented as

$$B_t = \int_0^t K_H(t,s) dW_s. \tag{19}$$

A d-dimensional fractional Brownian motion is a d-dimensional process where the components are independent 1-dimensional fractional Brownian motions.

Theorem 3.1 (Girsanov's theorem for fBm). *Let* $u = \{u_t, t \in [0,T]\}$ *be an* $\mathbb{R}^d$-*valued,* $\{\mathcal{F}_t\}_{t\in[0,T]}$-*adapted process with integrable trajectories and set* $\widetilde{B}_t = B_t + \int_0^t u_s ds, \quad t \in [0,T]$. *Assume that*

(i) $\int_0^{\cdot} u_s ds \in (I_{0+}^{H+\frac{1}{2}}(L^2([0,T])))^d$, *P-a.s.*

(ii) $E[\xi_T] = 1$ *where*

$$\xi_T := \exp\left\{ -\int_0^T K_H^{-1}\left(\int_0^{\cdot} u_r dr\right)(s) \cdot dW_s - \frac{1}{2}\int_0^T \left|K_H^{-1}\left(\int_0^{\cdot} u_r dr\right)(s)\right|^2 ds \right\}.$$

Then the shifted process $\widetilde{B}$ *is an* $\{\mathcal{F}_t\}_{t\in[0,T]}$-*fractional Brownian motion with Hurst parameter* H *under the new probability* $\widetilde{P}$ *defined by* $\frac{d\widetilde{P}}{dP} = \xi_T$.

Moreover, for every $p > 1$ *we have* $E[|\xi_T|^p] \leq C_p(\|b\|_{\infty})$, *where* $C_p(\cdot)$ *is an increasing function.*

For a proof we refer to [19]. In particular, the moment-estimate is found in the proof of Theorem 3, [19].

In the absence of the independent increments one has for $H = \frac{1}{2}$, we shall use the following fact (see [21, Theorem 3.1]).

Lemma 3.2. *The fractional Brownian motion is strong local non-deterministic, i.e. there exists a constant c such that*

$$Var(B_t : (B_s)_{s:|t-s|\geq r}) \geq cr^{2H}. \tag{20}$$

Given an m-dimensional Gaussian vector $Z \sim \mathcal{N}(0, \Sigma)$ it is well known that

$$|\Sigma| = Var(Z_m) Var(Z_{m-1}|Z_m) \ldots Var(Z_1|Z_m \ldots Z_2), \tag{21}$$

and so from Cramer's rule we get

$$(\Sigma^{-1})_{j,j} = (Var(Z_j|Z_1, \ldots, \widehat{Z_j}, \ldots, Z_m))^{-1} \tag{22}$$

From the above we can prove the following technical estimates on the fractional Brownian motion.

Proposition 3.3. *Given a fractional Brownian motion there exists C such that*

$$\int_{\mathbb{R}^m} \prod_{j=1}^m |v_j|^k \exp\left\{-\frac{1}{2}Var\left(\sum_{j=1}^m v_j B_{s_j}\right)\right\} dv_1 \ldots dv_m \leq C^m \sqrt{(km)!} \prod_{j=1}^m |s_j - s_{j-1}|^{-H(1+2k)} \qquad (23)$$

for all $(s_1, \ldots, s_m) \in \Delta^{(m)}(0, T)$, and we read $s_0 = 0$.

Proof. Define the matrix $A_{i,j} = E[B_{s_i} B_{s_j}]$, let $X \sim \mathcal{N}(0, A^{-1})$ and denote by $\tilde{X}$ the km-dimensional Gaussian vector

$$\tilde{X}_i = X_j \text{ for } (j-1)k + 1 \leq i \leq jk.$$

Rewrite the right hand side of (23) as

$$(2\pi)^{m/2}|A|^{-1/2}E[\prod_{j=1}^m |X_j|^k] = (2\pi)^{m/2}|A|^{-1/2}E[\prod_{i=1}^{km} |\tilde{X}_i|]$$

$$\leq (2\pi)^{m/2}|A|^{-1/2}\left(\sum_{\sigma \in S_{km}} \prod_{i=1}^{km} E[\tilde{X}_i \tilde{X}_{\sigma(i)}]\right)^{1/2}$$

$$\leq (2\pi)^{m/2}|A|^{-1/2}\left(\sum_{\sigma \in S_{km}} \prod_{i=1}^{km} E[\tilde{X}_i^2]^{1/2} E[\tilde{X}_{\sigma(i)}^2]^{1/2}\right)^{1/2}$$

$$= (2\pi)^{m/2}|A|^{-1/2}\left(\sum_{\sigma \in S_{km}} \prod_{j=1}^m E[X_j^2]^k\right)^{1/2} = (2\pi)^{m/2}|A|^{-1/2}\left((km)! \prod_{j=1}^m E[X_j^2]^k\right)^{1/2},$$

where we have used [15, Theorem 1] in the first inequality. Then we get from (22) that

$$(A^{-1})_{j,j} \geq c|s_{j+1} - s_j|^{2H} \wedge |s_j - s_{j-1}|^{2H} \geq c|s_j - s_{j-1}|^{4H}$$

where we have used (20) and $|s_{j+1} - s_j| \leq 1$ in the two last steps, respectively. Using (21) and (20) we get that

$$|A|^{-1/2} \leq c^{-m} \prod_{j=1}^m |s_j - s_{j-1}|^{-H}$$

The result follows. $\qquad \square$

As noted in Remark 2.7, it turns out that the structure of the noise will not see the full rough path lift of the fBm. Still we mention that the fractional Brownian motion can be lifted to a rough path, as was first done in [22]. We shall, however, refer to [20] for a different construction where the authors construct the iterated integrals using a Stratonovich-Volterra-type representation.

Theorem 3.4 (Theorem 1.1. in [20]). *Let B be a fractional Brownian motion admitting the representation (19). For $1 \leq n \leq \lfloor \frac{1}{H} \rfloor$ define*

$$B^{(n)} : \Delta^{(2)}(0, T) \to (\mathbb{R}^d)^{\otimes n}$$

component wise, i.e. for any tuple $\{i_1, \ldots i_n\}$ in $\{1, \ldots, d\}$, as the Stratonovich iterated integral

$$\langle B_{st}^{(n)}, e_{i_1} \otimes \cdots \otimes e_{i_n} \rangle = \sum_{j=1}^n (-1)^{j-1} \int_{A_j^n} \prod_{l=1}^{j-1} K(s, r_l)[K(t, r_j) - K(s, r_j)] \prod_{l=j+1}^n K(t, r_l) \circ dW_{r_1}^{i_1} \cdots \circ dW_{r_n}^{i_n}$$

where

$$A_j^n := \{(r_1, \ldots r_n) \in [0, t]^n : r_j = min(r_1, \ldots, r_n), r_1 > \cdots > r_{j-1} \text{ and } r_{j+1} < \cdots < r_n\}.$$

Then there exists a set $\Omega_{\mathbf{B}}$ with full measure such that

$$\mathbf{B}_{st} := (1, B_t - B_s, B_{st}^{(2)}, \ldots B_{st}^{(\lfloor 1/H \rfloor)})$$

satisfies (8) and (11) on $\Omega_{\mathbf{B}}$. Moreover, for $\gamma < H$ we have $|B_{st}^{(n)}| \lesssim |t - s|^{\gamma n}$.

Assume now that H is such that $\frac{1}{H}$ is not an integer. We can choose $\gamma < H$ such that $\lfloor \frac{1}{\gamma} \rfloor = \lfloor \frac{1}{H} \rfloor$, and from the above theorem we have, P-a.s., $\mathbf{B} \in \mathscr{C}_g^\gamma$.

Let us remark that for $H \in (\frac{1}{4}, \frac{1}{2})$ there exists a lift of B to a rough path building the iterated integral from linear interpolation of B. For the method of the current paper to work we need smaller H, see Section 4. When $H \in (0, \frac{1}{4})$ the dyadic interpolation fails to give a converging sequence of rough paths, see [8]. Nevertheless, the construction in [20] gives a geometric rough path so that we may approximate $\mathbf{B}$ by a sequence of lifted smooth paths, in the rough path topology.

4 Fractional Brownian motion SDE's

For this section we shall study a SDE driven by an additive fractional Brownian motion, i.e.

$$\phi_t(x) = x + \int_0^t b(r, \phi_r(x))dr + B_t. \tag{24}$$

Existence and uniqueness of a solution to this equation under low regularity on b was recently proved in [2] as demonstrated in the next Proposition. For proofs the reader is referred to [2].

Proposition 4.1 (Theorem 4.1 and Corollary 4.8 in [2]). *Assume* $H < \frac{1}{2(2d+1)}$. *Let* $\{b_n\}_{n \geq 0} \subset C_c^\infty([0,T] \times \mathbb{R}^d; \mathbb{R}^d)$ *be a sequence of functions such that*

$$\sup_{n \geq 0} (\|b_n\|_{\infty,1} \vee \|b_n\|_\infty) < \infty.$$

Denote by $\phi_n(t,x)$ *the solution to (24) when b is replaced by b_n. Then for fixed $(t,x) \in [0,T] \times \mathbb{R}^d$ the sequence is $\phi_n(t,x)$ is relatively compact in the* **strong** *topology of $L^2(\Omega)$.*

Furthermore, if $\lim_{n \to \infty} b_n(t,x) = b(t,x)$ *for almost all $(t,x) \in [0,T] \times \mathbb{R}^d$ for $b \in L^1(\mathbb{R}^d; L^\infty([0,T]; \mathbb{R}^d)) \cap L^\infty([0,T] \times \mathbb{R}^d; \mathbb{R}^d)$ then $\phi_n(t,x)$ is converging for every $(t,x) \in [0,T] \times \mathbb{R}^d$ to the unique solution of (24).*

The proof of this Proposition relies on a compactness criterion from [4] based on Malliavin calculus. Without going into too much detail there is compactness in $L^2(\Omega)$ if we can bound the Malliavin derivative of $\phi_n(t,x)$ by a constant depending only on $\|b_n\|_{L^1(\mathbb{R}^d; L^\infty([0,T]; \mathbb{R}^d))} \vee \|b_n\|_{L^\infty([0,T] \times \mathbb{R}^d; \mathbb{R}^d)}$.

Once one has strong convergence, one can use a somewhat standard trick, see e.g. [14] or [19], to show that $\int_0^t b_n(r, \phi_n(r,x))dr \to \int_0^t b(r, \phi_r(x))dr$ which gives that the limit solves (24).

Furthermore the following result shows how the fBm regularizes the flow of (24).

Lemma 4.2 (Theorem 5.1 in [2]). *Assume* $H < \frac{1}{(d-1+2k)}$ *and let p,k be integers, $p \geq 2$, $k \geq 1$. There exists an increasing function $C : [0,\infty) \to [0,\infty)$ only depending on H, d, p and k such that*

$$\sup_{t \in [0,T], x \in \mathbb{R}^d} E\left[\left| \nabla^k \phi_n(t,x) \right|^p \right] \leq C(\|b_n\|_{\infty,1} \vee \|b_n\|_\infty).$$

Using the two previous results together with weak compactness in $L^2(\Omega; W^{k,p}(U))$ for an open and bounded $U \subset \mathbb{R}^d$ we get the following result.

Theorem 4.3 (Theorem 5.2 in [2]). *Assume $H < \left(\frac{1}{2(2d+1)} \wedge \frac{1}{2(d-1+2k)} \right)$ and $b \in L^1(\mathbb{R}^d; L^\infty([0,T]; \mathbb{R}^d)) \cap L^\infty([0,T] \times \mathbb{R}^d; \mathbb{R}^d)$. For every open and bounded $U \subset \mathbb{R}^d$ the solution to (24) is k-times weakly differentiable in the sense that*

$$\phi_t \in L^2(\Omega; W^{k,p}(U))$$

for every $p > 1$. Moreover, $\phi_n(t)$ converges to ϕ_t in the weak topology of $L^2(\Omega; W^{k,p}(U))$.

4.1 The one-dimensional case

In this section we include a proof of Proposition 4.1 when $d = 1$ and $H < \frac{1}{6}$. From [19] it is already known that there exists a unique strong solution to this equation when b is of linear growth. From [19] it also becomes clear why the proof is simpler when $d = 1$ - one can use comparison of SDE's to generate the strong convergence as indicated in Section 4.1.3.

We shall restrict our attention to when b is bounded and integrable, but we are interested in how the solution depends on the initial value x. More specifically we will show the following.

Theorem 4.4. *Assume $b \in L^1(\mathbb{R}^d; L^\infty([0,T]; \mathbb{R})) \cap L^\infty([0,T] \times \mathbb{R})$. If $H < \frac{1}{6}$ there exists a unique strong solution to (24). Moreover the mapping $(x \mapsto \phi_t(x))$ is weakly differentiable in the sense that for fixed t we have*

$$\phi_t(\cdot) \in L^2(\Omega; W^{1,p}(U))$$

for all open and bounded $U \subset \mathbb{R}$.

This theorem is proved in three steps. In the first step we establish an integration by parts formula for the fractional Brownian motion. In the second step we assume that b is smooth and has compact support. It is then well known that $\phi_t(\cdot)$ is smooth, and we use the integration by parts formula to bound $\|\phi_t\|_{L^2(\Omega; W^{1,p}(U))}$ independently of b'. In the third step we approximate a general b by smooth functions. We use comparison to generate strong convergence in $L^2(\Omega)$ of the corresponding sequence of solutions. From step one and two we can bound the sequence in $L^2(\Omega; W^{1,p}(U))$ and argue via weak compactness to prove Theorem 4.4.

4.1.1 An integration by parts formula

The purpose of this section is to prove a integration by parts type formula involving a random variable inspired by local time calculus. More specifically, we have

$$\int_0^t b'(s, B_s) ds = -\int_{\mathbb{R}} \Lambda^b(t, y) dy \quad P - a.s. \tag{25}$$

where

$$\Lambda^b(t, y) = (2\pi)^{-1} \int_{\mathbb{R}} \int_0^t b(s, y) iu e^{-iu(B_s - y)} ds du. \tag{26}$$

We start by *defining* $\Lambda^b(t, z)$ as above, and prove that it is a well defined element of $L^p(\Omega)$ for every $p > 1$.

Lemma 4.5. *Assume b is bounded. Then $\Lambda^b(t, y)$ exists and all moments are integrable provided $H < \frac{1}{3}$. More precisely if m is an even integer*

$$E[|\Lambda^b(t, y)|^m] \leq \frac{C^m \|b(\cdot, y)\|_\infty^m m! \sqrt{m!}}{\Gamma(m(1 - 3H) + 1)}.$$

Proof. Since we assume m is an even integer, we may write

$$E[|\Lambda^b(t, y)|^m = (2\pi)^{-m} E|\int_{\mathbb{R}} \int_0^t b(s, y) iu \exp\{-iu(B_s - y)\} ds du|^m$$

$$= (2\pi)^{-m} m! \int_{\Delta^m(0,t)} \int_{\mathbb{R}^m} b^{\otimes m}(s, y) \prod_{j=1}^m iu_j E[\exp\{-iu_j(B_{s_j} - y)\}] du ds$$

$$\leq (2\pi)^{-m} m! \int_{\Delta^m(0,t)} \int_{\mathbb{R}^m} |b^{\otimes m}(s, y)| \prod_{j=1}^m |u_j| \exp\{-\frac{1}{2} Var(\sum_{j=1}^m u_j B_{s_j})\}] du ds$$

where for notational convenience we have used $B_{s_0} = y$ and $v_{m+1} = 0$, $ds = ds_1 \ldots ds_m$, $du = du_1 \ldots du_m$ and $b^{\otimes m}(s, y) := \prod_{j=1}^m b(s_j, y)$. Using (23) the above is bounded by

$$C^m m! \sqrt{m!} \|b(\cdot, y)\|_\infty^m \int_{\Delta^{(m)}(0,t)} \prod_{j=1}^m |s_j - s_{j-1}|^{-3H} ds \leq \frac{C^m m! \sqrt{m!} \|b(\cdot, y)\|_\infty^m}{\Gamma((1 - 3H)m + 1)}.$$

$\square$

From (26) we see that $\operatorname{supp} \Lambda^b(t, \cdot) \subset \bigcup_{s \leq t} \operatorname{supp} b(s, \cdot)$. In particular, if the latter set is bounded, $\Lambda^b(t, \cdot)$ is integrable P-a.s.

It remains to show that Λ^b satisfies the integration by parts formula (25). Notice that one has to be careful interchanging the order of integration in (26). Indeed, if $b = 1$, one should think of $\int_{\mathbb{R}} iu e^{-iu(B_s - y)} du = -\partial_y \delta_{B_s}(y)$ where $\delta_{B_s}(y)$ is the Donsker-Delta of B_s, which is not a random variable in the usual sense (one could introduce the Donsker-Delta as a generalized random variable in the sense of White Noise theory, but we shall avoid this).

To circumvent this difficulty we define an approximating sequence

$$\Lambda_K^b(t,y) := (2\pi)^{-1} \int_{-K}^{K} \int_0^t b(s,y) iu e^{-iu \cdot (B_s - y)} ds du.$$

It is immediate that

$$|\Lambda_K^b(t,y)| \leq C_K \int_0^t |b(s,y)| ds,$$

for an appropriate constant, so that $\Lambda_K^b(t,\cdot)$ is integrable if $\int_{\mathbb{R}} \int_0^t |b(s,y)| ds dy < \infty$. One can show that $\Lambda_K^b(t,y) \to \Lambda^b(t,y)$ in, say, $L^2(\Omega)$ for all t and y. To see this the reader is invited to modify the above proof to see that

$$E[|\Lambda_K^b(t,y) - \Lambda^b(t,y)|^2] \leq C\|b(\cdot,y)\|_\infty^2 \int_{\Delta^2(0,t)} \int_{\mathbb{R}^2} 1_{\{|u_1|>K\}} 1_{\{|u_2|>K\}} |u_1||u_2| e^{-\frac{1}{2}Var(u_1 B_{s_1} + u_2 B_{s_2})} du ds$$

which converges to zero as $K \to \infty$. In the above C is a constant that is independent of K. Now we have

$$\int_{\mathbb{R}} \Lambda_K^b(t,y) dy = (2\pi)^{-1/2} \int_{-K}^{K} \int_0^t (\mathcal{F}^{-1}b)(s,u) iu e^{-iu \cdot X_s} ds du$$

$$= \int_0^t (2\pi)^{-1/2} \int_{-K}^{K} (\mathcal{F}^{-1}b)(s,u) iu e^{-iu \cdot X_s} du ds.$$

Provided $b(s,\cdot) \in \mathcal{S}(\mathbb{R})$ we have

$$\lim_{K \to \infty} (2\pi)^{-1/2} \int_{-K}^{K} (\mathcal{F}^{-1}b)(s,u) iu e^{-iu X_s} du = (2\pi)^{-1/2} \int_{\mathbb{R}} (\mathcal{F}^{-1}b)(s,u) iu e^{-iu X_s} du$$

$$= \mathcal{F}(iu(\mathcal{F}^{-1}b)(s,u))(X_s) = -b'(s, X_s)$$

thus proving (25).

We summarize these considerations.

Lemma 4.6. *Let $b : [0,T] \times \mathbb{R} \to \mathbb{R}$ be such that $b(s,\cdot)$ is smooth for every s and $\bigcup_{s \leq T} \operatorname{supp} b(s,\cdot)$ is a bounded set. Then (25) holds on a set of measure 1.*

We can however extend (25) to bounded and differentiable b.

Lemma 4.7. *Assume $b \in L^\infty([0,T]; C_b^1(\mathbb{R}))$. Then (25) holds for b and we have P-a.s.*

$$\operatorname{supp} \Lambda^b(t,\cdot) \subset [-B_t^*, B_t^*]$$

where $B_t^* := \sup_{0 \leq s \leq t} |B_s|$.

Proof. Assume first that b satisfies the assumptions of Lemma 4.6, and let $\phi \in C_c^1(\mathbb{R})$. From (26) we have $\Lambda^{\phi b}(t,y) = \phi(y)\Lambda^b(t,y)$. Consequently, using (25)

$$\int_{\mathbb{R}} \phi(y)\Lambda^b(t,y) dy = \int_{\mathbb{R}} \Lambda^{\phi b}(t,y) dy = -\int_0^t (\phi(B_s)b(s, B_s))' ds$$

$$= -\int_0^t \phi'(B_s)b(s, B_s) ds - \int_0^t \phi(B_s)b'(s, B_s) ds,$$

so that for all $\phi \in C_c^1(\mathbb{R})$ such that $\operatorname{supp} \phi \cap [-B_t^*, B_t^*] = \emptyset$, we have $\int_{\mathbb{R}} \phi(y)\Lambda^b(t,y) dy = 0$. In particular, $\Lambda^b(t,\cdot)$ has compact support independent of b P-a.s.

From linearity of $b \mapsto \Lambda^b$ and Lemma 4.5 we may approximate a general b by smooth, compactly supported functions. The result follows by elementary calculations. $\square$

Using $\Lambda^{\phi b}(t,y) = \phi(y)\Lambda^b(t,y)$ as in the above proof we get that if b is time homogeneous, $\Lambda^b(t,y) = b(y)\partial_y L^B(t,y)$ where $L^B(t,y)$ denotes the local time of the fractional Brownian motion (which is well known to be differentiable when $H < \frac{1}{3}$, see [11]).

Proposition 4.8. *There exists a constant $C > 0$ such that for all even integers m*

$$E\left[\left(\int_{\mathbb{R}} |\Lambda^b(t,y)|dy\right)^m\right] \leq \frac{C^m \|b\|_{\infty,1}^m m^{m/2} \sqrt{(2m)!}}{\sqrt{\Gamma(m(1-3H)+1)}}.$$

Proof. We write

$$E\left[\left(\int_{\mathbb{R}} |\Lambda^b(t,y)|dy\right)^m\right] = \int_{\mathbb{R}^m} E[\prod_{j=1}^m |\Lambda^b(t,y_j)|]dy_1 \ldots dy_m \leq \int_{\mathbb{R}^m} \prod_{j=1}^m E[|\Lambda^b(t,y_j)|^m]^{1/m} dy_1 \ldots dy_m$$

$$\leq \frac{C^m m! \sqrt{m!}}{\Gamma((1-3H)m+1)} \int_{\mathbb{R}^m} \prod_{j=1}^m \|b(\cdot, y_j)\|_\infty dy_1 \ldots dy_m = \frac{C^m m! \sqrt{m!}}{\Gamma((1-3H)m+1)} \|b\|_{1,\infty}^m$$

for an appropriate constant C, where we have used Lemma 4.5. $\qquad\square$

4.1.2 Derivative free estimates

In this section we assume that $b \in L^\infty([0,T]; C_c^1(\mathbb{R}))$ and denote by $\phi_\cdot(x)$ the solution to (24). It is well known that $\phi_t(\cdot)$ continuously differentiable, and we have

$$\partial_x \phi_t(x) = 1 + \int_0^t b'(r, \phi_r(x))\partial_x \phi_r(x)dr \tag{27}$$

$$= \exp\left\{\int_0^t b'(r, \phi_r(x))dr\right\}. \tag{28}$$

We are ready to prove our main estimate on SDE's.

Theorem 4.9. *There exists an increasing continuous function $C : [0,\infty) \to [0,\infty)$ such that for all $b \in L^\infty([0,T]; C_b^1(\mathbb{R}))$*

$$\sup_{t\in[0,T], x\in\mathbb{R}} E\left[(\partial_x \phi_t(x))^2\right] \leq C(\|b\|_\infty \wedge \|b\|_{\infty,1}),$$

where $\phi_\cdot(x)$ is the unique solution of (24) driven by b.

Proof. Set $\theta_t := \left(K_H^{-1}\left(\int_0^\cdot b(r, \phi_r(x))dr\right)\right)(t)$ and consider the Doléans-Dade exponential

$$Z := \exp\left\{\int_0^T \theta_s dW_s - \frac{1}{2}\int_0^T \theta_s^2 ds\right\}.$$

Define the measure $\tilde{P}$ by

$$d\tilde{P} := ZdP.$$

Then $\tilde{P}$ is a probability measure and under $\tilde{P}$ the solution $\{\phi_t(x)\}_t$ is a fractional Brownian motion starting in x. From (28) we get

$$E[(\partial_x \phi_t(x))^2] = E\left[\exp\left\{2\int_0^t b'(r, \phi_r(x))dr\right\}\right] = \tilde{E}\left[\exp\left\{2\int_0^t b'(r, \phi_r(x))dr\right\} Z^{-1}\right]$$

$$\leq \left(\tilde{E}\left[\exp\left\{4\int_0^t b'(r, \phi_r(x))dr\right\}\right]\right)^{1/2} \left(\tilde{E}[Z^{-2}]\right)^{1/2}.$$

Now we write

$$\tilde{E}\left[\exp\left\{4\int_0^t b'(r, \phi_r(x))dr\right\}\right] = E\left[\exp\left\{4\int_0^t b'(r, x+B_r)dr\right\}\right] = E\left[\exp\left(\{4\int_{\mathbb{R}} \Lambda^b(t,y)dy\}\right]\right]$$

$$= \sum_{m\geq 0} \frac{4^m E\left[\left(\int_{\mathbb{R}} \Lambda^b(t,y)dy\right)^m\right]}{m!} \leq \sum_{m\geq 0} \frac{(4\|b\|_{1,\infty})^m C^m (2m!)^{1/4}\sqrt{(2m)!}}{m!\sqrt{\Gamma((1-3H)2m+1)}}$$

$$=: \tilde{C}(\|b\|_{\infty,1})$$

which converges by Stirling's formula.

From Theorem 3.1 we know that we can bound $\tilde{E}[Z^{-2}]$ by a function depending on $\|b\|_\infty$. The result follows. $\qquad\square$

4.1.3 Singular SDE's

For this section we shall consider a bounded and measurable $b : [0,T] \times \mathbb{R} \to \mathbb{R}$ and the corresponding SDE (24). As indicated above we shall use an approximation b_n of b and comparison to generate strong convergence in $L^2(\Omega)$. The technique is somewhat classical, and we refer to [19] for a proof, but let us briefly explain the idea:

Let b be bounded and measurable and define for $n \in \mathbb{N}$

$$b_n(t,x) := n \int_{\mathbb{R}} \rho(n(x-y))b(t,y)dy$$

where ρ is a non-negative smooth function with compact support in $\mathbb{R}$ such that $\int_{\mathbb{R}} \rho(y)dy = 1$. We let

$$\tilde{b}_{n,k} := \bigwedge_{j=n}^{k} b_j, \quad n \leq k, \qquad \text{and} \qquad \mathcal{B}_n = \bigwedge_{j=n}^{\infty} b_j,$$

so that $\tilde{b}_{n,k}$ is Lipschitz. Denote by $\tilde{\phi}_{n,k}(t,x)$ the unique solution to (24) when we replace b by $\tilde{b}_{n,k}$. Then one can use comparison to show that

$$\lim_{k \to \infty} \tilde{\phi}_{n,k}(t,x) = \phi_n(t,x), \qquad \text{in } L^2(\Omega)$$

where $\phi_n(t,x)$ solves (24) when we replace b by $\mathcal{B}_n$. Furthermore,

$$\lim_{n \to \infty} \phi_n(t,x) = \phi_t(x), \qquad \text{in } L^2(\Omega)$$

where $\phi_t(x)$ is a solution to (24). For details see [19].

We are ready to prove the main result of the section.

Proof of 4.4. Let $U \subset \mathbb{R}$ be open and bounded. We know from the discussion above that $\phi_n(t,x) \to \phi_t(x)$ in $L^2(\Omega)$. From Theorem 4.9 plus elementary bounds we see that $\phi_n(t, \cdot)$ is bounded in $L^2(\Omega; W^{1,2}(U))$. Consequently we may extract a subsequence $\{\phi_{n_k}(t, \cdot)\}_{k \geq 1}$ converging to an element f_t in the weak topology of $L^2(\Omega; W^{1,2}(U))$. Let $A \in \mathcal{F}$ and $\eta \in C^\infty(U)$. Using strong convergence coupled with weak convergence we get

$$E[1_A \int_U \phi_t(x)\eta'(x)dx] = \lim_{k \to \infty} E[1_A \int_U \phi_{n_k}(t,x)\eta'(x)dx] = -\lim_{k \to \infty} E[1_A \int_U \partial_x\phi_{n_k}(t,x)\eta(x)dx]$$
$$= -E[1_A \int_U \partial_x f_t(x)\eta(x)dx].$$

Consequently we have $\int_U \phi_t(x)\eta'(x)dx = -\int_U \partial_x f_t(x)\eta(x)dx$ on some $\Omega_\eta \in \mathcal{F}$ such that $P(\Omega_\eta) = 1$. Let now Ω^* be the intersection of a countable, dense in $W^{1,2}(U)$, set of η such that the above integration by parts formula holds. It is clear that $P(\Omega^*) = 1$ and that ϕ_t is weakly differentiable on this set. The result follows. $\qquad \square$

Remark 4.10. *For fixed $t_0 > 0$, consider the equation*

$$\psi_t^{t_0}(y) = y - \int_0^t b(t_0 - r, \psi_r^{t_0}(y))dr - (B_{t_0} - B_{t_0-t}).$$

Since the fractional Brownian motion has stationary increments the above equation is on the same form as (24) and we may apply the same machinery to obtain a sequence $\psi_t^{t_0,n}$ of corresponding smooth flows that converges in the weak topology of $L^2(\Omega; W^{1,p}(U))$ and $\psi_t^{t_0,n}(x)$ converges in the strong topology of $L^2(\Omega)$ to the solution of the above equation.

We have $\psi_{t_0}^{t_0} = \phi_{t_0}^{-1}$, so that ϕ_{t_0} is invertible with a Sobolev-differentiable inverse.

Let now $f \in C_b^1(\mathbb{R})$. For every $n \in \mathbb{N}$ we have

$$\partial_x f(\psi_t^{t,n}(x)) = f'(\psi_t^{t,n}(x))\partial_x\psi_t^{t,n}(x)$$

which is bounded in any $L^p(U)$ for $p > 1$, U open and bounded. Consequently, there is a weakly converging subsequence which by uniqueness must converge weakly in $L^p(U)$ and we have

$$\partial_x f(\psi_t^t(x)) = f'(\psi_t^t(x))\partial_x\psi_t^t(x), \qquad \text{for almost all } x \in \mathbb{R}.$$

since $\psi_t^{t,n}(x) \to \psi_t^t(x)$ strongly in $L^2(\Omega)$. When b is time-homogenuous we have the following representation

$$\phi_t^{-1}(y) = y - \int_0^t b(\phi_r^{-1}(y))dr - B_t.$$

4.2 Local time of the flow

We now return to the general case of $d \geq 1$.

In this section we develop a local time theory for the solutions $\phi_t(x)$ of (24). Assuming we have a solution to $\phi_t(x)$, the results here will rely only on Girsanov's theorem 3.1 meaning we only use boundedness of b. Let now $Q : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be given and define $q = D^\alpha Q$ for some multiindex, $\alpha = (\alpha^{(1)}, \ldots \alpha^{(d)})$. The main objective of this section is to prove that there exists a random field $\Lambda_\alpha^{\phi(x),Q}$ on $[0,T] \times \mathbb{R}^d$ such that

$$\int_0^t q(s, \phi_s(x))ds = (-1)^{|\alpha|} \int_{\mathbb{R}^d} \Lambda_\alpha^{\phi(x),Q}(t,y)dy,$$

and that the right hand side above can be bounded in terms of Q. Motivated by the previous subsection, we define

$$\Lambda_\alpha^{\phi(x),Q}(t,y) = (2\pi)^{-d} \int_{\mathbb{R}^d} \int_0^t (iu)^\alpha Q(s,y) \exp\{-iu \cdot (\phi_s(x) - y)\}dsdu.$$

We denote by $\Lambda_\alpha^Q(t,y)$ the random field obtained by choosing B instead of $\phi(x)$ in the above definition. Note that from Girsanov's theorem we have

$$E[f(\Lambda_\alpha^{\phi(x),Q}(t,y))] = E[f(\Lambda_\alpha^Q(t,y))\xi_T]$$

for any f such that the above expressions exists and ξ_T was defined in Theorem 3.1 and we have $E[|\xi_T|^2] \leq C(\|b\|_\infty)$. We get a similar result as Lemma 4.5.

Lemma 4.11. *Assume Q is bounded and $H < \frac{1}{d+2|\alpha|}$. We have the following moments estimates on Λ_α^Q*

$$E[|\Lambda_\alpha^Q(t,y)|^m] \leq \frac{C^m m! \prod_{k=1}^d \sqrt{(m\alpha^{(k)})!}}{\Gamma(m(1 - H(d + 2|\alpha|)) + 1)} \tag{29}$$

where $C = C(\alpha, H)$ does not depend on m or Q.

Proof. The proof follows the same lines as in the proof of Lemma 4.5. Begin by writing

$$E\left[|\Lambda_\alpha^Q(t,y)|^m\right] = \frac{m!}{(2\pi)^{dm}} \int_{(\mathbb{R}^d)^m} \int_{\Delta^{(m)}(0,t)} \prod_{j=1}^m (iu_j)^\alpha Q(s_j,y) \exp\{-\frac{1}{2} Var(u_j \cdot (B_{s_j} - y)\}dsdu$$

$$\leq \frac{m!}{(2\pi)^{dm}} \int_{(\mathbb{R}^d)^m} \int_{\Delta^{(m)}(0,t)} \prod_{j=1}^m \prod_{k=1}^d |u_j^{(k)}|^{\alpha^{(k)}} |Q(s_j,y)| \exp\{-\sum_{k=1}^d \frac{1}{2} Var(\sum_{j=1}^m u_j^{(k)} B_{s_j}^{(1)})\}dsdu$$

$$\leq \frac{m!\|Q(\cdot,y)\|_\infty^m}{(2\pi)^{dm}} \int_{\Delta^{(m)}(0,t)} \prod_{k=1}^d \int_{\mathbb{R}^m} \prod_{j=1}^m |u_j^{(k)}|^{\alpha^{(k)}} \exp\{-\frac{1}{2} Var(\sum_{j=1}^m u_j^{(k)} B_{s_j}^{(1)})\}dsdu_1^{(k)} \ldots du_m^{(k)}$$

where we have used the independence of the components of B in the second line. Using (23), the above is bounded by

$$C^m m! \|Q(\cdot,y)\|_\infty^m \prod_{k=1}^d \sqrt{(m\alpha^{(k)})!} \int_{\Delta^{(m)}(0,t)} \prod_{k=1}^d |s_j - s_{j-1}|^{-H(1+2\alpha^{(k)})} ds$$

$$\leq \frac{C^m m! \prod_{k=1}^d \sqrt{(m\alpha^{(k)})!} \|Q(\cdot,y)\|_\infty^m}{\Gamma(m(1 - H(d + 2|\alpha|)) + 1)}$$

provided $H < \frac{1}{d+2|\alpha|}$. $\qquad\square$

Using Theorem 3.1 we get

Corollary 4.12. *Let Q, H and α be as in the previous lemma. There exists a constant $C = C(\|b\|_\infty, H, d, \alpha)$ such that*

$$E[|\Lambda_\alpha^{\phi(x),Q}(t,y)|^m] \leq \frac{C^m \sqrt{(2m)! \prod_{k=1}^d \sqrt{(2m\alpha^{(k)})!}} \|Q(\cdot,y)\|_\infty^m}{\sqrt{\Gamma(2m(1 - H(d + 2|\alpha|)) + 1)}}.$$

If we assume integrability of Q in the spatial variable we see that we can define the stochastic process $\int_{\mathbb{R}^d} \Lambda_\alpha^{\phi(x),Q}(t,z)dz$.

Lemma 4.13. *If we assume $Q \in L^1(\mathbb{R}^d; L^\infty([0,T]))$, $|\alpha| \leq 1$ and $H < \frac{1}{d+2}$ we have*

$$E[\exp\{\int_{\mathbb{R}^d} |\Lambda_\alpha^{\phi(x),Q}(t,z)|dz\}] \leq C(\|Q\|_{\infty,1} \wedge \|b\|_\infty)$$

where C is an increasing function.

Proof. Begin by writing

$$\begin{aligned}
E\left[\left(\int_{\mathbb{R}^d} |\Lambda_\alpha^{\phi(x),Q}(t,z)|dz\right)^m\right] &= \int_{(\mathbb{R}^d)^m} E\left[\prod_{j=1}^m |\Lambda_\alpha^{\phi(x),Q}(t,z_j)|\right] dz_1 \ldots dz_m \\
&\leq \int_{(\mathbb{R}^d)^m} \prod_{j=1}^m E\left[|\Lambda_\alpha^{\phi(x),Q}(t,z_j)|^m\right]^{1/m} dz_1 \ldots dz_m \\
&\leq \frac{C^m \sqrt{(2m)! \prod_{k=1}^d \sqrt{(2m\alpha^{(k)})!}}}{\sqrt{\Gamma(2m(1-H(d+2|\alpha|))+1)}} \int_{(\mathbb{R}^d)^m} \prod_{j=1}^m \|Q(\cdot,z_j)\|_\infty dz_1 \ldots dz_m \\
&\leq \frac{C^m \sqrt{(2m)!\sqrt{(2m)!}}}{\sqrt{\Gamma(2m(1-H(d+2))+1)}} \|Q\|_{\infty,1}^m
\end{aligned}$$

where C is as in Corollary 4.12. We get

$$E[\exp\{\int_{\mathbb{R}^d} |\Lambda_\alpha^{\phi(x),Q}(t,z)|dz\}] = \sum_{m \geq 0} (m!)^{-1} E\left[\left(\int_{\mathbb{R}^d} |\Lambda_\alpha^{\phi(x),Q}(t,z)|dz\right)^m\right]$$

$$\sum_{m \geq 0} \frac{C^m \sqrt{(2m)!\sqrt{(2m)!}}}{\sqrt{\Gamma(2m(1-H(d+2))+1)}m!} \|Q\|_{\infty,1}^m$$

which converges as long as $H < \frac{1}{d+2}$ by Stirling's formula. $\qquad\square$

We now proceed to prove stability of the vector field $\Lambda_\alpha^{\phi(x),Q}$ in both Q and ϕ in the following way.

Remark 4.14. *We shall need stability of the mapping $(\phi(x),Q) \mapsto \int_{\mathbb{R}^d} \Lambda^{\phi(x),Q}(t,z)dz$, but we only need continuity in each variable separately. If $\phi^\epsilon(x)$ converges to $\phi_\cdot(x)$ in, say, Lebesgue measure over $[0,T]$ and Q is smooth, we immediately get*

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^d} \Lambda^{\phi^\epsilon(x),Q}(t,z)dz = \lim_{\epsilon \to 0} (-1)^{|\alpha|} \int_0^t q(s,\phi_s^\epsilon(x))ds = (-1)^{|\alpha|} \int_0^t q(s,\phi_s(x))ds = \int_{\mathbb{R}^d} \Lambda^{\phi(x),Q}(t,z)dz$$

by dominated convergence.

Stability in Q as a mapping $L^1(\mathbb{R}^d; L^\infty([0,T];\mathbb{R}^d)) \to L^m(\Omega)$ follows from the linearity of the mapping $Q \to \int_{\mathbb{R}^d} \Lambda^{\phi(x),Q}(t,z)dz$ as well as the bounds from Lemma 4.13.

4.3 Convergence in Hölder spaces

With the notation of Proposition 2.13 we shall need a result to ensure convergence of $\nu(f(\phi_\cdot^n))$ is uniform on a set of full measure.

Proposition 4.15. *Let $\gamma \in (0,H)$, $f \in C_b^1(\mathbb{R}^d;\mathbb{R}^d)$ and ν be a finite signed measure on $\mathbb{R}^d$. Then there exists a set $\Omega_{\gamma,\nu}$ of full measure such that on this set we have*

$$\lim_{n \to \infty} \nu(f(\phi_\cdot^n)) = \nu(f(\phi_\cdot))$$

in $C^\gamma([0,T];\mathbb{R}^d)$.

Proof. We begin by showing that $\nu(f(\phi^n_t)) \to \nu(f(\phi_t))$ in $L^2(\Omega)$ for every t. To see this, consider

$$E[|\nu(f(\phi^n_t)) - \nu(f(\phi_t))|^2] = E[|\nu(f(\phi^n_t) - f(\phi_t))|^2]$$
$$\leq |\nu|(\mathbb{R}^d)\|\nabla f\|_\infty \int_{\mathbb{R}^d} E[|\phi^n_t(x) - \phi_t(x)|^2] d\nu(x) \to 0$$

as $n \to \infty$ by dominated convergence, which proves the first claim.

Next we find a set universal in t for which we have pointwise in ω convergence. For the rest of the proof we will abuse notation and write $\phi(\omega)$, $\phi^n(\omega)$ etc, thus supressing the dependence on x. Denote by $\{q_j\}_{j=1}^\infty$ an enumeration of $[0,T] \cap \mathbb{Q}$. We may extract a subsequence $\{\nu(f(\phi^{n(k,1)}_{q_1}))\}_{k\geq 1} \subset \{\nu(f(\phi^n_{q_1}))\}_{n\geq 1}$ such that

$$\lim_{k\to\infty} \nu(f(\phi^{n(k,1)}_{q_1}(\omega))) = \nu(f(\phi_{q_1}(\omega)))$$

for $\omega \in \Omega_1$ with full measure. Furthermore, we define inductively a subsequence $\{\nu(f(\phi^{n(k,j+1)}_{q_{j+1}}))\}_{k\geq 1} \subset \{\nu(f(\phi^{n(k,j)}_{q_{j+1}}))\}_{k\geq 1}$ such that

$$\lim_{k\to\infty} \nu(f(\phi^{n(k,j+1)}_{q_{j+1}}(\omega))) = \nu(f(\phi_{q_{j+1}}(\omega)))$$

for $\omega \in \Omega_{j+1}$ with full measure. Let $\Omega_0 = \cap_{j=1}^\infty \Omega_j$, so that we have

$$\lim_{j\to\infty} \nu(f(\phi^{n(j,j)}_q(\omega))) = \nu(f(\phi_q(\omega)))$$

for all $\omega \in \Omega_0$ and q rational.

Now, we construct a set where $\{\nu(f(\phi^n))\}_{n\geq 1}$ is relatively compact in $C([0,T]; \mathbb{R}^d)$. Let $\epsilon > 0$ be such that $\gamma < H - \epsilon$ and choose a subset $\Omega_{H-\epsilon}$ with full measure such $\phi.$ satisfies (3) and for every $\omega \in \Omega_{H-\epsilon}$ we have

$$(t \mapsto B_t(\omega)) \in C^{H-\epsilon}([0,T]; \mathbb{R}^d).$$

Note that $\nu(f(\phi.))$ is continuous on this set.

From (3) we see

$$|\nu(f(\phi^n_t(\omega))) - \nu(f(\phi^n_s(\omega)))| \leq |\nu(f(\phi^n_t(\omega)) - f(\phi^n_s(\omega)))|$$
$$\leq |\nu|(\mathbb{R}^d)\|\nabla f\|_\infty (\|b_n\|_\infty |t-s| + |B_{st}(\omega)|)$$
$$\leq |\nu|(\mathbb{R}^d)\|\nabla f\|_\infty (\|b_n\|_\infty |t-s| + \|B(\omega)\|_{H-\epsilon} |t-s|^{H-\epsilon})$$

so that the uniform boundedness of b_n implies that $\{\nu(f(\phi^n.(\omega)))\}_{n\geq 1}$ is equicontinuous. Moreover, the sequence is bounded in $C([0,T]; \mathbb{R}^d)$ and from Arzela-Ascoli's theorem there exists a converging subsequence $\{\nu(f(\phi^{j(k,\omega)}.(\omega)))\}_{k\geq 1} \subset \{\nu(f(\phi^{n(j,j)}.(\omega)))\}_{j\geq 1}$. For $\omega \in \Omega_0 \cap \Omega_{H-\epsilon}$ - which has full measure - we see that the limit coincides with $\nu(f(\phi.(\omega)))$. Applying the above reasoning to any subsequence of $\{\nu(f(\phi^n.(\omega)))\}_{n\geq 1}$ we get a further subsequence that converges to $\nu(f(\phi.(\omega)))$ in $C([0,T]; \mathbb{R}^d)$. Since $C([0,T]; \mathbb{R}^d)$ is a Banach space this implies that the *full* sequence converges. By interpolation of Hölder spaces we see that the claim is true if we let $\Omega_{\gamma,\nu} := \Omega_0 \cap \Omega_{H-\epsilon}$. $\square$

5   Continuity Equation

In this section we study the measure valued rough linear continuity equation

$$\partial_t \mu_t + \text{div}(b\mu_t) + \text{div}(\mu_t d\mathbf{X}_t) = 0 \tag{30}$$

with given initial condition μ_0. The notion of solution is as follows.

Definition 5.1. *Let μ_0 be a finite signed measure on $\mathbb{R}^d$. A measure valued function $\mu : [0,T] \to \mathcal{M}(\mathbb{R}^d)$ is called a measure solution to (30) if*

$$\mu_t + \int_0^t div(b(r,\cdot)\mu_r)dr + \int_0^t div(\mu_r d\mathbf{X}_r) = \mu_0$$

holds weakly in $\mathcal{M}(\mathbb{R}^d)$ meaning for every $\eta \in C_c^\infty(\mathbb{R}^d)$ we have $\mu.(\nabla\eta) \in \mathscr{D}_{\mathbf{X}}^{p\gamma}$ and

$$\mu_t(\eta) = \mu_0(\eta) + \int_0^t \mu_r(b(r,\cdot)\nabla\eta)dr + \int_0^t \mu_r(\nabla\eta)d\mathbf{X}_r.$$

If we know that there exists a solution to

$$\phi_t(x) = x + \int_0^t b(r, \phi_r(x))dr + X_t,$$

then for any test function $\eta \in C_c^\infty(\mathbb{R}^d)$ we have from Lemma 2.12

$$\eta(\phi_t(x)) = \eta(x) + \int_0^t \nabla\eta(\phi_r(x))b(r, \phi_r(x))dr + \int_0^t \nabla\eta(\phi_r(x))d\mathbf{X}_r.$$

We integrate the equation w.r.t. μ_0 to see that $\mu_t := (\phi_t)_\sharp \mu_0$ solves (30) if we can use integration by parts for the rough path integral, namely

$$\int_{\mathbb{R}^d} \int_0^t \nabla\eta(\phi_r(x))d\mathbf{X}_r d\mu_0(x) = \int_0^t \mu_0(\nabla\eta(\phi_r))d\mathbf{X}_r.$$

Suppose now that $b \in C_b^1([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$ and $\mathbf{X} \in \mathscr{C}_g^\gamma$. Let $X^\epsilon \in C^1([0, T]; \mathbb{R}^d)$ be such that $\mathbf{X}^\epsilon \to \mathbf{X}$ in $\mathscr{C}^\gamma$. Using Section 2 we get

$$\int_0^t \mu_0(\nabla\eta(\phi_r))d\mathbf{X}_r = \lim_{\epsilon\to0} \int_0^t \mu_0(\nabla\eta(\phi_r^\epsilon))\dot{X}_r^\epsilon dr = \lim_{\epsilon\to0} \int_{\mathbb{R}^d} \int_0^t \nabla\eta(\phi_r^\epsilon(x))\dot{X}_r^\epsilon dr d\mu_0(x)$$
$$= \int_{\mathbb{R}^d} \int_0^t \nabla\eta(\phi_r(x))d\mathbf{X}_r d\mu_0(x).$$

We summarize the above in a lemma.

Lemma 5.2. *Suppose $b \in C_b^1([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$ and $\mathbf{X} \in \mathscr{C}_g^\gamma$. Then there exists a solution to (30) and the solution is given by $\mu_t := (\phi_t)_\sharp \mu_0$.*

Given the previous sections the reader will not be surprised that we can extend this to when the drift is discontinuous provided we choose the rough path to be the lift of a fractional Brownian motion with low Hurst index.

Lemma 5.3. *Assume $H < \frac{1}{2(2d+1)}$, $b \in L^\infty([0, T] \times \mathbb{R}^d; \mathbb{R}^d) \cap L^1(\mathbb{R}^d; L^\infty([0, T]; \mathbb{R}^d))$ and μ_0 a finite signed measure on $\mathbb{R}^d$. There exists a subset $\Omega^* \subset \Omega$ with full measure such that for every $\omega \in \Omega^*$ we have*

- *The fractional Brownian motion lifts to a geometric rough path $\mathbf{B}(\omega) \in \mathscr{C}_g^\gamma$, $\gamma < H$.*

- *There exists a solution $\mu_\cdot(\omega)$ to*

$$\mu_t(\omega) + \int_0^t div(b(r, \cdot)\mu_r(\omega))dr + \int_0^t div(\mu_r(\omega)d\mathbf{B}_r(\omega)) = \mu_0.$$

Proof. Denote by $\Omega_\mathbf{B}$ the set of $\omega \in \Omega$ such that $B(\omega)$ lifts to a rough path, $\mathbf{B}(\omega) \in \mathscr{C}_g^\gamma$.

Let $\eta \in C_c^\infty(\mathbb{R}^d)$. Consider the approximation from Section 4, i.e. we have $\Omega_{\gamma,\eta,\mu_0}$ such that $\lim_{n\to\infty} \mu_0(\nabla\eta(\phi_\cdot^n(\omega))) = \mu_0(\nabla\eta(\phi_\cdot(\omega)))$ in $C^\gamma([0, T]; \mathbb{R}^d)$. From Propositions 2.13 and 4.15 we get that

$$\int_0^\cdot \mu_0(\nabla\eta(\phi_r^n(\omega)))d\mathbf{B}_r(\omega) \to \int_0^\cdot \mu_0(\nabla\eta(\phi_r(\omega)))d\mathbf{B}_r(\omega)$$

on $\Omega_{\gamma,\eta,\mu_0} \cap \Omega_\mathbf{B}$.

For every n we have that $\mu_t^n := (\phi_t^n)_\sharp \mu_0$ satisfies

$$\mu_t^n(\eta) = \mu_0(\eta) + \int_0^t \mu_r^n(b_n(r, \cdot)\nabla\eta)dr + \int_0^t \mu_r^n(\nabla\eta)d\mathbf{B}_r$$

on $\Omega_\mathbf{B}$. Denote by Ω_{η,μ_0} the set of $\omega \in \Omega$ such that $\mu_t^n(\eta) \to \mu_t(\eta)$, so that we must have that all the above terms converges on $\Omega_{\eta,\mu_0} \cap \Omega_{\gamma,\eta,\mu_0} \cap \Omega_\mathbf{B}$, to

$$\mu_t(\eta) = \mu_0(\eta) + \int_0^t \mu_r(b(r, \cdot)\nabla\eta)dr + \int_0^t \mu_r(\nabla\eta)d\mathbf{B}_r.$$

Let now

$$\Omega^* := \Omega_\mathbf{B} \cap \bigcap_{k\geq1} \Omega_{\eta^k,\mu_0} \cap \Omega_{\gamma,\eta^k,\mu_0}$$

where $\{\eta^k\}_{k\geq1} \subset C_c^\infty(\mathbb{R}^d)$ is dense in $C_c^\infty(\mathbb{R}^d)$ equipped with the ususal test function topology. Then Ω^* is the desired set. $\square$

6 Transport Equation

In this section we want to study (5). Morally, the solution to this equation should be given by

$$u(t,x) = u_0(\phi_t^{-1}(x)) \exp\{-\int_0^t c(s, \phi_r(y))dr|_{y=\phi_t^{-1}(x)}\}.$$

When c is a distribution this expression does not make sense, but using Section 4.12 we can however *define* the solution to be

$$u(t,x) = u_0(\phi_t^{-1}(x)) \exp\{(-1)^{|\alpha|+1} \int_{\mathbb{R}^d} \Lambda_\alpha^{\phi(y),C}(t,z)dz|_{y=\phi_t^{-1}(x)}\}.$$

where $c = D^\alpha C$, i.e. the distributional derivative of a function C.

A question that arise in this setting is in what way does this function defined above satisfy (5). To answer this we should look for a spatially weak formulation of the equation, namely for every $\eta \in C_c^\infty(\mathbb{R}^d)$ the function should satisfy

$$\langle u(t), \eta \rangle + \int_0^t \langle b(r)\nabla u(r), \eta \rangle + \int_0^t \langle u(r)c(r), \eta \rangle + \int_0^t \langle \nabla u(r), \eta \rangle dB_r = \langle u_0, \eta \rangle.$$

In order to make sense of the stochastic integral term we need to guarantee that $\langle \nabla u(r), \eta \rangle$ is a path controlled by B as described in Section 2. Using integration by parts we get

$$\langle \nabla u(r), \eta \rangle = -\int_{\mathbb{R}^d} u_0(\phi_r^{-1}(x)) \exp\{(-1)^{|\alpha|+1} \int_{\mathbb{R}^d} \Lambda_\alpha^{\phi(y),C}(r,z)dz|_{y=\phi_r^{-1}(x)}\}\nabla \eta(x)dx$$

$$= -\int_{\mathbb{R}^d} u_0(y) \exp\{(-1)^{|\alpha|+1} \int_{\mathbb{R}^d} \Lambda_\alpha^{\phi(y),C}(r,z)dz\}\nabla \eta(\phi_r(y))|\nabla \phi_r(y)|dy$$

where we have used the change of variables $\phi_r(y) = x$. It is clear from Section 2.2 that $\nabla \eta(\phi_\cdot(y))$ can be regarded as a controlled path. However, the terms

$$\int_{\mathbb{R}^d} \Lambda_\alpha^{\phi(y),C}(\cdot,z)dz \qquad \text{and} \qquad |\nabla\phi_\cdot(y)| = \exp\{-\sum_{j=1}^d \int_{\mathbb{R}^d} \Lambda_{e_j}^{\phi(y),b_j}(\cdot,z)dz\}$$

are not expected to be more than $1 - H(2+d)$ regular in time (at least at the current level of knowledge) so we can not invoke Lemma 2.1 and it is not clear how to define the product as a controlled path. In fact this seems to require that also e.g. $|\nabla\phi_\cdot(y)|$ is controlled by B and we do not yet know how do this construction.

In its full generality we still cannot show that u defined as above solves the equation, but we provide some examples ($d = 1$, $\operatorname{div} b$ bounded, $c = \operatorname{div} b$ and time-homogenuous drift) where we can.

First, let us study the equation when the coefficients and the noise are regular.

6.1 Regular Case

Assume for a moment that the drift $b \in L^\infty([0,T]; C_b^1(\mathbb{R}^d))$ and we want to study the rough linear transport equation

$$\partial_t u + b\nabla u + cu + \nabla u d\mathbf{X}_t = 0 \tag{31}$$

with given initial condition $u|_{t=0} = u_0$. If we assume that $\mathbf{X}$ is the geometric lift of a smooth path $X \in C^1$, we may read (32) in a classical way:

$$\partial_t u(t,x) + b(t,x) \cdot \nabla u(t,x) + c(t,x)u(t,x) + \nabla u(t,x) \cdot \dot{X}_t = 0 \tag{32}$$

with initial condition $u(0,x) = u_0(x)$. To solve this equation, let us define

$$u(t,x) := u_0(\phi_t^{-1}(x)) \exp\left\{-\int_0^t c(r, \phi_r(y))dr|_{y=\phi_t^{-1}(x)}\right\}$$

where $\phi_t(x)$ is the solution to (16). Immediately, $u(t, \phi_t(x)) = u_0(x) \exp\{-\int_0^t c(r, \phi_r(x))dr\}$ and so

$$-c(t, \phi_t(x))u_0(x) \exp\left\{-\int_0^t c(r, \phi_r(x))dr\right\} = \frac{d}{dt}u(t, \phi_t(x))$$

$$= \partial_t u(t, \phi_t(x)) + \nabla u(t, \phi_t(x)) \cdot \dot{\phi}_t(x)$$

$$= \partial_t u(t, \phi_t(x)) + \nabla u(t, \phi_t(x)) \cdot b(t, \phi_t(x)) + \nabla u(t, \phi_t(x)) \cdot \dot{X}_t.$$

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Making a change of variables we see that $u(t,x)$ is indeed a solution of (32).

Integrating the above w.r.t. t and approximating a rough path $\mathbf{X}$ by smooth paths and taking the limit, it is reasonable that we should get

$$u(t,x) + \int_0^t b(r,x) \cdot \nabla u(r,x) dr + \int_0^t c(r,x) u(r,x) dr + \int_0^t \nabla u(r,x) d\mathbf{X}_r = u_0(x)$$

provided the solution is such that $\nabla u(\cdot, x)$ is controlled by $\mathbf{X}$. Unfortunately, to guarantee that $\nabla u(t,x)$ is a controlled path we need higher order differentiability of the solution than the regularization of the fractional noise can provide. To circumvent this we use a spatially weak notion of solution.

Definition 6.1. *Let $u_0, c : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ and $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be given locally integrable functions. Let $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be such that for all $t \in [0,T]$ we have $u(t,\cdot) \in W^{1,2}(U)$ for all open and bounded $U \subset \mathbb{R}^d$. We call u a weak controlled solution to (31) if for all $\eta \in C_c^\infty(\mathbb{R}^d)$ the path $\int_{\mathbb{R}^d} \nabla u(\cdot, x) \eta(x) dx$ is controlled by $\mathbf{X}$ and the following equality holds*

$$\int_{\mathbb{R}^d} u(t,x) \eta(x) dx + \int_0^t \int_{\mathbb{R}^d} \nabla u(r,x) \cdot b(r,x) \eta(x) dx dr + \int_0^t \int_{\mathbb{R}^d} u(r,x) c(r,x) \eta(x) dx dr \qquad (33)$$

$$+ \int_0^t \int_{\mathbb{R}^d} \nabla u(s,x) \eta(x) dx d\mathbf{X}_r = \int_{\mathbb{R}^d} u_0(x) \eta(x) dx.$$

Existence of such a solution when the drift is nice is relatively straightforward. The proof is a consequence of the discussion in Section 2.3 together with the above computations.

Lemma 6.2. *Assume $b, c \in L^\infty([0,T]; C_b^1(\mathbb{R}^d))$, and $\mathbf{X} \in \mathscr{C}_g^\gamma$. Then there exists a weak solution to (31).*

Proof. Consider a smooth approximation $\mathbf{X}^\epsilon$ of $\mathbf{X}$ and let

$$u^\epsilon(t,x) := u_0(\phi_t^{\epsilon,-1}(x)) \exp\left\{ -\int_0^t c(r, \phi_r^\epsilon(y)) dr|_{y = \phi_t^{\epsilon,-1}(x)} \right\}$$

so that u^ϵ satisfies

$$\int_{\mathbb{R}^d} u^\epsilon(t,x) \eta(x) dx + \int_0^t \int_{\mathbb{R}^d} \nabla u^\epsilon(r,x) \cdot b(r,x) \eta(x) dx dr + \int_0^t \int_{\mathbb{R}^d} c(r,x) u^\epsilon(r,x) \eta(x) dx dr$$

$$+ \int_0^t \int_{\mathbb{R}^d} \nabla u^\epsilon(r,x) \eta(x) dx \dot{X}_r^\epsilon dr = \int_{\mathbb{R}^d} u_0(x) \eta(x) dx.$$

Consider now $\int_{\mathbb{R}} \nabla u^\epsilon(r,x) \eta(x) dx$ as above. Using integration by parts we get

$$\int_{\mathbb{R}^d} \nabla u^\epsilon(r,x) \eta(x) dx = -\int_{\mathbb{R}^d} u_0(\phi_r^{\epsilon,-1}(x)) \exp\left\{ -\int_0^r c(r, \phi_s^\epsilon(y)) ds|_{y = \phi_r^{\epsilon,-1}(x)} \right\} \nabla \eta(x) dx$$

$$= -\int_{\mathbb{R}^d} u_0(y) \exp\left\{ -\int_0^r c(s, \phi_s^\epsilon(y)) ds \right\} |\nabla \phi_r^\epsilon(y)| \eta(\phi_r^\epsilon(y)) dy$$

where we have used a change of variable $y = \phi_r^{\epsilon,-1}(x)$ in the last equality. From Liouville's formula we get $|\nabla \phi_r^\epsilon(y)| = \exp\left\{ \int_0^r \operatorname{div} b(s, \phi_s^\epsilon(y)) ds \right\}$

From Section 2.2, if we can show that $\exp\{\int_0^r \operatorname{div} b(s, \phi_s^\epsilon(y)) - c(s, \phi_s^\epsilon(y)) ds\} \eta(\phi_r^\epsilon(y))$ converges in $\mathscr{D}_{\mathbf{X}}^{p\gamma}$ to $\exp\{\int_0^r \operatorname{div} b(s, \phi_s(y)) - c(s, \phi_s(y)) ds\} \eta(\phi_r(y))$, then it follows immediately that

$$\int_0^t \int_{\mathbb{R}^d} \nabla u^\epsilon(r,x) \eta(x) dx \dot{X}_r^\epsilon dr \to \int_0^t \int_{\mathbb{R}^d} \nabla u(r,x) \eta(x) dx d\mathbf{X}_r.$$

Towards this goal, we notice that from Lemma 2.1 it is enough to prove that $\int_0^{\cdot} \operatorname{div} b(s, \phi_s^\epsilon(y)) - c(s, \phi_s^\epsilon(y)) ds$ converges in C^β to $\int_0^{\cdot} \operatorname{div} b(s, \phi_s(y)) - c(s, \phi_s(y)) ds$. From Hölder's inequality we get

$$\left| \int_r^t \operatorname{div} b(s, \phi_s^\epsilon(y)) - c(s, \phi_s^\epsilon(y)) - \operatorname{div} b(s, \phi_s(y)) + c(s, \phi_s(y)) ds \right|$$

$$\leq |t - r|^\beta \| \operatorname{div} b(\phi^\epsilon(y)) - c(\phi^\epsilon(y)) - \operatorname{div} b(\phi(y)) + c(\phi(y)) \|_{L^{1/\beta}([0,T])}.$$

The result follows by dominated convergence and continuity (of c and $\operatorname{div} b$) as long as we choose $\beta = p\gamma < 1$.

Convergence of the remaining terms follows by similar considerations. $\qquad \square$

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6.2 Singular case

Motivated by the previous section we define our solution via the flow transformation.

Definition 6.3. *Let $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ be a given function and $c : [0, T] \to \mathcal{D}'(\mathbb{R}^d)$ be a distribution such that there exists functions $C_j : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ for $j = 1, \ldots, J$ and multiindices $\alpha_1, \ldots, \alpha_J$ satisfying $c(t) = \sum_{j=1}^J D^{\alpha_j} C_j(t)$ where D^{α_j} denotes spatial differentiation in the weak sense. We call the function*

$$u(t, x) = u_0(\phi_t^{-1}(x)) \exp \left\{ \sum_{j=1}^J (-1)^{|\alpha_j|+1} \int_{\mathbb{R}^d} \Lambda_{\alpha_j}^{\phi(y), C_j}(t, z) dz |_{y = \phi_t^{-1}(x)} \right\}.$$

a local time solution of (31) with initial condition u_0 provided all the terms exists as in Section 4.2

We go on to prove existence of such a solution for almost all sample paths of the fBm.

Theorem 6.4. *Assume we have*

- $b \in L^\infty([0, T] \times \mathbb{R}^d; \mathbb{R}^d) \cap L^1(\mathbb{R}^d; L^\infty([0, T]; \mathbb{R}^d))$

- *There exists smooth functions C_j^k such that $\int_{\mathbb{R}^d} \sup_{t \in [0,T]} |C_j(t, y) - C_j^k(t, y)| dy \to 0$ as $k \to \infty$ for all j,*

- u_0 *is continuous,*

- $|\alpha_j| \leq 1$

- B *is a fBm with Hurst parameter $H < \frac{1}{d+2}$.*

Then there exists a set of full measure, Ω_0 such that for every $\omega \in \Omega_0$ there exists a local time solution of (31).

Proof. The proof is done by approximation of b and then c as in the above assumptions. For notational simplicity we assume $J = 1$. Let $\Omega_{\gamma, \delta_x}$ be as in Proposition 4.15 where δ_x is the Dirac centered at x, so that we have $u_0(\phi_n(t, x, \omega)^{-1}) \to u_0(\phi(t, x, \omega)^{-1})$. For a fixed k we have

$$\lim_{n \to \infty} (-1)^{|\alpha|} \int_{\mathbb{R}^d} \Lambda_\alpha^{C^k, \phi_n(x)}(t, y) dy = \lim_{n \to \infty} \int_0^t c^k(s, \phi_n(s, x)) ds$$

$$= \int_0^t c^k(s, \phi_s(x)) ds = (-1)^{|\alpha|} \int_{\mathbb{R}^d} \Lambda_\alpha^{C^k, \phi(x)}(t, y) dy$$

on a set Ω_k of full measure. Finally, we notice that

$$E\left[\left(\int_{\mathbb{R}^d} \Lambda_\alpha^{C^k, \phi(x)}(t, y) dy - \int_{\mathbb{R}^d} \Lambda_\alpha^{C, \phi(x)}(t, y) dy \right)^m \right] = E\left[\left(\int_{\mathbb{R}^d} \Lambda_\alpha^{C^k - C, \phi(x)}(t, y) dy \right)^m \right]$$

$$\leq C_m \int_{\mathbb{R}^d} \|C^k(\cdot, y) - C(\cdot, y)\|_\infty dy \to 0$$

by assumption, and thus there exists a subsequence and a set of full measure, $\tilde{\Omega}$ such that we have $\lim_{k \to \infty} \int_{\mathbb{R}^d} \Lambda_\alpha^{C^k, \phi(x)}(t, y) dy = \int_{\mathbb{R}^d} \Lambda_\alpha^{C, \phi(x)}(t, y) dy$ on $\tilde{\Omega}$. The result follows when we choose $\Omega_0 = \Omega_{\gamma, \delta_x} \cap \tilde{\Omega} \cap \cap_{k \geq 1} \Omega_k$. □

Example 6.5 (The continuity equation revisited). *Let $c = \operatorname{div} b = \sum_{j=1}^d \frac{\partial b_j}{\partial x_j}$ where b is as before. We get from Theorem 6.4 that the solution to*

$$\int_{\mathbb{R}^d} u(t, x) \eta(x) dx + \int_0^t \int_{\mathbb{R}^d} \nabla u(r, x) b(r, x) \eta(x) dx dr + \int_0^t \int_{\mathbb{R}^d} \operatorname{div} b(r, x) u(r, x) \eta(x) dx dr$$

$$+ \int_0^t \int_{\mathbb{R}^d} \nabla u(s, x) \eta(x) dx d\mathbf{B}_r = \int_{\mathbb{R}^d} u_0(x) \eta(x) dx.$$

is given by (actually by definition of the solution)

$$u(t,x) = u_0(\phi_t^{-1}(x)) \exp\left\{\sum_{j=1}^{d} \int_{\mathbb{R}^d} \Lambda_{e_j}^{\phi(y),b_j}(t,z)dz|_{y=\phi_t^{-1}(x)}\right\}.$$

Rewriting the above equation

$$\int_{\mathbb{R}^d} u(t,x)\eta(x)dx + \int_0^t \int_{\mathbb{R}^d} \text{div}(u(r,x)b(r,x))\eta(x)dxdr$$
$$+ \int_0^t \int_{\mathbb{R}^d} \nabla u(s,x)\eta(x)dxd\mathbf{B}_r = \int_{\mathbb{R}^d} u_0(x)\eta(x)dx$$

this should give us the same solution as the continuity equation if $\frac{d\mu_0}{dx} = u_0(x)$, i.e. u_0 is the Radon-Nikodym of the measure μ_0 w.r.t. Lebesgue measure.

To see that this is indeed the case we consider again the approximation from Section 4. The solution of the continuity equation $\mu_t^n = (\phi_n(t,\cdot))_\sharp \mu_0$ so that for any $\eta \in C_c^\infty(\mathbb{R}^d)$ we have

$$\mu_t^n(\eta) = \int_{\mathbb{R}^d} \eta(\phi_n(t,y))u_0(y)dy = \int_{R^d} \eta(x)u_0(\phi_n^{-1}(t,x)) \exp\left\{-\int_0^t \text{div}(b_n(r,\phi_n(r,y)))dr|_{y=\phi_n^{-1}(t,x)}\right\}dx$$

where we have used the change of variable $y = \phi_n^{-1}(t,x)$. As in the proof of Theorem 6.4 we can let $n \to \infty$ and find a set of full measure for which

$$\mu_t(\eta) = \int_{R^d} \eta(x)u_0(\phi_t^{-1}(x)) \exp\left\{\sum_{j=1}^{d} \int_{\mathbb{R}^d} \Lambda_{e_j}^{\phi(y),b_j}(t,z)dz|_{y=\phi_t^{-1}(x)}\right\}dx = \int_{R^d} \eta(\phi_t(y))u_0(y)dy$$

since $|\nabla\phi_t(y)| = \exp\left\{\sum_{j=1}^{d} \int_{\mathbb{R}^d} \Lambda_{e_j}^{\phi(y),b_j}(t,z)dz\right\}$. The latter expression is obviously a controlled path. We conclude that our definitions 6.1 and 6.3 coincides in this case, justifying definition 6.3 as more than just a limit object, but something that actually satisfies the equation in a reasonable sense.

6.3 Local time solutions that are weak controlled solutions

In this section we look at examples of b and c for which we can show that the local time solutions in Definition 6.3 are also solutions in the sense of Definition 6.1.

We recall that we have to make sense of

$$\langle u(t),\eta\rangle = -\int_{\mathbb{R}^d} u_0(y)\eta(\phi_t(y))|\nabla\phi_t(y)| \exp\left\{\sum_{j=1}^{J}(-1)^{|\alpha_j|+1} \int_{\mathbb{R}^d} \Lambda_{\alpha_j}^{\phi(y),C_j}(t,z)dz\right\}dy$$

as a controlled rough path.

When c is a bounded function, the exponential term does not pose any problems: writing

$$\sum_{j=1}^{J}(-1)^{|\alpha_j|+1} \int_{\mathbb{R}^d} \Lambda_{\alpha_j}^{\phi(y),C_j}(t,z)dz = \int_0^t c(r,\phi_r(y))dr$$

shows that this term Lipschitz in t, and so using Lemma 2.1 it is clear that this term can always be considered a controlled path. For this reason, we shall for the rest of this section assume for simplicity that $c = 0$. The extension to bounded c is straightforward.

6.3.1 One spatial dimension

Consider the approximation $\phi_n(t,x)$ from section 4.1.3, i.e. we have $\phi_n(t,x) \to \phi_t(x)$ in $L^2(\Omega)$ and $\phi_n(t,\cdot) \to \phi_t$ weakly in $L^2(\Omega; W^{1,2}(U))$ (for simplicity we omit the subsequence).

Theorem 6.6. *Assume $b \in L^\infty([0,T] \times \mathbb{R}) \cap L^1(\mathbb{R}^d; L^\infty([0,T]))$, $u_0 \in C^1(\mathbb{R})$ such that $u_0' \in L^1(\mathbb{R})$ and $H < \frac{1}{6}$. There exists a subset with full measure $\Omega^* \subset \Omega$, such that there exists a weak controlled solution to (31) w.r.t. every $B.(\omega)$ for $\omega \in \Omega^*$.*

Proof. Define $u(t,x) := u_0(\phi_t^{-1}(x))$ and fix Ω_γ as in Proposition 4.15. We begin by showing that $\int_{\mathbb{R}} u(\cdot, x, \omega)\eta'(x)dx$ is controlled by $\mathbf{B}.(\omega)$ for every $\omega \in \Omega_\gamma$. It is enough to show

$$\int_{\mathbb{R}} u(\cdot, x, \omega)\eta'(x)dx = -\int_{\mathbb{R}} u_0'(y)\eta(\phi.(y, \omega))dy.$$

To this end, note that for every n we have

$$\int_{\mathbb{R}} u_n(t, x, \omega)\eta'(x)dx = -\int_{\mathbb{R}} \nabla u_n(t, x, \omega)\eta(x)dx = -\int_{R} u_0'(\phi_n^{-1}(t, x, \omega))\nabla\phi_n^{-1}(t, x, \omega)\eta(x)dx$$

$$= -\int_{\mathbb{R}} u_0'(y)\eta(\phi_n(t, y, \omega))dy.$$

where we have used the change of variables $y = \phi_n^{-1}(t, x, \omega)$. Letting $n \to \infty$ we get $\eta(\phi_n(t, x, \omega)) \to \eta(\phi_t(x, \omega))$ and $u_n(t, x, \omega) \to u(t, x, \omega)$ from Remark 4.10. The desired equality holds from dominated convergence.

For every n we have

$$\int_{\mathbb{R}} u_n(t, x, \omega)\eta(x)dx + \int_0^t \int_{\mathbb{R}} \nabla u_n(r, x, \omega)b_n(r, x)\eta(x)dxdr - \int_0^t \int_{\mathbb{R}} u_n(t, x)\eta(\phi_n(r, x, \omega))dxd\mathbf{B}_r(\omega)$$

$$= \int_{\mathbb{R}} u_0(x)\eta(x)dx$$

where $\mathbf{B}_{st}(\omega)$ is the geometric lift of the fractional Brownian motion. We see that

$$\lim_{n\to\infty} \int_{\mathbb{R}} u_n(t, x, \omega)\eta(x)dx = \int_{\mathbb{R}} u(t, x, \omega)\eta(x)dx$$

and

$$\lim_{n\to\infty} \int_0^t \int_{\mathbb{R}} u_0'(x)\eta(\phi_n(r, x, \omega))dxd\mathbf{B}_r(\omega) = \int_0^t \int_{\mathbb{R}} u_0'(x)\eta(\phi_r(x, \omega))dxd\mathbf{B}_r(\omega).$$

where we have used Proposition 2.13 since $u_0' \in L^1(\mathbb{R})$. Consequently, we must have that

$$\int_0^t \int_{\mathbb{R}} \nabla u_n(r, x, \omega)b_n(r, x)\eta(x)dxdr$$

is converging. Now we get

$$\int_{\mathbb{R}} \nabla u_n(r, x, \omega)b_n(r, x)\eta(x)dx = \int_{\mathbb{R}} \nabla u_n(r, x, \omega)(b_n(r, x) - b(r, x))\eta(x)dx + \int_{\mathbb{R}} \nabla u_n(r, x, \omega)b(r, x)\eta(x)dx.$$

For the first term, from Remark 4.10 we see that

$$\int_{\mathbb{R}} \nabla u_n(r, x, \omega)b(r, x)\eta(x)dx \to \int_{\mathbb{R}} \nabla u(r, x, \omega)b(r, x)\eta(x)dx,$$

for all $r \in [0, T]$.

For the second term we take the expectation

$$E[|\int_{\mathbb{R}} \nabla u_n(r, x)(b_n(r, x) - b(r, x))\eta(x)dx|] \leq \int_{\mathbb{R}} E[|\nabla u_n(r, x)|]|b_n(r, x) - b(r, x)||\eta(x)|dx$$

$$\leq \sup_{y \in \text{supp}\,\eta} \left(E[|\nabla u_n(r, y)|^2]\right)^{1/2} \int_{\mathbb{R}} |b_n(r, x) - b(r, x)||\eta(x)|dx$$

$$\to 0.$$

which shows that there exists a subsequence

$$\int_0^t \int_{\mathbb{R}} \nabla u_{n_k}(r, x)b_{n_k}(r, x)\eta(x)dxdr \to \int_0^t \int_{\mathbb{R}} \nabla u(r, x)b(r, x)\eta(x)dxdr$$

as $k \to \infty$ on some set Ω_1 which has full measure. The statement holds on $\Omega^* = \Omega_\gamma \cap \Omega_1$. $\qquad\square$

6.3.2 Divergence of b bounded

When the divergence of b is bounded, we can write

$$|\nabla\phi_t(x)| = \exp\left\{\int_0^t \operatorname{div} b(r, \phi_r(x))dr\right\},$$

and so the mapping $t \mapsto |\nabla\phi_t(x)|$ is of bounded variation. Using Lemma 2.1 we can show the following result.

Theorem 6.7. *Assume $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies the assumptions of Proposition 4.1 and has bounded divergence. Assume moreover that $u_0 : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is a bounded function. Then, if u is a local time solution it is also a weak controlled solution.*

Proof. We need to show that $u(t,x) = u_0(\phi_t^{-1}(x))$ is a weak controlled solution. Using Lemma 2.1 and Lemma 2.6 it is clear that

$$\langle u(t), \nabla\eta\rangle = \int_{\mathbb{R}^d} u_0(y)\nabla\eta(\phi_t(y))\exp\left\{\int_0^t \operatorname{div} b(r, \phi_r(y))dr\right\}dy$$

is controlled by $\mathbf{B}$. The proof follows the same lines as the proof of 6.6, using Proposition 4.1 to obtain strong $L^2(\Omega)$ convergence of $u_n(t,x)$ locally in x. $\qquad\square$

6.3.3 Time-homogenuous drift and smooth initial data

When b is time-homogenuous, we can write (see Remark 4.10)

$$\phi_t^{-1}(x) = x - \int_0^t b(\phi_r^{-1}(x))dr - B_t.$$

If now the initial condition is sufficiently regular, it is clear from Lemma 2.6 that $u_0(\phi_\cdot^{-1}(x))$ is controlled by $\mathbf{B}$.

Theorem 6.8. *Assume $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies the assumptions of Proposition 4.1 and assume that $u_0 \in C_b^k(\mathbb{R}^d; \mathbb{R})$ for some $k \geq \lfloor\frac{1}{H}\rfloor$. Then, if u is a local time solution it is also a weak controlled solution.*

Proof. Since $\mathscr{D}_{-\mathbf{B}}^{p(H-)} = \mathscr{D}_{\mathbf{B}}^{p(H-)}$, it is clear that

$$\langle u(t), \nabla\eta\rangle = \int u_0(\phi_t^{-1}(x))\nabla\eta(x)dx$$

is controlled by $\mathbf{B}$. The proof follows the same lines as the proof of 6.6, using Proposition 4.1 to obtain strong $L^2(\Omega)$ convergence of $u_n(t,x)$ locally in x. $\qquad\square$

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