Adaptive Backstepping Control of a 2-DOF Helicopter System in the Presence of Quantization

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Abstract

This paper studies the attitude tracking control for an uncertain 2-degrees of freedom helicopter system where the inputs and the states are quantized. An adaptive backstepping based control scheme is proposed to handle the effect of quantization for tracking of reference angles for pitch and yaw. All closed-loop signals are ensured uniformly bounded and the tracking errors will converge to a compact set containing the origin. Experiments on the helicopter system illustrate the proposed control scheme.

D.1 Introduction

The interest for wireless communication, remote controlled systems and other network control systems (NCSs) where the control loops are closed through a communication network has increased recent years. The network bandwidth might be limited and signals are required to be quantized before transmitted over the network. Then it is important to choose a quantization scheme that can reduce the communication burden over the network, and at the same time ensure sufficient precision for the system. Quantization introduces nonlinear errors in the control loop that may lead to degradation of system performance or even unstable control systems.

Various results have been reported for quantized feedback control systems with input quantization, see e.g [1–4], where only the information from controller to the plant is quantized, while the controller is designed by continuous measures of the state feedback. The feedback control problem of systems with state quantization has been studied in [5–8], where the system dynamics in these works are precisely known.

Uncertainties often appears in systems, and adaptive control is a control method that can be used to handle such uncertainties. Adaptive control schemes were developed in [2, 9, 10] for uncertain systems with input quantization. Adaptive backstepping technique was proposed in the 1990's in [11] to deal with plant nonlinearity and parameter uncertainties. The backstepping technique has several advantages over the conventional approaches such as providing a promising way to improve the transient performance of adaptive systems by tuning design parameters. Several results have been reported for adaptive backstepping control for systems with input quantization, e.g. in [12, 13] for uncertain nonlinear systems, in [14] for a 2-degrees of freedom (DOF) helicopter system, in [4] for tracking control for under-actuated autonomous underwater vehicles and in [15] for formation tracking control for a group of UAVs. Adaptive backstepping-based stabilization of uncertain systems with state quantization are very limited, since the backstepping technique requires differentiating the quantized states that are discontinuous. This problem was solved in [16] where the states were quantized by a static bounded quantizer for uncertain nonlinear systems. The solution in [16] to handle the discontinuous states was considered in [17] for attitude control of a rigid body.

Both inputs and states are in practice quantized due to actuator and sensor limitations, but there are only a few results handling both input and state quantization. In [18], trajectory tracking control for autonomous underwater vehicles with the effect of quantization was investigated using a sliding mode controller. In [19], adaptive attitude control for a rigid body with input and output quantization was studied. In [20], adaptive tracking control for nonholonomic mobile robots with input and state quantization was considered. In [21], an adaptive neural network controller was developed for a 2-DOF helicopter system with saturated input and quantized input and state.

In this paper we extend the results from [22] and [14], where the adaptive backstepping control of a 2-DOF helicopter was considered in [22] and with input quantization in [14], to now deal with both input and state quantization for the same helicopter system. The helicopter is a nonlinear multiple-input and multiple-output (MIMO) system, with challenges in controller design due to its nonlinear behavior, its cross coupling effect between inputs and outputs, and with uncertainties both in the model and the parameters. Based on Lyapunov stability theory, the stability of the helicopter system is analyzed, were the tracking errors are shown to converge to an ultimate bound. Experiments on the helicopter system illustrate the proposed control scheme.

The main contributions in this paper are summarized as follows.

- Compared to [14] where the problem of input quantization was considered, this paper studies the problem where both the inputs and the states are quantized. The main challenge is that the designed controller and virtual control can only utilize quantized states, and this problem is being addressed.

- The attitude, i.e. orientation, of a MIMO 2-DOF helicopter system is to be controlled, where the system has challenges due to uncertain parameters, there is a coupling between the inputs and the outputs that makes control more complicated, and quantization of both the inputs and the states introduce errors that need to be handled in the control design and in the stability analysis. We propose an adaptive control algorithm using the backstepping technique to deal with these problems.

The paper is organized as follows. In Section D.2, the system model, problem statement and the considered quantizer are presented. Section D.3 presents the adaptive control design based on backstepping technique. In Section D.4 a stability analysis is given, Section D.5 presents the results from experiment before a conclusion is given in Section D.6.

D.2 Dynamical Model and Problem Formulation

D.2.1 Notations

Vectors are denoted by small bold letters and matrices with capitalized bold letters. $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ denotes the maximum and minimum eigenvalue of the matrix $(\cdot)$, and $\|\cdot\|$ denotes the $\mathcal{L}_2$-norm and induced $\mathcal{L}_2$-norm for vectors and matrices, respectively.

D.2.2 System Model

The considered helicopter system is visualized in Fig. D.1 showing the body fixed coordinate frame. This is a two-rotor laboratory equipment for flight control-based experiments. The setup is a horizontal position of the main thruster and a vertical position of the tail thruster, which resembles a helicopter with two propellers driven by two DC motors. The main motor is producing a force in the z_b-direction that will give a positive pitch angle, and at the same time the rotation of the propeller generates a torque about the motor shaft causing a motion in the y_b-direction, meaning this will give a yaw angle. The tail motor is producing a force in the y_b direction and at the same time a torque changing the pitch angle. Thus, this is a MIMO system with 2 DOF, where each input will change both the pitch and the yaw angle. The helicopter model is considered as a rigid body and the equations of motion are derived using Euler-Lagrange equations as given in [22], where the system parameters are uncertain.

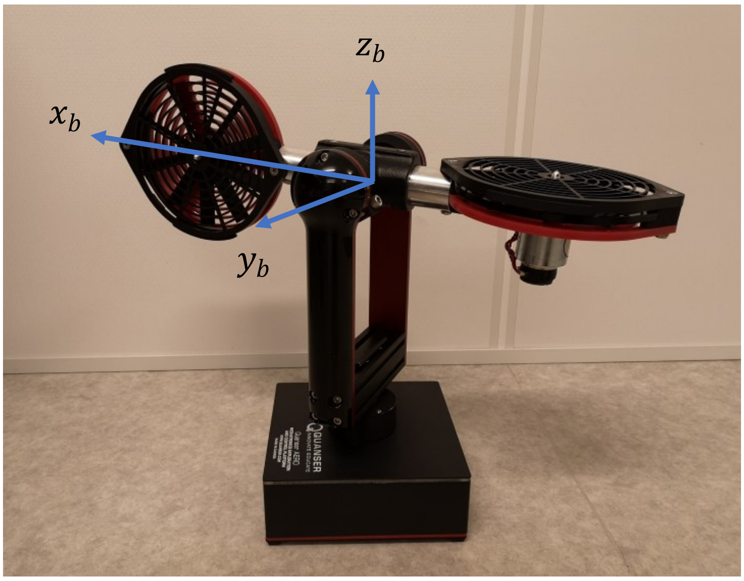

Figure D.1: Quanser Aero helicopter system with body coordinate frame

The state variables are defined as

$$\boldsymbol{x}_1 = [\vartheta(t)\ \psi(t)]^\top \in \mathbb{R}^2, \quad \boldsymbol{x}_2 = [\dot{\vartheta}(t), \dot{\psi}(t)]^\top \in \mathbb{R}^2, \tag{D.1}$$

where ϑ and ψ are pitch and yaw angles, and $\dot{\vartheta}$ and $\dot{\psi}$ are angular velocities of pitch and yaw. The nonlinear state space model is expressed as

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{x}_2 \\ \boldsymbol{\Phi}_1^\top \boldsymbol{\theta}_1 + \boldsymbol{\Phi}_2^\top \boldsymbol{\theta}_2 + \boldsymbol{K}\boldsymbol{u}^q \end{bmatrix} \in \mathbb{R}^4, \tag{D.2}$$

where

$$\boldsymbol{\Phi}_1 = \begin{bmatrix} -x_{2,1} & 0 \\ -\sin x_{1,1} & 0 \\ x_{2,2}^2 \cos x_{1,1} \sin x_{1,1} & 0 \end{bmatrix} \in \mathbb{R}^{3\times 2}, \tag{D.3}$$

$$\boldsymbol{\Phi}_2 = \begin{bmatrix} 0 & -x_{2,2} \\ 0 & -x_{1,2}x_{2,2} \cos x_{1,1} \sin x_{1,1} \end{bmatrix} \in \mathbb{R}^{2\times 2}, \tag{D.4}$$

are known nonlinear functions,

$$\boldsymbol{\theta}_1 = \frac{1}{I_p} \begin{bmatrix} d_p \\ mgr \\ mr^2 \end{bmatrix} \in \mathbb{R}^3, \quad \boldsymbol{\theta}_2 = \frac{1}{I_y} \begin{bmatrix} d_y \\ 2mr^2 \end{bmatrix} \in \mathbb{R}^2, \tag{D.5}$$

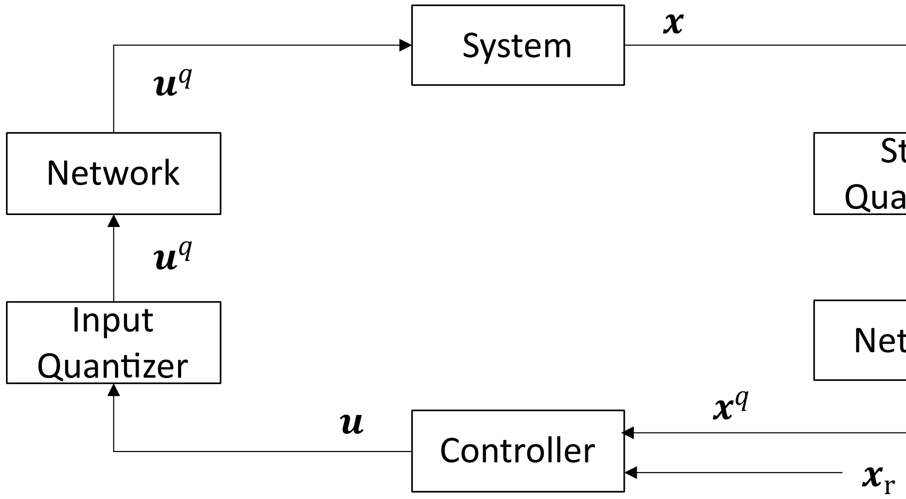

Figure D.2: Control system with input and state quantization over a network.

are unknown constant vectors,

$$\boldsymbol{K} = \begin{bmatrix} \frac{k_1}{I_p} & \frac{k_2}{I_p} \\ \frac{-k_3}{I_y} & \frac{k_4}{I_y} \end{bmatrix} \in \mathbb{R}^{2\times 2}, \tag{D.6}$$

is the control allocation matrix. The constants k_1 and k_4 are torque thrust gains from the main and the tail motors, k_2 is a cross-torque thrust gain acting on pitch from the tail motor, k_3 is a cross-torque thrust gain acting on yaw from the main motor, r is the distance between the center of mass and the origin of the body-fixed frame, I_p and I_y are the moments of inertia of pitch and yaw respectively, g is the gravity acceleration, m is the total mass of the Aero body, and d_y and d_p are damping constants.

D.2.3 Problem Statement

We consider a control system as shown in Fig. D.2, where the state vector $\boldsymbol{x}$ and the input vector $\boldsymbol{u}$ are quantized at the encoder side to be sent over a network. The network is assumed noiseless, so that the quantized state signal $\boldsymbol{x}^q$ is recovered and sent to the controller and the quantized input signal $\boldsymbol{u}^q$ is recovered and sent to the plant.
The quantizers for the state and control input are modeled as follows.

$$\boldsymbol{x}^q = Q_1(\boldsymbol{x}), \tag{D.7}$$

$$\boldsymbol{u}^q = Q_2(\boldsymbol{u}), \tag{D.8}$$

where the control input $\boldsymbol{u}$ can only use the quantized state as follows:

$$\boldsymbol{u} = [u_1(t, \boldsymbol{x}^q), u_2(t, \boldsymbol{x}^q)]^\top \in \mathbb{R}^2. \tag{D.9}$$

Given reference signal $\boldsymbol{x}_r(t)$, the control objective is to design a control law for $\boldsymbol{u} = \boldsymbol{u}(t, \boldsymbol{x}^q)$ by utilizing only quantized state $\boldsymbol{x}^q(t)$, to force the state $\boldsymbol{x}_1(t)$ to track the reference signal $\boldsymbol{x}_r(t)$ when the inputs are quantized, and to ensure that all the signals in the closed-loop system are uniformly bounded. To achieve the objective, the following assumptions are imposed.

Assumption 1. *The reference signal $\boldsymbol{x}_r$ and first and second order derivatives are known, piecewise continuous and bounded. Then there exists $k_{x_r}, k_{\dot{x}_r}, k_{\ddot{x}_r} > 0$ such that $\|\boldsymbol{x}_r\| < k_{x_r}$, $\|\dot{\boldsymbol{x}}_r\| < k_{\dot{x}_r}$ and $\|\ddot{\boldsymbol{x}}_r\| < k_{\ddot{x}_r}, \forall t \geq t_0$.*

Assumption 2. *The unknown parameter vectors $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ are bounded by $\|\boldsymbol{\theta}_1\| \leq k_{\theta_1}$, $\|\boldsymbol{\theta}_2\| \leq k_{\theta_2}$ where $k_{\theta_1}, k_{\theta_2}$ are positive constants. Also $\boldsymbol{\theta}_1 \in C_{\theta_1}$, $\boldsymbol{\theta}_2 \in C_{\theta_2}$ where C_{θ_1} and C_{θ_2} are known compact convex sets.*

Assumption 3. *The functions $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$ satisfy locally Lipschitz conditions such that*

$$\|\boldsymbol{\Phi}_1(t, \boldsymbol{y}_1) - \boldsymbol{\Phi}_1(t, \boldsymbol{y}_2)\| \leq L_{\Phi_1}\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|, \tag{D.10}$$

$$\|\boldsymbol{\Phi}_2(t, \boldsymbol{y}_1) - \boldsymbol{\Phi}_2(t, \boldsymbol{y}_2)\| \leq L_{\Phi_2}\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|, \tag{D.11}$$

where L_{Φ_1} and L_{Φ_2} are constants and $\boldsymbol{y}_1, \boldsymbol{y}_2$ are real vectors.

D.2.4 Quantizer

In this paper, a uniform quantizer is considered for both state quantization $Q_1(\boldsymbol{x})$ and input quantization $Q_2(\boldsymbol{u})$, which has intervals of fixed length and is defined as follows:

$$y^q = Q(y) = \begin{cases} y_i \operatorname{sgn}(y), & y_i - \frac{l}{2} < |y| \leq y_i + \frac{l}{2} \\ 0, & |y| \leq y_0 \end{cases}, \tag{D.12}$$

where $y_0 > 0, y_1 = y_0 + \frac{l}{2}$, $y_{i+1} = y_i + l$, $l > 0$ is the length of the quantization interval, $\operatorname{sgn}(y)$ is the sign function. The uniform quantization $y^q \in U = \{0, \pm y_i\}$, and a map of the quantization for $y_i > 0$ is shown in Fig. D.3. The quantizer considered in this paper has the following property

$$|y^q - y| = |d_y| \leq \delta_y, \tag{D.13}$$

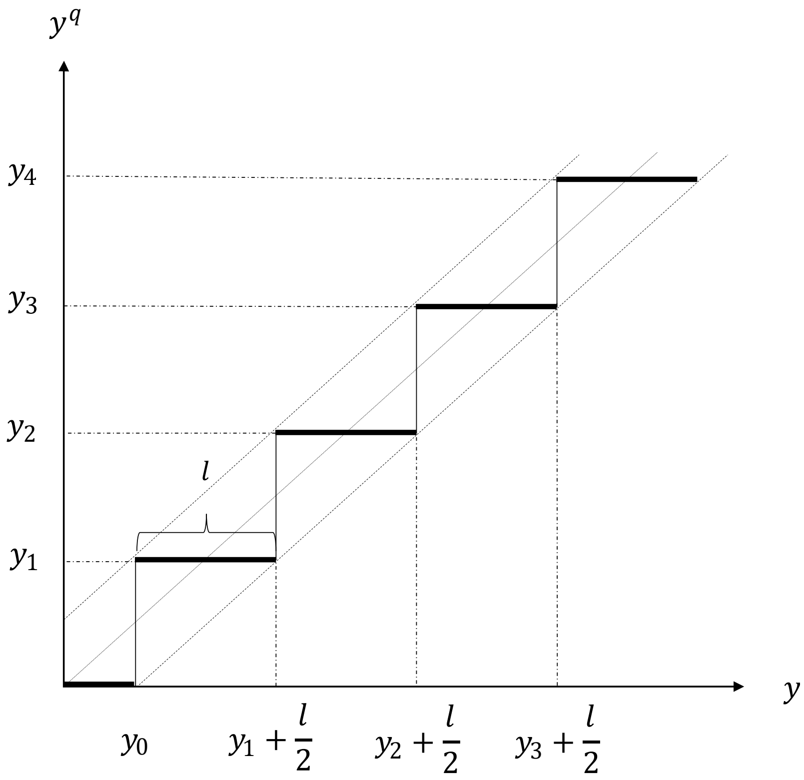

Figure D.3: Map of the uniform quantizer for $y > 0$.

where y is a scalar signal, d is the quantization error and $\delta_y > 0$ denotes the quantization bound. Clearly, the property in (D.13) is satisfied with $\delta_y = \max\{y_0, \frac{l}{2}\}$. When a vector is quantized, we have

$$\boldsymbol{y}^q = \begin{bmatrix} y_1^q & y_2^q & \cdots & y_n^q \end{bmatrix}^\top, \tag{D.14}$$

and so each vector element is bounded by (D.13), and we have $\|\boldsymbol{y}^q - \boldsymbol{y}\| = \|\boldsymbol{d}_y\| \leq \|\boldsymbol{\delta}_y\| \triangleq \delta_y$.

D.3 Adaptive Control Design

In this section we will design adaptive feedback control laws for the helicopter system using backstepping technique. For this model, two steps are included, where the control signal is designed in the last step. We first introduce the change of coordinates

$$\boldsymbol{z}_1 = \boldsymbol{x}_1 - \boldsymbol{x}_r, \tag{D.15}$$

$$\boldsymbol{z}_2 = \boldsymbol{x}_2 - \boldsymbol{\alpha} - \dot{\boldsymbol{x}}_r, \tag{D.16}$$

where $\boldsymbol{\alpha}$ is a virtual controller designed in the first step and chosen as

$$\boldsymbol{\alpha} = -\boldsymbol{C}_1\boldsymbol{z}_1, \tag{D.17}$$

where $\boldsymbol{C}_1 \in \mathbb{R}^{2\times 2}$ is a positive definite matrix. The derivative of (D.15) and (D.16) are given as

$$\dot{\boldsymbol{z}}_1 = \boldsymbol{x}_2 - \dot{\boldsymbol{x}}_r = \boldsymbol{z}_2 + \boldsymbol{\alpha}, \tag{D.18}$$

$$\dot{\boldsymbol{z}}_2 = \boldsymbol{\Phi}_1^\top \boldsymbol{\theta}_1 + \boldsymbol{\Phi}_2^\top \boldsymbol{\theta}_2 + \boldsymbol{K}\boldsymbol{u}^q - \dot{\boldsymbol{\alpha}} - \ddot{\boldsymbol{x}}_r. \tag{D.19}$$

To propose a suitable control scheme, the quantized input $\boldsymbol{u}^q(t)$ is decomposed into two parts

$$\boldsymbol{u}^q(t) = \boldsymbol{u}(t) + \boldsymbol{d}_u(t), \tag{D.20}$$

where $\boldsymbol{d}_u$ is the quantization error of the input, which is bounded by $\|\boldsymbol{d}_u\| \leq \|[\delta_{u_1}\ \delta_{u_2}]^\top\| = \|\boldsymbol{\delta}_u\| \triangleq \delta_u$, from (D.13).
The adaptive controller is designed as

$$\boldsymbol{u}(t) = \boldsymbol{K}^{-1}\Big[- \boldsymbol{z}_1^q - \boldsymbol{C}_2\boldsymbol{z}_2^q - \boldsymbol{\Phi}_1(\boldsymbol{x}^q)^\top \hat{\boldsymbol{\theta}}_1 - \boldsymbol{\Phi}_2(\boldsymbol{x}^q)^\top \hat{\boldsymbol{\theta}}_2 + \bar{\boldsymbol{\alpha}}^q + \ddot{\boldsymbol{x}}_r\Big], \tag{D.21}$$

$$\dot{\hat{\boldsymbol{\theta}}}_1 = \mathrm{Proj}\{\boldsymbol{\Gamma}_1\boldsymbol{\Phi}_1(\boldsymbol{x}^q)\boldsymbol{z}_2^q\}, \tag{D.22}$$

$$\dot{\hat{\boldsymbol{\theta}}}_2 = \mathrm{Proj}\{\boldsymbol{\Gamma}_2\boldsymbol{\Phi}_2(\boldsymbol{x}^q)\boldsymbol{z}_2^q\}, \tag{D.23}$$

where $\boldsymbol{C}_2, \boldsymbol{\Gamma}_2 \in \mathbb{R}^{2\times 2}$ and $\boldsymbol{\Gamma}_1 \in \mathbb{R}^{3\times 3}$ are positive definite gain matrices, $\hat{\boldsymbol{\theta}}$ is the estimated value of $\boldsymbol{\theta}$, the vector $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta} - \hat{\boldsymbol{\theta}}$, and where $\mathrm{Proj}\{\cdot\}$ is the projection operator given in [11] and where

$$\boldsymbol{z}_1^q = \boldsymbol{x}_1^q - \boldsymbol{x}_r, \tag{D.24}$$

$$\boldsymbol{z}_2^q = \boldsymbol{x}_2^q - \boldsymbol{\alpha}^q - \dot{\boldsymbol{x}}_r, \tag{D.25}$$

$$\boldsymbol{\alpha}^q = -\boldsymbol{C}_1\boldsymbol{z}_1^q, \tag{D.26}$$

$$\boldsymbol{\Phi}_1(\boldsymbol{x}^q) = \begin{bmatrix} -x_{2,1}^q & 0 \\ -\sin x_{1,1}^q & 0 \\ (x_{2,2}^q)^2 \cos x_{1,1}^q \sin x_{1,1}^q & 0 \end{bmatrix} \tag{D.27}$$

$$\boldsymbol{\Phi}_2(\boldsymbol{x}^q) = \begin{bmatrix} 0 & -x_{2,2}^q \\ 0 & -x_{1,2}^q x_{2,2}^q \cos x_{1,1}^q \sin x_{1,1}^q \end{bmatrix} \tag{D.28}$$

$$\bar{\boldsymbol{\alpha}}^q \triangleq -\boldsymbol{C}_1(\boldsymbol{x}_2^q - \dot{\boldsymbol{x}}_r). \tag{D.29}$$

Remark 1. *The projection operator* $\mathrm{Proj}\{\cdot\}$ *in* (D.22) *and* (D.23) *ensures that the estimates and estimation errors are nonzero and within known bounds, that is* $\|\hat{\boldsymbol{\theta}}\| \leq k_\theta$ *and* $\|\tilde{\boldsymbol{\theta}}\| \leq k_\theta$*, and has the property* $-\tilde{\boldsymbol{\theta}}^\top \boldsymbol{\Gamma}^{-1}\mathrm{Proj}(\boldsymbol{\tau}) \leq -\tilde{\boldsymbol{\theta}}^\top \boldsymbol{\Gamma}^{-1}\boldsymbol{\tau}$*, which are helpful to guarantee the closed-loop stability.*

Remark 2. *Only the quantized state can be used in the designed controller. Since* *the quantized state is used in the design of the virtual controller* $\boldsymbol{\alpha}^q$ *in* (D.26)*, the derivative of the virtual controller is discontinuous and can not be used in the design of the controller. Instead, a function* $\bar{\boldsymbol{\alpha}}^q$ *is used in* (D.29)*, which is designed as if the state is not quantized.*

D.4 Stability Analysis

To analyze the closed-loop system stability, we first establish some preliminary results as stated in the following lemmas.

Lemma 1. *The effects of state quantization are bounded by the following inequalities:*

$$\|\boldsymbol{z}_1^q - \boldsymbol{z}_1\| = \|(\boldsymbol{x}_1^q - \boldsymbol{x}_r) - (\boldsymbol{x}_1 - \boldsymbol{x}_r)\| \leq \delta_{x_1}, \tag{D.30}$$

$$\|\boldsymbol{\alpha}^q - \boldsymbol{\alpha}\| = \|-\boldsymbol{C}_1\boldsymbol{z}_1^q + \boldsymbol{C}_1\boldsymbol{z}_1\| \leq \lambda_{\max}(\boldsymbol{C}_1)\delta_{x_1} \triangleq \delta_\alpha \tag{D.31}$$

$$\|\boldsymbol{z}_2^q - \boldsymbol{z}_2\| = \|(\boldsymbol{x}_2^q - \boldsymbol{x}_2) + (\boldsymbol{\alpha} - \boldsymbol{\alpha}^q)\| \leq \delta_{x_2} + \delta_\alpha \triangleq \delta_{z_2} \tag{D.32}$$

$$\|\boldsymbol{\Phi}_1(\boldsymbol{x}^q) - \boldsymbol{\Phi}_1(\boldsymbol{x})\| \leq L_{\Phi_1}\|\boldsymbol{x}^q - \boldsymbol{x}\| = L_{\Phi_1}\delta_x \triangleq \delta_{\Phi_1} \tag{D.33}$$

$$\|\boldsymbol{\Phi}_2(\boldsymbol{x}^q) - \boldsymbol{\Phi}_2(\boldsymbol{x})\| \leq L_{\Phi_2}\|\boldsymbol{x}^q - \boldsymbol{x}\| = L_{\Phi_2}\delta_x \triangleq \delta_{\Phi_2} \tag{D.34}$$

$$\|\bar{\boldsymbol{\alpha}}^q - \dot{\boldsymbol{\alpha}}\| = \|-\boldsymbol{C}_1(\boldsymbol{x}_2^q - \dot{\boldsymbol{x}}_r) + \boldsymbol{C}_1(\boldsymbol{x}_2 - \dot{\boldsymbol{x}}_r)\| \leq \lambda_{\max}(\boldsymbol{C}_1)\delta_{x_2} \triangleq \delta_{\bar{\alpha}}, \tag{D.35}$$

where $\delta_{(\cdot)}$ *are positive constants.*

Proof: Using the property (D.13) of the quantizer, we have

$$\|\boldsymbol{x}_1^q - \boldsymbol{x}_1\| = \|\boldsymbol{d}_{x_1}\| \leq \|[\delta_{x_{1,1}}\ \delta_{x_{1,2}}]^\top\| = \|\boldsymbol{\delta}_{x_1}\| \triangleq \delta_{x_1}, \tag{D.36}$$

$$\|\boldsymbol{x}_2^q - \boldsymbol{x}_2\| = \|\boldsymbol{d}_{x_2}\| \leq \|[\delta_{x_{2,1}}\ \delta_{x_{2,2}}]^\top\| = \|\boldsymbol{\delta}_{x_2}\| \triangleq \delta_{x_2}, \tag{D.37}$$

$$\|\boldsymbol{x}^q - \boldsymbol{x}\| = \|[\delta_{x_1}\ \delta_{x_2}]^\top\| = \|\boldsymbol{\delta}_x\| \triangleq \delta_x. \tag{D.38}$$

Then from (D.15)-(D.18), (D.24)-(D.26), Assumption 3 and (D.36)-(D.38) the inequalities (D.30)-(D.35) holds.

Lemma 2. *The state* $\boldsymbol{x}$ *satisfies the following inequality:*

$$\|\boldsymbol{x}\| \leq k_{x_1} + k_{x_2}\|\boldsymbol{z}\|, \tag{D.39}$$

where $\boldsymbol{z} = [\boldsymbol{z}_1^\top\ \boldsymbol{z}_2^\top]^\top$.

Proof: From the definitions in (D.15)-(D.17) and Assumption 1 we have

$$\|\boldsymbol{x}_1\| \leq \|\boldsymbol{z}_1 + \boldsymbol{x}_r\| \leq k_{x_r} + \|\boldsymbol{z}_1\| \leq k_{x_r} + \|\boldsymbol{z}\|, \tag{D.40}$$

$$\|\boldsymbol{\alpha}\| \leq \lambda_{\max}(\boldsymbol{C}_1)\|\boldsymbol{z}_1\|, \tag{D.41}$$

$$\|\boldsymbol{x}_2\| \leq \|\boldsymbol{z}_2 + \boldsymbol{\alpha} + \dot{\boldsymbol{x}}_r\| \leq \|\boldsymbol{z}_2\| + \lambda_{\max}(\boldsymbol{C}_1)\|\boldsymbol{z}_1\| + k_{\dot{x}_r}$$

$$\leq k_{\dot{x}_r} + [1 + \lambda_{\max}(\boldsymbol{C}_1)]\|\boldsymbol{z}\|. \tag{D.42}$$

Then

$$\begin{aligned}\|\boldsymbol{x}\| &= \|[\boldsymbol{x}_1^\top \ \boldsymbol{x}_2^\top]^\top\| = \sqrt{(\|\boldsymbol{x}_1\|)^2 + (\|\boldsymbol{x}_2\|)^2} \\ &\leq (k_{x_r} + k_{\dot{x}_r}) + (2 + \lambda_{\max}(\boldsymbol{C}_1))\|\boldsymbol{z}\| \\ &\triangleq k_{x_1} + k_{x_2}\|\boldsymbol{z}\|. \end{aligned} \tag{D.43}$$

The main results are now stated in the following theorem.

Theorem 1. *Consider the closed-loop adaptive system consisting of the plant* (D.2) *with input and state quantization satisfying the bounded property* (D.13)*, the adaptive controller* (D.21)*, the parameter updating laws* (D.22)-(D.23) *and Assumptions 1-3. All signals in the closed-loop system are ensured to be uniformly bounded and the error signals will converge to a compact set, i.e.*

$$\|\boldsymbol{z}(t)\| \leq \sqrt{\frac{2a}{c_0}}, \tag{D.44}$$

where c_0 *is the minimum eigenvalue of* $\boldsymbol{C}_0 = \min\{\boldsymbol{C}_1, \boldsymbol{C}_2\}$, *and where*

$$a = \delta_{V_1} + \frac{1}{2c_0} d_{V_2}^2, \tag{D.45}$$

$$\delta_{V_1} = \delta_{\theta_{11}} + \delta_{\theta_{21}}, \tag{D.46}$$

$$\delta_{V_2} = \delta_{z_2} + \delta_{x_1} + \delta_{\bar{\alpha}} + \|\boldsymbol{K}\|\delta_u + \delta_{\theta_{12}} + \delta_{\theta_{22}}, \tag{D.47}$$

and is ultimately bounded. Tracking of a given reference signal is achieved, with a bounded error.

Proof: We choose a Lyapunov function candidate as

$$V = \frac{1}{2}\boldsymbol{z}_1^\top \boldsymbol{z}_1 + \frac{1}{2}\boldsymbol{z}_2^\top \boldsymbol{z}_2 + \frac{1}{2}\tilde{\boldsymbol{\theta}}_1^\top \boldsymbol{\Gamma}_1^{-1}\tilde{\boldsymbol{\theta}}_1 + \frac{1}{2}\tilde{\boldsymbol{\theta}}_2^\top \boldsymbol{\Gamma}_2^{-1}\tilde{\boldsymbol{\theta}}_2. \tag{D.48}$$

Following the controller design in (D.21)-(D.23), the derivative of (D.48) is

$$\begin{aligned}\dot{V} =& -\boldsymbol{z}_1^\top \boldsymbol{C}_1\boldsymbol{z}_1 + \boldsymbol{z}_1^\top \boldsymbol{z}_2 - \tilde{\boldsymbol{\theta}}_1^\top \boldsymbol{\Gamma}_1^{-1}\dot{\hat{\boldsymbol{\theta}}}_1 - \tilde{\boldsymbol{\theta}}_2^\top \boldsymbol{\Gamma}_2^{-1}\dot{\hat{\boldsymbol{\theta}}}_2 + \boldsymbol{z}_2^\top[\boldsymbol{\Phi}_1^\top\boldsymbol{\theta}_1 + \boldsymbol{\Phi}_2^\top\boldsymbol{\theta}_2 + \boldsymbol{K}\boldsymbol{u}^q - \dot{\boldsymbol{\alpha}} - \ddot{\boldsymbol{x}}_r] \\ =& -\boldsymbol{z}_1^\top \boldsymbol{C}_1\boldsymbol{z}_1 - \boldsymbol{z}_2^\top\boldsymbol{C}_2\boldsymbol{z}_2^q + \boldsymbol{z}_2^\top(\boldsymbol{z}_1 - \boldsymbol{z}_1^q) + \boldsymbol{z}_2^\top(\bar{\boldsymbol{\alpha}}^q - \dot{\boldsymbol{\alpha}}) + \boldsymbol{z}_2^\top\boldsymbol{K}\boldsymbol{d}_u \\ &+ \Big[\boldsymbol{z}_2^\top\Big(\boldsymbol{\Phi}_1(\boldsymbol{x})^\top\boldsymbol{\theta}_1 - \boldsymbol{\Phi}_1(\boldsymbol{x}^q)^\top\hat{\boldsymbol{\theta}}_1\Big) - \tilde{\boldsymbol{\theta}}_1^\top\boldsymbol{\Phi}_1(\boldsymbol{x}^q)\boldsymbol{z}_2^q\Big] \\ &+ \Big[\boldsymbol{z}_2^\top\Big(\boldsymbol{\Phi}_2(\boldsymbol{x})^\top\boldsymbol{\theta}_2 - \boldsymbol{\Phi}_2(\boldsymbol{x}^q)^\top\hat{\boldsymbol{\theta}}_2\Big) - \tilde{\boldsymbol{\theta}}_2^\top\boldsymbol{\Phi}_2(\boldsymbol{x}^q)\boldsymbol{z}_2^q\Big]. \end{aligned} \tag{D.49}$$

By using (D.10), (D.33), (D.38) and (D.43) and Assumption 2, The following inequality is satisfied for the terms in (D.49) containing $\boldsymbol{\theta}_1$ and $\hat{\boldsymbol{\theta}}_1$:

$$
\begin{aligned}
&\boldsymbol{z}_2^\top(\boldsymbol{\Phi}_1(\boldsymbol{x})^\top\boldsymbol{\theta}_1 - \boldsymbol{\Phi}_1(\boldsymbol{x}^q)^\top\hat{\boldsymbol{\theta}}_1) - \tilde{\boldsymbol{\theta}}_1^\top\boldsymbol{\Phi}_1(\boldsymbol{x}^q)\boldsymbol{z}_2^q\\
&= \boldsymbol{\theta}_1^\top\boldsymbol{\Phi}_1(\boldsymbol{x})\boldsymbol{z}_2 - \boldsymbol{\theta}_1^\top\boldsymbol{\Phi}_1(\boldsymbol{x}^q)\boldsymbol{z}_2 + \tilde{\boldsymbol{\theta}}_1^\top\boldsymbol{\Phi}_1(\boldsymbol{x}^q)\boldsymbol{z}_2 - \tilde{\boldsymbol{\theta}}_1^\top\boldsymbol{\Phi}_1(\boldsymbol{x}^q)\boldsymbol{z}_2^q\\
&\leq \|\boldsymbol{\theta}_1\|\|\boldsymbol{\Phi}_1(\boldsymbol{x}) - \boldsymbol{\Phi}_1(\boldsymbol{x}^q)\|\|\boldsymbol{z}_2\| + \|\tilde{\boldsymbol{\theta}}_1\|\|\boldsymbol{\Phi}_1(\boldsymbol{x}^q)\|\|\boldsymbol{z}_2 - \boldsymbol{z}_2^Q\|\\
&\leq k_{\theta_1}\delta_{\Phi_1}\|\boldsymbol{z}\| + k_{\theta_1}L_{\Phi_1}\|\boldsymbol{x}^{\boldsymbol{q}}\|\delta_{z_2}\\
&\leq k_{\theta_1}\delta_{\Phi_1}\|\boldsymbol{z}\| + k_{\theta_1}\delta_{z_2}L_{\Phi_1}(k_{x_1} + k_{x_2}\|\boldsymbol{z}\| + \delta_x)\\
&= [k_{\theta_1}\delta_{z_2}L_{\Phi_1}(k_{x_1} + \delta_x)] + [k_{\theta_1}\delta_{\Phi_1} + k_{\theta_1}\delta_{z_2}L_{\Phi_1}k_{x_2}]\|\boldsymbol{z}\|\\
&\triangleq \delta_{\theta_{11}} + \delta_{\theta_{12}}\|\boldsymbol{z}\|.
\end{aligned}
\tag{D.50}
$$

In a similar way, by using (D.11), (D.34), (D.38) and (D.43) and Assumptions 2, the following inequality is satisfied for the terms in (D.49) containing $\boldsymbol{\theta}_2$ and $\hat{\boldsymbol{\theta}}_2$:

$$
\begin{aligned}
&\boldsymbol{z}_2^\top(\boldsymbol{\Phi}_2(\boldsymbol{x})^\top\boldsymbol{\theta}_2 - \boldsymbol{\Phi}_2(\boldsymbol{x}^q)^\top\hat{\boldsymbol{\theta}}_2) - \tilde{\boldsymbol{\theta}}_2^\top\boldsymbol{\Phi}_2(\boldsymbol{x}^q)\boldsymbol{z}_2^q\\
&= [k_{\theta_2}\delta_{z_2}L_{\Phi_2}(k_{x_1} + \delta_x)] + [k_{\theta_2}\delta_{\Phi_2} + k_{\theta_2}\delta_{z_2}L_{\Phi_2}k_{x_2}]\|\boldsymbol{z}\|\\
&\triangleq \delta_{\theta_{21}} + \delta_{\theta_{22}}\|\boldsymbol{z}\|.
\end{aligned}
\tag{D.51}
$$

Using the properties (D.30), (D.32) and (D.35) in Lemma 1 together with (D.50) and (D.51) and Young's inequality, we have

$$
\begin{aligned}
\dot{V} \leq& -\boldsymbol{z}_1^\top\boldsymbol{C}_1\boldsymbol{z}_1 - \boldsymbol{z}_2^\top\boldsymbol{C}_2\boldsymbol{z}_2 + \|\boldsymbol{z}_2\|\delta_{z_2} + \|\boldsymbol{z}_2\|\delta_{x_1} + \|\boldsymbol{z}_2\|\delta_{\bar{\alpha}} + \|\boldsymbol{z}_2\|\|\boldsymbol{K}\|\delta_u + \delta_{\theta_{11}}\\
&+ \delta_{\theta_{12}}\|\boldsymbol{z}\| + \delta_{\theta_{21}} + \delta_{\theta_{22}}\|\boldsymbol{z}\|\\
\leq& -c_0\|\boldsymbol{z}\|^2 + \delta_{V_1} + \delta_{V_2}\|\boldsymbol{z}\|\\
\leq& -\frac{c_0}{2}\|\boldsymbol{z}\|^2 + \delta_{V_1} + \frac{1}{2c_0}\delta_{V_2}^2\\
=& -\frac{c_0}{2}\|\boldsymbol{z}\|^2 + a.
\end{aligned}
\tag{D.52}
$$

From (D.48) and (D.52) it is shown that $\dot{V} < 0\ \forall\|\boldsymbol{z}\| > \sqrt{\frac{2a}{c_0}}$, thus $\boldsymbol{z}(t)$ is ultimately bounded and satisfies (D.44). The boundedness of $\boldsymbol{z}$ and (D.30) and (D.32) ensure the boundedness of the quantized error states $\boldsymbol{z}_1^q$ and $\boldsymbol{z}_2^q$. Then $\boldsymbol{\alpha}^q$ in (D.31) is also bounded. Since $\boldsymbol{x}$ is bounded in (D.43), then from (D.38) also $\boldsymbol{x}^q$ is bounded. From the projection operator, $\hat{\boldsymbol{\theta}}_1$ and $\hat{\boldsymbol{\theta}}_2$ are ensured bounded. Then, together with Assumptions 1-3, $\boldsymbol{u}$ in (D.21) is also bounded, and so all the closed-loop signals are uniformly bounded. Tracking is achieved, where the tracking error is ultimately bounded by (D.44).

Table D.1: Helicopter Parameters and initial values.

Symbol	Value
$\boldsymbol{x}(t_0)$	$[0\ 0\ 0\ 0]^\top$
$\hat{\boldsymbol{\theta}}_1(t_0)$	$[0.3218\ 1.8423\ 0.0007]^\top$
$\hat{\boldsymbol{\theta}}_2(t_0)$	$[0.4374\ 0.0014]^\top$
$\boldsymbol{K}$	$\begin{bmatrix} 0.0506 & 0.0506 \\ -0.0645 & 0.0810 \end{bmatrix}$

D.5 Experimental Results

The proposed controller was simulated using MATLAB/Simulink and tested on the Quanser Aero helicopter system. The mathematical model is described by (D.2), and the initial states and parameters used for simulation and experiments are shown in Table D.1.

The objective was to track a reference signal chosen as $\boldsymbol{x}_r(t) = [40\pi/180 \sin(0.1\pi t)$ $100\pi/180 \sin(0.05\pi t)]^\top$ when both the inputs and the states were quantized, and to ensure that all the signals in the closed-loop system were uniformly bounded. The inputs have limits of ± 24 V. The quantization levels were chosen as $l_u = 0.3$ for both inputs, and $l_s = 0.02$ for all the states. The gain matrices were set to $\boldsymbol{C}_1 = 6\boldsymbol{I}_3$, $\boldsymbol{C}_2 = 3\boldsymbol{I}_2$, $\boldsymbol{\Gamma}_1 = \boldsymbol{I}_3$ and $\boldsymbol{\Gamma}_2 = \boldsymbol{I}_2$.

The trajectories of the quantized states $\boldsymbol{x}^q = [\vartheta^q(t), \psi^q(t), \dot{\vartheta}^q(t), \dot{\psi}^q(t)]^\top$ are shown in Fig. D.4, where the desired states are shown with a dotted line and measured values from test on the helicopter system are shown with a solid line. The error in states $\boldsymbol{x}_1^q - \boldsymbol{x}_r$ and $\boldsymbol{x}_2^q - \dot{\boldsymbol{x}}_r$ are shown in Fig. D.5, and Fig. D.6 shows the quantized input $\boldsymbol{u}^q$. The results here illustrate the theoretical findings in Theorem 1, where tracking is achieved and all signals are shown to be uniformly bounded.

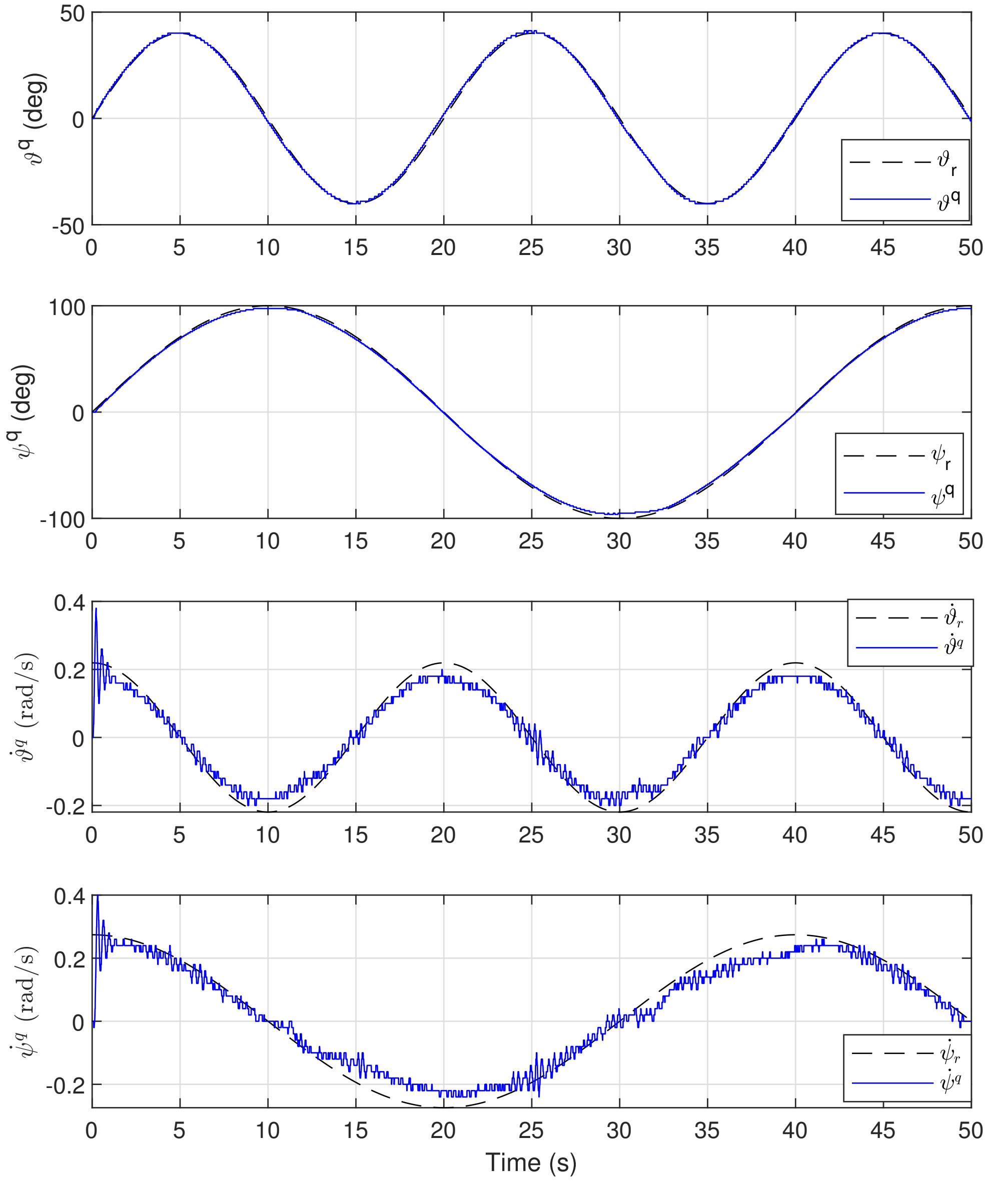

Figure D.4: Trajectories of the quantized states $\boldsymbol{x}^q$ from experiment.

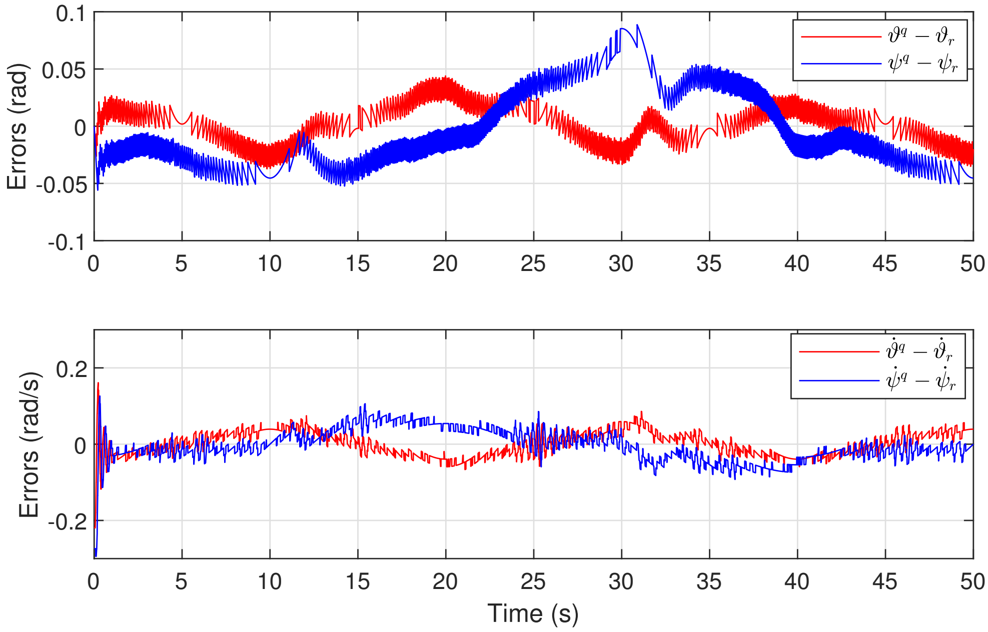

Figure D.5: Errors $\boldsymbol{x}_1^q - \boldsymbol{x}_r$ and $\boldsymbol{x}_2^q - \dot{\boldsymbol{x}}_r$.

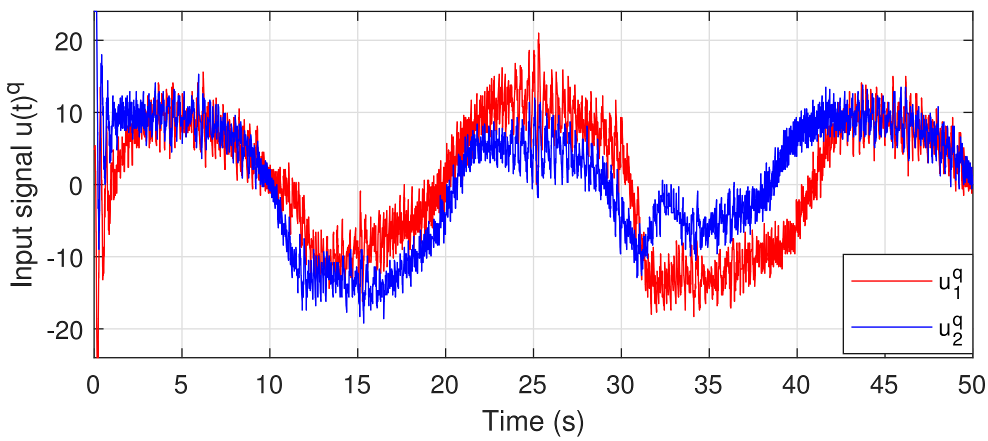

Figure D.6: Quantized input $\boldsymbol{u}(t)^q$.

D.6 Conclusion

In this paper, an adaptive backstepping control scheme for an uncertain nonlinear MIMO helicopter system with both input and state quantization was developed. The quantizer considered satisfies a bounded condition and so the quantization error is bounded. For the closed loop system, all signals are shown to be uniformly bounded where the error signals will converge to a compact set containing the origin. Tracking of a given reference signal is achieved, with a bounded error. Experiments on the helicopter system support the proof.

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