The development of students' mathematical competencies

The case of biology students

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Doctoral Dissertations at the University of Agder 396

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Abstract

This research focuses on the development of mathematical competencies of first-year university students in a Biology department as they engage in non-routine open-ended mathematical tasks set in the context of biology. The research employs a scaling instrument on a modified extant competence framework to identify evidence of student competency development by exploring situations where students activate particular mathematical competencies. The scaling instrument is designed to evaluate the quality of competence activation. The research creates individual competency profiles for volunteer students who participated in a series of calculus classes addressing several mathematical areas over one semester. The research also explores whether the activation of the distinct mathematical competencies are inter-related exploring instances of concurrent activation.

The findings suggest competency development for a subset of competencies. There is evidence that specific mathematical domains facilitate or hinder the development of specific mathematical competencies. Furthermore, there are indications that the activation of specific competencies hinder or favor the concurrent activation of specific other competencies.

This research highlights techniques that may advance the study of students' assessment and evaluation concerning the acquisition of mathematical knowledge. It provides evidence that the employment of a competency scaling scheme can provide information regarding the quality of students' competence activation This study also comprises implications for new student assessment methods where the teacher can obtain a more detailed picture of each student's performance by creating a competence profile where the weaknesses and strengths of the students are on full display.

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Glossary

Term	Definition/Description
Calculus unit	The set of calculus sessions focused on a particular mathematical context. There are four calculus units: Periodic Functions, Exponential Growth. & Regression, Population Dynamics, and Integrals & Modeling.
Competency activation	The instance where a student brings into action (provides evidence of activation) a particular competency while working on a mathematical task.
Competency aspect	Components of the five competencies of the competence framework. There are 16 aspects in total.
Domain of activation (or activation domain)	Areas of development where the competency activation occurs. There are three activation domains: T.S.V., M.L.V., and Pr.In.T.
Strength of evidence combination	A qualitative description of the summary of the three types of evidence regarding the progress of competency development. There are four qualitative descriptors for the strength of evidence combination: strong, moderate, weak, and negligible.
Turn	Used in tables of transcribed text from class utterances to display the episode's actor and the order that it took place.
Quality of competency activation	An evaluation of a set of attributes that characterize a mathematical competency activation. These attributes concern the student's awareness of the task solution steps, their use of mathematical language, and the amount of help they received from a fellow student or their teacher.
T.S.V.	Task Solving Vision
M.L.V.	Mathematical Language and Vocabulary
Pr.In.T.	Prompting and Independent Thinking

...In Loving Memory of my Father

1. Introduction

This dissertation explores the students' activation and development of mathematical competencies over time. In this research, I explore the progress of the individual development of these mathematical competencies that students bring into action while operating within the classroom environment of a Biology department. First-year undergraduate students in a department of Biology in a Norwegian university have been involved in the project and participated in a series of calculus sessions working on a set of non-routine (open-ended) mathematical tasks addressing several mathematical areas. This chapter briefly introduces the motivation and rationale for this research in Section 1.1, followed by an overview of the objectives and the research design planning in Section 1.2. Finally, the structure and outline of the dissertation are discussed in Section 1.3.

1.1 The motive and rationale of the research

The arena of STEM (Science, Technology, Engineering, and Mathematics) studies –apart from mathematics departments- is becoming gradually interdisciplinary (e.g., Drake & Burns, 2004), and mathematical skills and competencies are deemed necessary for all students in any area of these studies. Smith and Karr-Kidwell (2000) conceptualize the interdisciplinary nature of STEM as "[a] holistic approach that links the [individual] disciplines so that learning becomes connected, focused, meaningful, and relevant to learners." (p. 24). It is reasonable to assume that different disciplines will require a set of different mathematical competencies without, however, excluding the possibility that some mathematical competencies are deemed necessary for any discipline within the STEM field. Adding to this assumption, how students in biology departments develop their mathematical competencies will presumably differ from the development in other STEM education fields and from student to student.

The author of this dissertation expresses an interest in exploring the progress of students' competency activation and development during a series of calculus sessions where they will engage in solving non-routine mathematical tasks in the context of biology. It poses a great interest because such an endeavor will potentially provide valuable information regarding the students' difficulties and areas of growth concerning a set of particular mathematical competencies. These competencies could concern, for example, the ways students communicate their reasoning about mathematics and mathematical ideas or the ways they decode and interpret symbolic and formal mathematical language. Among other things, the information mentioned at the beginning of this paragraph could be used by the instructors to reshape their teaching, the students to gain awareness of their weaknesses and improve specific mathematical competencies, and the researchers to gain insight into students' learning processes.

Evaluating competencies in primary and secondary education has been the focus of several studies (Hartig et al., 2008). However, little has been done at the higher education level, and several studies have pointed out a need for competence models assessment (e.g., Blömeke et al., 2013). Several instruments for assessing competencies are available within the realm of teacher education studies (OECD, 2010; Blömeke et al., 2013; Hill et al., 2005.) However, the assessment of competencies at the undergraduate level is still an area of significant growth, and therefore a further study of the underlying competence structures is needed. The present study intends to add knowledge to this area: the exploration and assessment of competency activation and development in undergraduate studies and creating individual competency profiles for the students involved in this research.

This study met several challenges during the research design's conceptualization, the data generation, and the data analysis processes. Appropriate technical equipment was necessary to generate audio and video recordings and secure the collection of all written work that students produced during the calculus sessions. Accurate descriptions and definitions of concepts were as vital as challenging for this research's needs. I highlight the necessity of describing the notions of competency, competency activation, and competency development and what they mean for this research. These issues are addressed in the following chapter with the presentation of the Literature Review and at the beginning of Chapter 3 - Research design, where this thesis's epistemological and ontological stance is presented.

Additional challenges include the necessity of describing the concept of progress because it was essential to clarify that progress of competency development does not suggest advancement necessarily but instead describes the attempt to explore the course of competency development during a particular teaching intervention (the teaching of non-routine tasks in a Biology department). This description takes place in Chapter 4 - Data Analysis. To perform such an exploration, I propose a new way of exploring the progress of undergraduate students' competency development over time, suggesting the additional use of a scaling instrument presented in the same chapter. This suggestion is accompanied by the employment of an already existing model designed to monitor competency progression.

It is well documented that across OECD countries, large proportions of students reported low levels of enjoyment of mathematics; in Norway, these proportions are close to the average. For example, 53% of students in OECD countries agreed or strongly agreed that they are interested in the things they learn in mathematics. In Norway, 50% of students agreed or strongly agreed with this statement. In general, students who participated in PISA 2012 did not show high levels of enjoyment of mathematics and intrinsic motivation to learn mathematics. The author of this thesis is a member of the Centre for Research, Innovation, and Coordination of Mathematics Teaching (MatRIC), which in collaboration with the Centre for Excellence in Biology Education (bioCEED), developed a joint project aimed at improving biology students' motivation for, interest in, and perceived relevance of mathematics in biological studies. Within that context, first-year biology students at a Norwegian university participated in a series of calculus sessions for two months in which they engaged in solving non-routine tasks such as population dynamics and exponential growth and regression problems. In a more detailed fashion, the following subsection presents the objectives of this thesis.

1.2 Objectives of the research

As mentioned in the previous subsection, this research includes a series of calculus sessions conducted by the author of this dissertation in a Norwegian university's Biology department. Participants of this study were students of this department who followed their main calculus courses during their spring semester. The students agreed to attend a series of calculus sessions as part of an additional learning opportunity offered by their department. The participants would have the opportunity to work on non-routine mathematical problems within the context of biology. A non-routine problem "requires a novel idea from the student" (Milgram, 2007, p. 47), and as presented in TIMMS 2011 framework, "[N]on-routine problems are problems that are very likely to be unfamiliar to students. They make cognitive demands over and above those needed for solution of routine problems, even when the knowledge and skills required for their solution have been learned." (Mullis et al., 2009, p. 45). In the following paragraphs, the discussion continues on the objectives of this study in more detail to provide a clearer perspective about the aims of this thesis, and by the end of Chapter 2 - Literature Review, the research questions of this dissertation will be presented.

This thesis explores the progress of individual mathematical competency development of first-year biology students throughout a series of calculus sessions working on non-routine mathematical tasks within a biology context. This exploration focuses on identifying the instances in which a student brings any mathematical competency into action (competency activation) while working on the assigned tasks. Simply put, the focus here is on those moments where a student activates a mathematical competency or a set of mathematical competencies when asked to engage with the problem-solving activity. The competencies belong to a broader set of mathematical competencies that will constitute this study's competence framework, and to facilitate this exploration, I seek ways to gather and analyze relevant data. The element of time is an integral part of this study because the interest is on the progress over time to identify competency development characteristics. Particular analytical and scaling instruments were created and used to explore this inquiry. Data from the sessions were collected using several digital resources such as video recordings, researcher's observation/field notes, and smartpens to collect written work in class.

Another main objective of this research is to explore the recorded competency activations and search for instances where the students provide evidence (through his/her actions in the class) that signifies the concurrent activation of two mathematical competencies from the competence framework. An example of such an instance would be when a student communicated his/her reasoning about mathematics and at the same time decoded and interpreted symbolic and formal mathematical language. Such an instance would be considered an instance of concurrent competency activation. Within this objective, I address the notion of competencies interrelation, addressed in Chapter 4 – Data Analysis. The interest is in whether there may be any association between competencies; in other words, whether the activation of one competency often occurs concurrently with another or whether two competencies are never activated – or observed to be activated simultaneously. For such an exploration, it was necessary to obtain information about (i) the frequencies (of competencies activated in the task attempted by the student) and (ii) the frequencies of the simultaneous occurrence of competencies (competency pairs) activated during the students' work in the sessions.

Exploring individual progress of competency development offers opportunities to identify possible advances within a comprehensive set of mathematical skills for the student. Similarly, those mathematical skills that did not advance over time can also be tracked. As mentioned in the previous subsection, this information can be valuable for the instructor and the student and add knowledge to this research body on competency development assessment. Because this study focuses on one particular biology department class, it is only slightly possible to obtain some indications regarding the learning of mathematics in studies of biology for first-year students. I should repeat that by exploring the progress of competency development over time, I wish to describe a detailed observational process to record the course of competency development over a series of calculus sessions, working with non-routine mathematical tasks, over a university semester.

1.3 Structure and outline of the dissertation

The core content of the present study is organized into three parts; Part I: Setting the Scene, Part II: Results and Findings, and Part III: Discussion of Findings and Conclusions. Below is a brief outline of the chapters included in those three parts.

Part I: Setting the scene. This part sets the scene by introducing the research's motivation and rationale in Chapter 1 - Introduction. Chapter 2 - Literature Review, introduces the theoretical background and numerous critical concepts involved in this thesis. By the end of the chapter, I will address the research questions of this dissertation. The questions are accompanied by a set of sub-questions that will facilitate the conduct of this research. In Chapter 3 - Research Design, I discuss the design of this research and reflect on the choice of methods and their relationship with the research questions of this thesis. Finally, in Chapter 4 - Data Analysis, I elaborate on the methods employed to analyze the data and introduce the methodological tools used to facilitate the data analysis. Even though the methods for data analysis are usually included in the Research Design chapter, the extent of the analysis led to the decision to devote a separate chapter for the presentation of the analysis methods employed in this thesis.

Part II: Results and Findings: This part includes the results of this study in Chapter 5- Results: Competency activations, followed by Chapter 6- Competency development and interrelation, where the findings of this research are presented by revisiting the main Research Questions of the thesis.

Part III: Chapter 7-Discussion, provides a complete revision of the key concepts introduced in the first parts of the thesis by addressing the Research Questions of this dissertation. In the last chapter, Chapter 8-Conclusions, I present the key findings, the contributions, and the implications of this research. The limitations of this study and the suggestions for future research are also provided and discussed.

2. Literature Review

In the following sections of this chapter, I address and explore several issues from the existing literature regarding the notions of mathematical competence and the assessment of competency development. Section 2.1 discusses the notion of competence and in particular mathematical competence. In Section 2.2, existing competence frameworks are discussed and some additional definitions are provided for obtaining a comprehensive idea of the notion of mathematical competence. Section 2.3 discusses the efforts that have been done for the exploration and assessment of competency development. In Section 2.4, I discuss the 3-D competency progress model, an analytical tool that explores the developmental process of competencies. Finally, in Section 2.5, I present a summary of the chapter and this dissertation's research questions.

A note

The following sections include the description of a competence framework (i.e., the KOM model) and, as mentioned above, of an analytical tool for monitoring the progress of competency development (i.e., the 3-D tool). These two models facilitated the data analysis of this thesis. For that reason, in Chapter 4: Data Analysis, I revisit these constructions and elaborate on them in greater detail. One might say that this revisiting constitutes an additional part of this thesis's literature review. It is, therefore, presumable that a form of an added and more targeted literature review will take place in Chapter 4, where I elaborate on specific mathematical competencies (e.g., mathematical reasoning, mathematical thinking, and acting) and on the utilization of the 3-D analytical tool of competency progress.

2.1 Mathematical competence

One of the first references to the notion of competence can be found in McClelland's (1973) work, where he suggests alternative ways of testing a person's cognitive abilities. Within a psychological context, he criticizes the testing movement in the United States and wonders, "[h]ow would one test for competence […] for an alternative approach to traditional intelligence testing?" (p. 7). At the same time McClelland made these suggestions, the idea of focusing on competencies "gradually gained momentum" within a societal and economic context as part of the "so-called knowledge capitalism as well as in the educational paradigm," Geraniou and Jankvist (2019, p. 29).

At the beginning of the 1980s, the mathematics education community shared their doubts about the skill-focused philosophy of the 'back-to-basics' movement. As Abrantes (2001, p. 125) informs us, mathematics educators "started to emphasise the need to enlarge the components of what was generally considered as basic skills in mathematics," and he continues by pointing out that "problem solving, reasoning, applications and the use of technology" were considered essential suggestions in numerous programmatic documents. The latter is confirmed by the Agenda for Action's position (NCTM, 1980, p. 1), where they concluded that there was a need for encompassing "more than computational facility" within what is defined as basic mathematical skills. Many researchers criticized the 'back-to-basics' movement within this Agenda for Action because it appeared to "place a low ceiling on mathematical competence" (NCTM, 1980, p. 6). The notion of mathematical competence appeared to be something more than a simple listing of computational and memorizing skills within the mathematical activity.

In a broad sense, competence is defined as a complex capability related to performance in real-life situations, as Hartig et al. (2008) point out. In a study by Shavelson (2013), seven facets of competence and its assessment are identified: complexity, performance, standardization, fidelity, level, improvement, and disposition. The same study considers competence a theoretical construct that cannot be observed directly but can be inferred from the individual's performance on simple tasks. Finally, competence should not be considered the same as academic knowledge; the same applies to the connection between academic competence and professional competence (Oser, 2013), even though these two notions are not mutually exclusive.

Identifying key competencies for education and training was part of the European Commission's commitment, as Crick (2008) points out. This Commission defined competence as "a combination of knowledge, skills, and attitudes appropriate to a particular context" (p. 312). Several key competences were recommended by the EU working group and considered essential for lifelong learning (European Council, 2006). Communication in the mother tongue, communication in foreign languages, and digital competence were among these competencies. As seen above, the term competence is used and can be found in several aspects of our everyday lives. For example, a large body of research explores the notion of professional competences of mathematics teachers (Kaiser et al., 2017) in light of teachers' professional development.

Geraniou and Jankvist (2019) state that the educational systems in some countries, such as Denmark, integrated competency descriptions within their national curricula for all the educational levels that reach tertiary educational programs. This is also confirmed by Boesen et al. (2018, p. 109), who remark that "[m]any countries with longer traditions of national standards have also recently implemented new standards, two examples being Norway and Sweden, where educational goals have been even more explicitly described in competency terms." The same authors believe that these new standards and national curricula describe in various ways what it means to be mathematical competent when working on or with mathematics in any educational context. Being competent in mathematics gives a more comprehensive description of that competent person's skills and abilities when working with mathematics.

What constitutes mathematics as a subject? What do we mean when we say that someone knows mathematics? What mathematical skills are necessary for a biology classroom where students deal with non-routine mathematical problems? One attempt to answer these questions is to view mathematical knowledge as a set of various mathematical qualities, named here mathematical competencies, and assume that possessing and developing them at a certain level could mean that one can possess a certain 'amount' of mathematical knowledge. Niss and Højgaard (2011) describe mathematical competence as "having knowledge of, understanding, doing, using, and having an opinion about mathematics and mathematical activity in a variety of contexts where mathematics plays or can play a role" (p. 49). The emphasis here is on the mathematical activity and any process that could comprise it.

As Abrantes (2001) observes, mathematical competence can be considered as a set of all these cognitive, affective, and technical elements of mathematics required for a person to participate actively in society and lifelong learning. Similarly, mathematical competencies are seen as these complex abilities that are related to performance in real-life-situations (Hartig et al., 2008; Shavelson, 2013; Blömeke et al., 2015) and are also regarded as the willingness to solve future problems, act responsibly, and critically evaluate a problematic (task) situation (Weinert, 2001). Within this description, the focus is on the person's ability or abilities that brings into action to work on a particular situation (situation-related). This ability or set of abilities constitutes the competencies that a person possesses. This definition is less generative and more dependent on the situation that a person, the student for example, encounters while working on a mathematical task.

According to Niss (2003, p. 7), mathematical competence is "[…] the ability to understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role." Different mathematical competencies are expected to be possessed to a greater or lesser extent by each person that can be thought of as a set of individual characteristics or qualities. Lastly, any attempt to summarize some basic categories of mathematical competencies encapsulates a risk of leaving out some mathematical skills that could be important in a particular educational context.

This thesis focuses on the notion of mathematical competence, which has become, in the last 15 years, a central component of the educational milieu and a fundamental element of mathematical literacy (e.g., Sadler, 2013; Stacey & Turner, 2014). Many existing curricula are designed to address the issue of what it is to know mathematics by paying attention to competencies and including aspects of mathematical literacy (Burkhardt, 2014). Niss and Højgaard (2019) bring to our attention the report of the Survey Team (Niss et al., 2016) at the Thirteenth International Congress on Mathematical Education (ICME-13). This report clarifies that "notions such as mathematical proficiency, mathematical practices, fundamental mathematical capabilities, mathematical literacy, quantitative literacy and numeracy" (p. 10), even though related, cannot be considered identical to those of mathematical competence and mathematical competencies. The latter is an essential distinction because these notions, especially mathematical literacy, comprise elements in their definition that differ from those of mathematical competence.

The definitions presented above do not contradict but rather complement each other. It appears that the notion of mathematical competence is associated with the performance of any form of mathematical activity and the exercise of mathematics. When discussing mathematical competence, the focus is on what Niss and Højgaard (2019. p. 11) describe as “the enactment of mathematical activities and processes.” The following subsection discusses researchers’ endeavors to conceptualize mathematical competence by framing them within a set of fundamental aspects that characterize this notion. In addition to the latter, some considerations will be presented regarding the distinction between mathematical competenc*e* and mathematical competenc*y*. This distinction will be valuable for the work that will be presented in the chapters to come. The following section reports on several mathematical competence frameworks developed to describe a variety of cognitive skills considered relevant to any feature of mathematical activity.

2.2 Competence frameworks

Various attempts at framing the notion of mathematical competence have been presented in the literature (e.g., Maaß, 2006; Boesen et al., 2014). The ambiguity of the concept of competence is highlighted by Kilpatrick (2014, p. 85) who points out that the notion of competence “is one of the most elusive in the educational literature.” The increased focus on the notion of competence in mathematics education resulted in the construction of several competence frameworks that are sets of different cognitive skills and abilities related to the mathematical activity that constitute mathematical competence. In his study on competence frameworks, Kilpatrick (2014) discusses the fact that school mathematics is often “portrayed as a simple contest between knowledge and skill” and notices that, on the other hand, “[c]ompetency frameworks are designed to demonstrate to the user that learning mathematics is more than acquiring an array of facts and that doing mathematics is more than carrying out well-rehearsed procedures” (p. 87).

Approaching the notion of mathematical competence in the form of a framework that includes specific competencies such as problem-solving, reasoning, procedural, representation (use of symbolic forms), and communication competency appears to be a valid way to conceptualize what it means to be competent in mathematics (Boesen et al., 2014). In a similar fashion, Turner (2010) suggests another set of specific competencies: communication, mathematizing, representation, reasoning, devising strategies, and using symbolic, formal, and technical language, and operations. These competencies often share some common characteristics. For example, there are similarities within the actions that describe the processes of problem-solving and reasoning because the process of solving a mathematical task demands, at some point, some form of mathematical reasoning to produce an outcome. This observation will be revisited in the following subsection in the discussion of a particular competence framework, namely the KOM model (Niss & Højgaard, 2011).

There are several mathematical competence frameworks often used in the research literature. The main objective of such frameworks is “to communicate goals and means for educational development” Boesen (2014, p. 74). The Mathematics Learning Study of the US National Research Council’s report Adding It Up (Kilpatrick et al., 2001) identifies the five strands of mathematical proficiency. Similarly, the five components of mathematical problem-solving ability were identified in the Singapore mathematics framework (Ministry of Education, Singapore, 2006). Problem-solving, reasoning, and proof skills are some of the standards highlighted as essential and necessary for any student at the American Principles and Standards for school mathematics (NCTM, 2000).

The Swedish Mathematical Competencies: A Research Framework (MCFR) (Lithner et al., 2010), and the Danish KOM-project (Niss, 2003) focus on the competencies that are believed to be vital for any student at any educational level. The KOM competence framework (Niss & Højgaard, 2011), which will be further analyzed in the following sections, was implemented as the foundation of the PISA mathematical competencies framework (e.g., Stacey & Turner, 2014) and “influenced curricula reforms in several European countries, such as Denmark, Germany, Catalonia, Sweden, and Norway” as Pettersen and Braeken (2019, p. 406) remark. It appears that whenever the focus of a curriculum reform comprises content goals (e.g., arithmetic, symbolic) then this content is actually taught in the class (Boesen et al., 2014). However, when any curriculum reform effort incorporates competency such as mathematical reasoning, problem-solving ability, proof skills, then serious considerations on the teaching process should be made. The following subsections (2.2.1 and 2.2.2) will discuss a competence model that aimed at exploring what mastering mathematics means and how it can be described. .

2.2.1 The KOM competence model

The notion of mathematical competency attracted the attention of several researchers who wanted to explore ways of assessing students (e.g., Greer & Verschaffel 2007; Henning & Keune 2007; De Bock et al., 2007; Houston & Neill, 2003; Blomhøj & Jensen, 2007). During the summer of 2000, the project Competence Development and Mathematics Learning started. The initiative came from the Danish National Council for Science Education. This project took the name ‘KOM project’ Niss (2003) and focused on describing mathematics curricula primarily by using the notion of mathematical competency in a way that this framework could apply at any educational level. One of the main purposes of this project was to describe “characterize and measure students' mastering of mathematics" and address the "assessment problem" (Niss, 2003, p. 4) of reliable and valid ways to assess students' knowledge and skills in and about mathematics.

In the KOM competence model (Kompetencer Og Matematiklæring- Competencies And Mathematics Learning) by Niss and Jensen (2002) and Niss and Højgaard (2011), there is a distinction between competenc*e* and competenc*y* (competenc*ies*). Competenc*e* is used for the two overarching competenc*es*: *to ask and answer in, with, about mathematics* and *to deal with mathematical language and tools*. Each competence covers four competencies (in singular form is called competency). The resulting eight competencies have been visualized as in Figure 2.1, so that the possibility of overlapping between the competencies would be visually identifiable. As Niss (2003) points out, "[a] mathematical competency is a clearly recognizable and distinct, major constituent of mathematical competence." However, the fashion in which these competencies overlap has not been explored. The KOM model was based on research questions concerning students' needs and desired mathematical skills. Niss (2003) asks,

> Which mathematical competencies need to be developed in students at the different stages of the education system? and how does one ensure progression and coherence in mathematics teaching throughout the education system? and how does one measure mathematical competencies? (pp. 4-5)

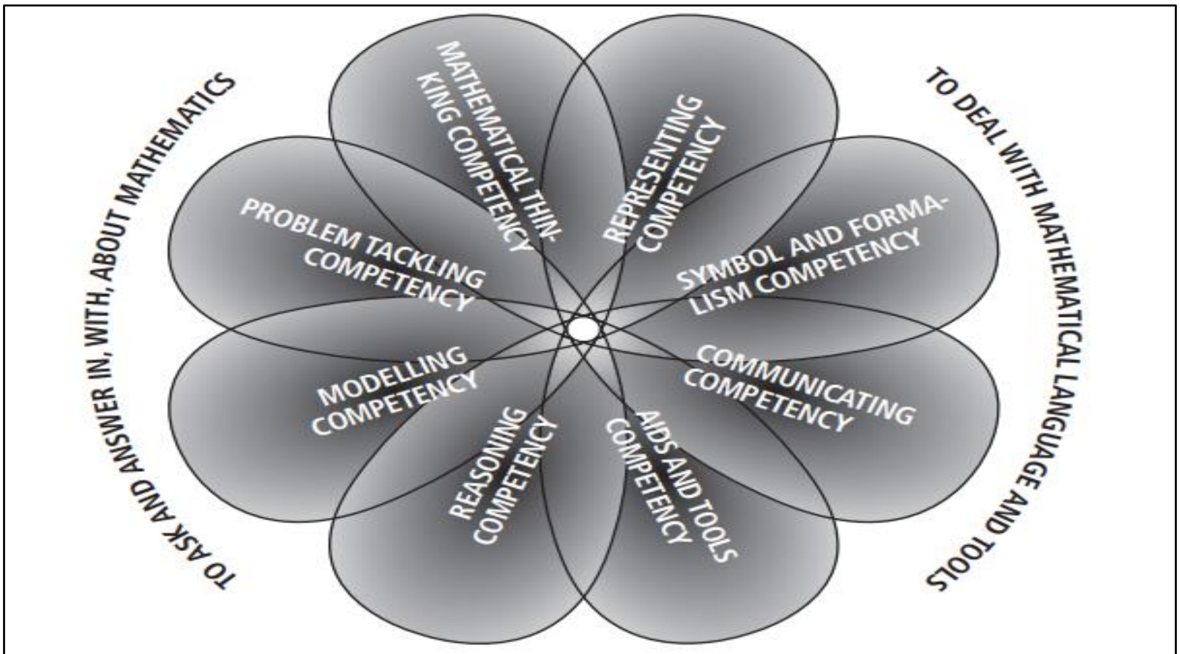

Figure 2.1. Visual representation – the "KOM flower" – presented and exemplified in the KOM report, Niss & Højgaard, 2011.

The authors approached the notion of mathematical competence, focusing on specific actions towards mathematics and mathematical activity, and attempted to define mathematical competency as a main constituent of mathematical competence. They considered that competency is "[…] a well-informed readiness to act appropriately in situations involving a certain type of mathematical challenge" (Niss & Højgaard, 2011, p.49). The same authors (Niss & Højgaard, 2019, p. 12) suggest an even more context-based definition of competence, claiming that "[c]ompetence is someone's insightful readiness to act appropriately in response to the challenges of given situations." In their study, they make clear that in their conceptualization, competence is a notion of cognitive nature and 'readiness' suggests that the person brings into action a set of "cognitive prerequisites" (Niss & Højgaard, 2019, p. 18) to work and engage in any activity. Therefore, mathematical competence is someone's insightful readiness to act and address all mathematical challenges relating to given situations. The following sub-section presents an overview of the KOM framework.

2.2.2 The KOM framework

The KOM competence framework, as displayed in Niss and Højgaard (2011), consists of eight primary competencies with 24 aspects (components of each competency) in total. The aspects are elements of the competencies that describe each one of the eight primary competencies. The eight competencies are:

1. thinking mathematically (mastering mathematical modes of thought)
    a. posing questions that are characteristic of mathematics, and knowing the kinds of answers (not necessarily the answers themselves or how to obtain them) that mathematics may offer
    b. understanding and handling the scope and limitations of a given concept
    c. extending the scope of a concept by abstracting some of its properties; generalizing results to larger classes of objects
    d. distinguishing between different kinds of mathematical statements (including conditioned assertions ('if-then'), quantifier laden statements, assumptions, definitions, theorems, conjectures, cases)
2. posing and solving mathematical problems
    a. identifying, posing, and specifying different kinds of mathematical problems pure or applied; open-ended or closed
    b. solving different kinds of mathematical problems (pure or applied, open-ended or closed), whether posed by others or by oneself, and, if appropriate, in different ways
3. modelling mathematically (i.e., analyzing and building models)
    a. analyzing foundations and properties of existing models, including assessing their range and validity

   b. decoding existing models, i.e. translating and interpreting model elements in terms of the 'reality' modelled
   c. performing active modelling in a given context
      i. structuring the field
      ii. mathematising
      iii. working with(in) the model, including solving the problems it gives rise to
      iv. validating the model, internally and externally
      v. analyzing and criticizing the model, in itself and vis-à-vis possible alternatives
      vi. communicating about the model and its results
      vii. monitoring and controlling the entire modelling process
4. reasoning mathematically
   a. following and assessing chains of arguments, put forward by others
   b. knowing what a mathematical proof is (not), and how it differs from other kinds of mathematical reasoning, e.g. heuristics
   c. uncovering the basic ideas in a given line of argument (especially a proof), including distinguishing main lines from details, ideas from technicalities
   d. devising formal and informal mathematical arguments, and transforming heuristic arguments to valid proofs, i.e. proving statements
5. representing mathematical entities (objects and situations)
   a. understanding and utilizing (decoding, interpreting, distinguishing between) different sorts of representations of mathematical objects, phenomena and situations
   b. understanding and utilizing the relations between different representations of the
   c. same entity, including knowing about their relative strengths and limitations
   d. choosing and switching between representations
6. handling mathematical symbols and formalisms
   a. decoding and interpreting symbolic and formal mathematical language, and understanding its relations to natural language
   b. understanding the nature and rules of formal mathematical systems (both syntax and semantics)
   c. translating from natural language to formal/symbolic language
   d. handling and manipulating statements and expressions containing symbols and formulae
7. communicating in, with, and about mathematics
   a. understanding others' written, visual or oral 'texts', in a variety of linguistic registers, about matters having a mathematical content;

    b. expressing oneself, at different levels of theoretical and technical precision, in oral, visual or written form, about such matters
8. making use of aids and tools (IT included)
    a. knowing the existence and properties of various tools and aids for mathematical activity, and their range and limitations
    b. being able to reflectively use such aids and tools

Niss (2003) clarifies that these eight competencies concern mental or physical processes, activities, and behaviors. These competencies comprise both analytical and productive characteristics. Within the analytical view, each competency focuses on understanding, interpreting, exploring, and assessing mathematical phenomena and processes. Examples from that view can be seen when a person follows a chain of mathematical arguments or when he/she understands the nature of mathematical representation. The productive view refers to the situations where there is an active construction of mathematical processes or when the student carries out such processes. For instance, when the student constructs a chain of arguments or employs some mathematical representation when tackling a mathematical task. Employing an analytical framework, such as the competence framework discussed above, offers opportunities for understanding how a competency or a set of competencies are developed over time. The following subsection discusses efforts from the relevant literature to explore the assessment and development of mathematical competencies.

2.3 Exploring and assessing the development of mathematical competencies

The exploration of mathematical competency development has been a primary research topic within mathematics (e.g., Hartig, 2007; Leuders, 2014) for the last two decades because of the recent implementation of competence-based mathematics curricula. Consequently, this implementation created the necessity for a shift in assessment practices so that assessment instruments can encapsulate all these components that describe the wide variety of mathematical competencies described within these competency-based curricula (Boesen et al., 2018; Lane, 2004; Niss, 2007). Nevertheless, the assessment of mathematical competencies is considered a challenging task (Koeppen et al., 2008; Blömeke et al., 2015; Niss et al., 2016) because of the complexity of what constitutes (forms of cognitive skills and abilities) a mathematical competence. In particular, Niss (2007) expresses his concerns about the suitability of existing assessment items regarding the higher-order competencies, and Lane (2004, p. 8) expresses similar concerns about the evaluation of “high-level thinking skills,” which are primary components of mathematical competence. For example, critical thinking is considered to belong within the set of these higher-order skills, playing a central role in problem-solving, logical thinking, and decision-making (Butler, 2012; Halpern, 2003).

Mathematical competencies and the construction of relevant analytical frameworks have been the focus of recent educational reform and development attempts by researchers, institutes, and organizations, even at a national level. The US National Council of Teachers of Mathematics Standards (National Council of Teachers of Mathematics [NCTM], 1989; 2000) initiated and inspired many of these attempts focusing on the construction of a competence framework and possible competence assessment (e.g., Kilpatrick et al., 2001; Niss & Jensen, 2002). Nevertheless, research focused on a complete description of mathematical competencies is still limited. Only a few studies have used a set of competencies as an analysis framework for empirical classroom data (Säfström, 2013).

As seen in previous sections and subsections of this chapter, the efforts to conceptualize the notion of mathematical competence led to the consideration of several descriptions and definitions. Consequently, imprecision and overlapping within these descriptions are unavoidable (Boesen et al., 2014). By proposing a conceptual framework, Yang et al. (2021) address these difficulties that the imprecision and overlapping cause and suggest a way to explore the development of mathematical competence. By adopting the term 'cores,' they present four non-overlapping components that characterize the notion of mathematical competence: learning, knowing, applying, and viewing. Within these cores and through epistemological, ontological, and pedagogical lenses, they explore the development of mathematical competence and "highlight the process-product nature of mathematical thinking" (p. 80). This effort suggests that when considering the nature and dimensions of the competence concept, it is feasible to construct a conceptual or analytical framework that will facilitate the exploration of competency development.

The strands of mathematical proficiency are specified in the US National Research Council's report Adding It Up (2001). This report mentions adaptive reasoning, strategic competence, conceptual understanding, procedural fluency (skill in carrying out procedures efficiently and appropriately), and productive disposition (i.e., "habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy" p. 116). Within this context, Boesen et al. (2018, p. 110) also notice that "[m]athematical processes, mathematical proficiencies, and mathematical competencies are different names for related ideas that have become more visual and present in many countries' mathematics curricula and syllabi."

The same study (Boesen et al., 2018) addresses the concept of competencies and the competency-related activities explored in the Swedish national mathematics tests. They introduce three activities in their analysis to clarify the manifestation of competency activation when a person engages with mathematics at any level or manner. The first competency-related activity concerns the process of interpretation of the information related to the competencies. The second competency-related activity discusses the ‘do and use’ phases in the task-solving process. The ‘do’ phase refers to developing students’ mathematical knowledge. The ‘use’ phase refers to applying that knowledge within and outside mathematics. Finally, the ‘judge’ competency-related activity concerns the evaluating, reflecting, forming opinions, and conclusions phases on mathematics and the activities related to learning, understanding, doing, and using mathematics. The results report a general agreement between the syllabi’s competencies and the competencies that the exams test. However, the authors also concluded that the national tests “do not capture the complex nature of the competencies that can be visualized and emphasized by the [competency-related activities] in the framework used in the analysis.” (Boesen et al., 2018, p. 120).

The need to develop competence-based assessment items is also demonstrated within the Norwegian educational context (Norwegian national exam for grade 10 students). Pettersen and Braeken (2019) explore the relationship between teacher-rated mathematical competency demands of assessment items and mathematics item difficulty (from the Programme for International Student Assessment (PISA) 2012 survey) to identify empirical evidence that indicates the relevance of the mathematical competencies for solving these items. They infer that the “[r]easoning and argument and symbols and formalism” competencies “were the two prominent [out of six] competency demands” (p. 418) and that within the Norwegian exam context, these two competencies seem to influence the difficulty of the items.

Studies have also focused on exploring particular mathematical competencies rather than the general notion of mathematical competence. In their systemic literature review on mathematical modeling competency, Cevikbas et al. (2021) consider that the focus on the notion of modeling competency and its measurement became more intense at ICTMA12 (Haines et al., 2007) and continued at ICTMA14 (Kaiser et al., 2011). Concerning the assessment of mathematical modeling competency, several studies (e.g., Frejd, 2011, and 2013) indicate that any effort to assess modeling competency is a challenging task. The focus of these studies is the uneven emphasis on different aspects of mathematical modeling and whether we can obtain a holistic view when studying modeling competency assessment. Blomhøj and Hoff Kjeldsen (2006, p. 166) state that “the pedagogical idea behind identifying mathematical modelling competency as a specific competency is exactly to highlight the holistic aspect of modelling.” Efforts were also made (Hankeln et al., 2019) to measure the modeling sub-competencies of simplifying, mathematizing, interpreting, and validating as independent dimensions. By constructing test items to assess each mathematical modelling sub-competency separately, the authors claim that differences from one sub-competency to another can be identifiable, suggesting that an evaluation at a sub-competencies level is preferable to building average scores.

The 2009 PISA (Programme for International Student Assessment) assessment framework focuses on the critical competencies in reading, mathematics, and science (OECD, 2009) and influenced school policies in several ways in many countries such as Denmark, Norway, and the Netherlands (Pettersson, 2008). The international study Trends in International Mathematics and Science (TIMSS) focuses on mathematical competencies as a general educational goal (Mullis, 2005), considering that the notion of mathematical knowledge could be translated as a process of acquirement of sets of several mathematical skills.

Focusing on the connection between educational objectives and mathematical competencies offers opportunities for changes and education reforms. Research interest in mathematical competencies as an educational and reformative goal can also be found in Australia (Chubb et al., 2012) and India. There, educators, state organizations, and institutes have implemented its National Curriculum Framework in mathematics (National Council of Educational Research and Training, 2005). Norway and Sweden are two countries with long traditions of national standards. The 2006 Norwegian (Kunnskapsløftet, LK06) school curriculum was based loosely on Niss's competence model (Kjærstad, 2011). Furthermore, they have also recently implemented new standards-oriented, in terms of educational goals, toward the notion and the characteristics of mathematical competencies (e.g., OECD, 2011).

As seen above, a large body of research exploring the notion of mathematical competence and its development gained significant relevance in the last decades. However, there is a concern that student achievement and mathematical competence are seen as two almost identical concepts. Niss et al. (2012) observe that the design and methodology for several international studies on the teaching and learning of mathematics consider student achievement on written tasks the main feature of their design. The authors express their concern that "the constraint spectrum of forms of achievement" could lead to "epitomize mathematical competence at large" (p. 999) and therefore deviate the necessary attention towards the multiple dimensions of mathematical competence. What is communicated here is that student achievement is a vital facet of any study's methodological design that focuses on the learning and teaching of mathematics. However, when no other exploration of the nature of mathematical competence is considered (e.g., what characteristics a competency displays when a student activates it?), then oversimplifications occur, and the exploration of the learning process appears as a purely quantifiable procedure.

More information is needed to gain insight into the mathematics learning process and exploring the progress of competency development can be a promising path to follow. The dynamic nature of any competency offers opportunities for exploring how (progress) a competency may be activated over time (development) within a particular context (e.g., classroom) where students are asked to work on certain mathematical activities for a period of time. The following section describes the properties of a particular theoretical model that functions as an assessment instrument constructed to explore the progress of competency development and compare this progression among students who work on a particular mathematical task or series of tasks.

2.4 The 3-D Model

In this section, I describe a theoretical model that explores the progression of competency development. This model is one of the main components of the methodological tools that will be used for the facilitation of the data analysis of this study. For that reason, in this section, I present the model as described within the relevant literature, and I will revisit it in the data analysis chapter with further exploration. This tool is a three-dimensional model designed by Niss and Højgaard (2011), and in Figure 2.2, one can see a visual representation of it. This theoretical construction is a holistic approach and functions as a form of visual presentation of the possible progression in a person's possession of a mathematical competency using three dimensions: the *Degree of Coverage,* the *Radius of Action,* and the *Technical Level.* Sub-sections 2.4.1 to 2.4.3 describe the dimensions of the model.

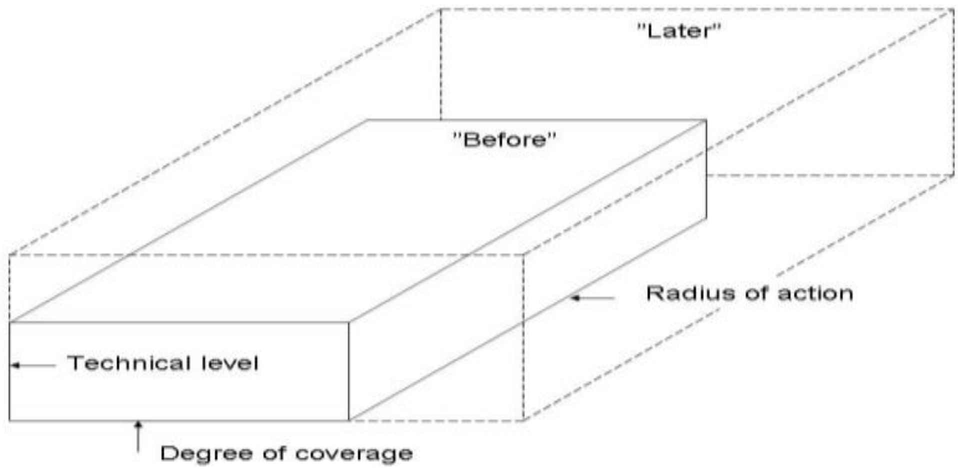

Figure 2.2. Visual representation of the three dimensions, Niss and Højgaard, 2011.

2.4.1 Degree of coverage

Niss (2003, p. 10) describes this dimension as the extent to which a person exhibits specific aspects of the competency in question. It is the breadth of a person's possession of the competency, "the extent to which the person masters the characteristic aspects of the competence at issue as indicated in the […] characterisation of it." As seen in previous sections, the KOM model includes eight primary competencies with 24 aspects. The more aspects of the competency a student exhibit, the higher the Degree of Coverage of that student's competency. Through this dimension, one seeks to understand and identify how many aspects the person can activate in various settings and to what extent independent activation occurs (Niss & Højgaard, 2011). To identify the nature of each competency's development, one can consider that if more aspects of a particular competency will be activated over time, then there is progression within the Degree of Coverage.

Within the content of mathematical modeling, Blomhøj and Jensen (2007) describe the Degree of Coverage as the dimension that monitors which parts of the modeling process students work with and their reflection level. Verschaffel et al. (2000) describe modeling as a cyclical process in which real-life problems are translated into mathematical language, solved within a symbolic system, and the solutions tested back within the real-life system. This process includes phases such as mathematizing, simplifying, and interpreting. This approach to describing the Degree of Coverage (Blomhøj & Jensen, 2007) can be expanded and include more than the characteristics of modeling competency and include the modeling competency and its aspects as a part of an extensive set of mathematical competencies. The expansion mentioned above is a theoretical elaboration by developing what Blomhøj and Jensen (2007) consider mathematical processes. Instead of monitoring just the modeling process (e.g., mathematizing, interpreting), we can investigate several aspects from a set of competencies.

Niss and Jensen (2006) provide descriptions of the 3-D model, which allow the possibility of a dynamic illustration of competency development. One should not necessarily expect a linear development of a student's competency profile but rather a dynamic activation that may provide a dynamic picture of overall competency development. In what follows, I will present examples of assessment of students' Degree of Coverage from the literature (Niss & Højgaard, 2011) for selected competencies.

<u>Some examples</u>

- A person who can often understand others' arguments but is seldom able to think out or carry out satisfactory justification himself/herself has less of a Degree of Coverage of the *reasoning competency* than the person who can more often do both.
- A person who can, clearly and in ordinary language, express the thought processes behind the solution to a mathematical problem and who is also able to explain the solution in technical terms, has a greater Degree of Coverage of the *communicating competency* than someone who is only able to do the latter.

The examples above assess which student has a higher Degree of Coverage. Using the examples mentioned above, tracking signs of the individual development of competency could be a plausible extension of this dimension's utilization. One can make the following hypothetical observation by rephrasing the examples above concerning individual competency development over time. In the beginning, the student may often be able to understand others' arguments but seldom think up or carry out a satisfactory justification himself/herself. Suppose the student can gradually improve her/his argumentation skill by providing suitable justification for himself/herself. In that case, we could claim that this student displays progress in the reasoning competency within the Degree of Coverage.

2.4.2 Radius of action

Niss (2003) interprets the Radius of Action as the spectrum of contexts and situations in which a student can activate a specific competency. As Højgaard (2007, p. 144) points out, "[t]he need for this dimension is a consequence of the contextual nature of competencies." A distinct characteristic of this dimension is the student's ability to work on routine-like situations (i.e., familiar tasks and exercises) while dealing successfully and with more unfamiliar tasks using the available mathematical tools. The latter could be an indication of a greater radius of action. The competency here is accessed through the filter of the student's ability to use symbols and formalisms (using the example of *handling mathematical symbols and formalisms* competency of the KOM framework) towards a mid-term or long-term result.

Niss and Højgaard (2019, p. 21) revisit their description of what Radius of Action represents. They point out that it signifies "the range and variety of different contexts and situations in which the individual can successfully activate the competency," suggesting that development within the Radius of Action can come with a successful competency activation across various mathematical contexts. While it remains within an interpretive area, 'successful activation' should describe this situation where the activated competency produced a mathematically legit action. For example, if a student claims that a function is continuous because it was proven to be differentiable while working on a mathematical task, then she/he provided evidence of a successful activation for the mathematical reasoning competency. The following examples provide a clearer perspective on comparing the students' Radius of Action when they work on a mathematical activity.

Some examples

One of the components of this dimension is students' handling of new situations, for example, when new symbols are introduced and used to tackle a task. We can also examine whether a student can handle situations where common and known symbols are replaced with other, less common. For instance, monitoring how a student engages with a situation where the traditional x, y, z are replaced by α, β, γ, or when simple letters like m, n, p are used to represent a whole function, in contrast to the traditional f, g, h (i.e., $m(x)$ instead of $f(x)$). Does the student lean more towards the traditional symbols, or does he show some adaptability to unfamiliar notation and formalisms? Should the student be initially able to do both, then he/she displays a greater radius of action than a student who can only work with the traditional symbols. Similarly, suppose a student would provide evidence of activation for the mathematical modeling competency, for example, within algebra, geometry, and probability theory. In that case, he or she has a greater radius of action than this student, who would provide evidence of competency activation only for algebra and geometry.

2.4.3 Technical level

Blomhøj and Jensen (2007) describe the Technical Level as a dimension that has to do with the flexibility students display when involved in problem-solving situations such as mathematical modeling activities. Niss (2003, p. 10) considers that this dimension is used to indicate "[…] how conceptually and technically advanced the entities and tools are with which the person can activate the competence." As Jensen (2007) points out, this dimension represents the size and content of the mathematical toolbox, and the levels of development of this dimension can be displayed by an example from the same study. Jensen (2007) considers that:

> Someone who can model a situation by means of establishing a functional relationship is more competent than another person who can only work with one variable 'tied up' by an equation, but less competent than someone who can also consider using differential equations. (pp. 144-145)

However, one can adapt this example by focusing on one student's development over time. Suppose the comparison mentioned in the example above can be seen as a comparison over different time points for the same student. In that case, it is possible to identify signs of progression within this dimension.

Some examples

- A person who is only able to calculate correctly in situations involving two- or three-digit whole numbers, has a lower Technical Level of his/her symbol and formalism competency than that person who can also cope with multi-digit numbers or decimals.
- At the beginning of the semester, a student may be able to sketch graphs for one-variable real functions only. If the same student will be able to design graphs of real functions of two variables or real functions with several parameters by the end of the semester, then there is development within the Technical Level of the representing competency of the student.

2.5 Summary and stating the research questions

Sections 2.2 and 2.4 gave the basis for the construction of this study's analytical framework. In this section, I provide a brief overview of the above literature review (subsection 2.5.1) highlighting critical features from the studies on mathematical competence, the assessment of competencies, and the functionality of the 3-D model that will be employed for the exploration of competency development. Consequently, subsection 2.5.2 includes the research questions of this dissertation. These questions are accompanied by a set of sub-questions that will facilitate addressing the main ones.

2.5.1 Revisiting the basic theoretical concepts

This research considers that mathematical competence can be developed depending on the fashion and the frequency of its activation. Mathematical competence for this study is seen as someone's insightful readiness (Niss & Højgaard, 2019) to take actions and appropriately address the challenges of given situations that describe any mathematical activity. By competency activation, I wish to describe this instance where a student provides evidence of bringing into action (activates) a particular competency while working on a mathematical task. For example, when a student shows evidence that he/she understands and handles the scope and limitations of a given concept, he/she activates the thinking mathematically competency of the KOM competence framework (subsection 2.2.2).

Competency development describes the situation where the person that activates a particular competency or a set of competencies gradually provides evidence that signifies some form of progress within this activation. The assessment of competencies is seen in this study as an effort to assess the development mentioned above. The relevant literature highlights that the assessment of mathematical competencies is facilitated better when employing competence frameworks that comprise a set of mathematical competencies. This literature review focused mainly on the KOM competence framework that comprises a set of mathematical competencies that characterize facets of any mathematical activity. I also drew attention to the 3-D model, which constitutes an analytical tool designed to explore the competency progress through specific dimensions (Degree of Coverage, Radius of Action, and Technical Level). These models will be used as analysis instruments for exploring the nature of competence activation and development.

Both the KOM competence framework and 3-D competence assessment model presented in this chapter will form the basis of this thesis analytical framework and will be developed in greater detail in Chapter 4. The following subsection addresses the research questions of this thesis that I aim to answer in the forthcoming chapters.

2.5.2 Research questions

Taking into consideration the concepts described in this chapter, I now present the research questions of this thesis. The questions concern the notions of competency development and competency activation. The first Research Question is accompanied with a set of four sub-questions and addresses the notion of progress of competency development over time. The second Research Question addresses the issue of competency overlapping discussed in the sections above (Section 2.2 and Section 2.3) and whether some competencies are interrelated in a particular fashion.

The first Research Question is the following: *What is the progress of individual competency development over time for a student who participates in a series of calculus sessions that comprise non-routine mathematical tasks?*

To address the general question above, four sub-questions need to be addressed. The first two sub-questions address the steps that need to be taken before tackling the main core of this question which is the progress of competency development addressed with the last two sub-questions. The four sub-questions are the following.

a. What set of competencies would be suitable to constitute the competence framework of this research?

b. How can we utilize the 3-D model to explore the progress of individual competency development over time?

c. What is the progress of individual competency development over time through each dimension of the 3-D model that will eventually form a competency development profile for each student?

d. What competencies from the competency development profile of each student display evidence of progression?

The second Research Question of this study is the following: *Are the competencies interrelated and, if so, in what fashion?*

Again, a number of sub-questions is used to address the general question above:

a. Can we create a table where all frequencies of concurrent competency activation are displayed for all possible pairs?

b. Is there a quantitative approach that could be adopted to track any statistically significant observations regarding the observed and expected frequencies of concurrently activated competency pairs?

3. Research Design

This chapter discusses the research design of this study, reflecting on the selected methodology and its relationship with the research questions of this thesis. There are numerous strategies (e.g., Crotty, 1998; Denzin & Lincoln, 2000; Flick et al., 2004; Cohen et al., 2011) for constructing the research design of a study. Crotty (1998) considers that for producing reliable and convincing inferences, a research design should comprise four elements that inform each other. These four elements are the epistemology, the theoretical perspective, the methodology, and the methods for data collection and analysis. This thesis will include these four parts with one modification. The theoretical perspective will be discussed by presenting, first, the ontological view of this thesis on competencies and competency development, and second, by discussing the adoption and adaptation of a known competence framework (Niss, 2003) employed in this thesis. Furthermore, because of the extent of the data analysis methods, a separate chapter (Chapter 4) will be dedicated to presenting approaches taken to analyze the generated data.

Section 3.1 reflects on the epistemological stance that this thesis adopts, and Section 3.2 discusses this research's ontological stance, examining in what sense competencies 'exist' in an ontological perspective. In Section 3.3, I present the competence framework that was customized and employed to facilitate the aims of this thesis. The competence framework is also summarized in a table at the end of this section. In Section 3.4, I discuss the methodology of the thesis, beginning with presenting the setting and the context of this research and concluding with a discussion on the non-routine mathematical tasks and the teaching structure selected for this project. Section 3.5 discusses the generation of data and the researcher's role as a teacher of the sessions of this project. Lastly, in Section 3.6, I reflect on the methods selected for this research design, discuss the research questions' relation to the data collection and the expected data analysis, and present some final considerations about the selected research design.

3.1 Epistemological stance

Given the research questions of this study, a qualitative approach was employed. That is because this study aims at exploring students' competency development over time while investigating their actions and their meanings when working on non-routine mathematical tasks. This section begins by briefly discussing the acquirement of knowledge from an epistemological stance and continues by presenting the author's view on the construction of knowledge.

Subjectivism, objectivism, constructivism, constructionism, realism, skepticism, and many other reality views have dominated the realm of epistemologies (Crotty, 1998; Candy, 1989; Higgs, 2001; Confrey & Kazak, 2006). There is an epistemological battle regarding the ongoing inquiry on acquiring knowledge of reality and the reliability and truth of that knowledge. It is no surprise that this 'battle' dates back many centuries. As Glasersfeld (1984) writes,

> The history of philosophy is a tangle of 'isms.' Idealism, rationalism, nominalism, realism, skepticism, and dozens more have battled with one another more or less vigorously and continuously during the 25 centuries since the first written evidence of Western thought. (…) The epistemological problem – how we acquire knowledge of reality, and how reliable and "true" that knowledge might be – occupies contemporary philosophy no less than it occupied Plato. (p. 2)

As Packer and Goicoechea (2000) point out, epistemology can be described as the systematic consideration of knowing, and when the outcome of knowing, that is, the knowledge, is valid, we should consider what counts as truth.

The author of this study considers that when individuals (the students in this research) are exposed to a learning environment, they construct knowledge and meaning of the world based on their own experiences. It is the observer (the learner) who has the leading role in the construction of the mental world, and when humans perceive their environment, they are the ones who invent it (Von Foerster, 1973). Creating a mental model of the world is a constructivist position. Constructivism conveys that the subject's mind actively constructs mental structures and processes rather than passively acquiring them. As Riegler (2012, p. 237) notices, "It [constructivism] refers to the idea that the mental world – or the experienced reality – is actively constructed or brought forward, and that the observer plays a major role in any theory." Constructivism comprises two principal schools of thought: radical constructivism and social constructivism. However, both schools' stance toward reality and truth comprises the idea that "[k]nowledge cannot be thought as a copy of an external reality and claims of truth cannot be grounded in claims about reality." (Thompson, 2014, p. 96).

Research on misconceptions, problem-solving and the conceptual development of mathematical ideas, influenced the evolution of constructivism (Confrey & Kazak, 2006), with the ideas of Piaget, Vygotsky, and von Glasersfeld being the most influencing within the field of mathematics education. Following Glasersfeld's (1987) position on knowledge being actively built up by the cognizing subject and not passively received, this thesis considers that competencies, especially competency development, describe this continuous build-up of some form of knowledge. Students' interaction with mathematical activities and other students activates the student's intuitive mathematical thinking, gradually becoming more abstract and powerful

(Clements & Battista, 2009). In tertiary education, where university students are assumed to be self-directed (Gibbs, 1992), there are numerous opportunities to develop their understanding. This thesis provided such opportunities to the students by getting them to work on mathematical tasks where mathematical thinking, problem-solving skills, and several other mathematical competencies are considered necessary to be brought into action for a successful engagement with these tasks.

Mental constructions (i.e., mathematical competencies) exist when those are activated. As mentioned in section 2.2.1, Niss and Højgaard (2011, p.49) defined competency as the “readiness to act appropriately in situations involving a certain type of mathematical challenge”. It is, therefore, necessary for the observer who aims to identify the activation of a mathematical competency or competencies, to obtain an insight into the learner’s mental world. A way to cope with this is to construct a model that fits and exposes students’ mental constructions. This model can be a framework that supports the exploration of students’ work in classroom activities and makes inferences about their knowledge and understanding of the world.

This study employed an analytical competence framework (Section 3.3) that comprises several competencies and sub-competencies (i.e., competency aspects for this study) to explore the student’s competency development. Therefore, from a constructivist approach, this framework (an adaptation of a known competence framework (Niss, 2003)) functions as a model that can describe the construction of knowledge by exploring the development of these mathematical competencies that the students will activate during their work on their assigned mathematical activities. To a certain extent, this framework can be seen as an analytical tool for conceptual analysis of mathematical thinking, reasoning, and even ideas that are developed within each student’s mathematical reality.

Several tools and constructions were created (e.g., the scaling instrument in Sections 4.1 and 4.2) or employed (i.e., the 3-D model of competence progression (Section 2.4)) during the research design and the data analysis process to facilitate the exploration of competency development. With its tools, the framework, discussed in the previous paragraph, functions as a ‘window’ that, I hold, provides a view of the learner’s mental world to the researcher. When referring to structures and mental objects, ‘constructing’ signifies a form of development, a developmental path from some initial state, rather than teleological progress towards some final state (Burman, 2007). The following section discusses the idea of knowledge acquirement through the study of the developmental process of a set of competencies.

3.2 Ontological stance: competency, development, knowledge

I begin by addressing this study’s view on the ontology of mathematical competency. I then consider the connection between competency development and knowledge, and I conclude with considerations on the existence of mental constructions (i.e., mathematical competencies) and a closing remark regarding the existence of mathematical competencies.

The ontological presupposition of this study is that the notion of mathematical competency is a mental construction. The existence of a mathematical competency can be experienced both by the competent person when acting on a situation that involves some form of mathematical activity or challenge and by the observer who aims at understanding and exploring this person’s willingness to take appropriate actions to address these situations. Therefore, this study’s position is that some form of competency activation (i.e., readiness to act) must occur to discuss mathematical competency and competency development.

This study follows the definition of competency provided by Niss and Højgaard (2011, p 49), who consider that competency is “[…] a well-informed readiness to act appropriately in situations involving a certain type of mathematical challenge.” I consider the ‘readiness to act’ information to be experienced both internal by the student and by external sources such as the teacher or the researcher. For example, the students may experience such readiness when they follow and assess a series of arguments put forward by their classmates. At the same time, the teacher can also experience this by seeing and evaluating how the students communicate their ideas while working on a mathematical task or discussing an issue that may concern them.

The above-mentioned mathematical task can be just one of these ‘situations’ that this definition describes. A mathematical activity, a group discussion regarding a newly-introduced concept, or a question asked by one student to the teacher or their fellow students can also be one of these situations in which the competent student can presumably display her/his readiness to act. An appropriate action within such situations would include reaching a solution to a mathematical problem, the advancement of a group discussion regarding a mathematical concept, or the suitable representation of a mathematical entity. As Leikin (2007 and 2014) points out, a challenge in mathematics can be any of the situations mentioned above that include an interesting and motivating mathematical difficulty for the student to overcome.

This study aims to connect the notion of competency with the process of knowledge acquisition. To be more specific, I advocate that developing a set of competencies indicates advancing some mathematical knowledge. Doing mathematics, thinking mathematically, providing mathematical reasoning, and manipulating a wide range of mathematical symbols can be knowledge itself. Competency, and especially mathematical competency, is not a stable and constant notion but a capacity for thinking and knowing that a person possesses. It is, therefore, a notion with dynamic characteristics that can be activated and possibly developed over time, provided that appropriate conditions are met.

The present study advocates the ontological view that knowledge advancement can occur when a competency or a set of competencies is activated and developed. The quality (a description of what *quality of activation* means can be found in the Glossary but will also be presented in the upcoming sections) and the frequency of competency activation (i.e., how many times a student will activate a particular competency) play a significant role for creating more opportunities to gain new knowledge or advance an existing one. A series of competency activations indicate I hold, a form of development and establish a basis for progression in a set of mathematical competencies within a person's competence profile. Enhancement -if not an improvement- of certain mathematical qualities or skills is knowledge itself because the person is learning how to learn. Learning new or better ways how to work on mathematical tasks can be a form of knowledge. For example, being able to pose questions with mathematical characteristics or articulate when providing a mathematical argumentation on a particular issue is a form of knowledge.

3.3 The study's competence framework

Incorporating an analytical framework that comprises a set of mathematical competencies facilitates identifying signs of competency activation in students' actions. This study's competence framework was mainly built on an existing framework, namely, the KOM model presented in Section 2.2. This framework includes various competencies (e.g., mathematical thinking, reasoning, and problem-solving skills) that are believed to be necessary for students who seek a STEM degree. Incorporating mathematics as part of any STEM activity is necessary to ensure that there will be consistency between the mathematics used and taught and the standards for the targeted grade level(s). As Larson (2017) points out, this consistency concerns both the content and the mathematical thinking level necessary for the mathematical activity.

The joint position statement on STEM studies (2018) from the National Council of Supervisors of Mathematics (NCSM), the National Council of Teachers of Mathematics (NCTM), and the SEFI Framework for Mathematics Curricula in Engineering Education (Alpers, 2013) constituted another inspiration for the creation of this study's framework. Both councils draw attention to the necessity of a well-designed and effective STEM program based on a strong mathematics and science component. They conclude that any STEM program should offer students (and teachers) many opportunities to use mathematical and scientific thinking, reasoning, and modeling across disciplines to work on real problems included in the STEM fields.

The present competence framework was designed as a set of competencies. Each of them includes several aspects that characterize each competency believed to be necessary as prior knowledge for a first-year biology student who will attend calculus courses. In Sections 2.1 and 2.2, I registered some of the central attempts that have been made, focusing on conceptualizing the notion of mathematical competency and creating competence frameworks that encapsulate the vast span of mathematical skills and qualities. This literature review regarding the notion of mathematical competency worked as a guide for the structure and creation of the final competence framework. However, it was not the only factor that informed the construction of this study's competence framework.

During the pilot study for this research, I decided to use the KOM model (Niss & Højgaard, 2011) in its original form to attempt an initial data analysis and test its functionality with the data I had obtained. Some initial observations highlighted the possibility of several episodes of competency overlapping. In simple words, more than one competency was activated at the same moment when a student was working on the tasks during the sessions. For example, one student *handled the scope and limitations of a given concept* (i.e., Fibonacci sequence). At the same time, she *followed and assessed the chains of arguments* put forward by her fellow students. That episode signified the activation of two mathematical competencies (i.e., thinking mathematically competency and reasoning mathematically competency) simultaneously. Niss (2003, p. 9) addresses that possibility and points out that this should not be seen as a malfunction of the KOM model, "[t]he competencies are closely related - they form a continuum of overlapping clusters - yet they are distinct in the sense that their centres of gravity are clearly delineated and disjoint." However, this overlapping phenomenon could cause a possible problem regarding the inter-coder reliability process when creating a robust framework for tracking students' competency development.

The initial eight (8) competencies (subsection 2.2.2) merged into five (5) based both on theoretical and empirical (observations from the pilot study) considerations. These five competencies are the following: Mathematical Thinking and Acting, Mathematical Modeling, Representing and Manipulating symbolic forms, Mathematical Reasoning and Communicating, and Use of Aids and Tools. I elaborate on how and why this merging occurred in subsections 3.3.1, 3.3.2, 3.3.3, 3.3.4, and 3.3.5.

In this study, I will use the terms 'competency' and 'competencies' when referring to the five competencies mentioned in the paragraph above. Each competency includes several aspects assigned, with a unique code (abbreviation) used to record the signs of competency activation when this occurs. By the end of this section, all competencies and their aspects will be grouped in a table. For the reader's convenience, I will refer to the study's framework as *competence framework*, to each of the five competencies as *competency*, and to the components of every competence as *aspects* or *codes* of the competency.

3.3.1 Mathematical Thinking and Acting competency

The KOM model offers two different competencies regarding the descriptions of actions required for mastering the modes of thought: thinking mathematically competency and posing and solving mathematical problems competency. Studies concerning several individual competencies have focused on phenomena on (or relating to) problem-solving situations (e.g., Farlow Morris & Speiser, 2010; Speiser et al., 2007). Considering the KOM model's approach and relevant theoretical approaches from the existing literature, the conceptualizations of the mathematical thinking competency and the posing and solving mathematical problems competency share some common ground. As Niss and Højgaard (2011, p. 51) point out, being able to "[…] ask and answer questions about and by means of mathematics […]" suggests that a person can pose questions and understand the available answers to the relevant question, connected to the mathematical thinking competency. As a continuation of the latter, answering such questions connects with the KOM model's problem tackling (solving mathematical problems) competency.

Mathematical thinking is not an unchanging process, especially when describing the transition to an advanced form of mathematical thinking. Tall (1992) describes advanced mathematical thinking as comprising two components: the specification of concepts using precise mathematical definitions or axioms and the logical deductions of theorems based upon them. Tall elaborates on the transition of students' thinking to an advanced level by pointing out that "[…] we should realize that the formalizing and systematizing of the mathematics is the final stage of mathematical thinking, not the total activity." (Tall, 1992, pp. 508-509). Tall's position relates to the aims of the present dissertation, where I seek ways to characterize and explore students' activities while working on mathematical tasks as they progress in their mathematical sophistication.

The focus of this study is the exploration of competency development over time. Therefore, mathematical thinking is considered a mental action in progress with dynamic characteristics, where there is no definite and final point of achievement, as discussed in the ontology section (i.e., Section 3.2). The interest here is on the nature of development (dynamic) as time progresses. Rasmussen et al. (2005), in their work in undergraduate mathematics education, avoid describing the nature of advanced thinking as a final state and claim that students (learners) develop ways of participating in mathematical activities in increasingly sophisticated ways. They use the term advanc*ing* rather than advanc*ed* because they want to address students' activity rather than a final state. Advancing suggests an ongoing process and focusing on students' development through a semester enabled the design of this research to use the element of time for the operationalization of the analysis and the exploration of competency development.

Focusing on thinking and acting within the realm of mathematics, I considered the study mentioned in the paragraph above, where the authors express their fears about neglecting the types of mathematical activities that foster and promote gradually sophisticated mathematical reasoning. They believe that as students engage in particular activities, they not only enact their understandings but also enlarge their thinking and ways of reasoning.

Furthermore, the literature (e.g., Rott et al., 2015; Schoenfeld, 1985) informs us about the close relationship between the notions of mathematical thinking, mathematical reasoning, and problem-solving situations. Therefore, it is reasonable to assume that any effort to identify distinguishable characteristics between these mathematical competencies will be feasible but rather challenging.

A student thinks and acts towards a new mathematical challenge or a problem (task) solving situation. This connection between thinking and acting resulted in merging these two competencies to facilitate the present study. By merging these two competencies, I combine aspects from both of them, and I do not disregard one in favor of the other. The students involved in the project would tackle non-routine mathematical tasks within a biological context, possibly for the first time in their student life. Mathematical thinking is a rather broad notion, and any attempt to encapsulate it within the narrow limits of a particular definition is hardly a straightforward task. Sternberg and Ben-Zeev's (2012) work on mathematical thinking includes numerous chapters where several authors were asked to focus on their approach to mathematical thinking and address a common core of issues such as the nature of mathematical thinking and locate similarities and differences from other kinds of the thinking process.

For this study, mathematical thinking and acting are explored through problem-solving situations where students engage in solving non-routine mathematical tasks. The National Council of Teachers of Mathematics (NCTM, 2000, p. 51) describes problem-solving as the situation where a person is "[…] engaging in a task for which the solution method is not known in advance." However, these tasks may require exploration (Niss & Jensen, 2002) before engaging with the solution process. Furthermore, acknowledging that certain limitations exist for a given concept is a powerful weapon for someone to solve a problem posed by others or oneself. Mathematical thinking, as a mathematical competency, could be regarded as the necessary step before one can pose or solve a mathematical problem, so merging these competencies (thinking mathematically competency and posing and solving mathematical problems competency) was considered a reasonable step.

The KOM model registers (subsection 2.2.2) the following competencies' aspects, which are not included in this study's competence framework: i) extending the scope of a concept by abstracting some of its properties; generalizing results to broader classes of objects, and ii) distinguishing between different kinds of mathematical statements. This decision (to not include these aspects in the final framework) is based on two main reasons. The first argument concerns the time the students had at their disposal to work on the tasks. Extending the scope of a concept or abstracting some of its properties while targeting to cover all four mathematical contexts (exponential growth, population, dynamics, periodic functions, and integration) could be a time-consuming process to be included in the time available. That said, I would welcome and consider, in the data analysis process, any question or comment from a student that would be relevant to the competency aspect (aspect i) mentioned in this paragraph. Secondly, this project's tasks did not include mathematical statements with conditioned assertions (if-then) and quantifier-laden statements or definitions, theorems, and conjectures as would exist in a traditional mathematical task. Therefore, I decided against including the second (aspect ii) competency aspect mentioned in this paragraph.

The following are the final aspects of the Mathematical Thinking and Acting competency:

- To pose questions that have a mathematical characteristic or to know the kinds of answers that mathematics may offer (entry). (code: P.Q.)
- To understand and handle the scope and limitations of a given concept (entry). (code: Lim.)
- To attack (take actions towards a solution) mathematical problems of various kinds. (code: →Sol.)

3.3.2 Mathematical Modeling competency

Working with non-routine mathematical tasks with first-year biology students presents an excellent opportunity to explore the development of modeling competency as one separate and distinctive mathematical quality. Modeling problems have been characterized (Hıdıroğlu et al., 2017) as non-routine and complex mathematical tasks where the students must be able to make assumptions and predictions to interpret real-world situations. It appears that the element of real-world situations is common in modeling activities and non-routine mathematical tasks.

Mathematical modeling is often portrayed as a cyclical process involving engagement in developing a (mathematical) abstraction of some form of a real situation so that a question that arises might be addressed (Wake et al., 2015). The modeling cycle presents a description of modeling activities as a step-by-step process. The first context is the real world, which includes real-world problems and situations, and the second is the milieu of mathematics. Blum and Leiβ (2007) consider that the modeling cycle informs the reader about separating the mathematical world from the extra-mathematical one. They also refer to the active parts of the modeling process being transitions between identifiable points of the real and the mathematical world. Voskoglou (2007) considers the modeling cycle a process that focuses on the transition between successive discrete stages where specific outcomes must be produced before moving from one stage to the next.

Siller and Maaβ (2009) point out that the starting point of the modeling process is an example from a real-world situation that could be procedures, scenes, or incidents in real life. A visualization (Figure 3.1) of the modeling process (following the approach by Blum 1996, p. 18) is given by Maaβ (2006), focusing on five main sub-processes that occur within the realm of mathematical modeling: *simplifying*, *mathematizing*, *working within mathematics, interpreting, and validating*. However, Maaβ points out that this illustration should not be considered a rigid algorithm that everyone should follow linearly but rather a simplified scheme that includes fundamental aspects of the modeling process. That said, some researchers (e.g., Houston &Neill, 2003; Frejd & Ärlebäck, 2011) draw attention to the difference between the mathematical and the extra-mathematical world, expressing their concern that it is not often distinct nor evident.

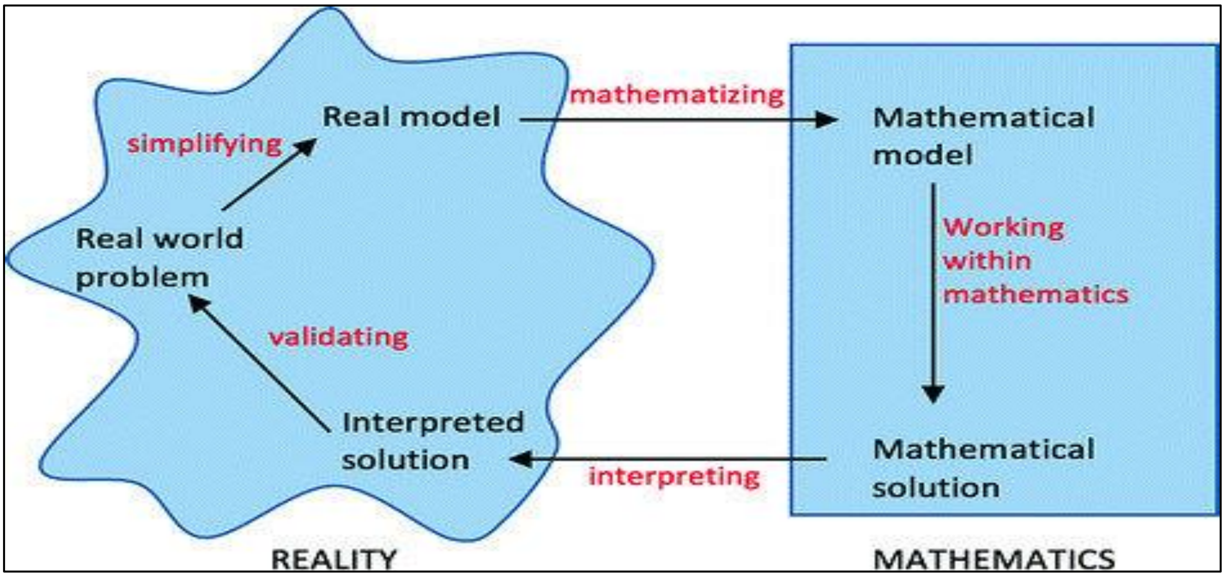

Figure 3.1. The Modeling Process (Maaβ, 2006, p. 115).

Several studies are focused on developing descriptions of mathematical modeling as an analytical tool for locating and classifying the necessary competencies involved in this process (Kaiser et al., 2006). This specific competency provides rich literature that does not embrace only the KOM project. Maaß's (2006, p. 139) work on modeling competencies influenced the conceptualization of the mathematical modeling competency of this study's competence framework. Her framework of modeling competencies included five basic categories:

i. competencies to carry out the single steps of the modeling process:
   a) Competencies to understand the real problem and to set up a model based on reality
   b) Competencies to set up a mathematical model from the real model
   c) Competencies to solve mathematical questions within this mathematical model
   d) Competencies to interpret mathematical results in a real situation
   e) Competencies to validate the solution
ii. Metacognitive modeling competencies
iii. Competencies to structure real-world problems and to work with a sense of direction for a solution.

iv. Competencies to argue about the modeling process and to write down this argumentation
v. Competencies to see the possibilities mathematics offers for the solution of real-world problems and to regard these possibilities as positive

It appears that within the modeling competency, several sub-competencies exist. While trying to specify the nature and aspects of modeling competency, Blum et al. (2002) address questions about the difference -if any- between modeling ability and modeling competency and if specific sub-skills and competencies of modeling competency can be identified. This study does not exclude the possibility that aspects of the modeling competency could reveal similarities with other competencies of the competence framework.

As illustrated in the paragraphs above, Maaß (2006) describes a whole set of competencies focused on the single steps of the modeling process. This research focuses on the notion of mathematical modeling being regarded as an ongoing procedure with several steps. As Geiger et al. (2021) point out, mathematical modeling competency is “the capacity to undertake all aspects of mathematical modelling in a holistic manner.” In conclusion, the mathematical modeling competency should portray a set of particular skills brought into action when one engages in the modeling process. As noticed before, it is what Blomhøj and Jensen (2003) describe (expanding on Niss and Højgaard’s (2011) competency definition) as someone’s insightful readiness to carry through the parts of a mathematical modeling procedure within a particular context.

A recent review of studies on the current discourse on modeling competencies (Cevikbas et al., 2021) revealed interesting observations. The authors argue that there is a need for more research on evaluating measurement instruments and approaches for implementing mathematical modeling in schools and universities. Assessment instruments were presented for measuring modeling competencies, highlighting the existence of such an instrument designed to analyze classroom modeling cultures. The authors also suggest using a particular instrument designed for analyzing the development of teachers’ noticing competencies. The importance of creativity in assessing modeling competencies was also identified in this study by highlighting the connection between fluency and originality. This review also examined various approaches to fostering modeling sub-competencies. Instructional modeling competence of in-service teachers, reading comprehension, teaching methods and styles, and practical sessions for improving preservice teachers’ pedagogical content knowledge of modeling were among these abovementioned approaches that could affect and possibly foster mathematical modeling competencies. This detailed review suggested that future research should focus on international comparative studies about modeling competencies.

Before displaying the aspects of the modeling competency of this study’s framework, some final considerations should be addressed. The participants of this project were first-year biology students with limited experience in modeling practices and tasks. For that reason, the main constituents of the modeling process, like mapping, interpreting, and criticizing (the model), even though they have been included as aspects of the modeling competency, have not been treated with absolute mathematical strictness. That said, the individual characteristics for each aspect of the modeling competency (and therefore for each code-assignment) should have been sufficiently distinct and recognizable.

Mathematical thinking and mathematical modeling competencies describe, to some extent, the cognitive process of "self-regulating learning," a term used within the context of the PISA study (Baumert et al., 2001, p. 271). Students are asked to check their solutions, chosen strategies, and methods when approaching mathematical tasks. It is a form of self-criticism towards their thinking and acting. Competency development is similar to the necessity of developing metacognition (e.g., Maaß 2006, Sjuts 2003; Baumert et al., 2001; Schoenfeld, 1992). The self-criticism mentioned above on one's modes of thought is closely related to what Sjuts (2003) describes as the notion of metacognition, where he claims that it is the process of thinking and management of one's thinking. Several studies (e.g., Sjuts 2003; Schoenfeld, 1992) advocate a connection between solving complex mathematical problems and the significance of metacognition within this process. Nevertheless, the notion of metacognition is a rather demanding cognitive 'accomplishment.' It is more easily traceable within the procedures of an expert than the one of a beginner because experts need less effort, at least timewise, to reflect on their thinking due to their experience in these situations where self-criticism is a vital component of their profession.

The process of mathematical modeling is a cognitively demanding activity (Blum, 2015) and this is due to the different aspects that belong to and define *mathematical modeling* as a competency. Some aspects are not strictly mathematical, and extra-mathematical knowledge is required. These cognitive necessities can create practical difficulties and are responsible for students' problematic use of modeling because the connection between the mathematical and the extra-mathematical world is not that distinct or evident (e.g., Houston & Neill 2003; Frejd & Ärlebäck 2011).

The literature and research focused primarily on the modeling competency, and the fact that the students worked on non-routine mathematical tasks within a biology context gave ground on the decision to consider the Mathematical Modeling competency as a discrete competence cluster.

The aspects of this competency are the following six:

- To be able to assess the range and validity of existing models. (code: R.+V.)
- To interpret and translate the elements of a model during the mapping process. In this situation, the student should be able to understand the real problem and adapt a model based on reality. (code: Int.El.)

- To interpret mathematical results in an extra-mathematical context and generalize the solutions developed for a special task-situation. (code: Int.Res.X.)
- To critique the model by reviewing or reflecting and questioning the results. (code: Cr.)
- To look and search for available information and to differentiate between relevant and irrelevant information. (code: Info)
- To represent situations graphically. (code: Graph.)

3.3.3 Representing and Manipulating symbolic forms competency

Words, phrases, and sentences are the essential elements of any spoken language that allow people to communicate. In the mathematical world's milieu, symbols, signs, and formalisms are the necessary components of the mathematical language, so those who engage in mathematics can communicate mathematical sentences, objects, theorems, axioms, and any other mathematical entity. Several forms of representations such as verbal representation, images, numeric, symbols, tables, and graphs are essential for thinking mathematically and communicating the mathematical ideas they need to represent. Using these various types of representation and translation between representations is crucial when someone attempts to express any mathematical idea (Bal, 2015). Presumably, representing (mathematically) could develop and optimize students' thinking skills since it is a process of construction and abstraction of mathematical knowledge.

The representing competency has been the focus of many studies in the last three decades (e.g., Hiebert & Carpenter, 1992; Sfard, 2008; Stylianou, 2011). As a notion, mathematical representation can be thought of as a way of capturing an abstract mathematical concept or relationship. Studies focused on developmental monitoring (e.g., Carpenter et al., 1988; Hiebert & Wearne, 1986) demonstrate students' tendency to often produce answers by manipulating and decoding symbols on paper according to memorized rules that they rarely engage in reflective activities. However, reflection is an integral part of all mathematical activities. The present research is based on the instruction of calculus through the engagement with mathematical tasks to explore competency development in a university classroom. For that reason, the abovementioned tendency has been considered while constructing the competence framework even though they are focused on the elementary level. In their study, Boesen et al. (2014) observe that the ability to represent is addressed on these occasions where the student has to consider the representation's meaning when interpreting, for example, a diagram or a table.

How a student manipulates and connects symbols and representations appears to have similarities to what Boesen et al. (2018, p. 112) describe as *connection competency*. In their study, they define this competency as: "[…] the ability to connect mathematical entities or representations of mathematical entities.” For the authors of this study, connecting represents the action of using a representation to create a link and establish a mathematical relationship. To some extent, linking representations of mathematical entities manifest a conceptual understanding of a specific mathematical notion or entity under study. Such ability appears necessary for a student who works on non-routine mathematical tasks where new mathematical entities may be introduced. As discussed in the following paragraph, connecting mathematical entities or representations of them suggests that the student can also decode or translate everyday language into a more symbolic one.

Task solving situations require a student to be able to translate and decode from natural language to a formal/symbolic one. Especially within the environment of non-routine mathematical tasks, this ability becomes even more necessary. For example, when studying the bacterial population focusing on the number of organisms added in each generation (growth rate), the student encounters expressions like “Bacteria reproduce by binary fission (splitting in half), and the time between divisions is about an hour for many bacterial species.”, “our bacterial population would have grown from 1000 to over 16 billion” or “what is the maximum population growth rate for these species who …?” These statements are expressed in natural language, which must be translated into symbolic ones. In the particular example mentioned before, some of the expected symbolic expressions that students will be asked to work with should be in the form of $\frac{dN}{dT} = r_{max} \cdot N$, for the maximum population growth where N is the population size and r or r_{max} represents the exponential growth in this case.

This form of translation, mentioned above, includes the ability to switch between representations while manipulating them. There is a connection to what Duval (2006) describes as *treatments* and *conversions*. Treatments are considered transformations of representations occurring within the same register, whereas conversions are transformations of representation that describe the change of register without changing the objects being denoted. It appears that the notions of *symbolism* and *representing* share some common characteristics. It would not be risky to assume a reliable connection between these characteristics that describe mathematical competencies of representing and handling of symbolic forms. For this study and for methodological reasons mainly (reliability of coding process), I narrowed the spectrum of mathematical competencies compared to what Niss did (2003), and I attempted a combination of the *representing mathematical entities* competency with that of *handling mathematical symbols and formalisms*, and that led me to the next step of creating a merged competency with the following aspects:

- Choose (or decode in) a representation. (code: C.D.Rep.)
- Switch between representations. (code: Rep.↔Rep.)
- Manipulate within a representation. (code: Man.Rep.)

3.3.4 Mathematical Reasoning and Communicating competency

Students often communicate their ideas within groups and the whole classroom. To do that, they need to provide justifications, explanations, and arguments to support their ideas. It is a form of communication about, with, and in mathematics. The importance of communication in the mathematics curriculum (NCTM, 2000) is well established. The literature informs us about the nature of argumentation construction (Mueller, 2009). It describes reasoning as a product in the form of sequences and the line of thought implemented to make assertions and deductions and reach conclusions in task-solving situations (Yackel & Hanna, 2003). Boesen (2014, p. 75) considers reasoning "the explicit act of justifying choices and conclusions by mathematical arguments," which is a continuous thinking process. As Lithner (2008, p. 257) points out, one can see reasoning as "[…] the line of thought adopted to produce assertions and reach conclusions in task solving [situations]." How mathematical argumentation is constructed and analyzed appeared to be an interesting research path for several researchers.

The analysis of mathematical argumentation is a core area of research within mathematics education. A prominent analytical tool used in several studies (e.g., Krummheuer, 1995; Evens & Houssart, 2004; Yackel, 2001) is Toulmin's (1958) model of analytical argument. This model includes six types of statement serving a specific role within an argument. These statements are the conclusion (C), the data (D), the warrant (D), the backing (B), the modal qualifier (Q), and the rebuttal (R). The conclusion is the statement that the one who argues aims to convince a person or an audience, the data is the evidence for the argument, and the warrant validates the data-conclusion connection. The backing statement supports the warrant with additional evidence, and the modal qualifier qualifies the conclusion by expressing degrees of confidence. Lastly, the rebuttal statement contests the conclusion by stating these conditions under which the conclusion would not be valid. However, the studies mentioned in this paragraph customized this model. They used a reduced version of it (Inglis et al., 2007), omitting the rebuttal and the modal qualifier statements because they were considered unrelated to mathematical arguments.

Reasoning sequences have the task as a starting point and the relevant solution as an ending point. Especially within the context of non-routine mathematical tasks, the literature (e.g., Öztürk et al., 2020; Jäder et al., 2017) informs us that working with this kind of mathematical problems facilitates the formation of a basis for the student to enhance their reasoning skills. As Niss and Jensen (2002) point out, there is a connection between the mathematical reasoning competency and the problem-solving and modeling competencies in the light of the reasoning definition given by the NCTM (2000) who considers that reasoning facilitates the efforts "to develop and evaluate mathematical arguments and proofs" (p. 55). Reasoning within mathematics includes discovery, organizing, explanation, verification, communication, and possible construction of a new theory (Yackel & Hanna, 2003.) In the present study, these functions within the reasoning process have been considered components of the reasoning competency.

In this thesis, communicating is considered a form of reasoning, or at least, a necessary step a student should take to reach a form of mathematical reasoning. The communication competency has also been the research focus of many recent studies, considering it a necessary learning skill and, therefore, a primary goal for mathematics education (Shein, 2012; Ryve, 2011; Truxaw & DeFranco, 2008). When considering communication as a form of addressing an individual or a party of individuals, it describes an attempt to broaden the notion of communication by considering it a general form of interaction. I avoid the description of mathematical reasoning as a mental action of high deductive-logical quality, and I approach the term with a task-solving perspective.

The act of reasoning is any attempt at argumentation and justification used to solve a mathematical task. In this direction, similar attempts have been made by Lithner (2003), who analyzes a sequence of reasoning based on a set of data comparable to those collected for the present thesis (text, pictures, video recordings). He reports that we cannot know the actual reasoning that took place in someone’s mind and that a task-solving attempt can be analyzed and classified in several ways depending on the data that manifest a form of mathematical reasoning. Bergqvist’s (2007) study suggests ways to advance mathematical reasoning. Examining Swedish university students’ reasoning in mathematics, the paper recommends ways to avoid imitative reasoning (e.g., copying algorithms or recalling facts) by giving them opportunities to learn creative reasoning. He suggests letting the students practice more on solving tasks that promote this type of reasoning.

The reasoning process that a biology student follows may differ from that of a student in a mathematics major. This difference may happen due to a -to some degree- unfamiliarity of the biology student with mathematical terms and even more when dealing with non-routine mathematical tasks. This possibility is expected to be more intense since the students were in their first semester as students appear to face difficulties in numerical problem-solving questions in high-school biology using a numeracy framework (Scott, 2015). A reasoning structure can include several distinct phases, and this is what Lithner (2004, p. 406) attempts by suggesting a four-step scheme structure of reasoning by highlighting the phenomena studied:

i. A problematic situation is met where it is not obvious how to proceed.
ii. Strategy choice: Try to choose (in a wide sense: choose, recall, construct, discover, etc.) a strategy that can solve the difficulty. This choice can be supported by predictive argumentation: Will the strategy solve the difficulty?
iii. Strategy implementation: This can be supported by verifying argumentation: Did the strategy solve the difficulty?

iv. Conclusion: A result is obtained.

Let us now stand to the other side of reasoning, to the side of the receiver, where the student follows a chain of arguments put forward by others. The focus is on the receiver's reaction when another group member attempts to share and communicate their ideas. In this case, one can study how well a student can assimilate suggestions from a student sitting next to him/her, a group working on the same task, or the teacher trying to guide the students during the session. Being able to understand others' reasoning and following and assessing their chains of arguments is considered (Niss & Højgaard, 2011) another component of the mathematical reasoning competency.

In conclusion, this study focuses on the two primary forms of mathematical reasoning and communication, and, therefore, two aspects characterize this competency:

- To understand others' written, visual or oral 'texts,' in a variety of linguistic registers, about matters having a mathematical content (incoming). To follow and assess chains of arguments, put forward by others. (code: Reason←)
- To express oneself, at various levels of theoretical and technical precision, in oral, visual or written form, about such matters (outgoing). To provide explanations or justifications to support your result or idea. (code: Reason→)

3.3.5 Use of Aids and Tools competency

The advancement of digital technologies during the last decades unavoidable entered and affected the educational milieu, influencing the way students understand and think mathematically (e.g., Noss & Hoyles, 1996; Monaghan et al., 2016). As seen in subsection 2.2.2, the KOM framework of mathematical competencies (Niss & Jensen, 2002; Niss & Højgaard, 2011) suggested introducing the making use of aids and tools competency. There are several situations where the students are asked to apply and use their mathematical knowledge and skills when engaging with digital resources (Geraniou & Jankvist, 2019), and this use is what the KOM framework describes as making use of aids and tools competency.

The use of aids and tools competency ranges from using calculators, computers, arithmetic, geometric, and computational software to using rulers, abacuses, and compasses. Someone may argue that we still deal with manipulation and find similarities with the Representing and Manipulating symbolic forms competency. However, here the focus is on aids and tools at students' disposal. Furthermore, Niss and Højgaard (2011, p. 69) point out that because there are aids that "often involves submitting to rather definite 'rules' and rests on particular mathematical assumptions, the aids, and tools competency is also linked to the symbol and formalism competency." For this thesis competence framework, the Making Use of Aids and Tools competency is taken from the KOM model unchanged with no addition or omission:

- Knowing about and being able to make use of various aids and tools (including information technology tools) that may assist mathematical activity. (code: Ext.T.)
- To be able to use aids and tools to develop insight or intuition (code: U.T.)

Table of competencies and their aspects

Table 3-1 below, illustrates the five main competencies of this thesis competence framework, including their aspects (a total of 16) with their codes (abbreviations) used during the coding process. One can notice that there are aspects that present similarities between them regarding their description. The second Research Question addresses this issue by asking which of these competency aspects appear more often than others and if the activation of one competency aspect favors or hinders the activation of another. The overlapping of the competencies and some of their aspects is not unexpected. Indeed, the findings from the second Research Question will facilitate a discussion about which aspects appear to activate simultaneously and whether refinement of the KOM model deems appropriate. Numerous examples of the coding process can be found in several tables of Chapter 4. To be more specific, Table 4-8 and Table 4-9 (subsection 4.2.1), Table 4-11 (subsection 4.2.2) and Table 4-12 (subsection 4.2.3) include several extracts from students' transcription where specific codes (competency activation) are assigned.

Table 3-1. The thesis competence framework.

Competency	Competency aspects	Codes (abbreviations)
Mathematical Thinking and Acting	To pose questions that have a mathematical characteristic or to know the kinds of answers that mathematics may offer. (1)	**P.Q.**
	To understand and handle the scope and limitations of a given concept. (2)	**Lim.**
	To (take actions towards a solution) mathematical problems of various kinds. (3)	**→Sol.**
Mathematical Modeling	To be able to assess the range and validity of existing models. (4)	**R.+V.**
	To interpret and translate the elements of a model during the mapping process. In this situation, the student should be able to understand the real problem and adapt a model based on reality. (5)	**Int.El.**
	To interpret mathematical results in an extra-mathematical context and generalize the solutions developed for a special task-situation. (6)	**Int.Res.X**

Competency	Competency aspects	Codes (abbreviations)
	To critique the model by reviewing or reflecting and questioning the results. (7)	**Cr.**
	To look and search for available information and to differentiate between relevant and irrelevant information. (8)	**Info**
Representing and Manipulating symbolic forms	To represent situations graphically. (9)	**Graph**
	To choose (or decode in) a representation. (10)	**C.D.Rep.**
	To switch between representations. (11)	**Rep↔Rep**
	To manipulate within a representation. (12)	**Man.Rep.**
Mathematical Reasoning and Communicating	To understand others' written, visual or oral 'texts', in various linguistic registers, about matters having a mathematical content. To follow and assess chains of arguments, put forward by others. (13)	**Reason←**
	To express oneself, at various levels of theoretical and technical precision, in oral, visual or written form, about such matters. To provide explanations or justifications to support your result or idea. (14)	**Reason→**
Use of Aids and Tools	To know the existence and properties of various tools and aids for mathematical activity. (15)	**Ext.T.**
	To be able to use aids and tools to develop insight or intuition. (16)	**U.T.**

3.4 Methodology

In this section, I discuss the setting and context of this study (3.4.1), followed by the presentation of the pilot study (3.4.2) that took place six months before the beginning of the project. In subsection 3.4.3, I discuss some decisions made regarding the study's design. A brief discussion on the theoretical underpinnings of what constitutes a case study is presented in subsection 3.4.4, followed by a presentation of the sessions' schedule and content (3.4.5). This section concludes with a discussion on the nature, design, and content of the non-routine mathematical tasks employed in this study in subsection 3.4.6.

3.4.1 Setting and context

This research took place in a department of Biology at a Norwegian university. For various reasons, the department was not at the same university I am employed in, but in a different city in Norway. Two Norwegian centers for excellence in higher education (CEHEs), the Centre for Research, Innovation, and Coordination of Mathematics Teaching (MatRIC) and the Centre for Excellence in Biology Education (bioCEED), developed a joint project aimed at improving biology students' motivation for, interest in, and perceived relevance of mathematics in biological studies using mathematical activities. In the following parts of this subsection, I describe the goals and context of the biology students' compulsory mathematics courses, and I continue by presenting several setting details. Following that, I discuss constraints presented during the planning and the project discussion with the department's administration. However, additional discussion about this study's constraints is discussed after the pilot study's presentation (subsection 3.3.2).

Objectives and content of mathematics courses

First-year students in the Biology department follow calculus courses as part of their mathematical training. The first course (MAT101) is offered during the autumn semester. It is an elementary introduction to one variable, exponential and trigonometric functions, differentiation and integration, vectors, simple differential equations, and extremal points for functions of two variables. An intermediate level of mathematics from Norwegian high school is considered the recommended previous knowledge.

The second course (MAT102) is a continuation of (MAT101) during the spring semester. The course deals with equation systems, determinants, matrix algebra, eigenvalues, and vectors. Furthermore, students are introduced to homogeneous linear differential equations, systems of differential equations, Population Dynamics, and functions of multiple variables. An introduction to Matlab is also given, focusing on numerical solving of algebraic and differential equations programs. Students are expected to use the Matlab program in their exercises.

Setting details

The bio students were attending the courses at the university's mathematics department facilities. Students from the chemistry and geology departments also attended the same course simultaneously. At the beginning of the courses, approximately 180-200 students were regularly in the classroom. From the information I have obtained from students and university administration members, I can confirm that many students progressively dropped out of the course. Several students gradually delivered their compulsory assignments online without attending the actual course, that is, by physical presence.

The agreement with bioCEED concerned only the biology students, and, in addition, the content of the non-routine mathematical tasks was colored within a biological context. Therefore, the interest was only in the biology students, so I decided to conduct my study and teach the calculus sessions at the facilities of the Biology department. At the beginning of the planning, no special requirement existed for the students' participation in the study. Every biology student who was taking MAT101 or MAT102 was invited. The number of students that would eventually participate would be determined by the end of the pilot study and maybe even after the first calculus session. One of the main factors that would decide this number would be the classroom's video and audio recording capability, an issue discussed in Section 3.4: Data generation.

Constraints and discussion with administration stuff

Finding a three-hour gap in the department's schedule for three consecutive hours of sessions within the academic week was challenging. By the time the project began, the Biology department already had a fixed program, which created an obstacle to locating this three-hour gap. Working with the administration and discussing with the professor responsible for teaching MAT101, we eventually found a three-hour gap every Monday from 09:00 to 12:00 for two months. The students could participate in any of the three 50-minute sessions offered: 09:00 - 09:50, 10:00 - 10:50, and 11:00 - 11:50. The aim was to have at least 50 minutes at my disposal for each session. We planned to use two large classrooms to avoid possible space problems, but it eventually proved to be more than enough.

During the first days of September 2016, I met with the mathematics professor who was teaching the calculus courses (MAT101 and MAT102), and we discussed my intentions, my project, and possible limitations and problems that may occur. I explained that I could use as many students as possible but only from the Biology department. At this point, the professor expressed his doubts about being biased when suggesting that only bio students can be part of the calculus sessions of this research in front of other departments' students. That was a reasonable argument, but it was impossible to include non-bio students because of the research center's agreement and the nature of this study. His suggestion (to include all students) would contradict the intervention's main idea, introducing non-routine mathematical tasks explicitly designed for their scientific needs.

The administration estimated 100-110 students in attendance. That said, the plan was from the beginning to focus on a significantly smaller number of students to get more accurate data from their actions and their written work in the class. It would be challenging and beyond the scope of the thesis to extract and analyze data from all students (100-110) participating in the project and even more time-consuming creating competence profiles for all these students. The main factor for determining which students would belong to the main body of the research focus would be the students' punctuality with the sessions' schedule. Because the focus is on exploring competency development over time, these students who would participate in all (or the majority of) the sessions during these two months would be preferred over those who miss several classes.

We agreed to participate in the students' official assignments in the meeting with the professor in charge. The students of MAT101 had to deliver 10 compulsory assignments almost every week. It has been agreed that for every second assignment, one of the tasks (the last in the paper sheet) that students must submit will be similar to the tasks I was teaching within the project. By the end of the semester, I had participated in 5 assignments. Eventually, data from these assignments was not used because it was impossible to confirm whether the students worked independently or received some form of assistance (e.g., using computational knowledge engines such as WolframAlpha). I shall now describe and discuss the pilot study conducted to test and evaluate the methodology and methods initially designed.

3.4.2 Pilot Study

During the spring semester of 2016, I conducted a pilot study at the same university, where the primary research project took place later. This thesis's research design was not fully formed then; therefore, this pilot study served several and less-specific purposes. Besides testing the research plan, I also needed to explore the existing conditions in the Biology department and establish personal connections with the faculty members I would work with in the future. I also wanted to obtain a relatively intuitive and rough estimation of the students' mathematical level in their first year of biology studies. However, that would be somehow problematic because the pilot students would not be the students involved in the main project and because they were in their second semester, having already attended a calculus (MAT101) course.

With my research team from MatRIC (Centre for Research, Innovation, and Coordination of Mathematics Teaching), we assembled a group of first-year students taking calculus courses (MAT102) for their spring semester. The plan was to create a three-hour calculus unit with three teachers: a mathematician, a mathematics educator, and a Ph.D. fellow (the author of this thesis). The course focused on different difficulty-level tasks and introduced real-world mathematics using the Fibonacci numbers as a central theme. By the time of this pilot study, the competence framework that would be later employed in this thesis was not entirely shaped. Therefore, I focused on general competencies by applying the KOM framework (Niss, 2003) in its original form and how the students approached their learning.

The pilot study's goal was to explore ways to identify signs of procedural or conceptual understanding before narrowing the foci to mathematical competencies and abandoning the idea of studying the differences between procedural and conceptual understanding. The idea of coding with specific mathematical competencies students' actions in the classroom took a more robust form during the pilot study. More specifically, I concluded that there was a need for a trustworthy, valid, and reliable coding method based on a customized framework of competencies. By coding, I describe the process of recognizing instances within a student's action in the class where they activate a particular competency or competencies and assigning a code that signifies that instance. As mentioned before, numerous examples of the coding process can be found in several tables of Chapter 4.

Table 3-2 below, illustrates the sessions' schedule with a short description of the students' tasks. The first column includes the time the sessions occurred, the second describes the instructor's specialty, and the third column includes the description of the tasks the students were asked to address during these sessions.

Table 3-2. Timetable with descriptions of the tasks for the pilot study.

Time	Instructor	Task(s)
10:10-11:00	Mathematics Educator	We are interested in the dynamics of the interaction between the predator foxes and the prey rabbits and investigate the long-term behavior of the two species. Since the forest is a complex ecosystem, the students will need to make assumptions to simplify the predator-prey model. Students are asked to construct a simple mathematical model based on one or several difference equations to describe the competition between the two species
11:10-12:00	Mathematician	The students are asked to suggest how the present model should be modified if a certain fixed number of prey hide from the predator in a place called refuges where the predator cannot eat the prey. How the model should be modified to describe coexistence of two herbivore species both eating grass on an island with limited area and food resources.
12:10-13:00	Ph.D. Fellow (The author of the present thesis)	Short introduction on the relationship between nature and mathematics titled: "Modeling Patterns in The Natural (and not only) World". Students asked to make a family tree of honeybees and observe number patterns.

Hand notes, provisional coding attempts, and general observations were the generated raw data material. With these initial observations and notes from this pilot study, I formed a basis for the next research steps. For the facilitation of this study, the following issues have been considered:

- Students were willing to participate in a project that combines mathematical tasks with biology but is specially designed for biology students.
- Working in groups activates discussions, question acquisition, and constructive criticism among the students (NCTM, 1989 and 2014).
- The classroom environment was suitable and could be easily monitored from multiple angles.

- The students need to see the connection of this project with their formal mathematics courses. That could offer motivation and attract more students to participate in the project.

The benefit from these observations regarding the research design of this thesis was manifold. Students' willingness and motivation to participate in the sessions would facilitate the production of important oral and written data from their work. The real-world context of every non-routine mathematical task should be apparent so that the students can value the tasks' practicality throughout their involvement with the project. An appropriate description of non-routine mathematical tasks will be given in subsection 3.4.6. Furthermore, monitoring the session from different angles would potentially capture all the critical moments during the session. For example, the teacher could write something on the whiteboard, and at the same time, the student may consider this vital information for the solution of the task at hand. This instance could suggest that the student activates a particular competency, which would be captured from at least one camera angle in the classroom.

Among the observations made during this pilot study, one concerned the coding process adopted for the data analysis of this project. The initial coding featured the possibility of concurrent activation of more than one competency. Coding as a process is an integral part of this research design. It describes the assignment of codes representing a particular mathematical competency to any student's action (verbal or written) in class. During the pilot study, these informal first attempts enhanced the argument that each competency (and, therefore, each assigned code) should be described and defined in a rather informative and analytical manner. All the above observations resulted in some decisions discussed below.

3.4.3 Decisions for the research design

This study aims at exposing first-year university biology students to a specific type of non-routine mathematical tasks to explore the development of these mathematical competencies that they bring into action. The decision to select these tasks (which will be described in an upcoming sub-section) was based on the fact that non-routine problems "do not yet have a known, routine solution strategy to the student, but which provide opportunities for the student to develop new solution strategies" (Doorman et al., 2007, p. 405), and potentially create more opportunities for the students to activate one or a set of mathematical competencies. For the exploration of competency development, the element of time is of great importance, and therefore a longitudinal design is deemed appropriate for this research. A naturalistic design might be preferred in a qualitative study such as the present. However, for the needs of this study, an interventionistic approach was preferred. This is because students were recruited to join a special class taught by the author of this thesis that suggests an "[intervention] in the classroom practises" (Gravemeijer & Prediger, 2019, p. 34) of the department.

Furthermore, it appeared appropriate that the students participating in this special class should work in groups, not individually. Two factors prompted this decision. First, because this study wants to understand and explore the development of mathematical competencies, creating opportunities to activate a large spectrum of them was vital. Working in groups, students would have the opportunity to express and address their ideas (i.e., mathematical reasoning competency, mathematical thinking competency), expose their thinking to their classmates, and understand and challenge ideas and arguments posed by others (e.g., teacher, other students). Second, the content of the selected non-routine mathematical tasks invites students to develop new approaches to problem-solving practices. In their study on the effectiveness of group work in mathematics, Sofroniou and Poutos (2016, p. 12) observed students "attempting to be critical and developing their analytical thinking" and that the collaboration with fellow students motivated them to discuss "the importance of different proposed solutions, searching for applicable problems" something that would possibly be hindered during individual work at this stage of development.

3.4.4 Case study

For the needs of this non-sampling study and the research questions addressed here, I decided to follow a qualitative case study approach because I attempt to locate these competency elements (e.g., mathematical thinking, understanding, and reasoning) within a real-life context. A qualitative case study is a research approach that explores a phenomenon within its context through various data sources, as Baxter and Jack (2008) point out. In this way, researchers can ensure that the topic under study is explored through more than one lens, creating more opportunities to reveal, discover, and understand the multiple facets of the phenomenon.

In this study, the context is non-routine tasks for biology studies, which deal with real-life problems and applications by its nature. Literature informs us that a case study approach is considered a highly suitable methodological approach for research that asks the 'hows' and 'whys' questions (Stake, 2005; Yin, 2009). This dissertation seeks patterns of expected or unanticipated interrelationships between the competencies and their development within a mathematical context where students are exposed to non-routine problems and ask if and how competency development occurs and why these possible correlations appear during the competency activation.

This research took place within a confined time frame in a specific university with a finite number of students, and, therefore, the aim was particularization rather than an overall generalization. Creswell's (2013) description of qualitative case studies' nature appears to fit my project's needs and characteristics with biology students. He portrays a model of exploring a bounded system or case over time through detailed, fine-grained data collection, including several sources of information, each with its sampling, data collection, and analysis strategies. Each 'focus group' (usually two or three students) from every classroom formed a single case study to address and value every case's uniqueness. Simultaneously, there was a supportive observation of the other groups too.

Because I followed the same research design and data generation procedures for all groups, I consider this study a multiple case study. Yin (2009, p. 18) defines a case study as: "an empirical inquiry that:

- investigates a contemporary phenomenon in depth and within its real-life context, especially when
- the boundaries between phenomenon and context are not clearly evident"

Regarding the connection between the thesis research design and the real-life context that Yin (2009) mentions, I argue that being the teacher of the calculus sessions did not affect the course's realistic characteristics. For the students, I was a lecturer in a project of additional teaching in their calculus courses. This dissertation concentrates on the dynamic nature of competency development during a series of calculus sessions within a group of students in a Biology department. Therefore, the boundaries between the setting and the case/phenomenon of study cannot be determined. More details about the research design of this thesis will follow in Section 3.6.

3.4.5 Sessions and participants

The fact that this was a two-month project led me to plan seven teaching sessions with the possibility of an additional one in case that would be necessary. The mathematical content of these sessions was determined from the content of the MAT101 course that included one-variable, exponential, and trigonometric functions, differentiation and integration, vectors, simple differential equations, and extremal points for functions of two variables, as mentioned in a previous subsection. I eventually decided that the mathematical tasks would derive from the following mathematical domains: Periodic Functions, Exponential Growth & Regression, Population Dynamics, and Integration & Modeling.

In the following parts of this subsection, I discuss an important incident during the project's beginning that reformed, to some extent, the planning of this study. I continue by presenting the sessions' schedule and some additional setting details, and I offer brief information regarding the students who formed the primary student body of this research.

A session before the first session

Before displaying more information regarding the sessions' schedule and the participants involved in the project, it is important to highlight an incident during this study's first scheduled session. As the administration confirmed, the first scheduled session was set for the 26th of September with an expected attendance of 100-110 students. However, significantly fewer students eventually attended this first session. Eventually, 15 students showed up, and an updated strategy had to be set. The main core of the students ultimately became smaller because many of the original 15 students came only to this first session. Six students attended four or more sessions of this project, and only assumptions can be made for the students' low attendance. However, it is probable that there was some communication failure between the administration and the students or that the original attendance estimations were inaccurate from the beginning.

As pointed out in subsection 3.4.1, the professor teaching the MAT101 course suggested inviting all the students attending MAT101 to the project, not just those from the Biology department. Dismissing this suggestion (for reasons explained in 3.4.1) could probably be why the professor's unwillingness to promote this project in his class led to this communication failure. That said, it is essential to state that these obstacles are not rare in any research endeavor and that there is no one to blame for this combination of events that led to this problematic situation.

The challenge described above was quickly resolved by incorporating some adaptions in the planning of the sessions. No changes regarding the selected classrooms and schedule occurred. The main core of the planning was not altered because the focus of the research was already on a smaller body of students, significantly smaller than the anticipated attendance. Naturally, one change that occurred was that there would be fewer student groups in the sessions that would have functioned as agents of interaction for the other students. Still, there were enough students to create working groups for the conduction of the sessions. Conclusively, this session on the 26th of September was not included in the data collection and analysis processes. However, it was constructive because of the information gathered regarding the actual students' participation in the project.

Due to the anticipated high attendance, I created three separate classrooms of 30-40 students, but as mentioned above, a smaller number of students eventually turned out. Nevertheless, the formation remained as planned, and there were three separate sessions (09:00-10:00, 10:00-11:00, 11:00-12:00) for around 10-15 students. There was a focus group and two or three more monitored from a distance in every session to capture the whole classroom's actions. The focus was on specific students of the focus groups, and pseudonyms were used instead of participants' real names. In the following part of this subsection, I will display a table with the sessions' schedule and some additional setting details.

Sessions' schedule

Table 3-3 displays the schedule of the calculus sessions with a description of the mathematical subject that was introduced to the students. As seen in the table, there was only one session devoted to the Periodic Functions.

Table 3-3. Table with the sessions' schedule.

Dates	Hours	Domain
3/10	09:00-12:00	Periodic Functions
10/10	09:00-12:00	Exponential Growth & Regression I
24/10	09:00-12:00	Exponential Growth & Regression II
31/10	09:00-12:00	Population Dynamics I
7/11	09:00-12:00	Population Dynamics II
14/11	09:00-12:00	Integrals & Modeling I
21/11	09:00-12:00	Integrals & Modeling II
28/11	10:00-12:00	Ending Session with Questionnaires

This research took place in the Biology department using classrooms of regular capacity that fit around 30-40 students. Seven calculus sessions were conducted with an additional one by the end of the project as an epilogue for the project. In this last session, the whole research team from MatRIC participated in a game with mathematics and biology. In the end, we distributed questionnaires to the students who participated in at least four sessions. All students who participated were Norwegians, but we decided to do the sessions in English. Below, I briefly describe these six students who participated in four or more sessions.

Students

Erika: Erika was a student very willing to participate in the project. Her mathematics background was from the second year of high-school (R1), which means that she entered the department with no mathematical training for one year. "*I only had R1 at high school (second year), but I think I got 4, so I wasn't very good, but not bad either [...] I ended up with a C at the exam for the university*."

Johannes: Johannes admitted that he faces some difficulties in mathematics - "*[...] most of the tasks seemed difficult to me since mathematics was my weak spot.*"- and especially with the definitions and the use of symbols. He attended an S1 mathematics course in his last high-school year, which is lower than the R1.

Kim: Kim was one of the students of the groups who had followed mathematics at the last high-school year too (R2). His attendance was not regular (four times), but he was very active in dealing with the tasks, "*[...] getting to understand the practical use of math.*"

Linnea: Linnea was present at all sessions, and her background in mathematics was from the second year of high-school (R1). She was very fond of the warming-up tasks because they helped her get in touch with the new concepts. Integrals and periodic functions, "*[...] just everything about trigonometry.*" was the field where she claimed that she faced difficulties in the past.

Maria: Maria had an R2 course a year before entering the Biology department, which provided her additional mathematics experience. She had never attended a mathematics class taught in English. "*[...] it was the first time I had to explain mathematics in English*." and that was a challenging task for her.

Rene: Rene was a very punctual but also a 'silent' student. He attended all sessions, and he was involved in all the activities where he sometimes had a hard time following the solution process. He mentioned that "*[...] it was difficult to find the proper info for the solution of the tasks*.". He was very eager to learn and focused on getting as much help as he could to be successful in his MAT101 exams.

3.4.6 Non-routine mathematical problems and teaching structure

Working on non-routine mathematical problems is highlighted by many researchers (Schoenfeld, 1985; Voss & Post, 1988; Ge & Land, 2003) as a meaningful way for the students to comprehend the relevance and meaning of what they learned and see how this knowledge can be transferred in authentic situations. Non-routine are these problems that reflect the relation between the real world and mathematics and concern a model of a real-life situation not readily solved by applying a formula. These problems require using specific mathematical skills such as analyzing relevant information, planning a solution strategy and determining and justifying the solution (Wathall, 2016). Furthermore, the Swedish Schools Inspectorate (2010) highlights the need to introduce to students mathematical problems that do not rely heavily on procedures to develop their problem-solving and reasoning skills.

Engaging students in non-routine mathematical tasks in any context play a key role in enhancing students' learning and understanding (Shimizu et al., 2010). The structure of the teaching sequences of the tasks is based -to some extent- on Blum and Ferri's (2009, p. 52) notion of "quality mathematics teaching." Their study, driven by empirical evidence, considers that teaching effects can be expected based on what could be referred to as *quality mathematics teaching*. In their project, they introduce a working definition as below:

- A demanding orchestration of teaching the mathematical subject matter (by giving students vast opportunities to acquire mathematical competencies and establishing connections within and outside mathematics)
- Permanent cognitive activation of the learners (by stimulating cognitive and meta-cognitive activities and fostering students' independence)
- An effective and learner-oriented classroom management (by varying methods flexibly, using time effectively, separating learning and assessment, etc.)

Inspired partly by the pilot study's observations (subsection 3.4.2), a specific time plan was constructed for every calculus session (50 minutes each session), where primary phases of a teaching sequence were included. These phases are i) initial problem statement, ii) assumptions, iii) interpretation of the model, iv) "attack" the problem, v) revising the model, and vi) conclusions. The term 'model' is used not necessarily within the well-defined context of mathematical modeling but within a more general context where a suitable equation or representation could be a model for a task that students are asked to address. However, there will be occasions where the term describes what model is within a modeling task. For example, students will need to choose between a discrete or a continuous model for population growth (e.g., $N(T) = N(0)e^{rT}$).

The other part of the design regarding the teaching sequences presents similarities with Berry and Le Masurier's (1984) study. They explore several issues that could be useful to teachers and teachers dealing with designing a mathematical modeling class setting. Even though the focus is not on teaching modeling activities, the authors' suggested teaching structure appears appropriate because it entails steps necessary for teaching non-routine mathematical tasks that ask the students to bring a real-world problem into a mathematical context. Therefore, the structure of the calculus sessions is based (to some degree) on the scheme/pattern below (I removed, though, their reference in a point system that is irrelevant to my study), constructed by Berry and Le Masurier (1984, p. 58):

1. Abstract: Include a statement of the original problem, the conclusion, and its significance. Indicate data sources.
2. Formulation: State assumptions, simplifications, and other features of the problem scenario.
3. Initial Model: Define variables, give the model and the solution. Interpret and critique the initial model.
4. Data: Explain how data was collected and its relevance. Devise a straightforward way of presenting the data.
5. Revised Model: Present revised models, explaining how they respond to criticism in stage 3. Present and criticize final model.
6. Conclusions: Summarize the outcome of the modeling assignment

Before describing the fashion the calculus sessions were designed, it is expected that not all of the steps mentioned above will necessarily be followed during the teaching of the sessions. Many factors such as time restrictions, students' attendance, and mathematical content misconceptions could possibly readjust the scheduled teaching structure. The following section discusses in detail some factors that were considered.

Structuring the calculus sessions

For the preparation of the teaching process, several factors were considered:

- The selected mathematical tasks are non-routine tasks set in a clear biological context, not modeling tasks. Therefore, some of the above steps may not be followed.
- The context of a first-year biology class: the pilot study and the first meeting with the students indicated a limited familiarity with non-routine mathematical tasks and calculus concepts (e.g., integration techniques).
- The fact that the students had just commenced their university studies: the autumn semester was their first engagement with a tertiary level of education, and it could be possible that the 'mathematical demands' within the tasks could be somehow discouraging and off-putting, especially during their first weeks in the university.
- An initial, informal, and intuitive evaluation of the students' level in mathematics.

The formation of every session consisted of four parts:

1. 10 minutes: introduction to the concept that we would study, providing the necessary formulas and definitions that students will have to use.
2. 10 minutes: assigning a warm-up task to the students to put their acquired knowledge into practical use. The task was usually not of high difficulty, so the student would not be discouraged from the beginning.
3. 20 minutes: students work on the main task. students work on the main task. This is a more demanding task in terms of difficulty; therefore, more time for thinking and writing was required.
4. 15 minutes: reflection time for the class. Discussion about the solution of the main task and the problems that came up during that process.

Some theoretical considerations

The decision to use warm-up tasks in the sessions is based on considering that students need time to adopt new knowledge. The decision mentioned before can be justified by borrowing some observations from literature related to modeling activities. Lesh et al. (2003) explore the benefits of such an approach. They focus on the teaching sequence that would probably be more valuable for the students dealing with mathematical activities, especially modeling activities. A modeling session should begin with a warm-up task/activity from their perspective. More specifically, they claim that because students use manipulatives (models to help reason about fractions in the multiple forms of representation), they should have time to play with them to get familiar with their properties and eventually learn them. In the same study, they introduced the notion of Model-Eliciting Activity (MEA), intended to create a feeling of necessity for students to solve a problem. Students can later elaborate and possibly generalize their model in this activity and often write a letter to an imaginable client detailing their solution, preferably in a generalized form. The latter offered a bridge between applied and industrial mathematics. On many occasions, a redesign of the tasks was necessary to adjust them to a more 'biological' classroom environment.

The tasks

In Table 3-4, I present the warm-up and main tasks (second and third column) introduced in the study. In the fourth column, I describe from which mathematical area was based.

Table 3-4. Sessions' tasks.

Warm up task(s)	Main task	Mathematical area
Airport Temperatures task (3.10): A table with the average monthly temperatures at an airport is given. The students are asked to produce a sinusoidal equation using the cosine and sine function. **Rabbits and Wolves task (3.10)**: Students were given two sinusoidal functions that modeled the population of rabbits and wolves and asked to interpret it orally and discuss it with their classmates. (No written work was asked).	**New Creature task (3.10)**: A team of biologists discovered a new creature. Information for the animal's temperature, through time, are given. The students are asked to provide an equation to model the creature's temperature.	**Periodic functions** Discussion on $g(x) = \mathrm{a}\cos[b(x-c)] + d$ and $g(x) = \mathrm{a}\sin[b(x-c)] + d$ functions
Barley Yield task (10.10): In 1960, an acre of barley in the U.S yielded 31.0 bushels (more than a cubic foot, about the size of a wastebasket) of grain. Students are informed about the yield growth (1.7%) and asked to estimate the approximate barley yields for years after 1960?	**Recyclables task (10.10)**: A table with the number of recyclables measured in tons in respect to a specific year, is given to the students. Students need to find at what year will recycle reach 3 million tons and to estimate the number of recyclables (tons) in a specific year.	**Exponential Growth and Regression I** Discussion on the rules of exponents and logarithms. Discussion on the exponential growth equations (e.g., $y = y_0(1+r)^x$)

Warm up task(s)	Main task	Mathematical area
Salmon task (10.10): Populations of chinook salmon along the west coast of North America have declined dramatically over the last century. The students were asked to predict when the salmons would disappear if no action would be taken. The students were given the average decline (18.1%) rate and the initial population (86.500) of them in 1967.		
Bacterial population task (24.10): A bacterial population follows geometric progression. Students asked to find generation time G. Time interval t, number of bacteria B at the beginning of a time interval, number of bacteria b at the end of the time interval and number n of generations are given	**Cancer Cells task (24.10)**: The students are assigned the following task: The very first malignant cell in the body of a patient appeared on the 1st of April 2016. Can you figure out the date (approximately) when the cluster will be detectable by x-ray?	**Exponential Growth and Regression II** Discussion on the cell's growth population curve and four phases (log phase, exponential phase, stationary phase, and death phase). Discussion on binary fission equation: $b = B \cdot 2^n$, and its elements (e.g., generation time G)
World Population task (31.10): Students asked to estimate how long would it take for a pair of individuals to produce the world population of today at	**E. Coli task (31.10)**: Reading the main theme of a science fiction thriller novel the students are asked to check if a single cell of the bacterium E. coli divides every 20 minutes, how many E. coli would be there in 24 hours.	**Population Dynamics I** Discussion on discrete and continuous models for population growth (e.g., $N(T) = N(0)e^{rT}$)

Warm up task(s)	Main task	Mathematical area
the present rate of growth r=2% per year.		
Population Doubled task (7.11): Students asked to find the date when the world's population will be doubled using any kind of information.	**Reversed Population Doubled task (7.11)**: A reverse task is given here. Assuming the age of human race (2 millions of years) students must find the excess rate.	**Population Dynamics II** Discussion on decline and increase in population. Reflecting on graphs with population curves.
Earth's Temperature task (14.11): According to one study, the temperature is rising at the rate of $0.014t^{0.04}$ degrees Fahrenheit per year, where t is the number of years since 2000. Given that the average surface temperature of the earth was 57.8 degrees Fahrenheit in 2000, predict the temperature in 2200.	**Note**: The students faced some difficulties when using integrals and I had to rearrange the day's planning by revisiting parts of integral calculus (mostly integration techniques) that resulted in having time only for the warm-up task)	**Integration & Modeling I** Discussion on integration techniques. Discussion on consumption rates and increase or decrease of rates.
Neuron Voltage task (21.11): Change in voltage V of a neuron with respect to time follows the differential equation over the course of 100 ms, where t is measured in milliseconds and v in millivolts (mV). Assume V (0) = −70 mV. What is the voltage after 100 $\boldsymbol{ms}$?	**Petroleum task (21.11)**: If the present worldwide rate of consumption increases by 2% every year, how long will it take to use up the earth's petroleum?	**Integration & Modeling II** Discussion on consumption rates and increase or decrease of rates.

3.5 Methods for data generation

This section describes the observation methods and tools applied for this study (subsection 3.5.1). In subsection 3.5.2, I discuss my involvement with the project during the calculus sessions as both the teacher and the researcher responsible for designing and selecting the non-routine mathematical tasks.

3.5.1 Monitoring the study

Traditional observation methods were followed to generate the necessary data to address the research questions of this study. These methods included video recordings of participants, the researcher's field notes, and using digital smartpens that secured students' written work data collection. A primary concern for monitoring the sessions was to find a way to capture as much information as possible. As mentioned above, one group in each classroom was the focus group, and one camera targeted them. Using a view from above, I captured all the written actions of the students of the group. More than one group in the classroom would interact, deciding to use another camera.

Before the beginning of the sessions, an issue considered was the simultaneous capturing of the teacher's involvement and groups' activities. When the teacher (author) had to write or explain something, the students interacted with each other, and these moments should have been captured in a synchronized manner. Therefore, I used a GoPro camera with a panoramic view of more than 200° facing the whiteboard and the groups. All audio recordings were obtained from the cameras after being tested during the pilot study for their quality. Studies (e.g., Spradley, 1980) have criticized this method concerning the researcher's involvement and intrusiveness due to the camera and his/her presence during data collection. In addition to the latter, "the effect of video becomes negligible in most situations after a certain phase of habituation" (Knoblauch et al., 2006, p.11), and the camera's intrusiveness gradually fades away.

Below one can see two tables (Table 3-5 and Table 3-6), with these students that came to the sessions at least four times. We can see that the central core consists of six students because we had several dropouts since there was no effective communication with the department of mathematics from the very beginning of the project. However, I maintained a core of six students who provided satisfactory data for analysis. During the sessions, we used two cameras, termed *Focus Camera* and abbreviated (F.C.), for close capture and a GoPro camera to capture the whole classroom. With this camera, capturing the student's work in great detail was feasible because it was set right on top of the students. That way, it was possible to identify what the student was writing or deleting when he/she was doing that and capture all the body gestures from a close distance. As seen in Table 3-5, only after the third session (i.e., 24/10) the Focus Camera was used for monitoring Linnea and Maria's work too. This happened because the Focus Camera was assigned to two other students who stopped coming to the sessions after the second one, and I, therefore, had to reconsider how to address this issue. For similar reasons, Kim was assigned the Focus Camera on the third session even though she eventually participated in only four sessions.

In Table 3-5, we can see the students and the dates that have attended (or not) the sessions and the kind of camera capturing them. The GoPro camera captured the whole classroom, so the F.C. abbreviation signifies that I also have a Focus Camera recording. The empty box stood for the dates when the student was absent.

Table 3-5. Student's monitoring.

Students	3/10	10/10	24/10	31/10	7/11	14/11	21/11
Rene	F.C.	F.C.		F.C.	F.C.	F.C.	F.C.
Linnea	GoPro	GoPro	F.C.	F.C.	F.C.	F.C.	F.C.
Maria	GoPro	GoPro	F.C.	F.C.	F.C.		F.C.
Erika	F.C.	F.C.	F.C.	F.C.	F.C.	F.C.	F.C.
Johannes	F.C.	F.C.	F.C.	F.C.	F.C.	F.C.	F.C.
Kim		GoPro	F.C.	F.C.	F.C.		

As mentioned above, the student's written work was essential to the data material needed for addressing the research questions. The students have been provided with the Livescribe 3 smartpens and smart notebooks, which use Bluetooth technology to send everything they write to an iPad. Therefore I was able to secure the written data without any delay. The vertical view from the Focus Camera could capture the moments when the student was erasing something in his/her notebook while, for example, the teacher was giving feedback on the whiteboard. The students who provided written work are displayed in Table 3-6. It displays which students used the LiveScribe (LScribe) pens and the relevant notebook since the data obtained through these pens was transferred directly to an iPad device in PDF format.

Table 3-6. Written data collection.

Students	3/10	10/10	24/10	31/10	7/11	14/11	21/11
Rene	LScribe	LScribe		LScribe	LScribe	LScribe	LScribe
Linnea			LScribe	LScribe	LScribe	LScribe	LScribe
Maria			LScribe	LScribe	LScribe		LScribe
Erika	LScribe	LScribe	LScribe	LScribe	LScribe	LScribe	LScribe
Johannes	LScribe	LScribe	LScribe	LScribe	LScribe	LScribe	LScribe
Kim		LScribe	LScribe	LScribe	LScribe		

3.5.2 Researcher's role

This subsection briefly discusses my involvement with the project during this intervention. Being both the researcher and the teacher of the sessions created a situation with certain advantages and possible shortcomings. In this dissertation, I will frequently refer to myself as the "teacher." The literature (e.g., Spradley, 1980; Gall et al., 1996) informs us that the researcher's role while observing participants (students in my case) varies according to their kind and level of involvement in the observed process. This deviation ranges from being a complete observer to being a complete participant.

My participation included discussion and suggestions with the students regarding the tasks, but there was an effort not to provide direct instructions concerning the solution. The focus was on prompting students to detail further their thinking (when needed), even if the student(s) may not have reached an outcome for the task. That said, the level of prompting was not always the same. There were instances where the students could activate a particular competency aspect with minimum assistance from the teacher or need continuous help. As I will explain in detail in subsection 4.1.7, the teacher-student interplay was considered by exploring the level of prompting and how independently the students performed their actions in the class.

3.6 Reflecting: research questions and design considerations

In this last section of the chapter, I revisit the research questions of this study in subsection 3.6.1, offering a reflection on the data sets generated to address these questions. The following subsection (3.6.2) discusses some additional considerations regarding the research design in the light of what has been discussed and presented in this chapter. Finally, this section ends with a brief summary of this chapter in subsection 3.6.3.

3.6.1 Reflection on data and research questions

In subsection 2.5.2, the two research questions of this thesis were stated. These questions were the following:

1. What is the progress of individual competency development over time for a student who participates in a series of calculus sessions that comprise non-routine mathematical tasks?
2. Are the competencies interrelated and, if so, in what fashion?

As mentioned in subsection 3.4.4, this research is a case study that seeks to explore a phenomenon within its context through various data sources, as Baxter and Jack (2008) point out. The exploration concerns the progress of individual competency development over time, as the first Research Question of this study indicates. The term progress here describes the process of tracking and documenting the course of competency development over time and does not necessarily imply advancement. The factors that shaped this thesis's design were i) the focus on each student as a single subject of study and ii) the notion of time. Section 2.5, I presents a set of sub-questions that facilitate exploring the main two research questions.

As illustrated in Table 3-7 below, each sub-question addresses a particular part of the exploration. The first column of the table indicates the particular sub-question, and the second displays the data needed to address it. The third column describes the expected data analysis incorporated to address the sub-question. Finally, the fourth column displays the expected product after the expected analysis. The order of the sub-questions is not random. Every expected product obtained from the data analysis of the relevant sub-question is used for addressing the next sub-question. For example, only after creating the updated version of the competence framework (the thesis framework) can the coding of all the sessions' observations follow. The coding results from observations of all sessions will also facilitate exploring the second Research Question, but the analysis will follow a different path.

Table 3-7. Data and research questions.

Statements of Research Questions	Data needed	Expected data analysis	Expected product
RQ 1.1 What set of competencies would be suitable to constitute the competence framework of this research?	Oral and written data from the first three sessions	Coding based on KOM competence model	Updated competence framework
RQ 1.2 How can we utilize the 3-D model to explore the progress of individual competency development over time?	Oral and written data from all sessions	Coding based on updated (thesis) competence framework	Methodological tools for exploring the progress of competency development
RQ 1.3 What is the progress of individual competency development over time through each dimension of the 3-D model that will eventually form a competency development profile for each student?	Expected product from data analysis for RQ 1.2	Using the expected product for analysis of each one dimension of the 3-D model	Summaries of observations and evidence for each competency aspect
RQ 1.4 What competencies from the competency development profile of each student display evidence of progression?	Summaries of observations and evidence from RQ 1.3	Categorization of these summaries by the competency of the competence framework	Table of assessment of progress; individual competence profile
RQ 2.1 Can we create a table where all frequencies of concurrent competency activation are displayed for all possible pairs?	Coding results from observations of all sessions	Documenting these instances where a student activates two competency aspects concurrently	Table of frequencies of concurrent activation by pair of competency aspects
RQ 2.2 Is there a quantitative approach that could be adopted to track any statistically significant observations regarding the observed and expected frequencies of concurrently activated competency pairs?	Expected product from data analysis of RQ 2.1	Run the binomial test of significance	Documentation of these competency aspects pairs with higher (or lower) than expected concurrent activation frequency

3.6.2 Final considerations about the research design

The study of individual competency development focuses on the notion of progression and changes over time and comprises dynamic characteristics. The single-subject research provides data on single individuals, such as the exploration of learning development, where “a person-by-person analysis is needed.” (Creswell, 2002, p. 317). The changes over time should concern some areas of development that can provide us with information about the progress of competency development. Subsection 4.1.7 will address a particular set of such areas of development. To better communicate the results of the data analysis, I considered the importance of the appropriate visualization of such results. The selected visualization should communicate in a helpful way any evidence entailed there. In subsection 4.2.3, I introduce and describe the selected visualization for this thesis. The following paragraphs of this subsection discuss similar studies that resemble the research design of this dissertation. I also address the similarities with a particular type of research design.

Within the realm of research designs, experimental research has a long tradition in various fields, and during the 90s, several studies (e.g., Bausell, 1994; Boruch; 1998; Reichardt & Mark, 1998) focus on experimental design for the conduction of meaningful research designs on planning and evaluation. Experimental is intervention research where one focuses on explaining whether an intervention influences an outcome. Shadish et al. (2002) advance the discussions on experimental designs and establish the basic designs, the visual representation, the notation, the limitations (threats to designs), and the statistical procedures of educational experiments. Creswell (2002, p. 307) suggests that although experiments have common characteristics, “their use and applications vary depending on the type of design used.” The same study suggests two main designs for educational research, namely Between Group Designs and Within Group or Individual Designs. The following paragraph discusses the latter design type.

Within Group or Individual Designs include time-series experiments (interrupted, equivalent), repeated measures experiments, or single-subject experiments. This study aims to explore the progress of competency development for two students and provide summarized findings for four more. For that reason, I believe the Within Group or Individual design characteristics, especially the Single-Subject Design, bear close similarities with the characteristics of this thesis design. I will elaborate further on these similarities. The Single-Subject Design comprises the situation where the researcher “[…] assume[s] that you [he] seek[s] to learn about the behavior of single individuals rather than groups.” (Creswell, 2002, p. 316). This thesis seeks to investigate individual rather than group competency development. Furthermore, as Creswell (2002) points out, this type of design facilitates observation of development over time, which this dissertation aims to explore because the focus is on the dynamic nature of competency development.

As mentioned before, this study follows a general qualitative approach. Lincoln and Guba (1985) consider qualitative investigation a research method that relies on collecting, analyzing, and interpreting non-numeric data that occur in a given context. This qualitative method was conducted with the addition of a particular quantitative method with specific characteristics and tools to address a particular part of the research questions. That is because this investigation explores these instances of concurrent activation of two competencies simultaneously and if the activation of a particular competency facilitates or hinders the activation of another competency. Chapter 4 discusses the methods for data analysis, and I introduce a scaling instrument that categorizes the quality of competency activation in different development levels through three different domains. These levels are in ordinal form with individual qualitative characteristics.

In Sections 5.7 and 6.2, I address the second Research Question of this study, applying a particular statistical test. By incorporating an ordinal qualitative measurement to explore the progress of competency development (first Research Question) and a statistical test for the study of competencies interrelation (second Research Question), I incorporate a qualitative method enriched with quantitative characteristics. In conclusion, to address the epistemological stance that informed this research, I first provide a review of the relevant literature's views concerning the blend of these two methods.

There is an ongoing discussion between researchers, educators, and policymakers concerning the most appropriate research approach to use for research in mathematics education. Ross and Onwuegbuzie (2012) point out that this debate led to a research identity crisis. This discussion addressed the possibility of combining qualitative and quantitative methods in mathematics education. However, skepticism regarding the implementation of quantitative methods in mathematics education research is less intense in our days than in the past. Those who disapproved of these techniques argued that a quantitative approach could not address questions of what works. Indeed, over 50 years ago, Scandura (1967) promoted and encouraged the idea of a journal devoted solely to mathematics education research. He writes,

> Many thoughtful people are critical of the quality of research in mathematics education. They look at tables of statistical data and they say, "So what!" They feel that vital questions go unanswered while means, standard deviations, and t-tests pile up. (p. iii)

Lester and Lambdin (2003) note that during the 1960s and 1970s, experimental and quasi-experimental techniques in mathematics education research were criticized.

Research since 2000 has increasingly appropriated mixed methods research. Members of the National Research Council (NRC) entered the conversation regarding the using and applying mixed methods by publishing a consensus statement, Scientific Research in Education (NRC, 2002). They encouraged and supported the utilization of mixed methods research. Eisenhart and Towne (2003) refer to the NRC report and point out that it supports the inclusion of several research designs (e.g., experimental, case study, survey) and the implementation of qualitative and quantitative methods. Finally, it is reasonable to assume that any research design (including the present) depends on the research questions under investigation. Nevertheless, there is no claim that the present thesis follows a mixed-method research path. The following sub-section summarizes what was discussed in this chapter.

3.6.3 Summary of Chapter 3

In this chapter, there was a presentation of issues that concerned the research design of this study. The present research design comprises four elements that inform each other: epistemology, theoretical perspective, methodology, and methods for data collection and analysis. The notion of mathematical competency and competency development was considered from an ontological and epistemological stance. Following that, the analytical competence framework of this thesis was presented and explored. These considerations were followed by a thorough presentation of the methodological issues of this thesis. Setting and context details were discussed, and information about the pilot study and the decisions that followed it were presented. An important note from the first meeting with the students was also discussed. This meeting, which was initially planned to be the first calculus session of the intervention, was facilitated as an indicator regarding the students' actual attendance and what would be the main core of students that would be observed. Through this observation, the data of this study would be generated.

The last part of this chapter discussed issues regarding the methods adopted for this study's data generation. Tables with information about the students' type of data were displayed, providing information about what type of camera was used to monitor them and whether written data was also selected from them using a particular digital tool. This discussion was followed by considering the researcher's role as the researcher and the teacher of the calculus sessions. In concluding this chapter, I revisited the research questions of this study in a table form that illustrates the data needed to address these questions, the expected data analysis that will be followed, and the expected product of such an analysis. The following chapter addresses the data analysis process followed in this thesis. Because the length of this part of the thesis is substantial, I decided to dedicate one chapter exclusively to describing and analyzing the methods of analysis and several issues regarding the utilization of analysis tools used to facilitate this research.

4. Data Analysis

This chapter discusses the methods employed to analyze the data material generated for this thesis. This material includes the transcribed texts from all the discussions during the sessions (from the video and audio recordings) and the students' written work during these sessions. I begin with a prologue that will include a visualization of the analysis process and summarize all the steps.

Section 4.1 addresses the methods used to analyze the obtained data, focusing on the transcription and coding, and constructing a scaling instrument to facilitate this analysis. Section 4.2 is a continuation of the previous section, where I discuss the functionality of the scaling instrument providing examples from extracts from students' discussions in the classroom while working on the non-routine mathematical tasks. In Section 4.3, I describe how the 3-D model of competency progression will be utilized to provide information about the progress of competency development. Sections 4.2 and 4.3 introduce three types of evidence that will facilitate the exploration of students' competency development over time. Section 4.4 describes the method adopted to analyze competencies interrelation, presenting the Binomial test for two proportions. Lastly, sections 4.5 and 4.6 discuss the validity and reliability of the thesis and the ethical issues considered for the conduction of this research.

Prologue of the analysis

The following sections describe the tools and analytical frameworks incorporated to facilitate the aims of this research. To explore the development of competencies that students bring into action (activating the competencies), some questions needed to be addressed: How do we identify that a student activates a particular competency? In what ways can a competency be developed? Are there different levels of competency development, and how can we determine such levels? To address these challenging questions, it appeared necessary to incorporate a certain level of complexity in the analysis process. However, before delving into the description of all the analysis components, I wish to summarize and visualize the data analysis process for the reader's convenience.

To visualize a summarized version of the analysis process, it is necessary to briefly introduce some terms and constructions that will be defined and described in the following sections. The analysis begins by incorporating an analytical competence framework comprising a set of competencies consisting of several components, termed *competency aspects*. This framework will be used to identify which competencies the students bring into action (coding process of the transcribed oral and written students' work) while engaging with the mathematical tasks of the calculus sections. As it will be displayed, the competence framework will comprise five mathematical competencies, and each competency will include several competency aspects.

In this study, each competency aspect activation (i.e., when a student provides evidence of activating a competency) is explored through three different areas of activation and development:

1. The students' awareness of the steps that need to be followed for the correct task solution (at the moment they activated the competency), which will be termed *Task Solving Vision* abbreviated (T.S.V.).
2. The level of articulation (mathematical vocabulary and language) that the students bring when they provide evidence of competency activation, which will be termed *Mathematical Language and Vocabulary* abbreviated (M.L.V.).
3. How much help did they need to bring into action the competency, the level of independence when engaging with the tasks, which will be termed as *Prompting and Independent Thinking* abbreviated (Pr.In.T.).

For this research, these three areas of development are now termed *domains of competency activation*, and their abbreviations will be used thoroughly in the upcoming sections and chapters.

A qualitative scaling instrument will be constructed and used to evaluate the domains (discussed in the previous paragraph), which will lead to evaluate the quality of competency activation (an accurate description of quality will be presented in the upcoming sections). This instrument will foster the evaluation of the competency activation level (quality of activation) through these three domains. As it will be shown, there will be three levels of competency activation: Initial (I), Intermediate (II), and Mature (III). Each observed competency activation will be evaluated, and the evaluation will concern these three domains (T.S.V., M.L.V., and Pr.In.T.). The following paragraph presents a brief example of what was discussed in the two paragraphs above.

The *posing questions* competency aspect of the *thinking mathematically* competency of the KOM model (see subsection 2.2.2) may be observed in 15 occurrences within a student's oral or written work. Each of these 15 occurrences will be evaluated in terms of their activation quality through the T.S.V., M.L.V., and Pr.In.T., domains of competency activation. This will result in a total of 45 evaluations. For example, the evaluations of the 12th activation (occurrence) could be the following:

- for the T.S.V. domain, the Initial level (I) is assigned
- for the M.L.V., the Mature level (III) is assigned
- for the Pr.In.T., the Intermediate level (II) is assigned.

The 12th activation can be symbolized as (I, III, II), and one can assume that there will be 14 similar but not identical assessments symbolized as seen here.

These evaluations will be translated in a diagrammatical form accompanied by a table that will illustrate several types of evidence. Therefore, a corresponding diagram and evaluation table will be provided for each competency aspect of the competence framework. Finally, the three-dimensional model (see section 2.4), designed by Niss and Højgaard (2011), will be applied to determine the students' competency development in these calculus sessions. Thus, while the scaling instrument determines the quality of competency activation, the 3-D model for competency progression will determine the quality of competency development. Figure 4.1 summarizes in a visual manner what has been discussed in this prologue, beginning from the generated data and then the analysis phases of this process that will eventually produce inferences about the nature of students' competency development.

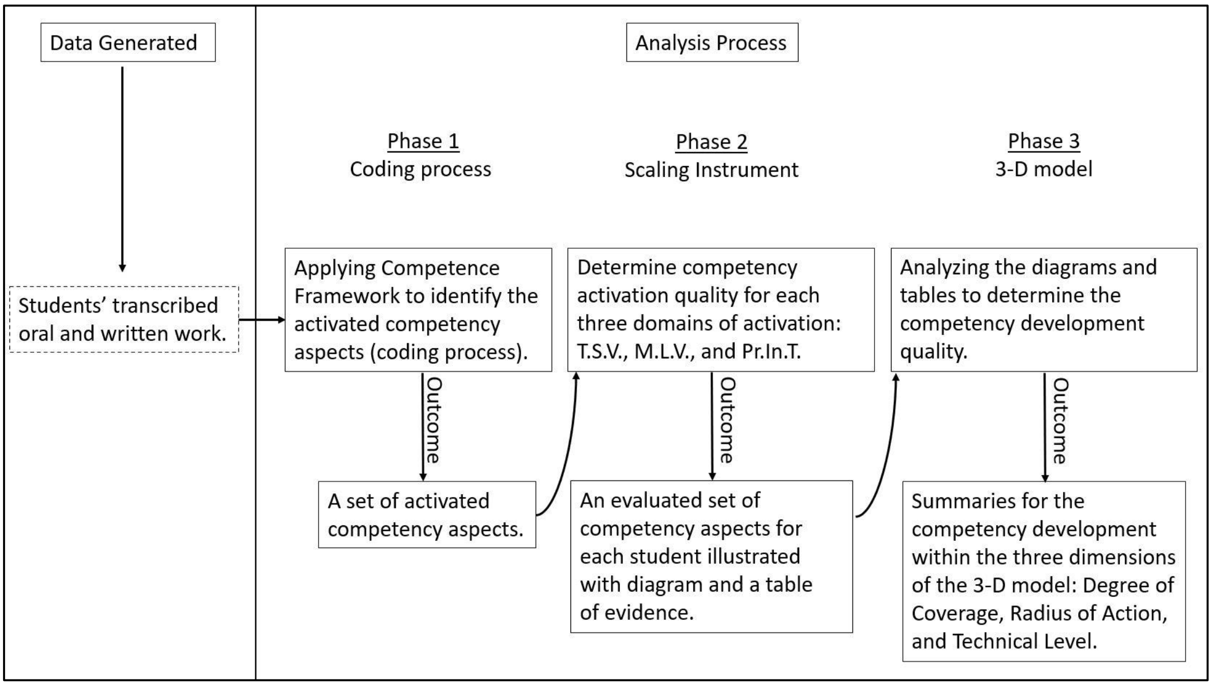

Figure 4.1. Visualization of analysis process.

4.1 Methods for data analysis

In this section, I describe the methods implemented to analyze the obtained data. Subsection 4.1.1 discusses the transcription of the data, and subsection 4.1.2 the coding process adopted for this study. Subsection 4.1.3 introduces and discusses the concept of *quality of competency activation* and the considerations for exploring the progress of competency development. This discussion will include a presentation of examples to elaborate on the quality of competency activation. Next, in subsection 4.1.4, the considerations and the origins of the construction of the scaling instrument are presented. In subsection 4.1.5, I discuss the use of rubrics for constructing the scaling instrument that facilitated the exploration of competency development. Subsections 4.1.6 and 4.1.7 include a detailed presentation of the scaling instrument's characteristics that will constitute vital analytical tools for this thesis. The final subsection (4.1.8) presents the scaling instrument in the form of a concise table.

In this thesis, I will often use the term 'fine-grained analysis' to describe the method I implemented to perform my analysis. For that reason, a suitable definition of this term deems necessary. I use this term to describe the attention in detail to create several constructs (concepts) throughout the thesis.

Introducing newborn concepts is always a dangerous and challenging endeavor in any form of writing, and for that reason, a solid foundation is more than necessary. Lastly, the initial indications and observations collected from the first sessions of this study led to the construction of the whole analytical apparatus that will be presented in the present chapter. Students' attendance, difficulties, problems they faced while working with the tasks, and time allocation issues were among these indications. However, the essential factor that determined and resulted in the construction of the analytical tools presented in this chapter was the observations from the initial coding process (subsection 4.1.2) applied to identify the competency activations.

4.1.1 Data transcription

For this research analysis, I started with an initial hand-written data analysis to maintain simultaneous control of the students' oral and written actions and understand how the coding will be performed. Figures of this initial hand-written data analysis draft (Figure 10.6 and Figure 10.7) can be found in Appendix G (the figures' quality is not optimal) for the data analysis planning. The idea was to have the student's written work on one single page, the transcription of the part that I wanted to code, and the specific session's tasks. Also, I wanted to connect the timing of a specific moment where a code (competency) has been activated with the specific part of the student's written work. Every student was participating in a group discussion, but at the same time, they had to write their solution to their notebook. Therefore, for the first transcription step, I used A1 papers for the written data to include the task, pictures, tables, and parts of discussions targeted to a specific part of the task solution process. In this way, it was easier -visually-to assign the codes without losing the chronological order. The benefits of this method were several and especially when:

- The students were referring to a specific part of the task.
- The students posed a question while pointing out a particular part of a graph, equation, number.
- The students were connecting various parts of the task in different moments.
- The students were writing in their notebooks and commenting at the same time.

At the same time, all discussions and comments had been transcribed in a standard A4 paper. Later, I used Microsoft Word and Microsoft Excel to transfer and analyze the hand-written data to a digital form. The parts where the students have been talking Norwegian have been translated into English by the author of this thesis. Two video views (Figure 4.2) are provided. While transcribing, I used two laptops playing the videos from the same session simultaneously but from different angles to capture as much information as possible. The left image in Figure 4.2 displays the class view from the GoPro camera, and the right image illustrates the view from the Focus Camera where a much more detailed observation was possible.

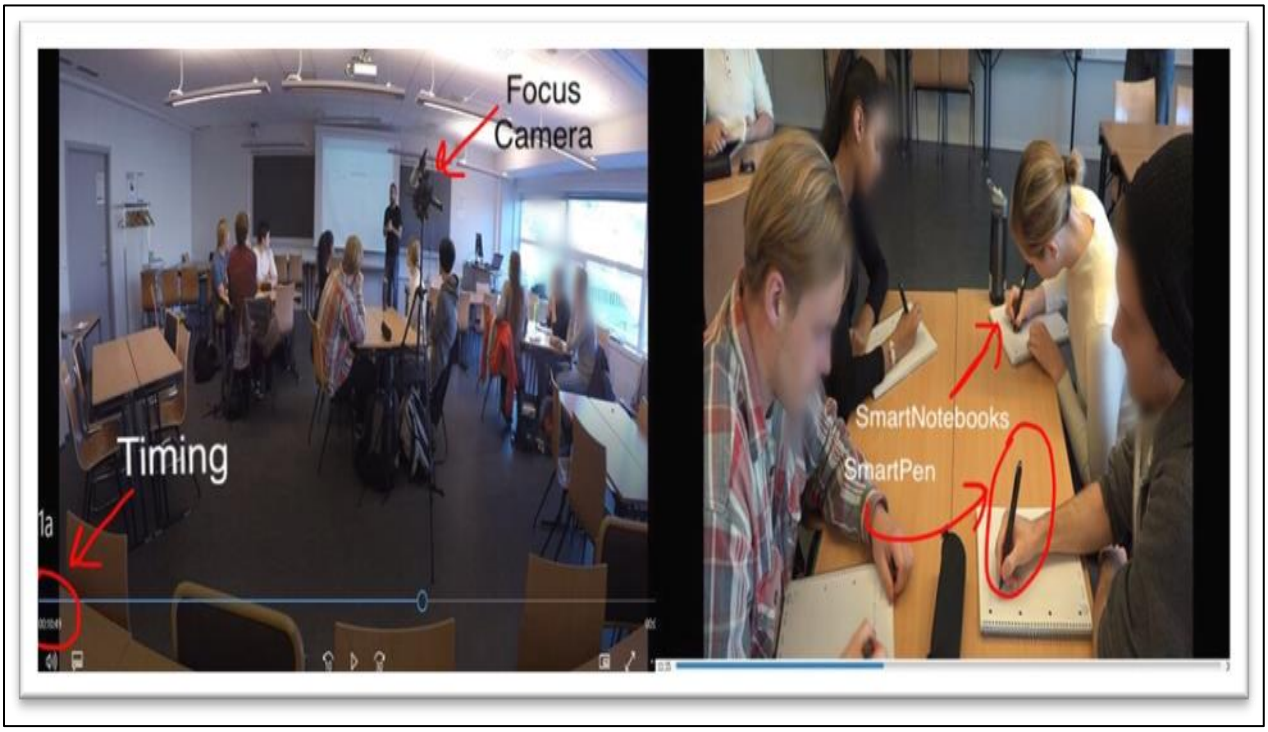

Figure 4.2. Screenshots from double-screen data analysis.

4.1.2 The coding process

A core element of the data analysis process was identifying evidence that would signify the activation of a particular competency in any student action performed in the class. These student actions include statements, comments, discussions between students or the teacher, written work, and body gestures (e.g., showing something on the board, nodding, pointing at a particular part of a text). This process constituted the coding process of this thesis. It is, in fact, a form of thematic analysis applied in a set of seven transcribed texts for each student that represent each of the seven calculus sessions.

As seen in Section 3.3, where the competence framework of the thesis was explored, a set of codes (abbreviations) was assigned to each competency aspect. Using the description for each aspect, I aimed to identify these instances where the students' actions would suggest that the student brought into action one or more competency aspects. I would assign the relevant code or codes to the transcribed text when this identification was complete. By the end of this process, a competency activation student account was available where one could see what competency aspects were activated, where (i.e., in which calculus session), and how often (i.e., frequency of activation) for each session and in the total of the seven sessions.

Table 4-1 below shows a transcription extract from the fourth calculus session: Population Dynamics I. In particular, this extract is from the main task the students were assigned to work on and concerned the population of a malicious bacterium. Reading the main theme of a science fiction thriller novel, the students were asked to check if a single cell of the bacterium E. coli divides every 20 minutes and how many E. coli would be there in 24 hours. They were also asked to check if the author's claim that "in a single day, one cell of E. coli could produce a super-colony equal in size and weight to the entire planet Earth" was accurate. The first column of the table shows the turn of the transcribed text, and the second column displays the utterance (i.e., the verbatim transcription) or the description of the student's written work or gesture. The third column includes the codes assigned to the second column's content. The fourth and last column comprises the context of the utterance under analysis and the rationale for the coding, explaining why the particular codes were assigned to this specific student action. In bold text, one can see that the language used to justify the coding resembles the language used in Section 3.3 to describe the competency aspects of the competence framework.

Table 4-1. Extract from transcription with the code assignment.

Turn	**Speaker:** Utterance [gesture or written work]	Code	Coding Rationale
1	**K:** Shall we see how many 20 minutes we have in 24 hours? There are 72 sets of 20 minutes in 24 hours we will use the same model here because we have similar information	**P.Q.** **Reason→** **→Sol.**	The student **posed a question that had a mathematical characteristic**. She then **justified and supported her idea** and at the same time **took actions towards the solution** of this part of the E. Coli task.
2	**M:** The starting point [looks at her notes] is one bacterium as we .	**Info** **Reason→**	The student **searched for available information** in his notes to infer what the starting point would be and **expressed herself with some theoretical and technical precision, supporting her idea** based on her previous notes.
3	**K:** The increase is 100% since it is double all the time	**Reason→**	The student **provided argumentation to support the result** she found.
4	**L:** What then should be the right rate?	**P.Q.** **Reason←**	The student **posed a question that had a mathematical characteristic** and also **reacted** to K's **oral 'text' about a matter that had mathematical content**.
5	**M:** So, we have the time which is 24 hours, but we don't have the rate. At first, we have to find how many E. Coli will be in 24 hours ... maybe it means that they will be divided 72 times, 72 doublings so maybe2^{72}?	**Lim.** **Int.El.** **Reason→** **→Sol.** **P.Q.**	The student **understood the limitation of the** 'rate' **concept** and **provided her reasoning** about how to proceed and **take actions towards the solution** of the task. She ended her observation by **posing a question that had a mathematical characteristic**.
6	**K:** Is it 72^2 or 2^{72} [checks calculator] seems more right the second-choice even though it is a big number?	**P.Q.** **Graph** **U.T.**	The student **posed a question that had a mathematical characteristic.** She **represented the expression graphically** in her calculator which she was **efficient in using it** and creating graphs.

Some observations can be made in these six turns of Table 4-1. First, one student's utterance or written work or gesture can provide evidence for identifying more than one competency aspect activation, and this is expected as the competency overlap is predicted (see also Section 3.3) by the theoretical construction of the competence framework. Second, to describe the rationale for assigning a particular code to the utterance under examination, I did not necessarily use the complete description of that aspect as offered in Section 3.3. For example, in turns 1 and 4, the justification for assigning the code (P.Q.) was that the student posed a question that had a mathematical characteristic. However, the complete description of this aspect is *'To pose questions that have a mathematical characteristic or to know the kinds of answers that mathematics may offer.'* However, the second part of this description has little relevance to these particular utterances. It is therefore expected that it is not necessary to use all the features of the competency aspect description to assign a code during the coding process. Lastly, it must be noted that the way the 'turns' are determined is relatively straightforward. These occasions where the student was interrupted by the teacher or a fellow student or finished his/her statement were considered a complete turn.

As noted at the beginning of this section, all these observations significantly determined the construction of this thesis's whole analytical apparatus. A discussion on the reliability of the coding process can be found in Section 4.6 and Appendix F. The following subsection discusses why the coding process does not provide all the information needed for exploring competency development over time and highlights the necessity of considering the construction of a new concept which will be termed as *quality of competency activation*.

4.1.3 Quality of competency activation

One of the core endeavors of this research approach is to explore the progress of competency development over time. This exploration comprises the assessment of the *quality of competency activation* under study. Presumably, this quality may vary from student to student and session to session, and to clarify what quality of activation describes, an example will follow. A student may pose a question (P.Q., competency aspect '1' from the competence framework, Table 3-1) that has a mathematical characteristic using an articulated mathematical vocabulary, or he/she may use everyday expressions or gestures with limited or no mathematical language. In both situations, the (P.Q.) competency is activated, but this activation's quality is relatively different. The latter example foreshadows the need to develop a way to identify and explore this qualitative difference regarding different competency activation levels. Exploring the progress of competency development comprises assessing this quality of competency activation over time. The notion of qualitative differences in mathematical competencies has been addressed before, as shown in the following paragraph.

In their study, Siller et al. (2015) present a content and action-related competency level model to facilitate the evaluation and comparability of mathematics examination questions. This model is a competency four-level grid that comprises three domains of mathematical competencies that describe three mathematical operations, namely, Operating, Modelling, and Reasoning (O-M-A), believed to describe the main facets of the mathematical work at school. The authors point out that “the levels of the competency model postulate what skills are needed to solve them [the tasks]” (Siller et al., 2015, p. 2718) and consider that the description of these competency levels should be based on a hierarchical structure of cognitive actions. By identifying these levels within these three domains, they explore qualitative differences for each competency. The latter adds another argument to the belief that mathematical competencies possess dynamic characteristics and that the notion of quality regarding competencies can be addressed from different perspectives.

A preliminary data analysis took place by the end of the data collection. This analysis highlighted instances where the same competency aspect (code) activation differed in activation quality. The interpretative element of the present study is evident. Therefore, it is necessary to establish persuasive argumentation for the origins and functions of the different levels of competency activation. For this study, these instances where the researcher/ teacher is the leading actor in the activity have not been coded. The latter does not suggest that the interplay between the teacher and the student was not analyzed. As I will show in subsection 4.1.7, the teacher’s interaction was considered a factor for determining the quality of activation by identifying the level of prompting, that is, the student received help to activate the particular aspect of the competency. That said, help could also come during a student-student interaction during group work or class discussion.

During the first session’s data analysis, the coding of students’ transcriptions and written material revealed the qualitative difference in competency activation discussed in the first two paragraphs of this subsection. More specifically, there were many instances where the same codes appeared (i.e., competency activation) within the same session in different turns. However, they did not necessarily illustrate the same competency activation quality. Different turns had different tasks, levels of difficulty, and context, and therefore different assessments took place to explore competency development. Below, I present an extract from the sessions’ data to illustrate the incidents of same-code activation more elaborately.

An example

Table 4-2 is a part of a transcription of the third calculus session on the 31st of October. The group consists of two students, Johannes and Erika, and the teacher (Yannis). In the first column, I use the term ‘turns’ to display the episode’s actor and the order that it took place. The second column includes the names of the episodes’ actors and the actual transcription of the episode. The transcriptions display verbal or written actions from the students. Finally, the third column includes the competency code (P.Q.), in this case, for the posing questions aspect of *mathematical thinking* competency. As mentioned above, the focus here is on students' actions and their competency development over time; therefore, the codes are assigned only to students; hence the grey shaded boxes when the teacher participates in the discussion.

Table 4-2. Extract from transcription with the same code activation.

Turn	**Speaker:** Utterance [gesture or written work]	Code
1	**Y**: So, what do we count as part of Norway's population in 2017? [Pointing to the following formula: $N_{t+1} = N_t + B + D + I - E$, where N_{t+1} is the population at $t+1$ years, N_t is the population at t year, B is the number of births, D is the number of deaths, I is the number of people immigrated, and E is the number of people emigrated]	
2	**J**: Why do we have '+' at deaths?	**P.Q.**
3	**J**: Don't we use ln for all sides so we get t down? (He refers to this equation $N_t = N_0 \times e^{rt}$)	**P.Q.**
4	**J**: Since we have lne can we just delete it?	**P.Q.**
5	**J**: Can I take this down now? (Johannes is writing down the following: $\ln(N_t) = \ln(N_0) + lne^{rt}$, and he is asking about the 'rt' part)	**P.Q.**
6	**J**: Shouldn't we use the same method we used last Monday? (This question was posed when I introduced the main task for the first time)	**P.Q.**
7	**Y**: How many doublings do we have?	
8	**J**: 24 hours divided by 20 minutes?	**P.Q.**

Table 4-2 displays a specific extract from the third session's transcription (Exponential Growth & Regression), where the (P.Q.) code was activated six times. However, the frequency of the appearance gives only a part of the information needed. In this case, the main point of inquiry is the quality of activation for every code and if that varies considerably and holds different qualitative characteristics every time the student activates it. The (P.Q.) code is an aspect of the Mathematical Thinking and Acting competency. Even though all questions have a mathematical characteristic, not all appeared to have the same level of articulation when posed. For example, Johannes (Turn 2) asks, "Why do we have '+' at deaths?" referring to the $N_{t+1} = N_t + B + D + I - E$ equation where D is the number of deaths in the population model. The latter is a question with a mathematical characteristic but also implies that Johannes was presumably confused regarding the $(+)$ sign since he was expecting that deaths should be assigned with the $(-)$ symbol. In a different part of this extract (Turn 5), Johannes wonders, "Can I take this down now?" referring to the 'rt' part of $\ln(N_t) = \ln(N_0) + lne^{rt}$. Again, this is a question with mathematical characteristics, but it lacks any formal mathematical language. This difference regarding the quality of these two activations of the (P.Q.) aspect revealed the need to evaluate and display these qualitative diversities.

4.1.4 Foreshadowing the construction of a scaling instrument

The 3-D model by Niss and Højgaard (2011) is a visual representation of the progression in a person's possession of a mathematical competency using three dimensions: the Radius of Action, the Degree of Coverage, and the Technical Level. The introduction of this model took place in Section 2.4. Progress within the Degree of Coverage can be identified if we provide evidence that a student gradually activates more aspects of a particular competency over time. However, identifying the activation of more aspects (codes) over time is hardly a straightforward process. The following paragraph addresses this issue.

The competence framework (Section 3.3) includes some competencies that consist of two or three aspects. The latter constitutes a problem if a student can activate these two or three aspects from the beginning of the sessions. It appeared to be challenging to track progress in the Degree of Coverage if there are no more aspects to be activated over time. For example, the Mathematical Reasoning and Communicating competency includes two aspects (Reason→ and Reason←), and the question is how we can identify if more aspects are activated over time if (Reason→) and (Reason←) are both activated from the beginning of the sessions. This obstacle was a primary reason for introducing and discussing the concept of quality activation of a competency aspect in the subsection above. This discussion resulted in constructing a scaling instrument in a rubric form. Consequently, before utilizing the 3-D model, the following subsections will discuss the conceptualization and construction of this study's scaling instrument.

4.1.5 A scaling instrument: rubrics

The exploration of the progress of competency development could be facilitated by some form of qualitative scaling for every aspect of the competence framework. This scaling should comprise a classification by levels of competency activation (quality of activation) when this competency is activated. These levels should function as qualitative scoring scales representing some form of assessment with specific criterion-referenced measures. Carr

(2000, p. 55) notices that "[d]ifferent rubric 'scores' represent a difference in the quality of the student work, not the quantity." In this thesis, an assessment is performed to provide each student's formal academic record of a competence profile. For the reasons mentioned above, I have considered using rubrics to construct a scaling instrument. As Goodrich (1996) points out, a rubric is a scoring tool where specific criteria/dimensions are listed for a particular piece of work. It articulates gradations of quality for each criterion, from excellent to poor.

Rubrics are related to attempts to explore students' performance and possible development and help teachers and researchers form a framework of desired expectations and outcomes. The most common argument for using rubrics is that they help define what development and quality mean (e.g., Andrade & Ying, 2005; Wolf & Stevens, 2007) regarding any subject of study, in this case, quality of competency activation. A common mistake should be avoided and concerns the focus of the assessment. Brookhart (2013) points out that the biggest mistake a teacher or a researcher can make when using rubrics for performance assessment concerns the subject of their focus. They (the teachers) tend to emphasize the task or the product rather than the actual learning outcome or proficiency that the task is supposed to get students to demonstrate. This observation was taken into consideration when constructing the scaling instrument.

A rubric should be both reliable and valid. Reliability concerns the extent to which an instrument (a test or rubric) gives the designer of the rubric consistent results when repeatedly used under the same measurement conditions. For example, a scale is unreliable if a person is weighed three times in three minutes and gets a different weight each time. The validity of an instrument measures what it is supposed to measure. For example, a traditional paper-and-pencil test may not be a valid instrument to judge how well pilots fly during adverse weather conditions. In contrast, a test in a flight simulator would be more valid. The reliability and validity of this study's scaling instrument (rubric) were tested by two individuals, the author, and the principal supervisor of this study. The results of the inter-coder reliability process are presented in Section 4.5. To construct the rubric, I also considered the advantages and disadvantages of such a tool. The analytic rubric provides feedback on areas of strength or weaknesses, and each criterion can be weighted to reflect its relative importance. At the same time, this type of rubric is more time-consuming to construct, and it might not be used consistently by those who use it unless it is well defined.

For the technical construction of a rubric, five main items must be considered apart from the reliability and validity discussed in the previous paragraph. The first paragraph discussed the first item and concerned the definition of a purpose for constructing and using the instrument. The second item concerns selecting the appropriate type of rubric that will satisfy that purpose. In particular, I considered the use of an analytic rubric. In this type of rubric, scores are provided for several different dimensions or component skills.

An analytic rubric resembles a grid with the dimensions (criteria) for a student product listed in the leftmost column and scales or levels of performance listed across the top row, often using numbers and descriptive tags. Table 4-3 is an example of such a rubric with three levels of performance where the first column includes the criteria (i.e., the different dimensions). For this study, the criteria will be the domains of activation (a construct that will be described in subsection 4.1.7), and instead of performance levels, I constructed competency activation levels (detailed description in subsection 4.1.6), which are the third and fourth items that need consideration. A more detailed presentation of the characteristics of the scaling instrument will follow in the following subsections.

Table 4-3. Example of a rubric.

	Initial or Beginning Level (I)	**Intermediate or Developing Level (II)**	**Mature or Accomplished Level (III)**
Criteria #1	Description reflecting Criteria #1 for this level of assessment.	Description reflecting Criteria #1 for this level of assessment.	Description reflecting Criteria #1 for this level of assessment.
Criteria #2	Description reflecting Criteria #2 for this level of assessment.	Description reflecting Criteria #2 for this level of assessment.	Description reflecting Criteria #2 for this level of assessment.
Criteria #3	Description reflecting Criteria #3 for this level of assessment.	Description reflecting Criteria #3 for this level of assessment.	Description reflecting Criteria #3 for this level of assessment.

Regarding the fifth item that needs consideration, the cells within the center of the rubric may be left blank or contain descriptions of the specified criteria for each performance level. These descriptions are the rubric descriptors and should ideally be observable and measurable, written in a consistent and parallel language across the scale. When scoring with an analytic rubric, each criterion is scored individually, and this is the method followed for the construction and use of this study’s analytic rubric. The levels of competency activation and the descriptors are discussed in the following subsections.

4.1.6 A scaling instrument: levels of competency activation and labelling

An initial coding attempt (from the first three sessions) indicated that students activated their mathematical competencies as they gradually became more familiar with the environment of non-routine mathematical tasks. However, there is no implication that competency activation was always evident since familiarity does not necessarily indicate development. In conclusion, this linear way of development is not absolute for any reason, and it was just an indication of how the levels should be constructed, that is, from an initial to a more mature level of activation. Considering the above, some primary reflections should be made to construct a reliable scaling instrument. I needed to determine these levels of competency activation based on two questions that needed to be answered: i) how many levels to include? and ii) how to label these levels?

The number of levels: The qualitative nature of this scaling attempt led to a two-choice dilemma. I considered a three-level system in favor of a four-level one. The advantage of a three-level scale is based on its speed of use and simplicity. A large amount of data could be better categorized on a three-level scale since every level would be noticeably different. For that reason, the need for the construction of distinctive levels of competency activation was imperative. As O'Connor (2009) points out, there must be a noticeable qualitative difference between each level, described in words and assigned with a number or a symbol as a label.

Labeling: The naming of the levels derives from characterizing the quality of competency activation when activated. Their names should represent the level of possible competency activation that occurs. This is a goal-orientated scaling instrument, and the labeling process connects to the main objectives of this study. The first level should describe the starting stage of activation; the second should portray the phase where students reach an intermediate level while developing their competencies or some expectations met. The third level should express this situation where the student approaches a mature activation level if not proficient. For these reasons, the following terminology has been decided:

- Initial Level of Activation (I)
- Intermediate Level of Activation (II)
- Mature Level of Activation (III)

4.1.7 A scaling instrument: domains of activation and descriptors

Considering the focus on the development progress of each competency aspect, appropriate and reliable criteria must be constructed and defined. These criteria should facilitate the effort to answer how the student activates a particular competency. As mentioned earlier (subsection 4.1.3), the exploration of the progress of competency development comprises assessing the quality of competency activation over time. Therefore, I define these development criteria as domains of activation where competency activation occurs. I can only assume that there can be several such domains and that we cannot know the exact number. In this thesis, I identify just a number of them. These domains concern the student's vision towards the solution of the mathematical task, the student's use of mathematical language and vocabulary, and the amount of help the student receives when activating a particular competency aspect (or aspects) of the competence framework. In what follows, I elaborate on these domains of activation graded in the three levels of competency activation described in subsection 4.1.6 (Initial Level of Activation, Intermediate Level of Activation, and Mature Level of Activation).

Task Solving Vision (T.S.V.): A student's purpose, I assume, when dealing with mathematical tasks is to reach a concluding solution. We, as researchers, can and want to study the whole problem-solving process in every step. The student's primary concern is to find or approach a solution to the problem that he/she works on without that being necessary from the researcher's point of view. This activation domain focuses on the student's awareness and understanding of the task's solving steps and his/her perspective on the solution. However, it must be clear that each calculus session did not consist of just task-solving situations. For example, many discussions (all part of the obtained data) before the actual task took place regarding the mathematical context we were about to study.

The possibility of providing competency activation evidence on an instance not directly related to the task's solution we were focusing on has also been considered. For example, even though a particular task in Integration & Modeling did not ask for sketching a graph, one student attempted to sketch one, hoping that it could provide her some insight into the problem. During the sessions, I noticed that there were moments where a student was activating a specific competency aspect without considering any future steps of the solving process. Furthermore, there were episodes where the student activated a particular aspect while fully portraying the solving process. The spectrum of students' vision regarding the solving steps was broad, ranging from unclear and ambiguous to explicit.

At this point, I illustrate the approach followed to make this activation domain functional in assessment and evaluation. I needed to locate these instances of students' actions in class that could afford valuable insight into students' understanding of the particular task. The focus was on how students expressed their ideas when trying to communicate their perspectives on the task-solving steps. In combination with suitable questions addressed by the teacher to the group, students' reasoning was considered the primary source of evidence about their perspective toward the task's concluding solution. It is what Lithner (2000) points out while addressing the notion of reasoning where he claims that to reason (in mathematics) is to follow a line of thought, a way of thinking adopted to produce assertions and reach conclusions. He also considers argumentation as substantiation and part of the reasoning that aims to convince oneself or someone else that the reasoning is appropriate. This definition aligns with the reasoning competency description incorporated in the competence framework (subsection 3.1.4).

The progress of competency development is not expected to be linear. It will probably display diversity depending on the moment of activation, task difficulty, and probably other factors impossible (or at least time-consuming) to identify. If the student has a clear, consistent long-range perspective toward the solution while activating a code (aspect), it does not necessarily suggest that he/she will not display a less clear one in the future.

Table 4-4 below displays three examples from the data where the student provided evidence of activating a particular competency aspect through his or her actions in the class. In these examples, I wish to demonstrate how these activations differ regarding students' vision toward the task solution or the issue under discussion. The first column includes the competency aspect that the student activated, and the second column comprises the transcribed part ("utterance" or [written work or gesture]) of what the student said or did that constitute evidence for this activation. The last column provides the context of this activation, describing the relevance of the student's actions to the solution of the task or the issue under discussion. The text in bold indicates why the specific evidence (i.e., utterance) signified the activation of the particular competency aspect using the language used to describe the competency aspects of the framework, as seen in Section 3.3.

Table 4-4. Examples from evidence of competency activations (T.S.V.).

Competency Aspect	**Evidence** "utterance" or [written work or gesture])	**Context and Rationale**
Reason←	"I see you started your graph on January one point to the right of the zero"	Erika, **aimed to understand the teacher's graph** and reacted **by providing argumentation** regarding her interpretation of the graph during the Airport Temperatures task. She made this observation without knowing what the next steps should be as she continued asking the teacher about how to procced.
Int.El.	"So for *d* we take this point (showing the lowest point on the graph) lowest point plus the amplitude"	Johannes **interpreted the elements of the equation $g(x) = a\cos[b(x-c)] + d$**, while working on the Airport Temperatures task. He was aware of the next step of the solution process but not to an extent that could lead him to the final steps of

Competency Aspect	Evidence "utterance" or [written work or gesture])	Context and Rationale
		the solution as he later posed some questions on what the next steps should be (e.g., "Do we have to use the amplitude again?")
→Sol.	"131.9 represents the 20 minutes sets so we need to divide this by three"	Erika **took actions towards the solution** of the E. Coli task. She had to calculate time *t* from the equation: $2^t = 3.5 \cdot 10^{39}$. She was aware of the next steps she had to take (i.e., "we need to divide this by three") and the future ones because she used this info for the final parts of the solution.

The examples from the data extracts above illustrate that there is some form of layering regarding the student's vision towards the solution of the task at hand when they activate a particular competency aspect. There are times when the students are fully aware of the steps they need to take and when they have a good but incomplete idea about what to do to solve the task or address the discussed issue. There are also instances where the students have a short-range perspective towards the solution of the task, not knowing what they need to do to continue with the following steps. Analyzing through this domain and considering the three levels of competency activation of the scaling instrument (subsection 4.1.6), I considered three gradations that could capture the more significant -if not all- part of this domain of activation:

- The student activates the competency by having a short-range perspective toward the solution (**Initial Level of Activation, Level I**)
- The student activates the competency by having a mid-range perspective toward the solution (**Intermediate Level of Activation, Level II**)
- The student activates the competency by having a long-range perspective toward the solution (**Mature Level of Activation, Level III**)

Use of Mathematical Language & Vocabulary flexibility (M.L.V.): The handling of concepts, algorithms, procedures, and computations, problem-solving, and the use of language are some of the critical component skills that need to be combined when working on and with mathematics (Riccomini et al. 2008). The language of mathematics can be confusing (Rubenstein & Thompson, 2002), especially for students who do not have mathematics as their main subject. However, students dealing with non-routine mathematical tasks for biology (the same applies, for example, to students in chemistry or geology departments) must communicate their ideas in some form of reasoning. There is a significant difference between getting the correct answer and explaining how you got it. There are many ways in which students can express themselves while they activate their reasoning competency (e.g., Reason→) or when they try to critique the model (e.g., Cr.) by reviewing or reflecting and questioning the results. They can use a vocabulary rich in mathematical terms or just familiar, everyday expressions and gestures where we may not recognize any mathematical language signs.

For this study, the mathematical language includes all possible forms besides the oral one. The way students deal with their writing work is also a form of mathematical language. For the theoretical construction of this activation domain, body language and gestures have also been considered as part of the language the students use to express their thoughts and ideas. Therefore, I broadly describe Mathematical Language and Vocabulary, considering its flexibility a primary factor for evaluating this activation domain. Each session was goal-oriented since the groups had to provide an estimated solution for their tasks. To meet these goals, one must be able to understand and apply mathematical words, symbols, and diagrams. An undeveloped mathematical language can cause a deceleration of mathematics learning (Van der Walt et al., 2008). Difficulties in using and learning of mathematical vocabulary are evident (Riccomini et al., 2015), and language development can be challenging.

Communicating mathematically is a demanding task for all students, even those with a more advanced familiarity with mathematical concepts and notions. Most competencies' aspects from the competence framework are activated through verbal or written communication. A student uses mathematical language to pose questions (P.Q.) with mathematical characteristics or when attempting to interpret and translate the elements (Int.El.) of a model during the mapping process. This direct connection between the competence framework and mathematical language was considered a significant reason for introducing Mathematical Language and Vocabulary as the second domain of competency activation of the scaling instrument.

Table 4-5 displays three examples from the data where the student provided evidence of activation of a particular competency aspect through his or her actions in the class. Some of these examples are the same as those used in the T.S.V. domain. However, now I wish to demonstrate how these activations differ in terms of the mathematical language and vocabulary use the student demonstrated while bringing into action this particular competency.

Table 4-5. Examples from evidence of competency activations (M.L.V.).

Competency Aspect	**Evidence** "utterance" or [written work or gesture])	**Context and Rationale**
Reason←	"I see you started your graph on January one point to the right of the zero"	Erika, **aimed to understand the teacher's graph** and reacted **by providing argumentation** regarding her interpretation of the graph during the Airport Temperatures task. She made this observation using everyday expressions with limited mathematical language and not bringing into discussion the "starting point of the graph" that was discussed before.
Int.El.	"Yes, so we have the following expression: $1.02 = e^r$, and need to find the *r*."	Johannes, while he was working on the Petroleum Consumption Rate task, replied to a teacher's question **providing his interpretation of the expression** '$C(t) = C(0) \cdot e^{rt}$', having for $C(0)$, the value: 1.02. In his reply, he used mathematical language (i.e., "Yes, so we have the following expression"). However, he did not name r, as the rate. He also displayed familiarity when he wrote down the expression: $1.02 = e^r$, with no hesitation.
→Sol.	"131.9 represents the 20 minutes sets so we need to divide this by three"	Erika **took actions towards the solution** of the E. Coli task. She had to calculate time *t* from the equation: $2^t = 3.5 \cdot 10^{39}$. She conveyed her thinking by making regular but also correct use of expressions with mathematical language.

The examples from the data extracts above illustrate that one student may be able to use only familiar or everyday expressions with no mathematical content regardless of if he/she works in the right direction towards the solution of the task. Students may regularly use expressions containing mathematical terms and language but still not master how they express themselves. It is also possible that a student may display regular and correct use of expressions with mathematical language and vocabulary. Taking into consideration the previous, the following three gradations of mathematical language use have been considered:

- The student activates the competency using familiar and occasional use of everyday expressions or gestures with no or limited mathematical language and vocabulary (**Initial Level of Activation, Level I**)
- The student activates the competency using regular use of expressions with mathematical language and vocabulary (**Intermediate Level of Activation, Level II**)
- The student activates the competency using regular and correct use of expressions with mathematical language and vocabulary (**Mature Level of Activation, Level III**)

Prompting/Independent Thinking (Pr.In.T.): Several studies have shown that prompting results in higher mathematical performance (e.g., Kramarski & Gutman, 2006; Kramarski & Zeichner, 2001; Schoenfeld, 1987). In any intervention with students, the teacher's involvement needs consideration. Identifying and estimating the teacher's involvement creates an opportunity to explore the student's independence during the session, especially while working on tasks within the context of their study (i.e., biology). This subsection connects with subsection 3.5.2, where I described the role of the researcher in this study. Within this competency activation domain, I aim to analyze the breadth of prompting and scaffolding that emerged in the sessions.

Wood et al. (1976) characterize scaffolding as an interactive exchange system in which the tutor operates with an implicit theory on the learner's acts to capture his attention. They believe that this is a way to reduce the degrees of freedom in the task to manageable limits, maintain direction in the problem-solving process, mark critical features, and even control frustration and demonstrate solutions where the learner can recognize them. They continue by identifying scaffolding as the process that enables a child or novice to solve a problem, carry out a task, or achieve a goal beyond his/her efforts if not assisted. The latter definition could raise some justified arguments since we can never be sure about our students' responses to the assigned tasks.

Beyond their complementary nature, the non-routine tasks were designed to introduce the students to using and applying mathematics for studying the natural world. Also, the project was taking place at the same time as their "Calculus I" course, which resulted in simultaneously new obtained knowledge. The students were in their first semester in the Department of Biology, attending this calculus course with two more university departments (chemistry and geology) when the calculus sessions took place. For the students who participated in this study, mathematics has not been their primary focus, so it is reasonable to assume that working on non-routine mathematical problems could be challenging for them.

In this competency activation domain, I analyze the extent to which the teacher used probing questions and assisted the students. The focus here is the level of independence in students' thinking and actions. The traditional mathematics classroom portrays the mathematics teachers asking several questions of no explanatory nature. It is relatively common that these questions have a single correct answer, which teachers already know. On the contrary, the number of questions that demand some explanation is relatively small (e.g., Kawanaka & Stigler, 1999; Myhill & Dunkin, 2005; Sahin & Kulm, 2008; and Viirman, 2015).

The request for more emphasis on using probing questions in the classroom, where the teacher will ask students to explain their thinking, is evident in the relevant literature (e.g., Hufferd-Ackles et al., 2004; Kazemi & Stipek, 2001). Current efforts have been made toward promoting inquiry-based mathematics teaching regarding the elaboration of different ways of asking students to give explanations (e.g., Artigue & Blomhøj, 2013; Hähkiöniemi, 2017). Inquiry-based undergraduate mathematics methods have been considered to be effective by several researchers and practitioners (e.g., Freeman et al., 2014; Rasmussen & Kwon, 2007; Kogan & Laursen, 2014). However, studies on using such methods have not yielded reliable findings about the relationship between inquiry-based learning and student outcomes (e.g., Freeman et al., 2014; Laursen et al., 2014; Sonnert et al., 2015; Rasmussen & Ellis, 2013). Studies using meta-analysis synthesis (e.g., Hattie, 2009) claim that teacher questioning has a medium effect on student learning. It appears that probing questions play a significant role in the student's construction of mathematical ideas and the development of teaching toward inquiry-based mathematics (Hähkiöniemi, 2017; Hufferd-Ackles et al., 2004).

A variety of questions can stimulate, to some degree, mathematical thinking and activate students' involvement in the learning activity. It is possible -if not sure- that at some point, the teacher may have to simplify the task by reducing the number of component acts required to reach a solution. Bernstein (1967) refers to the importance of reducing alternative actions during skill acquisition, and he considers it an essential process to adjust and regulate feedback to use it for correction. The latter is strongly related to this study's focus because I attempt to explore competency development progress when activated through problem (task) solving. The quality of scaffolding can be determined by assessing the types of questions used in each case to bring the competency into action:

i. A starter question usually takes an open-ended form and is used to give the students a starting point. Some examples of these questions could be the following: What happens when we …? What can we make from …? How could you sort these…? If the student could only activate the competency with this type of question, then the level of scaffolding is relatively low.
ii. A stimulative question focuses on specific strategies and assists the students in recognizing patterns and possible relationships. This kind of

question can serve as a prompt when the student or the group of students cannot continue (stuck) with the solution of the task. It is worthwhile to mention that teachers are often tempted to turn these questions into instructions. The latter could lead to a non-stimulating environment for mathematical thinking and even remove the student's responsibility to "do mathematics" and investigate on his/her own. These episodes, where the question turns into pure guiding, describe high-level prompting.

Some examples of these questions could be the following: Can you spot any difference? What is the same? Can you see the pattern here? What would happen if …?

Analyzing this activation domain, I need to consider these instances that occurred before the specific episode under study. It is essential to consider what provoked the activation of a particular competency aspect and especially what assistance the student received before bringing into action this competency aspect. Table 4-6 below again displays examples from the data focusing on the layering of the student's amount of help to activate the competency under observation.

Table 4-6. Examples from evidence of competency activations (Pr.In.T.).

Competency Aspect	**Evidence** "utterance" or [written work or gesture])	**Context and Rationale**
Reason←	"This is the starting point that we will be asked to use at some point."	Johannes followed a comment made by his teacher when working on the Earth's Temperature task. When the teacher mentioned that he was giving them the "temperature at 2000 degrees Fahrenheit" Johannes **followed that providing his argumentation**. To do that though, he frequently received hints and prompting questions from the teacher because he was not able to begin the solving process of the task.
C.D.Rep.	She wrote [$\frac{df}{dt} = f'(t)$] mentioning that "this is the same thing."	Erika **decoded the representation** mentioned in the left cell while discussing a given rate with the expression '$\frac{df}{dt} = 0.014 \cdot t^{0.004}$. To that she needed occasional prompting from the teacher who was regularly providing her hints regarding the next steps she needed to take.

Competency Aspect	**Evidence** "utterance" or [written work or gesture])	**Context and Rationale**
Int.El.	"Yes, so we have the following expression: $1.02 = e^r$, and need to find the *r*."	Johannes, while he was working on the Petroleum Consumption Rate task, replied to a teacher's question **providing his interpretation of the expression** '$C(t) = C(0) \cdot e^{rt}$', having for $C(0)$, the value: 1.02. To provide this reply, Johannes received minimal prompting from the teacher who just posed the question and let the students to figure out the answer.

It appears that a student may receive some form of assistance from the teacher or a classmate to activate a particular competency. This assistance could be in the form of continuous help with hints and instructions that suggest the student would not have been able to provide evidence of such activation if it was not for this assistance. However, a student may receive assistance with occasional prompting but with regular instances that indicate independent thinking. Lastly, a student may bring into action a particular competency aspect with minimal or no prompting acting completely independent.

The following three gradations of prompting and independent thinking have been considered:

- The student activates the competency with frequent prompting with no signs of independent thinking from the student (**Initial Level of Activation, Level I**)
- The student activates the competency with occasional prompting with regular instances of independent thinking from the student (**Intermediate Level of Activation, Level II**)
- The student activates the competency with minimal or no prompting; the student consistently demonstrates independent thinking (**Mature Level of Activation, Level III**)

4.1.8 The Scaling instrument, the table

Table 4-7 illustrates, in a compact form, the scaling instrument that will be used in this study. The last three columns describe the levels of competency activation (I, II, and III) with a description of what should be, somehow, expected from the student who functions in that level for each one of the competency domain of activation (T.S.V., M.L.V., and Pr.In.T) displayed in the first column. For all three activation domains, (T.S.V., M.L.V., and Pr.In.T.) one could argue that the layering could be performed in less or in more than three levels. As discussed before, the decision to categorize the activation domains in three gradation levels was based on simplicity reasons in terms of design and on literature reports on the use of rubrics.

Table 4-7. The scaling instrument.

Activation Level / Activation Domain	**Initial (I)**	**Intermediate (II)**	**Mature (III)**
Task Solving Vision (T.S.V.)	The student activates the competency by having a short-range (local) perspective toward the solution.	The student activates the competency by having a mid-range (tactical) perspective toward the solution.	The student activates the competency by having a long-range (strategic) perspective toward the solution.
Mathematical Language and Vocabulary (M.L.V.)	The student activates the competency making familiar and occasional use of everyday expressions or gestures with limited or no mathematical language and vocabulary.	The student activates the competency making regular use of expressions with mathematical language and vocabulary.	The student activates the competency making regular and correct use of expressions with mathematical language and vocabulary.
Prompting and Independent Thinking (Pr.In.T.)	The student activates the competency with frequent prompting with no signs of independent thinking from the student.	The student activates the competency with occasional prompting with regular instances of independent thinking from the student.	The student activates the competency with minimal or no prompting; student consistently demonstrates independent thinking.

4.2 Functionality of the scaling instrument

This section describes how the scaling instrument works using episodes from the obtained data and includes three parts. The first subsection (4.2.1) describes the coding of one competency aspect and the assignment of activation levels to this competency aspect (code) based on the scaling instrument. Subsection 4.2.2 describes the coding of two competency aspects within the same turn and the relevant assignment of activation levels to these aspects. The last subsection (4.2.3) presents and describes a visual tool (i.e., diagrams), that will be used to present the results from the data analysis.

4.2.1 First stage: one-competency aspect coding

The first stage of the analysis is assigning the competency aspect code for each student. As I will illustrate in a different example, more than one competency aspect (code) activation is expected to be identified in the same instance (turn in the tables). Table 4-8 describes an episode from the class (Exponential Growth & Regression unit). It includes five turns where the activations of the reasoning competency aspects were observed in two of them (turns 3 and 5). In turn 3 of the example, Erika (E) understands the probing question, which has mathematical content, from the teacher, Yannis (Y). Based on the description of the first aspect of the Mathematical Reasoning and Communicating competency of the competence framework (complete description of the competency aspect in Table 3-1), the (Reason←) code assigned for this unit. Later on, in turn 5 of the same table, Johannes (J) engages in the discussion providing his argumentation by answering another question from the teacher. Based on the description of the second aspect of the mathematical reasoning competency of the competence framework, the (Reason→) code was assigned for this unit. The last column will include the levels of activation assigned for each of the activation domains: T.S.V., M.L.V., and Pr.In.T.

Table 4-8. Extract from transcription with one-code activation.

Turn	**Speaker:** Utterance	Code	Level of Activation by Domain
1	**E**: I don’t know how to find this number (showing the $\frac{0.181}{100}$)… I mean the percentage that we are looking for		
2	**Y**: Do you mean the rate of growth?		
3	**E**: Yes, the growth rate, this is what we are looking for	**Reason←**	
4	**Y**: Do you see any increase or decrease?		
5	**J**: Yes, it is increasing and therefore the rate should be positive(looking at the table and indicating that the number are getting bigger)	**Reason→**	

Second stage: assigning levels of competency activation

The second stage of the analysis is to identify and assign a level of competency activation for each aspect using the scaling instrument presented in the previous section. First, I demonstrate the scaling process (assigning levels from the scaling instrument) for Erika (E) and then for Johannes (J). The scaling process for each competency aspect is analyzed through the three competency activation domains presented before, that is, the Task Solving Vision (**T.S.V.**), the Mathematical Language and Vocabulary (**M.L.V.**), and the Prompting and Independent Thinking (**Pr.In.T.**) The scaling instrument's levels (I, II, or III) are assigned at the end of every domain. For example, the first bullet below explains why the level II of the scaling instrument was assigned for the activation of the (Reason⟵) competency aspect, observed in turn 3 in Table 4-8 above, regarding the T.S.V. domain of activation.

Scaling Erika (turn 3)

- **T.S.V.**: Erika activated (Reason⟵) having a mid-range (tactical) perspective towards the solution knowing that this (i.e. the rate of growth) is what we were looking for ⟶ II
- **M.L.V.**: Erika used familiar expressions, repeating what the teacher said, with limited mathematical language and vocabulary while activating the (Reason⟵) aspect ⟶ I
- **Pr.In.T.**: the probing question (Y: *Do you mean the rate of growth?*) that preceded the (Reason⟵) activation was directive and not a stimulative one ⟶ I

Scaling Johannes (turn 5)

- **T.S.V.**: Johannes had a short-range perspective towards the solution with no reference on the growth rate ⟶ I
- **MLV**: Johannes showed regular use of expressions with mathematical language (e.g., increasing and therefore the rate should be positive) while activating the (Reason⟶) aspect ⟶ II
- **Pr.In.T.**: Johannes needed one necessary probing question (stimulative) to activate the (Reason⟶) aspect ⟶ II

Following the second step of analysis the previous table will now take the following form as seen in Table 4-9:

Table 4-9. One-code activation with assigned activation level.

Turn	**Speaker:** Utterance	**Code**	**Level of Activation by Domain**
1	**E**: I don't know how to find this number (showing the $\frac{0.181}{100}$)… I mean the percentage that we are looking for		

Turn	**Speaker:** Utterance	Code	Level of Activation by Domain
2	**Y**: Do you mean the rate of growth?		
3	**E**: Yes, the growth rate, this is what we are looking for	**Reason←**	**II, I, I**
4	**Y**: Do you see any increase or decrease?		
5	**J**: Yes, it is increasing and therefore the rate should be positive(looking at the table and indicating that the number are getting bigger)	**Reason→**	**I, II, II**

4.2.2 First stage: two-competency aspect coding

The initial coding attempt and the primary data analysis confirmed what Niss (2013) describe as cluster overlapping, meaning that it is possible to observe two or more aspects of the same or different competencies to be activated within the same turn. Table 4-10 Table 4-10includes an extract from an episode where Johannes activated two competencies within the same turn on two occasions. This episode includes six turns.

Table 4-10. Extract from transcription with one and two-code activation.

Turn	**Speaker:** Utterance	Code	Level of Activation by Domain
1	**Y**: So, what do we count as part of Norway's population in 2017? (Pointing to the following formula: $N_{t+1} = N_t + B + D + I - E$)		
2	**J**: Why do we have the "+" symbol in front of deaths? Is it maybe because the number of deaths counts as negative	**P.Q.** **Reason→**	
3	**Y**: So, in this formula ($N_t = N_0 \times e^{rt}$) N_0 represents what?		
4	**J**: N_0, is the population at time 0, at the starting point	**C.D.Rep.**	
5	**Y**: So, we need to have t alone somehow		
6	**J**: We can get t down using ln or log	**→Sol.** **C.D.Rep.**	

Second stage: assigning levels of competency activation

The activation of two or more competencies within the same turn did not alter the functionality of the scaling analysis process. It is rare but possible that a different scaling may occur within the same episode, so I perform two separate analyses. In turn 2, I identified the activation of two codes from different competencies. Johannes posed a question with mathematical characteristics. Based on the description of the first aspect of the Mathematical Thinking and Acting competency of the competence framework (Table 3-1) the (P.Q.) code was assigned. In the same turn, Johannes attempted to provide argumentation for his thinking, and the (Reason →) code was also assigned. In turn 4, Johannes chose and decoded a specific representation. Based on the description of the first aspect of the Representing and Manipulating symbolic forms competency of the competence framework, the (C.D.Rep.) was assigned. Similar actions took place in turn 6, and two codes were activated, (→Sol.) by taking actions towards the solution and (C.D.Rep.) by decoding the representation. In this example, I analyze these units (i.e., the first and the third) where two codes were activated within the same turn.

Scaling Johannes (P.Q.) (turn 2)

- **T.S.V.**: Johannes had a short-range perspective towards the solution focusing mainly on the validity of the formula → I
- **M.L.V.**: He showed regular use of expressions with mathematical language with a certain degree of fluency → II
- **Pr.In.T.**: There was no direct probing question when "J" activated the (P.Q.) code but rather an introduction of the formula, so he had a minor hint at his disposal → III

Scaling Johannes (Reason →) (turn 2)

- **T.S.V.**: Johannes had a short-range perspective towards the solution focusing mainly on the validity of the formula → I
- **M.L.V.**: He provided reasoning by demonstrating regular use of expressions with mathematical language with a certain degree of fluency → II
- **Pr.In.T.**: There was no direct probing question when "J" activated the (Reason →) code but rather an introduction of the formula, so he had a minor hint at his disposal → III

Scaling Johannes (→Sol.) (turn 6)

- **T.S.V.**: Johannes recognized that finding t is very important for the final solution → III
- **M.L.V.**: the "get t down" is an everyday expression, he also used gestures to explain his thinking → I
- **Pr.In.T.**: He activated (→Sol.) with no direct help but just with a minor hint, demonstrating independent thinking → III

Scaling Johannes (C.D.Rep.) (turn 6)

- **T.S.V.**: Johannes recognized that finding t is very important for the final solution and suggests a further step for it → II
- **M.L.V.**: the "get t down" is an everyday expression, he also used gestures to explain his thinking → I
- **Pr.In.T.**: He activated (C.D.Rep.) with no direct help but just with a minor hint, demonstrating independent thinking → III

Following the second step of analysis, Table 4-10Table 4-10 will now take the following form of Table 4-11.

Table 4-11. Extract from transcription the second step of the analysis.

Turn	**Speaker:** Utterance	**Code**	**Level of Activation by Domain**
1	**Y**: So, what do we count as part of Norway's population in 2017? (Pointing to the following formula: $N_{t+1} = N_t + B + D + I - E$)		
2	**J**: Why do we have the "+" symbol in front of deaths? Is it maybe because the number of deaths counts as negative	**P.Q.** **Reason→**	**I, II, III** **I, II, III**
3	**Y**: So, in this formula ($N_t = N_0 \times e^{rt}$) N_0 represents what?		
4	**J**: N_0, is the population at time 0, at the starting point	**C.D.Rep.**	**II, II, II**
5	**Y**: So, we need to have t alone somehow		
6	**J**: We can get t down using ln or log	**→Sol.** **C.D.Rep.**	**III, I, III** **II, I, III**

4.2.3 Visualization of the results

In this section, I introduce and describe the diagrams, a visual tool that will be used to present the results from the data analysis. These diagrams visualize the outcomes from the analysis process described in the previous subsections. I explain how to interpret these diagrams for the two case studies of Erika and Johannes, both belonging to the same working group. I conclude this subsection by introducing the first type of evidence to explore the progress of competency development, namely the *activation frequency*.

Firstly, to explain the diagram, I present in Table 4-12 the activations levels activation of the (C.D.Rep.) competency aspect for each domain (T.S.V., M.L.V., and Pr.In.T.), across all calculus sessions, based on the scaling instrument. The first column includes the session where the code was activated. The second column shows which code was activated and the frequency *n* of the activation in the particular session. The third column includes the assignment of the levels of competency activation by activation domain: Task Solving Vision (T.S.V.), Mathematical Language and Vocabulary (M.L.V.), and Prompting and Independent Thinking (Pr.In.T.), based on the scaling instrument.

Table 4-12. Grading (C.D.Rep.) for Johannes.

Calculus Unit	**Code activated, frequency *n***	**Activation Level by Domain (T.S.V., M.L.V., Pr.In.T.)**
Periodic functions	C.D.Rep., *1*	(II, II, III)
Exponential Growth and Regression	C.D.Rep., *6*	(I, II, II), (I, III, II), (II, III, III), (I, II, III), (II, II, III), (I, II, III)
Population Dynamics	C.D.Rep., *6*	(II, II, II), (II, I, III), (III, III, II), (III, III, III), (II, III, III), (II, III, III)
Integrals & Modeling	C.D.Rep., *9*	(I, III, II), (I, II, III), (I, I, I), (II, II, II), (I, II, I), (I, I, II), (I, I, II), (II, I, II), (II, II, III)

The table above can be improved by showing the changes from one activation level to a different one in a graphical fashion. Therefore, a decision was made to visualize the obtained data in a diagram where the progress of the competency aspect development, in that case, the (C.D.Rep.) code, would be illustrated more coherently. Figure 4.3 below, displays this diagram with descriptions of its characteristics.

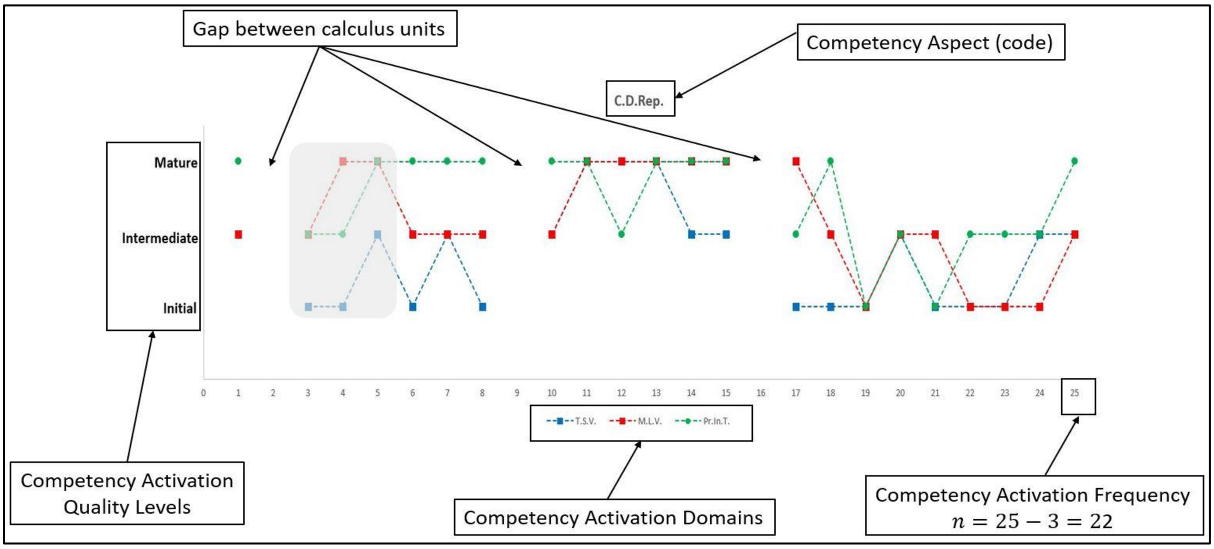

Figure 4.3. Interpretation of the diagram.

Each diagram consists of three distinct dotted lines (blue, red, and green) representing each activation domain of the scaling instrument. The dotted blue line represents the Task Solving Vision (T. S. V.), the dotted red line the Mathematical Language and Vocabulary (M. L. V.), and finally, the dotted green line, the Prompting (Pr.In.T.) domain. The vertical axis represents the three levels (Initial, Intermediate, and Mature) of competency activation of the scaling instrument. Every point on the diagram's horizontal axis shows the number-frequency of competency activation. For future reference, for capturing a particular activation or a sequence of activations as shown in the grey-shaded area (points 3, 4, and 5 on the x-axis) in Figure 4.3, I will use the following notation: [x-y], which for the case of the shaded area is [3-5]. A gap between the lines is used to represent the four calculus units. For example, in Figure 4.3, the four calculus units (Periodic Functions, Exponential Growth & Regression, Population Dynamics, and Integrals & Modeling) are represented by particular points on the x-axis; specifically: Periodic Functions [1] (only one recorded activation), Exponential Growth & Regression [3-8], Population Dynamics [10-15], and Integrals & Modeling [17-25]. The following subsection describes the first type of evidence that will be used to explore the progress of competency development, namely the *competency activation frequency*.

First type of evidence: competency activation frequency

The first type of evidence that will facilitate the exploration of the progress of competency development across all domains (T.S.V, M.L.V., and Pr.In.T.) is the competency activation frequency. More accurately, this type of evidence concerns the frequency of evidence the student provided regarding the activation of each competency aspect of the competence framework. In every diagram displayed, the total frequency n of competency activation is calculated by subtracting the number of gaps (always three) by the final number displayed at the horizontal axis.

In the example in Figure 4.3, the frequency n is 22. It is possible that a particular code will not be activated in a session, and a single dot on the x-axis will depict this instance. In this case, the number of instances in which there was no activation must be subtracted to calculate the total frequency of activation. The frequencies for each of the four calculus units were 1, 6, 6, and 9. For notation purposes, each unit's activation frequencies will be displayed by the following notation [x, y, z, w]. Therefore, in the example in Figure 4.3, the activation frequencies for each calculus unit will be symbolized as [1, 6, 6, 9].

The frequency of activation is not the only type of evidence that this study employed to explore the progress of competency development. This type of evidence offers a numeric approach to this exploration, but, as I mentioned in the introduction of the scaling instrument, the quality of activation must be considered too. The following subsection presents the methodological approach and exploration of the 3-D model (Section 2.4), which led to the conceptualization and introduction of two additional types of evidence that will support the exploration of the progress of competency development.

4.3 Utilizing the 3-D Model

In section 2.4, a brief presentation of the three-dimensional model, designed by Niss and Højgaard (2011), took place. This model will facilitate the exploration of competency development for the needs of the present thesis. Each competency aspect will be analyzed through each dimension of the 3-D model: Degree of Coverage, Radius of Action, and Technical Level. The scaling instrument (sub-section 4.1.8) will utilize the theoretical properties of the 3-D model and provide evidence for the progress of competency development. This research explores students' competency development over time through each dimension of the 3-D model. The objective is to use the data analysis results that the scaling instrument will provide by interpreting them through all three dimensions of the model. This section describes how this interpretation will function.

Reader's orientation

The following parts of the subsection discuss two new types of evidence (i.e., the first being the competency activation frequency) of competency development within the Degree of Coverage (subsection 4.3.1), Radius of Action (subsection 4.3.2), and Technical Level (subsection 4.3.3) of the 3-D model. I begin by describing the development within the Degree of Coverage by highlighting a methodological issue when attempting to explore competency development over time. In the same section, I suggest a way to resolve this issue, which facilitated the creation of two additional types of evidence to explore competency development over time. These additional types of evidence will be termed: *changes between activation levels* and *competency activation consistency*. Following that, I describe how to interpret competency development over time within the Radius of Action and the Technical Level. This description includes an example and an overview.

Each part of this subsection (Development for Degree of Coverage, Development for Radius of Action, and Development for Technical Level) ends with an overview that describes the accumulation of all the evidence obtained by the data analysis, which facilitated the exploration of the progress of competency development. This overview will be used to present the findings of this thesis. The overviews describe one competency aspect (e.g., Int.El.) from the competence framework. Exploring each competency aspect is necessary to draw inferences for the development of a mathematical competency as a whole. For example, to provide inferences regarding the development of the Mathematical Thinking and Acting competency, the overviews of the (P.Q.), (Lim.), and (→Sol.) aspects in any dimension of the 3-D model must and will be considered.

4.3.1 Development for Degree of Coverage

Looking back at the description of this dimension (subsection 2.4.1), we saw that Niss (2003) describes the Degree of Coverage as the extent to which a person (student in our case) exhibits the characteristic aspects of the competency at issue. It is seen as the breadth of a person's possession of the competency he/she covers all the aspects involved in the definition and delineation of that competency. Rephrasing the latter, one can claim that the more aspects of the competency a student possesses, the higher the Degree of Coverage of that competency with that student.

Focusing on each stage of the modeling cycle, Haines and Crouch (2001) introduce a method to assess the activation of the competency aspects through a dissected approach using multiple-choice questions. In later work, they suggest that this approach can be used to develop rating scales for this dimension and apply them to students at postgraduate, undergraduate, pre-university, and school levels (Haines & Crouch, 2007). These scales are closely related to the levels of competency activation of the scaling instrument of this thesis.

A methodological issue

What proved problematic for this research's specific needs was that there are competencies with very few aspects from the competence framework. For example, in subsection 3.3.4, one can see that Mathematical Reasoning and Communicating competency consist of two aspects:

i. To understand others' written, visual or oral 'texts,' in a variety of linguistic registers, about matters having a mathematical content (incoming). To follow and assess chains of arguments, put forward by others (code: Reason←)
ii. To express oneself, at various levels of theoretical and technical precision, in oral, visual, or written form, about such matters (outgoing). To provide explanations or justifications to support your result or idea. (code: Reason→)

Through their actions in the class, students may provide evidence of competency activation of both aspects from the first session. The latter creates a challenging situation regarding identifying the change in the competency's breadth. We cannot identify the activation of more aspects of the particular competency because there are no more aspects to be activated from one session to the next. The following paragraphs describe a way to overcome this obstacle.

The scaling instrument provides a qualitative approach to the methodological issue presented in the paragraph above. I can now track changes in the progress of competency development over time by identifying changes from a lower to a higher level of activation (or vice versa) from one competency activation to the next. Changes do not necessarily suggest a shift from one activation level to another. Part of the exploration of the progress of competency development comprises the situation where two or more activations are identified at the same activation level. The following paragraph includes an example for exploring the competency development over time within the Degree of Coverage.

Figure 4.4 displays the analysis results of Erika's (Int.El.) competency aspect activation of the Mathematical Modeling competency.

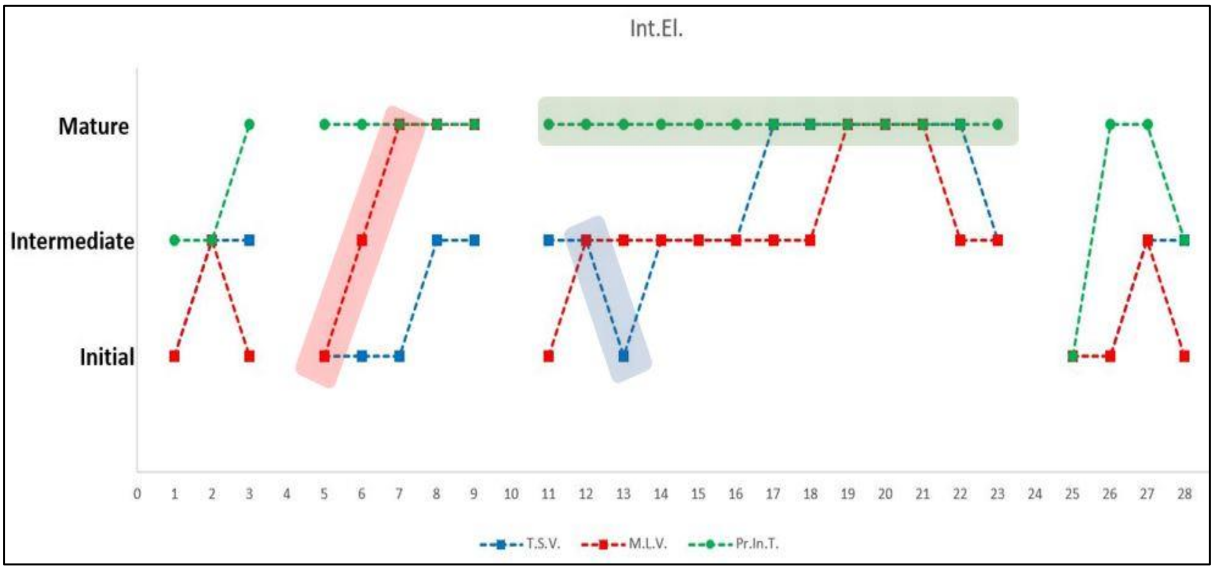

Figure 4.4. Diagram design: activation level changes and consistency.

The calculus units in the diagram of Figure 4.4 are the following: Periodic Functions [1-3], Exponential Growth & Regression [5-9], Population Dynamics [11-23], and Integrals & Modeling [25-28].The red shaded area indicates a change in the activation level across three different activations [5-7] for the M.L.V. competency activation domain (dotted red line). The change occurs from the Initial level (I) to the Intermediate (II) and ends at the Mature level (III). This progress of development can be illustrated as (I→II→III). The blue shaded area indicates a change in the activation level across two different activations [12, 13] for the T.S.V. competency activation domain. The change occurs from the Intermediate (II) level to the Initial level (I). This progress of development can be illustrated as (II→I). Finally, the green shaded area indicates a situation where the level of activation of the Pr.In.T. domain does not change. The progress of development in the [11-23] activations can be illustrated as (III→III→…→III).

The observations mentioned in the previous paragraph led to the conceptualization of the two following types of evidence for competency development: i) the changes between activation levels and ii) the levels of consistency of competency activation. Both types of evidence are described in this subsection because they were conceptualized while considering the methodological issue mentioned above. However, for the same reasons, these issues apply to the other two dimensions of the 3-D model: Radius of Action and Technical Level. The following sub-section discusses the second type of evidence (the first type being the frequency of activation) that will be considered to explore competency development.

Second type of evidence: Changes between activation levels

The progress of competency development reflects the dynamic nature of competency activation. A change from a lower to a higher activation level (or vice versa) indicates such progress. Changes can be tracked only during the same calculus unit (e.g., Periodic Functions) and not during all seven sessions. Different mathematical contexts of the units, conditions that prevail from unit to unit, such as time restraints, the number of students in the groups, and the nature of the course itself, are the primary reasons for focusing only on the changes within the same unit. The latter, however, implies an unavoidable limitation. Focusing on a smaller number of calculus sessions suggests that the exploration of competency development will occur over a narrower time period and, therefore, may concern a smaller number of competency activations.

Looking at Figure 4.4, progress from a lower to a higher level of activation for the T.S.V. domain is displayed in the following activations: [1-2], [7-8], [13-14], [16-17], and [26-27]. Similarly one can identify the progress from a lower to a higher activation level for the other activation domains. Progress from a higher to a lower activation level for the M.L.V. activation domain is displayed in the following activations: [2,3], [21,22], and [27,28]. Table 4-13 below summarizes the number of changes from a lower to a higher (and vice versa) competency activation level for every activation domain. The frequency of these changes is the second type of evidence that will be used to explore the progress of competency development. If the frequency of changes from a lower to a higher activation level is comparatively higher than the frequency of the opposite direction, that could suggest some form of competency development.

Table 4-13. Summarizing the changes by domain.

Domain of Activation	**Lower to Higher**	**Higher to Lower**
T.S.V	5	2
M.L.V.	6	3
Pr.In.T.	2	1

In the following paragraph, I describe the third and last type of evidence, used in this study, for the exploration of competency development. I will elaborate on these situations where the assigned activation level within any activation domain (T.S.V., M.L.V., and Pr.In.T.) does not change from one or more activation to the next.

Third type of evidence: Consistency of competency activation

The diagram displayed in Figure 4.4 and any diagram used to present this analysis's results offers one additional type of evidence that Table 4-13 does not capture. Looking at Table 4-13, one can notice that the frequency of total changes within the Pr.In.T. domain is relatively low (3) in comparison to that of M.L.V. (9) and T.S.V. (7). A low number of changes between activation levels can be an indication of either a low total frequency of competency activation or that there are activations that are identified within the same activation level for two or more consecutive times. For example, activations [12-18] for the M.L.V. domain in the diagram of Figure 4.4 are identified at the Intermediate Level (II) for seven consecutive activations. Similarly, activations [4-8] for the Pr.In.T. domain are identified at the Mature Level for five consecutive activations.

Interestingly, looking at activations [11-23] at the Pr.In.T. domain, one can notice that the activation was identified 13 consecutive times at the Mature Level (III), which justifies the aforementioned low frequency of total changes within this domain of competency activation. Considering the above, and for lack of a better word, this thesis adopts the term: *consistency of competency activation* to describe the third type of evidence that will facilitate the exploration of competency development. Similar to the changes between different levels of competency activation, consistency can be tracked only during the same unit (e.g., Population Dynamics) and not during all seven sessions.

The last two examples (regarding the Pr.In.T. domain, activations [4-8] and [11-23]) suggest some discrepancy between the consistencies of activation from one situation to another. This discrepancy concerns the frequency of consecutive activations (five for the first example and 13 for the second). Furthermore, consistency can be tracked at all levels of competency activation of the scaling instrument (Initial, Intermediate, or Mature). To address this discrepancy and incorporate it in the analysis results, it seems suitable to create some form of scaling (i.e., levels) to describe these different types of consistency. This study will integrate a descriptive approach to scaling the consistency levels. Therefore, I will adopt a similar, to the scaling instrument, approach using three different levels of consistency. Table 4-14 shows that the levels are based on the proportionality of the total frequency of the activation in the particular calculus unit. In the following paragraph, I provide instances of identified consistencies using Figure 4.4 as an example.

Table 4-14. Levels of consistency.

% of total unit activation frequency	**Consistency Level**
<33.3%	Low
33.3%-66.6%	Moderate
66.6%-100%	High

Using the diagram from Figure 4.4 and applying the considerations of Table 4-14, one can notice that activations [19-21] at the M.L.V. domain present a proportion of $\frac{3}{13} = 23.1\%$ (three consecutive activations in a total of 13 for the Population Dynamics unit), and the level of consistency is, therefore, low. Activations [5-7] at the T.S.V. domain have a proportion of $\frac{3}{5} = 60\%$, and the level of consistency is, therefore, moderate. Lastly, activations [11-23] at the Pr.In.T. domain have a proportion of $\frac{13}{13} = 100\%$ and, therefore, the level of consistency is high. Before displaying an example of a table depicting how consistency levels will be presented, an additional consideration was taken for conceptualizing the third type of competency development evidence. The following two paragraphs concern this consideration.

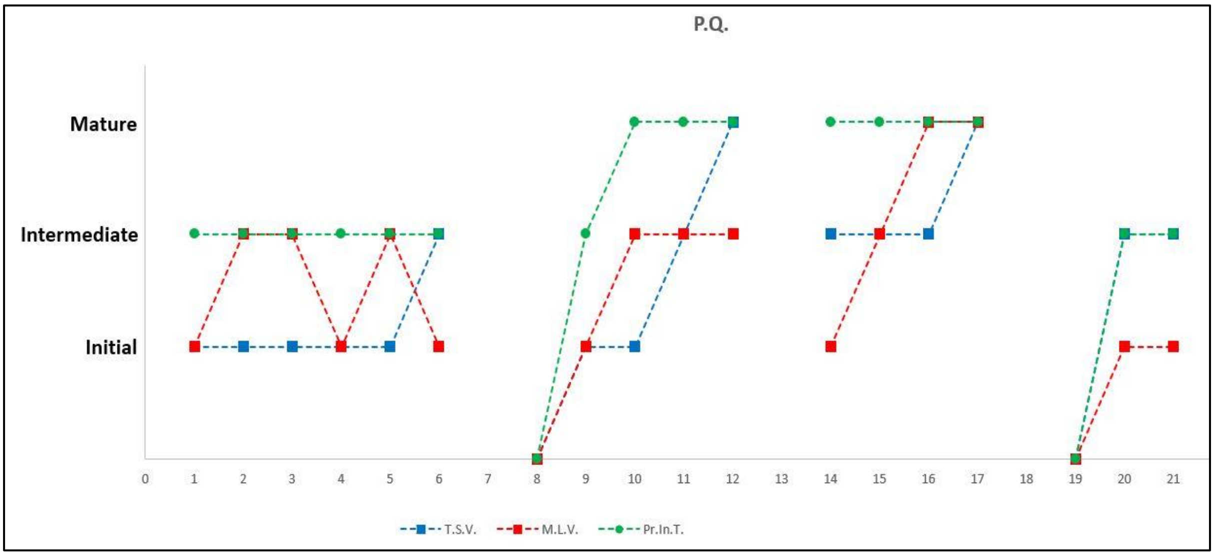

Figure 4.5. Diagram example for the (P.Q.) competency activation.

As mentioned above, the consistency of activation is assessed by examining the proportion of continuous activations frequency within the same activation level by the total activation frequency in the same calculus unit. It was also mentioned before that evidence of no competency activation is possible as seen in points [8] and [19] in the diagram in Figure 4.5. These points show that there was no activation of the (P.Q.) competency aspect during the first sessions of the Exponential Growth & Regression (2^{nd} session) and Integrals & Modeling units (6^{th} session). To calculate the consistency of the Pr.In.T. domain during the Exponential Growth & Regression unit (activations [9-12]), it is seen that three out of four activations occurred within the Mature level. Therefore the proportion is 75% of the total activation frequency which suggests evidence of high consistency based on the information from Table 4-14.

The fact that there was no activation during the first session (activation [8]) of the Exponential Growth & Regression unit will be considered for the final competency development assessment. During the Integrals & Modeling unit, it is seen that there were only two activations [20-21] that were recorded. Both activations occurred within the Intermediate activation level, which would suggest high consistency because two out of two activations (100%) were at the same level. However, discussing consistency with only two activations would be somewhat speculative. Having three levels of consistency would demand a frequency of at least three activations. Therefore, the additional consideration is the following: to calculate the consistency of activation for a competency, at least three activations must be recorded in the relevant diagram.

Table 4-15. Consistency levels from diagram in Figure 4.5.

Consistency Level	T.S.V	M.L.V.	Pr.In.T.
High	1(1)	1	3
Moderate	(1)	1	0
Low	0	1	0

Table 4-15 provides information about all types of consistency from the diagram in Figure 4.5. However, low consistency does not communicate any additional information for exploring the progress of competency development. Low consistency suggests that there should be several changes between different levels of competency activation These changes are captured from the second type of evidence (changes between activation levels). Therefore, the focus is on capturing only the instances of moderate and high consistency. It is possible to identify high or moderate consistency at the Initial Level (I) of activation, which suggests that the student had some difficulties activating the competency to a higher activation level.

In Table 4-15, at the high consistency level row and T.S.V. column, the notation ‘1(1)’ is used to indicate that there were two (1+1) instances of high consistency within the T.S.V. domain, and one of them was at the Initial Level (I). This information is captured because it provides insight into the student’s difficulty in advancing to a higher level of competency activation. As I will show in the following subsection, it must be clear that analyzing one type of evidence can produce inaccurate inferences regarding competency development. One student may display changes from a lower to a higher activation level more often than the other way around, but at the same time, display consistency at the Initial Level (I) of activation. Therefore, it is vital to consider all types of evidence combined to make inferences concerning the competency development.

Combination of evidence

The following three types of evidence have been presented and described: i) the frequency of competency activation (described in subsection 4.2.3), ii) the changes between competency activation levels, and iii) the consistency of competency activation. Combining these types of evidence will facilitate this study’s aims, that is, to explore the progress of students’ competency development over time. Table 4-16 is a combination of Table 4-13 and Table 4-15 that includes these three types of evidence. More specifically, these types of evidence concern the competency aspects (codes), which are the competencies’ components. Combining these types of evidence provides a qualitative description of the dynamic nature (progress) of competency development.

Table 4-16. *Summarized evidence for the (P.Q.) competency activation.*

Competency domain / Type of evidence	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	6	7	3	
Changes from higher to lower	0	2	0	
High consistency	1(1)	1	3	1(1)
Moderate consistency	(1)	1	0	(1)
Frequency of activation				16

This data interpretation method, visualized by the diagrams, can provide inferences regarding the progress of students' competency development only by combining the information provided by these three types of evidence. The frequency of competency activation concerns the number of times a particular competency aspect has been activated across all four units (seven sessions), but this activation's quality is unknown. Thus, the activation frequency provides only a part of the information available from the data. A more comprehensive approach is achieved by incorporating these additional types of evidence (changes between competency activation levels and consistency within competency activation level). The following overview describes the development within the Degree of Coverage of the 3-D model using as an example the (Int.El.) competency aspect. It displays how this combination of evidence is incorporated into the results and facilitates this study's findings.

Overview for Degree of Coverage for the (Int.El.) aspect

Regarding the first type of evidence (i.e., frequency of activation), the diagram in Figure 4.6 (repetition of Figure 4.4) informs that there is a total activations frequency of $n = 25$. For each of the four calculus units, the frequencies were [3, 5, 13, 4].

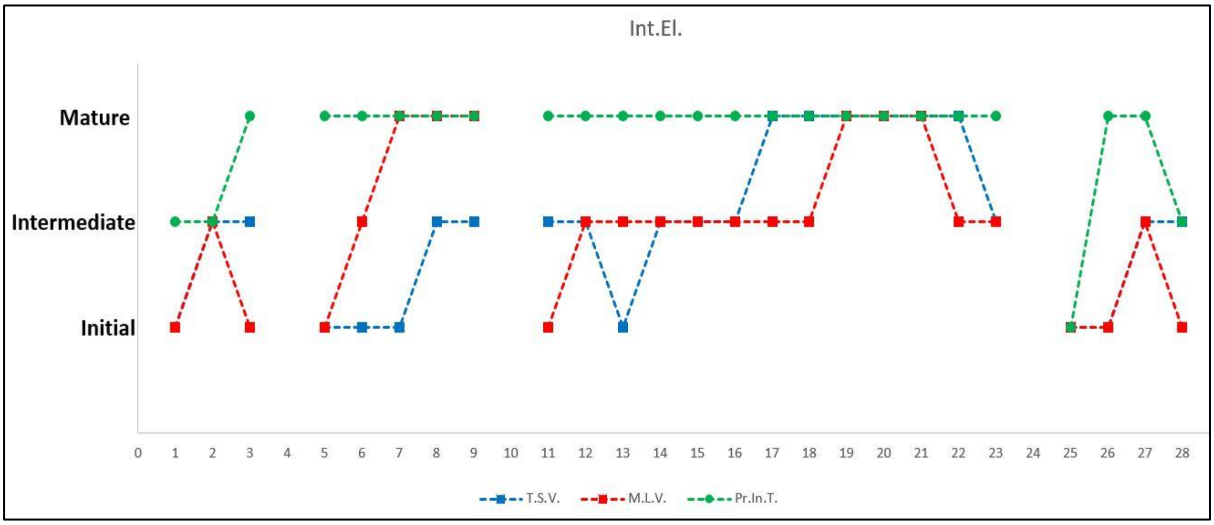

Figure 4.6. Activation for competency aspect (Int.El.).

Frequency of competency activation

Regarding the first type of evidence, namely the frequency of competency activation, a gradual increase until the third unit is noticeable, followed by an evident drop by the last unit. The increase is spotted from one unit to the next. As mentioned above, the frequencies were [3, 5, 13, 4], that is, three instances of competency activation during the Periodic Functions unit, five during the Exponential Growth & Regression unit, 13 during the Population Dynamics unit, and four during the Integrals & Modeling unit.

Changes between activation levels

By incorporating the second type of evidence, changes between activation levels, the information from Table 4-13 informs us that the changes from lower to higher and higher to lower levels of competency activation by domain were: (5, 2) for T.S.V., (6, 3) for M.L.V., and (2, 1) for Pr.In.T. These observations suggest that the progress was more often from a lower to a higher level of activation than the other way around and shows a dynamic of development over time.

Levels of consistency

Lastly, regarding the third type of evidence, levels of consistency, the diagram in Figure 4.6 informs us that there was high consistency at the Mature level for the Pr.In.T. domain twice, and moderate consistency for the T.S.V., M.L.V., and Pr.In.T. domains six, three and two times respectively. All this amount of evidence can be summarized in Table 4-17 below.

Any competency development overview for the Degree of Coverage will end with a table like the one below.

Table 4-17. Summarized evidence.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	5	6	2	
Changes from higher to lower	2	3	1	
High consistency	0	0	2	
Moderate consistency	4(2)	2(1)	2	
Frequency of activation				25

4.3.2 Development for Radius of Action

As mentioned in Section 2.4.2, the Radius of Action deals with the context in which the students will bring the competency into action. Niss and Højgaard (2011, p. 72) define the Radius of Action of competency as the spectrum of contexts and situations in which the person can activate the competency. The latter concerns the “mathematical contexts and situations (both internal mathematical ones and applied topics) but also [relates] to contexts and situations determined by problem formulations and challenges.” In this thesis, the four calculus units include different problem formulations that create, to some extent, different mathematical contexts.

Several theoretical modifications regarding the notion of the Radius of Action and adjustments to the characteristics of this thesis data were necessary. More specifically, I aim to explore the competency development within the Radius of Action of one student over time and not compare him/her with another student. Niss and Højgaard (2011) focus on comparing students regarding their Radius of Action, but here the focus is on the progress of development within the same individual over time. The calculus sessions have focused on the following mathematical areas: Periodic Functions (first), Exponential Growth & Regression (second and third), Population Dynamics (fourth and fifth), and Integrals & Modeling (sixth and seventh). These are four different mathematical contexts - whether there is a relation between exponential growth and regression and population dynamics - where I intend to explore the progress of competency development.

The same method adopted for exploring competency development within the Degree of Coverage was followed here to explore and identify evidence for competency development within the Radius of Action, but with one modification. The attention here is on each calculus unit, constituting a different mathematical context. Progress within the Radius of Action will be determined by the student's progress of competency development in each of the four units. The more contexts the student activated and developed a particular competency aspect, the greater the Radius of Action within this competency. In conclusion, within the Radius of Action, I focus on the spectrum of contexts and situations where the student activated the competency and to what degree compared to other contexts. The following subsection provides an example of how a competency aspect's development within the Radius of Actions is determined.

Overview for Radius of Action for the (Reason →) aspect

For this example of an overview, the diagram in Figure 4.7 will be used. This diagram displays the activation of the (Reason →) aspect of the Mathematical Reasoning and Communicating competency.

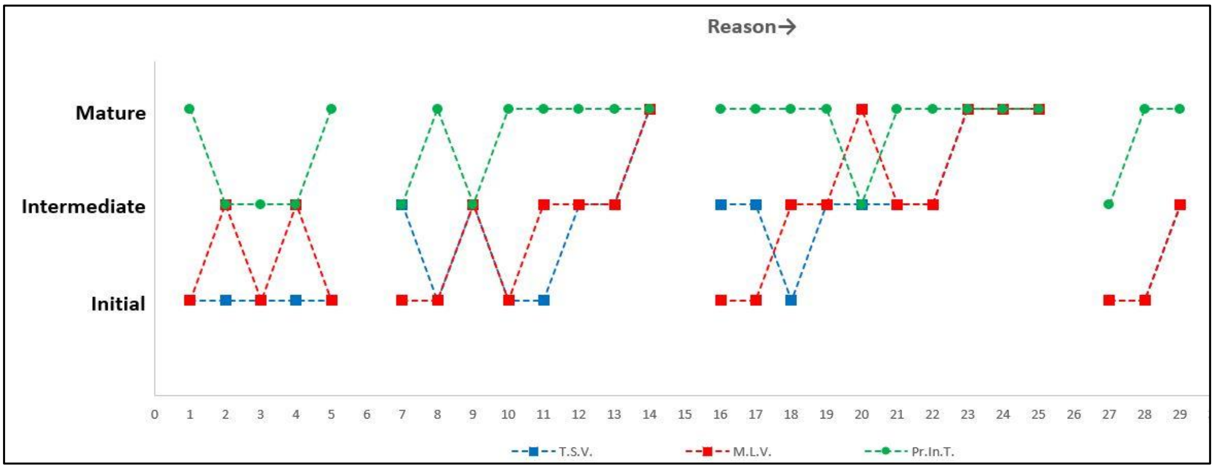

Figure 4.7. Example of diagram for the (Reason→) aspect.

Regarding the first type of evidence, frequency of activation, the diagram informs that there is a total frequency of $n = 26$. The frequencies for each of the four units were [5, 8, 10, 3]. Similar to the overview for the Degree of Coverage, I will provide a table that will summarize the evidence provided by the illustration of the data analysis through the diagram. Table 4-18 provides this evidence, and the only structural difference with Table 4-17 is that the summary of the evidence is displayed by calculus unit (1^{st}, 2^{nd}, 3^{rd}, and 4^{th} unit) because the interest here is on the different contexts, as mentioned in the previous subsection. For example, the information in the light, gray-shaded cell indicates that there were no changes from a lower to a higher level of activation, within the T.S.V. domain, in the Periodic Functions unit, three changes in the Exponential Growth & Regression unit, two changes in the Population Dynamics unit, and one change in the Integration & Modeling unit. Similarly, the information in the dark, gray-shaded cell indicates one instance of moderate consistency within the M.L.V. domain during the Exponential Growth & Regression and one (within the initial level) during the Integrals & Modeling unit.

Table 4-18. Summary of evidence by calculus unit for the (Reason→) aspect.

Type of evidence	**T.S.V.**	**M.L.V.**	**Pr.In.T.**
Changes from lower to higher	(0, 3, 2, 1)	(2, 3, 3, 1)	(1, 2, 1, 1)
Changes from higher to lower	(0, 2, 1, 0)	(2, 1, 1, 0)	(1, 1, 1, 0)
High consistency	((1), 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)
Moderate consistency	(0, 0, 1, (1))	(0, 1, 0, (1))	(1, 1, 2, 1)

Taking into consideration the information provided by the evidence above, the way in which competency development within Radius of Action will be determined will follow the subsequent structure:

Frequency of competency activation

As shown in the diagram, the frequencies were [5, 8, 10, 3], that is five instances of competency activation during the Periodic Functions unit, eight during the Exponential Growth & Regression unit, ten during the Population Dynamics unit, and three during the Integrals & Modeling unit. A more frequent activation was noticed during the Periodic Functions and the Exponential Growth & Regression units. Less frequent activation was noticed during the Periodic Functions and the Integrals & Modeling units.

Changes between activation levels

As seen in the table, there were more changes from a lower to a higher level of activation than the other way around for all calculus units except the Periodic Functions. This observation could signify a dynamic of competency development for the calculus units of Population Dynamics and Exponential Growth & Regression. It is difficult though to draw any inference, by employing this type of evidence, about the development of the (Reason→) aspect during the Integrals & Modeling unit because of the low frequency of activation that was observed.

Levels of consistency

Evidence of high consistency was only observed within the Initial activation level during the Periodic Functions unit for the T.S.V. domain. This finding suggests that the student had difficulties to visualize the solutions steps for the task she was working on. There were instances of moderate consistency at the Intermediate or Mature level of activation during the Population Dynamics and the Exponential Growth & Regression units which signifies a dynamic of competency development during these two units. Erika was able to express herself and provide justification for her ideas and writings having a clear perspective towards the solution of the tasks or the issue that was discussed and she did that while using expressions with mathematical language and with limited prompting from the teacher. During the Integrals & Modeling unit there were instances of moderate consistency within the Initial activation level (visualized with parenthesis (1) at the T.S.V. and M.L.V. domain). This suggests that there were limited evidence of competency development during this unit.

Concluding remarks to section 4.3.2

The overview that took place in the previous section concerns the competency development within the Radius of Action of the (Reason→) competency aspect. I can infer that there is evidence for competency development for the Population Dynamics and the Exponential Growth & Regression units but not for the Periodic Functions unit. Even though there were identifiable activations during the Periodic Functions unit, the changes from a lower to a higher activation level were not comparatively more than those in the opposite direction. Furthermore, the student struggled to imagine the solution steps of the relevant tasks. This difficulty is displayed in Table 4-18 with the symbolism ((1), 0, 0, 0) at the High Consistency row at the T.S.V. column.

The same analysis methods and tools used for the competency development assessment within the Degree of Coverage (subsection 4.3.1) are used for the competency development assessment that concerns the Radius of Action of the 3-D Model. The only change here is that these tools are used differently to navigate the focus of analysis at the different mathematical contexts that the four calculus units represent. Again, the combination of the three types of evidence is essential to make inferences regarding the development within the Radius of Action of the 3-D model.

4.3.3 Development for Technical Level

Looking back at subsection 2.4.3, we saw that Blomhøj and Jensen (2007) describe the Technical Level as a dimension that concerns students' flexibility when involved in problem-solving situations. Within this dimension, the focus is on these indications that will inform the study on how conceptually and technically advanced the mathematics is; that the student can integrate appropriately during the competency activation. Here, I explore the student's conceptual and technical comprehension and focus on how well developed the entities and tools are when they bring a competency into action. Haines and Crouch (2010) point out that the Technical Level is usually regarded as sufficiently evaluated outside the actual mathematical activities. Addressing the nature of development within the Technical Level of each competency is an attempt to reveal and display the flexibility of the student's use of mathematics while activating a specific mathematical competency.

For this research, the development within the Technical Level will be determined by exploring the students' flexibility when working with the technical parts of the tasks. A student may activate two or more competency aspects when working on the same technical part of a task, for example, while working on the exponential growth formulas. A visualization of the sets of technical parts of every session is displayed in Table 4-19.

Table 4-19. Table of tasks and their representative technical parts (a).

Dates	Warm up task	Main task	Technical Part
3/10	A table with the average monthly temperatures at an airport is given. The students are asked to produce a sinusoidal equation using the cosine function and after that using the sine one.	A team of biologists has discovered a new creature. Information for the animal’s temperature, through time, are given. The students are asked to provide an equation which models the creature’s temperature in a day.	Real functions, sinusoidal functions, variables and constants in functions (parameters), graphs and Cartesian coordinate system, symbols.
10/10	In 1960, an acre of barley in the U.S yielded 31.0 bushels (more than a cubic foot, about the size of a wastebasket) of grain. Students are informed about the yield growth and are asked to estimate the approximate barley yields for years after 1960?	A table with the number of recyclables measured in tons in respect to a specific year, is given to the students. Students need to find at what year will recycle reach 3 million tons and to estimate the number of recyclables (tons) in a specific year.	Exponents, logarithms, symbols and formulas, variables and constants in functions, exponential functions, percentages, graphs.
24/10	Intro to exponential growth by binary fission. The increase in a bacterial population is by geometric progression. The students are asked to find generation time G when the time interval t, the number of bacteria B at the beginning of a time interval, the number of bacteria b at the end of the time interval and number n of generations are given.	The students are assigned the following task: The very first malignant cell in the body of a patient appeared on the 1st of April 2016. Can you figure out the date (approximately) when the cluster will be detectable by x-ray?	

Dates	Warm up task	Main task	Technical Part
31/10	The groups must estimate how long would it take for a pair of individuals to produce the world population of today at the present rate of growth r=2% per year.	Reading the main theme of a science fiction thriller novel the students are asked to check if a single cell of the bacterium E. coli divides every 20 minutes, how many E. coli would be there in 24 hours.	Symbol and formula rich problems, logarithms, variables and constants, differentiation, exponents, exponential functions large decimal numbers and fractions, graphs.
7/11	The students are asked to find the date the world's population will be doubled using any kind of information that is available online.	A reverse task is given here. Assuming the age of human race (2 millions of years) students must find the excess rate.	
14/11	According to one study, the temperature is rising at the rate of $0.014t^{0.04}$ degrees Fahrenheit per year, where t is the number of years since 2000. Given that the average surface temperature of the earth was 57.8 degrees Fahrenheit in 2000, predict the temperature in 2200.	(* students faced some serious difficulties when using integrals and I had to rearrange the day's planning by making a small introduction to basic parts of integral calculus that resulted in having time only for the warm-up task)	Symbol and formula rich problems, logarithms, variables and constants, differentiation, integration, exponents, large decimal numbers and fractions, graphs.
21/11	Assume the change in voltage V of a neuron with respect to time follows the differential equation over the course of 100 ms, where t is measured in (ms) and V in millivolts (mV). Assume $V(0) = -70mV$. What is the voltage after 100 ms?	If the present worldwide rate of consumption increases by 2% every year, how long will it take to use up the earth's petroleum?	

Niss and Højgaard (2011) present a series of examples about tracking development for each dimension of the 3-D model. To explore the competency development within the Technical Level, they analyze the technical parts of every competency's activation and provide a qualitative description of the students' development concerning this dimension. For example, this kind of analysis includes assessing the use of numbers, the decimal position system, and the use of geometric statements. A similar method is used in this thesis to evaluate the development of the Technical Level but with a significant addition: the implementation of the M.L.V. activation domain.

While constructing the scaling instrument and creating the M.L.V. domain (Mathematical Language and Vocabulary), I observed several theoretical similarities with the description of the Technical Level of the 3-D model. Both M.L.V. and Technical Level describe a competency dimension relevant to the flexibility of the mathematics in use and address the students' conceptual and technical comprehension. However, there are evident differences between these two layers. The Technical Level not only focuses on the flexibility of mathematics in use. It also focuses on whether this use is directed towards the correct path of the solution process. The latter resembles the T.S.V. (Task Solving Vision) domain of the scaling instrument of this thesis. Niss and Højgaard (2011, p.158), in their evaluation of students' Technical Level of symbol and formalism competency, notice the following:

> The Technical Level of F's symbol and formalism competence is linked (…) to real functions of one or two variables, as well as linear algebra in number spaces. She can cope with both symbol and formula rich problems relating to concrete exemplars of functions, vectors, matrices, eigenvalues (…) she has a solid grasp of symbol and formula based handling of differentiation, integration, series expansion, series convergence, determination of eigenvalues, diagonalization of matrices (…) if originality and inventiveness are the order of the day because the situation is not completely covered by familiar conditions and procedures, she can normally not work out what to do.

In conclusion, for this research, I combine the information obtained from the data analysis and illustrated in the diagrams - using only the M.L.V dimension - while providing an accumulative qualitative description for the technical parts of each of the five competencies of the competence framework, following Niss and Højgaard's (2011) method as described above. To perform such a description, I follow a session-to-session evaluation of the Technical Level for every competency activated during the calculus units. The same technical parts can be observed in different competencies while exploring the Technical Level.

Therefore it is possible to use the same data for a different analysis. More than one competency can be activated in a single turn, which suggests that it is expected to use the same technical parts for a different competency analysis.

Overview for Technical Level for the (→Sol.) aspect

In what follows, I provide an example of an overview regarding the development of the (→Sol.) aspect of the Mathematical Thinking and Acting competency within the Technical Level. This overview consists of two parts: the overview for technical parts and the exploration of the M.L.V. domain of activation. To draw inferences regarding the development of the Mathematical Thinking and Acting competency within the Technical Level, all aspects of this competency (i.e., (→Sol.), (Lim), and (P.Q.)) should be considered as it will displayed in detail in Chapter 6.

<u>Overview for technical parts</u>

Erika exhibited a grasp of symbol and formula-based handling and manipulation of differentiation, but her handling skills appeared somewhat limited when performing integration. During the solution process (i.e., activation of the (→Sol.) competency aspect), Erika faced some problems in comprehending the difference between constants and variables. This problem was mainly exhibited at the first and at the beginning of the second session, but she gradually grasped the notion of variables and constants within exponential and logarithmic functions. However, the problem reappeared when she failed to see the use of constants a and b within the definite integral $\int_a^b f(x)dx$.

<u>Exploring the M.L.V. domain</u>

As mentioned above, for the purposes of this example of analysis, I present here only the diagram for the (→Sol.) competency aspect (Figure 4.8) of the Mathematical Thinking and Acting competency and provide the evidence focusing on the M.L.V. domain of activation. In Chapter 6, I include all the aspects of each competency and provide a final assessment regarding the competency development of all competencies within the Technical Level.

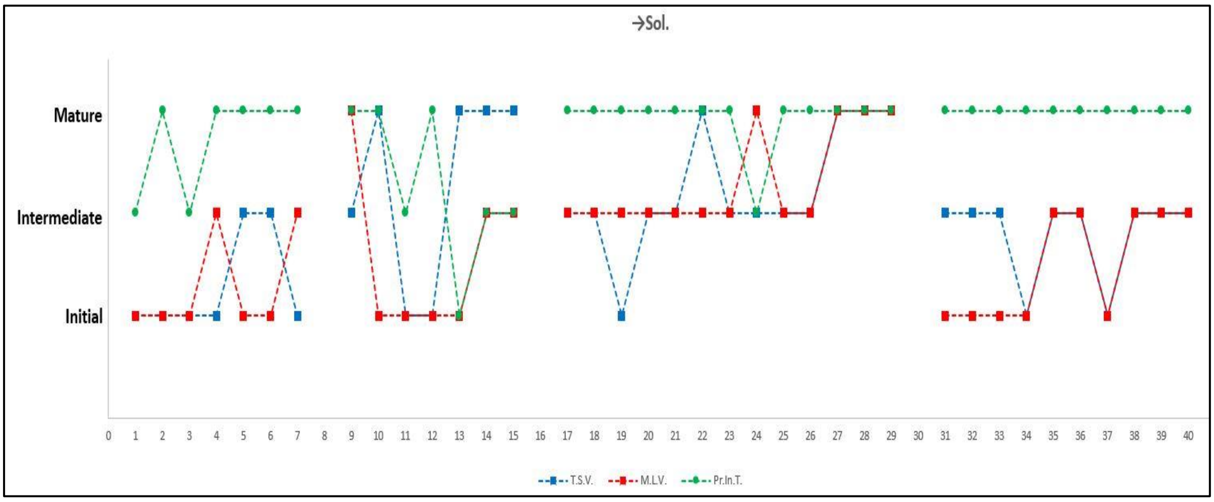

Figure 4.8. Example of diagram for the (→Sol.) Aspect.

Table 4-20 below summarizes the two types of evidence (changes between activation levels and consistency of activation) within the M.L.V. domain for the (→Sol.) aspect of the Mathematical Thinking and Acting competency. Interestingly, one can notice in the last row of the table (gray-shaded cell) that Erika displayed moderate consistency at the Initial Level of activation three times and only once above it (symbolized as ‘1(3)’) while activating the (→Sol.) aspect. The evidence from this table suggests that Erika activated the (→Sol.) aspect more often from a lower to a higher level of activation than the other way around, which indicates a dynamic of development over time. However, as mentioned above, she displayed a consistent activation within the Initial activation level on multiple occasions except during the Population Dynamics unit. Therefore the combination of evidence derived from the overview for technical parts and the exploration of the M.L.V. domain suggests a *moderate combination of evidence* for the development of the (→Sol.) aspect of the Mathematical Thinking and Acting competency within the Technical Level of this competency.

Table 4-20. Evidence for Mathematical Thinking and Acting competency.

Type of evidence	**Lim.**	**P.Q.**	**→Sol.**
Changes from lower to higher			7
Changes from higher to lower			4
High consistency			0
Moderate consistency			1(3)

As seen in the table above, the cells for the (Lim) and the (P.Q.) aspects are empty. This is because here, for the purposes of presenting an example of how the development with the Technical Level will be determined, the focus is narrowed in one competency aspect, the (→Sol.). To make inferences for the development of the Mathematical Thinking and Acting competency, all aspects need to be considered. Therefore these cells will be filled with the relevant information in the Findings chapter comprising a final assessment concerning the quality of the combination of evidence that characterizes the nature of the competency development.

4.4 Analyzing the competencies interrelation

This section includes three subsections where I first provide a reminder of the second Research Question of this study (subsection 4.4.1). Following that, in subsection 4.4.2, I elaborate on the phenomenon of concurrent competency activation. Finally, in subsection 4.4.3, I describe the properties and the notation of the hypothesis test adopted to explore the concurrent competency activation.

4.4.1 A reminder of the second Research Question

This study's aims include the exploration of competencies interrelation as stated in subsection 2.5.2, where the second Research Question is addressed. I seek to identify whether the competencies are interrelated and, if so, how. In subsection 4.2.2 and Table 4-10, instances of concurrent competency activation were presented. Concurrent activation occurs when the activation of two or more competency aspects is identified within the same turn. For the needs of this study, I narrowed my interest to these instances where two competency aspects are activated simultaneously. Concurrent activation of two competency aspects would form a *competency pair*.

Exploring these instances of concurrent activation would give possible insight into whether competency aspects appear to coexist with other aspects (i.e., two aspects activated within the same student's utterance) more often or less often than others. To address this issue, I needed to search for a quantitative tool that could be adopted to track any statistically significant observations regarding the observed and expected frequencies of concurrently activated competency pairs. To address this issue, I decided to apply a hypothesis test using the Binomial Test presented in the following subsections.

4.4.2 Competencies interrelation

Concurrent competency activation is by no means an outcome of a poorly designed competence framework. Regarding their work on the KOM model, Niss and Højgaard (2011) are prepared for that possibility, and they point out that:

> The fact that such competencies are independent and relatively distinct does not imply that the different competencies are unrelated to each other or that they are so sharply defined that there is no overlap. Let us instead think of a competency as a "centre of gravity" in a "cluster" of things that are dense near the middle and sparser towards the edges, and which is partly interwoven with other clusters. (p.49).

The interest is whether there may be any association between the competency aspects of the framework; in other words, where the activation of one competency often occurs at the same time as another, or whether two competencies are never observed to be activated at the same time. The competence framework of this thesis comprises 16 competency aspects, as seen in Section 4.1. Therefore, 120 ($\frac{16!}{2!(16-2)!} = \frac{16!}{2!14!} = \frac{15\times16}{2} = 120$) possible pairs of competencies (i.e., competency aspects) could be observed and examined if any pair presents a statistically significant frequency of appearance. As discussed in the following paragraphs, statistically significant frequency could be significantly low or high.

Approaching the second sub-question of the second Research Question (subsection 2.5.2) I consider a null hypothesis H_0 (I will revisit the null hypothesis concept in the following subsection too) that the observation of activations shows no association nor positive (evidence of a pair being activated at the same time), or negative (evidence that the activation of one competency aspect appears to exclude the other). Another way of expressing the null hypothesis is that simultaneous activation occurs randomly.

For each of the 120 possible pairs of concurrent activation, the number of turns in which an incidence of concurrent activation is observed is counted. The frequency of competency activation (i) and the frequency of concurrent activation of the competencies pair (i, j) will be represented respectively as: $f_a(i)$ and $f_{ca}(i, j)$. It is necessary to show whether $f_{ca}(i, j)$ is sufficiently large or small to reject the null hypothesis and conclude that the alternative hypothesis – there is an association (positive or negative) should be accepted. It seems important, I hold, to find a way to determine the value of $f_{ca}(i, j)$ is statistically significant.

Due to the large number of possible pairs to be tested, it is necessary to take a rather small 'p-value' before a statistically significant deviation from the null hypothesis is accepted. This is referred to as a 'Bonferroni correction.' Given over 100 tests, even at the 1% level ($p > 0.01$) at least one 'significant' result would be anticipated. Therefore the significance level is set at 0,1% ($p < 0{,}001$) – it would be better still to set the level at 0,01% ($p < 0{,}0001$), but the data is probably not strong enough to support this and it likely that all pairs would satisfy the null hypothesis.

It is necessary to compute the number of turns in which pairs appear together and the probability of common occurrence taking place randomly. An a-priori approach would start from the known number of turns N and the known number of observed occurrences of each competency activation: $n_1, n_2, n_3, \ldots n_{16}$. It is possible to compute a theoretical probability for a turn including the activation of a competency if this is a purely random event. Probability that turn 'X' include activation of competency 'c' is $P(X_c) = \frac{n_x}{N}$. However, the calculation of the theoretical probability of two activations in a single turn is more difficult to compute, and more complex still would be the computation of probabilities for 1, 2, 3, 4 etc. such co-occurrences. This calculation is believed to be possible, but it has not been attempted. In the following subsection, I present the statistical test employed to address the second Research Question, namely, the Binomial Test.

4.4.3 The Binomial Test

The Binomial Test is employed when an experiment has two possible outcomes (i.e., success/failure), and there is an idea about the probability of success. A Binomial Test is run to see if observed test results differ from what was expected. The observed test results for this study will be proportions that will derive from the analysis of the coding process. This proportion should be a fraction of the observed frequency of concurrent activation of two competency aspects over the total frequency of the analyzed students' utterances (i.e., turns). The expected test results for this study are proportions that will derive again from the coding process. However, the proportion here is a product of two fractions that represent the probability of appearance of each one of these two competency aspects. I elaborate on these two proportions in the following paragraph. The null hypothesis H_0 for this test is that these results do not differ significantly from the expected ones.

To simplify the notation, I number all aspects of the competence framework (Table 3-1) in the following order: (P.Q.) as 1, (Lim.) as 2…, (U.T.) as 16. For example, the competency pair of (Lim.) and (Reason←) competencies will be represented as: $(2{,}13)$. The two proportions mentioned in the previous paragraph are: $p_{e(i,j)}$, the theoretical (expected) probability of concurrent activation and $p_{o(i,j)}$, the actual (observed) concurrent activation. As mentioned in the previous subsection, the frequency of activation of competency (i) and the frequency of concurrent activation of the competencies pair (i,j) will be represented respectively as $f_a(i)$ and $f_{ca}(i,j)$ respectively. For example, $f_a(5)$ represents the frequency activation of the (Int.El.) competency aspect (fifth in the competence framework), and $f_{ca}(10{,}16)$ represents the frequency of concurrent activation for (C.D.Rep.) and (U.T.) competencies (10th and 16th competency aspects of the competence framework.

The values for the *expected probabilities* can be seen as a fraction of:

$$p_{e(i,j)} = \frac{frequency\ of\ comp.\ activation\ (i)}{total\ number\ of\ discource\ units} \times \frac{frequency\ of\ comp.activation\ (j)}{total\ number\ of\ discource\ units} = \frac{f_a(i)}{n} \times \frac{f_a(j)}{n}\ (1)$$

The values for the *observed concurrent activation* as a fraction of:

$$p_{o(i,j)} = \frac{frequency\ of\ concurrent\ activation\ of\ comptencies\ (i,j)}{total\ number\ of\ discource\ units} = \frac{f_{ca}(i,j)}{n}\ (2)$$

The formula for the Binomial Test is: $P(x) = \frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}$ (3), where,

$P(x)$ is the binomial probability
x is the total number of expected 'successes'
p is the probability of a success on an individual trial, observed proportion
n is the number of trials

Using the notation introduced above from (1) and (2), formula (3) will now take the following form: $P(x) = \frac{n!}{x!(n-x)!}{p_0}^x(1-p_0)^{n-x}$, for this study, the total number of expected successes 'x' is given by the formula: $p_{e(i,j)} \times n$, where n is the sample size. Since there are 16 competencies aspects, we have 120 possible pairs ($\frac{16!}{2!(16-2)!} = \frac{16!}{2!14!} = \frac{15\times 16}{2} = 120$). This suggests that the test should be run 120 times. The sample size n is 341 turns (i.e., students' utterances in all seven sessions). The values for the numerator of the $p_{e(i,j)}$ proportion and the values for the numerator of the $p_{o(i,j)}$ proportion will be extracted from tables that will be produced in the next chapter (subsection 5.7.1). Three different types of binomial tests were conducted: greater, less, and two-tailed at a significant level of 0.05. These tests were run using R software, and the results will be presented in subsection 5.7.2.

Assumptions for the Binomial Test

Below I list the primary assumptions that one must make to apply the Binomial Test. These assumptions will be also discussed in Section 8.4 where I present the limitations of this work.

i. Items are dichotomous (i.e. there are two of them) and nominal.
ii. The sample size is significantly less than the population size.
iii. The sample is a fair representation of the population.
iv. Sample items are independent (one item has no bearing on the probability of another).

The null hypothesis

The null hypothesisH_0, in this case, is that the results do not differ significantly from what is expected, that is, the observed proportions do not differ significantly from the expected. The alternative hypothesis H_a is that there is a significant difference between these two proportions. I am assuming a significance level alpha: $a = 0.05$, which means that p-values $= p$ of the test (for some competency pair) less than the significance level would be significant. In these cases, I can conclude that the proportion (probability) of the frequency of appearance of the specific pair is significantly different from the expected with a p-value $= p$. There are some possible meta-interpretations of the situations where the p-value is lower than the significance level a. These interpretations are discussed in the following paragraph.

One interpretation could be connected with a form of competency dependency and mutual exclusion. To be more specific, in these cases where $p \leq 0.05$, I can assume that the activation of a particular competency of the framework excludes with a 95% confidence interval the concurrent activation of another competency. Also, this method could give information regarding the overall level of independence that a specific competency has concerning the activation of other competencies. There is an opportunity to examine which of the 16 of the competency aspects tends to exclude, more often, the concurrent activation of another competency. Lastly, we can conclude which pairs appear less or more frequently than expected.

4.5 Validity and reliability

This section displays the ways adopted to secure acceptable inter-coder reliability. To check the reliability of the coding schema (based on the competence framework) and the scaling instrument that I had developed and applied during the data analysis, my principal supervisor of this research and I had a meeting. Working on different parts of the sessions' transcriptions, we attempted a separate coding and grading (assigning activation levels using the scaling instrument). We aimed at a significant number of turns and codes to reach a certain level of reliability. To be more specific, we coded 11 turns separately where 23 different competency aspect activations were identified, achieving 87% (20 out of 23) matching. There were cases where the coding included a concurrent activation of more than one competency, and the percentage mentioned before includes these cases.

The grading inter-coder reliability (assigning competency aspect activation levels based on the scaling instrument) took place by the end of the competency coding process. Three grades were assigned for every code activation, reaching a total of 60 gradings. I do mention 60 grades since my principal supervisor and I had observed 20 code activations ($20 \times 3 = 60$) in common. The results were quite similar to the competencies intercoder process: 86.7% (52 out of 60). An identical process has been applied in several coding stages and for several months, and the inter-coder agreement never dropped below 83%. The competence framework appeared to be somewhat reliable and, therefore, suitable to incorporate it into this study.

In Appendix F, there are two tables (Tables F-1 and F-2) with extracts from different transcriptions with the coding process that illustrates the instances where there was a coding matching. I should mention that the three cases where there was a disagreement regarding the coding did not cause concern about different coding incidents. I considered that a competency had been activated in these three instances, and my principal supervisor considered that the specific competency was not. There is only one difference between these two tables. In Table F-1, the last two columns display the codes, and in Table F-2, the contents of the last two columns are changed and replaced by their grades assigned by applying the scaling instrument.

4.6 Ethical issues

In this section, I refer to the invitation issued to the students (subsection 4.6.1), where I introduce the goals and the people involved in the project. Subsection 4.6.2 describes the discussions and agreements made with the administration and academic staff of the universities included in this study. This is a brief section because a lot of the material that concerned the ethical issues (e.g., Data Protection approval document) is located at Appendix G of this thesis.

4.6.1 Students invitation

Before starting the sessions, I considered it essential to send a formal invitation to the students that would present the project with a description of the program's goal and a brief introduction to the notion of mathematical modeling its goal, the research team, and some other important information. The relevant text can be found in Appendix G. The same day with the invitation; I delivered the letters of consent to the students where the MatRIC team was asking for students' consent for filming them during the project.

4.6.2 Discussions with academic and administrative staff and data protection.

This research was part of a MatRIC (Centre for Research, Innovation, and Coordination of Mathematics Teaching) project. Therefore, approval from the Norwegian Data Protection Agency has been requested and granted before the beginning of the project. Anonymity and confidentiality have been secured, and the students were informed on multiple occasions that their anonymity would be guaranteed. Attention was also given to minimizing the risk of harm for the participants. Any situation where the participants could be harmed or put in a position of discomfort or distress has been considered.

Before the beginning of the project, there have been several meetings with the university administration and faculty members. One of the critical meetings for the facilitation of the sessions that would take place was with the professor of mathematics responsible for teaching the calculus course that the students would participate in the project. Syllabus and technical issues have been discussed, and there was a general agreement regarding the goals of the implementation and how it will be connected with their formal calculus courses. A second meeting took place, which included several administration staff members. The purpose of that meeting was to determine the time frame of the calculus sessions, the room booking for the courses, the composition of a student invitation that would describe and explain the project and few other logistic issues (e.g., recording equipment, data storage gear) so the higher degree of preparation would be achieved.

5. Results: Competency activations

This chapter begins by presenting an overall summary of all seven calculus sessions in Section 5.1 with a description of the tasks and the students' and teacher's activities that took place during these sessions; this provides the reader with background information on the sessions in order to follow the case studies presented in section 5.6. A reminder of the thesis's competence framework is provided in section 5.2. The initial results regarding the competencies activations for all students that participated in the sessions are included in section 5.3. A reminder of the scaling instrument can be found in Section 5.4, and Section 5.5 comprises a guide for the four steps that will be followed for the presentation of the results of the two case studies of Erika and Johannes displayed in Section 5.6. Finally, Section 5.7 includes the results from the competencies interrelation analysis.

5.1 The sessions

This section presents a descriptive account of this project's calculus sessions during an Autumn semester. This account does not include an interpretation of students' actions. Instead, it describes what happened in the sessions and how the students engaged and interacted with the non-routine mathematical tasks. To keep this account concise and in a summarized form, I decided to include and present here a reduced number of the sessions and not all seven of them. The complete account of the seven calculus sessions is presented in Appendix B. In this account here, I will present the second session (first session of the second calculus unit) on Exponential Growth and Regression, held on the 10th of October, and the fifth session (second session of the third calculus unit) on Population Dynamics, held on the 7th of November.

In this account, I describe the sessions' tasks (see also Table 3-4), the classroom actions, and discussions between the students and between the students and the teacher. Here, I follow the actions of all students participating in the sessions. I begin by listing the students present in the sessions. Not all students who participated in this project managed to attend every session, and not all students were monitored by the Focus camera. This account concludes with a summary of all sessions that will enable the reader to move to the next section that discusses some provisional results from the students' work. The summary is accompanied by a table that recaps the sessions' notable occurrences, such as difficulties the students faced while engaging with their assigned tasks, duration of discussions on a particular concept or task, and student's interest in particular task themes.

Second session: Exponential Growth and Regression I (10.10, second unit)

Students present: Erika, Johannes, and Rene (Focus Camera), Kim, Linnea, and Maria (GoPro camera).

In this session, the students were reintroduced to the notion of exponential growth and regression. This session was the first part of this calculus unit and began with a short introduction to the exponents and logarithms rules and properties, and then the class moved on to the warm-up task, the Burley Yield task (Table 3-4). After consulting his notes, Rene began working by categorizing the relevant information from the task. Without using the notation given to him, he wrote "$y = 1961 = 31.0 + 0.017 \cdot 31.0 = 31.522$", which showed that he struggled with the task's notation. He solved the task after some discussion with the teacher. Linnea and Maria started by working on understanding what '1.7%' represents and how to incorporate into their model. The teacher asked questions that could help the students envision how the model's equation should look like, "how can we call the expression '1.7%' that we are dealing with here?" and Linea and Maria eventually wrote down the expression: $y = 31.0(1.017)^t$.

Kim found it difficult to translate the percentage (1.7%) to 1.017, and she spent considerable time on this task. The teacher worked with her on the solution due to time restrictions and proceeded to the main task. Kim followed what Linea and Maria did and copied the expression '$y = 31.0(1.017)^t$' at her notes. Erika and Johannes started discussing the task, and Erika asked Johannes whether he remembered the 'farmer task' that they did in their calculus course. Both students translated the percentage (1.7%) to 1.017 quickly and exchanged ideas about the solution's final steps. Interestingly, Erika was not happy with her result even though it was correct, "I [may] do something wrong with my calculations with the calculator," but this issue was resolved. Before proceeding to the next task, the teacher projected on the board how an exponential graph should look and explained why the expression '1.7%' was the growth rate.

The students continued on another warm-up task, the Salmon task (Table 3-4). Rene noticed many similarities with the task, "harder but the same, I think," and began working on his notebook. Linnea looked at her notes from the previous task so she could compare them with the new task. Maria started working on the expression "average decline of 18.1% per year," and Kim copied the necessary information into her notes. Erika and Johannes both experienced difficulties understanding that this is a task about a declining population. They both said that they were expecting a "negative percentage." With some guidance from the teacher, Erika (even though struggling again with the calculator) announced her solution, '$86.500(1 - \frac{0.181}{100})^{15}$'.

Linnea and Maria solved the task, with Kim having some difficulties following due to difficulties understanding the behavior of the a^xgraph for $a < 1$, but she eventually understood of the task. Johannes, working closely with Erika at all times for this task, reached the same expression that Erika presented. At this point, the teacher initiated a discussion about the difference between '$1 + r$' and '$1 - r$' when using the following equation:

$y = y_0(1 + r)^t$ and explained these differences graphically. The class moved then to the main and final task of the session.

During the Recycles task (Table 3-4). Linnea and Maria initially struggled with the task because no rate was given. Kim looked at her previous notes and wrote down the table elements with the years and the relative number of tons. The teacher pointed out that "the fact that you don't see any rate at the task does not mean that you cannot find one based on what is given to you" in an attempt to motivate the students to work on what they have. They (Linnea, Maria, and Kim) first noticed an increase in the number of recyclables each year and then tried to find the "pattern of the increase." Kim was relatively silent during this process. Working together, Linnea and Maria suggested a solution for the task with minimal help from the teacher and presented their results. Kim did not produce a solution, and the teacher explained the task-solving process.

Erika and Johannes compared the previous task with the new one and realized that even though there was no rate, they should find one. Johannes wondered if they are "supposed […] to estimate the rate on [their] own?" and Erika agreed that this was the way to solve the task. With some teacher assistance, Erika and Johannes realized that they were dealing with an increase in the recycles each year and solved the task faster than the previous tasks. By that time, the session was already 10 minutes over schedule, and the teacher mentioned that they would continue exploring the notion of exponential growth and regression in their next meeting.

Fifth session: Population Dynamics II (7.11, third unit)

Students present: Erika, Johannes, Kim, Linnea, Maria, and Rene (Focus Camera).

This session began with the teacher notifying the students that "today [they] are going to be the researchers" and that they "should come up with the results that satisfy [their] logic" and began working on the warm-up task, the Population Doubled task (Table 3-4). Rene wrote down what the task was asking him to find and began looking online for the available information. After some searching, the teacher suggested a site where he could find the needed information. Linnea and Maria started working on the concepts of 'births rate' and 'deaths rate' while Kim transferred the information projected onto the screen into her notebook. Johannes had some difficulties handling the concepts of births and death rates, saying that "we may have to divide them and find their proportion." However, he overcame this issue after discussing it with the teacher and Erika. All students needed more time than the teacher planned, and the teacher decided to present the solution to the task, with most of the students being close to reaching a solution themselves.

The class moved to the main task, the Reversed Population Doubled task. Rene had to leave the class for personal reasons and did not work on this task. Linnea and Maria worked on the notation of the task (i.e., $P(t) = P(0)e^{rt}$), and Kim realized, "we are looking for the *r* in our case." Erika and Johannes found the correct excess rate (i.e., 0.001%) with minimal prompting from the teacher, and Johannes concluded that giving just a single number for the rate would not convince someone that the rate is not constant, “I don’t think it would not help [anyone] just giving a number.” The class ended with a discussion about factors that significantly influenced changes in births and death rates. The teacher finished the session by projecting a graph of the future world population until 2100, which provoked comments both from the past (e.g., the baby boom period) and for the future (e.g., the annual growth rate of the world population would be close to 0.08% by the year 2100).

A summary of all sessions

Across all the calculus sessions, two students, Erika and Johannes, attended all seven of them, and their actions were captured with the Focus and the GoPro camera. Rene attended six sessions, all captured with the Focus and the GoPro camera. Linnea and Maria attended seven and six sessions, respectively, but two were captured only with the GoPro camera. Finally, Kim attended four sessions, one of them was captured with the GoPro camera only (see also subsection 3.5.1 and Table 3-5). The most complete and promising data would be derived from Erika and Johannes, who attended all seven sessions, and their actions were captured from both cameras. That was the main reason for deciding to maintain the focus on these two students and explore their competency development throughout these seven sessions.

All students who participated in the sessions showed willingness and enthusiasm to work with the tasks, which could be attributed to the fact that they encountered a series of mathematical tasks within a biological context. As was expected, the students approached the tasks in different ways. Some students displayed familiarity with new mathematical concepts, symbols, and graphs, and there were students who needed more time to adapt to this relatively new classroom environment and were asked to work on mathematical tasks relevant to the field of their study. Due to scheduling restrictions, there was only one session for the Periodic Functions unit. However, the rest of the units (i.e., Exponential Growth and Regression, Population Dynamics, and Integration & Modeling) consisted of two sessions each.

Table 5-1 below displays in a concise form notable incidents or observations that occurred in each section and the difficulties that students appeared to experience during their engagement with the non-routine mathematical tasks. The table also includes the session number, the names of the calculus units, the warm-up and main tasks, and the number of students participating in each session.

Table 5-1. Observations from the sessions.

Session/Unit/Warm up/main task/Number of students present/	Notable occurrences	Content difficulties
1st/Periodic Functions/Airport Temperatures task/New Creature task/5/	• Use and translation of graphs for every task. • Different but correct graphs for the same task. • All students constructed equations that modeled the temperatures from the table. • Some students were able to track down and correct their own mistakes without teacher's intervention.	• Notation in periodic equations was challenging. • Graphical depiction of main task.
2nd/Exponential Growth and Regression/Barley Yield and Salmon tasks/Recyclables task/6/	• Lengthy discussion and work on the two warm-up tasks. • Session lasted 10 minutes longer than scheduled. • Reintroduction of exponents and logarithms rules.	• Rules of exponents and logarithms. • Translation of percentages to decimal form. • Growth rate concept.
3rd/Exponential Growth and Regression/Bacterial population task/Cancer Cells task/5	• Riddle use appeared to motivate students. • Frequent internet use for searching task-relevant information. • Great interest for the cancer cells task.	• Properties of natural logarithm. • Finding the correct and appropriate information from the web.
4th/Population Dynamics/World Population task/ E. Coli task/6	• Introduction lasted longer the previous sessions. • Students showed familiarity with the structure of the sessions. • Population dynamics tasks were appealing to students.	• Notation of growth rates. • Differences between discrete and continuous models. • Notation for main task.
5th/Population Dynamics/Population Doubled task/ Reversed	• No introduction needed; students engaged with the warm-up task instantly. • Solution for warm-up task given from the teacher.	• Concepts of births and death rates. • Notation for main task.

Session/Unit/Warm up/main task/Number of students present/	Notable occurrences	Content difficulties
Population Doubled task/6	• More than 35 minutes of class time spent on main task. • Accurate interpretation of future population graphs.	
6th/Integration and Modeling/Earth's Temperature task/no main task/4	• Session's plan rearranged due to difficulties in integration techniques. • Time only for the warm-up task.	• Integration techniques. • Notation for warm-up task.
7th/Integration and Modeling/Neuron Voltage task/ Petroleum task/5	• Teacher's prompting was more often than any other session. • Students engaged in discussion about applications in mathematics.	• Integration techniques. • Notation for both tasks. • The concept of increasing rate of consumption.

5.2 A reminder of the competence framework

This section functions as a reminder of the competence framework presented in Section 4.1. The competences are presented in the same order as they are in each case study (Erika and Johannes). The framework is the following:

1) Mathematical Thinking and Acting, including three basic competencies (aspects):
    - To pose questions that have a mathematical characteristic or to know the kinds of answers that mathematics may offer (abbreviation: P.Q.) (1)
    - To understand and handle the scope and limitations of a given concept (abbreviation: Lim.) (2)
    - To attack (take actions towards a solution) mathematical problems of various kinds (abbreviation: →Sol.) (3)
2) Mathematical Modeling, including six basic competencies (aspects):
    - To be able to assess the range and validity of existing models (abbreviation: R.+V.) (4)
    - To interpret and translate the elements of a model during the mapping process. In this situation, the student should be able to understand the real problem and adapt a model based on reality (abbreviation: Int.El.) (5)
    - To interpret mathematical results in an extra-mathematical context and generalize the solutions developed for a special task-situation (abbreviation: Int.Res.X) (6)
    - To critique the model by reviewing or reflecting and questioning the results (abbreviation: Cr.) (7)
    - To look and search for available information and to differentiate between relevant and irrelevant information (abbreviation: Info) (8)
    - To choose appropriate mathematical notations and to represent situations graphically (abbreviation: Graph) (9)
3) Representing and Manipulating symbolic forms, including three basic competencies (aspects):
    - To choose (or decode in) a representation (abbreviation: C.D.Rep.) (10)
    - To switch between representations (abbreviation: Rep.↔Rep.) (11)
    - To manipulate within a representation (abbreviation: Man.Rep.) (12)
4) Mathematical Reasoning and Communicating, including two basic competencies (aspects)
    - To understand others' written, visual or oral 'texts', in a variety of linguistic registers, about matters having a mathematical content and to follow and assess chains of arguments, put forward by others (abbreviation: Reason←) (13)

- To express oneself, at various levels of theoretical and technical precision, in oral, visual, or written form, about such matters. To provide explanations or justifications to support your result or idea (abbreviation: Reason→) (14)

5) Use of Aids and Tools, including two basic competencies (aspects)
   - To know the existence and properties of various tools and aids for mathematical activity (abbreviation: Ext.T) (15)
   - To be able to use such aids and tools to develop insight or intuition (abbreviation: U.T.) (16)

5.3 Initial results

In this section, I report on the results from the exploration of competencies activations for the students who participated in the sessions. This section aims to inform the reader about the initial results and indications of the analysis, such as competency aspects that appeared to be invoked most frequently and possible interrelations between them. As noted at the beginning of Section 4.1 and subsection 4.1.2, these initial indications led to the construction of the whole analytical apparatus (i.e., applying the scaling instrument, exploration of competency development using the 3-D model) presented in Chapter 4. Therefore, the presentation of the students' results in this section does not follow the entire analytical apparatus adopted for the two case studies of Erika and Johannes (Section 5.6).

In subsection 5.3.1, I select one student (Rene) to demonstrate the analysis results regarding his work during the project. This student was selected because he participated in six sessions (out of the total seven), and data from the Focus Camera (see also subsection 3.4.1) was selected from all these six sessions. Some students participated in six sessions too, but the Focus Camera data was selected from fewer sessions. Provisional results for all students (Kim, Linnea, Maria, Erika, and Johannes) can be found in Appendix C. Subsection 5.3.2 presents in tabular form results from all students regarding all competencies of the competence framework. Five tables for each mathematical competency are provided, describing the coding process and giving the context and rationale of these competency activations. The following subsection (5.3.3) discusses the indications regarding these instances where one action from a student (e.g., something they said, did, or wrote) signified evidence for the simultaneous activation of two competency aspects. In the last subsection (5.3.4), I offer some conclusive remarks regarding the results presented in this section.

Subsection 5.3.2 resembles 5.3.1 but offers a more comprehensive description of the coding process for all competency aspects. It is an account of all competency aspects of the competence framework where examples from all the students are given and concern the way their actions in the class offered evidence of competency (aspect) activation for all the aspects of each of the five main competencies of the framework: Mathematical Thinking and Acting;

Mathematical Modeling; Representing and Manipulating symbolic forms; Mathematical Reasoning and Communicating; and Use of Aids and Tools. This second part of this section explains how the data was used and analyzed. In other words, why a particular code (that represents a particular competency aspect) was assigned to a particular utterance, gesture, or written work of a student.

5.3.1 Rene's work

Rene participated in six sessions, and data from the Focus Camera was collected from all these sessions. Regarding the frequency of competency activation, that is, how frequently he brought into action (i.e., activated) a particular competency aspect, Rene's work in class facilitated the identification and recording of several aspects. Table 5-2 below includes these aspects (first column) that Rene frequently provided evidence of activation. The second column includes the evidence of this activation which could be something that he said (i.e., utterance) or did (i.e., gesture and written work). Finally, the third column displays the source of this evidence: the session and task that this activation occurred and the context of this activation that briefly justifies why this utterance, written work, or gesture from the student was considered as evidence of activation of the particular competency aspect.

Table 5-2. Rene's activated competencies.

Competency Aspect	**Evidence** "utterance" or [written work or gesture])	**Context** Rationale (Session/Task)
P.Q.	"Why do we need to divide by 2 this?"	Rene **posed a question** on why to divide by 2 the amplitude a when they identified the table's highest and lowest temperatures. (1^{st}/Airport Temperatures)
→Sol.	[estimates the mass of the colony writing $2^{72} \times 1.7 \times 10^{-12} \approx 8 \cdot 10^{9}$]	Rene **attacked the problem** beginning with finding the total cell colony's mass. (4^{th}/E. Coli)
Int.El.	"So t is for the years like after 2 years $t = 2$"	Rene **translated an element** of the $y = 31.0(1.017)^{t}$ expression. (2^{nd}/Barley Yield)
Info	[writes in his notes $a = \frac{17.9 - 9.5}{2} = 4.2$]	Rene **searched** the task's available **information** to calculate the amplitude. (1^{st}/Airport Temperatures)
C.D.Rep.	"So, we need to find the $V(100)$"	Rene needed to find the change in voltage V of a neuron with respect to time after 100 ms and **decoded the original expression** of $V(t)$ for $t = 100$.

Competency Aspect	**Evidence** “utterance” or [written work or gesture])	**Context** Rationale (Session/Task)
		(7th/Neuron Voltage)
Reason→	“We need to take out [subtract] the birth rate with the death rate to find the rate of change”	Rene **provided a justification to support his idea** of subtracting two rates, when working on the task that asked when the current population of Earth will be doubled. (5th/ Population Doubled)
U.T.	[used graphic calculator Desmos, to produce two sinusoidal functions graphs]	Rene **used a graphic calculator** to check the two sinusoidal graphs and make inferences regarding the population of the two species. (1st/ Rabbits and Wolves)

Evidence was also found for the activation of the competency aspects (Int.Res.X) of the Mathematical Modeling competency and (Reason←) of the Mathematical Reasoning and Communicating competency. Less frequent evidence was identified for the activation of the competency aspects (Lim.) of the Mathematical Thinking and Acting competency, (Cr.) of the Mathematical Modeling competency, and (Rep.↔Rep.) of the Representing and Manipulating symbolic forms competency. Lastly, limited or no evidence was found for the activation of the (Graph) and (R.+V.) competency aspects of the Mathematical Modeling competency and the (Ext.T.) of the Aids and Tools competency.

During the E. Coli task (Table 3-4), Rene began his work by saying that “at first, we have to find how many E. Coli will be in 24 hours (…) like they will be divided 72 times,” indicating that he grasped the concept of “division every 20 minutes,” providing evidence for the activation of (Reason→) because he provided justification to support his idea. At the same time, he provided evidence for the activation of the (→Sol.) competency aspect by taking actions towards the solution of the task, and for the activation of (Int.El.) competency aspect by translating the elements of the expression “division every 20 minutes” regarding the 2^t part of the expression.

These instances of activation of the same competency aspects (i.e., (Reason→), (→Sol.), and (Int.El.)) in different moments and tasks highlighted an indication that was observed on multiple occasions for all students. I refer to the quality of competency activation presented and explored in subsection 4.1.3, where the evidence for the activation of the same competency aspects variated. As discussed in subsection 4.1.7, this variation concerned the solving process (leading to conceptualizing and constructing the Task Solving Vision (T.S.V.) competency domain). It also concerned the articulation (leading to conceptualizing and constructing the Mathematical Language and Vocabulary

(M.L.V.) competency domain) and the amount of help that the student received when the competency activation occurred (Prompting and Independent Thinking (Pr.In.T.) competency domain).

In terms of units (Periodic Functions, Exponential Growth & Regression, Population Dynamics, Integrals & Modeling), Rene's work in class provided evidence of competency activation more frequently during the Population Dynamics unit. However, this could also have happened because he missed one session during the Exponential Growth & Regression unit. Provisional indications were also identified regarding the evidence Rene provided (with his actions in the class) for the simultaneous activation of particular competency aspects. For example, the activation of the (Int.El.) competency aspect was observed at the same turn (the term *turn* is presented and explained in subsection 4.1.2) with the (→Sol.) and the (U.T.) competency aspects.

Summing up, Rene's engagement in the calculus sessions facilitated the identification of evidence for the activation of a wide range of competency aspects of the competence framework across all six sessions in which he participated. As the sessions progressed, the evidence for competency activation became more frequent. In the last session of the Population Dynamics unit (5th session), evidence was found for the most frequent activation in a single session. Difficulties were observed during the Integrals & Modeling unit sessions, where less evidence of competency activation was observed.

5.3.2 Class work: results for all competency aspects

This account consists of five tables corresponding to the competence framework's five main competencies: Mathematical Thinking and Acting; Mathematical Modeling; Representing and Manipulating symbolic forms; Mathematical Reasoning and Communicating; and Use of Aids and Tools. As in the previous section (Rene's work), several examples from the data are displayed. These examples concern all students and all aspects of the competence framework. Every table is followed by a documentation of the main observations from these results and indications that led the author of this thesis to employ a complete analytical apparatus on the two case studies: Erika and Johannes.

The Mathematical Thinking and Acting competency consisted of three aspects, as seen in Table 5-3 below. The first concerned the ability to pose questions with a mathematical characteristic or know the kinds of answers that mathematics may offer coded with the abbreviation (P.Q.). The second aspect concerned the ability to understand and handle the scope and limitations of a given concept coded with the abbreviation (Lim.). Lastly, the third aspect refers to these actions that a student takes to attack mathematical problems of various kinds coded with the abbreviation (→Sol.). There was evidence of activation for all three competency aspects but not in the same frequency, that is, there were actions from the students that signified the activation of a particular aspect more often than others. In particular, the activation of the (P.Q.) and the (→Sol.) competency aspects was observed more often than the (Lim.) aspect by all six students at the sessions that they attended.

Students were posing questions that concerned both technical and conceptual parts of the tasks they had to work on, and these questions were addressed to the teacher and their classmates. It has been observed that these questions differed in relevance to the task solution, the vocabulary, and the mathematical language used. It was also interesting to observe what provoked these questions. On some occasions, the teacher's interventions provoked additional questions. On other occasions, students came up with questions just by trying to understand what the task was asking them to find or calculate.

The students were observed to solve the tasks or parts of the tasks by analyzing their written work and statements that concerned actions they were about to take to attack the problem. Again, these actions differed in how students expressed themselves (written or orally) and how relevant these actions were to the solution of the task. In addition, the help the student(s) received from the teacher or a classmate while taking these actions was also a point of further exploration. In the examples in the previous table, Maria wrote on her notes '$1.9\% - 0.8\% = 1.1\%$' independently and without assistance from anyone in the class. On the other hand, Rene (from the same table) took actions when the teacher asked him a question regarding his next steps. In comparison to the first two aspects of this competency (i.e., P.Q. and the →Sol.), not many actions from the students signified that they attempted to understand and handle the scope and limitations (Lim.) of a given concept.

Table 5-3: Mathematical Thinking and Acting competency.

Aspect	**Evidence** "utterance" or [written work or gesture] (student)	**Context** Rationale (Session/Task)
P.Q.	• "Shouldn't be minus $(-)$ in front of deaths?"(Rene) • "But are all cancer cells the same size?"(Linnea) • "Are we supposed to find the rate here?" (Johannes)	• Rene **questioned** why there was a $(+)$ and not a $(-)$ before deaths in the following equation: "$N_{t+1} = N_t$ + births +deaths + immigration – emigration." (4th/World Population) • Linnea **posed a question with a mathematical characteristic** (i.e., comparison between sizes).(3rd/Cancer Cells) • Johannes **addressed a question with mathematical characteristics** to Erika regarding the table's values on the amount of recyclables. (2nd/Recyclables)
Lim.	• "[She] should not have the same equation because [she had] a different starting point." (Erika) • "Probably not the same." (Linnea)	• Erika referred to the work of a classmate on the construction: $g(x) = a\cos[b(x-c)] + d$, because she had sketched a different (**understood the limitation**) yet correct graph. (1st/Airport's Temperature) • Linnea replied to a question set by her classmate who asked whether all cancer cells have the same size **realizing the limitations of the cell size concept**. (3rd/Cancer Cells)
→Sol.	• [writes on her notes '$1.9\% - 0.8\% = 1.1\%$'] (Maria) • "Now, I need to find C" (Rene) • "We now multiply n with 2" (Johannes)	• Maria **took actions towards the solution** by subtracting the two rates (i.e., birth and death rates). (5th/Population Doubled task) • Rene **attacks the problem** by responding to a teacher's question and aimed to find the next missing piece of the periodic function $g(x) = a\cos[b(x-c)] + d$. (1st/Airport's Temperature task) • Johannes **took actions** to create the equation for binary fission that would calculate the generation time G required for the cells population to be divided. (3rd/Bacterial population task)

The Mathematical Modeling competency (Table 5-4) consisted of six aspects. The first aspect concerns the ability to assess the range and validity of existing models coded with the abbreviation (R.+V.). The second aspect described the ability to interpret and translate the elements of a model where the student should be able to understand the real problem and adapt a model based on reality coded with the abbreviation (Int.El.). The third and fourth aspects concerned the ability to interpret mathematical results in an extra-mathematical context and generalize the solutions and the ability to critique the model by reviewing or reflecting and questioning the results. These aspects were coded with the abbreviations (Int.Res.X) and (Cr.), respectively. The last two aspects describe the situation where the student looks and searches for available information and can differentiate between relevant and irrelevant information (Info) and the ability to represent situations graphically coded with the abbreviation (Graph).

Some actions from the students provided evidence of activation for several competency aspects of the Mathematical Modeling competency, with some aspects being activated more often than others. There were not many instances from students' actions where they attempted to assess the range and validity of existing models (i.e., activation of the (R.+V.) aspect) or aimed at representing situations graphically (i.e., activation of the (Graph) aspect). This lack of evidence resulted in a limited activation of these two aspects.

Across all sessions, almost all students were observed to make efforts to interpret and translate the elements of the mathematical expressions that modeled the problem (i.e., activation of the (Int.El.) aspect). They were also observed to make interpretations of their results in a real-world context (i.e., activation of the (Int.Res.X) aspect) and generalize their solutions (on some occasions) for the next steps in their tasks. Students were also seen searching their curriculum materials (e.g., textbooks, past notes) and the internet for information that was relevant to the features and the solution of the task (i.e., activation of the (Info) aspect). Interestingly, in a few instances, some students critiqued the mathematical expressions (i.e., activation of the (Cr.) aspect) that described the model and reflected on the results but mainly during the teaching of the Exponential Growth & Regression and the Population Dynamics sessions.

As discussed in the previous table, all the above actions that signified the existence of evidence for the activation of these competency aspects presented some differences. Activations of the same competency aspect differed regarding how much assistance the students received, how they expressed themselves while activating the competency, and how close or relevant these actions were to the task's solution they were working on. This observation is repeated in the following tables, and it played a vital role in constructing a more comprehensive analytical apparatus that was eventually employed.

Table 5-4:Mathematical Modeling competency.

Aspect	**Evidence** "utterance" or [written work or gesture] (student)	**Context** Rationale (Session/Task)
R.+V.	• "So, we have to find which of the models…equations we must choose [referring to $N_{t+1} = R \cdot N_t + N_t$ and $N(T) = N(0) \cdot e^{rT}$] (Maria) • "I remember when we discussed about doublings in general that we used an expression with initial and final number of bacteria." (Johannes)	• Maria attempts to **assess the range and validity** of these two existing **models**. (4th/World Population) • Johannes searched for his notes from previous sessions to **identify which model should be valid** for the needs of the particular task. (3rd/Bacterial population)
Int.El.	• "This time, the $N(0)$ of the first task is this one [showing the colony's mass]" (Maria) • "We are looking for the r in our case." (Kim) • "So the interval is [0, 200] so we write $\int_0^{200} 0.014 \cdot t^{0.04}$." (Rene)	• Maria **interpreted and translated the $N(0)$ element** of the '$N(0) \cdot e^{rT}$' expression which she used it in the warm-up task to adjust it to the new one. (4th/E.Coli) • Kim **interpreted the element** r of the '$P(t) = P(0)e^{rt}$' model **during the mapping process** when they discussed about the rate of change. (5th/Reversed Population) • Rene **interpreted the element of time** within the '$\frac{dF}{dt} = 0.014 \cdot t^{0.04}$' expression and **the reality of the task** to produce the intervals for the integral. (6th/ Earth's Temperature)
Int.Res.X	• "If the population of wolves decreases then the population of rabbits increases." (Rene)	• Rene **interpreted the result** depicted in the populations graph **within a real world context**. (1st/Rabbits and Wolves)

Aspect	**Evidence** "utterance" or [written work or gesture] (student)	**Context** Rationale (Session/Task)
	• "But since we say growth, we have to write 1.02." (Linnea)	• Linnea **interpreted a result** (i.e., 2% $=$ 0.02) that the class found regarding the growth rate, and **generalized the solution** for her next move. (4^{th}/World Population)
	• "And then you see that the number $8 \cdot 10^9$, is not even close to the earth's mass. (Maria)	• Maria **interpreted the result** of the cell's colony mass to **an extra-mathematical context** (i.e., world) when she compared it with the earth's mass. (4^{th}/World Population)
Cr.	• "Maybe I have a bit different number because I used 7.95 instead of 8" (Linnea)	• Linnea **reviewed and questioned the results** she **looked back at the model** she applied produced because she had different outcomes from her classmates. (4^{th}/E.Coli)
Info	• "I found the size of a normal cell." (Kim) •	• Kim **searched for the available information** in the web **differentiating it from irrelevant information** such as any other kind of cell. (3^{rd}/Cancer Cells)
	• "I am trying to remember [searching in her notes] the riddle lake." (Maria)	• Maria **searched for available information**, she recalled she had in her notes from a previous task. (4^{th}/E.Coli)
	• "I remember when we used this model [finds it in his notes] when we looked for doublings, but this is a bit different." (Johannes)	• Johannes **searched his past notes** to track down the model that described a similar to the Reversed Population Doubled task. (5^{th}/ Reversed Population Doubled)
Graph	• [he produces a graph from the '$y = 31.0 \cdot (1 + 0.017)^t$' expression for different t] (Rene)	• Rene **was able to represent graphically** the equation that estimated approximate barley yields for years after 1960. (2^{nd}/Barley Yield)

The Representing and Manipulating symbolic forms competency comprised three aspects, as seen in Table 5-5 below. The first aspect concerned the ability to choose and decode any mathematical representation coded with the abbreviation (C.D.Rep.) The second aspect described the ability to switch between these representations coded with the abbreviation (Rep.↔Rep.), and the third aspect concerned the student's ability to manipulate within a representation, coded with the abbreviation (Man.Rep.). Some actions from the students provided evidence of activation for all competency aspects of this mathematical competency, with some aspects being activated more often than others. In particular, students often provided evidence regarding the activation of the (C.D.Rep.) and (Man.Rep.) competency aspects but also for the activation of the (Rep.↔Rep.) aspect.

The students were observed attempting to decode and manipulate numerical and graphical representations (i.e., activation of the (C.D.Rep.) and (Man.Rep.) aspects) to adapt them to the needs of each task. The students needed to connect and change symbols and representations in all sessions and work within the same representations to attack the problems. Students were observed to decode and translate a representation set in natural language to a symbolic one. For example, expressions such as "our bacterial population would have grown from 1000 to over 16 billion" or "what is the maximum population growth rate for these species who …?" were decoded and translated from their natural language representation to a mathematical one. On fewer occasions, students were observed to attempt to switch between different kinds of representations (e.g., text, numerical, graphical), and there were even fewer sessions with evidence regarding the activation of the (Rep.↔Rep.) aspect.

As noted before, the students' actions that provided evidence that signified activation of these aspects differed in various factors. On some occasions, students could decode or manipulate a representation without help from the teacher or a fellow student. For example, the two activations of the (C.D.Rep.) aspect, as seen in Table 5-5, differ regarding the assistance the students received. Erika needed some hints from the teacher to proceed with decoding the graphical representation. On the other hand, Maria worked independently when he decoded the '2^x' representation and adapted it to the 'cells doublings' part of the task. The differences in these activations also concerned how the students expressed themselves while decoding, for example, a representation. There were instances where they used formal mathematical language and instances where they used everyday expressions.

Table 5-5: Representing and Manipulating symbolic forms competency.

Aspect	**Evidence** "utterance" or [written work or gesture] (student)	**Context** Rationale (Session/Task)
C.D.Rep.	• "So it is $C(1) = 1.02 \cdot C(0)$" (Erika)	• Erika **decoded a graphical representation** displayed by the teacher that concerned the increase of the worldwide rate petroleum consumption by 2% every year. (7th/Petroleum consumption)
	• "The cell will be divided 72 times so we can write 2^{72}." (Maria)	• Maria **decoded the** '2^x' **representation** and adapted it to the 'cells doublings' case of the task. (4th/E.Coli)
Rep↔Rep	• [she writes "$\frac{\Delta f}{\Delta t}$" then switches to "$\frac{df}{dt}$" and eventually to "$f'(t)$"] (Erika)	• Erika initially wrote '$\frac{\Delta f}{\Delta t}$' to express the rate of the rising temperature but she **switched to a different representation** '$\frac{df}{dt}$' and eventually to '$f'(t)$' (6th/ Earth's Temperature)
	• [he writes '$N_{t+1} = R \cdot N_t + N_t$'] (Rene)	• Rene **switched between the text representation of the model** (i.e., $N_{t+1} = N_t + B + D + I - E$) **to the symbolic one**. B, D, I, and E, represented the births, deaths, immigration, and emigration within a year. (4th/E.Coli)
Man.Rep.	• [he writes on his notes $\frac{dF}{dt} = 0.014 \cdot t^{0.04}$] (Rene)	• Rene **manipulated the** '$0.014 \cdot t^{0.04}$' **representation** of the rate equation from the task to create the differential equation. (6th/Earth's Temperature)
	• [she writes "$\int_a^x f(t)\,dt = F(x) = G(x) + c$" on her notes] (Erika)	• Erika **worked on the expression**: '$F(x) = \int_a^x f(t)\,dt$' during a discussion on the Fundamental Theorem of Calculus reaching the '$F(x) = G(x) + c$' expression. (6th/Discussion on the Fundamental Theorem of Calculus)

The Mathematical Reasoning and Communicating competency consisted of two aspects. The first described the student's ability to understand others' written, visual or oral 'texts' about matters that have a mathematical content and to follow and assess chains of arguments put forward by others. This aspect was coded with the abbreviation (Reason←). The second aspect concerned expressing oneself at various levels of theoretical and technical precision, in oral, visual, or written form, and providing explanations or justifications to support a result or idea. The abbreviation (Reason→) was used for the coding process. Students' actions signified the existence of evidence for the activation of both competency aspects of the Mathematical Reasoning and Communicating competency.

Students were observed providing their argumentations and justifications for their actions, thoughts, and decisions to their fellow students and the teacher. Again, these justifications were expressed in various levels of language use (e.g., formal mathematical, everyday expressions) and the assistance received before the beginning of the student's argumentation. Similarly, students were observed to follow explanations and justifications of their fellow students. As seen in the examples from Table 5-6, to verify that a student followed or assessed arguments put forward by others, it was essential to explore the immediate students' reaction to the justification that took place by their fellow students or the teacher and this reaction should have been clearly related to the argumentation that was expressed.

Table 5-6: Mathematical Reasoning and Communicating competency.

Aspect	**Evidence** "utterance" or [written work or gesture] (student)	**Context** Rationale (Session/Task)
Reason←	• "So, we are looking for T" (Kim)	• Kim **followed an argument** from Maria regarding the "$N(T) = N(0) \cdot e^{rT}$" expression and what is they should be looking for. (4th/World Population)
	• "This is the starting point that we will be asked to use at some point." (Johannes)	• Johannes **followed a comment made by his teacher**, "temperature at 2000 degrees Fahrenheit" and added his interpretation. (6th/Earth's Temperature)
	• "I see you started your graph/January one point to the right of zero" (Erika)	• Erika **understood the teacher's written text (the graph) following his argumentation** on why he sketched it in that way. (1st/Airport Temperatures)
Reason→	• "If you know how many individuals you start with and if the population is two then probably $N(0) = 2$" (Maria)	• Maria **justified her choice** on assigning the value 2 at $N(0)$ for $T = 0$. (4th/World Population)
	• "If the population of wolves decreases then the population of rabbits increases." (Rene)	• Rene **provided an explanation** on what the population graphs were depicting. (1st/Rabbits and Wolves)
	• "We need generation time and the number of bacteria at the end" (Linnea)	• Linnea **provided an explanation** on what is needed for exploring a bacterial population that follows geometric progression. (3rd/Bacterial)

The Use of Aids and Tools competency (Table 5-7) comprised two aspects. The first aspect concerned the situation where the students show awareness of the existence and properties of various tools and aids for mathematical activity coded with the abbreviation (Ext.T.). The second aspect described the ability to use aids and tools to develop insight or intuition coded with the abbreviation (U.T.). Interestingly, there were very few occasions (limited evidence) where students expressed their awareness regarding the existence of an aid or a digital tool, and therefore this aspect presented limited activation. On the other hand, students were observed to use mainly their calculators and online computing machines (e.g., Wolfram Alpha) to make numerical or trigonometrical calculations during the solving process or to confirm their differentiations or integrations that they attempted.

In the following section, I provide some indications that concern the objective of the second research question of this thesis regarding the simultaneous activation of two competency aspects. The complete results that illustrate if and how the competency aspects are interrelated are displayed in the final section of this chapter.

Table 5-7: Use of Aids and Tools.

Aspect	**Evidence** "utterance" or [written work or gesture] (student)	**Context** Rationale (Session/Task)
Ext.T.	• "We could use Desmos to see how it looks like" (Rene)	• Rene **displayed awareness of a digital tool** that could help the class see how the graph that described the annual airport's temperatures would look like if instead of 'sin' they used the 'cos' expression. (1st/Airport's Temperature)
U.T.	• [he uses calculator to find the value of $log10.000$] (Kim)	• Kim **used this digital tool** to understand how the log function works. (3rd/Bacterial population)
	• [he uses Wolfram Alpha to calculate $\int_0^{200} 0.014 \cdot t^{0.04} dt$] (Rene)	• Rene **used the Wolfram Alpha computational engine** to calculate the integral that would give the rise of Earth's temperature. (6th/Earth's Temperature)

5.3.3 Class work: indications about the competency interrelations

Another main objective of this research (as expressed in the second Research Question of this thesis) is to explore the recorded competency activations and search for instances where the student provides evidence (through his actions in the class) that signifies evidence for the activation of two mathematical competencies from the competence framework simultaneously. The purpose of this section is to provide examples of such instances of simultaneous activations that describe the notion of competency interrelation. These examples are derived from the tables of the above section and constitute indications that suggest that some competencies may be activated more often with some and less often with others. These indications resulted in the decision to explore the notion of competency interrelation with two students (Erika and Johannes) who attended all project sessions. The following two paragraphs discuss two examples of simultaneous activation.

In Table 5-3 and the (Lim.) row (first bullet), Erika refers to the written work of a fellow student, "[she] should not have the same equation because [she had] a different starting point." This utterance signified evidence for the activation of the (Lim.) competency aspect of the Mathematical Thinking and Acting competency. However, Erika, apart from understanding the limitation of the concept of the periodic function she also explained her thinking regarding the writing of the equation, an action that signified evidence for the activation of the Mathematical Reasoning competency (Reason→). This observation is described in this thesis as concurrent activation. All aspects of the competence framework are numbered in the following order: (P.Q.) as 1, (Lim.) as 2,…, and (U.T.) as 16, as seen in Table 3-1. Therefore, the competency pair discussed above regarding the simultaneous activation of (Lim.) and (Reason→) competency aspects is represented as: $(2{,}14)$.

During the population dynamics session, Maria attempted to assess the range and validity of two existing models as seen in the first row of Table 5-4 when she said, "So, we have to find which of the models…equations we must choose [referring to $N_{t+1} = R \cdot N_t + N_t$ and $N(T) = N(0) \cdot e^{rT}$], signifying evidence for the activation of the (R.+V.) competency aspect of the Mathematical Modeling competency. At the same time, though, Maria took actions towards the solution of the task, which signified evidence for the activation of the (→Sol.) aspect of the Mathematical Thinking and Acting competency, and this competency pair is represented as: $(3{,}4)$.

Summarizing these indications, it has been observed that it is not rare to locate instances of simultaneous competency activations. The interest, though, is not whether competencies interrelate or not because interrelations are expected to happen as Niss (2003, p. 9) points out, "[t]he competencies are closely related - they form a continuum of overlapping clusters - yet they are distinct in the sense that their centres of gravity are clearly delineated and disjoint." This thesis was interested in the frequency of these interrelations and whether two competency pairs appear less or more often than expected. As I mentioned at the end of the previous section, the complete results that illustrate if and how the competency aspects are interrelated are displayed in the final section of this chapter. The following and last subsection of this section provides some conclusive remarks regarding the provisional results discussed above.

5.3.4 Some conclusive remarks and comments

Section 5.3 provided some provisional results regarding the analysis of competency activation for all students who participated in the sessions and described the context and rationale of the coding process. It has been mentioned before that not all six students participated in all seven sessions and that not all students were recorded with the Focus Camera (see also Table 3-5) that captured closely and in much detail the student's actions. Actions such as note-taking, the timing of writing, and gestures were captured only with the camera close to the students (i.e., Focus Camera). Students originally planned to be recorded with the Focus Camera and collect their written data with the LScribe smartpen tool unexpectedly dropped out of the project after one or two sessions, and I, therefore, had to rearrange the data collection process and adapt to this new situation. Three students (Linnea, Erika, and Johannes) participated in all seven sessions of this project but Focus Camera data for all seven sessions were collected for Erika and Johannes. That said, this section included and presented results regarding the evidence of competency activation for all students for the sessions they participated in.

The first part of this section included the provisional results for another student (Rene) who participated in six out of seven sessions with Focus Camera data available for all these sessions. These results concerned i) the identification of these competency aspects that Rene appeared to provide evidence of activation, and ii) the possible interrelations between these aspects that appeared to be formed, that is, which competency aspects appeared to be activated simultaneously with others. Provisional inferences regarding the frequency of this evidence will be provided (i.e., most, or least frequently and limited evidence). These competency aspects that Rene appeared to bring into action more frequently were presented in tabular form. I continued by identifying these calculus units (Periodic Functions, Exponential Growth & Regression, Population Dynamics, Integrals & Modeling), in which Rene's actions allowed me to recognize evidence of competency activation.

The first part of this section was followed by an account (subsection 5.3.2) of five tables corresponding to the five main competencies of the competence framework. It displayed the functionality of the coding process by providing the context of all the competency aspects of the framework with examples of competency activations from all students. It appeared that it was feasible to identify evidence that signified competency activations by applying this coding process. However, as discussed below each table, it was evident that activations of the same competency aspect characterized by differences in terms of the language and vocabulary the students used to activate a particular competency, the assistance they received to do that, and the relevance of their actions (that signified activation) to the solution of the task at hand. Lastly, in subsection 5.3.3, cases from the previous tables were used, to provide some examples of competency aspects activated concurrently.

The above-reported indications and provisional results led me to focus on the two students (i.e., Erika and Johannes) and construct this research's complete analytical apparatus. Using data from all seven sessions for these two students allowed me to explore the concept of competency development over time, something that would not be accomplished to the same extent with these four students who did not participate in all seven sessions. The following sections provide a reminder of the table of the scaling instrument (Section 5.4) that will be used to display the results of the two case studies (i.e., Erika and Johannes); and the description of the diagrams (Section 5.5) that will be used in Section 5.6, where the results of the analysis of the two case studies are displayed. The last two sections of this chapter include a summary of these results (Section 5.7) and the results (Section 5.8) from the competencies interrelation (concurrent activation) analysis.

5.4 A reminder of the scaling instrument

In this section and prior to the case studies' results, I briefly reintroduce the levels of the scaling instrument (Table 5-8, next page) that was explored and analyzed in Section 4.1.

Table 5-8. The scaling instrument (reminder).

Activation Level / **Activation Domain**	**Initial (I)**	**Intermediate (II)**	**Mature (III)**
Task Solving Vision (T.S.V.)	The student activates the competency by having a short-range (local) perspective toward the solution.	The student activates the competency by having a mid-range (tactical) perspective toward the solution.	The student activates the competency by having a long-range (strategic) perspective toward the solution.
Mathematical Language and Vocabulary (M.L.V.)	The student activates the competency making familiar and occasional use of everyday expressions or gestures with limited or no mathematical language and vocabulary.	The student activates the competency making regular use of expressions with mathematical language and vocabulary.	The student activates the competency making regular and correct use of expressions with mathematical language and vocabulary.
Prompting and Independent Thinking (Pr.In.T.)	The student activates the competency with frequent prompting with no signs of independent thinking from the student.	The student activates the competency with occasional prompting with regular instances of independent thinking from the student.	The student activates the competency with minimal or no prompting; student consistently demonstrates independent thinking.

5.5 Visual guide

This guide introduces the four steps that will be followed for the presentation of the results. These steps are the following: (1) the diagram; (2) the competency aspect activations: location and frequencies; (3) data extracts from students' competency activations; and (4) the summary of evidence.

Step 1: The diagram

The first step for the display of the results is to present the diagram for each competency aspect that is under study. Figure 5.1 below, is a reminder on how one should interpret the diagram and what information it provides. Detailed information about the interpretation of the diagrams can be found in subsection 4.2.3 (Visualization of the results). However, the following step explains again how the competency activations should be located and symbolized.

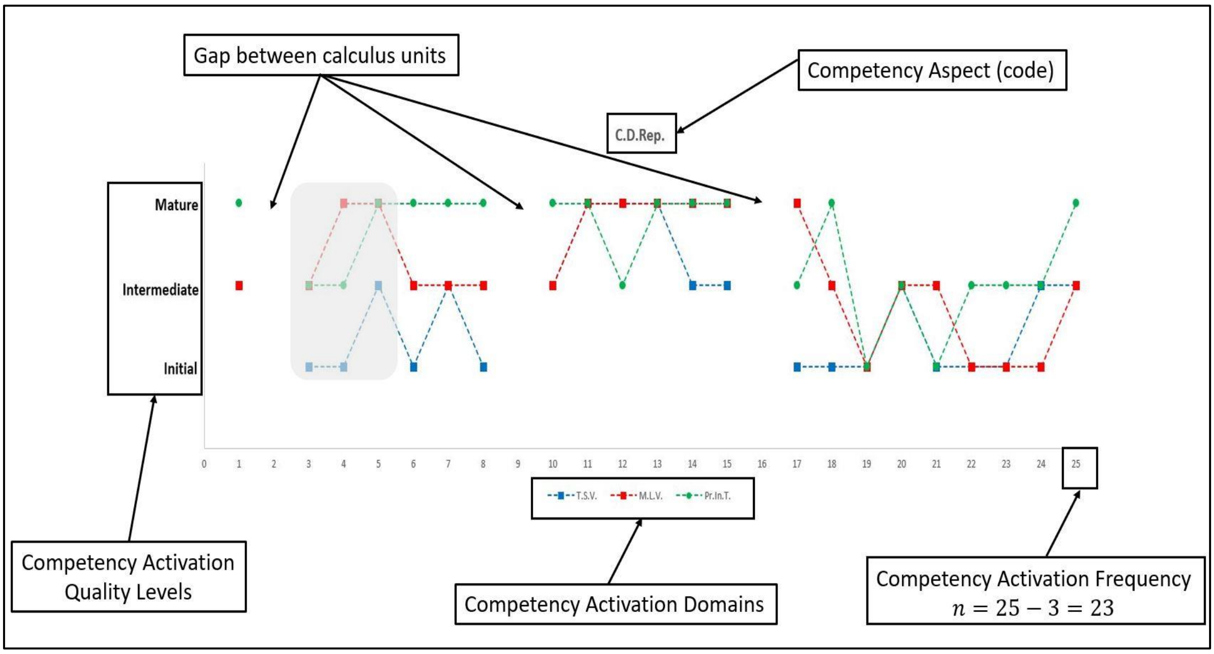

Figure 5.1. Interpretation of the diagram (repetition).

Step 2: Competency aspect activations: location and frequencies

In the case studies report, the diagram will be followed by a paragraph that includes information about the competency aspect activations and where they are located by calculus unit. The frequency of competency activation (first type of evidence) per unit and the total frequency is also included, in the case studies, by the end of that paragraph. I use the notation [x-y] to refer to a particular activation or a sequence of activations. For example, the grey shaded area in Figure 5.1 would be referred to by [3-5]. A gap between the lines is used to separate (to make clear to the reader) the four calculus units. For example, in Figure 5.1, the four units (Periodic Functions, Exponential Growth & Regression, Population Dynamics, and Integrals & Modeling) are represented by particular points or point on the x-axis; specifically: Periodic Functions [1] (because there was only one identified activation of the (C.D.Rep.) aspect), Exponential Growth & Regression [3-8] (six activations), Population Dynamics [10-15] (six activations), and Integrals & Modeling [17-25] (nine activations). The activation frequency per unit is [1, 6, 6, 9], and the total frequency of activation $n = 22$.

Step 3: Examples from activations

In this step, I provide example(s) from the data displayed in the relevant diagram regarding the competency activations. To draw attention to a particular activation, I will use (as mentioned above) the [ ] symbol. For example, the activation on point 3 of the diagram's x-axis will be displayed as follows: [3]. The examples from the data include quotes from the students while working on the non-routine mathematical tasks, short descriptions of the tasks, explanations about why the particular code (competency aspect) was assigned, and why this code was assigned with a particular level of activation (from the scaling instrument). The activation level assignment concerns each of the three domains of activation. The activation domains are the following: Task Solving Vision (T.S.V.) displayed by the dotted blue line, Mathematical Language and Vocabulary (M.L.V.) displayed by the dotted red line and Prompting and Independent Thinking (Pr.In.T.) displayed by the dotted green line.

Step 4: Summary of evidence

In this last step, I will provide a table that summarizes the two types of evidence (discussed in subsection 4.4.1), namely the changes between activation levels and the levels of consistency of competency activation.

5.6 The case studies

This section includes the results of the two case studies of Erika (5.6.1) and Johannes (5.6.2). In each case study, there is a presentation of the results regarding the evidence of competency activation for all 16 competency aspects of the competence framework. As mentioned in Section 5.5, this presentation includes the diagrams of activations, information for the location and frequency of activations, examples from the activations, and a summary of evidence. Due to space reasons, in the following two subsections and for each student, I present a selection of competency aspects and not all 16 competency aspects that constitute the competence framework. However, from all the examples form Erika and Johannes, the vast majority of the competency aspects are covered. Appendix D includes the remaining competency activations results that were not included here. This section is completed with a summary of the two case studies in subsection 5.6.3.

5.6.1 Case study 1: Erika

Diagrams 5-1 to 5-5 illustrate the development of each competency aspect by displaying the development of the three competency domains of development (T.S.V., M.L.V., and Pr.In.T.) across all seven calculus sessions. Because evidence of activation for two competency aspects (R.+V. and Ext.T.)

was limited, no diagram was constructed for these cases but only a description of what occurred in these instances. Tables 5-9 to 5-13 summarize the evidence regarding the changes within the competency activation levels and the consistency levels.

1) Mathematical Thinking and Acting

To pose questions that have a mathematical characteristic or to know the kinds of answers that mathematics may offer. (code: P.Q.) (1)

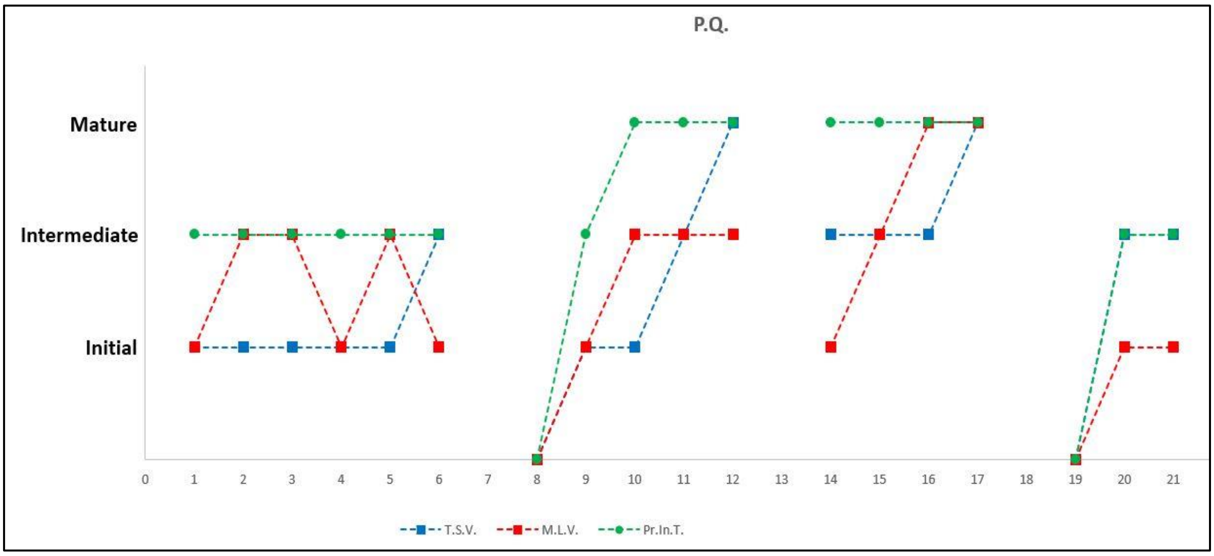

Diagram 5-1. Activation for competency aspect (P.Q.).

Competency aspect activations: location and frequencies

The competency activations by units were as follows: Periodic Functions [1-6], Exponential Growth & Regression [9-12], Population Dynamics [14-17], and Integrals & Modeling [20-21]. There was no activation of the competency aspect (P.Q.) during the first sessions of the Population Dynamics [8] and the Integrals & Modeling [19] units. The activation frequency per unit was [6, 4, 4, 2], and the total activation frequency was *n=16*.

Examples from the activations

Activation [1]: Evidence for the activation of the competency aspect (P.Q.) was provided by Erika's question, "so, we start from 15 °F, we put this value there?" when asked to fill in a table of monthly temperatures to produce a periodic function. This question was addressed both to the teacher and her classmate, Johannes. She posed this question having a mid-range perspective towards the solution (T.S.V., at Initial Activation Level, I) because she was aware only of the first step she needed to take to address the task having a short-range perspective towards the solution of the task. She expressed herself making familiar use of everyday expressions (M.L.V., at Initial Activation Level, I). All that occurred with occasional prompting (Pr.In.T., at Intermediate Activation Level, II) because the teacher said to the students that "we need the equation that describes the temperature during the year and the graph," which suggests that he did give some hints but let the students to discover mostly on their own how to proceed.

Activation [11]: Evidence for the activation of the competency aspect (P.Q.) was provided by Erika's question on how to work on a particular equation ($N_t = N_0 e^{rt}$) during the Exponential Growth & Regression unit. This evidence came from her written work when she asked, "how should I work with this formula if I replace the N_t and N_0 with their values?" She posed this question having a mid-range perspective towards the solution (T.S.V., at Intermediate Activation Level, II) being close to estimating how long it would take for a pair of individuals to produce the world population of today at the present rate of growth r=2% per year. She expressed herself making regular use of expressions with mathematical language (M.L.V., at Intermediate Activation Level, II) with minimal prompting from the teacher (Pr.In.T., at Mature Activation Level, III).

Summary of evidence

The summary of evidence regarding the changes within the competency activation levels and the consistency levels is displayed below.

Table 5-9. Summarized evidence for (P.Q) activation.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	6	7	3	
Changes from higher to lower	0	2	0	
High consistency	1(1)	1	3	
Moderate consistency	(1)	1	0	
Frequency of activation				16

2) Mathematical Modeling

To be able to assess the range and validity of existing models. (code: R.+V.)(4)

Competency aspect activations: note for location and frequencies

Erika provided evidence for the activation of the competency aspect (R.+V.) only on three occasions. Twice during the Population Dynamics unit (once in the first and once in the second session of this unit) and once at the last session of the Integrals & Modeling unit. There was no evidence of competency activation during the Periodic Functions and the Exponential Growth & Regression units.

Examples from the activations

During the Petroleum task, Erika provided evidence for the activation of the competency aspect (R.+V.) when while consulting her notes from the previous session, she considered the expression '$C(t) = C(0) \cdot e^{rt}$', as valid for this task. She had a mid-range perspective towards the solution (T.S.V., at Intermediate Activation Level, II) close to estimating the consumption rate. She expressed herself making regular and use of expressions with mathematical language (M.L.V., at Mature Activation Level, III) with limited prompting from the teacher (Pr.In.T., at Mature Activation Level, III) who asked if they could, “remember the model that includes change in rate.”

Summary of evidence

The low frequency of activation and the fact that there was no evidence of competency activation during the Periodic Functions and the Exponential Growth & Regression units resulted in considering that Erika provided limited evidence regarding the activation of (R.+V.) competency aspect.

To interpret and translate the elements of a model during the mapping process. In this situation, the student should be able to understand the real problem and adapt a model based on reality. (code: Int.El.) (5)

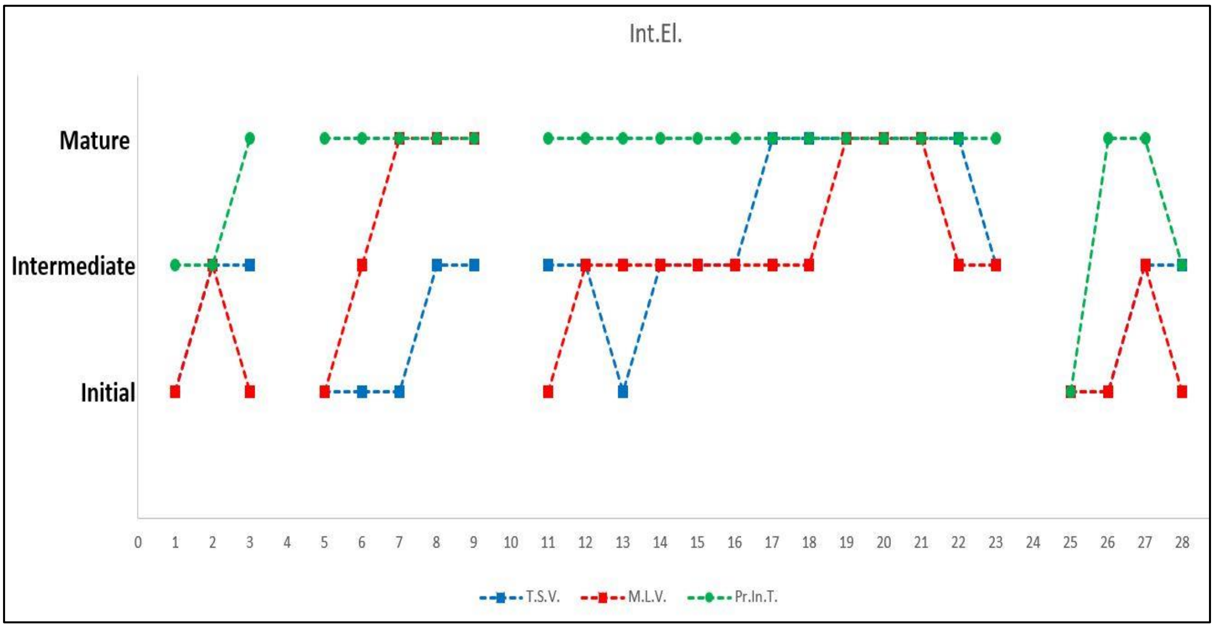

Diagram 5-2. Activation for competency aspect (Int.El.).

Competency aspect activations: location and frequencies

The competency activations by units were as follows: Periodic Functions [1-3], Exponential Growth & Regression [5-9], Population Dynamics [11-23], and Integrals & Modeling [25-28]. The frequency of activation per units was [3, 5, 13, 4] and the total activation frequency was *n=25*.

Examples from the activations

Activation [6]: Evidence for the activation of the competency aspect (Int.El.) was provided by Erika when she replied to a teacher’s question. The question asked students what was different in the new task compared to the previous one where they were informed about the yield growth and were asked to estimate the approximate barley yields for years after 1960. For this task, the students needed to find at what year will recycle reach 3 million tons and estimate the number of recyclables (tons) in a specific year. The teacher asked, “comparing to the previous task, what is not given in this one?” and Erika replied, “we don’t have the percentage [as] we had it in the previous task.” She interpreted the elements of the previous model, understood the new real problem, and adapted to the reality of this new task. She interpreted these elements having a short-range perspective towards the solution (T.S.V., at Initial Activation Level, I) because she was unaware of how to proceed after providing this response. She expressed herself using expressions with mathematical language (M.L.V., at Intermediate Activation Level, II) but did not realize that the ‘percentage’ was actually the growth rate we studied in the previous course. However, all that occurred with minimal prompting (Pr.In.T., at Mature Activation Level, III) because the teacher simply asked a question to navigate the students.

Activation [25]: Evidence for the activation of the competency aspect (Int.El.) was provided by Erika when she replied to a teacher’s question after some lengthy discussion between all members of the group (teacher, Erika, Johannes). This was the beginning of the Integrals & Modeling unit (first session), and the students faced difficulties understanding basic concepts from integration. When the discussion reached the question, “Do you agree that we should use the following expression: $f(t) = f(0) \cdot e^{rt}$?” Erika replied, “Yes, based to previous courses this is what we were doing.” She interpreted these elements without any vision towards the solution (T.S.V., at Initial Activation Level, I) without knowing why the teacher asked these questions. She expressed herself using everyday expressions with no evidence of mathematical language (M.L.V., at Initial Activation Level, I). The teacher guided the discussion at this instance, and the prompting was frequent (Pr.In.T., at Initial Activation Level, I).

Summary of evidence

The summary of evidence regarding the changes within the competency activation levels and the consistency levels is displayed below.

Table 5-10. Summarized evidence for (Int.El.) activation.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	5	6	2	
Changes from higher to lower	2	3	1	
High consistency	0	0	2	
Moderate consistency	4(2)	2(1)	2	
Frequency of activation				25

3) Representing and Manipulating symbolic forms

Choose (or decode in) a representation. (code: C.D.Rep.) (10)

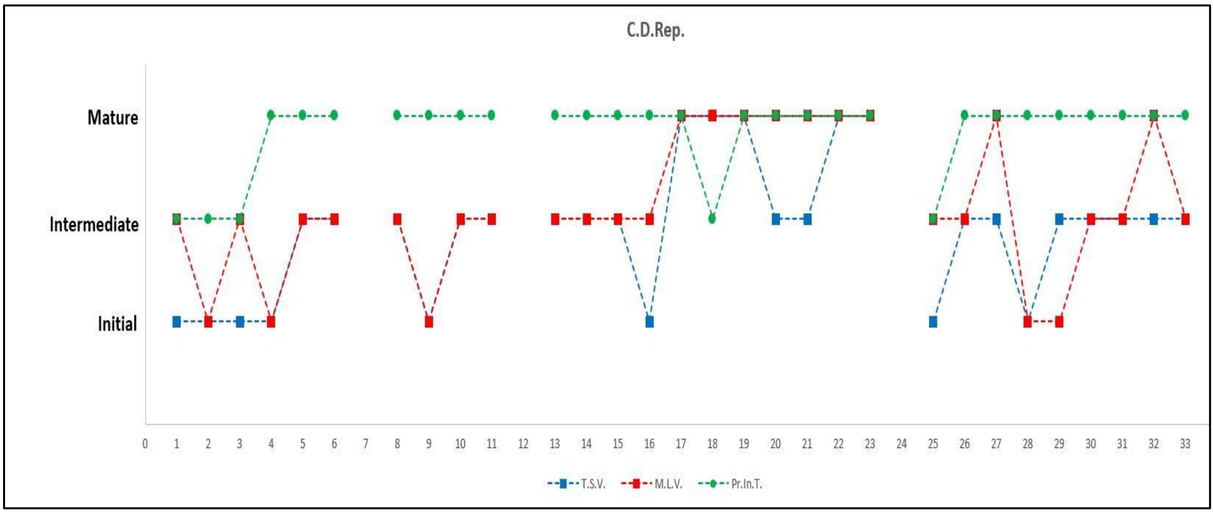

Diagram 5-3.Activation for competency aspect (C.D.Rep.).

Competency aspect activations: location and frequencies

The competency activations by units were as follows: Periodic Functions [1-6], Exponential Growth & Regression [8-11], Population Dynamics [13-23], and Integrals & Modeling [25-33]. The frequency of activation per units was [6, 4, 11, 9] and the total frequency of activation was, *n=30*.

Examples from the activations

Activation [8]: Evidence for the activation of the competency aspect (C.D.Rep.) was provided by Erika when she attempted to decode a representation '$31.0 + 1.7/100$' that would lead to an equation working on a task where students were informed about the yield growth and are asked to estimate the approximate barley yields for years after 1960. She had a mid-range perspective towards the solution (T.S.V., at Intermediate Activation Level, II), narrating while writing that "we need the Initial yield and this percentage" making regular use of expressions with mathematical language but not referring to the percentage as *rate* (M.L.V., at Intermediate Activation Level, II). All that occurred with no prompting (Pr.In.T., at Intermediate Activation Level, III) from the teacher or any classmate.

Activation [25]: Evidence for the activation of the competency aspect (C.D.Rep.) was provided by Erika while discussing a given rate with the expression '$\frac{df}{dt} = 0.014 \cdot t^{0.004}$' and she wrote "$\frac{df}{dt} = f'(t)$" mentioning that "this is the same thing." She decoded this representation having a short-range perspective towards the solution (T.S.V., at Initial Activation Level, I) being uncertain for the purpose of such an expression. She expressed herself making regular use of expressions with mathematical language and being familiar with the notation she used (M.L.V., at Intermediate Activation Level, II) with occasional prompting from the teacher (Pr.In.T., at Intermediate Activation Level, II).

Summary of evidence

The summary of evidence regarding the changes within the competency activation levels and the consistency levels is displayed below.

Table 5-11. Summarized evidence for (C.D.Rep.) activation.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	6	7	3	
Changes from higher to lower	4	5	1	
High consistency	0	0	2	
Moderate consistency	2(1)	3	4	
Frequency of activation				30

4) Communicating and reasoning mathematically

To express oneself, at various levels of theoretical and technical precision, in oral, visual or written form, about such matters. To provide explanations or justifications to support your result or idea. (code: Reason→) (14)

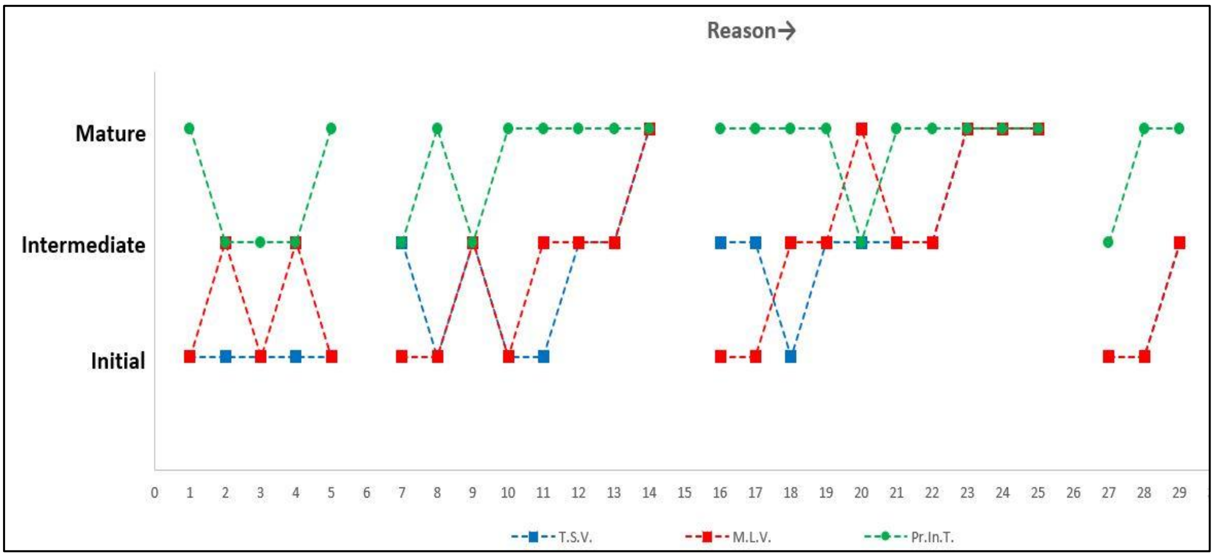

Diagram 5-4. Activation for competency aspect (Reason→).

Competency aspect activations: location and frequencies

The competency activations by units were as follows: Periodic Functions [1-5], Exponential Growth & Regression [7-14], Population Dynamics [16-25], and Integrals & Modeling [27-29]. The frequency of activation per units was [5, 8, 10, 3] and the total frequency of activation was, *n=26*.

Examples from the activations

Activation [2]: Evidence for the activation of the competency aspect (Reason→) was provided by Erika when she assisted her classmate on an issue regarding the amplitude of the periodic function of the Airport Temperatures task. She said, “I mean, to find how many degrees are between the highest and the lowest point," without mentioning that she will need to subtract these two numbers to find the amplitude of the "$g(x) = a\cos[b(x - c)] + d$" function. She provided her reasoning having a short-range perspective towards the solution (T.S.V., at Initial Activation Level, I) because she had begun her own steps (including her suggestion) towards the solution before her classmate. She expressed herself making regular use of expressions with mathematical language (M.L.V., at Intermediate Activation Level, II). All that occurred with occasional prompting (Pr.In.T., at Intermediate Activation Level, II) because her reasoning originated from a question from her classmate, "do we say basically the highest and the lowest point?"

Activation [28]: Evidence for the activation of the competency aspect (Reason→) was provided by Erika's explanation while helping her classmate on a calculation of the following expression: $\int_0^{200} \frac{df}{dt} = \int_0^{200} 0.014 \cdot t^{0.04}$. She said, "you need to take out that [showing the constant '0.014' of the expression] to continue." She reasoned having a short-range perspective towards the solution (T.S.V., at Initial Activation Level, I) because she was unsure of how to proceed after that. She expressed herself occasional use of everyday expressions and gestures with no mathematical language (M.L.V., at Initial Activation Level, I). However, all this occurred with no prompting from the teacher (Pr.In.T., at Mature Activation Level, III).

Summary of evidence

The summary of evidence regarding the changes within the competency activation levels and the consistency levels is displayed below.

Table 5-12. Summarized evidence for (Reason→) activation.

Competency domain / Type of evidence	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	6	9	4	
Changes from higher to lower	3	4	3	
High consistency	(1)	0	0	
Moderate consistency	1(1)	1(1)	5	
Frequency of activation				26

5) Use of Aids and Tools

To know the existence and properties of various tools and aids for mathematical activity. (code: Ext.T.) (15)

Note

For various reasons, that will be discussed in the Findings chapter, no evidence for competency activation was provided by Erika at any of the seven sessions.

To be able to use aids and tools to develop insight intuition. (code: U.T.) (16)

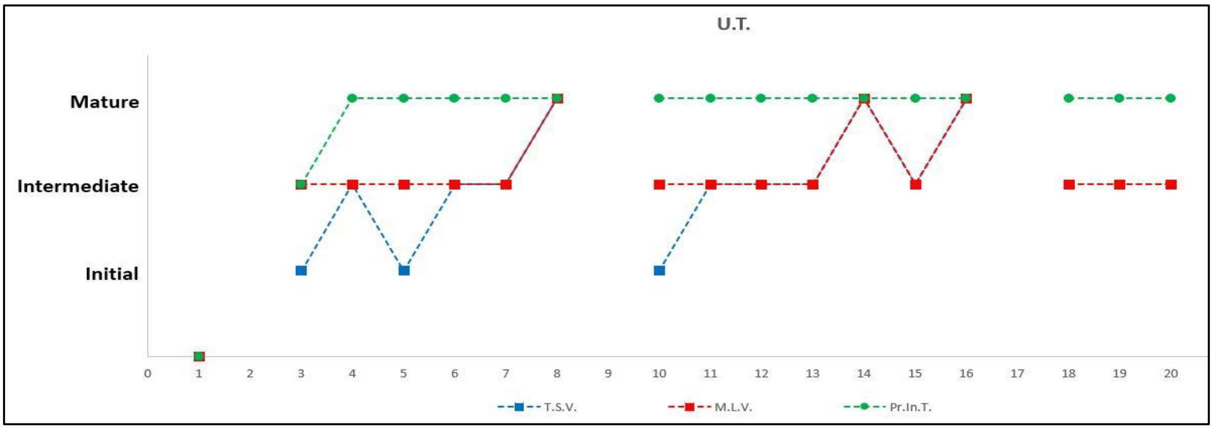

Diagram 5-5. Activation for competency aspect (U.T.).

Competency aspect activations: location and frequencies

The competency activations by units were: Exponential Growth & Regression [3-8], Population Dynamics [10-16], and Integrals & Modeling [18-20]. No activation was identified during the Periodic Functions. The activation frequency per unit was [0, 6, 7, 3], and the total activation frequency was *n=16*.

Examples from the activations

Activation [4]: Erika provided evidence for the activation of the (U.T.) competency aspect when she used the calculator to estimate the salmon population that was declining due to overfishing in the Salmon task. She failed to realize that the population was declining, probably because of previous examples of increasing populations. She used the tool having a mid-range perspective towards the solution (T.S.V., at Intermediate Activation Level, II), showing awareness of what the next step should be, "and as in the previous task with the bushels […]" while using the calculator and (M.L.V., at Intermediate Activation Level, II). All that occurred with minimal prompting (Pr.In.T., at Mature Activation Level, III) from the teacher.

Summary of evidence

The summary of evidence regarding the changes within the competency activation levels and the consistency levels is displayed below.

Table 5-13. Summarized evidence for (U.T.) activation.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	6	3	1	
Changes from higher to lower	2	1	0	
High consistency	1	2	3	
Moderate consistency	1	1	0	
Frequency of activation				16

5.6.2 Case study 2: Johannes

Diagrams 5-6 to 5-11 illustrate the development of each competency aspect by displaying the development of the three competency domains of development (T.S.V., M.L.V., and Pr.In.T.) across all seven calculus sessions. Johannes (similarly to Erika) provided limited evidence regarding the activation of two competency aspects (R.+V. and Ext.T.) and therefore no diagram was constructed for these cases but only a description of what occurred in these instances. Tables 5-14 to 5-19 summarize the evidence regarding the changes within the competency activation levels and the consistency levels.

1) Mathematical Thinking and Acting

To attack (take actions towards a solution) mathematical problems and tasks of various kinds. (code: →Sol.) (3)

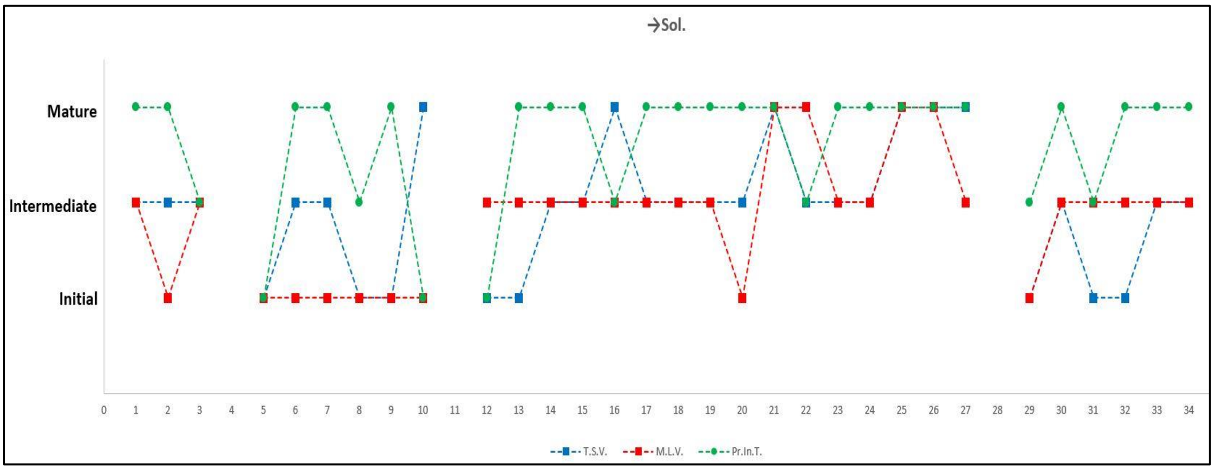

Diagram 5-6. Activation for competency aspect (→Sol.).

Competency aspect activations: location and frequencies

The competency activations by units were as follows: Periodic Functions [1-3], Exponential Growth & Regression [5-10], Population Dynamics [12-27], and Integrals & Modeling [29-34]. The frequency of activation per unit was [3, 6, 16, 6] and the total activation frequency was *n=31*.

Examples from the activations

Activation [14]: Evidence for the activation of the competency aspect (→Sol.) was provided by Johannes when he suggested that "[they] can get the *t* down using the *ln* or the *log* [functions]" while working on a particular equation ($N_t = N_0e^{rt}$) during the Exponential Growth & Regression unit. He had a mid-range perspective towards the solution (T.S.V., at Intermediate Activation Level, II) while expressing himself making regular use of expressions with mathematical language (M.L.V., at Intermediate Activation Level, II). All that occurred with minimal prompting (Pr.In.T., at Mature Activation Level, III) because the teacher said to the students that "we need the equation that describes the temperature during the year and the graph," which suggests that he gave some hints but let the students discover primarily on their own how to proceed.

Activation [22]: Evidence for the activation of the competency aspect (→Sol.) was provided by Johannes while he was working on the World's Population task. He replied to a question from his teacher, who asked about the kind of information he should look for when Johannes found out an estimation of the Earth's population. He said, "Ok, first I have to find the current population which I did, then I need to find how much was increasing or decreasing over the years." He had a mid-range perspective towards the solution (T.S.V., at Intermediate Activation Level, II), being close to estimating how long it would take for the world population to be doubled. He expressed himself making regular and correct use of expressions with mathematical language and vocabulary with clearly stated steps (M.L.V., at Mature Activation Level, III) with minimal prompting from the teacher (Pr.In.T., at Mature Activation Level, II), who asked him about the kind of information he should be looking for.

<u>Summary of evidence</u>

The summary of evidence regarding the changes within the competency activation levels and the consistency levels is displayed below.

Table 5-14. Summarized evidence for (→Sol.) activation.

Type of evidence / **Competency domain**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	8	4	7	
Changes from higher to lower	4	4	6	
High consistency	1	1(1)	0	
Moderate consistency	0	1	4	
Frequency of activation				31

2) Mathematical Modeling

<u>Look and search for available information and to differentiate between relevant and irrelevant information. (code: Info) (8)</u>

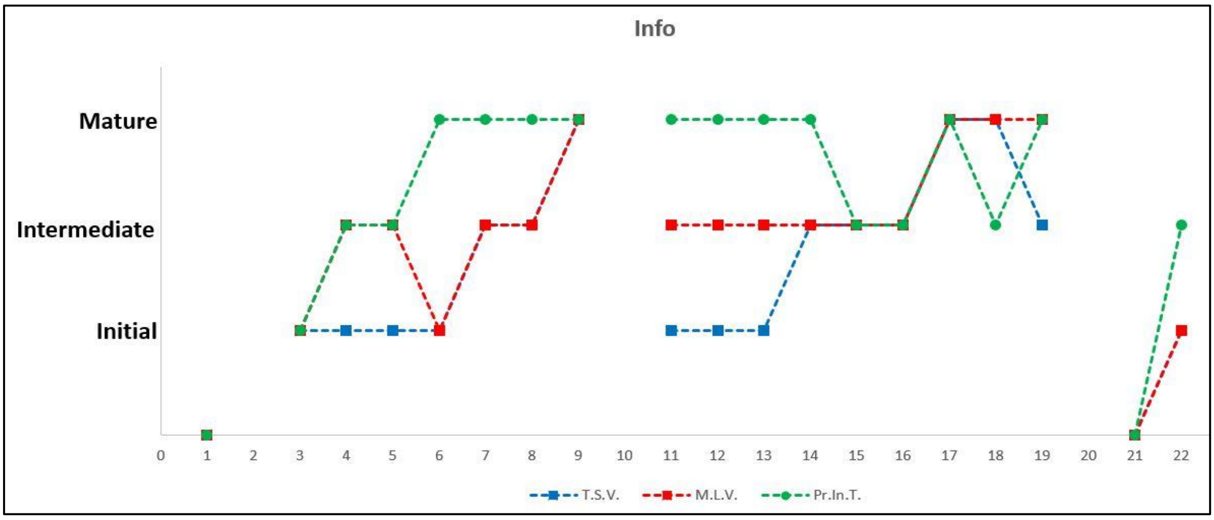

Diagram 5-7. Activation for competency aspect (Info).

Competency aspect activations: location and frequencies

The competency activations by units were as follows: Exponential Growth & Regression [3-9], Population Dynamics [11-19], and Integrals & Modeling [22]. There was no activation of the (Info) competency aspect during the Periodic Functions and the first session [21] of the Integrals & Modeling units. The activation frequency per unit was [0, 7, 9, 1], and the total activation frequency was $n=17$.

Examples from the activations

Activation [14]: Evidence for the activation of the competency aspect (Info) provided by Johannes when he pointed out that "If [they] find or know the mass of one bacterium then [they] can find [the whole mass]" when asked from the teacher if they can find the mass of the cells when they found that there are 2^{72} cells in the colony when working on the E. Coli task. He searched for this information with a mid-range perspective towards the solution (T.S.V., at Intermediate Activation Level, II) while expressing himself making regular use of expressions with mathematical language (M.L.V., at Intermediate Activation Level, II). All that occurred with minimal prompting (Pr.In.T., at Mature Activation Level, III) from the teacher who posed the abovementioned question.

Activation [19]: Evidence for the activation of the competency aspect (Info) was provided by Johannes when he searched for the appropriate population model to use for the Population Doubled task. He said, "I remember when we used this model (finds it in his notes) when we looked for doublings, but this is a bit different." He had a mid-range perspective towards the solution (T.S.V., at Mature Activation Level, II), being close estimating the date when the world's population will be doubled. He expressed himself making regular and correct use of expressions with mathematical language and searching for the appropriate model from previous notes (M.L.V., at Mature Activation Level, III) with minimal prompting from the teacher (Pr.In.T., at Mature Activation Level, III).

Summary of evidence

The summary of evidence regarding the changes within the competency activation levels and the consistency levels is displayed below.

Table 5-15. Summarized evidence for (Info) activation.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	5	5	5	
Changes from higher to lower	1	1	2	
High consistency	0	0	0	
Moderate consistency	(1)	1	2	
Frequency of activation				17

To choose appropriate mathematical graphical notations during the task solution process. (code: Graph) (9)

Competency aspect activations: note for location and frequencies

Johannes provided evidence for the activation of the competency aspect (Graph) only on two occasions. Once during the first session of the Exponential Growth & Regression unit and once during the second session of the Population Dynamics unit. There was no evidence of competency activation during the Periodic Functions and the Exponential Growth & Regression units.

Examples from the activations

Johannes provided evidence for the activation of the(Graph) competency aspect by the end of the second session of the Population Dynamics unit when the class was discussing a graph on world population from 1750 to a projection of 2100 where the annual growth rate of world population varied across periods of time in history (e.g., baby boomer generation). In particular, before showing the graph on the screen, the teacher asked the class to imagine how such a graph would be. Johannes started sketching a graph explaining that it should start "very carefully and at some point, maybe explodes and jumps." He expressed himself having a mid-range perspective on how the actual graph was (T.S.V., at Intermediate Activation Level, II), making regular use of expressions with mathematical language (M.L.V., at Intermediate Activation Level, II). All that occurred with minimal prompting (Pr.In.T., at Intermediate Activation Level, III) because the teacher asked the class to imagine such a graph.

Summary of evidence

The frequency of activation of the "*To choose appropriate mathematical graphical notations during the task solution process*" competency (code: Graph) is low. It is impossible to locate any sign of consistency having such a low frequency. This competency is considered *non-activated*. However, some task-related characteristics will be considered in the Conclusions chapter.

3) Representing and Manipulating symbolic forms

Switch between representations. (code: Rep.↔Rep.) (11)

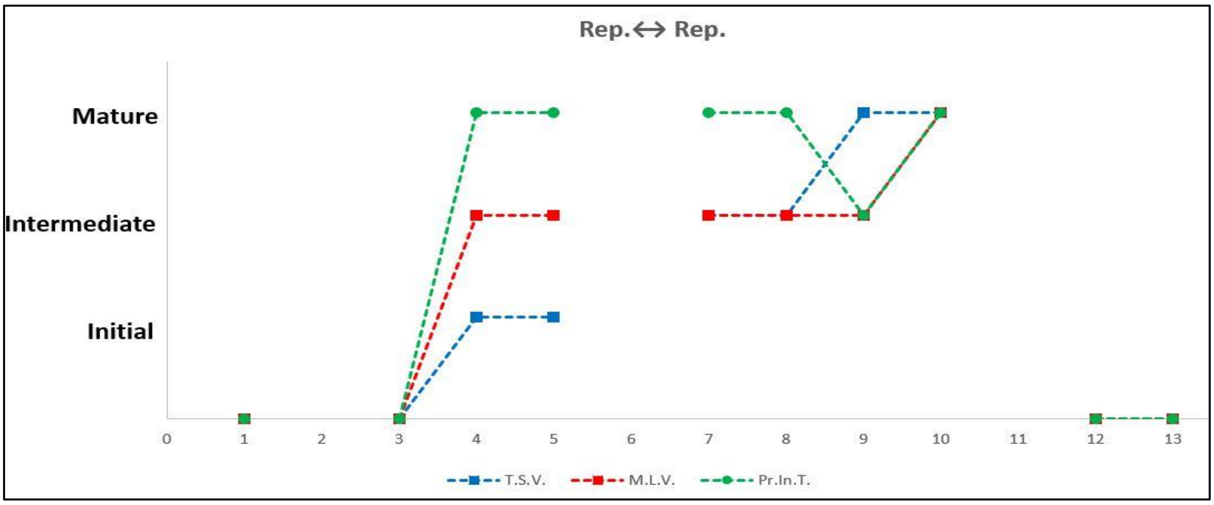

Diagram 5-8. Activation for competency aspect (Rep.↔Rep.).

Competency aspect activations: location and frequencies

The competency activations by unit were: Exponential Growth & Regression [4-5] and Population Dynamics [7-10]. No activation of the (Rep.↔Rep.) aspect was identified during the Periodic Functions, the first session [3] of the Exponential Growth & Regression, and the Integrals & Modeling units. The activation frequency per unit was [0, 2, 4, 0] and the total frequency of activation was, $n=6$.

Examples from the activations

Activation [10]: Johannes provided evidence for the activation of the competency aspect (Rep.↔Rep.) when working on the Population Doubling task when he switched between representations (i.e., $P(t) = 2 \cdot P(0)$). He switched between the representations having a long-range towards the solution (T.S.V., at Mature Activation Level, III) while writing down his work, making regular and correct use of expressions with mathematical language (M.L.V., at Mature Activation Level, III). All that occurred with no prompting (Pr.In.T., at Mature Activation Level, III), working on his own until reaching the solution.

Summary of evidence

The summary of evidence regarding the changes within the competency activation levels and the consistency levels is displayed below.

Table 5-16. Summarized evidence for (Rep.↔Rep.) activation.

Type of evidence \ Competency domain	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	2	2	2	
Changes from higher to lower	0	0	1	
High consistency	0	1	0	
Moderate consistency	2	0	1	
Frequency of activation				6

Manipulate within a representation. (code: Man.Rep.) (12)

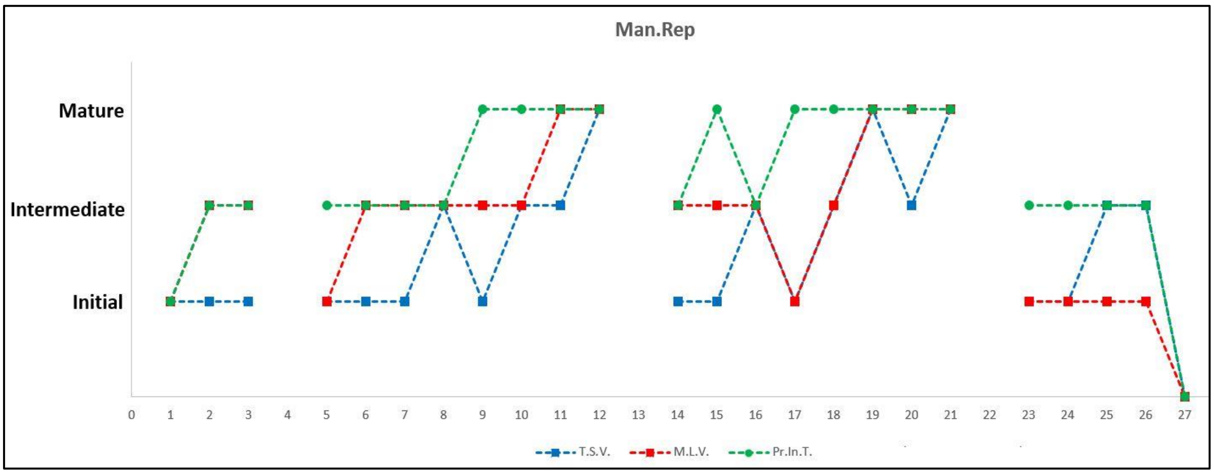

Diagram 5-9. Activation for competency aspect (Man.Rep.).

Competency aspect activations: location and frequencies

The competency activations by units were as follows: Periodic Functions [1-3], Exponential Growth & Regression [5-12], Population Dynamics [14-21], and Integrals & Modeling [23-26]. There was no activation of the (Man.Rep.) competency aspect during the last session [27] of the Integrals & Modeling unit. The activation frequency per unit was [3, 8, 8, 4], and the total activation frequency was *n=23*.

Examples from the activations

Activation [2]: Evidence for the activation of the competency aspect (Man.Rep.) was provided by Johannes when he worked on finding the amplitude a of the $g(x) = a \cdot \cos[b(x-c)] = b$, function and handled the symbols of this representation. He wrote in his notes the maximum and minimum values of the graph and subtracted them, "$17.9 - 9.5 = 8.4$" and then stopped because he was not happy with the result. He had a short-range perspective towards the solution (T.S.V., at Initial Activation Level, I) while expressing himself, making regular use of expressions with mathematical language (M.L.V., at Intermediate Activation Level, II). All that occurred with occasional prompting (Pr.In.T., at Intermediate Activation Level, II) because Erika helped him see that he needed to divide his result (i.e., 8.4) by two.

Activation [21]: Evidence for the activation of the competency aspect (Man.Rep.) was provided by Johannes when he was working on the Population Doubling task when he worked and manipulated the system of equations (1) $P(t) = 2 \cdot P(0)$ and (2) $P(t) = P(0) \cdot e^{rt}$. He had a long-range perspective towards the solution (T.S.V., at Mature Activation Level, III), being close estimating the date when the world's population will be doubled. He expressed himself with regular use of expressions with mathematical language in his writing (M.L.V., at Mature Activation Level, III) with minimal prompting from the teacher (Pr.In.T., at Mature Activation Level, III).

Summary of evidence

The summary of evidence regarding the changes within the competency activation levels and the consistency levels is displayed below.

Table 5-17. Summarized evidence for (Man.Rep.) activation.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	8	5	4	
Changes from higher to lower	4	2	2	
High consistency	(1)	(1)	1	
Moderate consistency	1(2)	4(1)	4	
Frequency of activation				23

4) Communicating and reasoning mathematically

To understand others' written, visual or oral 'texts', in a variety of linguistic registers, about matters having a mathematical content. To follow and assess chains of arguments, put forward by others. (code: Reason⟵) (13)

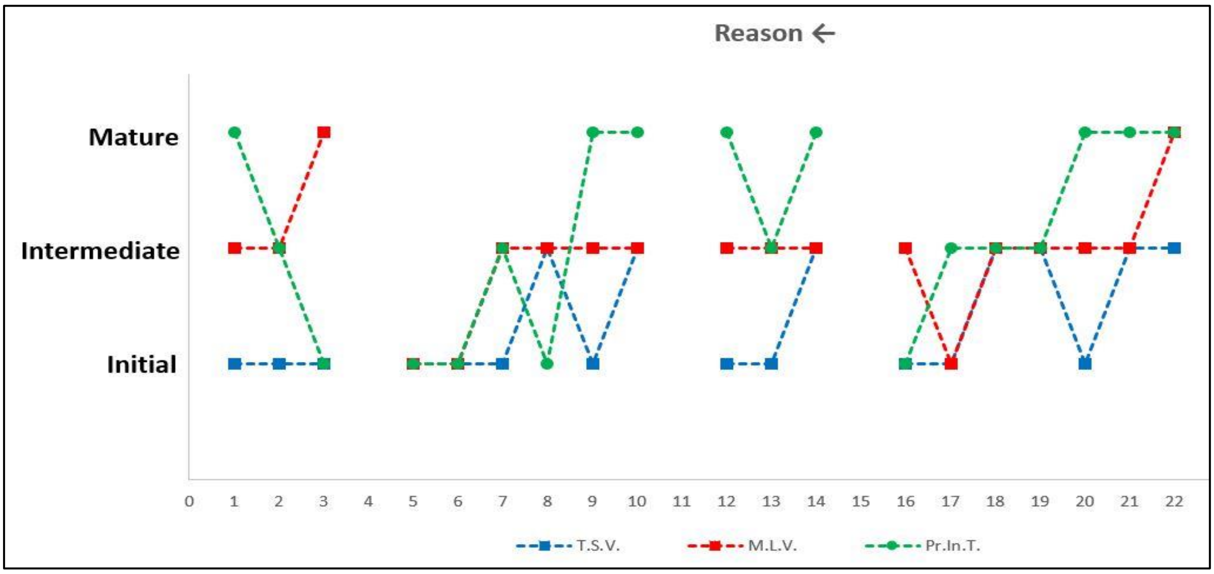

Diagram 5-10. Activation for competency aspect (Reason⟵).

Competency aspect activations: location and frequencies

The competency activations by units were as follows: Periodic Functions [1-3], Exponential Growth & Regression [5-10], Population Dynamics [12-14], and Integrals & Modeling [16-22]. The frequency of activation per unit was [3, 6, 3, 7] and the total activation frequency was $n=19$.

Examples from the activations

Activation [7]: Evidence for the activation of the competency aspect (Reason⟵) was provided by Johannes when he followed arguments put Erika when they were working on the World Population task. The discussion concerned the "starting point" that Erika mentioned that she needed to find to begin solving that task (i.e., the $N(0)$ value on the $N(t) = N(0)\, e^{rt}$ model). Johannes, following that argument, pointed out that "2 is the starting point because we had two people in the beginning so $N(0) = 2$" and Erika agreed. Johannes followed this argumentation having a short-range perspective towards the solution (T.S.V., at Initial Activation Level, I) just following the discussion initiated by Erika, while expressing himself making regular use of expressions with mathematical language (M.L.V., at Intermediate Activation Level, II). All that occurred with occasional prompting (Pr.In.T., at Intermediate Activation Level, II) because both Erika and the teacher contributed to the discussion.

Activation [16]: Evidence for the activation of the competency aspect (Reason⟵) was provided by Johannes when he followed a comment made by his teacher when working on the Earth's Temperature task. When the teacher mentioned that he was giving them the "temperature at 2000 degrees Fahrenheit," Johannes followed that by mentioning that "this is the starting point that we will be asked to use at some point." He had a short-range perspective towards the solution (T.S.V., at Initial Activation Level, I) and expressed himself making regular use of expressions with mathematical language (M.L.V., at Intermediate Activation Level, II) with frequent prompting from the teacher (Pr.In.T., at Initial Activation Level, I).

Summary of evidence

The summary of evidence regarding the changes within the competency activation levels and the consistency levels is displayed below.

Table 5-18. Summarized evidence for (Reason←) activation.

Competency domain / Type of evidence	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	5	4	5	
Changes from higher to lower	2	1	4	
High consistency	(1)	1	0	
Moderate consistency	(2)	3	2	
Frequency of activation				19

5) Use of Aids and Tools

To know the existence and properties of various tools and aids for mathematical activity. (code: Ext.T.) (15)

Note

For various reasons, that will be discussed in the Conclusions chapter, no evidence for competency activation was provided by Erika at any of the seven sessions.

To be able to use such aids and tools to develop insight intuition. (code: U.T.) (16)

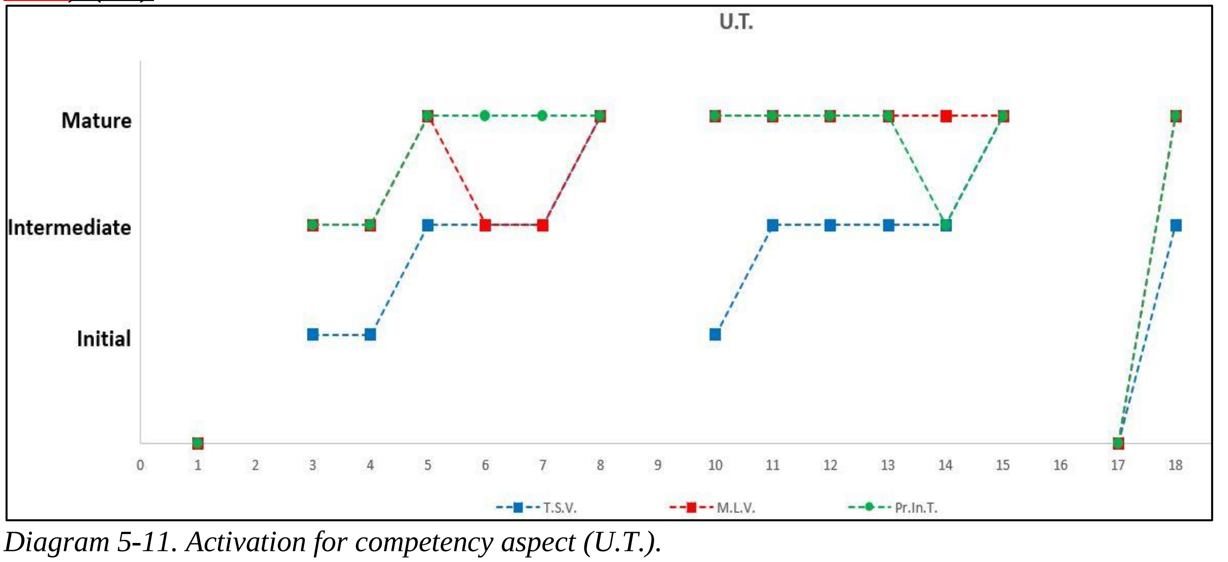

Diagram 5-11. Activation for competency aspect (U.T.).

Competency aspect activations: location and frequencies

The competency activations by units were as follows: Exponential Growth & Regression [3-8], Population Dynamics [10-15], and Integrals & Modeling [18]. There was no activation of the (U.T.) competency aspect during the Periodic Functions and first session [17] of the Integrals & Modeling units. The activation frequency per unit was [0, 6, 6, 1], and the total activation frequency was *n=13*.

Examples from the activations

Activation [18]: Evidence for the activation of the competency aspect (U.T.) was provided by Johannes during the Petroleum task in the last session of the Integrals & Modeling unit. He used his calculator and his phone to check how to calculate the expression '$1.02 = e^{r}$' and especially the value of $ln0.02$. He used these tools having a mid-range perspective towards the solution and to find the rate *r* (T.S.V., at Intermediate Activation Level, II) while handling the phone (i.e., being aware of how to track the relevant information) and the calculator efficiently (M.L.V., at Mature Activation Level, III). All that occurred with no prompting (Pr.In.T., at Mature Activation Level, III) from the who was just observing the solving process of Johannes and Erika.

Summary of evidence

The summary of evidence regarding the changes within the competency activation levels and the consistency levels is displayed below.

Table 5-19. Summarized evidence for (U.T.) activation.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	5	3	3	
Changes from higher to lower	0	1	1	
High consistency	0	1	0	
Moderate consistency	2	0	2	
Frequency of activation				13

5.6.3 Summary for the two case studies

There appears to be significant evidence of competence activation and possible development for Erika and Johannes regarding the following competencies: Mathematical Thinking and Acting, Representing and Manipulating symbolic forms, and Mathematical Reasoning and Communicating. It is possible, however, to infer that additional or fewer competencies displayed evidence of development for each student. Furthermore, until this point, there are only indications regarding the assessment of their competency development because further analysis is needed. The evidence displayed above must be combined to produce more reliable inferences regarding the individual competency development throughout these seven sessions. This additional analysis will take place in the following chapter.

Both students exhibited a more frequent competency activation during the Population Dynamics unit and a relatively less frequent during the Integrals & Modeling one. Finally, it appears that Erika displayed evidence of activation more frequently than Johannes across all competency aspects. However, as mentioned before, this observation cannot produce conclusive inferences until further analysis is undertaken. The following subsection presents the results that will be used to address the second Research Question of this thesis.

5.7 Competencies interrelation

This section contains the Binomial Test results applied to analyze the frequencies of concurrent competency activation. As stated in subsection 2.5.2, where the second Research Question is addressed, I seek to identify whether the competencies are interrelated and, if so, how. Subsection 5.7.1 displays the tables that include data from the coding process for running the Binomial Test. These test results are presented in subsection 5.7.2, where the competency pairs that recorded a statistically significant frequency are displayed in a table. This section is significantly smaller than the previous one and concerns the second Research Question. This is because the central part of the analysis is undertaken by applying the Binomial Test. However, it was impossible to produce the tables displayed below without the work undertaken in the previous sections.

5.7.1 Tables used for the Binomial Test

Table 5-20Table 5-20 includes the frequencies of competency activation for each competency aspect. The columns illustrate each aspect of the framework, and in the first two rows, the individual (by student) activation frequency. For example, at the (Int.El.) column and Johannes's row, one can see the number 44. This is the total frequency of the (Int.El.) competency aspect activation for Johannes across all seven calculus sessions. In other words, Johannes was observed to activate the (Int.El.) competency aspect 44 times during the project. The last row displays the total activation frequency for both students. A total number of $n = 341$ turns (i.e., students' utterances) was observed.

Table 5-20. Competency activation frequencies for each aspect.

	1 P. Q.	**2 Lim.**	**3 → Sol.**	**4 R. + V.**	**5 Int. El.**	**6 Int. Res. X**	**7 Cr**	**8 Info**	**9 Graph**	**10 C. D. Rep.**	**11 Rep. ↔ Rep.**	**12 Man . Rep.**	**13 Reason ←**	**14 Reason →**	**15 Ext. T.**	**16 U. T.**
Erika	16	8	37	3	25	11	9	14	4	30	8	31	15	26	0	16
Johannes	24	8	31	3	44	12	10	17	2	22	6	23	19	23	0	13
Totals	40	16	68	6	69	23	19	31	6	52	14	54	34	49	0	29

Table 5-21 illustrates the results from the observation of the coding process. The frequencies of competency pairs are displayed in the cells. For example, at the (C.D.Rep.) column and (Int.El.) row the number 10 indicates that across all students' utterances (i.e., 341 turns), there were 10 turns where the (C.D.Rep.) and (Int.El.) competency were observed to be activated concurrently.

Table 5-21. Concurrent activation frequencies for each competency pair.

	1 P. Q.	2 Lim.	3 → Sol.	4 R. + V.	5 Int. El.	6 Int. Res. X	7 Cr.	8 Info	9 Graph	10 C. D. Rep.	11 Rep. ↔ Rep.	12 Man. Rep.	13 Reason ←	14 Reason →	15 Ext. T.	16 U. T.
1 P.Q.		1	5	0	4	0	1	1	0	0	0	0	0	2	0	0
2 Lim.			1	1	3	1	0	0	0	2	0	0	0	0	0	0
3 → Sol.				0	13	5	1	0	1	7	0	5	1	4	0	7
4 R.+V.					0	0	0	1	0	1	0	0	0	0	0	0
5 Int.El.						1	0	1	0	10	0	1	6	9	0	0
6 Int.Res.X							4	0	0	2	0	0	0	0	0	0
7 Cr.								0	0	1	0	0	1	0	0	0
8 Info									0	1	0	0	0	2	0	2
9 Graph										1	0	0	0	0	0	0
10 C.D.Rep.											7	9	1	5	0	0
11 Rep. ↔ Rep.												4	0	0	0	1
12 Man.Rep.													4	2	0	4
13 Reason←														7	0	1
14 Reason→															0	0
15 Ext.T.																0
16 U.T.																

Let us remember here the proportions that were tested with the Binomial Test, and the formula of the test, originally presented in subsection 4.5.3. The values for the expected probabilities can be seen as a fraction of:

$$p_{e(i,j)} = \frac{frequency\ of\ comp.\ activation\ (i)}{total\ number\ of\ discource\ units} \times \frac{frequency\ of\ comp.activation\ (j)}{total\ number\ of\ discource\ units} = \frac{f_a(i)}{n} \times \frac{f_a(j)}{n} \quad (1)$$

and the values for the observed concurrent activation as a fraction of:

$$p_{o(i,j)} = \frac{frequency\ of\ concurrent\ activation\ of\ comptencies\ (i,j)}{total\ number\ of\ discource\ units} = \frac{f_{ca}(i,j)}{n} \quad (2)$$

Finally, the formula for the Binomial Test is: $P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$ (3), where:

$P(x)$ is the binomial probability, x is the total number of expected 'successes,' p is the probability of a success on an individual trial, observed proportion, and n is the number of trials. The values for the numerator of the $p_{e(i,j)}$ proportion have been extracted from Table 5-20 and the values for the numerator of the $p_{o(i,j)}$ proportion have been extracted from Table 5-21.

5.7.2 Results from running the Binomial Test

Using the information from the tables from the previous subsection, I proceeded to run the Binomial (two-tailed) Test with the R programming language for statistical computing and graphics. The raw results from this test are displayed in Figures G-1 and G-2 included in Appendix G because both figures are screenshots that illustrate the code written for the application of the test using the information from Table 5-20 and Table 5-21 and equations (1), (2), and (3) from the previous subsection. In particular, Figure G-2 includes the outcome from the test, listing a set of competency pairs that the application of the Binomial Test recorded p-value $p \leq 0.05$.

Fifteen (15) competency pairs were found to present statistical significance: (P.Q. & C.D.Rep.), (P.Q. & Man.Rep.), (P.Q. & Reason←), (→Sol. & Info), (→Sol. & Reason←), (Int.El. & Cr.), (Int.El. & Info), (Int.El. & Man.Rep.), (Int.El. & U.T.), (Int.Res.X & Cr.), (Info & Man.Rep.), (C.D.Rep. & Rep.↔Rep.), (C.D.Rep. & U.T.), (Man.Rep. & Reason→), (Reason→ & U.T.).

Table 5-22 below, is a repetition of Table 5-21 displaying the information obtained from the Binomial Test. The green shaded cells show the competency pairs where the application of the Binomial Test recorded p-value $p \leq 0.05$.

Table 5-22. Green shaded cells after running the Binomial Test.

	1 P.Q.	**2** Lim.	**3** →Sol.	**4** R.+V.	**5** Int.El.	**6** Int.Res.X	**7** Cr.	**8** Info	**9** Graph	**10** C.D.Rep.	**11** Rep.↔Rep.	**12** Man.Rep.	**13** Reason←	**14** Reason→	**15** Ext.T.	**16** U.T.
1 P.Q.		1	5	0	4	0	1	1	0	0	0	0	0	2	0	0
2 Lim.			1	1	3	1	0	0	0	2	0	0	0	0	0	0
3 →Sol.				0	13	5	1	0	1	7	0	5	1	4	0	7
4 R.+V.					0	0	0	1	0	1	0	0	0	0	0	0
5 Int.El.						1	0	1	0	10	0	1	6	9	0	0
6 Int.Res.X							4	0	0	2	0	0	0	0	0	0
7 Cr.								0	0	1	0	0	1	0	0	0
8 Info									0	1	0	0	0	2	0	2
9 Graph										1	0	0	0	0	0	0
10 C.D.Rep.											7	9	1	5	0	0
11 Rep.↔Rep.												4	0	0	0	1
12 Man.Rep.													4	2	0	4
13 Reason←														7	0	1
14 Reason→															0	0
15 Ext.T.																0
16 U.T.																

6. Competency development and interrelation

This chapter presents the findings of this study regarding the competency development of the students and the observed interrelation of the competency aspects of the competence framework that was employed for this study. Section 6.1 displays the findings regarding the progress of students' competency development over time for each of the five mathematical competencies using the 3-D theoretical model that explores the progression of competency development. This section concerns the issues addressed within the first Research Question (subsection 2.5.2) in Chapter 2. Section 6.2 concerns the issues addressed within the second Research Question (subsection 2.5.2) and displays the findings regarding the inferences about the competencies' interrelation, the structure of the competence framework, and which competency aspects' activation may influence the activation of another competency aspect.

6.1 Competency development

This section is structured in five subsections (6.1.1, 6.1.2, 6.1.3, 6.1.4, and 6.1.5). The first subsection (6.1.1) revisits the first Research Question by addressing the nature of competency development. Within this subsection, I also discuss the notion of 'strength of evidence combination' that will be used to determine the nature of competency development that Erika and Johannes displayed for all competencies. The second subsection (6.1.2) includes a visual guide that illustrates the method followed to produce the concluding inferences regarding the competency development within the 3-D competency development model (Niss & Højgaard, 2011). Subsections 6.1.3 and 6.1.4 focus on Erika and Johannes's work, respectively, and the evidence they provided regarding the development of their mathematical competencies within each of the three dimensions of the 3-D model. In subsection 6.1.4, I discuss the competency development results by displaying overall inferences. Subsection 6.1.5 summarizes what was discussed in the previous subsections and presents in tabular form the findings of this thesis regarding the issues addressed in the first Research Question.

6.1.1 Revisiting the first Research Question

In Chapter 2 and Section 2.5.2, the following general research question was set: *What is the progress of individual competency development over time for a student who participates in a series of calculus sessions that comprise non-routine mathematical tasks? What is the competence profile of each student after these series of calculus sessions?* The term 'progress of individual competency development' does not necessarily suggest a positive development. That is, a student becomes more competent concerning a particular competency over time. By progress, I wish to highlight the dynamic nature of development, and by exploring its progress, the focus is on the evolution or not of student's competency development during the seven calculus sessions. It would be somewhat uncertain or at least speculative if I were to claim that I can track signs of complete development within each student's competence profile throughout just one semester and seven calculus sessions over two teaching months.

This research aims to provide inferences regarding students' competency development over time. To provide such inferences, an evaluation of the available evidence collected and analyzed (i.e., frequency of competency activation, changes between levels of activation, and consistency within a level of activation) throughout these seven calculus sessions is performed. For that reason, the findings here in this section are presented by reflecting on what the combination of evidence suggests regarding students' competency development progress. It is possible for a student that he/she will display a strong combination of evidence that suggests development within the Degree of Coverage of Mathematical Thinking and Acting competency. At the same time, it is also possible that the same student will present a weak combination of evidence for the development within the Radius of Action of the same competency. The following subsection below discusses the functionality of this evaluation of evidence combination by adopting the term *strength of evidence combination*.

Strength of evidence combination

As discussed above, the combination of evidence can provide different inferences about the nature of competency development. This is not unexpected because two types of evidence (i.e., changes between levels of activation and consistency within a level of activation) were created based on the properties of the three-level scaling instrument (subsection 4.1.8) of this study and the frequencies of competency activation (the first type of evidence) can naturally differ from one competency to another. The latter suggests that the three types of evidence comprise an expected variation. This consideration led to the conceptualization of the strength of evidence combination that can qualitatively differ from student to student and from one mathematical competency to another. The strength of evidence combination is not an additional scaling instrument but a way to express the observations in an orderly way and in alignment with the properties of the three-level scaling instrument of this study. For that reason, three strength levels have been considered, with the addition of a fourth level for these instances where limited or non-activated competencies were identified. Again, each one of the levels presented below is connected to those of the main scaling instrument:

- ***Strong***: this is the strength level of evidence combination related to the Mature Level of Activation (III) of the scaling instrument.
- ***Moderate***: this is the strength level of evidence combination related to the Intermediate Level of Activation (II) the scaling instrument.

- ***Weak***: this is the strength level of evidence combination related to the Initial Level of Activation (I) the scaling instrument.
- ***Negligible***: this is the strength level of evidence combination that represents the situation where partially or non-activated competencies have been located and identified.

Example for the determination of the strength levels

Before commencing with the presentation of the findings for Erika and Johannes, I provide an example that illustrates the sequence for determining the strength levels of the combination of evidence. As mentioned at the beginning of Chapter 4 (prologue of the analysis) and visualized in Figure 4.1, the diagrams and the summarized tables produced for Erika and Johannes in subsections 5.6.1 and 5.6.2 are used to provide overviews to evaluate the strength levels of the evidence combination for each mathematical competency of the competence framework.

A combination of the three types of evidence (i.e., frequency of activation, changes between activation levels, and levels of consistency) must be considered to make inferences regarding the competency development. It is possible that one type of evidence would not be sufficient to provide enough evidence to make inferences about the development of a particular competency aspect. For example, one can imagine a situation where the number of changes from a lower to a higher level of activation and the number of changes from a higher to a lower one is the same or nearly the same. Therefore, no direction of change (i.e., from higher to lower or vice versa) is recorded (evidently) more often than the other. The latter would suggest that if the number of changes is small (in both directions), there should be some form of consistency regarding the activation levels. If, however, the number of changes (to both directions) is extensive, then no instance of identifiable consistency may be recorded. It is, therefore, possible to face a situation where no reliable inferences can be made.

The following is an example of how the inferences regarding the competency development of a particular competency aspect are produced. In subsection 4.3.1, I provided an example of an overview regarding the development of the (Int.El.) aspect of the Mathematical Modeling competency. To this overview, I now add the evaluation of the summarized data (diagrams and summarized table) to make inferences regarding the strength level of the evidence combination for this competency. Diagram 6-1 displays the progress of the (Int.El.) competency aspect development, and Table 6-1 exhibits the summarized evidence of this activation.

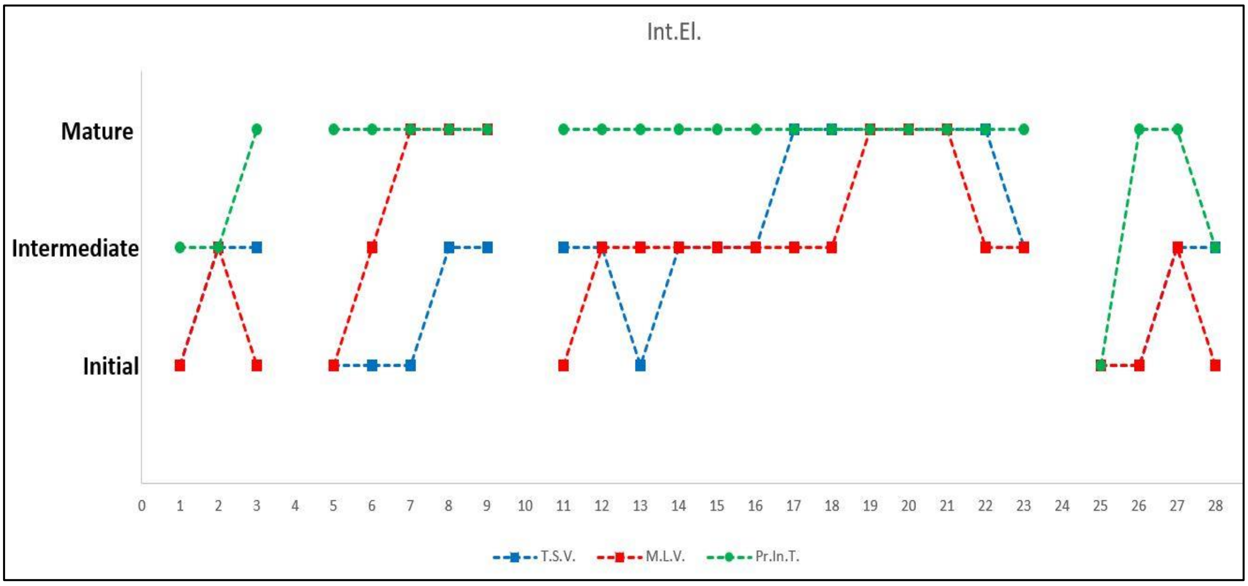

Diagram 6-1. Activation of competency aspect (Int.El) (repetition of Diagram 5-2).

Table 6-1. Summarized evidence for (Int.El.) activation. (repetition).

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	5	6	2	
Changes from higher to lower	2	3	1	
High consistency	0	0	2	
Moderate consistency	4(2)	2(1)	2	
Frequency of activation				25

Frequency of competency activation

Regarding the first type of evidence, namely the frequency of competency activation, a gradual increase until the third calculus unit is noticeable, followed by an evident drop by the last unit. The increase is spotted from one calculus unit to the next. As shown in the diagram, the frequencies were [3, 5, 13, 4], that is, three instances of competency activation during the Periodic Functions unit, five during the Exponential Growth & Regression unit, 13 during the Population Dynamics unit, and four during the Integrals & Modeling unit. The total frequency of activation is 25.

Changes between activation levels

By incorporating the second type of evidence, changes between activation levels, the information from Table 6-1 informs us that the changes from lower to higher and higher to lower levels of competency activation by domain were: (5, 2) for T.S.V., (6, 3) for M.L.V., and (2, 1) for Pr.In.T. These observations show that the progress was more often from a lower to a higher level of activation, double or more, than the other way around suggesting a dynamic of development over time.

Levels of consistency

Lastly, regarding the third type of evidence, levels of consistency, Diagram 6-1 and Table 6-1 informs us that there was high consistency at the Mature level for the Pr.In.T. domain twice, suggesting a continuous work with limited prompting, and moderate consistency for the T.S.V., M.L.V., and Pr.In.T. domains six, three and two times respectively. However, this moderate consistency was also noticed two out of six times at the Initial level for the T.S.V. domain (hence the 4(2) symbolism) and once for the M.L.V domain.

Combination of evidence: final evaluation

Based on the summaries above for all three types of evidence, I infer that there is a *strong combination of evidence* for the development of the (Int.El.) aspect of the Mathematical Modeling competency.

A Note

To determine the strength level of evidence combination regarding the Degree of Coverage of one of the five mathematical competencies of the competence framework, evidence from all competency aspects that constitute one mathematical competency must be evaluated. For example, to determine the strength level of evidence combination of the Representing and Manipulating symbolic forms competency regarding the Degree of Coverage, findings from all three aspects (i.e., C.D.Rep., Rep.↔Rep., and Man.Rep.) will be taken into consideration. It is evident that for the determination of the competency development over time for Erika and Johannes, a combination of evidence for all 16 competency aspects of the five mathematical competencies must occur. Following that, this combination will be interpreted through the three dimensions of the 3-D model of competency development (Niss & Højgaard, 2011) to produce inferences regarding the nature of development they manifested. The following subsection summarizes this methodological process and offers illustrations that will help the reader obtain a comprehensive idea of how inferences about competency development were made.

6.1.2 A visual guide

In this subsection, I display a set of illustrations that are intended to help the readers visualize the methodological pattern adopted to determine the nature of competency development for Erika and Johannes. Below each figure, an appropriate explanation will follow.

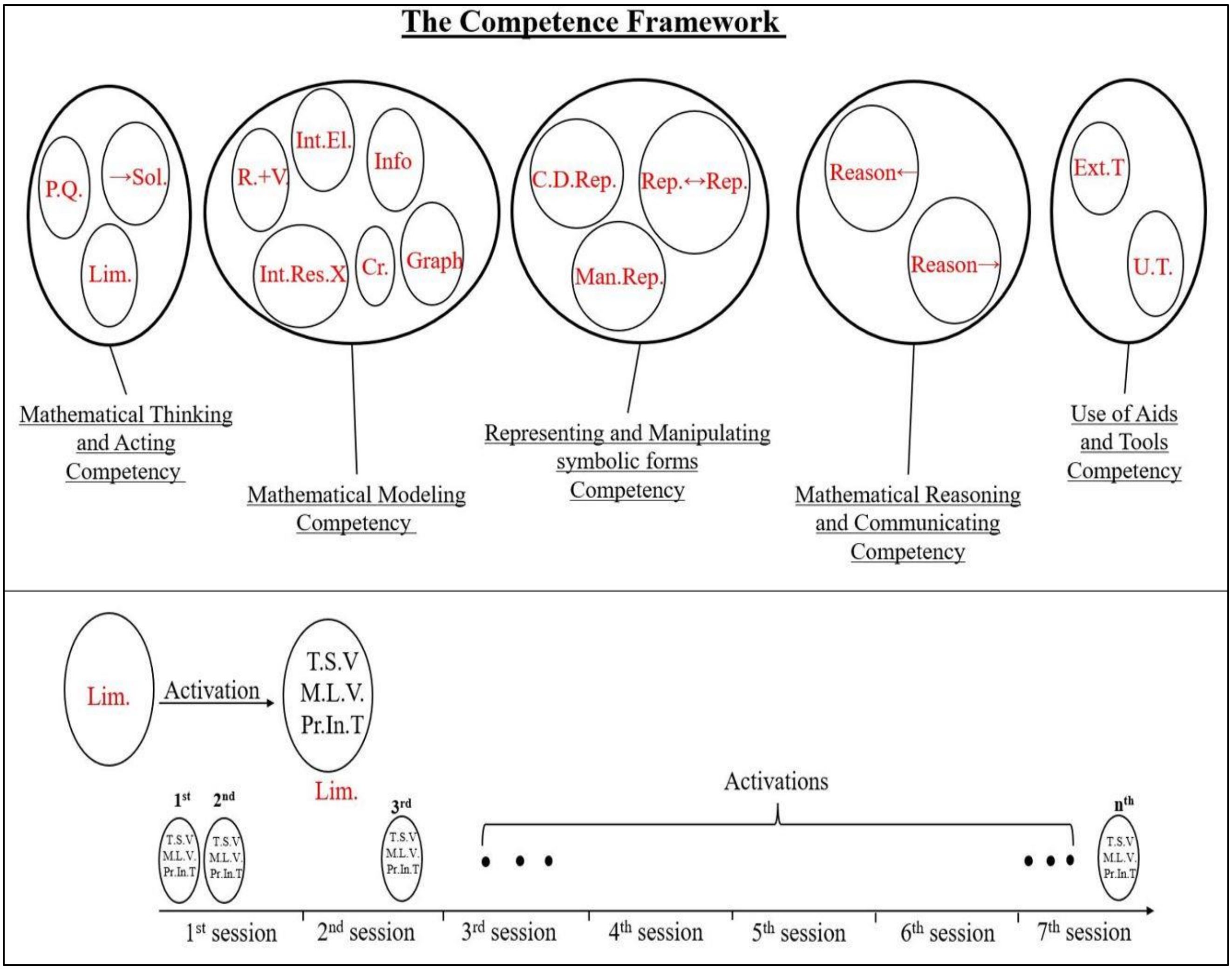

Figure 6.1. Competence framework and activation.

Figure 6.1 above comprises two parts. The top part of the figure depicts the competence framework of this thesis (see also Section 3.3) with its five mathematical competencies and the competency aspects (16 in total) that each competency includes. The bottom part of the figure shows how competency aspect activation occurs during the seven sessions. Using as an example the (Lim.) competency aspect of the Mathematical Thinking and Acting competency, the three domains of activation (i.e., T.S.V., M.L.V., and Pr.In.T., see also subsection 4.1.7) are shown when the (Lim.) aspect is activated. As shown in the timeline of the seven sessions, evidence of several activations is identified. However, there may be a session or sessions where a particular competency aspect activation may not be identified. The order of these activations is also important to be recorded, as I will show in the following illustrations.

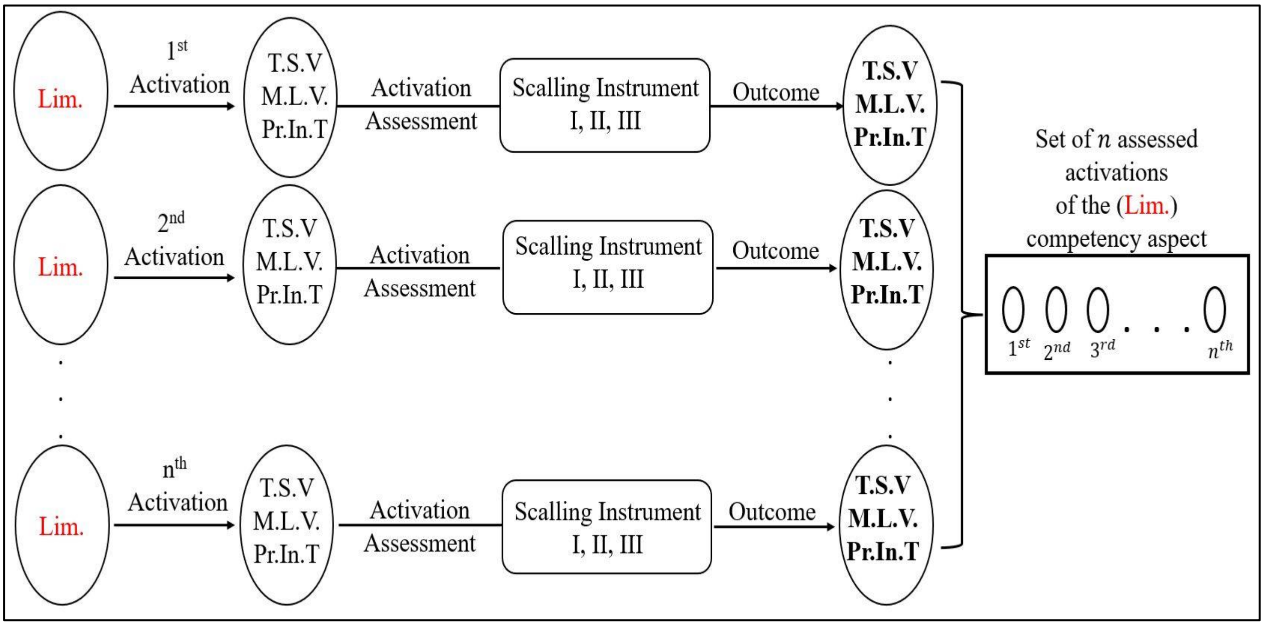

Figure 6.2. Competency activation assessment and outcome.

As seen in the previous illustration a set of activations of the (Lim.) competency aspect were identified across all these seven sessions. Figure 6.2 now illustrates that each of these recorded activations is evaluated during the activation assessment process by applying the properties of the scaling instrument (see also subsections 4.1.4 to 4.1.8). The assessment concerns the evaluation of the activation domains (i.e., T.S.V., M.L.V., and Pr.In.T.) of the competency aspect by assigning to these domains one of the three levels of activation (Initial (I), Intermediate (II), and Mature (III)) of the scaling instrument. As seen in the figure above, the outcome is signified by symbolizing in bold these domains (i.e., **T.S.V.**, **M.L.V.**, and **Pr.In.T.**). When the n^{th} activation is evaluated, a set of n assessed activations of the (Lim.) competency aspect is produced. This set of assessed activations will be used and explored to determine the strength of the evidence combination that will be applied. This is illustrated in the following figure.

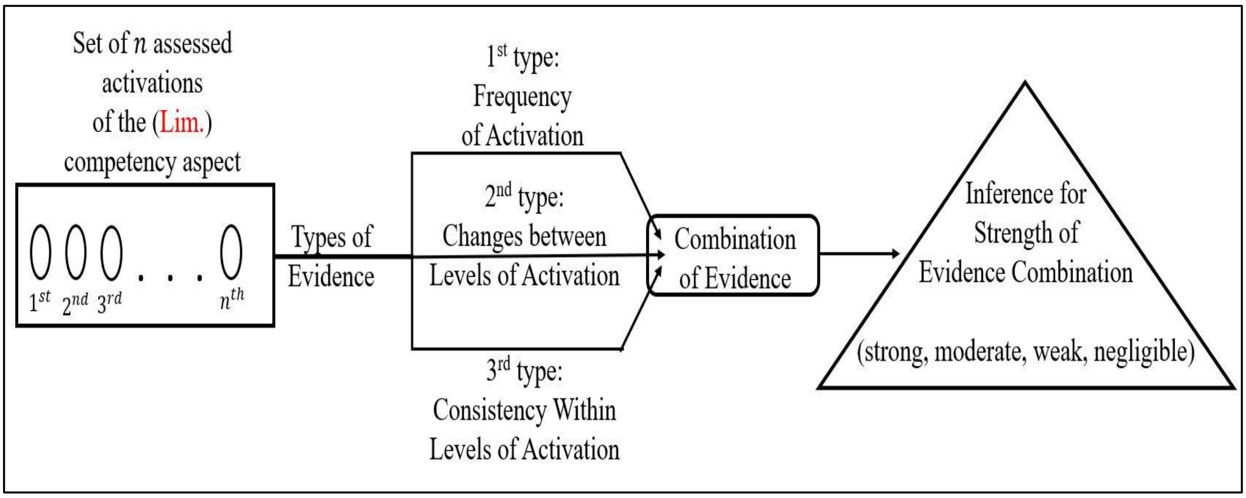

Figure 6.3. Producing inferences for stregnth of evidence combination.

As mentioned above, this set of n activations of the (Lim.) competency aspect is explored by employing, as seen in Figure 6.3, the three types of evidence: i) frequency of activation (subsection 4.2.3), ii) changes between levels of activation (subsection 4.3.1), and iii) consistency within the levels of activation (subsection 4.3.1). Once these three types of evidence are recorded, inferences are made for indications about competency development. These three types of evidence are combined by summarizing these indications. The outcome of this combination constitutes the final inference (triangle shape) regarding the strength of evidence (strong, moderate, weak, or negligible) of this combination that suggests development of the competency aspect that is under study, which in this example is the (Lim.) competency aspect. The following illustration portrays the process's final steps that determine the nature of competency development for the five competencies of the competence framework.

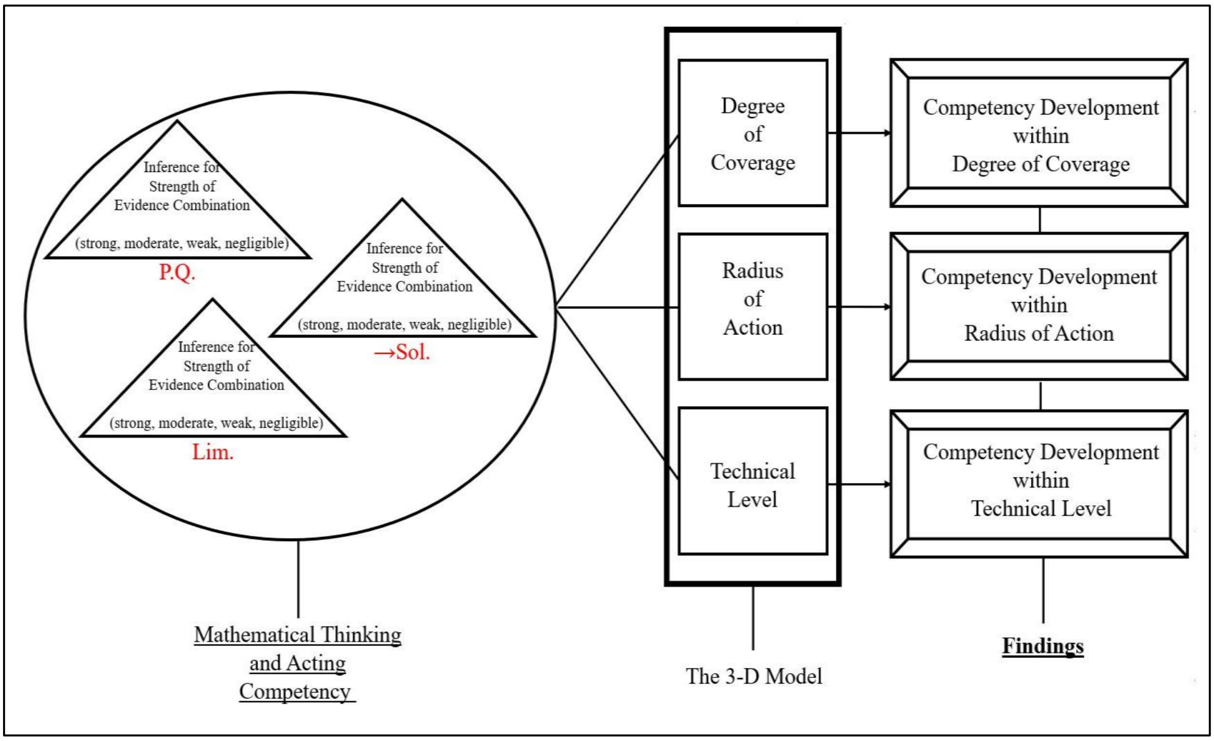

Figure 6.4. Producing the findings for the competency development.

The above processes were followed to produce inferences regarding the competency development concerned only one competency aspect of the Mathematical Thinking and Acting competency (i.e., the (Lim.) aspect). The same process will be followed to produce inferences regarding the strength of evidence combination for the remaining two aspects (i.e., (P.Q.) and (→Sol.)) of this competency. As shown in Figure 6.4 above, once all three final inferences are produced (i.e., the three triangles in the left circle), the concluding inferences regarding the development within the three dimensions of the 3-D competency development model (Niss & Højgaard, 2011) can be made. These concluding inferences are written in the same language used before. That is, they refer to the strength of evidence that was accumulated. For example, for the development within the Radius of Action of Mathematical Thinking competency, the language that will be used is the following: Based on the three overviews regarding the combination of evidence for the progress of development of the three competency aspects that constitute the Mathematical Thinking and Acting competency, I infer that there is a *moderate combination of evidence* that suggests development within the Radius of Action of the competency.

As seen in Figure 6.4, these concluding inferences (i.e., the three rectangles in the right part of the figure) constitute the findings of this thesis that concern the first Research Question that was posed: What is the progress of individual competency development over time for a student who participates in a series of calculus sessions that comprise non-routine mathematical tasks? What is the competence profile of each student after these series of calculus sessions? The findings will be presented in a summarized tabular form by the end of Section 6.1, in subsection 6.1.5.

6.1.3 Competency development for the case study of Erika

This subsection presents Erika's competency development throughout the seven calculus sessions. The results concern the progress of competency development by considering the three dimensions of the 3-D model (Niss & Højgaard, 2011), namely, Degree of Coverage, Radius of Action and Technical Level, and the strength of evidence combination. By the end of the presentation of each of the five competencies, there will be a final assessment of the relevant competency that will include a description of the strength of evidence combination (i.e., strong, moderate, weak, negligible) about the development of that competency.

This is an extensive part of this thesis because there must be an analysis for each competency aspect within the three dimensions of the 3-D progress model (parts 6.1.3.1, 6.1.3.2, and 6.1.3.3 for Erika and 6.1.4.1, 6.1.4.2, and 6.1.4.1 for Johannes). Therefore, the following analysis does not include all competency aspects of each competency but selected ones. However, to provide an accurate depiction of how the findings on competency development were produced, I decided to present the Mathematical Thinking and Acting competency in a complete manner, that is, to present the analysis for all three competency aspects (P.Q., Lim., and →Sol.) that constitute this competency. Therefore, I present all three aspects of this competency which implies how the analysis worked for all the remaining competencies. These competency aspects that were not included in the main body of this thesis can be found in Appendix E.

6.1.3.1 Development for Degree of Coverage

In section 4.3.1, I elaborated on how the development within the Radius of Action will be determined. Before delving into the exploration of each competency development, it is appropriate to make a brief revisit in the description of the Degree of Coverage. Looking back at subsection 2.4.1, the Degree of Coverage is described as the dimension that concerns the extent to which a person masters the aspects that characterize the competency, and as Niss and Højgaard (2011, p. 72) point out, it aims to explore “how many of these aspects the person can activate in the different situations available, and to what extent independent activation takes place.” Rephrasing the latter, the more aspects of the competency a student possesses over time, the higher the Degree of Coverage, and this could be an indication of competency development throughout the seven calculus sessions.

Mathematical Thinking and Acting competency

This competency comprises three aspects:

- To pose questions that have a mathematical characteristic or to know the kinds of answers that mathematics may offer (entry). (code: P.Q.) (1)
- To understand and handle the scope and limitations of a given concept (entry)’ competency (code: Lim) (2)
- To attack (take actions towards a solution) mathematical problems and tasks of various kinds. (code: →Sol.) (3)

Overview for (P.Q.)

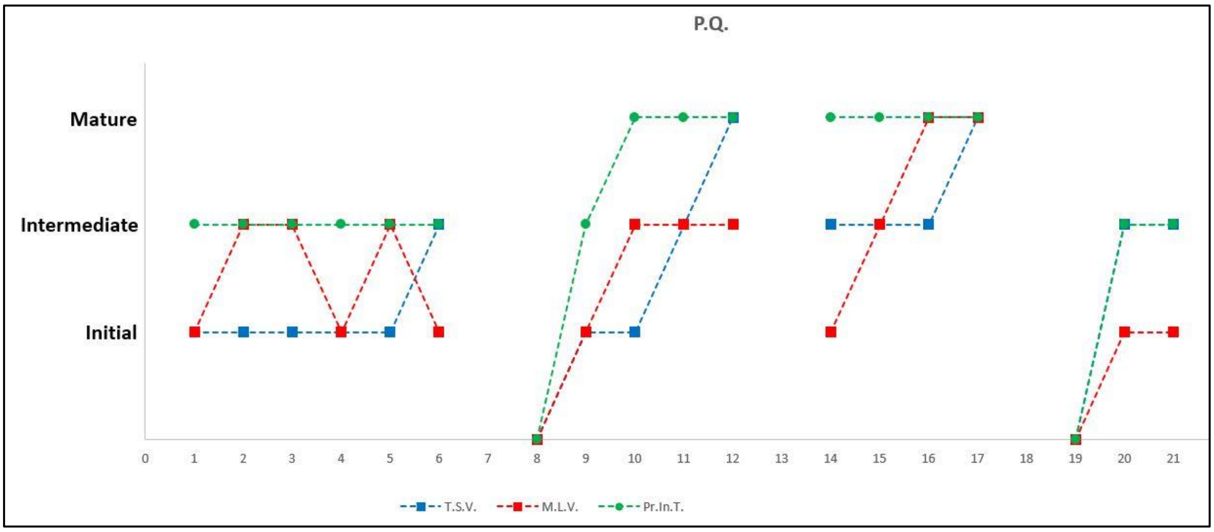

Diagram 6-2. Activation for competency aspect (P.Q.) (repetition of Diagram 5-1).

Table 6-2. Summarized evidence for (P.Q) activation (repetition).

Competency domain / Type of evidence	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	6	7	3	
Changes from higher to lower	0	2	0	
High consistency	1(1)	1	3	
Moderate consistency	(1)	1	0	
Frequency of activation				16

Frequency of competency activation

Regarding the first type of evidence, namely the *frequency of competency activation*, the more frequent activation located during the Periodic Functions unit and the less frequent activation located during the Integrals & Modeling sessions as seen in Diagram 6-2. As shown in the diagram, the frequencies were [6, 4, 4, 2], that is, six instances of competency activation during the Periodic Functions unit, four during the Exponential Growth & Regression unit, four during the Population Dynamics unit, and two during the Integrals & Modeling unit. The total activation frequency was 16 and there were two sessions where no evidence of competency activation was observed.

Changes between activation levels

By incorporating the second type of evidence, *changes between activation levels*, the information from Table 6-2 informs us that the changes from lower to higher and higher to lower levels of competency activation by domain were: (6, 0) for T.S.V., (7, 2) for M.L.V., and (3, 0) for Pr.In.T. These observations suggest that the progress was evidently more often from a lower to a higher level of activation, than the other way around which suggests a dynamic of development over time. Furthermore, in the cases of T.S.V. and Pr.In.T. domains no change from a higher to a lower level of activation was observed.

Levels of consistency

Regarding the third type of evidence, *levels of consistency*, Diagram 6-2 and Table 6-2 informs us that there was high consistency at the Mature level for the Pr.In.T. domain at the first three units, suggesting that Erika worked independently to activate the competency aspect. There were instances of moderate and high consistency for the M.L.V. domain at the Intermediate or Mature level. However, there were instances of high and moderate consistency at the Initial level regarding the T.S.V. domain. These last two observations suggest that Erika consistently used expressions with mathematical language to activate the (P.Q.) aspect but she did not have a clear perspective towards the solution of the tasks she had to work with, especially during the Population Dynamics unit.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, it is noticed that there was identifiable competency activation for all four calculus units, the changes between activation levels were evidently more often from a lower to a higher level of activation, than the other way around, and that there were many occasions where Erika consistently activated the (P.Q.) aspect above the initial level of activation. I, therefore, infer that there is a *strong combination of evidence* for the development of the (P.Q.) aspect of the Mathematical Thinking and Acting competency.

Overview for (Lim.)

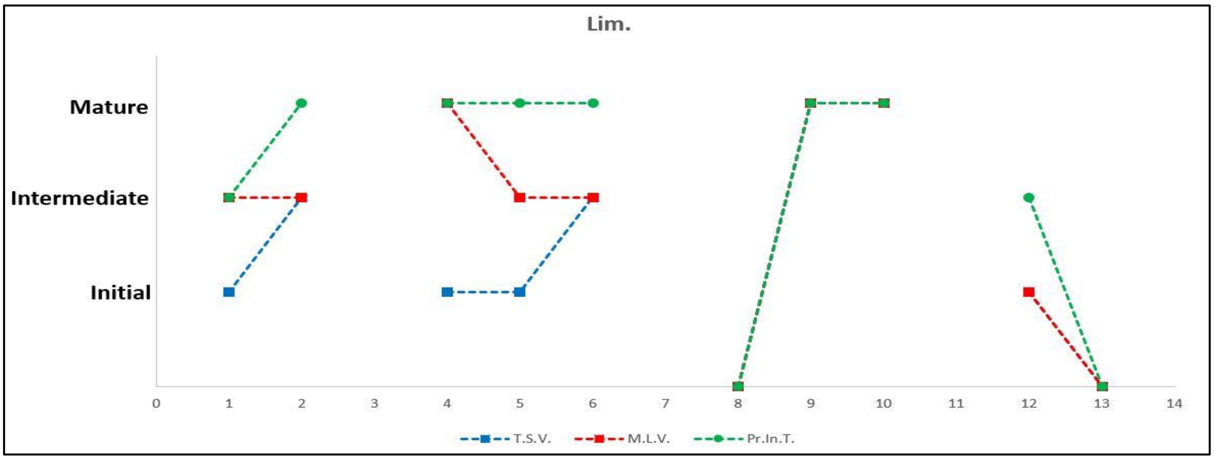

Diagram 6-3. Activation for competency aspect (Lim.).

Table 6-3. Summarized evidence for (Lim.) activation. (repetition).

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	3	1	2	
Changes from higher to lower	1	2	1	
High consistency	0	0	1	
Moderate consistency	(1)	1	0	
Frequency of activation				8

Frequency of competency activation

Regarding the first type of evidence, namely the *frequency of competency activation*, the more frequent activation was located during the Exponential Growth & Regression unit. The rest of the units display a few instances of competency activation, and there were calculus sessions (4th and 7th) in the Population Dynamics and Integrals & Modeling units, where there was no traceable activation. Diagram 6-3 shows that the frequencies were [2, 3, 2, 1], and the total activation frequency was 8. There were two instances of competency activation during the Periodic Functions unit, three during the Exponential Growth & Regression unit, two during the Population Dynamics unit, and one during the Integrals & Modeling unit. The total activation frequency was smaller (half) than the total activation of the (P.Q.) competency aspect.

Changes between activation levels

By incorporating the second type of evidence, *changes between activation levels*, the information from Table 6-3 informs us that the changes from lower to higher and higher to lower levels of competency activation by domain were: (3, 1) for T.S.V., (1, 2) for M.L.V., and (2, 1) for Pr.In.T. These observations suggest that there is a dynamic of development mainly within the T.S.V.

domain. Erika gradually had a clearer perspective towards the solution of the tasks, but at the same time, she did often used everyday expressions to activate the particular competency aspect.

Levels of consistency

Regarding the third type of evidence, *levels of consistency*, Diagram 6-3 and Table 6-3 informs us that instances of consistency occurred only during the Exponential Growth & Regression unit because of the observed low activation frequency during the other units. These few instances of high or moderate consistency occurred within the Mature or Intermediate level. However, the total activation frequency was relatively small, suggesting that we cannot make any strong inferences regarding the development dynamic using this type of evidence.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, it is noticed that on two units there were only two identifiable activations and that on one unit only one. The changes between activation levels were more often from a lower to a higher level of activation, than the other way around, and that there were occasions where Erika consistently activated the (Lim.) aspect above the initial level of activation. However, the fact there were only eight instances ([2, 3, 2, 1]) where Erika activated this aspect is taken into consideration. I, therefore, infer that there is a *weak combination of evidence* for the development of the (Lim.) aspect of the Mathematical Thinking and Acting competency.

Overview for (→Sol)

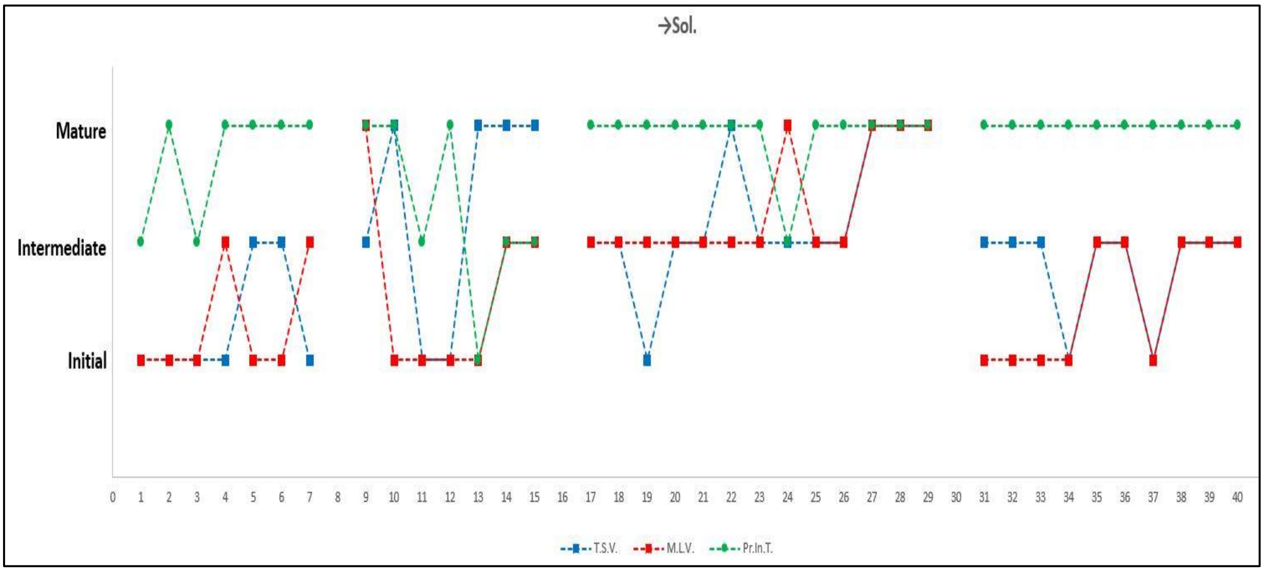

Diagram 6-4. Activation for competency aspect (→Sol.).

Table 6-4. Summarized evidence for (→Sol.) activation. (repetition).

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	8	7	5	
Changes from higher to lower	6	4	4	
High consistency	0	0	1	
Moderate consistency	1(1)	1(3)	3	
Frequency of activation				37

Frequency of competency activation

Regarding the first type of evidence, namely the *frequency of competency activation*, the more frequent activation was located during the Population Dynamics unit, but frequent activation occurred during all calculus units. As shown in Diagram 6-4, the frequencies were [7, 7, 13, 10]. That is, seven instances of competency activation during the Periodic Functions and the Exponential Growth & Regression unit, 13 during the Population Dynamics unit, and 10 during the Integrals & Modeling unit. The total activation frequency was 37, evidently higher than the total activation of the two other competency aspects (P.Q.) and (Lim.) of the Mathematical Thinking and Acting competency.

Changes between activation levels

By incorporating the second type of evidence, *changes between activation levels*, the information from Table 6-4 informs us that the changes from lower to higher and higher to lower levels of competency activation by domain were: (8, 6) for T.S.V., (7, 4) for M.L.V., and (5, 4) for Pr.In.T. These observations suggest that the progress was more often from a lower to a higher level of activation than the other way around which suggests a dynamic of development over time. Notably, within the M.L.V. domain, this dynamic is explained by the fact that Erika displayed multiple times consistency (that I discuss below) at the Initial level, which explains why the changes from a lower to a higher level of activation are relatively more often.

Levels of consistency

Regarding the third type of evidence, *levels of consistency*, Diagram 6-4 and Table 6-4 informs us that there were instances mainly of moderate consistency for all the domains (i.e., T.S.V., M.L.V., and Pr.In.T.) at all levels of the scaling instrument. What is noticeable, though, is that Erika (except during the Population Dynamics unit) regularly expressed herself (symbolized as ‘1(3) at M.L.V.’), during the activation of the competency, with everyday expressions and gestures and used less often formal mathematical language.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, it is noticed that there was frequent identifiable competency activation for all four calculus units and all seven sessions, the changes between activation levels were more often from a lower to a higher level of activation, than the other way around, and that there were many occasions where Erika consistently activated the (→Sol.) aspect at all levels of activation. I, therefore, infer that there is a *moderate combination of evidence* for the development of the (→Sol.) aspect of the Mathematical Thinking and Acting competency.

Final assessment for the Mathematical Thinking and Acting competency

Table 6-5 below summarizes the concluding inferences regarding the final evaluation of the strength of evidence combination for the three aspects of the Mathematical Thinking and Acting competency within the Degree of Coverage (D.o.C.) of the competency.

Table 6-5. Strength of evidence combination for Mathematical Thinking and Acting competency (D.o.C.).

Mathematical Thinking and Acting competency	
Aspect	Strength of evidence combination
P.Q.	Strong
Lim.	Weak
→Sol.	Moderate

Based on the three overviews of the three types of evidence that provided indications regarding the dynamic of competency development of the three aspects that constitute the Mathematical Thinking and Acting competency, I infer that there is a *moderate combination of evidence* that suggests development within the Degree of Coverage of the competency.

<u>Mathematical Modeling competency</u>

This competency comprises six aspects:

- To be able to assess the range and validity of existing models. (code: R.+V.) (4)
- To interpret and translate the elements of a model during the mapping process. In this situation, the student should be able to understand the real problem and adapt a model based on reality. (code: Int.El.) (5)
- To interpret mathematical results in an extra-mathematical context and generalize the solutions developed for a special task-situation. (code: Int.Res.X) (6)
- To critique the model by reviewing or reflecting and questioning the results. (code: Cr.) (7)
- To look and search for available information and to differentiate between relevant and irrelevant information. (code: Info) (8)
- To represent situations graphically. (code: Graph) (9)

Overview for (R.+V.)

The low activation frequency and the fact that there was no evidence of competency activation during the Periodic Functions and the Exponential Growth & Regression units resulted in considering that Erika provided limited evidence, regarding the activation of (R.+V.) competency aspect. I therefore infer that there is a *negligible combination of evidence* for the development of the (R.+V.) aspect of the Mathematical Modeling competency.

Overview for (Int.El.)

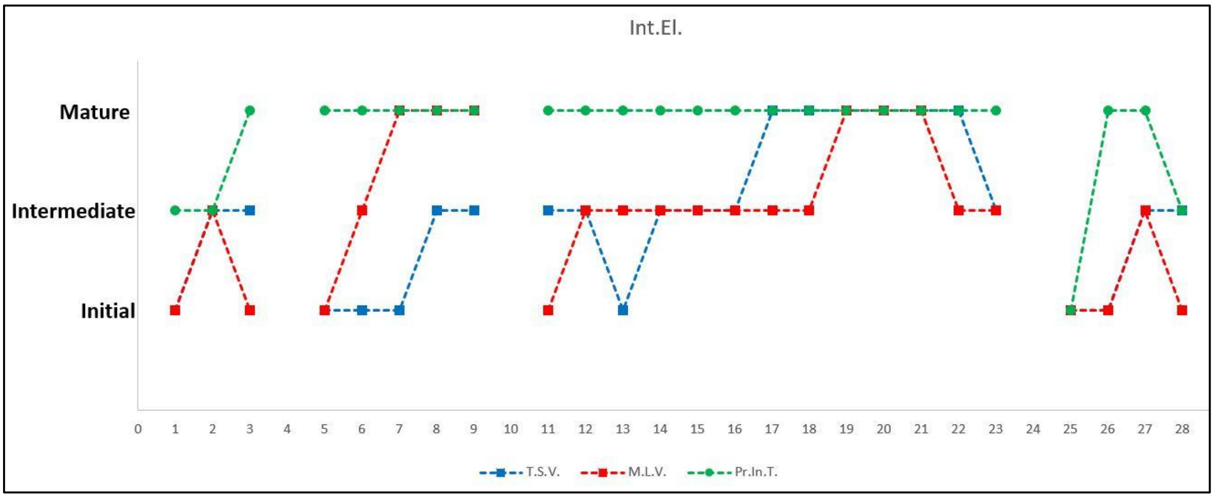

Diagram 6-5. Activation for competency aspect (Int.El.) (repetition of Diagram 5-2).

Table 6-6. Summarized evidence for (Int.El.) activation (repetition).

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	5	6	2	
Changes from higher to lower	2	3	1	
High consistency	0	0	2	
Moderate consistency	4(2)	2(1)	2	
Frequency of activation				25

Frequency of competency activation

Regarding the first type of evidence, namely the *frequency of competency activation*, the more frequent activation located during the Population Dynamics unit and the less frequent activation located during the Periodic Functions and the Integrals & Modeling units. As shown in Diagram 6-5, the frequencies were [3, 5, 13, 4], and the total activation frequency was 25. This was the most frequently activated competency aspect of the Mathematical Modeling competency.

Changes between activation levels

By incorporating the second type of evidence, *changes between activation levels*, the information from Table 6-6 informs us that the changes from lower to higher and higher to lower levels of competency activation by domain were: (5,

2) for T.S.V., (6, 3) for M.L.V., and (2, 1) for Pr.In.T. These observations suggest that the progress was more often from a lower to a higher level of activation, than the other way around and shows a dynamic of competency development over time.

Levels of consistency

Regarding the third type of evidence, *levels of consistency*, Diagram 6-2 and Table 6-2 informs us that there was high consistency at the Mature level for the Pr.In.T. domain twice, suggesting that Erika worked consistently independently to bring into action this competency aspect. There were more instances of moderate and high consistency for all activation domains at the Intermediate or Mature level than in the Initial level of the scaling instrument.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, it is noticed that there was identifiable competency activation for all four calculus units and all seven sessions, the changes between activation levels were evidently more often from a lower to a higher level of activation, than the other way around, and that there were many occasions where Erika consistently activated the (Int.El.) aspect above the initial level of activation. I, therefore, infer that there is a *strong combination of evidence* for the development of the (Int.El.) aspect of the Mathematical Modeling competency.

Overview for (Info)

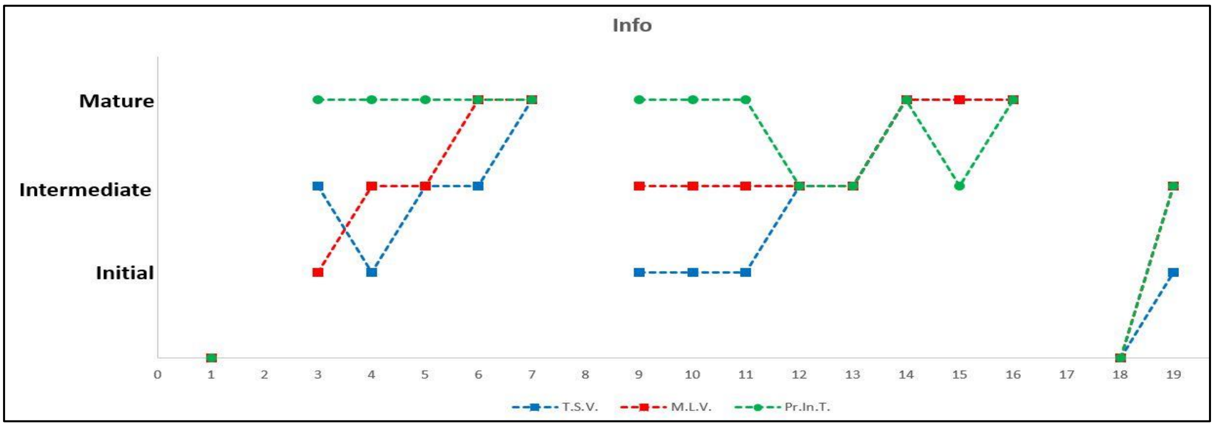

Diagram 6-6. Activation for competency aspect (Info).

Table 6-7. Summarized evidence for (Info) activation.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	5	4	3	
Changes from higher to lower	1	0	2	
High consistency	0	0	1	
Moderate consistency	2(1)	4	1	
Frequency of activation				14

Frequency of competency activation

Regarding the first type of evidence, namely the *frequency of competency activation*, activations were located mainly during the Exponential Growth & Regression and the Population Dynamics units. No activation has been observed during the Periodic Functions. As shown in Diagram 6-6, the frequencies were [0, 5, 8, 1], and the total activation frequency was 14.

Changes between activation levels

By incorporating the second type of evidence, *changes between activation levels*, the information from Table 6-7 informs us that the changes from lower to higher and higher to lower levels of competency activation by domain were: (5, 1) for T.S.V., (4, 0) for M.L.V., and (3, 2) for Pr.In.T. These observations suggest that the progress was more often from a lower to a higher level of activation, than the other way around, suggesting a dynamic of development over time.

Levels of consistency

Regarding the third type of evidence, *levels of consistency*, Diagram 6-6 and Table 6-7 informs us that there were several occasions of moderate consistency, mainly at the Intermediate and Mature levels of the scaling instrument. In particular, there were several occasions where Erika consistently expressed herself in oral or written form using expressions with mathematical language and vocabulary during the Exponential Growth & Regression and the Population Dynamics units.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, it is noticed that there was identifiable competency activation mainly for two units, the changes between activation levels were evidently more often from a lower to a higher level of activation, than the other way around, and that there were many occasions where Erika consistently activated the (Info) aspect above the initial level of activation. I, therefore, infer that there is a *moderate combination of evidence* for the development of the (Info) aspect of the Mathematical Modeling competency.

Final assessment for the Mathematical Modeling competency

Table 6-8 below, summarizes the concluding inferences regarding the final evaluation of the strength of evidence combination for the six aspects of the Mathematical Modeling competency.

Table 6-8. Strength of evidence combination for Mathematical Modeling competency (D.o.C.).

Mathematical Modeling competency	
Aspect	Strength of evidence combination
R.+V.	Negligible
Int.El.	Strong
Int.Res.X	Moderate

Mathematical Modeling competency	
Cr.	Moderate
Info	Moderate
Graph	Weak

Based on the six overviews of the three types of evidence that provided indications regarding the dynamic of competency development of the six competency aspects that constitute the Mathematical Modeling competency, I infer that there is a *moderate combination of evidence* that suggests development within the Degree of Coverage of the competency.

Representing and Manipulating symbolic forms competency

This competency comprises three aspects:

- To choose (or decode in) a representation. (code: C.D.Rep) (10)
- To Switch between representations. (code: Rep↔Rep) (11)
- To Manipulate within a representation. (code: Man.Rep) (12)

Overview for (Man.Rep.)

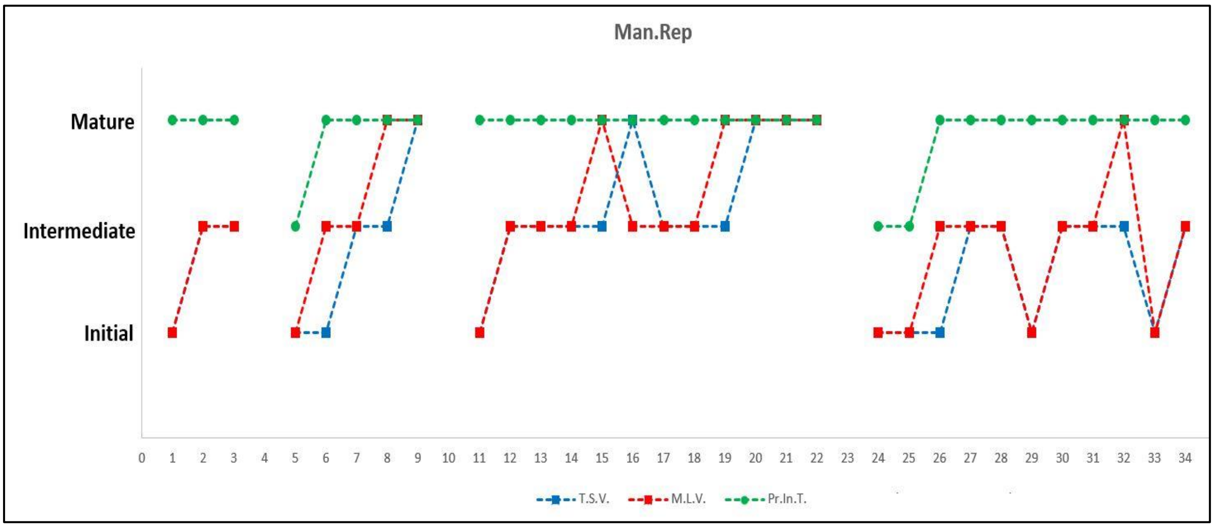

Diagram 6-7. Activation for competency aspect (Man.Rep.).

Table 6-9. Summarized evidence for (Man.Rep) activation.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	9	10	2	
Changes from higher to lower	3	3	0	
High consistency	0	0	4	
Moderate consistency	2(1)	3	0	
Frequency of activation				31

Frequency of competency activation

Regarding the first type of evidence, namely the *frequency of competency activation*, the most frequent activation was observed during the Population Dynamics and the Integrals & Modeling units. Diagram 6-7, shows that the frequencies were [3, 5, 12, 11], and the total activation frequency was 31. This was the most frequently activated competency aspect of the Representing and Manipulating symbolic forms competency.

Changes between activation levels

By incorporating the second type of evidence, *changes between activation levels*, the information from Table 6-9 informs us that the changes from lower to higher and higher to lower levels of competency activation by domain were: (9, 3) for T.S.V., (10, 3) for M.L.V., and (2, 0) for Pr.In.T. These observations suggest that the progress was evidently more often from a lower to a higher level of activation, for all the domains of activation, suggesting a dynamic of competency development over time.

Levels of consistency

Regarding the third type of evidence, *levels of consistency*, Diagram 6-7 and Table 6-9 informs us that there were several instances of high consistency at the Mature level for the Pr.In.T domain, suggesting that Erika worked consistently independently to activate the (Man.Rep.) aspect. Several occasions of moderate consistency were also observed for the other two domains of activation (i.e., T.S.V., M.L.V.). The latter suggests that on several occasions, Erika activated the (Man.Rep.) aspect having a clear view towards the solution of the tasks making regular use of expressions with mathematical language and vocabulary.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, it is noticed that there was frequent competency activation for all sessions of the calculus units, the changes between activation levels were evidently more often from a lower to a higher level of activation, than the other way around, and that there were many occasions where Erika consistently activated the (Man.Rep.) aspect above the initial level of activation. I, therefore, infer that there is a *strong combination of evidence* for the development of the (Man.Rep.) aspect of the Representing and Manipulating symbolic forms competency.

Final assessment for the Representing and Manipulating symbolic forms competency.

Table 6-10 below, summarizes the concluding inferences regarding the final evaluation of the strength of evidence combination for the three aspects of the Representing and Manipulating symbolic forms competency.

Table 6-10. Strength of evidence combination for the Representing and Manipulating symbolic forms competency (D.o.C.).

Representing and Manipulating symbolic forms competency	
Aspect	Strength of evidence combination
C.D.Rep.	Strong
Rep.↔Rep.	Weak
Man.Rep.	Strong

Based on the three overviews of the three types of evidence that provided indications regarding the dynamic of competency development of the three competency aspects that constitute the Representing and Manipulating symbolic forms competency, I infer that there is a *strong combination of evidence* that suggests development within the Degree of Coverage of the competency.

<u>Mathematical Reasoning and Communicating competency</u>

This competency comprises three aspects:

- To understand others' written, visual or oral 'texts', in a variety of linguistic registers, about matters having a mathematical content. To follow and assess chains of arguments, put forward by others. (code: Reason←) (13)
- To express oneself, at various levels of theoretical and technical precision, in oral, visual or written form. To provide explanations or justifications to support your result or idea. (code: Reason→) (14)

Overview for (Reason←)

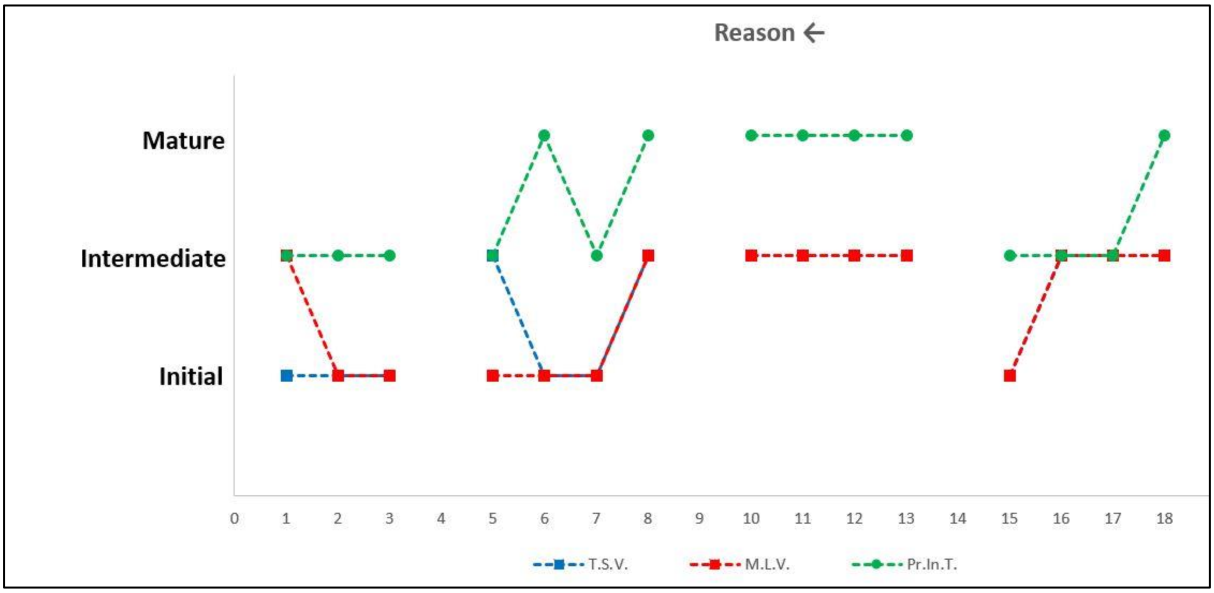

Diagram 6-8. Activation for competency aspect (Reason←).

Table 6-11. Summarized evidence for (Reason⟵) activation.

Competency domain / Type of evidence	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	2	2	3	
Changes from higher to lower	1	1	1	
High consistency	2(1)	2(1)	3	
Moderate consistency	(1)	(1)	0	
Frequency of activation				15

Frequency of competency activation

Regarding the first type of evidence, namely the *frequency of competency activation*, there was evidence of competency activation in all calculus units and every session of these units. As shown in Diagram 6-8, the frequencies were [3, 4, 4, 4], and the activation frequency was 15.

Changes between activation levels

By incorporating the second type of evidence, *changes between activation levels*, the information from Table 6-11 informs us that the changes from lower to higher and higher to lower levels of competency activation by domain were: (2, 1) for T.S.V., (2, 1) for M.L.V., and (3, 1) for Pr.In.T. These observations suggest that the progress was more often from a lower to a higher level of activation, for all the domains of activation and shows a dynamic of competency development over time. Especially for the Pr.In.T. domain, the changes suggest that Erika gradually worked more independently while activating this competency aspect.

Levels of consistency

Regarding the third type of evidence, *levels of consistency*, Diagram 6-8 and Table 6-11 informs us that there were several occasions of moderate or high consistency for the domains of activation (i.e., T.S.V., M.L.V.) at the Initial activation level during the first two calculus units. The latter suggests that on several occasions, Erika occasionally activated the (Reason⟵) aspect without a clear view towards the solution of the tasks using everyday expressions with limited mathematical vocabulary. However, there were several instances of high consistency at the Mature level for the M.L.V. domain during the Population Dynamics and Integrals & Modeling units suggesting that Erika gradually used expressions with mathematical language consistently to activate the (Reason⟵) aspect. For the Pr.In.T. domain, the information from the table, suggests that Erika continuously worked independently with limited need for prompting or hints from the teacher.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, it is noticed that there was identifiable competency activation for all sessions of the calculus units, the changes between activation levels were more often from a lower to a higher level of activation, than the other way around, and that there were occasions where Erika consistently activated the (Reason⟵) aspect at all levels of activation. I, therefore, infer that there is a *strong combination of evidence* for the development of the (Reason⟵) aspect of the Mathematical Reasoning and Communicating competency.

Final assessment for the Mathematical Reasoning and Communicating competency.

Table 6-12 below, summarizes the concluding inferences regarding the final evaluation of the strength of evidence combination for the two aspects of the Mathematical Reasoning and Communicating competency.

Table 6-12. Strength of evidence combination for the Mathematical Reasoning and Communicating competency (D.o.C.).

Mathematical Reasoning and Communicating competency	
Aspect	Strength of evidence combination
Reason⟵	Strong
Reason⟶	Strong

Based on the two overviews of the three types of evidence that provided indications regarding the dynamic of competency development of the two competency aspects that constitute the Mathematical Reasoning and Communicating competency, I infer that there is a *strong combination of evidence* that suggests development within the Degree of Coverage of the competency.

<u>Use of Aids and Tools</u>

This competency comprises two aspects:

- To know the existence and properties of various tools and aids for mathematical activity. (code: Ext.T.)
- To be able to use aids and tools to develop insight intuition. (code: U.T.)

Overview for (U.T.)

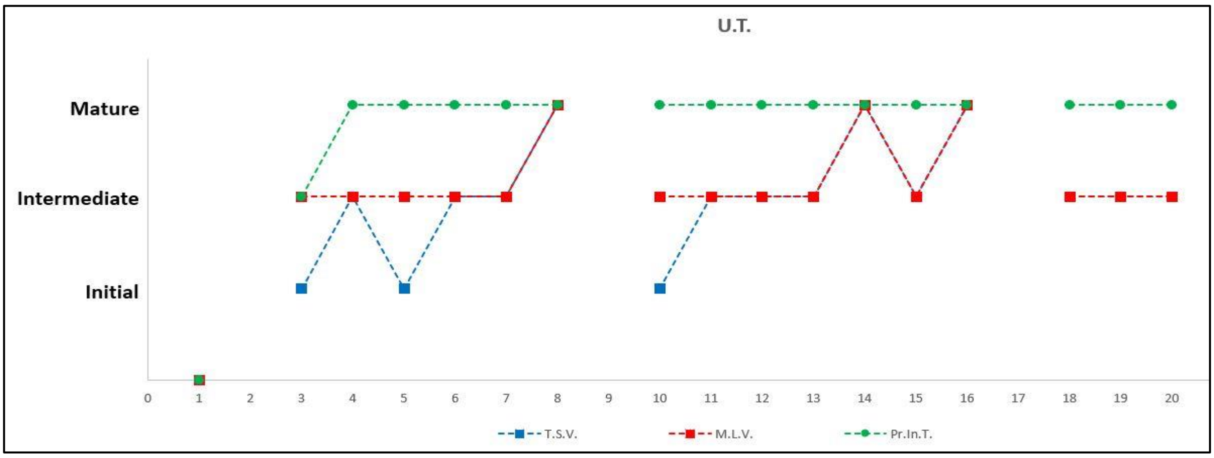

Diagram 6-9. Activation for competency aspect (U.T) (repetition of Diagram 5-5).

Table 6-13. Summarized evidence for (U.T.) activation.

Type of evidence \ **Competency domain**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	6	3	1	
Changes from higher to lower	2	1	0	
High consistency	1	2	3	
Moderate consistency	1	1	0	
Frequency of activation				16

Frequency of competency activation

Regarding the first type of evidence, namely the *frequency of competency activation*, there was evidence of competency activation in all calculus units except the Periodic Functions unit. As shown in the diagram, the frequencies were [0, 6, 7, 3], and the total activation frequency was 16. This was the only activated competency aspect of the Use of Aids and Tools competency.

Changes between activation levels

By incorporating the second type of evidence, *changes between activation levels*, the information from Table 6-13 informs us that the changes from lower to higher and higher to lower levels of competency activation by domain were: (6, 2) for T.S.V., (3, 1) for M.L.V., and (1, 0) for Pr.In.T. These observations suggest that the progress was more often from a lower to a higher level of activation, for all the domains of activation and shows a dynamic of competency development over time Especially for the T.S.V. domain, the changes suggest that Erika gradually had a clear perspective towards the solution of the task(s) she worked on while activating this competency aspect.

Levels of consistency

Regarding the third type of evidence, *levels of consistency*, Diagram 6-9 and Table 6-13 displays that were several instances of consistency for all domains of activation with none of these instances in the Initial level of activation. She consistently used tools efficiently, showing familiarity with the calculator and its features, and she continuously worked independently with a limited need for prompting or hints from the teacher. It is crucial, though, to notice that the use of tools does not suggest the use of multiple tools because the primary tool Erika used was her calculator and, in a few instances, the Desmos software.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, it is noticed that there was identifiable competency activation for all sessions of the calculus units except the first, the changes between activation levels were evidently more often from a lower to a higher level of activation, than the other way around, and that there were many occasions where Erika consistently activated the (U.T.) aspect above the initial level of activation. I, therefore, infer that there is a *strong combination of evidence* for the development of the (U.T.) aspect of the Use of Aids and Tools competency.

Final assessment for the Use of Aids and Tools competency.

Table 6-14 below, summarizes the concluding inferences regarding the final evaluation of the strength of evidence combination for the two aspects of the Use of Aids and Tools competency.

Table 6-14. Strength of evidence combination for the Use of Aids and Tools competency (D.o.C.).

Use of Aids and Tools competency	
Aspect	Strength of evidence combination
Ext.T	No evidence
U.T.	Strong

Based on the two overviews of the three types of evidence that provided indications regarding the dynamic of competency development of the two competency aspects that constitute the Use of Aids and Tools competency, and because one of these aspects was not observed to be activated, I infer that there is a *moderate combination of evidence* that suggests development within the Degree of Coverage of the competency. The fact that the discussion on Erika's use of tools mainly refers to the use of a calculator and not a variety of tools also attributed to the determination of this final assessment.

6.1.3.2 Development for Radius of Action

Before exploring each competency development, it is appropriate to revisit this dimension's description briefly. Niss (2003) describes the Radius of Action as this dimension that focuses on the spectrum of contexts and situations in which a student can activate the specific competency. The more contexts a student can activate a set of competencies, the bigger the Radius of Action. In section 4.3.2, I elaborated on how the development within the Radius of Action will be determined. The attention here is on each calculus unit, constituting a different mathematical situation. Progress within the Radius of Action will be determined by the progress of competency development that the student exhibited in each of the four calculus units. The more contexts the student activated and developed a particular aspect of a competency, the greater the Radius of Action within this competency.

In what follows, I present the findings (i.e., overviews) on the case study of Erika for the five main competencies of the relevant framework concerning the competency development within the Radius of Action. As mentioned in section 4.3.2, a modified version of the tables summarizing the evidence will be used in the following overviews. These tables will display the evidence by calculus unit (by mathematical context) and not as a whole. Within these overviews, I refer to the diagrams used in the previous part of this section (i.e., Development for Degree of Coverage). Therefore, I will not display them here to avoid any unnecessary repetition.

Mathematical Thinking and Acting competency

This competency consists of three aspects and to avoid repetition, I only mention here the relevant abbreviations (codes) that represent these aspects: (P.Q.), (Lim), and (→Sol.). Information about the activations frequencies by calculus unit are derived from previous diagrams to avoid any unnecessary repetition.

Overview for (P.Q.)

Table 6-15. Summarized evidence for (P.Q.) activation by unit.

Competency domain / Type of evidence	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	(1,3,1,0)	(2,2,2,1)	(0,2,0,1)	
Changes from higher to lower	(0,0,0,0)	(2,0,0,0)	(0,0,0,0)	
High consistency	((1),0,1,0)	(0,1,0,0)	(1,1,1,0)	
Moderate consistency	(0,(1),0,0)	(0,0,1,0)	(0,0,0,0)	
Frequency of activation				16

Frequency of competency activation

As shown in Diagram 5-1, the frequencies were [6, 4, 4, 2], that is, six instances of competency activation during the Periodic Functions unit, four during the Exponential Growth & Regression unit, four during the Population Dynamics unit, and two during the Integrals & Modeling unit. A more frequent activation was noticed during the Periodic Functions unit, followed by the Exponential Growth & Regression and the Population Dynamics units. Less frequent activation was noticed during the Integrals & Modeling unit.

Changes between activation levels

As seen in Table 6-15, changes from a lower to a higher activation level are identified mainly at the first three calculus units. Changes from a higher to a lower activation level are identified only during the first calculus unit and only at the M.L.V. domain. These findings could signify a development dynamic for the first three calculus units but mainly for the Exponential Growth & Regression and the Population Dynamics units.

Levels of consistency

Evidence of high consistency within the Intermediate or Mature activation level was identified mainly during the Population dynamics and the Exponential Growth & Regression units. There was also an instance of moderate consistency at the Initial activation level during the Exponential Growth & Regression unit. It is challenging to make inferences for the consistency during the Integrals & Modeling unit because of the small frequency of activation (i.e., two activations were identified). It is also notable that during the Periodic Functions unit, there was high consistency at the Initial level of the scaling tool where the student could not further explore the solving process of the task at hand, but she hardly sought any help.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, there is evidence of development mainly for three calculus units: Population Dynamics, Exponential Growth & Development, and Periodic Functions. I, therefore, infer that there is a *strong combination of evidence* within the Radius of Action for the development of the (P.Q.) aspect of the Mathematical Thinking and Acting competency.

Overview for (Lim.)

Table 6-16. Summarized evidence for (Lim.) activation by unit.

Type of evidence \ Competency domain	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	(1,1,1,0)	(0,0,1,0)	(1,0,1,0)	
Changes from higher to lower	(0,0,0,1)	(0,1,0,1)	(0,0,0,1)	
High consistency	(0,0,0,0)	(0,0,0,0)	(0,1,0,0)	
Moderate consistency	(0,(1),0,0)	(0,1,0,0)	(0,0,0,0)	
Frequency of activation				8

Frequency of competency activation

As shown in the diagram, the frequencies were [2, 3, 2, 1], that is, two instances of competency activation during the Periodic Functions unit, three during the Exponential Growth & Regression unit, two during the Population Dynamics unit, and one during the Exponential Growth & Regression unit but the low activation frequency during all calculus units does not allow any strong inference to be made for the development within the Radius of Action using this type of evidence.

Changes between activation levels

As seen in Table 6-16, changes from a lower to a higher activation level are identified at the first three calculus units but none at the Integrals & Modeling unit. This suggests some form of development dynamic over time for these first three units, but still, it is vital to consider the low frequency of activation that has been observed.

Levels of consistency

Instances of consistency occurred only during the Exponential Growth & Regression unit because of the observed low activation frequency during the other units. These few instances of high or moderate consistency occurred within the Mature or Intermediate level for the M.L.V. and the Pr.In.T. domain. It is difficult to make any inference for the consistency during the other units.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, there is evidence of development mainly for two calculus units: Exponential Growth & Development and Population Dynamics units. I, therefore, infer that there is a *weak combination of evidence* within the Radius of Action for the development of the (Lim.) aspect of the Mathematical Thinking and Acting competency.

Overview for (→Sol)

Table 6-17. Summarized evidence for (→Sol.) activation by unit.

Type of evidence \ Competency domain	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	(1,2,3,2)	(2,1,2,2)	(2,2,1,0)	
Changes from higher to lower	(1,1,2,2)	(1,1,1,1)	(1,2,1,0)	
High consistency	(0,0,0,0)	(0,0,0,0)	(0,0,0,1)	
Moderate consistency	((1),1,0,0)	((1),(1),1,(1))	(1,0,2,0)	
Frequency of activation				37

Frequency of competency activation

As shown in the diagram, the frequencies were [7, 7, 13, 10], that is, seven instances of competency activation during the Periodic Functions unit and during the Exponential Growth & Regression unit, 13 during the Population Dynamics unit, and 10 during the Integrals & Modeling unit[1]. The more frequent activation was noticed during the Population Dynamics unit, followed by the Integrals & Modeling unit. Frequent activation has been observed in the Exponential Growth & Regression and the Periodic Functions units.

Changes between activation levels

As seen in Table 6-17, changes from a lower to a higher level of activation are identified at all four calculus units. There are fewer changes from a higher to a lower activation level identified in the same units. These two observations suggest a dynamic of competency development across all sessions.

Levels of consistency

Evidence of high consistency within the Intermediate or Mature activation level was identified only during the Integrals & Modeling unit. Furthermore, there were several instances of moderate consistency during the Population dynamics unit. There were instances of moderate consistency at the Initial level of activation during the Periodic Functions and the Exponential Growth & Regression units too. Erika consistently took actions towards the Population

[1] From this point on I will only provide the [x,y,z,w] notation for the frequencies per calculus unit because it is now self-explanatory.

Dynamic task's solution with minimum help, having a clear perspective towards the solution of the tasks, and expressing herself using mathematical language.

Strength of evidence combination: final evaluation

Based on the summaries for all three types of evidence, there is evidence of development for all calculus units. I, therefore, infer that there is a *strong combination of evidence* within the Radius of Action for the development of the (→Sol.) aspect of the Mathematical Thinking and Acting competency.

Final assessment for the Mathematical Thinking and Acting competency

Table 6-18 summarizes the concluding inferences regarding the final evaluation of the strength of evidence combination for the three aspects of the Mathematical Thinking and Acting competency within the Radius of Action.

Table 6-18. Strength of evidence combination for Mathematical Thinking and Acting competency (R.o.A.).

Mathematical Thinking and Acting competency	
Aspect	Strength of evidence combination
P.Q.	Strong
Lim.	Weak
→Sol.	Strong

Based on the three overviews of the three types of evidence that provided indications regarding the dynamic of competency development of the three aspects that constitute the Mathematical Thinking and Acting competency, I infer that there is a *strong combination of evidence* that suggests development within the Radius of Action of the competency. This is explained by the fact that evidence of competency development was provided for the Population Dynamics, Exponential Growth & Regression, and Periodic Functions units but also for the sessions of the Integrals & Modeling unit.

Mathematical Modeling competency

This competency consists of six aspects presented with their abbreviations (codes): (R.+V.), (Int.El.), (Int.Res.X), (Cr.), (Info), and (Graph).

Overview for (Int.El.)

Table 6-19. Summarized evidence for (Int.El.) activation by unit.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	(1,1,2,1)	(1,2,2,1)	(1,0,0,1)	
Changes from higher to lower	(0,0,2,0)	(1,0,1,1)	(0,0,0,1)	
High consistency	(0,0,0,0)	(0,0,0,0)	(0,1,1,0)	
Moderate consistency	(1,1(1),1,1(1))	(0,1,1,(1))	(1,0,0,1)	
Frequency of activation				25

Frequency of competency activation

As shown in the diagram, the frequencies were the following: [3, 5, 13, 4]. A more frequent activation was noticed during the Population Dynamics unit. Erika attempted to interpret and translate an element of a mathematical expression (e.g., a model, an equation) more often throughout the sessions of the Population Dynamics unit.

Changes between activation levels

As seen in Table 6-19, changes from a lower to a higher level of activation are identified at all four calculus units and are comparatively more than those from a higher to a lower level. This suggests that there is a dynamic of development for all units. These findings could signify a development dynamic for all four calculus units but mainly for the Exponential Growth & Regression and the Population Dynamics units.

Levels of consistency

Evidence of high consistency within the Intermediate or Mature activation level was identified during the Exponential Growth & Regression and the Population Dynamics units and concerned the amount of help Erika received to activate the (Int.El.) aspect. During these two units, she worked independently while interpreting the elements of the models and any mathematical expression she encountered. There were instances of moderate consistency (above the Initial level of activation) but only during the Population Dynamics and Periodic Functions units.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, there is evidence of development mainly for two calculus units: Population Dynamics and Exponential Growth & Development. I, therefore, infer that there is a *moderate combination of evidence* within the Radius of Action for the development of the (Int.El.) aspect of the Mathematical Thinking and Acting competency.

Overview for (Info)

Table 6-20. Summarized evidence for (Info.) activation by unit.

Competency domain / Type of evidence	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	(0,2,2,1)	(0,2,1,1)	(0,0,2,1)	
Changes from higher to lower	(0,1,0,0)	(0,0,0,0)	(0,0,2,0)	
High consistency	(0,0,0,0)	(0,0,0,0)	(0,1,0,0)	
Moderate consistency	(0,1,1(1),0)	(0,2,2,0)	(0,0,1,0)	
Frequency of activation				14

Frequency of competency activation

The diagram shows that the frequencies were [0, 5, 8, 1]. Again, it is clear that Erika provided evidence of competency activation mainly for two calculus units: Population Dynamics and Exponential Growth & Regression.

Changes between activation levels

As seen in Table 6-20, there were comparatively more changes from a lower to a higher level of activation than the other way around for the units where activations were observed. This suggests a dynamic of development for the calculus units of Population Dynamics and Exponential Growth & Regression.

Levels of consistency

There was evidence of moderate consistency within the Intermediate or Mature activation level for the Population Dynamics and the Exponential Growth & Regression units. This suggests that Erika consistently was able to locate the relevant information that would assist her in addressing the tasks at hand, having a clear perspective towards the solution steps most of the time while regularly using expressions with mathematical language. Erika also displayed high consistency for the Pr.In.T. domain during the Exponential Growth & Regression unit, which suggests she worked independently to activate the (Info) aspect.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, there is evidence of development mainly for two calculus units: Population Dynamics and Exponential Growth & Regression. I, therefore, infer that there is a *moderate combination of evidence* within the Radius of Action for the development of the (Info) aspect of the Mathematical Thinking and Acting competency.

Final assessment for the Mathematical Modeling competency

Table 6-21 below, summarizes the concluding inferences regarding the final evaluation of the strength of evidence combination for the six aspects of the Mathematical Modeling competency.

Table 6-21. Strength of evidence combination for Mathematical Modeling competency (R.o.A.).

Mathematical Modeling competency	
Aspect	Strength of evidence combination
R.+V.	Negligible
Int.El.	Moderate
Int.Res.X	Weak
Cr.	Moderate
Info	Moderate
Graph	Weak

Based on the six overviews of the three types of evidence that provided indications regarding the dynamic of competency development of the six competency aspects that constitute the Mathematical Modeling competency, I infer that there is a *moderate combination of evidence* that suggests development within the Radius of Action of the competency. This is explained by the fact that evidence of competency development was provided mainly for the units of Exponential Growth & Regression and Population Dynamics.

Representing and Manipulating symbolic forms competency

This competency consists of three aspects and to avoid repetition, I only mention here the relevant abbreviations (codes) that represent these aspects: (C.D.Rep.), (Rep.↔Rep.), and (Man.Rep.).

Overview for (C.D.Rep.)

Table 6-22. Summarized evidence for (C.D.Rep.) activation by unit.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	(1,1,2,2)	(2,1,1,3)	(1,0,1,1)	
Changes from higher to lower	(0,1,2,1)	(2,1,0,2)	(0,0,1,0)	
High consistency	(0,0,0,0)	(0,0,0,0)	(0,1,0,1)	
Moderate consistency	((1),1,0,1)	(0,1,2,0)	(2,0,2,0)	
Frequency of activation				30

Frequency of competency activation

The diagram shows that the frequencies were [6, 4, 11, 9]. A more frequent competency activation was observed during the Population Dynamics unit, followed by the Integrals & Modeling and the Periodic Functions units. A less frequent activation was noticed during the Exponential Growth & Regression unit.

Changes between activation levels

During the Periodic Functions, Population Dynamics, and Integrals & Modeling units, the changes from a lower to a higher level of activation were comparatively more than the changes to the opposite direction. These findings could signify a development dynamic for the calculus units mentioned above. Especially for the Integrals & Modeling unit, the changes from a lower to a higher level of activation were comparatively more than the other way around for all activation domains. This suggests that Erika activated the (C.D.Rep.) aspect by developing her perspective towards the solution of the tasks and her use of mathematical language by gradually needing less assistance and prompting from the teacher.

Levels of consistency

As seen in Table 6-22, there were several occasions of moderate consistency for all calculus units and all activation domains Even during the Exponential Growth & Regression unit, where a low activation frequency was observed, Erika worked independently and regularly using expressions with mathematical language and vocabulary while being aware of the next steps of the solution process. This suggests that there was a dynamic of development for all calculus units.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, there is evidence of development for all calculus units. I, therefore, infer that there is a *strong combination of evidence* within the Radius of Action for the development of the (C.D.Rep.) aspect of the Mathematical Thinking and Acting competency.

Final assessment for the Representing and Manipulating symbolic forms competency

Table 6-23, summarizes the concluding inferences regarding the final evaluation of the strength of evidence combination for the three aspects of the Representing and Manipulating symbolic forms competency.

Table 6-23. Strength of evidence combination for Representing and Manipulating symbolic forms competency (R.o.A.).

Representing and Manipulating symbolic forms competency	
Aspect	Strength of evidence combination
C.D.Rep.	Strong
Rep.↔Rep.	Moderate
Man.Rep.	Strong

Based on the three overviews of the three types of evidence that provided indications regarding the dynamic of competency development of the three competency aspects that constitute the Representing and Manipulating symbolic forms competency, I infer that there is a *strong combination of evidence* that suggests development within the Radius of Action of the competency. This is explained by the fact that evidence of competency development was provided mainly for the Population Dynamics, Integrals & Modeling, and Exponential Growth & Regression units. In addition, there was also evidence of competency development during the sessions of the Periodic Functions unit.

Mathematical Reasoning and Communicating competency

This competency consists of two aspects and to avoid repetition, I only mention here the relevant abbreviations (codes) that represent these aspects: (Reason←) and (Reason→).

Overview for (Reason←)

Table 6-24. Summarized evidence for (Reason ←) activation by unit.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	(0,1,0,1)	(0,1,0,1)	(0,2,0,1)	
Changes from higher to lower	(0,1,0,0)	(1,0,0,0)	(0,1,0,0)	
High consistency	((1),0,1,1)	(0,(1),1,1)	(1,0,1,1)	
Moderate consistency	(0,(1),0,0)	((1),0,0,0)	(0,0,0,0)	
Frequency of activation				15

Frequency of competency activation

As shown in the diagram, the frequencies were [3, 4, 4, 4], which suggests that evidence of competency activation was observed at all calculus units.

Changes between activation levels

As seen in Table 6-24, changes from a lower to a higher activation level were identified only at the Exponential Growth & Regression and the Integrals & Modeling units. Fewer were the instances of a change from a higher to a lower level of activation during the Exponential Growth & Regression unit. These observations suggest a development dynamic regarding the abovementioned calculus units.

Levels of consistency

Evidence of high consistency within the Intermediate or Mature level of activation was observed during the Population dynamics and the Integrals & Modeling units, and this is an indication of competency development over time regarding these units. On the other hand, there were several instances of high and moderate consistency at the initial activation level for the Periodic Functions and the Exponential Growth & Regression units. This observation suggests that Erika regularly had difficulties following the argumentation of the teacher or a fellow student during the first two calculus units of this project.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, there is evidence of development for two calculus units: Population Dynamics and Integrals & Modeling. I, therefore, infer that there is a *moderate combination of evidence* within the Radius of Action for the development of the (Reason←) aspect of the Mathematical Thinking and Acting competency.

Final assessment for the Mathematical Reasoning and Communicating competency

Table 6-25 below, summarizes the concluding inferences regarding the final evaluation of the strength of evidence combination for the two aspects of the Mathematical Reasoning and Communicating competency.

Table 6-25. Strength of evidence combination for Mathematical Reasoning and Communicating competency (R.o.A.).

Mathematical Reasoning and Communicating competency	
Aspect	Strength of evidence combination
Reason←	Moderate
Reason→	Moderate

Based on the two overviews of the three types of evidence that provided indications regarding the dynamic of competency development of the two competency aspects that constitute the Mathematical Reasoning and Communicating competency, I infer that there is a *moderate combination of evidence* that suggests development within the Radius of Action of the competency. This is explained by the fact that evidence of competency development was provided mainly for the units of Population Dynamics, Integrals & Modeling, and Exponential Growth & Regression.

Use of Aids and Tools competency

This competency consists of three aspects and to avoid repetition, I only mention here the relevant abbreviations (codes) that represent these aspects: (Ext.T.) and (U.T.).

Overview for (U.T.)

Table 6-26. Summarized evidence for (U.T.) activation by unit.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	(0,3,3,0)	(0,1,2,0)	(0,1,0,0)	
Changes from higher to lower	(0,1,1,0)	(0,0,1,0)	(0,0,0,0)	
High consistency	(0,0,0,1)	(0,1,0,1)	(0,1,1,1)	
Moderate consistency	(0,0,1,0)	(0,0,1,0)	(0,0,0,0)	
Frequency of activation				16

Frequency of competency activation

The diagram shows that the frequencies were [0, 6, 7, 3]. A more frequent activation was noticed during the Population Dynamics followed by the Exponential Growth & Regression unit. A less frequent activation was observed during Erika's work on the Integrals & Modeling unit. No evidence of activation was observed during the Periodic Functions unit.

Changes between activation levels

As seen in Table 6-26, there were more changes from a lower to a higher activation level than the other way around for the calculus units where activation was observed. This observation could signify a dynamic of competency development for the calculus units of Population Dynamics and

Exponential Growth & Regression. It is difficult though to draw any inference, by employing this type of evidence, about the development of the (U.T.) aspect during the Integrals & Modeling unit because of the low frequency of activation that was observed. That said, the next type of evidence (i.e., levels of consistency) could provide more insight regarding the competency for this unit.

Levels of consistency

Evidence of high consistency within the Intermediate or Mature activation level was identified at all units where activation was observed. In particular, during the Integrals & Modeling unit, high consistency was observed for all activation domains. These observations suggest a dynamic of competency development over time for the Population Dynamics, the Exponential Growth & Regression units, and the Integrals & Modeling unit, even though a less frequent activation was observed. At all these units, Erika appeared to develop her use of tools while working on the tasks, and she did that with limited prompting, using expressions with mathematical language, and having a clear perspective towards the solution of the tasks she was working on.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, there is evidence of development for three calculus units: Population Dynamics, Exponential Growth & Regression, and Integrals & Modeling. I, therefore, infer that there is a *strong combination of evidence* within the Radius of Action for the development of the (U.T.) aspect of the Mathematical Thinking and Acting competency.

Final assessment for the Use of Aids and Tools competency

Table 6-27 below, summarizes the concluding inferences regarding the final evaluation of the strength of evidence combination for the two aspects of the Use of Aids and Tools competency.

Table 6-27. Strength of evidence combination for the Use of Aids and Tools competency (R.o.A.).

Use of Aids and Tools competency	
Aspect	Strength of evidence combination
Ext.T	No evidence
U.T.	Strong

Based on the two overviews of the three types of evidence that provided indications regarding the dynamic of competency development of the two competency aspects that constitute the Use of Aids and Tools competency, and because one of these aspects was not observed to be activated, I infer that there is a *moderate combination of evidence* that suggests development within the Radius of Action of the competency. This is explained by the fact that evidence of competency development was provided for the units of Population Dynamics, Integrals & Modeling, and Exponential Growth & Regression but only regarding the second aspect of this competency, the (U.T.) aspect.

6.1.3.3 Development for Technical Level

Before getting deeper into the discussion regarding the development within the Technical Level of each competency, it would be helpful to revisit some key features of this dimension -presented in subsections 2.4.3 and 4.3.3 of this thesis- of the 3-D competency progression model. Within this dimension, the focus is on indications concerning the conceptual and technical advancement of the mathematics that a student uses and can integrate appropriately during the activation of the competency under study. Development within the Technical Level addresses the issue of flexibility while students use the technical parts offered from a task when they provide evidence of competency activation through their actions in the class.

In what follows, I will present the findings regarding the development of the mathematical competencies at the Technical Level. The structure of this presentation will include three parts and will concern all the aspects of the relevant mathematical competency. The first part concerns the overview for the technical parts, where a description of how the student worked with the technical parts of the tasks takes place. This description adopts the methodological approach taken by Niss and Højgaard (2011) as described in subsections 2.4.3 and 4.3.3. The second part comprises an exploration of the M.L.V. domain for each aspect of the competency. This part will include the diagrams for each aspect and a table summarizing the evidence obtained from these diagrams, as seen in the previous sections (i.e., findings for Degree of Coverage and Radius of Action). The third and last part is the final assessment for the quality of the combination of evidence that will determine the nature of competency development.

As mentioned before, for space reasons, I present the development results for the Mathematical Thinking and Acting and Mathematical Modeling competencies here. However, I will include here the final assessments for the remaining competencies. The complete findings for the Technical Level for these competencies not included in the main body of the thesis can be found in Appendix E. The same also applies to Johannes's work.

<u>Mathematical Thinking and Acting competency</u>

This competency consists of three aspects, and to avoid repetition, I only mention here the relevant abbreviations (codes) that represent these aspects: (P.Q.), (Lim), and (→Sol.).

Overview for technical parts

When it comes to the Technical Level in which Erika activated the competency, there were situations (contexts) where she dealt with handling and interpreting graphs. She sketched the graphs quite fast and recognized what every axis represented when she worked on Periodic Functions and exponential growth. Erika tracked the limitations (Lim.) of the graphs, "oh, I have a different starting point, but the graph is correct," when she saw that her graph for an equation that modeled the temperature of a creature was different from the one her classmate has drawn. Occasionally, she posed questions (P.Q.) that had mathematical characteristics, "What does a stands for? Is it for the amplitude?" when she saw the $g(x) = a \cdot \cos[b(x-c)] = b$ equation. Erika found no difficulties when she had to deal with large decimal numbers using exponents, and she was successful when she had to transform percentages and incorporate them into the growth rate in a continuous or discrete model.

Erika exhibited a grasp of symbol and formula-based handling and manipulation of differentiation, but her handling skills appeared somewhat limited when performing integration. The latter is an indication that I will use for the evaluation of the Technical Level. During the solution process (→Sol.), Erika faced some problems in comprehending the difference between constants and variables. This problem was mainly exhibited at the first and at the beginning of the second session. However, she gradually grasped the notion of variables the notion of variables and constants within exponential and logarithmic functions. However, the problem reappeared when she failed to see the use of constants a and b within the definite integral $\int_a^b f(x)dx$.

Exploring the M.L.V. domain

In this part of the overview, I present the assembled diagrams from the (P.Q.), (Lim.), and (→Sol.) aspects of the Mathematical Thinking and Acting competency and provide the evidence focusing only for the M.L.V. domain of activation (dotted red line). Figure 6-5 below assembles all three diagrams that represent the activations of the three competency aspects of the Mathematical Thinking and Acting competency. The focus here is on the red-dotted line corresponding to the M.L.V. activation domain.

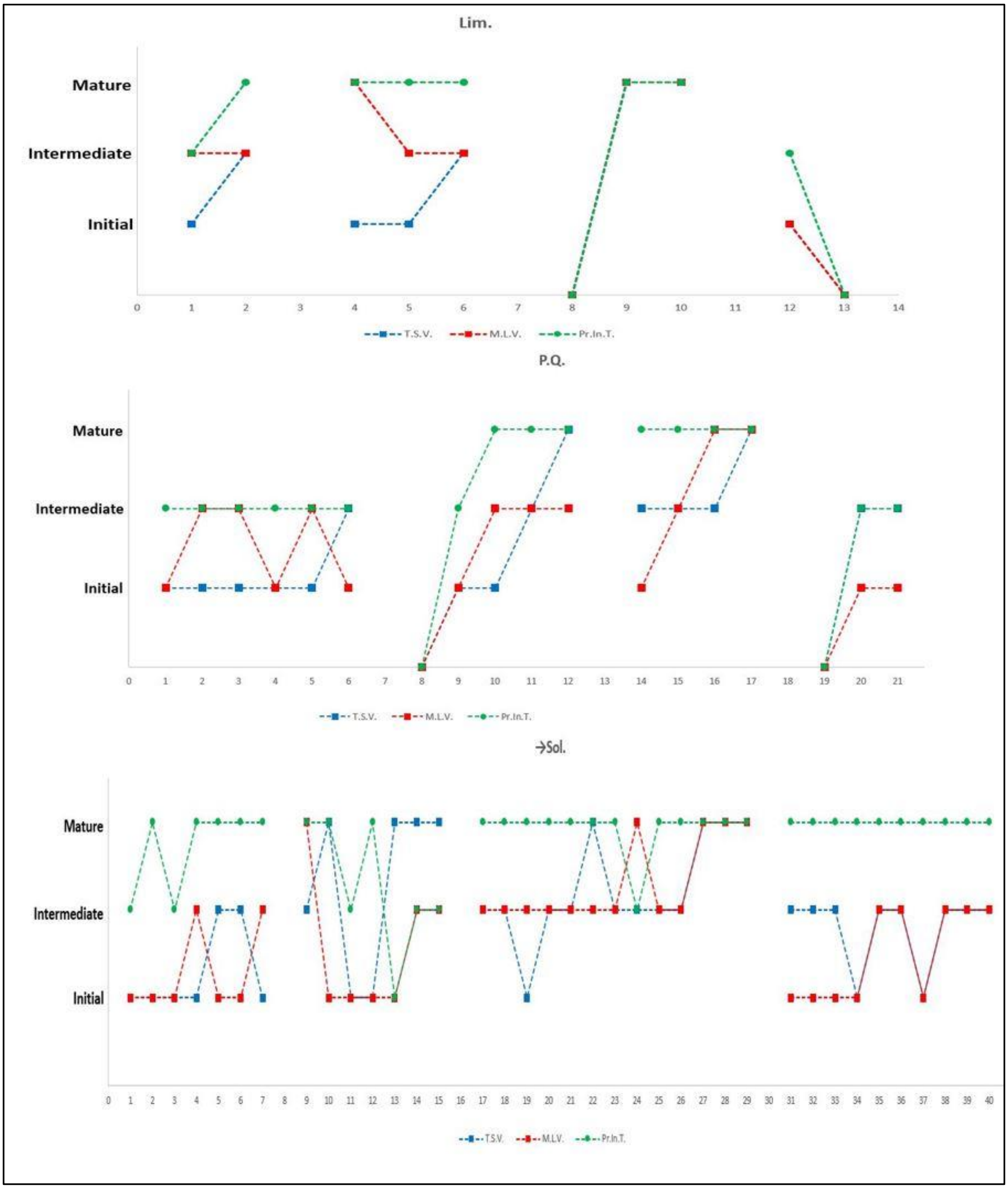

Figure 6.5. Assembled diagrams: Mathematical Thinking and Acting competency (Erika).

Table 6-28 summarizes the two types of evidence (changes between activation levels and consistency of activation) within the M.L.V. domain for the three aspects of the Mathematical Thinking and Acting competency. For two aspects of this competency, it is evident that the activations were observed to occur more often from a lower to a higher level of activation. Interestingly, one can notice on the last row of the table, that Erika displayed several times moderate consistency at the Initial Level of activation while activating the (→Sol.) competency aspect.

Table 6-28. Summary of evidence for M.L.V. domain, Mathematical Thinking and Acting competency.

Type of evidence \ Competency aspect	P.Q.	Lim.	→Sol.
Changes from lower to higher	7	0	7
Changes from higher to lower	2	2	4
High consistency	1	0	0
Moderate consistency	1	1	1(3)

Final assessment for the Mathematical Thinking and Acting competency

Based on the overviews for the technical parts and the exploration of the M.L.V. domain regarding the combination of evidence for the progress of development of the three competency aspects that constitute the Mathematical Thinking and Acting competency, I infer that there is a *moderate combination of evidence* that suggests development within the Technical Level of the competency.

Mathematical Modeling competency

This competency consists of three aspects and to avoid repetition, I only mention here the relevant abbreviations (codes) that represent these aspects: (R.+V.), (Int.El.), and (Int.Res.X), (Cr.), (Info), and (Graph).

Overview for technical parts

As mentioned before, the low activation frequency of the (R.+V.) aspect and the fact that there was no evidence of competency activation during the Periodic Functions and the Exponential Growth & Regression units resulted in considering that Erika provided limited evidence regarding the activation of (R.+V.) competency aspect.

In these instances where Erika provided evidence of activation of the '*interpret and translate the elements of a model during the mapping process*' competency (Int.El.), she displayed a firm grasp of the periodic functions during the homonymous calculus session. She understood the difference between a variable and a constant and recognized the elements of the equations. She also recognized the elements of the equations. For example, in the $g(x) = acos[b(x-c)] + d$ equation, she realized that the variable 'c' represents the starting point on a graph, and for the particular case of the task, she should have written $c = 0$, as she did. Erika displayed some problems at the beginning of the introduction of the exponential functions, but she gradually overcame that problem. Her interpretation of the difference while sketching the fiction $(1+r)^t$ when $t > 0$ and when $t < 0$ was acceptable. Erika also showed familiarity with using percentages when she had to transform them into a growth rate form. She displayed flexibility while working with symbol-rich formulas and differentiation. For example, she had no problem locating the difference between the $N(t)$ and $N(0)$ in the Population Dynamics model.

Lastly, there was a general inefficient use of the integration rules for various reasons. Nevertheless, she was able to follow the rule of $\int_0^{100} V'(t)dt = V(100) - V(0)$ during the Integrals & Modeling unit.

During the activation of the '*interpret mathematical results in an extra-mathematical context*' competency aspect (Int.Res.X), Erika displayed a relatively firm grasp of the periodic functions, the difference between the resulted graphs, a variable and a constant, and understood what the parameters and variables stand for. It is important to notice that more details regarding the technical parts can be found in the previous paragraph because the (Int.Res.X) competency activation occurred within the same turn with that of the (Int.El.) aspect. Erika also portrayed an understanding of percentages and their connection with the growth rate. She did not face issues while working with symbol-rich formulas, logarithms, and differentiation. However, during the Integrals & Modeling unit, she failed to activate the competency more than one time. With that being said, she was able to comprehend the application (not the Theorem itself) of the second part of the Fundamental Theorem of Calculus [if $G(x)$ is any antiderivative of $f(x)$ on I, so that $G'(x) = f(x)$ on I, then for any *b* on *I* we have: $\int_a^b f(x)dx = G(b) - G(a)$].

Regarding the analysis of the Technical Level for the '*critique the model by reviewing or reflecting and questioning the results*' competency aspect (Cr.), Erika argued with Johannes on why the formulas they came up with were different, and her points were valid. Apart from the fact that there was no competency activation during the Integrals & Modeling, the rest of the analysis is analogous to the(Int.Res.X) competency analysis. Following the same manner of analysis and regarding the '*look and search for available information and to differentiate between relevant and irrelevant information*' competency (Info), Erika displayed similar to the (Int.Res.X) competency development within the Technical Level. However, unlike the Int.Res.X competency analysis, there was no competency activation during the Periodic Functions.

Lastly, during the '*to choose appropriate mathematical and graphical notations during the task solution process*' competency (Graph), it has been stated several times before that the low activation frequency resulted in a generally limited competency activation. Nevertheless, during the first session (Periodic Functions), Erika manifested the higher frequency ($n = 3$) of activation regarding this competency which had an overall frequency of $n = 4$. She sketched the graphs quickly and recognized what every axis represented when she worked on the relevant task. For example, when attempting to understand the units of a function, she noted, "Ok, so …a function over time in minutes and instead of months (showing the previous graph) we will have minutes." Erika showed sketching skills used almost exclusively during the Periodic Functions unit. It is essential to point out that the tasks included in the Periodic Functions unit have been designed so that situations for graphical representations would appear more often than in the other tasks of different calculus units.

Exploring the M.L.V. domain

In this part of the overview, I present the assembled diagrams from the (Int.El.), (Int.Res.X), (Cr.), and (Info) aspects of the Mathematical Modeling competency and provide the evidence focusing only on the M.L.V. domain of activation (red dotted line). Because of the identified low frequency of activation (as mentioned multiple times before), there are no diagrams for the (R.+V.) and (Graph) aspects. Figure 6.6 below assembles all four diagrams representing the activations of the four activated competency aspects of the mathematical thinking and acting competency.

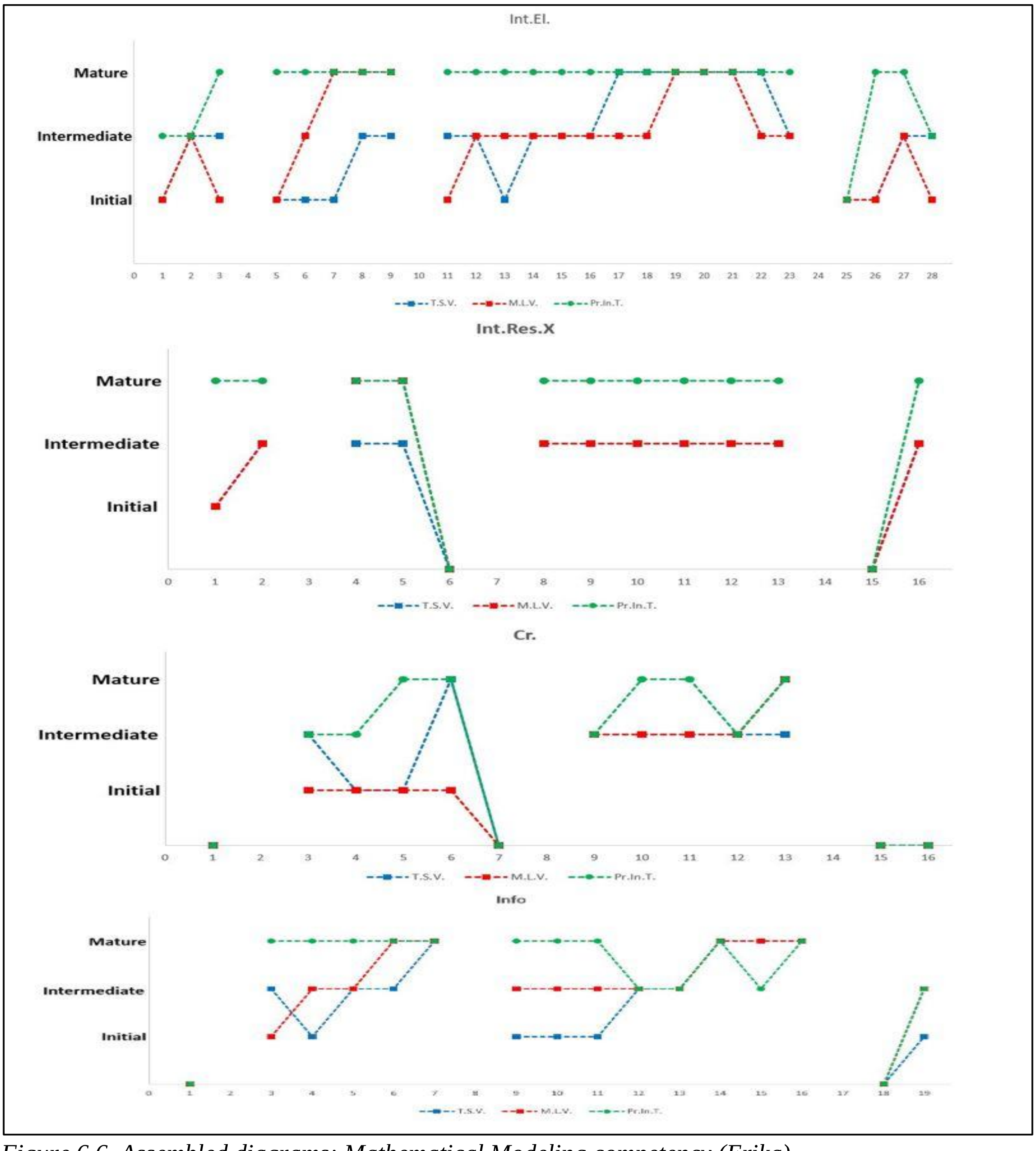

Figure 6.6. Assembled diagrams: Mathematical Modeling competency (Erika).

Table 6-29 below, summarizes the two types of evidence (changes between activation levels and consistency of activation) within the M.L.V. domain for the four aspects of the Mathematical Modeling competency. For three aspects of this competency it is evident that the activations were observed to occur more often from a lower to a higher level of activation. Interestingly, there were many instances of moderate consistency (i.e., activating the competency within the Intermediate of Mature level of activation) for the (Info) competency aspect.

Table 6-29. Summary of evidence for M.L.V. domain, Mathematical Modeling.

Competency aspect / Type of evidence	Int.El.	Int.Res.X	Cr.	Info
Changes from lower to higher	6	2	1	5
Changes from higher to lower	3	1	1	0
High consistency	0	1	1(1)	0
Moderate consistency	2(1)	0	0	4

Final assessment for the Mathematical Modeling competency

Based on the overviews for the technical parts and the exploration of the M.L.V. domain regarding the combination of evidence for the progress of development of the six competency aspects that constitute the Mathematical Modeling competency, I infer that there is a *strong combination of evidence* that suggests development within the Technical Level of the competency.

Final assessment for the Representing and Manipulating symbolic forms competency

Based on the overviews for the technical parts and the exploration of the M.L.V. domain regarding the combination of evidence for the progress of development of the three competency aspects that constitute the Representing and Manipulating symbolic forms competency, I infer that there is a *strong combination of evidence* that suggests development within the Technical Level of the competency.

Final assessment for the Mathematical Reasoning and Communicating competency

Based on the overviews for the technical parts and the exploration of the M.L.V. domain regarding the combination of evidence for the progress of development of the two competency aspects that constitute the Mathematical Reasoning and Communicating competency, I infer that there is a *moderate combination of evidence* that suggests development within the Technical Level of the competency.

Final assessment for the Use of Aids and Tools competency

Based on the overview for the technical parts and the exploration of the M.L.V. domain regarding the combination of evidence for the progress of development of the (U.T.) aspect and due to the fact that there was no evidence of activation for the (Ext.T.) aspect, I infer that there is a *moderate combination of evidence* that suggests development within the Technical Level of the competency.

6.1.4 Competency development for the case study of Johannes

This subsection presents the findings from Johannes's work and follows the same presentation fashion as in Erika's case. The development for each dimension of the 3-D model is displayed in three parts again (6.1.4.1, 6.1.4.2, and 6.1.4.3). To avoid unnecessary repetition, the presentation here follows a more concise form. For example, here, I do not present all the aspects that comprise each of the five competencies but selected ones. I, therefore, begin directly by presenting the overview for each aspect. I also present each type of evidence (i.e., frequency of activation, changes between activation levels, and levels of consistency) in a more summarized manner. Again, all competency aspects that were not included here can be found in Appendix E.

6.1.4.1 Development for Degree of Coverage

In this section, I present the findings concerning the competency development within the Degree of Coverage of each mathematical competency of the competence framework.

Mathematical Thinking and Acting competency

Similar to Erika's case, I follow the same structure of presentation here. The overview of the three types of evidence (i.e., frequency of activation, changes between activation levels, and levels of consistency) results in a final assessment of the combination of the evidence mentioned above.

Overview for (→Sol)

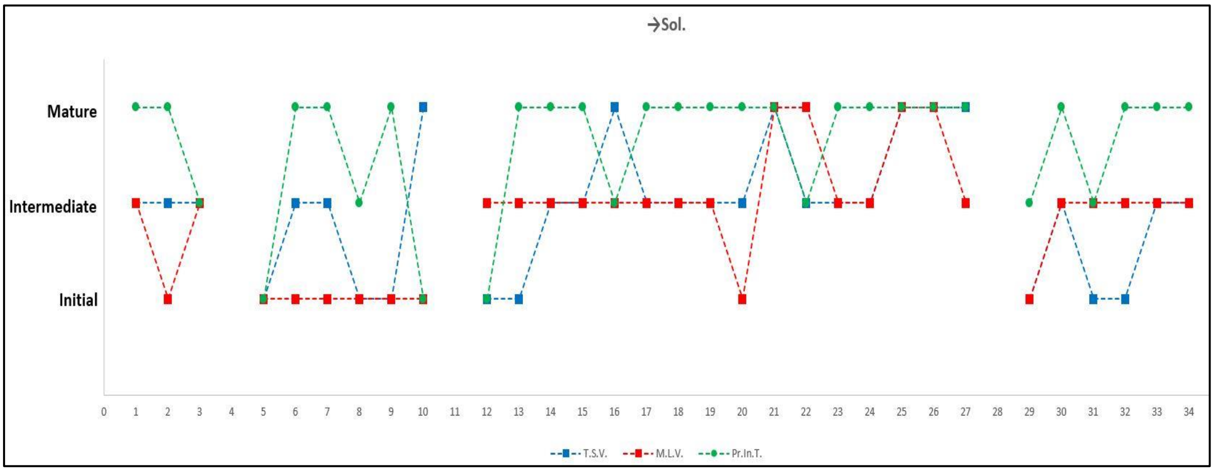

Diagram 6-10. Activation for competency aspect (→Sol.) (repetition of Diagram 5-6).

Table 6-30. Summarized evidence for (→Sol.) activation. (repetition).

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	8	4	7	
Changes from higher to lower	4	4	6	
High consistency	1	1(1)	0	
Moderate consistency	0	1	4	
Frequency of activation				31

Frequency of competency activation

As shown in Diagram 6-10, the frequencies were [3, 6, 16, 6], and the total activation frequency was 31. The most frequent activation was observed during the Population Dynamics unit, followed by Exponential Growth & Regression and Integrals & Modeling units. This was the most frequent activated aspect among the three competency aspects that constitute the Mathematical Thinking and Acting competency.

Changes between activation levels

As seen in Table 6-30, the changes from lower to a higher level of activation by domain were more often than the other way around: (8, 4) for T.S.V., (4, 4) for M.L.V., and (7, 6) for Pr.In.T. This finding suggests a dynamic of development because the changes from a lower to a higher level of activation were more than the other way around for two of the activation domains.

Levels of consistency

There were several instances of moderate consistency for all the domains (i.e., T.S.V., M.L.V., and Pr.In.T.) at the Intermediate of Mature level of the scaling instrument. There were also few instances of high consistency at the same levels, and there was only one instance of consistency within the Initial level of activation. However, what is noticeable is that Johannes continuously worked independently with limited need for help using expressions with mathematical language when attacking (→Sol.) the problem.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, it is noticed that there was frequent competency activation for all sessions of the calculus units, the changes between activation levels were more often from a lower to a higher level of activation, than the other way around, and that there were many occasions where Johannes consistently activated the (→Sol.) aspect above the initial level of activation. I, therefore, infer that there is a *strong combination of evidence* for the development of the (→Sol.) aspect of the Mathematical Thinking and Acting competency.

Final assessment for the Mathematical Thinking and Acting competency

Table 6-31 below, summarizes the concluding inferences regarding the final evaluation of the strength of evidence combination for the three aspects of the Mathematical Thinking and Acting competency.

Table 6-31. Strength of evidence combination for Mathematical Thinking and Acting competency (D.o.C.).

Mathematical Thinking and Acting competency	
Aspect	Strength of evidence combination
P.Q.	Moderate
Lim.	Moderate
→Sol.	Strong

Based on the three overviews of the three types of evidence that provided indications regarding the dynamic of competency development of the three aspects that constitute the Mathematical Thinking and Acting competency, I infer that there is a *moderate combination of evidence* that suggests development within the Degree of Coverage of the competency.

Mathematical Modeling competency

Overview for (Int.Res.X)

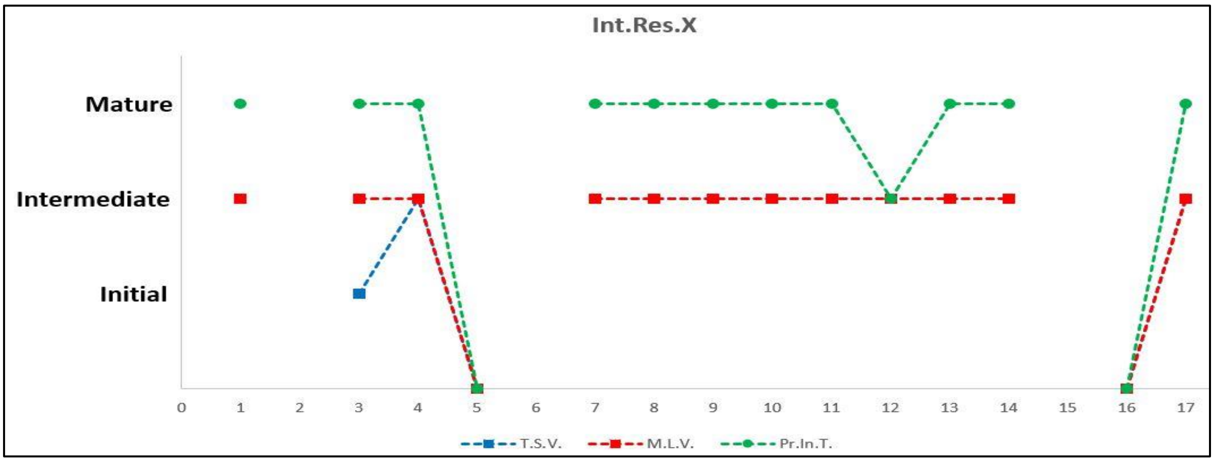

Diagram 6-11. Activation for competency aspect (Int.Res.X).

Table 6-32. Summarized evidence for (Int.Res.X) activation. (repetition).

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	2	1	3	
Changes from higher to lower	1	1	3	
High consistency	1	1	0	
Moderate consistency	0	0	1	
Frequency of activation				12

Frequency of competency activation

Regarding the first type of evidence, namely the *frequency of competency activation*, the more frequent activation located during the Population Dynamics unit. The rest of the sessions displayed less frequent activation. As shown in the diagram, the frequencies were [1, 2, 8, 1], and the total activation frequency was 12. There were two sessions where no evidence of competency activation was observed.

Changes between activation levels

As seen in Table 6-32, the changes from lower to higher and higher to lower levels of competency activation by domain were: (2, 1) for T.S.V., (1, 1) for M.L.V., and (3, 3) for Pr.In.T. These observations cannot provide reliable inferences because no direction (from lower to higher activation level or vice versa) was observed to occur more often than the other.

Levels of consistency

There were instances of high and moderate consistency at the Mature or Intermediate level of the scaling instrument for all the activation domains (i.e., T.S.V., M.L.V., and Pr.In.T.). Importantly, no instance of consistency was observed within the Initial level of activation. This suggests that Johannes was continuously activating this competency aspect having a clear perspective towards the solution of the task, using mathematical language, needing limited help to work with the tasks he was assigned to cope with.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, it is noticed that there was frequent competency activation mainly during one calculus unit, but activations were tracked at the other units too. The changes between activation levels were equally observed to both directions (from a lower to a higher level of activation and the other way around), and there were occasions where Johannes consistently activated the (Int.Res.X) aspect above the initial level of activation working almost independently. I, therefore, infer that there is a *moderate combination of evidence* for the development of the (Int.Res.X) aspect of the Mathematical Modeling competency.

Overview for (Cr.)

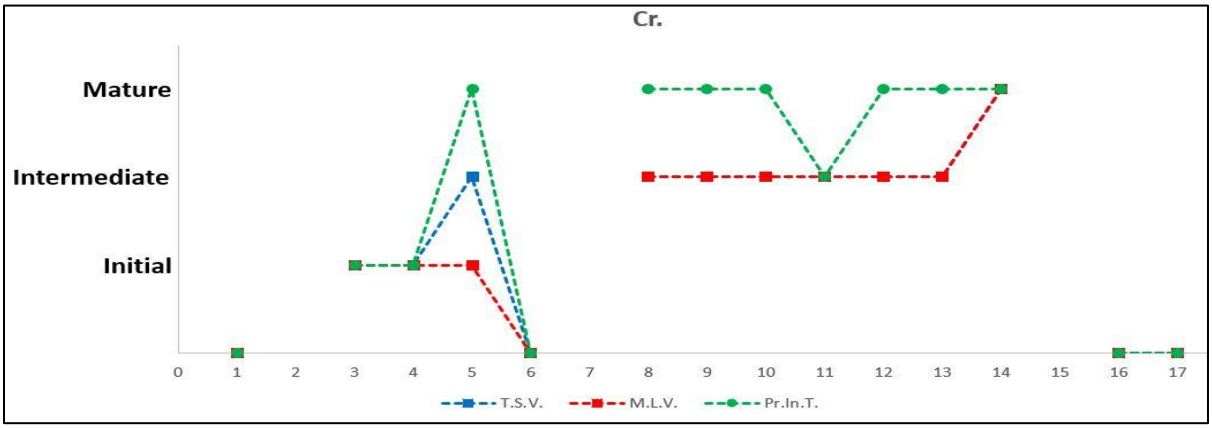

Diagram 6-12. Activation for competency aspect (Cr.).

Table 6-33. Summarized evidence for (Cr.) activation. (repetition).

Type of evidence \ **Competency domain**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	2	1	2	
Changes from higher to lower	1	1	2	
High consistency	1	1(1)	0	
Moderate consistency	(1)	0	2(1)	
Frequency of activation				10

<u>Frequency of competency activation</u>

The only observed activations have been located during the Exponential Growth & Regression (only during the first session of this unit) and the Population Dynamics units. The frequencies were [0, 3, 7, 0], and the total activation frequency was 10. There were four sessions where no evidence of competency activation was observed. Interestingly, the similarities of the above diagram with the diagram that described Erika's activation in this competency aspect are apparent.

<u>Changes between activation levels</u>

By incorporating the second type of evidence, *changes between activation levels*, the information from Table 6-33 informs us that the changes from lower to higher and higher to lower levels of competency activation by domain were: (2, 1) for T.S.V., (1, 1) for M.L.V., and (2, 2) for Pr.In.T. These observations cannot provide reliable inferences regarding the progress of competency development because no direction (from lower to higher activation level or vice versa) was observed to be more often than the other.

<u>Levels of consistency</u>

Regarding the third type of evidence, *levels of consistency*, Diagram 6-2 and Table 6-33 informs us about instances of high or moderate consistency at all levels of the scaling instrument. It was also observed that Johannes was able to work independently with limited prompting using expressions of mathematical language and in a continuous way during the Population Dynamics unit. This is also attributed to the fact that this was the only calculus unit where Johannes activated the (Cr.) competency aspect in both sessions of the unit.

<u>Strength of evidence combination: final evaluation</u>

Based on the summaries above for all three types of evidence, it is noticed that there was identifiable competency activation for two calculus units (three sessions), the changes between activation levels were equally observed to both directions (from a lower to a higher level of activation and the other way around), and that there were occasions where Johannes consistently activated the (Cr.) aspect at all activation levels. I, therefore, infer that there is a *weak combination of evidence* for the development of the (Cr.) aspect of the Mathematical Modeling competency.

Overview for (Graph)

Johannes provided evidence for the activation of the (Graph) aspect only on two occasions. Once during the first session of the Exponential Growth & Regression unit and once at the last session of the Population Dynamics unit. There was no evidence of competency activation during the Periodic Functions and the Exponential Growth & Regression units. Therefore, as mentioned in the Results chapter, this competency aspect is considered *non-activated*.

Final assessment for the Mathematical Modeling competency

Table 6-34 below, summarizes the concluding inferences regarding the final evaluation of the strength of evidence combination for the six aspects of the Mathematical Modeling competency.

Table 6-34. Strength of evidence combination for Mathematical Modeling competency (D.o.C.).

Mathematical Modeling competency	
Aspect	Strength of evidence combination
R.+V.	Negligible
Int.El.	Moderate
Int.Res.X	Moderate
Cr.	Weak
Info	Moderate
Graph	Non-activated

Based on the six overviews of the three types of evidence that provided indications regarding the dynamic of competency development of the six competency aspects that constitute the Mathematical Modeling competency, I infer that there is a *weak combination of evidence* that suggests development within the Degree of Coverage of the competency.

Representing and Manipulating symbolic forms competency

Overview for (C.D.Rep)

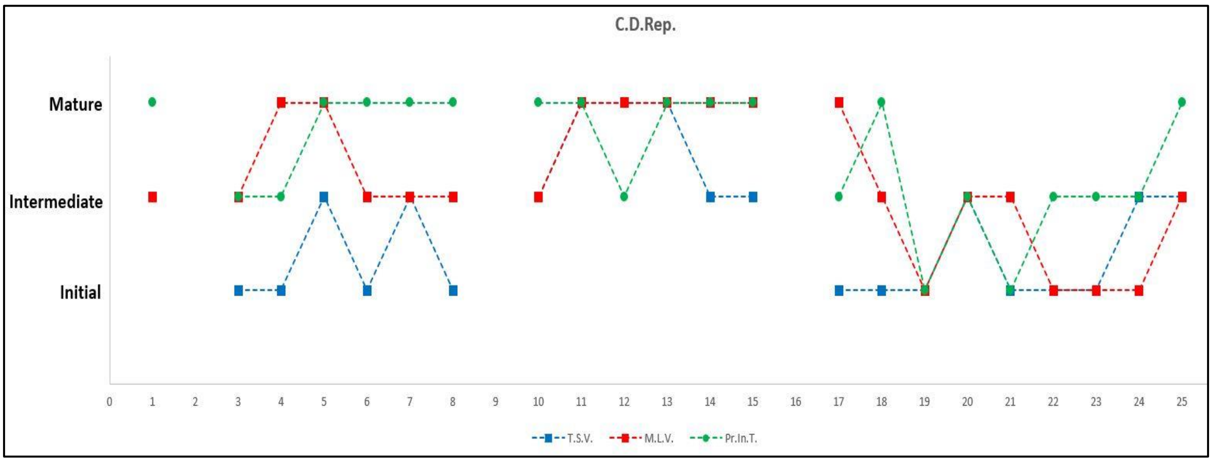

Diagram 6-13. Activation for competency aspect (C.D.Rep.).

Table 6-35. Summarized evidence for (C.D.Rep.) activation. (repetition).

Competency domain / Type of evidence	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	5	4	6	
Changes from higher to lower	4	4	3	
High consistency	0	1	0	
Moderate consistency	1	1	2	
Frequency of activation				22

Frequency of competency activation

The activation frequencies by unit were [1, 6, 6, 9], and the total activation frequency was 22. The most frequent activation was located during the Integrals & Modeling unit, followed by the Population Dynamics and Exponential Growth & Regression units.

Changes between activation levels

As seen in Table 6-35, the changes from lower to higher and higher to lower levels of competency activation by domain were: (5, 4) for T.S.V., (4, 4) for M.L.V., and (6, 3) for Pr.In.T. These observations, especially within the Pr.In.T. domain, suggest that the progress was more often from a lower to a higher level of activation, than the other way around and shows a dynamic of development over time.

Levels of consistency

There were instances of high and moderate consistency only at the Mature or Intermediate level of activation and none within the Initial one. This suggests that Johannes consistently decoded or chose appropriate representations with limited assistance while using expressions of mathematical language.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, it is noticed that there was frequent competency activation for all sessions of the calculus units except the first session, the changes between activation levels were more often from a lower to a higher level of activation, than the other way around, and that there were occasions where Johannes consistently activated the (C.D.Rep.) aspect above the initial level of activation. I, therefore, infer that there is a *strong combination of evidence* for the development of the (C.D.Rep.) aspect of the Representing and Manipulating symbolic forms competency.

Final assessment for the Representing and Manipulating symbolic forms competency.

Table 6-36 below summarizes the concluding inferences regarding the final evaluation of the strength of evidence combination for the three aspects of the Representing and Manipulating symbolic forms competency.

Table 6-36. Strength of evidence combination for Mathematical Modeling competency (D.o.C.).

Representing and Manipulating symbolic forms competency	
Aspect	Strength of evidence combination
C.D.Rep.	Strong
Rep.↔Rep.	Weak
Man.Rep.	Strong

Based on the three overviews of the three types of evidence that provided indications regarding the dynamic of competency development of the three competency aspects that constitute the Representing and Manipulating symbolic forms competency, I infer that there is a *strong combination of evidence* that suggests development within the Degree of Coverage of the competency.

Mathematical Reasoning and Communicating competency

Overview for (Reason→)

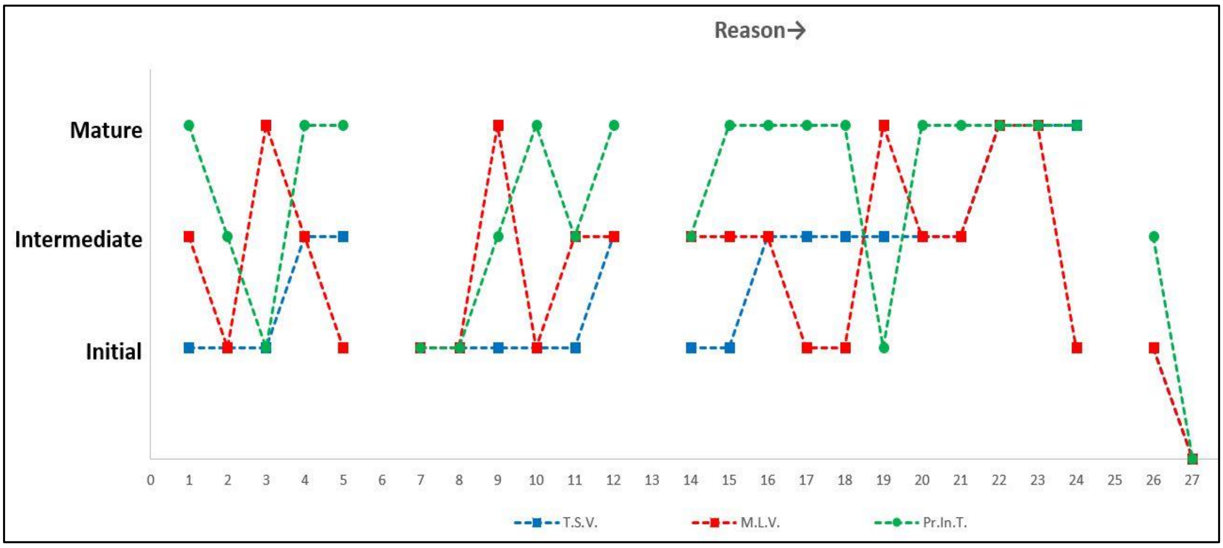

Diagram 6-14. Activation for competency aspect (Reason→).

Table 6-37. Summarized evidence for (Reason→) activation. (repetition).

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	4	5	6	
Changes from higher to lower	0	8	5	
High consistency	(1)	0	0	
Moderate consistency	2(1)	0	3	
Frequency of activation				23

Frequency of competency activation

As shown in the diagram, the frequencies were [5, 6, 11, 1], and the total activation frequency was 23. The most frequent activation was observed during the Population Dynamics unit, followed by the Exponential Growth & Regression and Periodic Functions units. This was the most frequently activated competency aspect of the Mathematical Reasoning and Communicating competency.

Changes between activation levels

As seen in Table 6-37, the changes from lower to higher and higher to lower levels of competency activation by domain were: (4, 0) for T.S.V., (5, 8) for M.L.V., and (6, 5) for Pr.In.T. These observations suggest that the progress was more often from a lower to a higher level of activation, for the T.S.V. and Pr.In.T. domains of activation (mainly for the T.S.V. though), however, for the M.L.V domain, it was the other way around. This suggests that even though Johannes showed evidence of developing his view towards the solution of the tasks and gradually needing less prompting from the teacher, he faced difficulties developing his use of mathematical language to express his arguments and justify his decisions.

Levels of consistency

There were several instances of moderate consistency for the Pr.In.T. domain of activation suggesting that on these instances, Johannes managed to work independently with limited prompting. Johannes also faced difficulties in maintaining a clear perspective towards the solution of the tasks while providing his argumentations and justifications of his thoughts or actions.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, it is noticed that there was frequent competency activation for all sessions of the calculus units except the last. Regarding the changes between activation levels, Johannes showed evidence of developing his view towards the solution needing less prompting but he faced difficulties developing his use of mathematical language. Lastly, there were occasions where Johannes consistently activated the (Reason→) aspect above the initial level of activation but faced difficulties in maintaining a clear perspective towards the solution of the tasks. I, therefore, infer that there is a *moderate combination of evidence* for the development of the (Reason→) aspect of the Mathematical Reasoning and Communicating competency.

Final assessment for the Mathematical Reasoning and Communicating competency.

Table 6-38 below summarizes the concluding inferences regarding the final evaluation of the strength of evidence combination for the two aspects of the Mathematical Reasoning and Communicating competency.

Table 6-38. Strength of evidence combination for Mathematical Reasoning and Communicating competency (D.o.C.).

Mathematical Reasoning and Communicating competency	
Aspect	Strength of evidence combination
Reason←	Strong
Reason→	Moderate

Based on the two overviews of the three types of evidence that provided indications regarding the dynamic of competency development of the two competency aspects that constitute the Mathematical Reasoning and Communicating competency, I infer that there is a *strong combination of evidence* that suggests development within the Degree of Coverage of the competency.

Use of aids and tools

Overview for (Ext.T.)

As mentioned in Section 5.6, no evidence was provided by Johannes at any of the seven sessions regarding the activation of the (Ext.T) aspect. As with Erika's case, it is possible the content of the sessions and the tasks did not provoke a discussion regarding the existence and properties of any tool or aid for mathematical activity. However, there were few instances from other students (e.g., Rene) who expressed their knowledge about using Desmos software for graphing activities. That said, it should also be considered that the description of this competency aspect may lead to difficulties in tracking instances of activation while engaging in mathematical activities.

Final assessment for the Use of Aids and Tools competency.

Table 6-39 below, summarizes the concluding inferences regarding the final evaluation of the strength of evidence combination for the two aspects of the Use of Aids and Tools competency.

Table 6-39. Strength of evidence combination for the Use of Aids and Tools competency (D.o.C.).

Use of Aids and Tools competency	
Aspect	Strength of evidence combination
Ext.T	No evidence
U.T.	Moderate

Based on the two overviews of the three types of evidence that provided indications regarding the dynamic of competency development of the two competency aspects that constitute the Use of Aids and Tools competency, and because one of these aspects was not observed to be activated, I infer that there is a *weak combination of evidence* that suggests development within the Degree of Coverage of the competency. The fact that the discussion on Erika's use of tools mainly refers to the use of a calculator and not a variety of tools also attributed to the determination of this final assessment.

6.1.4.2 Development for Radius of Action

In what follows, I present the findings (i.e., overviews) on the case study of Erika for the five main competencies of the relevant framework concerning the competency development within the Radius of Action of each of these competencies. As mentioned in section 4.3.2, a modified version of the tables summarizing the evidence will be used in the following overviews. These tables will display the evidence by calculus unit (by mathematical context) and not as a whole following, that way, the features attributed to this dimension by Niss and Højgaard (2011). Within these overviews, I refer to the diagrams used in the previous part of this section (i.e., Development for Degree of Coverage). Therefore, I will not display them here to avoid any unnecessary repetition. Following the same method as before, there will be a selection of competency aspects for each competency. A detailed analysis for each of these aspects not included here can be found in Appendix E.

Mathematical Thinking and Acting competency

This competency consists of three aspects and to avoid repetition, I only mention here the relevant abbreviations (codes) that represent these aspects: (P.Q.), (Lim), and (→Sol.). As mentioned before in Erika's case, the information about the activations frequencies by calculus unit are derived from previous diagrams to avoid any unnecessary repetition. The aforementioned repetition will be avoided for the rest of the competencies too.

Overview for (→Sol)

Table 6-40. Summarized evidence for (→Sol) activation by unit.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	(0,2,4,2)	(1,0,2,1)	(0,2,3,2)	
Changes from higher to lower	(0,1,2,1)	(1,0,3,0)	(1,2,2,1)	
High consistency	(1,0,0,0)	(0,(1),0,1)	(0,0,0,0)	
Moderate consistency	(0,0,0,0)	(0,0,1,0)	(1,0,2,1)	
Frequency of activation				31

Frequency of competency activation

The activation frequencies were [3, 6, 16, 6]. There most frequent activation frequency was observed during the Population Dynamics. Frequent activation was also observed during the Exponential Growth & Regression and the Integrals & Modeling unit. However, a less frequent activation has been observed in the Periodic Functions unit.

Changes between activation levels

As seen in Table 6-40, overall, more changes from a lower to a higher level of activation than the other way around were identified at the last three calculus units. This observation suggests that there is a dynamic of competency development over time for these units.

Levels of consistency

There was evidence of high and moderate consistency within the Intermediate or Mature activation level at all calculus units. Notably, the only instance of consistency within the Initial level of activation and the M.L.V. domain was observed during the Exponential Growth & Regression unit. Furthermore, several instances of moderate consistency for the M.L.V. and Pr.In.T. activation domains were observed during the Population dynamics unit. This suggests that Johannes attacked the problems he faced using expressions of mathematical language with limited prompting.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, there is evidence of development mainly for the Population Dynamics unit but also for the units of Exponential Growth & Regression and Integrals & Modeling. I, therefore, infer that there is a *strong combination of evidence* within the Radius of Action for the development of the (→Sol.) aspect of the Mathematical Thinking and Acting competency.

Final assessment for the Mathematical Thinking and Acting competency

Table 6-41 below summarizes the concluding inferences regarding the final evaluation of the strength of evidence combination for the three aspects of the Mathematical Thinking and Acting competency.

Table 6-41. Strength of evidence combination for Mathematical Thinking and Acting competency (R.o.A.).

Mathematical Thinking and Acting competency	
Aspect	Strength of evidence combination
P.Q.	Moderate
Lim.	Weak
→Sol.	Strong

Based on the three overviews of the three types of evidence that provided indications regarding the dynamic of competency development of the three aspects that constitute the Mathematical Thinking and Acting competency, I infer that there is a *moderate combination of evidence* that suggests development within the Radius of Action of the competency. This is explained from the fact that evidence of competency development was provided mainly for the Population Dynamics unit but also for the sessions of the Integrals & Modeling and Exponential Growth & Regression units.

Mathematical Modeling competency

This competency consists of six aspects and to avoid repetition, I only mention here the relevant abbreviations (codes) that represent these aspects: (R.+V.), (Int.El.), (Int.Res.X), (Cr.), (Info), and (Graph).

Overview for (Int.Res.X)

Table 6-42. Summarized evidence for (Int.Res.X) activation by unit.

Type of evidence \ Competency domain	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	(0,1,0,1)	(0,0,0,1)	(0,0,2,1)	
Changes from higher to lower	(0,1,0,0)	(0,1,0,0)	(0,1,2,0)	
High consistency	(0,0,1,0)	(0,0,1,0)	(0,0,0,0)	
Moderate consistency	(0,0,0,0)	(0,0,0,0)	(0,0,1,0)	
Frequency of activation				12

Frequency of competency activation

The activation frequencies were [1, 2, 8, 1]. The only noticeable activation frequency was observed during the Population Dynamics unit. The low activation frequency during the other calculus units does not allow any strong inference to be made for the development within the Radius of Action using this type of evidence.

Changes between activation levels

As seen in Table 6-42, it is not easy to draw any inference by employing this type of evidence for the other units besides the Population Dynamics sessions. The only changes between activation levels for the Population Dynamics unit were observed within the Pr.In.T. domain. This observation suggests that Johannes consistently activated the competency aspect within a particular level of activation, as shown below.

Levels of consistency

As implied above, evidence of high consistency within the Intermediate or Mature activation level was identified only during the Population dynamics unit, suggesting a dynamic of competency development over time. It is difficult to draw any inferences using this type of evidence for the other calculus units due to the observed low activation frequency.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, there is evidence of development mainly for one calculus unit: Population Dynamics. I, therefore, infer that there is a *weak combination of evidence* within the Radius of Action for the development of the (Int.Res.X) aspect of the Mathematical Thinking and Acting competency.

Final assessment for the Mathematical Modeling competency

Table 6-43 below summarizes the concluding inferences regarding the final evaluation of the strength of evidence combination for the six aspects of the Mathematical Modeling competency.

Table 6-43. Strength of evidence combination for Mathematical Modeling competency (R.o.A.).

Mathematical Modeling competency	
Aspect	Strength of evidence combination
R.+V.	Negligible
Int.El.	Strong
Int.Res.X	Weak
Cr.	Weak
Info	Moderate
Graph	Non-activated

Based on the six overviews of the three types of evidence that provided indications regarding the dynamic of competency development of the six competency aspects that constitute the Mathematical Modeling competency, I infer that there is a *weak combination of evidence* that suggests development within the Radius of Action of the competency. This is explained by the fact that evidence of competency development was provided mainly for the units of Population Dynamics and occasionally for the Exponential Growth & Regression units.

Representing and Manipulating symbolic forms competency

This competency consists of three aspects and to avoid repetition, I only mention here the relevant abbreviations (codes) that represent these aspects: (C.D.Rep.), (Rep.↔Rep.), and (Man.Rep.).

Overview for (C.D.Rep.)

Table 6-44. Summarized evidence for (C.D.Rep.) activation by unit.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	(0,2,1,2)	(0,1,1,2)	(0,1,1,4)	
Changes from higher to lower	(0,2,1,1)	(0,1,0,3)	(0,0,1,2)	
High consistency	(0,0,0,0)	(0,0,1,0)	(0,0,0,0)	
Moderate consistency	(0,0,1,0)	(0,1,0,0)	(0,1,1,0)	
Frequency of activation				22

Frequency of competency activation

The activation frequencies were [1, 6, 6, 9]. A more frequent competency activation was observed during the Integrals & Modeling unit, followed by the Population Dynamics and Exponential Growth & Regression units.

Changes between activation levels

As seen in Table 6-44, for all sessions of the above-mentioned units, the changes from a lower to a higher level of activation were comparatively more than the changes to the opposite direction. These findings could signify a dynamic of development for these units. Especially during the Integrals & Modeling unit, there were many instances where Johannes managed to decode representations that he had to work on while gradually needing less assistance and prompting from the teacher or a classmate.

Levels of consistency

There were occasions of moderate and high consistency for two calculus units (i.e., Exponential Growth & Regression and Population Dynamics) within all activation domains. The latter indicates that Johannes worked independently and regularly using expressions with mathematical language and vocabulary for these units while having consistently a clear view of the task solution. The latter suggests a dynamic of development for these calculus units.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, there is evidence of development for three calculus units: Integrals & Modeling, Population Dynamics, and Exponential Growth & Regression. I, therefore, infer that there is a *strong combination of evidence* within the Radius of Action for the development of the (C.D.Rep.) aspect of the Mathematical Thinking and Acting competency.

Final assessment for the Representing and Manipulating symbolic forms competency

Table 6-45 summarizes the concluding inferences regarding the final evaluation of the strength of evidence combination for the three aspects of the Representing and Manipulating symbolic forms competency.

Table 6-45. Strength of evidence combination for Representing and Manipulating symbolic forms competency (R.o.A.).

Representing and Manipulating symbolic forms competency	
Aspect	Strength of evidence combination
C.D.Rep.	Strong
Rep.↔Rep.	Weak
Man.Rep.	Moderate

Based on the three overviews of the three types of evidence that provided indications regarding the dynamic of competency development of the three competency aspects that constitute the Representing and Manipulating symbolic forms competency, I infer that there is a *moderate combination of evidence* that suggests development within the Radius of Action of the competency. This is due to the evidence of competency development was provided mainly for the units of Population Dynamics and Exponential Growth & Regression. In addition, there was also evidence of competency development during the sessions of the Periodic Functions unit.

Mathematical Reasoning and Communicating competency

This competency consists of two aspects: (Reason⟵) and (Reason⟶).

Overview for (Reason⟶)

Table 6-46. Summarized evidence for (Reason→) activation by unit.

Competency domain / Type of evidence	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	(1,1,2,0)	(1,2,2,0)	(1,3,2,0)	
Changes from higher to lower	(0,0,0,0)	(3,1,3,1)	(2,1,1,1)	
High consistency	(0,(1),0,0)	(0,0,0,0)	(0,0,0,0)	
Moderate consistency	(1(1),0,1,0)	(0,0,0,0)	(1,0,2,0)	
Frequency of activation				23

Frequency of competency activation

The activation frequencies were [5, 6, 11, 1]. A more frequent activation was noticed during the Population Dynamics followed by the Exponential Growth & Regression and Periodic Functions units. Limited activation was observed during Johannes's work on the Integrals & Modeling unit.

Changes between activation levels

As seen in Table 6-46, the changes from a lower to a higher level of activation were identified to be comparatively more than those that occurred towards the other direction during the Population Dynamics and the Exponential Growth & Regression units. This observation could signify a dynamic of competency development for these units. However, this was not the case for the Periodic Functions unit. In this unit, the changes from a higher to a lower activation level were comparatively more than those in the opposite direction.

Levels of consistency

Instances of moderate consistency only within the higher levels of activation were observed only during the Population Dynamics unit. Johannes expressed his argumentation, maintaining a clear view towards the solution of the tasks of this unit while needing limited assistance or prompting. Consistency was also observed during the last part of the Periodic Functions unit.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, there is evidence of development for two calculus units: Population Dynamics and Exponential Growth & Regression. I, therefore, infer that there is a *moderate combination of evidence* within the Radius of Action for the development of the (Reason→) aspect of the Mathematical Thinking and Acting competency.

Final assessment for the Mathematical Reasoning and Communicating competency

Table 6-47 below summarizes the concluding inferences regarding the final evaluation of the strength of evidence combination for the two aspects of the Mathematical Reasoning and Communicating competency.

Table 6-47. Strength of evidence combination for Mathematical Reasoning and Communicating competency (R.o.A.).

Mathematical Reasoning and Communicating competency	
Aspect	Strength of evidence combination
Reason←	Moderate
Reason→	Moderate

Based on the two overviews of the three types of evidence that provided indications regarding the dynamic of competency development of the two competency aspects that constitute the Mathematical Reasoning and Communicating competency, I infer that there is a *moderate combination of evidence* that suggests development within the Radius of Action of the competency. This is explained by the fact that evidence of competency development was provided mainly for the Exponential Growth & Regression unit. Evidence of competency development was also identified for the Population Dynamics and Integrals & Modeling units.

Use of Aids and Tools competency

This competency consists of two aspects presented here with their relevant abbreviations (codes): (Ext.T.) and (U.T.).

Overview for (U.T.)

Table 6-48. Summarized evidence for (U.T.) activation by unit.

Type of evidence / **Competency domain**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	(0,2,2,1)	(0,2,0,1)	(0,1,1,1)	
Changes from higher to lower	(0,0,0,0)	(0,1,0,0)	(0,0,1,0)	
High consistency	(0,0,0,0)	(0,0,1,0)	(0,0,0,0)	
Moderate consistency	(0,1,1,0)	(0,0,0,0)	(0,1,1,0)	
Frequency of activation				13

Frequency of competency activation

As shown in the diagram, the frequencies were [0, 6, 6, 1], and it is evident that competency activation was observed mainly during the sessions of Exponential Growth & Regression and Population Dynamics units.

Changes between activation levels

As seen in Table 6-48, for these sessions where evidence of competency activation was observed, the changes from a lower to a higher level of activation were identified to be comparatively more than those to the other direction (i.e., from a higher to a lower level). This suggests a dynamic of competency development during the calculus units of Exponential Growth & Regression and Population Dynamics.

Levels of consistency

There were several instances of moderate consistency at the higher activation levels for the T.S.V. and Pr.In.T. domains of activation. At all these units, Johannes appeared to develop his use of tools while working on the tasks maintaining a clear perspective towards the solution steps he needed to take, and he did that with limited prompting. Notably, during the Population Dynamics unit, Johannes exhibited high consistency within the M.L.V domain, suggesting that he was continuously able to handle the tools he was using easily.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, there is evidence of development for two calculus units: Population Dynamics and Exponential Growth & Regression. I, therefore, infer that there is a *moderate combination of evidence* within the Radius of Action for the development of the (U.T.) aspect of the Mathematical Thinking and Acting competency.

Final assessment for the Use of Aids and Tools competency

Table 6-49 below, summarizes the concluding inferences regarding the final evaluation of the strength of evidence combination for the two aspects of the Use of Aids and Tools competency.

Table 6-49. Strength of evidence combination for Mathematical Reasoning and Communicating competency (R.o.A.).

Use of Aids and Tools competency	
Aspect	Strength of evidence combination
Ext.T	No evidence
U.T.	Moderate

Based on the two overviews of the three types of evidence that provided indications regarding the dynamic of competency development of the two competency aspects that constitute the Use of Aids and Tools competency, and because one of these aspects was not observed to be activated, I infer that there is a *weak combination of evidence* that suggests development within the Radius of Action of the competency. This is explained from the fact that evidence of competency development was provided for the units of Population Dynamics and Exponential Growth & Regression but only regarding the second aspect of this competency, the (U.T.) aspect.

6.1.4.3 Development for Technical Level

As mentioned in Erika's case about the nature of competency development within the Technical Level, it is expected that the discussion on Johannes's handling of the technical parts of the tasks he worked on will present similarities across the various competency aspects. On several occasions. Johannes provided evidence of activation for two or more competency aspects within the same turn (utterance). For example, there were instances where while attacking the problem of a task (→Sol.), Johannes interpreted the elements (Int.El.) of a mathematical expression at the same time. Therefore, the discussion on how Johannes handled the technical parts of a particular part of a task concerns both the (→Sol.) and the (Int.El.) competency aspects.

<u>Mathematical Thinking and Acting competency</u>

This competency consists of three aspects, and to avoid repetition, I only mention here the relevant abbreviations (codes) that represent these aspects: (P.Q.), (Lim), and (→Sol.).

Overview for technical parts

Regarding the technical parts, while posing questions (P.Q.) that had mathematical characteristics, Johannes occasionally grasped what each parameter of an equation represented. When he looked at the table of temperatures of the Airport Temperatures task during the Periodic Functions session, he asked, "are we just supposed to use '17.9' or […] is that the amplitude?" trying to understand what does a stands for, at the $g(x) = a \cdot \cos[b(x - c)] = b$ equation. During the Population Doubled task, Johannes realized that the answer to the main question of the task could be in several units (e.g., days, hours, years) by posing a relevant question to the teacher. There were occasions also where Johannes faced difficulties grasping the use of symbols in some equations (e.g., $N_{t+1} = N_t + B + D + I - E$, where N_{t+1} is the population at $t + 1$ years, N_t is the population at t year, B is the number of births, D is the number of deaths, I is the number of people immigrated, and E is the number of people emigrated) when he asked, "Why do we have "+"[in front of] deaths?"

In instances, Johannes tracked the limitations (Lim.) of several concepts that he encountered quickly and proceeded to the solution of tasks. For example, during the sessions of the Exponential Growth & Regression unit, he realized that a cluster of malignant cells must be first visible when it reaches a particular size, "oh, it must be 1 cm^3," and he did that quite fast. When addressing the solution of the tasks (→Sol.), Johannes was able to handle expressions with exponents and their properties. For example, he understood the notion of binary fission when he realized that in 100 days, the '2^n' expression must be written as

‘2^{100}’ when working on the Bacterial Population task. Johannes exhibited a grasp of symbol and formula-based handling and manipulation of differentiation. However, as in Erika’s case, his handling skills were somewhat limited when performing integration, which indicates how developed the Technical Level was when he was attacking the problems of the Integrals & Modeling unit. Johannes faced difficulties on understanding the $'dx'$ part of the definite integral $\int_a^b f(x)dx$.

Exploring the M.L.V. domain

In this part of the overview, I present the assembled diagrams from the (P.Q.), (Lim.), and (→Sol.) in aspects of the Mathematical Thinking and Acting competency in Figure 6.7, and provide the evidence focusing only for the M.L.V. domain of activation (red dotted line).

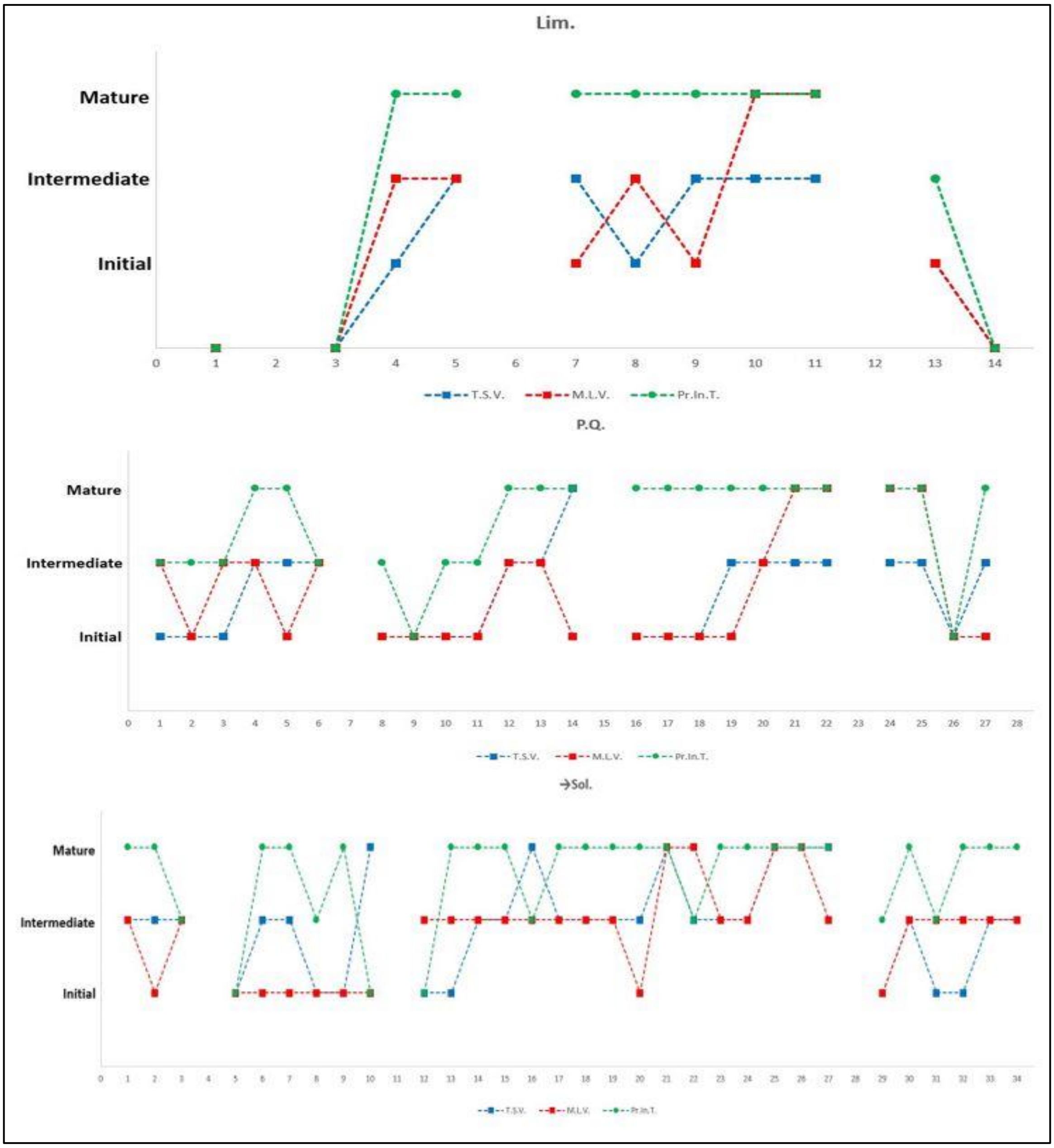

Figure 6.7. Assembled diagrams: Mathematical Thinking and Acting competency (Johannes).

Table 6-50 below summarizes the two types of evidence (changes between activation levels and consistency of activation) within the M.L.V. domain for the three aspects of the Mathematical Thinking and Acting competency. For two aspects of this competency, it is evident that the activations were observed to occur more often from a lower to a higher level of activation than to the other direction. Two observations can be made regarding the last two rows of the table (consistency of activation). First, Johannes faced difficulties in posing questions (activations of (P.Q.) aspect) regularly using expressions of mathematical language. There were three occasions (symbolized as 1(3) at the (P.Q.) column) where Johannes activated the competency continuously at the Initial level of activation of the scaling instrument and only once at a higher level. Second, Johannes was able to find information relevant to the tasks he needed to address, as seen in the (Lim.) column of the table below.

Table 6-50. Summary of evidence for M.L.V. domain, Mathematical Thinking and Acting competency.

Competency aspect / Type of evidence	P.Q.	Lim.	→Sol.
Changes from lower to higher	5	3	4
Changes from higher to lower	4	1	4
High consistency	0	0	1(1)
Moderate consistency	1(3)	1	1

Final assessment for the Mathematical Thinking and Acting competency

Based on the overviews for the technical parts and the exploration of the M.L.V. domain regarding the combination of evidence for the progress of development of the three competency aspects that constitute the Mathematical Thinking and Acting competency, I infer that there is a *moderate combination of evidence* that suggests development within the Technical Level of the competency.

Mathematical Modeling competency

This competency consists of three aspects and to avoid repetition, I only mention here the relevant abbreviations (codes) that represent these aspects: (R.+V.), (Int.El.), and (Int.Res.X), (Cr.), (Info), and (Graph).

Overview for technical parts

As mentioned before, Johannes only provided evidence for the (R.+V.) competency aspect activation on three occasions. Twice during the Population Dynamics unit and once at the last session of the Integrals & Modeling unit. Therefore, it is unfeasible to draw any conclusions regarding the development of this competency aspect within the Technical Level.

On these instances where Johannes provided evidence of activation of the (Int.El.) competency aspect Johannes showed awareness of what each parameter (i.e., a, b, c, and d) of the $g(x) = acos[b(x - c)] + d$, equation represented, "so, now d is about up or down with the graph" during the Periodic Functions session. He needed little assistance in handling the exponential expressions he worked on during the Exponential Growth & Regression unit, but there were occasions where he faced difficulties working with the lnx and the log functions. Johannes was able to work efficiently with the properties of the continuous model for unrestricted growth population (i.e., $N(T) = N(0)e^{rT}$) during the Population Dynamics sessions but again when he had to use the logarithmic functions to provide a final outcome, he faced some difficulties even though mentioning that "[they] can get the *t* down using the *ln* or the *log* [functions]." Lastly, and similarly to Erika's work, Johannes faced difficulties using the integration rules for various reasons when activating the (Int.El.) aspect.

On these occasions where Johannes provided evidence for the activation of the '*interpret mathematical results in an extra-mathematical context*' competency aspect (Int.Res.X), he displayed a relatively firm grasp of the periodic functions, the elements, and information provided by a temperature table, and the difference between variables and constants. Because the (Int.Res.X) competency activation occurred many times within the same turn with that of the (Int.El.) aspect, the analysis of the technical parts for the (Int.Res.X) competency aspect entails similarities with that of the (Int.El.) aspect.

Regarding the analysis of the Technical Level for the "*critique the model by reviewing or reflecting and questioning the results*" competency aspect (Cr.) Johannes argued with Erika about the formulas they came up with were different, and both of their points were valid. For these activations of the (Cr.) aspect during the Population Dynamics sessions, Johannes did not refer to technical parts of the mathematics he worked on but mainly displayed an understating of the concepts he encountered. There was no competency activation during the Integrals & Modeling, and the rest of the analysis is analogous to that of the Int.Res.X competency. Regarding the technical parts analysis for the '*look and search for available information and to differentiate between relevant and irrelevant information*' (Info) competency aspect, Johannes displayed similar to the Int.Res.X, competency development within the Technical Level.

Lastly, during the '*to choose appropriate mathematical and graphical notations during the task solution process*' competency (Graph), it has been stated Johannes provided evidence for the activation of the competency aspect (Graph) only on two occasions. Once during the first session of the Exponential Growth & Regression unit and once during the second session of the Population Dynamics unit. It is, therefore, unfeasible to draw any inferences regarding the development of this aspect within the Technical Level.

Exploring the M.L.V. domain

In this part of the overview, I present the assembled diagrams from the (Int.El.), (Int.Res.X), (Cr.), and (Info) aspects of the Mathematical Modeling competency in Figure 6.8, and provide the evidence focusing on the M.L.V. activation domain (red dotted line). Because of the identified low frequency of activation there are no diagrams for the (R.+V.) and (Graph) aspects.

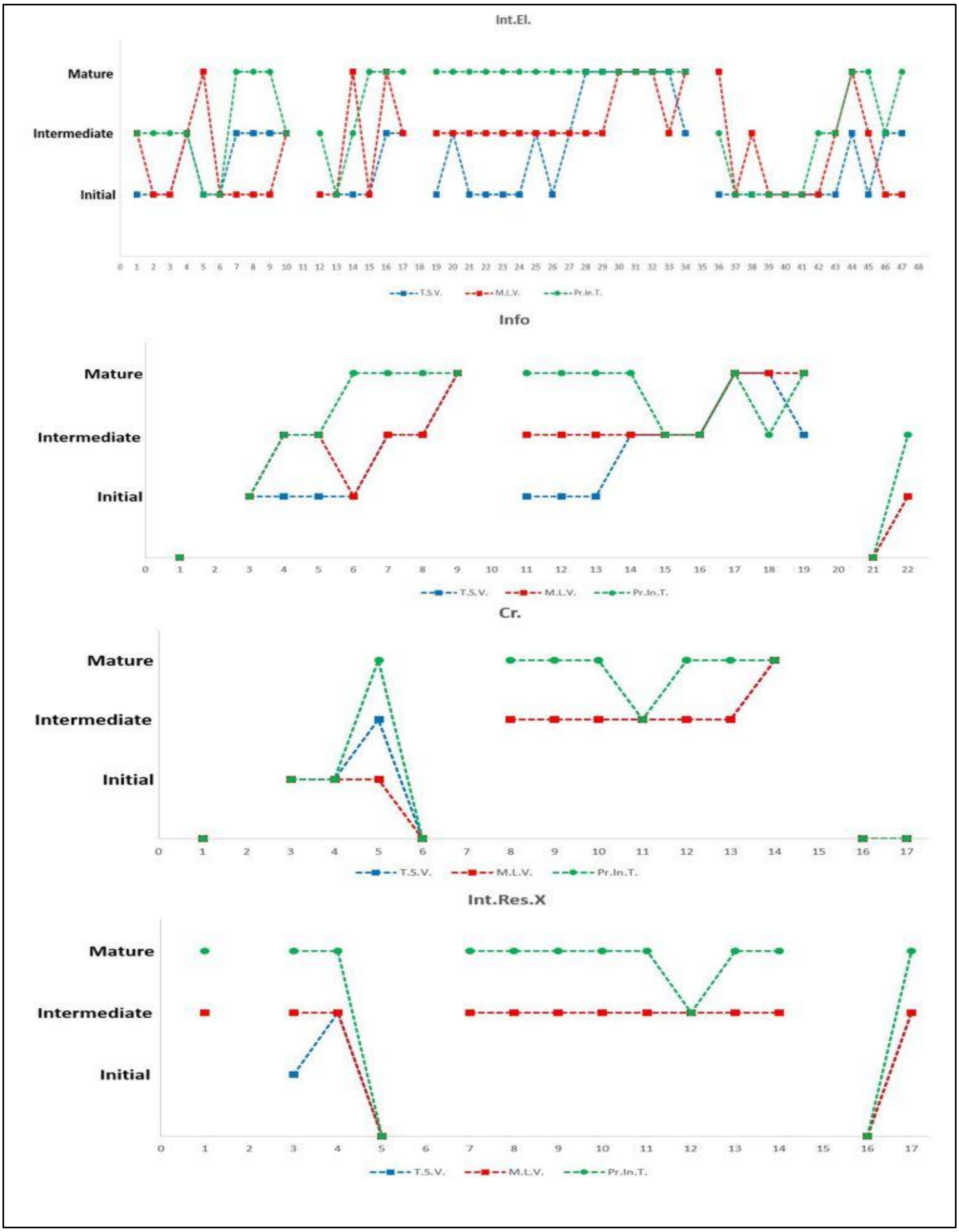

Figure 6.8. Assembled diagrams: Mathematical Modeling competency (Johannes).

Table 6-51 below summarizes the two types of evidence (changes between activation levels and consistency of activation) within the M.L.V. domain for the four aspects of the Mathematical Modeling competency. For three aspects of this competency, it is evident that the activations were observed to occur more often from a lower to a higher level of activation. Interestingly, there were many instances of moderate consistency (i.e., activating the competency within the Intermediate of Mature level of activation) for the (Info) competency aspect.

Table 6-51. Summary of evidence for M.L.V. domain, Mathematical Modeling competency.

Competency aspect / Type of evidence	Int.El.	Int.Res.X	Cr.	Info
Changes from lower to higher	10	1	1	5
Changes from higher to lower	9	1	1	1
High consistency	1	1	1(1)	0
Moderate consistency	(2)	0	0	1

Final assessment for the Mathematical Modeling competency

Based on the overviews for the technical parts and the exploration of the M.L.V. domain regarding the combination of evidence for the progress of development of the six competency aspects that constitute the Mathematical Modeling competency, I infer that there is a *moderate combination of evidence* that suggests development within the Technical Level of the competency.

Final assessment for the Representing and Manipulating symbolic forms competency

Based on the overviews for the technical parts and the exploration of the M.L.V. domain regarding the combination of evidence for the progress of development of the three competency aspects that constitute the Representing and Manipulating symbolic forms competency, I infer that there is a *strong combination of evidence* that suggests development within the Technical Level of the competency.

Final assessment for the Mathematical Reasoning and Communicating competency

Based on the overviews for the technical parts and the exploration of the M.L.V. domain regarding the combination of evidence for the progress of development of the two competency aspects that constitute the Mathematical Reasoning and Communicating competency, I infer that there is a *moderate combination of evidence* that suggests development within the Technical Level of the competency.

Final assessment for the Use of Aids and Tools competency

Based on the overview for the technical parts and the exploration of the M.L.V. domain regarding the combination of evidence for the progress of development of the (U.T.) aspect and due to the fact that there was no evidence of activation for the (Ext.T.) aspect, I infer that there is a *weak combination of evidence* that suggests development within the Technical Level of the competency.

6.1.5 Findings: Tables for competency development assessment

This last subsection of Section 6.1 comprises the findings of this thesis regarding the progress of competency development over time for Erika and Johannes. These findings concern the first Research Question of this thesis: "What is the progress of individual competency development over time for a student who participates in a series of calculus sessions that comprise non-routine mathematical tasks? What is the competence profile of each student after these series of calculus sessions?" This subsection summarizes what was presented in the previous subsections and presents the findings of competency development in tabular form.

Both students manifested evidence of competency development, although at different levels and mathematical competencies, as shown in tables 6-52 and 6-53. For the reader's convenience, I display again here the levels of evidence strength (subsection 6.1.1) regarding the competency development assessment: *strong* (related to the Mature Level of Activation (III) of the scaling instrument), *moderate* (related to the Intermediate Level of Activation (II)), *weak* (related to the Initial Level of Activation (I)), and *negligible* (the strength level of evidence combination that represents the situation where partially or non-activated competencies have been identified).

This thesis explores the progress of competency development of Erika and Johannes. This exploration is facilitated by evaluating the strength of the combination of evidence they provided regarding the indications about competency activation and development. The following tables provide information about this strength of evidence. Table 6-52 and Table 6-53 below display all 16 competency aspects of the competence framework for Erika and Johannes. Each aspect is positioned in its representative cell corresponding to the strength level it was assigned to after the assessment process (subsections 6.1.3 and 6.1.4) and to each of the three dimensions of the 3-D model (Degree of Coverage, Radius of Action, and Technical Level). These tables give information about which competency aspects displayed the strongest or the weakest evidence of competency development over time.

Table 6-52. Findings for Erika's competency aspect development.

		Strength level of evidence combination			
		Strong	Moderate	Weak	Negligible
The 3-D Model	Degree Of Coverage	P.Q. Int.El. C.D.Rep., Man.Rep. Reason →, Reason← U.T.	→Sol. Int.Res.X, Cr., Info	Lim Graph Rep.↔Rep.	R.+V. Ext.T
	Radius Of Action	P.Q., →Sol. C.D.Rep., Man.Rep. U.T.	Int.El., Cr., Info Rep.↔Rep. Reason←, Reason→	Lim Graph, Int.Res.X	R.+V. Ext.T
	Technical Level	Mathematical Modeling, Representing and Manipulating symbolic forms	Mathematical Thinking and Acting, Mathematical Reasoning and Communicating, Use of Aids and Tools		

Table 6-53. Findings for Johannes's competency aspect development.

		Strength level of evidence combination			
		Strong	Moderate	Weak	Negligible
The 3-D Model	Degree Of Coverage	→Sol. C.D.Rep., Man.Rep. Reason←	P.Q., Lim Info, Int.El., Int.Res.X Reason → U.T.	Cr. Rep.↔Rep.	Graph, R.+V. Ext.T
	Radius Of Action	→Sol. Int.El. C.D.Rep.	P.Q. Info Man.Rep. Reason←, Reason→ U.T.	Lim Cr., Int.Res.X Rep.↔Rep.	R.+V., Graph Ext.T
	Technical Level	Representing and Manipulating symbolic forms	Mathematical Thinking and Acting, Mathematical Modeling, Mathematical Reasoning and Communicating,	Use of Aids and Tools	

Table 6-54 and Table 6-55 below present the findings for the final inferences regarding the progress of competency development over time for Erika and Johannes for each of the five competencies of the competence framework. They display the three dimensions of the 3-D model (Degree of Coverage, Radius of Action, and Technical Level) and the five competencies of the framework. These tables constitute the competency development profiles of Erika and Johannes.

Table 6-54. Erika's competency development profile.

		The mathematical competencies				
		Mathematical Thinking and Acting	Mathematical Modeling	Representing and Manipulating symbolic forms	Mathematical Reasoning and Communicating	Use of Aids and Tools
The 3-D Model	Degree Of Coverage	Moderate	Moderate	Strong	Strong	Moderate
	Radius Of Action	Strong	Moderate	Strong	Moderate	Moderate
	Technical Level	Moderate	Strong	Strong	Moderate	Moderate

Table 6-55. Johannes's competency development profile.

		The mathematical competences				
		Mathematical Thinking and Acting	Mathematical Modeling	Representing and Manipulating symbolic forms	Mathematical Reasoning and Communicating	Use of Aids and Tools
The 3-D Model	Degree Of Coverage	Moderate	Weak	Strong	Strong	Weak
	Radius Of Action	Moderate	Weak	Moderate	Moderate	Weak
	Technical Level	Moderate	Moderate	Strong	Moderate	Weak

6.2 Competency interrelation

This section presents the findings regarding the work undertaken to address the second Research Question of this thesis and comprises two subsections. This is a comparatively smaller section than the previous one where the first Research Question was addressed. Most of the work included a computational statistical approach where details of it can be found in Appendix F. Subsection 6.2.1 revisits the second Research Question, reminding the reader what this question aims to address and answer. Deriving information from the results displayed in Section 5.7, subsection 6.2.2 presents the findings of this study about these competency aspects that appeared to be activated concurrently with a lower or higher-than-expected activation frequency.

6.2.1 Revisiting the second Research Question

By the end of Chapter 2, the second Research Question addressed the issue of concurrent competency activation by asking if any competencies are interrelated and, if so, how. More specifically, the question concerned the competence framework’s competency aspects (16 in total). As seen in subsection 2.5.2, the question entailed two sub-questions to address the main research question. The first sub-question asked if we could create a table where all frequencies of concurrent competency activations are displayed for all possible compete pairs. This table (Table 5-21) was presented in Section 5.7. The information from this table was derived from all the recorded observations where concurrent competency activation was identified.

The second sub-question asked if there is a quantitative approach that could be adopted to track any statistically significant observations regarding the observed and expected frequencies of concurrently activated competency aspects. The interest is in examining whether a particular competency's activation affects another competency's activation or non-activation. A possible rephrasing of the original research question can be the following: *Does the activation of a particular competency 'prevent' or 'favor' the concurrent activation of another competency?* In other words, where the activation of one competency often occurs at the same time as another, or whether two competencies are never activated – or observed to be activated at the same time. To address this second sub-question, I performed a hypothesis test using the *Binomial Test* by comparing two proportions. The results of this test can be found in Table 5-22 in subsection 5.7.2. The following paragraph presents the findings of this test and how they could be interpreted in the light of the second Research Question. Table 5-22 will be repeated in this following subsection for the reader's convenience.

6.2.2 Findings: competency pairs

Table 6-56 (next page), displays the results of the Binomial (two-tailed) Test. This table is a repetition of Table 5-22 for the reader's convenience where the new green-shaded cells are the competency pairs which the application of the Binomial Test recorded p-value $p \leq 0.05$. There were 15 competency pairs that presented statistical significance regarding their co-activation frequency: (P.Q. & C.D.Rep.), (P.Q. & Man.Rep.), (P.Q. & Reason←), (→Sol. & Info), (→Sol. & Reason←), (Int.El. & Cr.), (Int.El. & Info), (Int.El. & Man.Rep.), (Int.El. & U.T.), (Int.Res.X & Cr.), (Info & Man.Rep.), (C.D.Rep. & Rep.↔Rep.), (C.D.Rep. & U.T.), (Man.Rep. & Reason→), (Reason→ & U.T.).

Table 6-56. Competency pairs (green shaded) with <0.05 recorded p-value (two-tailed) (repetition).

	1 P. Q.	**2** Lim.	**3** → Sol.	**4** R. + V.	**5** Int. El.	**6** Int. Res. X	**7** Cr.	**8** Info	**9** Graph	**10** C. D. Rep.	**11** Rep. ↔ Rep.	**12** Man. Rep.	**13** Reason ←	**14** Reason →	**15** Ext. T.	**16** U. T.
1 P.Q.		1	5	0	4	0	1	1	0	0	0	0	0	2	0	0
2 Lim.			1	1	3	1	0	0	0	2	0	0	0	0	0	0
3 → Sol.				0	13	5	1	0	1	7	0	5	1	4	0	7
4 R.+V.					0	0	0	1	0	1	0	0	0	0	0	0
5 Int.El.						1	0	1	0	10	0	1	6	9	0	0
6 Int.Res.X							4	0	0	2	0	0	0	0	0	0
7 Cr.								0	0	1	0	0	1	0	0	0
8 Info									0	1	0	0	0	2	0	2
9 Graph										1	0	0	0	0	0	0
10 C.D.Rep.											7	9	1	5	0	0
11 Rep. ↔ Rep.												4	0	0	0	1
12 Man.Rep.													4	2	0	4
13 Reason←														7	0	1
14 Reason→															0	0
15 Ext.T.																0
16 U.T.																

In what follows, I provide a visualization of Table 6-56. Figure 6.9 (next page) displays a network of these 15 pairs (green-shaded cells). Using this form of visualization, one can identify how these competency aspects are interconnected within these 15 pairs. The (Int.El.) and (Man.Rep.) competency aspects are connected with four different competencies in this figure, and the (C.D.Rep.), (Info), and (U.T.) competencies with three different competencies. A discussion regarding the (P.Q.) competency aspect will take place in the last paragraph of this subsection. The rest of the competencies of this network record two or one connection. These interconnections suggest that the particular pairs present a statistical significance regarding their interrelation. Some pairs may appear more often than expected, and some may appear less often than expected.

Based on these findings, the (Int.El.), (Man.Rep.), (C.D.Rep.), (Info), and (U.T.) competencies (competency aspects to be more accurate) exhibit significantly different frequencies of concurrent activation with other competencies. However, discussion can be made for the other pairs too. The latter suggests that these competencies mentioned before are more likely to present significant differences concerning their expected and their observed proportions of concurrent activation. For example, looking at Figure 6.9, it is seen that the (Int.El.) is connected with four other competency pairs. As shown in the following paragraphs, the connection with these four other competency pairs may suggest a lower-than-expected frequency of concurrent activation or a higher one.

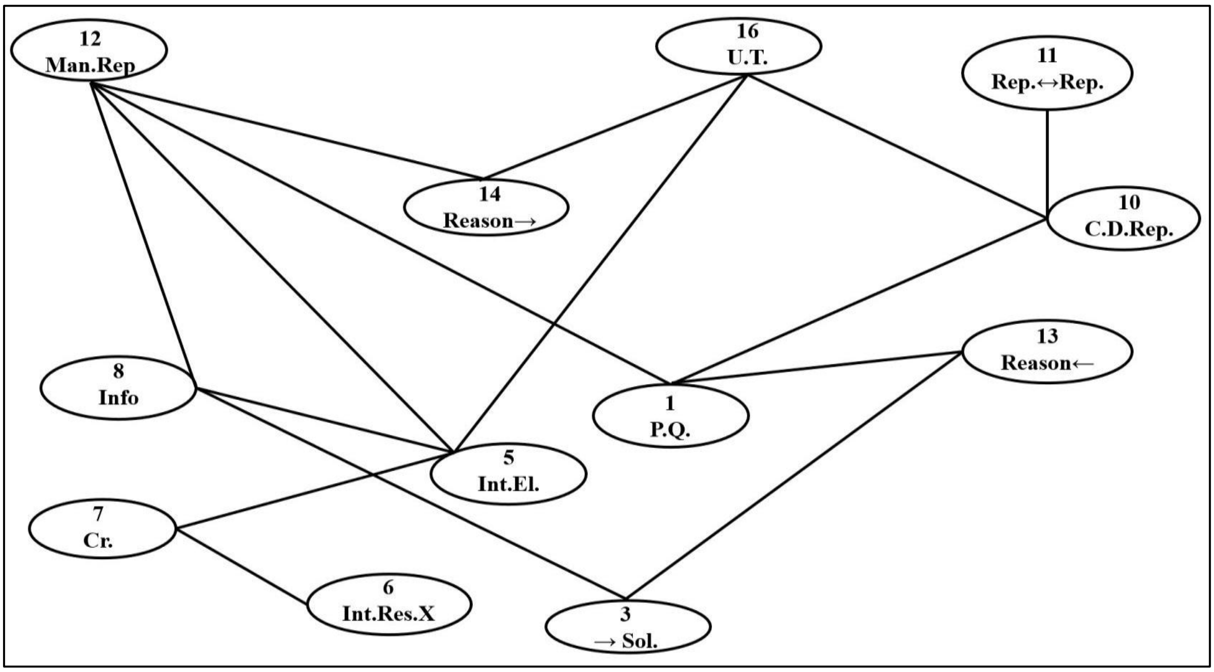

Figure 6.9. Network of competency pairs.

Figure 6.10 (next page) is a repetition of the previous figure, but it now provides additional information. First, it shows which competency pairs recorded a frequency of concurrent activation higher than expected, which is illustrated in the figure with a bold black line. These pairs are the following: (Int.Res.X & Cr.), and (C.D.Rep. & Rep.↔Rep.). Second, it displays which competency pairs recorded a frequency of concurrent activation that was less than expected, illustrated in the figure with a dotted black line. These pairs are the following: (Info & Man.Rep.), (Int.El. & Man.Rep.), (→Sol. & Reason←), (Reason→ & Man.Rep.), (C.D.Rep. & U.T.), (→Sol. & Info), (Int.El. & Info), (Reason→ & U.T.), (Int.El. & Cr.), and (Int.El. & U.T.).

Looking at the above, one can notice that these findings concern 12 and not 15 competency pairs. The pairs not included in either of the above groups (i.e., higher-than-expected or less-than-expected) can be seen with the red-dotted line in Figure 6.10 and are the following: (P.Q. & Reason←), (P.Q. & C.D.Rep.) and (P.Q. & Man.Rep.). These pairs were excluded because they did not belong at the intersection of the two-tailed and the one-tailed (less-than-expected) tests. When conducting binomial tests, the significance level is allocated differently based on the types of tests conducted. Two-tailed tests allocate 0.025 on each end of the tail, whereas one-tailed tests allocate the full 0.05 onto one end of the tail. An additional reflection and discussion on these results will be provided in Chapter 8 in Section 8.1, where I discuss the key findings of this study.

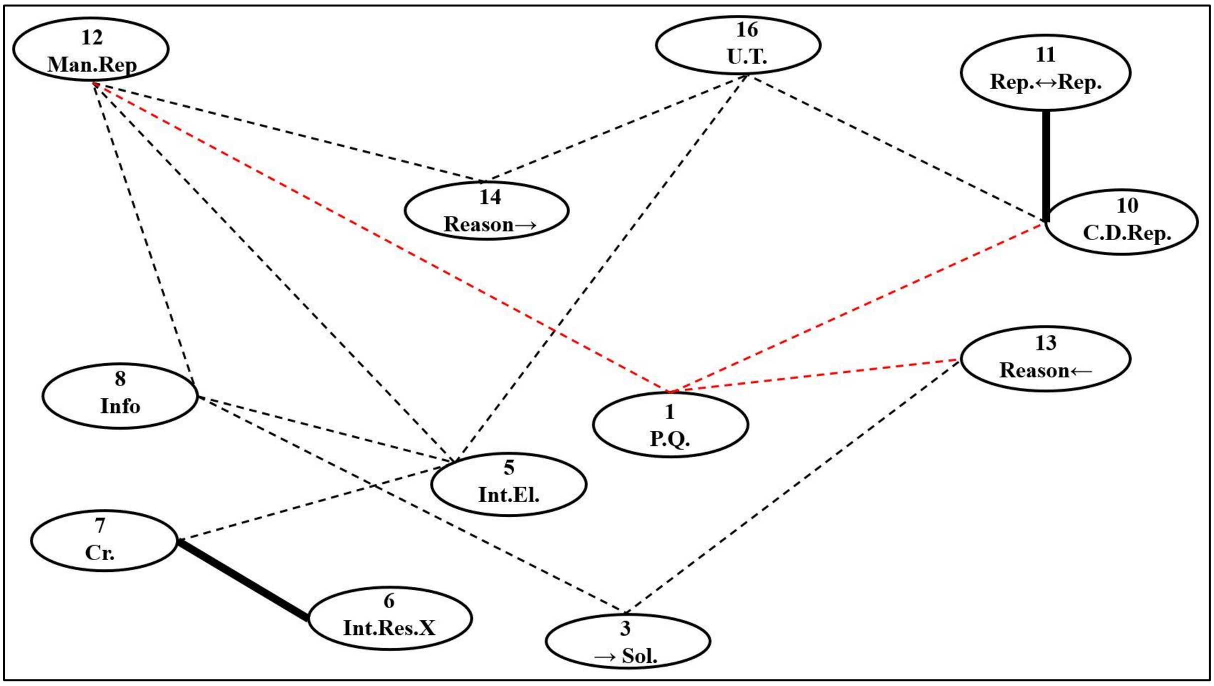

Figure 6.10. Pairs with higher (bold) or lower (dotted)-than-expected frequency of concurrent activation.

7. Discussion

This chapter discusses the findings of this thesis that were presented in Chapter 6. The discussion consists of three parts that formed three sections. In Section 7.1, the first Research Question is addressed by revisiting key concepts and constructs of this thesis and the findings that concern this question. Similarly, Section 7.2 discusses the findings regarding the second Research Question and the ideas that have been developed. The last section (Section 7.3) offers a brief discussion on the notion of competency development when non-routine mathematical tasks are introduced to students of a Biology department.

7.1 Addressing the First Research Question

There are four subsections in this section where key concepts of this thesis are revisited in light of the findings produced. Subsection 7.1.1 discusses the KOM competence model that was the main inspiration for constructing this thesis's competence framework. In subsection 7.1.2, the functionality of the 3-D model is discussed by reflecting on each of its three dimensions. The scaling instrument constructed and employed for producing inferences regarding the quality of competency activation and development is discussed in subsection 7.1.3. This section concludes (subsection 7.1.4) by discussing the findings regarding the nature of competency development that the students displayed during the conduction of this research.

The following discussion concerns the findings for the case studies of Erika and Johannes regarding their competency development over time. As mentioned before, this study provided some initial results that concerned the competency activations for the other four students that participated in this study. However, due to reasons explained before (subsections: 3.4.5; 5.1: A summary for all sessions, and 5.3.4), Erika and Johannes were the two students that provided the necessary data that could provide inferences regarding their competency development and not just their competency activations.

7.1.1 Revisiting the KOM competence model

The first sub-question of this thesis asked what set of competencies would be suitable to constitute the competence framework of this research. First, the word 'suitable' was used to highlight the necessity to construct a competence framework with these competencies that are thought to be necessary for any biology student to possess. Second, because this study's competence framework would build on an existing framework, namely, the KOM model (Niss, 2003), some adjustments must be considered. The KOM model comprises eight competencies with several aspects within these competencies. It was deemed appropriate to merge or leave out some of these competencies (or aspects) for contextual and coding reasons. By coding, I describe the process of recognizing instances within students' actions in the class where they activate a particular competency or competencies and assigning a code (corresponds to this particular competency) that signifies that instance.

By employing the KOM model (Niss, 2003) as an (initial) analytical framework and using data from the first three sessions, observations were made about which mathematical competencies appeared to be activated when students engaged with the mathematical tasks of the sessions. As discussed above (and thoroughly in Section 3.3), some competency aspects of the KOM model (e.g., 'extending the scope of a concept by abstracting some of its properties; generalizing results to larger classes of objects' of the *mathematical thinking* competency) appeared to be less relevant to the needs of a student in a Biology department. Even though all mathematical competencies are potentially necessary for any student working with mathematics at any educational level or field, some of them might be more relevant than others depending on the field of study (e.g., physics, engineering, biology). These observations led to constructing a competence framework similar to the KOM model with some alternations.

After several considerations, the competence framework of this study (introduced and presented in Section 3.3) took its final shape and numbered five main competencies:

1. Mathematical Thinking and Acting
2. Mathematical Modeling
3. Representing and Manipulating symbolic forms
4. Mathematical Reasoning and Communicating
5. Use of Aids and Tools

The competencies mentioned above included several aspects referred to as *competency aspects*. The Mathematical Thinking and Acting competency of this study's framework is an amalgamation of the two first competencies of the KOM model (Appendix A), namely the *thinking mathematically* and the *posing and solving mathematical problems* competencies and comprised three competency aspects (coded with the following abbreviations: P.Q., Lim., and →Sol.). The process of assigning these codes during the thematical analysis of the transcribed oral and written work of the students was not hindered by the merging of the two competencies of the KOM model or from the fact that some competency aspects of the KOM model were left out during the construction of this thesis's competence framework.

Some inferences can be drawn regarding the Mathematical Modeling competency of the competence framework. The six aspects of this competency (coded with the following abbreviations: R.+V., Int.El., Int.Res.X, Cr., Info, and Graph) were the unification of several competencies from the relevant literature on modeling competencies and what is defined as modeling process (e.g., Maaß, 2006; Blum et al., 2007; Kaiser & Schwartz, 2006). These six aspects were sufficiently distinct and therefore facilitated the coding process of identifying the activation of these aspects within the students' actions in the classroom. The

(Graph.) competency aspect was initially abbreviated as (Not.+Graph.) to describe the competency where a student chooses appropriate mathematical notations and (or) represents situations graphically. However, during the coding process of the first three sessions, it appeared there were very few differences between the description of this competency and the aspects of the Representing and Manipulating symbolic forms competency. Similarities among the characteristics that define the (Not.+Graph.) and (C.D.Rep.) competencies made the coding process tricky. Therefore, the (Not.+Graph.) was redefined as (Graph) to include only the circumstances where a student represents situations graphically.

The framework's Mathematical Reasoning and Communicating competency is a unification of the *reasoning mathematically* and the *communicating in, with, and about mathematics* competencies of the KOM model. Initial observations from the first three sessions suggested that the last three aspects of the *reasoning mathematically* competency of the KOM model (Appendix A) would not fit into the context of this study: first-year students of a Biology department. For example, the non-routine mathematical tasks incorporated in this work were designed or selected for first-year biology students. The focus was primarily on a situation where there is a transition between the real and the mathematical world. Designing a task that would demand rigorous mathematical proofs from the students was not within this study's scope. Consequently, the 'knowing what a mathematical proof is (not), and how it differs from other kinds of mathematical reasoning' competency aspect of the KOM model was not included in the competence framework of this research.

Continuing the discussion regarding the Mathematical Reasoning and Communicating competency, the decision to merge the first aspect of the *reasoning mathematically* competency (i.e., 'following and assessing chains of arguments, put forward by others') and the first aspect of the *communicating in, with, and about mathematics* competency (i.e., 'understanding others' written, visual or oral 'texts', in a variety of linguistic registers, about matters having a mathematical content') of the KOM model did not hinder the functionality of the coding process of this study. Both aspects describe the action of following argumentations from others, and this similarity could have created difficulties during the coding process.

The Use of Aids and Tools competency of the KOM model comprised two aspects, and it was incorporated into the competence framework of this study with no alternation. The two competency aspects (Ext.T. and U.T.) described the situations where i) a person knows the existence and properties of various tools and aids for mathematical activity and their range and limitations, and ii) where a person is able to reflectively use such aids and tools. Indications from the first three sessions regarding the inactivation of the (Ext.T.) competency aspect of the Use of Aids and Tools competency suggested that a different description may be more suitable for facilitating any coding process. An additional discussion regarding the lack of activation evidence for the (Ext.T.) competency aspect will occur in subsection 7.1.2, where I discuss these aspects that displayed limited or no activation during the seven calculus sessions of this project.

Summarizing the above and reflecting on the thematical analysis employed during the coding process, some observations can be made. During the coding process, competencies and their aspects were treated as mental or physical processes, activities, and behaviors following the conceptualization of Niss (2003). Furthermore, and still in agreement with Niss's conceptualization, the competencies selected to constitute the competence framework of this study comprised both analytical and productive characteristics. During these seven sessions, students were seen attempting to interpret, explore, and assess the mathematical phenomena they encountered (i.e., analytical view) but also actively constructing mathematical, carrying out such processes (i.e., productive view) when for example, they constructed their own chains of arguments to justify decisions and actions. Evidence of competency activation was any identifiable instance where a student was eager to act and address the mathematical tasks. The latter agrees with what Niss and Højgaard (2019, p. 12) defined as competence,"[c]ompetence is someone's insightful readiness to act appropriately in response to the challenges of given situations," which constitutes a more context-based definition.

In conclusion, it appears that the way the KOM model was customized for the needs and aims of this research by merging or leaving out specific mathematical competencies (i.e., aspects of the KOM competencies) facilitated the endeavor to explore competency activations and set the base for exploring the notion of competency development. However, additional analytical tools needed to be introduced and employed to facilitate such an exploration. The following subsection discusses how such a tool, namely the 3-D model (Niss & Højgaard, 2011), was incorporated into the data analysis process of this study.

7.1.2 Revisiting the 3-D model

A fundamental analysis tool of this study was the 3-D model developed by Niss and Højgaard (2011), first introduced in Section 2.4. This model was constructed to explore competency progress and compare students' competency development. The second sub-question of this study asked whether there is a way to utilize the 3-D model to explore the progress of individual competency development over time. This thesis agrees with Niss and Højgaard's (2011) considerations of what it means to master a competency. Therefore, I adopted their suggestions to incorporate the three dimensions, namely the Degree of Coverage, the Radius of Action, and the Technical Level, as the dimensions through which inferences on competency development would be drawn. They recognize these dimensions as partial non-quantitative ordering principles and offer extensive qualitative descriptions for each one of them.

As mentioned in Section 3.1, competency is considered a notion of dynamic nature, which complies with the considerations that Niss and Højgaard made regarding the dynamic nature of these three dimensions. Reflecting on the fashion in which this thesis utilized the model's three dimensions, there is an agreement with Niss and Højgaard (2011). They point out that by incorporating these three dimensions, it is feasible "to characterize a student's actual mastery of a specific competency" because this tool (i.e., the 3-D model) dynamically describes "how the respective student develops the respective competency over time" (Niss & Højgaard, 2011, p. 139). Furthermore, this thesis takes a step forward by suggesting a particular way of using this competency progress tool.

Reflecting on the method that was adopted in this thesis to produce inferences regarding the individual competency development over time and on the findings themselves, this study holds that the exploration of competency development using the three dimensions of the 3-D model should follow a specific order. The competency activations should be first analyzed through their Degree of Coverage, then through their Radius of Action, and lastly, their Technical Level. This study adds to the position held by Niss and Højgaard (2019, p. 22), who claimed that "[t]he three dimensions of competency possession can be used to define and characterize progression in an individual's competency possession." The exploration of students' competency development is at the core of this thesis, and in what follows, I discuss every dimension of the 3-D model.

Degree of Coverage (D.o.C.)

This dimension is defined as the breadth of the aspects that characterize each competency, that is, how many of these aspects (competencies) the person can activate in the different situations available and to what extent independent activation occurs. The competence framework of the present study includes five primary competencies, each having several aspects (competency aspects). Utilizing this dimension brought to the light a possible inherent dysfunctionality of this 3-D model. The obstacle can be described by the following question: How can we observe progression -more aspects of the competency gradually activated- if there is a limited number of aspects and if these are activated -to some extent- from the beginning of the monitoring process? For example, the Mathematical Reasoning and Communicating competency has two aspects (Reason← and Reason→), and it was not feasible to track signs of development within this dimension when these aspects were activated from the first calculus session.

Subsection 4.1.3 introduced the notion of *quality of competency activation* to tackle this problem. Following that, a scaling instrument (scheme) that evaluated this activation quality was constructed and employed in the analysis process. This instrument comprises three domains of competency activation: *Task Solving Vision* (T.S.V.), *Mathematical Language and Vocabulary* (M.L.V.), and *Prompting/Independent Thinking* (Pr.In.T.). This construction aimed to offer a different approach to utilizing this dimension of the 3-D model. Instead of tracking changes in the number of aspects activated over time, a different path was pursued. I recorded and analyzed the activation quality over time and offered an overall evaluation of that quality. This evaluation determined the progress of competency development over time within the Degree of Coverage of that competency. I do not claim, by any means, that the Degree of Coverage is a non-functional tool for the study of competency development. Instead, I suggest incorporating the competency activation quality notion to increase the utility of the 3-D tool when the objective is to track evidence of individual competency development over time.

Radius of Action (R.o.A.)

The Radius of Action addresses the domain of situations in which someone can perform mathematical activities by activating a mathematical competency. It is the spectrum of contexts where the student brought into action a particular or a set of competencies. By assessing the competency development through the Radius of Action in this thesis, it became evident that the need for constructing this dimension pertains to the fact that any mathematical competency has contextual characteristics, as Jensen (2007) points out. This study has shown that it is crucial to maintain a specific order when examining and assessing the activation and development of the competency when applying the 3-D model. Because of the contextual nature of the Radius of Action, it was necessary that exploring students' competency development through their Degree of Coverage should precede the exploration of their Radius of Action. Conclusively, each competency activation was assessed by analyzing each calculus unit (based on the different mathematical context they comprised) separately.

It is possible that differences between contexts, in terms of competency activation quality, will occur in any study of competency development throughout different mathematical domains. This study showed that the students were comfortable when dealing with the mathematical tasks of the Population Dynamics sessions. The students engaged with this calculus unit and provided the strongest evidence of competency development than in other units. On the other hand, the Integrals & Modeling unit was a domain where the students displayed the weakest combination of evidence regarding their competency development. How this study drew inferences regarding the students' competency development within the Radius of Action was facilitated by the detailed description given by Niss and Højgaard (2019, p. 21) when revisiting their work on mathematical competencies. Considering the Radius of Action as this dimension that signifies "the range and variety of different contexts and situations in which the individual can successfully activate the competency," they suggest that development within the Radius of Action can come with a successful competency activation within these range of different contexts. It is, therefore, reasonable to assume that this is a functional description that could facilitate a research endeavor oriented towards the exploration of competency development.

Technical Level (T.L.)

The Technical Level of each competency addresses the issue of flexibility regarding the technical parts of the mathematical task, "[…] meaning what kind of mathematics they use and how flexible they do it." (Blomhøj & Jensen, 2007, p. 51). Examining the nature of competency activation within its Technical Level is a study of how conceptually and technically advanced the entities and tools are when activated in the relevant competency. The process of tracking evidence of progression within this dimension presupposes the understanding of the term 'flexibility' in the use of mathematics, which entails a certain amount of vagueness. In order to tackle this problematic situation, a combined assessment approach was followed. In addition to Niss and Højgaard's (2011) approach, I suggest incorporating the Mathematical Language and Vocabulary (M.L.V.) activation domain in the analysis process. The (M.L.V.) domain (subsection 4.1.7) from the scaling instrument (subsection 4.1.8) describes the way the students express themselves in terms of mathematical language and vocabulary use while activating a particular competency.

In their work, Niss and Højgaard (2011) provide qualitative descriptions and examples of how one can explore the progress of competency within the Technical Level. However, they do that in a comparative fashion, "[t]he person who can sketch graphs for real functions of one variable, but not for real functions of two variables, has a representing competency at a lower Technical Level than the person who can attain both." (Niss and Højgaard, 2011, p. 73). In the present dissertation, instead of comparing two students' Technical Levels, the focus is on the development of the Technical Level of the same student over a specific period. The element of time (i.e., progression over time) is integrated into the evaluation process by incorporating the overall assessment of the (M.L.V.) domain into the analysis process. Reflecting on how this analysis was conducted to produce inferences about the strength of evidence that determined the competency development through the Technical Level suggests that this is a practical approach to be undertaken.

A final remark

All three dimensions of the Niss and Højgaard (2011) 3-D model proved to be suitable tools for the individual assessment of the student's competency development over time. The present study suggests the addition of particular notions and techniques (i.e., competency activation quality, incorporation of M.L.V domain) that could facilitate the exploration and analysis of competency development. Modifying this model's properties for this study's needs resulted in constructing a scaling instrument that determines the quality of competency activation through three domains every time a student provided evidence of such an activation. The discussion about this instrument takes place in the following subsection.

7.1.3 The scaling instrument

The previous subsection discussed how modifying the 3-D model's properties resulted in constructing a scaling instrument. This instrument was designed to assess the quality of competency activation through three domains every time a student provided evidence of such an activation. The central thinking behind this construction was to explore the dynamic nature of competencies (Niss & Højgaard, 2011) by assessing the *quality of competency activation* (detailed description of the term in subsection 4.1.3). Therefore I created a rubric comprising three levels of competency activation explored through three domains. The three levels were the following: Initial Level of Activation (I), Intermediate Level of Activation (II), and Mature Level of Activation (III). The activation domains concerned i) the student's vision towards the solution of the task (T.S.V.), ii) the use of mathematical language and vocabulary when evidence of activation was provided (i.e., when a competency was activated) (M.L.V.), and iii) the amount of help he/she received through prompting questions from the teacher or discussion with a fellow classmate (Pr.In.T.). As discussed in the following paragraph, the construction of the scaling instrument resulted in the conceptualization of two additional types of evidence.

For this research, I introduced three types of evidence that would facilitate the analysis for producing inferences regarding the competency activation and development of Erika and Johannes: *frequency of activation*, *changes between activation levels*, and *consistency of competency activation*. The last two types of evidence were direct outcomes of the scaling instrument built to utilize the instrument's features. The changes between activation levels indicated the development dynamic and whether the student was activating a competency aspect more often from a lower to a higher activation level or if that occurred in the opposite direction. The consistency of competency activation provided an additional indication regarding the students' ability to activate a competency aspect within a certain activation level on two or more consecutive instances of competency activation. If the student could activate a particular competency aspect consistently within the Intermediate (II) or Mature (III) activation level, that would constitute another indication of competency development. Conclusively, this research used the combination of these three types of evidence to draw inferences regarding the nature of competency development for each student.

Reflecting on the work undertaken to provide inferences regarding the competency development for each student using this scaling instrument, I can claim that this method of analysis facilitated the aims of this thesis. Assigning a level of activation (i.e., I, II, or III) within each activation domain (T.S.V., M.L.V., and Pr.In.T.) appeared to be a feasible task because the descriptions for each activation level were designed to be as distinct as possible to avoid potential incidents of ambiguity. Analyzing the competency activations through these three domains showed that two (or more) students may activate the same competency simultaneously within the same context (e.g., providing mathematical reasoning for the same argument) but in a different fashion. Closing this discussion on the scaling instrument, it is necessary to point out that the employment of three activation domains does not suggest that there are no other domains or dimensions in which competencies can be explored in terms of their activation and development.

7.1.4 Discussing the competency development

The last two sub-questions (R.Q. 1.3 and R.Q. 1.4, Table 3-7 in subsection 3.6.1) address the main core of the first Research Question. They ask what the progress of individual competency development over time is through each dimension of the 3-D model (that will eventually form a competency development profile for each student) and what competencies from the competence profile of each student display evidence of progression. As seen in Chapter 6 and subsection 6.1.5, Table 6-52 and Table 6-53 comprise the final inferences about each competency aspect of the five main competencies of the framework. In the same subsection, Table 6-54 and Table 6-55 present the findings for the final inferences regarding the progress of competency development over time for Erika and Johannes for each one of the five competencies of the competence framework and constitute the competency profiles of Erika and Johannes.

In the following paragraphs, I discuss these competencies (including their aspects) that Erika and Johannes displayed mainly strong or moderate evidence combination of development over time. I also discuss these competencies that appeared to be partially or non-activated during these seven sessions (i.e., weak or negligible evidence combination of development). This subsection ends with a discussion on the final competency profiles that Erika and Johannes displayed concerning these competencies that manifested evidence of progression. Let us remember here that by competenc*ies*, I refer to the five main competencies of the competence framework.

Competency aspects with strong evidence of development

Deriving information from Table 6-52 and Table 6-53, Erika provided mainly strong evidence of competency development for the following competency aspects: P.Q., →Sol., Int.El., Info, C.D.Rep., Man.Rep., and U.T. Respectively, Johannes provided similar evidence for the following competency aspects: →Sol., C.D.Rep., Man.Rep., Reason←. The following paragraphs are focused on the competency aspects mentioned above in the light of what was discussed in the relevant literature in Chapter 2 and Section 3.3 of Chapter 3 about competencies and this thesis competence framework.

Both students demonstrated strong evidence of development for the (→Sol.) competency aspect (*to attack and (take actions towards a solution) mathematical problems of various kinds.*) from the Mathematical Thinking and Acting competency. This is explained, to some extent, by the fact that this research dealt with problem-solving situations focused on non-routine mathematical tasks that asked students to be critical in their thinking. This agrees with the findings of Rott et al. (2015). They mention that the level of critical thinking in problem-solving situations grows alongside the sophistication of students' critical thinking and research on problem-solving. Furthermore, Schoenfeld (1985) points out that any action related to a step or a set of steps of the solution process indicates activation of the relevant problem-solving skill.

The activation of the (P.Q.) competency aspect (*to pose questions that have a mathematical characteristic or to know the kinds of answers that mathematics may offer*) from the Mathematical Thinking and Acting competency also displayed strong evidence of development through Erika's work. Introducing mathematical tasks within a biological context appeared to be new for the students. Erika (and Johannes, to a certain extent) reacted to this new learning context by asking and answering questions in a mathematical context. She could progressively pose such questions and understand the available answers to the relevant questions (see also subsections 5.6.1 and 6.1.3.1 on the relevant overview). For Niss and Højgaard (2011), these abilities mentioned above that Erika displayed belong within the realm of actions of a person who shows a progression of his/her *mathematical thinking* competency.

As seen in the two previous paragraphs, there was evidence of development for two aspects of the Mathematical Thinking and Acting competency: the (→Sol.) and the (P.Q.). It appears that the students developed their mathematical thinking while engaging in these non-routine mathematical activities. This dynamic characteristic of mathematical thinking is also highlighted by Rasmussen et al. (2005), who avoid describing the nature of advanced thinking as a final state. They claim that students develop ways of participating in mathematical activities in increasingly sophisticated ways. For that reason, they preferred using the term advanc*ing* rather than advanc*ed* because they want to address students' activity rather just than a final state. This development is seen in the competency development profiles of Erika and Johannes (Table 6-54 and Table 6-55) that include the final inferences regarding the progress of competency development over time for each of the five competencies of the competence framework. In these tables, it is seen that there was moderate or strong evidence of development for the Mathematical Thinking and Acting competency.

As mentioned in subsection 3.1.2, working with non-routine mathematical tasks with first-year biology students presented an excellent opportunity to explore the development of Mathematical Modeling competency as one separate and distinctive mathematical ability. Regarding the aspects of this competency, the students provided strong or moderate evidence development for the (Int.El.) (*to interpret and translate the elements of a model during the mapping process*) and the (Info) (*to look and search for available information* *and to differentiate between relevant and irrelevant information*) competency aspects. The mapping process of the modeling scheme (Maaβ, 2006,) as illustrated in Figure 3.1 and discussed in subsection 3.1.2, was observed multiple times in Erika (subsection 5.6.1) and Johannes's (Appendix D) work. This repetition of the mapping procedure resulted in numerous occasions where the students performed interpretations and translations (activation of the(Int.El.) aspect) of the elements of the models (e.g., the continuous model with r as the exponential per capita rate of growth: $\frac{dN}{dt} = rN(t)$ during the Population Dynamics sessions) in the study.

It also appeared that the context and content of these non-routine mathematical tasks (e.g., subsection 5.6.2 activation of the (Info) aspect) mobilized the students to look for any available information that would be relevant to the task they had to address. This observation agrees with studies (Wathall, 2016; Öztürk et al., 2020), who observe that working with non-routine mathematical tasks requires specific mathematical skills, including the ability to search and analyze information pertinent to the task content successfully. Lastly, in contrast to what was observed for the development of the Mathematical Thinking and Acting competency, where both students displayed strong or moderate evidence of development and looking again at Tables 6-54Table 6-54 and 6-55, it is seen that Erika provided relatively stronger evidence of development than Johannes.

Regarding the Representing and Manipulating symbolic forms competency, the findings suggest this was the competency that the students displayed the stronger evidence of development among all five competencies of the competence framework. This finding agrees with Duval's (2006) position (subsection 3.1.3) that any mathematical activity involves the transformation of representations which he categorized in two types: *treatments* and *conversions*. Erika and Johannes transformed representations within the same register (treatment) and when the register changed (conversion). Both (C.D. Rep.) (*choose or decode in, a representation*) and (Man.Rep.) (*manipulate within representation*) competency are seen to display (Table 6-52 and Table 6-53) mainly strong evidence of development at all three dimensions of the 3-D model. Both students engaged in the task-solving process by attempting to produce answers while manipulating and decoding symbols on paper, a tendency that several studies (e.g., Carpenter et al., 1988; Hiebert & Wearne, 1986) observe.

When engaging with the assigned tasks, Erika and Johannes often dealt with several representations, asking them to consider tables and diagrams that displayed temperatures, population rates, and other kinds of information. It is what Boesen et al. (2014) observe when they highlighted (subsection 3.1.3) that the representation ability is brought into action every time a student is about to take the meaning of a representation (e.g., table, diagram) into consideration.

The way Erika and Johannes manipulated and connected the symbols and representations they encountered resembled what Boesen et al. (2018, p. 112) described as *connection competency*. Both students were observed to connect mathematical entities or their representations, suggesting a manifestation of some form of conceptual understanding of a specific mathematical notion or entity under study.

Erika and Johannes's work provided strong or moderate evidence of development of the Mathematical Reasoning and Communicating competency. The task design and the adopted teaching method offered opportunities for recurrent interaction between the teacher and the groups but also among the members of the group itself. The literature (e.g., Lithner, 2003; Bergqvist's 2007; Boesen et al., 2010; Palm et al., 2011) includes studies where researchers attempt to determine the kind of mathematical reasoning required in order to solve tasks included in teacher-made tests and national tests.

The present work shows that both forms of mathematical reasoning of the competence framework displayed strong or moderate evidence of competency development through Erika and Johannes's work. Both (Reason→) (*to express one's reasoning*) and (Reason←) (*to understand the reasoning of others*) competency aspects are deemed necessary mathematical skills for a first-year biology student when engaging in non-routine mathematical activities and tasks. This observed development of their reasoning skills verifies what Öztürk et al. (2020) claim about enhancing the individual's reasoning skills when working with these kinds of mathematical problems.

This thesis considers that students' engagement with the solving process can provide valuable information for exploring of the reasoning sequence. Evidence from literature (e.g., Lithner, 2003; Bergqvist, 2007; Boesen et al., 2014) concerning mathematical reasoning and problem-solving situations agrees with the latter. In particular, Lithner (2003), analyzing a sequence of reasoning based on a set of data (text, pictures, video recordings), states that it is difficult to identify the actual reasoning that took place in someone's mind. To address this difficulty, he considers that one can analyze and classify a task-solving procedure in several ways depending on the data indicating a form of mathematical reasoning. Similar to Lithner (2003), the data provided by Erika and Johannes (subsections 5.6.1 and 5.6.2 on the relevant competency) while activating the Mathematical Reasoning and Communicating competency gave insight into the reasoning sequence they followed that proved to be necessary for the evaluation of the quality of activation that resulted in the assessment of the competency development over time.

Competency aspects with weak or negligible evidence of development

From the same tables above, inferences can be drawn about these competency aspects that Erika and Johannes provided weak or negligible evidence of development. These aspects were correspondingly the following: (Erika) Lim., Graph, R.+V., and Ext.T., and (Johannes) Cr., Rep.↔Rep., Graph, R.+V., and Ext.T. The following discussion concerns these aspects that both Erika and Johannes provided weak or negligible evidence of activation and development.

A general comment can be made regarding the activation of the (Ext.T.) (*to know the existence and properties of various tools and aids for mathematical activity*) competency aspect. The results indicated that no student appeared to activate this competency whatsoever. Reflecting on this finding, I consider it necessary to comment on the description of this competency aspect. The (Ext.T.) description from the competence framework suggests that to activate this competency, the student does not necessarily have to use a specific tool but, instead, must mention that he/she is aware of its existence. From the students' perspective, it proved unfeasible to mention a tool that they could use for tackling the task without actually using it. The latter could happen if the teacher posed a relevant question during the activity, e.g., "what kind of digital tool could you use in this situation to address the task?" The design of all activities facilitated using several tools and aids (e.g., calculators, rulers, and the internet) to be brought into action during the solving process. In conclusion, regardless of the reasons that could justify the issue discussed in this paragraph, I can claim that the (Ext.T.) competency aspect is not included in the competence profile of students' activated competency aspects.

Similar conclusions can be drawn for the (R.+V.) (*to be able to assess the range and validity of existing models*) competency aspect because there was negligible evidence combination for its activation and consequently for its development. Therefore, this competency aspect is considered non-activated for both groups of students. This observation is not entirely unexpected because most of the tasks were designed to facilitate using specific models relevant to the mathematical concepts introduced in the sessions. However, in the Periodic Functions and Population Dynamics sessions, the students had the opportunity to choose between several models. Eventually, they chose not to deviate from the basic model that had been described. For example, during the Periodic Functions, the students were asked to provide an equation that models a creature's temperature in a day. They could choose between $y = Acos[B(x \pm C)] + D$ or $y = Asin[B(x \pm C)] + D$. However, they insisted on using the first in all cases because that was the function we used for the 'warm-up' task. However, I should mention that the two functions mentioned before may not be considered two different models since they differ only in translation. That being said, the students did not indicate any intention to use different kinds of periodic functions.

Weak or negligible evidence of activation (and subsequently development) was observed for the (Graph.) (*to choose appropriate mathematical notations and to represent situations graphically*) competency aspect for Erika and Johannes. It appeared that the students felt unfamiliar with the use of graphs as a task-solving approach. Apart from the Integrals & Modeling unit, all the other sessions included tasks that could be approached using graphs to some extent.

Competency profiles: evidence of progression

In this final part of this subsection, I wish to venture some conclusive remarks regarding the individual competency assessment for Erika and Johannes. In particular, deriving information from Table 6-54 and Table 6-55, it is possible to make claims of competency progression for each student. The term 'progression' here is used differently than when discussing the progress of competency development over time which ascribes a dynamic characteristic to the notion of competency development. Progression constitutes what Niss and Højgaard (2011, p. 30) described as "individual's mastery of mathematics" and comprises the "growth in mathematical competence" as seen through the progress (growth) within the three dimensions of the 3-D model of the same authors.

Considering the above and consulting Tables 6-54 and 6-55, Erika displayed evidence of progression for the following competencies: Representing and Manipulating symbolic forms; Mathematical Reasoning and Communicating; Mathematical Thinking and Acting; and Mathematical Modeling. For Johannes, progression was observed for the following competencies: Representing and Manipulating symbolic forms; Mathematical Reasoning and Communicating. Even though it is beyond the scope of this research, it appears that it is also possible to compare two students' individual competency development over time. This thesis could testify that if that was the case, Erika appeared to develop more mathematical competencies than her classmate Johannes.

Further, in subsections 6.1.3 and 6.1.4, the competency development for Erika and Johannes was explored through all three dimensions of the 3-D model of competency progression. Interestingly, reflecting on what was observed during the exploration of competency development through the Radius of Action of each competency, it is feasible to provide an additional observation regarding the findings. As seen in the subsections mentioned above, to make inferences regarding the competency development with the Radius of Action, it was necessary to examine the quality of activations by each calculus unit: Periodic Functions, Exponential Growth & Regression, Population Dynamics, and Integrals & Modeling. It can be observed that the strongest evidence of competency development was provided during the Population Dynamics sessions.

7.2 Addressing the Second Research Question

This section comprises three parts. Subsection 7.2.1 offers a brief discussion on the notion of *competency interrelation*. Following that, subsection 7.2.2 discusses the findings of the second Research Question by reflecting on these competency pairs that displayed 'interesting' frequencies of concurrent activation. Lastly, subsection 7.2.3 discusses the rationale for choosing a statistical test in a qualitative study and reflects on some undeniably existing limitations.

7.2.1 Competency interrelation and the competence framework

The conceptualization of the second Research Question originated while constructing the methodological tools of analysis that would be used to address the first Research Question and the notion of competency development. From the first attempts to identify within students' activity which competencies were activated, it was apparent that more than one competency aspect could be activated simultaneously, that is, within the same student's action, such as an utterance or gesture. This phenomenon was described as *concurrent activation*, and the two competency aspects that appeared to be activated concurrently formed a *competency pair*. This study found many instances of concurrent activation resulted in the formation of several competency pairs. There were 155 recorded instances of concurrent activation that formed 47 competency pairs.

The definitions adopted to describe each of the competency aspects of the framework may overlap on some occasions. However, imprecision and overlapping within competency descriptions cannot be avoided (Boesen et al., 2014), and overlapping was correctly expected and observed. The observation of the concurrent activation phenomenon confirms what the relevant literature predicted regarding the possibility of overlapping between the competencies when brought into action by the students. Niss (2003, p. 9) points out that this should not be seen as a malfunction of the KOM model because "[t]he competencies are closely related - they form a continuum of overlapping clusters - yet they are distinct in the sense that their centres of gravity are clearly delineated and disjoint." The observation of concurrent activations and the fact that the fashion in which these competencies overlapped had yet to be explored resulted in formatting the second Research Question.

The findings presented in subsection 6.2.2 display 12 competency pairs that recorded statistically significant concurrent activation frequencies, higher or lower than expected. Looking back at Section 3.3, where the construction of this thesis competence framework took place, the main concern was to build an analytical framework where each competency aspect would be as distinct as possible. The fact that there could have been multiple possible competency pairs between these 16 competency aspects, but only two were tracked to demonstrate a concurrent activation with a higher-than-expected frequency suggests specific considerations about the functionality of the competence framework. It appears that the competency aspects were described in such a way that facilitated the coding process (i.e., identifying which competency aspect is activated) and reduced the phenomenon of concurrent activations to an expected frequency. The description given to each competency aspect allowed a functional coding process, and very few aspects may have entailed descriptions that were eventually similar to some extent. The following subsection discusses these pairs where 'interesting' frequencies of concurrent activation were recorded.

7.2.2 The 'interesting' competency pairs

Several assumptions can be made regarding the findings from subsection 6.2.2 concerning these competency pairs that recorded a higher or lower-than-expected frequency of concurrent activation. These assumptions can shed light on the fashion in which competency aspects of the competence framework overlap. Regarding these competency aspects that appeared to be activated together more often than (statistically) expected, it was found that the (Int.Res.X & Cr.), and (C.D.Rep. & Rep.↔Rep.) pairs recorded a higher-than-expected frequency of appearance (i.e., concurrent activation). Notably, the first pair concerns mainly Erika's work because Johannes provided weak evidence of activation for the particular competency aspect.

It appears that when the students attempted to interpret the mathematical results in an extra-mathematical context or when they attempted to generalize the solutions developed for a particular task situation (activation of the Int.Res.X. competency aspect), they, at the same time, often offered their critique for that model by reviewing or reflecting and questioning these results (activation of the Cr. competency aspect). Both competency aspects belong to the Mathematical Modeling competency. The concurrent activation of these two aspects could be attributed to the position advocated by several studies (e.g., Sjuts 2003; Schoenfeld, 1992) who highlight a connection between solving complex mathematical problems and the significance of metacognition within this process where a person may need to interpret and reflect on their thinking.

The other two competency aspects that recorded a higher-than-expected frequency of appearance were the (C.D.Rep. & Rep.↔Rep.) pair that again belongs to the same competency of the framework, the Representing and Manipulating symbolic forms competency. Students appeared to choose or decode representations simultaneously when switching between them to solve the assigned tasks. As stated in subsection 3.1.3, the Representing and Manipulating symbolic forms competency was an amalgamation of two competencies of the KOM model (i.e., *representing mathematical entities competency* and *handling mathematical symbols and formalisms competency*), and this could be one reason for observing the frequent concurrent activation of these competency aspects. Entailing characteristics from both competencies of the KOM model (Niss & Højgaard, 2011) could result in some similarities concerning their descriptions. Consequently, evidence of activation for the (C.D.Rep.) and (Rep.↔Rep.) aspects could be derived from the same student's utterance of action in the class.

There were 10 'interesting' competency pairs (subsection 6.2.2) that recorded a less-than-expected frequency of concurrent activation. These pairs were the following: (Info & Man.Rep.), (Int.El. & Man.Rep.), (→Sol. & Reason←), (Reason→ & Man.Rep.), (C.D.Rep. & U.T.), (→Sol. & Info), (Int.El. & Info), (Reason→ & U.T.), (Int.El. & Cr.), and (Int.El. & U.T.). What is suggested from the above is that the activation of one of these competency aspects may hinder the concurrent activation of the other and that these competency pairs appear to overlap less often than expected. For example, the (→Sol. & Reason←) pair suggests that it is less likely to observe a student taking actions towards a solution (attacking the problem) while at the same time following a chain of arguments put forward by others.

Similar descriptions could be produced for the other pairs mentioned above; I have not done this as one example provides a template for the other descriptions. Interestingly, the (Int.El.) and (Man.Rep.) competency aspects were present in four and three of these pairs. The latter suggests that these two competency aspects tend to hinder the activation of more aspects than other competency aspects did. The following and last subsection of this section will discuss the rationale for selecting a statistical test for addressing the second Research Question. Furthermore, limitations that these findings comprise will be highlighted and discussed.

7.2.3 Choosing a quantitative approach: The Binomial Test

By the time the data collection process was completed, it was evident that the transcribed work from a significant number of students would be available. Even when narrowing the focus to two students (i.e., Erika and Johannes), there were still more than 300 transcribed utterances (subsection 5.7.1) from the students' activities in the class. Within these utterances, evidence of competency activations was found. As highlighted in subsection 7.2.1, it was also evident that from the same utterance, there could be evidence that would signify the activation of more than one competency (i.e., competency aspects from the framework). Having a fixed number of transcribed utterances (341) and a fixed number of observed competency activations (including the concurrent ones) resulted in the assumption that a statistical significance test could shed light on whether there is any deviation between the expected and the observed concurrent activation. For that reason, the Binomial Test was selected.

The Binomial Test is employed when an experiment has two possible outcomes (i.e., success/failure), and there is an idea about the probability of success. A Binomial Test is run to see if observed test results differ from what was expected. The fact that the data could entail both expected and observed outcomes rendered the Binomial Test appropriate as a statistical approach. This approach could offer a more accurate idea about how the competencies overlap, as seen in the previous subsection. For that reason, the null hypothesisH_0, used for this analysis, is that the results do not differ significantly from what is expected, that is the observed concurrent activations do not differ significantly, in terms of frequency of appearance, from the expected. However, it is indisputable that certain limitations exist when such an approach is taken to understand a competence framework.

Approaching the second Research Question with a statistical method entails several limitations and generalizations (e.g., Matthews et al., 2017). The analysis (use of binomial distribution) may enshroud the substantive issues with a cloud of statistical/mathematical complexity. The 10 competency pairs that recorded a less-than-expected frequency of concurrent activation cannot provide information regarding which competency aspect hindered the concurrent activation of another one. For example, the fact that the (Reason→ & U.T.) competency pair belongs in these 10 pairs mentioned above does not provide information on whether the (Reason→) might have hindered the activation of the (U.T.) aspect (or the other way around). What is likely to be observed is that these two competency aspects appear to activate simultaneously less often than expected.

Similarly, the two competency pairs that recorded a higher-than-expected frequency of concurrent activation (Int.Res.X & Cr.) and (C.D.Rep. & Rep.↔Rep.) cannot provide information on which competency activation triggered the activation of the other aspect. What is possible to be observed is that these competency aspects appear to be activated simultaneously more often than expected. Further considerations will be discussed in the final chapter of this thesis, where the limitations of this research are presented.

7.3 Competency development and non-routine mathematical problems

In this last section of this chapter, I comment on some indications regarding incidents that occurred in the classroom where students engaged with non-routine mathematical tasks. While in routine tasks, the student has encountered them before and has a ready algorithm to apply for solving them; when students encounter non-routine mathematical tasks, they have to create an original solution method to solve the task (Jäder et al., 2017). These indications concern the possible relationship between the classroom environment (i.e., a biology classroom working on non-routine mathematical tasks) and the development and activation of mathematical competencies, and they are generated by going over and reviewing the provisional results from Section 5.3 and the findings regarding the competency development from Chapter 6. It should be noted that the following discussion will occur in a purely speculative fashion (hence indications) because the relationship between the particular classroom environment and competency development was not the focus of this thesis.

I first draw attention to the Mathematical Reasoning and Communicating competency, where the students provided strong (mainly) evidence of development as discussed in subsection 7.1.3. Several assumptions can be made regarding this finding. The observed development could be partly attributed to the teaching approach that has been followed. The participants were encouraged to work in groups while engaging with the tasks. They were working in continuous interaction and communication with the teacher and the other members of the groups. This continuous interaction, I hold, set in motion the students to provide justifications for their thoughts and decisions concerning the strategy they would follow to address the questions raised from attempting to address the non-routine mathematical tasks. Using probing questions (addressed by the teacher) when necessary, the students had to provide valid reasoning to persuade both the teacher and the other students in the group. Using probing questions in a teaching process plays a significant role in the student's construction and understanding of mathematical ideas (Hähkiöniemi, 2017; Hufferd-Ackles et al., 2004). The latter may suggest a connection between incorporating probing questions in the teaching process and developing mathematical reasoning competency that may lead to constructing these mathematical ideas and concepts.

In subsection 3.1.4, evidence from the literature (Lithner, 2003; Bergqvist, 2007) highlighted the connection between types of mathematical reasoning, task solving situations, and types of mathematical tasks. The teaching approach adopted for this study (teacher-student and student-student interaction) appeared to be an appropriate path to follow to facilitate and exploit the unique characteristics (e.g., real-world mathematics) of selected non-routine mathematical tasks of this thesis. By focusing on students' solving processes, one can obtain insight into how the student communicates mathematically and engages in and with mathematical reasoning. The indication here is that the observed development of students' mathematical reasoning competency can be affected by the nature of the tasks they have engaged in during these seven calculus sessions.

Following the comments above, the observed development of the Mathematical Reasoning and Communicating competency can be partly attributed also to the observation that a broader range of reasoning is used when students work on non-routine tasks (Boesen et al., 2010). As seen in subsection 3.3.4, there are several studies (e.g., Öztürk et al., 2020; Jäder et al., 2017) suggesting that working with non-routine mathematical problems offers opportunities for the students to enhance their mathematical reasoning skills and that these skills could be dependent on the type of task they encounter. According to the Swedish Schools Inspectorate (2010), opportunities for the advancement of mathematical reasoning and problem-solving skills can be provided by tasks that deviate from the textbook problems that rely heavily on more procedural solving processes. The following paragraph concerns the development of Mathematical Thinking and Acting competency.

As discussed in subsection 7.1.3, the students provided mainly moderate (mainly) evidence of development for the Mathematical Thinking and Acting competency. That being said, I wish to express my reluctance to attribute an association between the non-routine tasks' environment and the nature of the development of this particular competency. This hesitancy is driven by the assumption that the same competencies could display similar development within a different teaching context and not only when students are engaged in solving non-routine mathematical tasks. For example, there was strong evidence of development for both students' (→Sol.) competency aspect and within all three dimensions of the 3-D model. However, this competency aspect could be activated (i.e., students can provide evidence of activation) in any task-solving situation regardless of the nature of the chosen teaching activity.

It is also hypothetical that the strong and moderate evidence that Erika and Johannes provided regarding the development of the (P.Q.) competency aspect could be attributed to the fact that the students had to work on non-routine mathematical tasks. Engaging with tasks adapted within a biological context and beyond that routine textbook material may have given opportunities to the students to pose questions related to the solution of the tasks. The following paragraph discusses the findings for aspects of the Mathematical Modeling competency.

As seen in subsections 6.1.5 and 7.1.3, the students provided strong or moderate evidence development for the (Int.El.) and (Info) competency aspects of the Mathematical Modeling competency. Although there is no suggestion that the students were working on modeling tasks or that the sessions concerned the teaching of mathematical modeling, examining the development of the Mathematical Modeling competency provides an opportunity to discuss some similarities between the solving process of non-routine mathematical tasks and modeling activities. The mapping process of modeling connects the extra-mathematical world to a (or a set of) mathematical domains and encapsulates several sub-processes. Such connection also exists within the non-routine mathematical problems because they are related to real-life situations (Hıdıroğlu et al., 2017) and require the student (in our case) real-world knowledge and some realistic considerations to resolve them.

As discussed in subsection 3.1.2, Maaβ (2006) describes some key sub-processes (simplifying, mathematizing, working within mathematics, interpreting, and validating) that occur within the realm of mathematical modeling. Similar processes could potentially be observed when students work on non-routine mathematical tasks. However, the rigidity applied to describe these sub-processes within the mathematical modeling context is not the same when the discussion concerns these processes for non-routine mathematical tasks. The similarity may be debatable, but it is worth mentioning because it may partially explain or provide implications for the recorded evidence of development that concerns some aspects of the Mathematical Modeling competency. For example, the interpretation and mathematizing processes may provide an explanation for the recorded evidence of the development of the (Int.El.) aspect of the Mathematical Modeling competency.

In conclusion, in this section, there is no suggestion that introducing non-routine mathematical problems in STEM studies signifies a strong connection with the recorded development of several competencies. A correlation between the teaching of non-routine mathematical tasks in STEM fields and the development of mathematical competencies is beyond the scope of this thesis. What can be derived from the abovementioned indications is that students were motivated to work on tasks where the content was related to the focus of their study. Erika, Johannes, and the other students show eagerness to work on the tasks related to their Biology studies for seven sessions, giving opportunities to the author of this thesis to observe the activation of several mathematical competencies within a range of different quality of activation as discussed on multiple occasions before.

8. Conclusions

This chapter opens with an overview of the thesis' key findings in Section 8.1, which leads to Section 8.2, where a discussion of the contributions and implications of this thesis in research takes place. Following that, in Section 8.3, I discuss the limitations and obstacles of this research, and I conclude this final chapter with Section 8.4, where I suggest further research steps that can be taken for the study of competency development and assessment.

8.1 Key findings of this research

This thesis aimed to explore the progress of competency development over time for students who participated in several calculus sessions where they were asked to work on non-routine mathematical tasks. This research also investigated how competencies of a competence framework are interrelated by exploring how the students activate them during these calculus sessions. Employing a competence framework mainly built on an already existing one, namely the KOM competence framework (Niss & Højgaard, 2011) and constructing an assessment instrument that explored the quality of competency activation, I set the following two research questions accompanied with their sub-questions:

1) What is the progress of individual competency development over time for a student who participates in a series of calculus sessions that comprise non-routine mathematical tasks?

a. What set of competencies would be suitable to constitute the competence framework of this research?

b. How can we utilize the 3-D model to explore the progress of individual competency development over time?

c. What is the progress of individual competency development over time through each dimension of the 3-D model that will eventually form a competency development profile for each student?

d. What competencies from the competency development profile of each student display evidence of progression?

2) Are the competencies interrelated and, if so, how?

a. Can we create a table where all frequencies of concurrent competency activation are displayed for all possible pairs?

b. Is there a quantitative approach that could be adopted to track any statistically significant observations regarding the observed and expected frequencies of concurrently activated competency pairs?

While the thesis' findings have been discussed in Chapter 7, I briefly summarize, in this section, the key findings that emerged in addressing the two research questions (and their sub-questions). In what follows, I present these key findings and their derivatives in bullets and sub-bullets. Wherever is possible, I will accompany every bullet with a particular reference to the chapters where each finding can be located. I will, nevertheless, direct the reader to a section or subsection of the thesis. The key findings were the following:

- Both students provided evidence of competency development (subsections 6.1.5 and 7.1.4) throughout the sessions for some competencies of the competence framework. Erika displayed evidence of progression for the following competencies: Representing and Manipulating symbolic forms; Mathematical Reasoning and Communicating; Mathematical Thinking and Acting; and Mathematical Modeling. Johannes, progression was observed for the following competencies: Representing and Manipulating symbolic forms and Mathematical Reasoning and Communicating. These findings agree with the position of the (NCSM) and (NCTM) councils (2018) in the fields of STEM studies (see Section 3.3), which highlight the necessity for strong skills regarding mathematical and scientific thinking, reasoning, and modeling across the disciplines of the STEM fields.
    - Even though it is beyond the scope of this research, it appears that it is also possible to compare two students' individual competency development over time. This thesis could testify that Erika appeared to develop more mathematical competencies than her classmate Johannes if that was the case.
    - The students provided the strongest evidence of competency development during the Population Dynamics sessions. It appears that this is the mathematical topic where the students displayed an understanding of the notions that we were studying. The latter agrees with Ekici and Plyley's (2019) study on inquiry-based modeling of Population Dynamics. In their study, they conclude that when students work with models from both discrete and continuous perspectives, and by using multiple representations, "…students can see their solutions clearly from different contexts and develop deeper understandings." (Ekici & Plyley, p. 14).
    - Among all calculus units, both Erika and Johannes provided the weakest evidence of competency development when they entered the Integrals & Modeling unit. However, we should consider the fact that the project took place at the beginning of the semester. Many students were unfamiliar with the concept of integral and integration rules even though this was included in their syllabus and had attended some courses already.
- There were also some competency aspects for which the students provided weak or negligible evidence of development (subsections 6.1.5 and 7.1.4). These aspects were correspondingly the following: (Erika) Lim., Graph, R.+V., and Ext.T., and (Johannes) Cr., Rep.↔Rep., Graph, R.+V., and Ext.T.

- The way the KOM model (Niss & Højgaard, 2011) was customized for constructing this thesis's competence framework by merging or leaving out specific mathematical competencies (i.e., aspects of the KOM competencies) facilitated the endeavor to explore competency activations. Furthermore, it set the base for exploring the notion of competency development.
- All three dimensions of the Niss and Højgaard (2011) 3-D model proved to be suitable tools for the individual assessment of the student's competency development over time. However, this study showed that the Degree of Coverage dimension needs to be further explored for any endeavor to analyze individual competency development.
    - To facilitate the exploration and analysis of competency development, the present study suggests that it is necessary to explore the quality of competency activation over time. For that purpose, this study suggests the employment of a particular scaling instrument (Section 4.1) that could assess the quality of each competency activation using three levels of assessment through three domains (i.e., dimensions) of activation.
- This study found (subsections 6.2.2 and 7.2.2) that few competency aspects appeared to be activated concurrently more often than (statistically) expected. The (Int.Res.X & Cr.), and (C.D.Rep. & Rep.↔Rep.) pairs recorded a higher-than-expected frequency of appearance (i.e., concurrent activation). Several pairs (the exact pairs can be found in subsection 7.2.2) recorded a less-than-expected frequency of concurrent activation. This study claims that by identifying these competency pairs, it is feasible to describe how a competence framework functions regarding the fashion in which the competencies overlap, a phenomenon that is expected to happen (Niss, 2003) but has not been yet described outside of this thesis.
    - This study found that employing a statistical method (i.e., The Binomial Test) to explore how competency aspects of a competence framework are interrelated can give promising results. However, certain limitations should be taken into consideration.

This research was conducted within a classroom environment that has specific characteristics, that is, a classroom of first-year students in a Biology department who were willing to participate in a project where they would work on their mathematics by engaging in solving non-routine mathematical tasks that addressed four areas: Periodic Functions, Exponential Growth & Regression, Population Dynamics, and Integrals & Modeling. Focusing meticulously on two students of this class, inferences regarding their competency development were derived. Despite this specific research setting, contributions and implications to various research areas can be made. These contributions and implications are discussed next.

8.2 Contributions and implications of this research

This research aims to contribute and add to existing knowledge in research areas related to i) the study of mathematical competencies, ii) the exploration of students' competency development in STEM fields, and iii) the understanding of the functionality of competence frameworks. This study also comprises implications for new student assessment methods where the teacher can obtain a more detailed picture of each student's performance by creating a competence profile where the weaknesses and strengths of the students are on full display. Furthermore, the methodological and data analysis tools and methods designed and employed in this thesis add to existing knowledge regarding methodologies for exploring the development of mathematical competencies. The main elements of this research, identified as potential contributions, are discussed next.

Several studies advocate that the possession and mastery of mathematical competencies are closely related to acquiring knowledge (Niss & Højgaard, 2011). More specifically, mathematical knowledge is seen as a set of various mathematical qualities, and the assumption is that maintaining and developing them at a certain level could mean that some mathematical knowledge has been acquired. Hartig et al. (2008) describe competence as a complex ability closely related to performance in real-life situations. It is what Niss and Højgaard (2011) point out when they argue about what mathematical competence embraces. They claim that being competent in mathematics suggests having the knowledge of something and understanding, doing, and using mathematics, or having an opinion about a particular mathematical activity in several contexts where mathematics plays or can play a role. To that view, this study suggests that mathematical competence should be seen as a notion comprising dynamic characteristics that change over time. I argue that the contribution of this research is that we should refer to mathematical competencies as a set of mathematical skills susceptible to changes and that this is a suitable way to conceptualize this notion should one attempt to study any form of competency development over time.

This thesis provides an account of competence profiles of the students and argues that when there is a strong or a moderate combination of evidence of competency development, that would suggest progression within this competency or competencies. This combination of evidence comprises the frequency of competency activation, the evaluation of the quality of this activation, and exploring the dynamics of this quality as time passes during an academic semester. The contribution of this observation is that competency progression suggests acquiring new knowledge and that students who display such progression acquire new knowledge in a specific mathematical domain or domains. Also, the possession of new knowledge is not connected exclusively with a specific mathematical field, e.g., Population Dynamics or Periodic Functions. This research proposes that the possession and development of a particular mathematical competency or set of competencies is knowledge itself. Simply put, knowing how to do and learn is knowledge by itself.

Another contribution of this study is facilitating the exploration of mathematical competencies in the light of student assessment. This research attempted to contribute to the students' evaluation by suggesting the construction and investigation of an individual competence profile for each student. Such profiles can be considered an additional tool in the already existent 'toolbox' of educational systems' assessment methods. The suggested way of assessment could be seen as an attempt to answer the following question: *what are the areas where the student is more competent?* There is a dualistic approach to the term 'areas' since I intend to include the realm of mathematical fields on the one hand and the set of mathematical competencies on the other. For example, a student can be competent in his/her mathematical reasoning and mathematical thinking or be competent when working on Population Dynamics or Exponential Growth & Regression or even Algebra and Geometry. Apart from the comparative nature of any evaluation attempt, I consider this research contribution a suggestion to avoid rough comparisons among students and offer a fine-grained competence profile for each student. With that being said, it is not within this study's intentions to denounce that there are students that are more competent than others regarding the quality of mathematical skills (set of mathematical competencies) they possess.

This study also contributes to the understanding of competence frameworks by employing one that was mainly built on the already existing KOM model (Niss & Højgaard, 2011, 2019). This research suggests that the KOM model can be refined and customized depending on the field of study and the nature of mathematical tasks the students will have to explore. In this thesis's case, the field of study was biology, and the problems the students were asked to work on were non-routine mathematical tasks within the context of biology. There were competencies within the KOM model not incorporated into the competence framework of this dissertation because they did not appear to be relevant to the study's aims and primarily to what the students of a Biology department would need to develop. For example, the competency (subsection 2.2.2, the KOM framework) under the description "*distinguishing between different kinds of mathematical statements (including conditioned assertions ('if-then'), quantifier laden statements, assumptions, definitions, theorems, conjectures, cases)*" was not included in the final form of the framework. This competency belongs to the thinking mathematically competency of the KOM model. However, the tasks in the sessions did not entail this form of mathematical knowledge as a prerequisite.

Moreover, the Biology department students were more interested in the relevance of the tasks to the biology world; therefore, conditional assertions in mathematical statements, theorems, and conjectures were outside their study interest. It appears that the students focused more on applying formulas rather than the mathematics behind them. The latter assumption agrees with the research (e.g., NCTM, 2014; Hiebert, 1999), which suggests that once students have memorized and practiced procedures, they fail to understand, they have less motivation to understand their meaning or the reasoning behind them. In conclusion, this research contributes to the body of research that considers the employment of competence frameworks essential for exploring competency development.

An additional contribution to the understanding of competence frameworks concerns how competencies overlap, a phenomenon described by Niss (2003) when defining the KOM competence framework and illustrating it in a 'flower shape' as seen in Figure 2.1. However, the fashion in which these competencies are interrelated is an area that has not been explored yet. This study found that some competencies tend to be brought into action simultaneously more frequently than others within a specific classroom context by applying a statistical method. It was also found, though, that there were competencies that their activation hindered the activation of another. The classroom context mentioned above entails several features such as the number of students, area of study, teaching approach, and type of mathematical task the students were working on and exploring. The implication is that a combination of mathematical competencies may be necessary for the student to activate them to address a mathematical task. Therefore, a set of developed competencies appears to be vital for students who engage with mathematical (non-routine in our case) tasks in any STEM field. The latter is hardly an original observation because a set of mathematical competencies is a valuable tool for any student (as discussed in Chapter 2). However, this study implies that this set of mathematical competencies may feature different competencies depending on the study area and the mathematical content taught.

A methodological contribution of this study concerns the suggestion of employing a scaling instrument to facilitate exploring how competencies are developed over time. Understanding mathematical competencies as a notion with dynamic characteristics was the main reason for the suggestion above. This instrument supported and promoted the functionality of the 3-D model (Niss & Højgaard, 2011), especially while studying the competence activation and development of the Degree of Coverage (Section 4.4). In particular, it appears that the 3-D model was a suitable tool for the study of competency development with the help of the scaling instrument introduced in this study. This research recommended a refinement for describing one of the model's dimensions (Degree of Coverage) by introducing the notion of 'quality of competency activation.' This refinement eventually eliminated difficulties during the exploration of "the extent to which the person masters the characteristic aspects of the competence at issue as indicated in the […] characterisation of it" as Niss (2003, p. 10) described this dimension. The following paragraph discusses the Degree of Coverage of the 3-D model.

While striving to explore students' use of symbols and formalisms (Representing and Manipulating symbolic forms competency), I questioned whether they were merely 'playing' with symbolic forms or reached a conceptual understanding of them. Studying their discourse activities, I acknowledge similarities with the work of Viirman and Nardi (2019). They aim to contribute to a stronger operationalization of the commognitive constructs of ritual and exploration (Lavie et al., 2019). It implies that some possible refinement is somehow necessary -for contextual reasons- before applying or using these frameworks for formal student assessment.

Continuing from a methodological perspective, and in particular, from a data collection view, this thesis suggests ways of monitoring the classroom's activities. Three sources of interaction have been considered essential to be monitored. The first source was the group of students whom I planned to analyze. For that reason, a camera and an audio recording device were used to capture all students' actions of the specific group. The second source was the written work of the students. The students were provided with LiveScribe pens and notebooks to obtain all the data from their written work. These dynamic tools supported this study by gathering the students' written work and instantly uploading them on a different device in PDF form. The third interaction source was the other groups of students and the teacher. For that reason, a GoPro camera was placed to capture the students' actions who did not belong to the focus group and the teacher's actions when presenting the material or writing something on the whiteboard. The coordination of these sources was essential for this thesis's data generation and analysis.

Many assumptions can be made about the conditions that prevail when a student advances from a particular competency activation level (i.e., I, II, or III) of the scaling instrument to a higher or lower one. These assumptions were considered "weaknesses" when Siller et al. (2015, p. 2717) discussed about competency level models and that it "remains undetermined how a change to the next level can be accomplished and what conditions are necessary for this." However, this thesis aimed to study and determine if competency development occurred rather than exploring what conditions facilitated this development. I can only assume that providing students with mathematical tasks linked to their everyday lives could constitute a challenging educational environment that motivates them to engage actively with these kind of mathematical problems. By 'everyday life,' I refer to problem-solving situations and activities that are likely for the students to face during their studies in a department of Biology. Such activities can be found that are of high relevance to the characteristics of non-routine mathematical tasks.

As I will mention in Section 8.4, larger-scale research regarding time (number of semesters) and student participation could produce more solid indications and conclusions. With more activations' evidence of the same competency from more students and more than seven sessions, I could obtain stronger indications that identify activation and development patterns. In Chapter 4, I had to conduct a detailed analysis for every competency domain (T.S.V., M.L.V., and Prompt) to justify my results. Being aware of the patterns I mentioned above, I could facilitate the acceleration of any combination of evidence (i.e., frequency of competency activation, changes between levels of activation, and consistency within a level of activation) and therefore produce reliable results with less effort in terms of time and writing volume. The more reliable the scaling instrument may become; the less time and workload are needed to draw reliable conclusions.

8.3 Limitations of this research

This research has several limitations to bring forward and discuss in this section. I can only assume that some boundaries are expected to be found in -to some extent- any similar study.

Firstly, although the project aimed to follow many participants, the fine-grained analysis was applied to two students. The apparent reason for this decision was that many students gradually dropped out of this project, and only three students came to more than six sessions. However, this limitation was addressed by the methodological approach applied in this study. The way the analysis was performed and the analytical tools employed to explore the students' competency development would be the same regardless of the number of participants if there were no limitations on the number of students and time restrictions.

Section 5.3 offers an extensive report on the provisional results of exploring competencies activations for all the students who participated in the sessions and all competency aspects of the competence framework. This section (5.3) includes a student's account who did not belong in the focus group, where an interim analysis of his competency activations is presented. This account also includes an extensive report for each competency aspects of the framework. It comprises examples of how these competency activations were identified from all students who worked during these sessions they participated. Lastly, Appendices B and C include all sessions' accounts from all six students and provisional results for all six students regarding their work in the class and the indications they offered regarding the competencies they appeared to activate.

Following the limitation mentioned above, a reference to the difficulties that occurred while applying the Binomial Test must be stated. By implementing this statistical test, I strived to translate qualitative data using a quantitative approach. There should be some considerations regarding the number of turns (i.e., the transcribed units derived from a student's spoken or written utterances or actions, see also subsection 4.1.2) used to apply the statistical test. The test's validity and the results' reliability would be significantly enhanced if the total number of turns was relatively higher than the existing one. In subsection 4.4.3. a set of primary assumptions one must make to apply the Binomial Test was listed. One could challenge some of these assumptions, especially the claim that the sample size is significantly less than the population size.

As discussed in subsection 7.1.3, three types of evidence were introduced, which combined would provide a profile of the student's competency development over time. These types of evidence were the following: frequency of activation, changes between activation levels, and consistency of competency activation. These types of evidence function as analytical tools that can provide indications of competency development. It was relatively straightforward to recognize which competency aspects were activated more frequently by employing the frequency of activations in the analysis. However, more evidence was needed because the activation frequency did not fully portray the student's competency activation profile.

By employing the scaling instrument, two new types of evidence were added to the methodological apparatus of this study. Still, what would be considered 'high' or 'low' consistency (see Table 4-14) could be debatable, and I acknowledge that. That said, I believe that employing a combination of evidence (instead of a single one) to draw inferences regarding the nature of students' competency development adds to the reliability of the findings. For example, in subsection 6.1.3, the (→Sol.) aspect displayed the most frequent activation among all 16 competency aspects for Erika. However, by employing all three types of evidence it has been inferred that there was a moderate (and not a strong) combination of evidence regarding the development of this aspect. Furthermore, it is possible that a particular aspect could be identified to be activated less frequently than others but display an activation consistently at the higher levels of activation of the scaling instrument.

The duration of the project (seven one-hour sessions in 7 weeks for each working group) can be considered an additional limitation. In this thesis, I repeatedly referred to competence as a notion with dynamic characteristics. These characteristics could be explored further and possibly more accurately if the study was to take place for a more significant number of sessions or a whole academic year, for example. For that reason, this thesis adopted a language that described the quality of the evidence combination that could suggest an indication of competency development instead of discussing development itself. Nevertheless, this thesis considered that identifying a strong or moderate combination of evidence could indicate competency progression.

Lastly, I address a consideration regarding the coding process performed in this study. In Section 4.6 and Appendix E, I discussed and presented results from the intercoder reliability process of this research. Agreement percentages were above 80%, which led me to consider the coding process relatively reliable and that the customization of the competence framework (excluding some competency aspects of the KOM model) would be appropriate for the needs of this thesis. However, as a future step, I would consider applying additional methods of intercoder reliability to improve or re-examine the set of codes.

8.4 Suggestions for further research

Research informs us about a decline in the rate of high school students studying mathematics and science (e.g., Wilson and Mack, 2014; Wienk, 2015; PCAST STEM Undergraduate Working Group, 2012) and the poor mathematical skills of students who are about to attend STEM degree programs. A first step that one could take to address this problematic situation is to explore the necessary conditions for further motivation for the students to do mathematics. In this thesis, implementing non-routine mathematical tasks within a biological context was assumed to motivate these students to do mathematics.

The findings suggested that both Erika and Johannes displayed evidence of competency development, although for a different set of competencies. Whether the implementation mentioned before was the factor for this observed competency development is a question that this thesis cannot and did not aim to answer. The latter presents an opportunity for promising research endeavors that connect competency development with the types of mathematical tasks and activities taught in the classroom. An additional hypothesis worthy of further investigation is that different types of mathematical tasks and teaching approaches may result in developing a different set of mathematical competencies.

The scientific advances of our era have transformed the life sciences. However, this transformation had little influence on undergraduate training, and the gap between teaching and research is evident (Abell & Lederman, 2007; Kerfeld & Simon, 2007). The previous concerns justify the need for continuous teachers' training in various educational approaches, such as implementing non-routine mathematical tasks (adjusted to the student's field of study) in teaching activities. Even though this study was focused on a small sample of students in a single department of Biology studies, the findings and indications obtained are encouraging because competency development was observed and identified. Considering this study's findings, contributions, and limitations, some possible paths could be followed for future research on teachers' training on implementing non-routine mathematical tasks in their regular mathematics courses. This research attempt could concern the content and methods of this training, the mathematical content of these tasks, and how it could be connected with aiming at students' mathematical competency development.

It is tempting, if not reasonable, to assume that a larger sample of students could provide even stronger evidence regarding the nature of students' competency development. The fine-grained analysis of Erika and Johannes and the abridged study of the other four students offered some valuable observations that have been considered indications of several phenomena, as mentioned in Chapters 5, 6, and 7. However, having a larger sample of students could result in greater confidence. For example, the finding that the students involved in this project appeared to display the strongest evidence of competency development during the Population Dynamics unit (Section 8.1) could be investigated further if data from more students from other groups were analyzed. Having a more significant or representative sample of students is not the only change we could implement for future research. This is discussed in the following paragraph.

As mentioned in the previous section, the number of sessions and the project's duration could be additional limitations. The analysis of seven one-hour calculus sessions provided a range (in terms of strength) of evidence regarding the students' competency development. However, further research in this direction is needed, focusing on obtaining and attaining samples of students over a more extended period. An academic semester could include more than seven weekly sessions, allowing the researchers to spend more time on the selected mathematical areas that they wished to explore their students' competency development. The possible findings could be of great interest and may manifest patterns of competence activation and development that will reinforce some key findings of the present study or reveal new original indications and phenomena.

My future research endeavors (Mesa et al., 2019) will also focus on the advances in Topic Modeling in text and textbook analysis. The transcription process of this study was a demanding and time-consuming procedure. Using machine learning approaches (Topic Modeling) for a larger sample of transcriptions, I will propose the understanding of text's content in a scalable and efficient manner. Machine learning is the study of algorithms and statistical models computer systems use to perform a specific task. The innovation here is that this model does not demand explicit instructions but relies on patterns and inference. Topic Modeling is a text-mining tool focusing on discovering hidden semantic structures in a text body and abstract 'topics' that occur in a collection of documents. I am focusing on the GENSIM library approach (e.g., Koštial and Dařena, 2018), a production-ready open-source Python library for unsupervised topic modeling and natural language processing using modern statistical machine learning.

Following the references on Topic Modeling mentioned in the previous paragraph, I consider it essential to mention the author's particular interest in a specific model for natural language processing. I am referring to the Latent Dirichlet allocation (LDA), a generative statistical model that allows sets of observations to be explained (Blei et al., 2003). This method incorporates unobserved groups that explain why some parts of the data are similar. Since the observations are words (large, transcribed documents from students' discourse) collected into documents, it posits that each document is a mixture of a small number of topics and that each word's presence is attributable to one of the document's topics. The LDA algorithm aims to map the documents (e.g., transcriptions, text from textbooks, or students' responses in questionnaires) to a predefined number of topics. The topics can be annotated by codes that represent particular competencies. Each topic can be mapped to a number of keywords that facilitate the description of the topic. Any form of script will repeatedly apply the topic modeling algorithm based on a range of topic numbers that I can specify beforehand using the competence framework of this or any future study. The evaluation scores obtained could lead to a suitable model to be picked. I consider the Topic Modeling approach as a promising research path that could reveal tendencies and patterns in students' learning of mathematics at the undergraduate level. Providing a solid theoretical framework that will describe the set of topics of the model could offer opportunities for new studies on the development of mathematical competencies in the arena of Mathematics Education.

It is within the author's future research intentions to further utilize the scaling instrument (subsections 4.1.3 - 4.1.8) designed to assess the quality of competency activation, a core research design and analysis tool of this study. This endeavor will be facilitated by the affordance of statistical tests that can blend qualitative data and quantitative analysis methods. Should the total number of turns be higher (i.e., more recorded and transcribed students' actions), the findings regarding the students' competency development and the patterns of concurrent competence activation might be more apparent? The latter could suggest that during the teaching of calculus in STEM studies, we could identify what kind of competence profiles tend to be shaped individually and collectively for each STEM field. Such suggestions or indications could add to existing knowledge of what competencies are deemed necessary, depending on the area of study, for the learning of mathematics in tertiary education.

9. References

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10. Appendices

The Appendices chapter consists of seven parts:
- **Appendix A** presents the complete KOM competence framework
- **Appendix B** provides the sessions accounts for all participants of the project.
- **Appendix C** includes provisional results for all the students.
- **Appendix D** includes all the competency activation results that were not included in the main body of the thesis.
- **Appendix E** includes the competency development findings for all competency aspects.
- **Appendix F** provides extracts from transcriptions that I and my supervisor (Pr. John Monaghan) used during the process of intercoder reliability. In addition, I include the students' invitation for their participation to the project and the letter of approval form the Norwegian Centre for Research Data (NSD).
- **Appendix G** includes the students' invitation, the letter of approval, and the figures not included in the main body of the thesis

APPENDIX A: The KOM competence framework

1. *Thinking mathematically (mastering mathematical modes of thought)*
   - posing questions that are characteristic of mathematics, and knowing the kinds of answers (not necessarily the answers themselves or how to obtain them) that mathematics may offer
   - understanding and handling the scope and limitations of a given concept
   - extending the scope of a concept by abstracting some of its properties; generalising results to larger classes of objects
   - distinguishing between different kinds of mathematical statements (including conditioned assertions ('if-then'), quantifier laden statements, assumptions, definitions, theorems, conjectures, cases)
2. *Posing and solving mathematical problems*
   - identifying, posing, and specifying different kinds of mathematical problems pure or applied; open-ended or closed
   - solving different kinds of mathematical problems (pure or applied, open-ended or closed), whether posed by others or by oneself, and, if appropriate, in different ways
3. *Modelling mathematically (i.e. analysing and building models)*
   - analysing foundations and properties of existing models, including assessing their range and validity
   - decoding existing models, i.e. translating and interpreting model elements in terms of the 'reality' modelled
   - performing active modelling in a given context

i. structuring the field
ii. mathematising
iii. working with(in) the model, including solving the problems it gives rise to
iv. validating the model, internally and externally
v. analysing and criticizing the model, in itself and vis-à-vis possible alternatives
vi. communicating about the model and its results
vii. monitoring and controlling the entire modelling process

4. *Reasoning mathematically*
   - following and assessing chains of arguments, put forward by others
   - knowing what a mathematical proof is (not), and how it differs from other kinds of mathematical reasoning, e.g. heuristics
   - uncovering the basic ideas in a given line of argument (especially a proof), including distinguishing main lines from details, ideas from technicalities
   - devising formal and informal mathematical arguments, and transforming heuristic arguments to valid proofs, i.e. proving statements
5. *Representing mathematical entities (objects and situations)*
   - understanding and utilizing (decoding, interpreting, distinguishing between) different sorts of representations of mathematical objects, phenomena and situations
   - understanding and utilizing the relations between different representations of the same entity, including knowing about their relative strengths and limitations
   - choosing and switching between representations
6. *Handling mathematical symbols and formalisms*
   - decoding and interpreting symbolic and formal mathematical language, and understanding its relations to natural language
   - understanding the nature and rules of formal mathematical systems (both syntax and semantics)
   - translating from natural language to formal/symbolic language
   - handling and manipulating statements and expressions containing symbols and formulae
7. *Communicating in, with, and about mathematics*
   - understanding others' written, visual or oral 'texts', in a variety of linguistic registers, about matters having a mathematical content
   - expressing oneself, at different levels of theoretical and technical precision, in oral, visual or written form, about such matters
8. *Making use of aids and tools (IT included)*
   - knowing the existence and properties of various tools and aids for mathematical activity, and their range and limitations
   - being able to reflectively use such aids and tools

APPENDIX B: Sessions accounts for all participants

Periodic Functions: one session (3.10, first unit)

Students present: Erika, Johannes, and Rene (Focus Camera), Linnea, and Maria (GoPro camera).

In this session, the students were introduced to the notion of periodic functions and mainly the trigonometric functions, which we occasionally named "trigonometric models." After a description of the equations and their parameters, $g(x) = a\cos[b(x - c)] + d$ and the $g(x) = a\sin[b(x - c)] + d$, the students encountered their first warm-up task, the Airport Temperatures task (Table 3-4, p. X). The class started working on finding the elements of the g function, that is the amplitude a, the period b, the phase shift c, and the vertical shift d. Rene looked at the table with the monthly temperatures and wrote down relatively fast that "$a = \frac{17.9 - 9.5}{2} = 4.2$" while saying, "from what we know, we will have to take out the maximum minus the minimum value from the table."

Linnea and Maria worked together on identifying the maximum and minimum values of the table. While working with the table of average monthly temperatures, Erika noticed that she had "a different starting point" from the other group member (Johannes) and that the graph could be different. She referred to the construction of the following equation: $g(x) = a\cos[b(x - c)] + d$, because she had sketched a different (understood the limitation) yet correct graph. Johannes (in discussions with Erika) faced some difficulties regarding the notion of vertical shift d but eventually managed to produce the correct equation that would solve the task. All five students managed to produce an equation that modeled the temperatures from the table.

Following the warm-up task and before introducing the main task, the teacher proceeded to a presentation of a predator-prey task. The task was talking about a group of environmentalists who used sinusoidal functions to model populations of predators and prey in the environment. The model was described by the following equation: $1500\, sin(\frac{\pi}{2}(x - \frac{\pi}{2})) + 25000$, for the population of rabbits (prey) and the following equation: $2000\, sin(\frac{\pi}{2}\, x) + 5000$, for the population of the wolves (predators). The teacher discussed the relationship between the number of rabbits and the number of wolves by using the graphs of these two sinusoidal equations.

The class moved on to the main task, the New Creature task (Table 3-4, p. X). Rene was again focused on finding the model's primary elements (i.e., amplitude a, period b, phase shift c, and vertical shift d), having some problems with the graphical depiction of the creature's temperature over minutes. He eventually solved the problem with help from the teacher at some points. Linea and Maria (working again together) focused on finding how far the graph had to be shifted horizontally (phase shift). Erika and Johannes worked mostly together and, looking at the previous task for relevant information, started by finding the period b. There were some misconceptions regarding finding b because, at some point, Erika and Johannes realized that they had found “the half of the period” and needed to go over their work to track down where was the mistake in their written work. The session ended with the teacher projecting the solving process and the final results on the screen.

Exponential Growth and Regression I (10.10, second unit)

Students present: Erika, Johannes, and Rene (Focus Camera), Kim, Linnea, and Maria (GoPro camera).

In this session, the students were introduced to the notion of exponential growth and regression. This session was the first part of this calculus unit. The session began with a short introduction to the rules of exponents and logarithms, and then the class moved on to the warm-up task, the Burley Yield task (Table 3-4, p. X). Rene, after consulting his notes, begun working by categorizing the relevant information from the task. Without using the notation given to him, he wrote “$y = 1961 = 31.0 + 0.017 \cdot 31.0 = 31.522$”, which showed that he struggled with the task’s notation. He solved the task after some discussion with the teacher. Linnea and Maria started by working on understanding what ‘1.7%’ represents and how to incorporate it in their model. The teacher asked questions that could help the students envision how the model’s equation should look like, “how can we call the expression ‘1.7%’ that we are dealing with here?” and Linea and Maria eventually wrote down the expression: $y = 31.0(1.017)^t$.

Kim found it difficult to translate the percentage (1.7%) to 1.017, and she spent a considerable amount of time on this task. The teacher worked with her on the solution due to time restrictions and proceeded to the main task. Kim[2] followed what Linea and Maria did and copied the expression ‘$y = 31.0(1.017)^t$’ at her notes. Erika and Johannes started discussing the task, and Erika asked Johannes whether he remembered the “farmer task” that they did in their calculus course. Both students translated the percentage from 1.7% to 1.017 quickly and exchanged ideas about the solution’s final steps. Interestingly, Erika was not happy with the result she had even though it was correct, “I [may] do something wrong with my calculations with the calculator” but this issue was resolved. Before proceeding to the next task, the teacher projected on the board how an exponential graph should look and explained why the expression ‘1.7%’ was the growth rate.

The students continued on another warm-up task, the Salmon task (Table 3-4, p. X). Rene noticed many similarities with the task “harder but the same, I think” and began working on his notebook. Linnea looked at her notes from the

[2] Kim participated in the same class with Maria and Linnea after the first course

previous task so she could compare them with the new task. Maria started working on the expression “average decline of 18.1% per year” and Kim was copying the necessary information to her notes. Erika and Johannes both experienced difficulties in understanding that this is a task about a declining population. They both said that they were expecting a “negative percentage.” With some guidance from the teacher, Erika (even though struggling again with the calculator) announced her solution, ‘$86.500(1 - \frac{0.181}{100})^{15}$’.

Linnea and Maria both reached to the solution of the task with Kim having some difficulties following due to difficulties in understanding the behavior of the a^xgraph for $a < 1$, but she eventually reached an understanding of the task. Johannes, working closely with Erika at all times for this task, reached the same expression that Erika presented. At this point, the teacher initiated a discussion about the difference between ‘$1 + r$’ and ‘$1 - r$’ when using the following equation: $y = y_0(1 + r)^t$ and explained these differences graphically. The class moved then to the main and final task of the session.

The students moved on to the main task, the Recycles task (Table 3-4, p. X). Linnea and Maria initially struggled with the task because no rate was given. Kim looked at her previous notes and wrote down the table elements with the years and the relative number of tons. The teacher pointed out that “the fact that you don’t see any rate at the task does not mean that you cannot find one based on what is given to you” in an attempt to motivate the students to work on what they have. They (Linnea, Maria, and Kim) first noticed an increase in the number of recyclables each year and then tried to find the “pattern of the increase.” Kim was relatively silent during this process. Working together, Linnea and Maria reached a solution to the task with minimal help from the teacher and presented their results. Kim did not produce a solution, and the teacher walked her through the task solving process.

Erika and Johannes compared the previous task with the new one and realized that even though there is no rate, they should find one. Johannes wondered if they are “supposed […] to estimate the rate on [their] own?” and Erika agreed that this is the way to the solution of the task. With some teacher assistance, Erika and Johannes realized that they were dealing with an increase in the recycles each year and solved the task faster than the previous tasks. By that time, the session was already 10 minutes over schedule, and the teacher mentioned that in their next meeting, they would continue their exploration on the notion of exponential growth and regression.

Exponential Growth and Regression II (24.10, second unit)

Students present: Erika, Johannes, Kim, Linnea, and Maria (Focus Camera).

In this second session on exponential growth and regression, the teacher begun by discussing about the growth rate of cells and bacterial population. The discussion included a presentation on the cell’s growth curve where the population of cells in every four phases (log phase, exponential phase, stationary phase, and death phase) was illustrated. The teacher said that “when growing exponentially by binary fission, the increase in a bacterial population is by geometric progression,” and, to explain what binary fission is, he gave a riddle about a lake where lotuses in the lake doubled every minute. The goal for presenting this riddle was to introduce students to several concepts that would be useful for the upcoming tasks (e.g., generation time, time interval, number of generations).

Following the introduction, the teacher presented the warm-up task, the Bacterial task (Table 3-4, p. X). Linnea started working on the notation of the expression ‘$b = B \cdot 2^n$’, and how to translate it as a function of the number *n* of generations. Maria was searching her notes, looking for an example on the natural logarithm *ln*, “I know how to bring this [the exponent n] down” for the same reason. The teacher discussed with Kim the riddle and how it was relevant to the task at hand. Linnea and Maria wrote down the expression ‘$n = \frac{log b - log B}{log 2}$’ and provided the solution for this warm-up task. Erika and Johannes worked together on this task with the teacher prompting on several occasions when he asked what “the necessary information” are for this task and with Erika pointing out that they need the “generation time and the number of bacteria at the end.” Both Erika and Johannes needed more time than Linnea and Maria to complete this task, but they both provided the correct result.

The class moved then to the main task, the Cancer Cells task (Table 3-4, p. X). Before presenting this task, the students were informed about the phenomenon of mitosis, the process by which two cells divide in half, producing two daughter cells identical to one another and the original one. The students were informed that the amount of time it takes for a malignant cell to divide into two (mitosis) malignant cells was approximately three months. Linnea and Maria worked together on finding the necessary information to tackle this task. Linea said, “we need to find how a cancer cell looks [like] and see what we can learn from that,” with Maria using her phone to start her search. Kim noticed that the mitosis phenomenon was similar to the riddle that the teacher posed at the beginning of the session. For this task, the teacher helped Linnea, Maria, and Kim to find the necessary information for the task (e.g., cancer cell size), and by the end of the session, Linnea and Maria provided a solution which was not correct due to the different information they selected from the internet. The process they followed, though, was the correct one and the teacher pointed this out. Kim needed more help and followed the teacher’s presentation of the solution on the board.

Erika and Johannes asked, “how big should a cluster of cancer cells should be to be visible?” when they both began working on the task, and they spent sufficient time on the web looking for relevant information. Erika pointed out that “different cancers [have] different cells, different size,” and they reached the task’s solution by the end of the session. The teacher presented an image of an x-ray to illustrate to the students how the cluster looked like and to convey how important it was (and is) for the patient to have a model and a technology that would detect the cluster as fast as possible.

Population Dynamics I (31.10, third unit)

Students present: Erika, Johannes, Kim, Linnea, Maria, and Rene (Focus Camera).

In this first session of the Population Dynamics unit, the students were introduced to concepts such as population sizes, immigration, and emigration. The teacher presented the model that could calculate (or predict) the size of a population:
$N_{t+1} = N_t + births + deaths + immigration - emigration$,.at a certain time *t+1*. The presentation included the description of unrestricted growth of a population when the rate growth *R* is constant, where the simplest discrete model is: $N_{t+1} = R \times N_t + N_t$, or $N_t = N_0 (R + 1) T$, if we start with N_0 individuals at time *t=0* and we are looking for the number of individuals at time *T*. The presentation ended with introducing the continuous model with *r* as the exponential per capita rate of growth: $\frac{dN}{dt} = rN(t)$, which for *N(0)* individuals at time *t=0*, the number of individuals is: $N(T) = N(0)\, e^{rT}$. Rene did not pose any questions during the presentation. Linnea and Maria were confused about the '$\frac{dN}{dt}$' expression, and Kim was taking notes. Erika and Johannes followed the presentation closely, with Johannes wondering why we use the minus (-) for emigration in the population model.

The class moved on to the warm-up task, the World Population task (Table 3-4, p. X). Rene was unsure why the model is using (+) for the deaths and not minus (-), which provoked a discussion. Maria said, "we need to find which of the models [discrete or continuous] equations we must choose." Linnea was confused about what "rate of growth 2% per year" meant and said that "it [would] be weird if every 100 persons there is going to be a 2% rate." Kim suggested that the growth rate "could increase to 25%," which indicated that she realized that the task was dealing with a constant growth rate, but this growth could easily change. Erika and Johannes worked together on finding the "starting point" for their model referring to *N(0)*, which was 2 in this case. All students reached a result not aligned with the reality (e.g., 1101 years) and realized that more considerations should be made.

The class moved on to the main task, the E. Coli task (Table 3-4, p. X). The students were referred to the main theme of a science fiction thriller novel. The novel claimed that one cell of E. coli could produce a super-colony equal in size and weight to the entire planet Earth in a single day. The students were informed that a single cell of the bacterium E. coli divides every 20 minutes (i.e., 72 times in 24 hours), and information about the bacterium and Earth size was given. The students needed to find if the novel's claim was accurate and, if not, how much time should be allowed for this claim to be correct.

Rene began by saying that "at first, we have to find how many E. Coli will be in 24 hours […] like they will be divided 72 times," indicating that he grasped the concept of "division every 20 minutes." Maria was trying to remember the lake riddle of the previous session because it was dealing with similar concepts (i.e., doublings), and Linnea was following Maria tried to find a model that should include "the mass of Earth in one side and the colony's mass on the other." Kim faced difficulties with the notation of the task (i.e., 2^t, 2^{3x}) and needed assistance from the teacher. Erika and Johannes were struggling with the concept or "division every 20 minutes." A prompting question from the teacher, "how many sets of 20 minutes are in an hour?" seemed to help both of them to realize that they need to divide time by 3. All students except Kim, solved, or came close to the solution, of the task and when the teacher presented the solution it appeared that all of them followed the solving process.

Population Dynamics II (7.11, third unit)

Students present: Erika, Johannes, Kim, Linnea, Maria, and Rene (Focus Camera).

This session begun with the teacher notifying the students that "today you are going to be the researchers" and that they "should come up with the results that satisfy [their] logic" and began working on the warm-up task, the Population Doubled task (Table 3-4, p. X). Rene wrote down what the task was asking him to find and began looking online for the available information. After some searching, the teacher suggested a site where he could find the information that was needed. Linnea and Maria started working on the concepts of "births rate" and "deaths rate" while Kim transferred the information projected onto the screen into her notebook. Johannes had some difficulties handling the concepts of births and death rates, saying that "we may have to divide them and find their proportion," but after some discussion with the teacher and Erika, he overcame this issue. All students needed more time than the teacher planned, and the teacher decided to present the solution of the task, with the majority of the students being close to reaching a solution themselves.

The class moved to the main task; the Reversed Population Doubled task (Table 3-4, p. X). Rene had to leave the class for personal reasons and therefore did not work on this task. Linnea and Maria worked on the notation of the task (i.e., $P(t) = P(0)e^{rt}$), and Kim realized, "we are looking for the *r* in our case." Erika and Johannes found the correct excess rate (i.e., 0.001%) with minimal prompting from the teacher, and Johannes concluded that giving just a single number for the rate would not convince someone that the rate is not constant, "I don't think it would not help [anyone] just giving a number." The class ended with a discussion about factors that played and play a significant role in changes in births and deaths rates, and the teacher finished the session by projecting a graph of the future world population until 2100, which provoked comments both from the past (e.g., baby boom period) and for the future (e.g., the annual growth rate of the world population would be close to 0.08% by the year 2100).

Integration and Modeling I (14.11, fourth unit)

Students present: Erika, Johannes, Linnea, and Rene (Focus Camera).

In the first part of this unit, the students were introduced to integration rules (it was not yet taught in their regular calculus session) and exponential growth in resources' consumption rate. The teacher began by pointing out that for this session (and the next), they will need all the knowledge that they obtained in their last five sessions (i.e., knowledge on exponential growth, properties of logarithms, and derivatives). Because the students were just briefly introduced to the concept of integration, the teacher spent time explaining and providing examples of integration rules, and therefore, some readjustment of the schedule had to be made.

The class worked on one task only, the Earth's Temperature task (Table 5, p. X). Rene wrote down on his notes the necessary information and started working on the expression '$\frac{dF}{dt} = 0.014 \cdot t^{0.04}$' and he correctly found the interval of the integration (i.e., 0 and 200). With some help from the teacher on the notation he reached a solution of the task. Linnea after some initial difficulties with the integration rules also reached a solution of the task. Erika and Johannes spent time on getting familiar with the new notation and what the integration interval represented. The integration rules and techniques were a point of friction for both of them during the whole session. They eventually worked on the task with help from the teacher. The planned main task was postponed for the next week's session, which would be the last session of this classes.

Integration and Modeling II (21.11, fourth unit)

Students present: Rene, Johannes, Erika Linnea, and Maria (Focus Camera).

In this seventh and last session of the calculus sessions, the students continued their work from the previous week. The plan was to work on two tasks and then continue with an overall discussion about the project. In this first task, the Neuron Voltage task (Table 3-4, p. X), Rene had some difficulties with the notation of the task and used the expression '$0.014 \cdot t^{0.04}$' without the appropriate symbolism. Linnea and Maria also faced difficulties with the task's notation, and the teacher guided them through the solution of the task. Erika and Johannes both remembered what they have done in the previous session and worked with the teacher at all times for this task. The teacher reminded them how to work with a constant value when integrating, and as with Linnea and Maria, he helped them on numerous occasions during the solving process.

The class moved on to the course's last task, the Petroleum Task (Table 5, p. X). Rene was aware of the model that he wanted to use writing down "$P(t) = P(0)\, e^{rt}$" but he initially faced some difficulties in using the '2% increase' expression at this model. Linnea and Maria continued working together on the notation of the amount of petroleum *M(t)* and the rate of the amount $\frac{dM}{dt} = -C(t)$, and with occasional prompting from the teacher, they came close to the solution of the task. Erika and Johannes worked together, and Erika mentioned similarities with the “bacterium task” and worked with the teacher during the whole solving process. Both students had trouble with the notation of the task, although they were continually trying to use their previous knowledge to facilitate their solving process. The class ended with a discussion on modeling and the real-world applications, and the connection between biology and mathematics.

APPENDIX C: Provisional results for all students

Kim's work

Kim participated in four (2^{nd}, 3^{rd}, 4^{th}, and 5^{th}) out of the total seven sessions, and data from the Focus Camera was collected from three (3^{rd}, 4^{th}, and 5^{th}) sessions. Regarding the frequency of evidence of competency activation, Kim's actions facilitated the construction of competency activation evidence for the following competency aspects:

- (P.Q.) and (→Sol.) of the Mathematical Thinking and Acting competency;
- (Int.El.) and (Info) of the Mathematical Modeling competency; and (U.T.) of the Aids and Tools competency, as being the most frequent.

Evidence was also identified for the activation of the competency aspects (Int.Res.X) of the Mathematical Modeling competency and (Reason←) and (Reason→) of the Mathematical Reasoning and Communicating competency. Less frequent evidence was found for the activation of the competency aspects (Lim.) of the Mathematical Thinking and Acting competency, and (C.D.Rep.), (Man.Rep.), and (Rep.↔Rep.) of the Representing and Manipulating symbolic forms competency. Lastly, limited or no evidence was found for the activation of the (Cr.), (Graph) and (R.+V.) competency aspects of the Mathematical Modeling competency and the (Ext.T.) of the Aids and Tools competency.

While working on the World Population task (Table 3-4) of the Population Dynamics, Kim suggested that the growth rate "could increase to 25%," providing evidence for the activation of (Int.El.) competency aspect of the Mathematical Modeling competency interpreting the elements of the model but also reflecting on a real-world situation where "this rate could change for some reason." When the class moved to the main task, the Reversed Population Doubled task (Table 3-4), Kim replied to a question posed by a fellow student, "we are looking for the r in our case which is the change in the increase" while working on the notation of the task (i.e., $P(t) = P(0)e^{rt}$) which provided evidence for the activation of the (Int.El.) and (Reason→) interpreting the elements of the model but also providing justification to support her idea.

In terms of units (Periodic Functions, Exponential Growth & Regression, Population Dynamics, Integrals & Modeling), Kim's work in class, provided evidence of competency activation almost exclusively during the Population Dynamics unit because that was the unit where data from the Focus Camera were collected from both sessions (i.e., 4^{th} and 5^{th}). Less evidence was found regarding the indications that suggested concurrent activation of particular competency aspects. Still, some inferences can be made regarding the interrelation of the (→Sol.) the (U.T.) and the (Reason→) competency aspects that were observed at the same turn.

Summing up, Kim's engagement in the sessions facilitated the identification of evidence for the activation of a more limited range of competency aspects (i.e., compared to Rene as seen above) of the competence framework across the four sessions that she participated. She was more engaged in the second session of the Population Dynamics unit (5th session) but displayed some difficulties during the second session of the Exponential Growth & Regression unit (3rd session), where she mainly engaged in discussions with the teacher about concepts that were covered in the first session of that unit (2nd session). Similarly to Rene, limited or no evidence was found, from Kim's work, for the activation of the (Graph) and (R.+V.) competency aspects of the Mathematical Modeling competency and the (Ext.T.) of the Aids and Tools competency.

Linnea's work

Linnea participated in all seven sessions, and data from the Focus Camera was collected from five (3rd, 4th, 5th, 6th, and 7th) sessions. Regarding the frequency of evidence of competency activation, Linnea's actions facilitated the construction of competency activation evidence for the following competency aspects:

- (P.Q.) and (→Sol.) of the Mathematical Thinking and Acting competency;
- (Int.El.), (Int.Res.X), and (Info) of the Mathematical Modeling competency;
- (C.D.Rep.) and (Man.Rep.) of the Representing and Manipulating symbolic forms competency;
- (Reason→) and (Reason←) of the Mathematical Reasoning and Communicating competency; and
- (U.T.) of the Aids and Tools competency, as being the most frequent.

Evidence was also found for the activation of the competency aspect (Graph) of the Mathematical Modeling competency. Less frequent evidence was found for the activation of the competency aspects (Lim.) of the Mathematical Thinking and Acting competency, (Cr.) of the Mathematical Modeling competency, and (Rep.↔Rep.) of the Representing and Manipulating symbolic forms competency. Lastly, limited or no evidence was found for the activation of the (R.+V.) competency aspect of the Mathematical Modeling competency and the (Ext.T.) of the Aids and Tools competency.

While working on the Population Dynamics unit's tasks, Linnea provided evidence for the activation of the (P.Q.) competency aspect of the Mathematical Thinking and Acting competency. She addressed the group, "I am just wondering [if we] should write the '2%' as '2' or as '0.02' here" while working on the '$N(T) = N(0)\, e^{rT}$' equation for the World Population task. When working on the Petroleum Task (Table 3-4), Kim addressed her fellow student and the teacher on multiple occasions regarding the notation of the task, providing additional evidence for the activation of the (P.Q.) competency aspect. Following an explanation from Maria by the end of the E Coli task (Table 3-4), Linnea provided evidence for the activation of the (Reason←)

competency aspect "I can see how Maria used the number '72' as a set of 20 minutes, 72 sets of 20 minutes" and continued with providing evidence for the activation of the (→Sol.) and the (Int.El.) aspects when she suggested to what part of the expression '2^{3x}' should the number '72' fit in.

In terms of units (Periodic Functions, Exponential Growth & Regression, Population Dynamics, Integrals & Modeling), Linea's work in class provided evidence of competency activation more frequently during the Population Dynamics unit. However, evidence was also found during the Exponential Growth & Regression and Integrals & Modeling units. Similar to what was noticed with Rene, Linnea's work provided provisional indications for the concurrent activation of particular competency aspects. For example, the concurrent activation of the (Int.El.) competency aspect with the (→Sol.) and the (U.T.) competency aspects and the concurrent activation of the (Int.Res.X) and the (Cr.) competency aspects.

Summing up, Linnea's engagement in the sessions facilitated the identification of evidence for the activation of an extensive range of competency aspects of the competence framework across all five sessions that she participated. She was gradually more engaged in the sessions providing evidence of competency activation in both sessions of the Population Dynamics unit (5th session) as well as in the sessions of the Exponential Growth & Regression (one session) and Integrals & Modeling units (two sessions). However, during the Integrals & Modeling unit sessions (6th and 7th sessions), the evidence for competency activation was less frequent than those in the Population Dynamics unit. Linnea provided evidence that could support the claim that she demonstrated: i) a more expansive (than Rene's and Kim's) range of activation regarding the competency aspects of the competence framework (evidence for activation of more aspects), and ii) more frequent activation of these competency aspects.

Maria's work

Maria participated in six (1st, 2nd, 3rd, 4th, 5th, and 7th) out of the total seven sessions, and data from the Focus Camera was collected from four (3rd, 4th, 5th, and 7th) sessions. Regarding the frequency of evidence of competency activation, Maria's actions facilitated the construction of competency activation evidence for the following competency aspects:

- (P.Q.), (Lim.), and (→Sol.) of the Mathematical Thinking and Acting competency;
- (Int.El.), (Int.Res.X), and (Info) of the Mathematical Modeling competency;
- (C.D.Rep.) and (Man.Rep.) of the Representing and Manipulating symbolic forms competency;
- (Reason→) and (Reason←) of the Mathematical Reasoning and Communicating competency; and

- (U.T.) of the Aids and Tools competency, as being the most frequent.

Evidence was also found for the activation of the competency aspects (R.+V.) and (Graph) of the Mathematical Modeling competency. Less frequent evidence was found for the activation of the competency aspects (Cr.) of the Mathematical Modeling competency and (Rep.↔Rep.) of the Representing and Manipulating symbolic forms competency. Lastly, no evidence was found for the activation of the (Ext.T.) competency aspect of the Aids and Tools competency.

At the beginning of the first session of the Population Dynamics unit, Maria provided evidence for the activation of the (R.+V.) and (→Sol.) competency aspects of the Mathematical Modeling and Mathematical Thinking & Acting competencies. She addressed both the teacher and the group saying, “so we have to find which of the models…equations we must choose between these two” referring to the discrete and continuous models (: $N_{t+1} = R \times N_t + N_t$, or $N_t = N_0 (R + 1) T$ and $\frac{dN}{dt} = rN(t)$) that were discussed in the introduction of this fourth session, assessing the range and validity of these models as well as taking actions for the solution of the task. During the Periodic Functions unit (1st session), Maria (working with Linnea) focused on finding how far the graph for the New Creature task (Table 5, p. x) has shifted horizontally (phase shift) but also identified the maximum and minimum values of the table that was given for the Airport Temperatures task. These and similar actions provided evidence for the activation of the (Graph), (Int.El.), (Int.Res.X), and (Info) of the Mathematical Modeling competency.

In terms of units (Periodic Functions, Exponential Growth & Regression, Population Dynamics, Integrals & Modeling), Maria’s work provided evidence of competency activation more frequently during the Population Dynamics unit (as Linnea did), but evidence was also provided for all the other units too. Similar to what was noticed with Linnea, Maria’s work provided provisional indications for the concurrent activation of particular competency aspects. For example, the concurrent activations of the (P.Q.) and (→Sol.) competency aspects and the (Int.El.) competency aspect with the (→Sol.) and the (U.T.) aspects were also observed.

To sum up, Maria’s engagement in the sessions facilitated the identification of evidence for the activation of an extensive range of competency aspects of the competence framework across all six sessions that she participated. As with the case of Linnea, she was gradually more engaged in the sessions providing evidence of competency activation in both sessions of the Population Dynamics unit (two sessions) as well as in the sessions of the Exponential Growth & Regression (two sessions) and Integrals & Modeling units (one session) and the Periodic Functions (one session). During the one session of the Integrals & Modeling unit (7th session), the evidence for competency activation was less frequent than those in the Population Dynamics and the Exponential Growth &

Regression units. Maria provided evidence that could support the claim that she demonstrated: i) a more expansive (than any other student discussed above) range of activation regarding the competency aspects of the competence framework (evidence for activation of more aspects), and ii) more frequent activation of these competency aspects.

Erika's work

Erika participated in all seven sessions, and data from the Focus Camera was collected from all these sessions. Regarding the frequency of evidence of competency activation, Erika's work and actions in the class facilitated the construction of competency activation evidence for the following competency aspects:

- (P.Q.), (Lim.), and (→Sol.) of the Mathematical Thinking and Acting competency;
- (Int.El.), (Int.Res.X), and (Info) of the Mathematical Modeling competency;
- (C.D.Rep.) and (Man.Rep.) of the Representing and Manipulating symbolic forms competency;
- (Reason→) and (Reason←) of the Mathematical Reasoning and Communicating competency; and
- (U.T.) of the Aids and Tools competency, as being the most frequent.

Evidence was also found for the activation of the (Cr.) competency aspect of the Mathematical Modeling competency and (Rep.↔Rep.) of the Representing and Manipulating symbolic forms competency. Less frequent evidence was found for the activation of the (Graph) and (R.+V.) competency aspects of the Mathematical Modeling competency. Lastly, no evidence was found for the activation of the (Ext.T.) competency aspect of the Aids and Tools competency.

Evidence for the activation of the competency aspect (P.Q.) of the Mathematical Thinking and Acting competency was provided by Erika's question on how to work on a particular equation ($N_t = N_0 e^{rt}$) during the Exponential Growth & Regression unit. This evidence came from her written work when she asked, "how should I work with this formula if I replace the N_t and N_0 with their values?" For the same mathematical competency but a different aspect, Erika has been given a table with the average monthly temperatures at an airport (Airport Temperatures task of the Periodic Functions unit) and asked to produce a sinusoidal equation using the cosine function. Evidence for the activation of the competency aspect (Lim.) was provided by Erika when she noticed that "[she] should not have the same equation because [she had] a different starting point" as the other member of the group (Johannes) had. She referred to the construction of the following equation: $g(x) = a\cos[b(x - c)] + d$, because she had sketched a different (understood the limitation) yet correct graph. Evidence for the activation of the competency aspect (Reason→) was provided by Erika's explanation while helping her fellow classmate on a calculation of the following expression: $\int_0^{200} \frac{df}{dt} = \int_0^{200} 0.014 \cdot t^{0.04}$. She said, "you need to take out that [showing the constant "0.014" of the expression] to continue."

In terms of units (Periodic Functions, Exponential Growth & Regression, Population Dynamics, Integrals & Modeling), Erika's work provided evidence of competency activation more frequently during the Population Dynamics and the Exponential Growth & Regression units. However, evidence was also provided for all the other units too. Similar to what was noticed with Linnea and Maria's work, Erika provided provisional indications for the concurrent activation of particular competency aspects. For example, the concurrent activations of the (P.Q.) and (→Sol.); (C.D.Rep.) and (Rep.↔Rep.) competency aspects and the (Int.El.) competency aspect with the (→Sol.) and the (U.T.) aspects.

To sum up, Erika's engagement in the sessions facilitated the identification of evidence for the activation of an extensive range of competency aspects of the competence framework across all sessions of this study. She was gradually more engaged in the sessions providing evidence of competency activation in both sessions of the Population Dynamics and the Exponential Growth & Regression units, and during the Periodic Functions unit. Her work during the Integrals & Modeling unit indicated that there was -comparatively to the other units- fewer evidence of competency activation. Erika provided evidence that could support the claim that she demonstrated: i) an expansive range of activation regarding the competency aspects of the competence framework (evidence for activation of more aspects), and ii) more frequent activation of these competency aspects compared to Rene, Kim, and Linnea.

Johannes's work

Johannes participated in all seven sessions, and data from the Focus Camera was collected from all these sessions. Regarding the frequency of evidence of competency activation, Johannes's work and actions in the class facilitated the construction of competency activation evidence for the following competency aspects:

- (P.Q.), (Lim.), and (→Sol.) of the Mathematical Thinking and Acting competency;
- (Int.El.), (Int.Res.X), (Cr.), and (Info) of the Mathematical Modeling competency;
- (C.D.Rep.) and (Man.Rep.) of the Representing and Manipulating symbolic forms competency;
- (Reason→) and (Reason←) of the Mathematical Reasoning and Communicating competency; and
- (U.T.) of the Aids and Tools competency, as being the most frequent.

Evidence was also found for the activation of the (Rep.↔Rep.) of the Representing and Manipulating symbolic forms competency. Less frequent evidence was found for the activation of the (Graph) and (R.+V.) competency aspects of the Mathematical Modeling competency. Lastly, no evidence was found for the activation of the (Ext.T.) competency aspect of the Aids and Tools competency.

Evidence for the activation of the competency aspect (Lim.) was provided by Johannes's work on the E. Coli task. When the teacher repeated the question of the task, "If a single cell of the bacterium E. coli divides every 20 minutes, how many E. coli would be there in 24 hours?" he indicated that he grasped the scope of the exponential growth concept during this task. When he began to work on the task, he said, "the starting point is a single cell…there will be 2, 4, 8… so we have to find what happens by the end of 24 hours." Evidence for the activation of the competency aspect (Int.El.) was provided by Johannes's work during the first session. He interpreted the elements of the equation $g(x) = a\cos[b(x - c)] + d$, while working on the Airport Temperatures task, saying that "So for d we take this point (showing the lowest point on the graph) lowest point plus the amplitude." Lastly, evidence for the activation of the competency aspect (Reason→) was provided by Johannes's participation during a discussion in class about an example with a closed environment with wolves and rabbits, which was not part of their warm-up or main tasks. The teacher asked what could happen in such an environment, and Johannes reasoned that "when the population of wolves is sinking, then the population of rabbits is rising and vice versa."

In terms of units (Periodic Functions, Exponential Growth & Regression, Population Dynamics, Integrals & Modeling), Johannes's work provided evidence of competency activation more frequently during the Population Dynamics and the Exponential Growth & Regression units. However, evidence was also provided for all the other units too. Similar to what was noticed with Linnea, Maria, and Erika's work, Johannes provided provisional indications for the concurrent activation of particular competency aspects. For example, the concurrent activations of the (P.Q.) and (→Sol.); (C.D.Rep.) and (Rep.↔Rep.) competency aspects and the (Int.El.) competency aspect with the (→Sol.) and the (U.T.) aspects. These last indications were similar to those of Erika's.

To sum up, as in Erika's work, Johannes's engagement in the sessions facilitated the identification of evidence for the activation of an extensive range of competency aspects of the competence framework across all sessions of this study. He was gradually more engaged in the sessions providing evidence of competency activation in both sessions of the Population Dynamics and the Exponential Growth & Regression units and during the Periodic Functions unit. During the Integrals & Modeling unit, his work indicated that there was -comparatively to the other units- less evidence of competency activation. Erika provided evidence that could support the claim that she demonstrated: i) an expansive range of activation regarding the competency aspects of the competence framework (evidence for activation of more aspects), and ii) more frequent activation of these competency aspects compared to Rene, Kim, and Linnea.

APPENDIX D: Students' activation results (remaining aspects)

In this appendix, I include all these activation results from the competency aspects that were not included in the main body of the thesis in Section 5.6 due to page restrictions and to avoid any unnecessary repetition.

Erika

Mathematical Thinking and Acting competency

To understand and handle the scope and limitations of a given concept. (code: Lim.) (2)

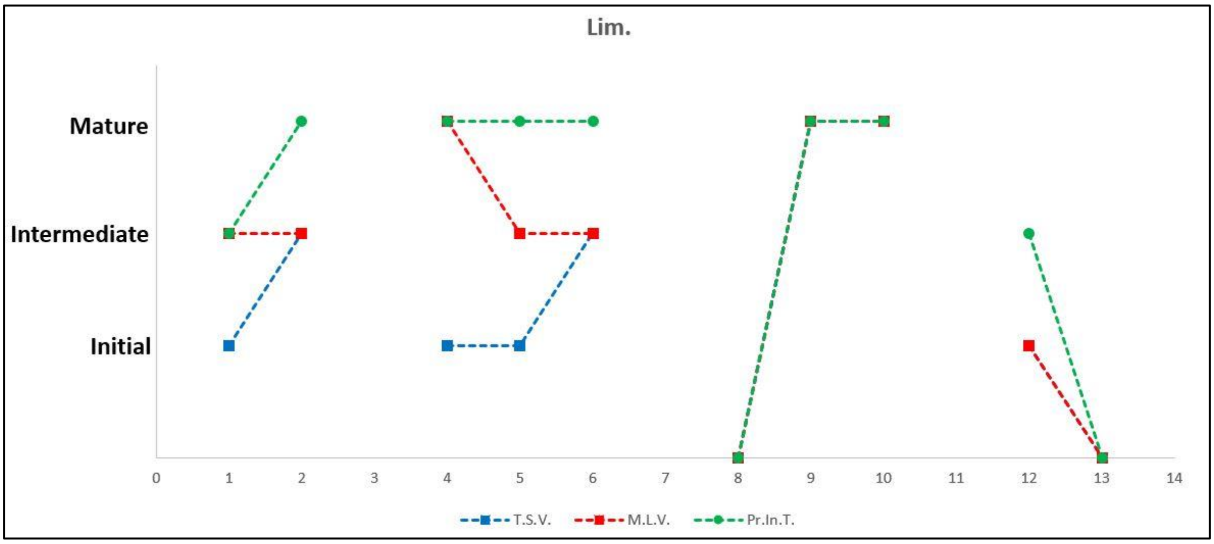


Diagram 10-1. Activation for competency aspect (Lim.).

Competency aspect activations: location and frequencies

The competency activations by modeling units were as follows: Periodic Functions [1-2], Exponential Growth & Regression [4-6], Population Dynamics [9-10], and Integrals & Modeling [12]. There was no activation of the competency aspect (Lim.) during the first session [8] of the Population Dynamics and the last session [13] of the Integrals & Modeling [13] units. The frequency of activation per units was [2, 3, 2, 1] and the total frequency of activation was, $n=8$.

Examples from the activations

Activation [2]: Erika has been given a table with the average monthly temperatures at an airport (Airport Temperatures task) and asked to produce a sinusoidal equation using the cosine function. Evidence for the activation of the competency aspect (Lim.) was provided by Erika when she noticed that "[she] should not have the same equation because [she had] a different starting point" as the other member of the group (Johannes) had. She referred to the construction of the following equation: $g(x) = a\cos[b(x - c)] + d$, because she had sketched a different (understood the limitation) yet correct graph. She addressed this limitation having a mid-range perspective towards the solution (T.S.V., at Intermediate Activation Level, II) while expressing herself

“I have a different starting point (pointing at the graph)” making regular use of expressions with mathematical language (M.L.V., at Intermediate Activation Level, II). All that occurred with no prompting (Pr.In.T., at Mature Activation Level, III) acting independently at this part of the solution process.

Activation [12]: Evidence for the activation of the competency aspect (Lim.) was provided by Erika when she replied to a question posed by the teacher (Yannis). The question asked whether the expression “$0.014t^{0.04}$” is a function or not. This expression concerned particular consumption rate on earth’s petroleum. After skimming through her notes, she considered the limitations of the concept *function* and replied, “yes this is a function.” She addressed this limitation having a short-range perspective towards the solution (T.S.V., at Initial Activation Level, I) not knowing what is that we were looking for with this task as a final product. She expressed herself with limited mathematical language and vocabulary (M.L.V., at Initial Activation Level, I) with occasional prompting from the teacher (Pr.In.T., at Intermediate Activation Level, II).

Summary of evidence

The summary of evidence regarding the changes within the competency activation levels and the consistency levels are displayed bellow.

Table 10-1. Summarized evidence for (Lim.) activation.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	3	1	2	
Changes from higher to lower	1	2	1	
High consistency	0	0	1	
Moderate consistency	(1)	1	0	
Frequency of activation				8

To attack (take actions towards a solution) mathematical problems and tasks of various kinds. (code: →Sol.) (3)

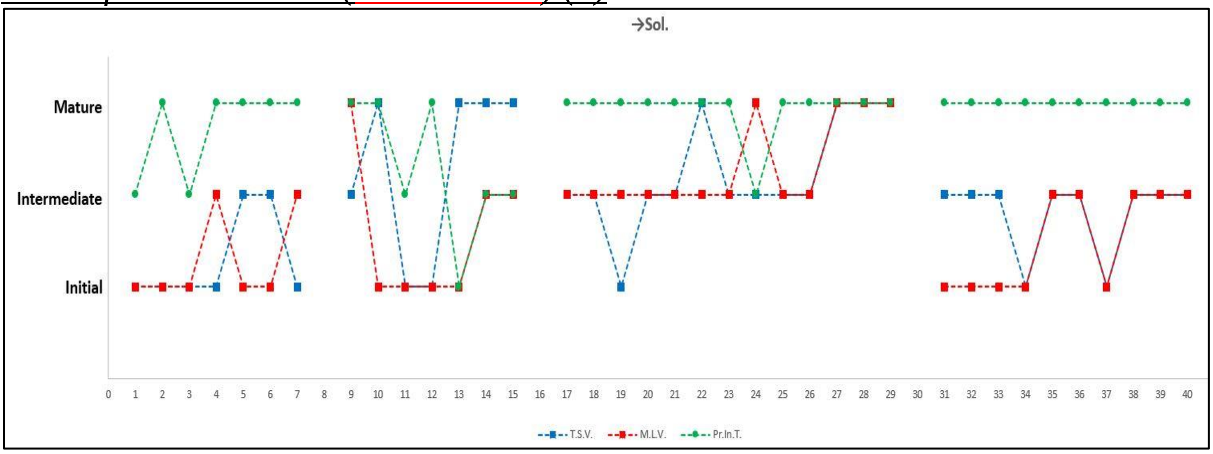


Diagram 10-2. Activation for competency aspect (→Sol.).

Competency aspect activations: location and frequencies

The competency activations by calculus units were: Periodic Functions [1-7], Exponential Growth & Regression [9-15], Population Dynamics [17-29], and Integrals & Modeling [31-40]. The frequency of activation per units was [7, 7, 13, 10] and the total frequency of activation was, *n=37*.

Examples from the activations

Activation [28]: Evidence for the activation of the competency aspect (→Sol.) was provided by Erika when she worked on a challenging part of the task that asked to check if a single cell of the bacterium E. Coli divides every 20 minutes, how many E. Coli would be there in 24 hours. She had to calculate time *t* from the equation: $2^t = 3.5 \cdot 10^{39}$, but from the description of the problem, time *t* was given in sets of 20 minutes. Erika, when calculated time *t*, said that, "131.9 represents the 20 minutes sets so we need to divide this by three" identifying the aforementioned challenging part. She had a long-range perspective towards the solution (T.S.V., at Mature Activation Level, III) while making regular and correct use of expressions with mathematical language (M.L.V., at Mature Activation Level, III). All that occurred while she was working with no prompting from the teacher (Pr.In.T., Activation Level, III).

Activation [37]: Evidence for the activation of the competency aspect (→Sol.) was provided by Erika when she begun to solve the Petroleum Consumption Rate task. In particular she was trying to translate the expression: $C(t) = C(0) \cdot e^{rt}$, having for $C(0)$, the value: 1.02. She wrote the following: $C(t) = C(0) \cdot 1.02 \cdot e^{rt}$, which was incorrect. She had a short-range perspective (she was copying from the whiteboard the $C(t)$ expression) towards the solution (T.S.V., at Initial Activation Level, I). She expressed herself using everyday expressions, "I can use this (pointing at the $C(t)$ expression) to start with" with limited mathematical language and vocabulary (M.L.V., at Initial Activation Level, I). However she was working with minimal prompting from the teacher (Pr.In.T., at Mature Activation Level, III).

Summary of evidence

The summary of evidence regarding the changes within the competency activation levels and the consistency levels are displayed bellow.

Table 10-2. Summarized evidence for (→Sol.) activation.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	8	7	5	
Changes from higher to lower	6	4	4	
High consistency	0	0	1	
Moderate consistency	1(1)	1(3)	3	
Frequency of activation				37

Mathematical Modeling competency

To interpret mathematical results in an extra-mathematical context and generalize the solutions developed for a special task-situation. (code: Int.Res.X) (6)

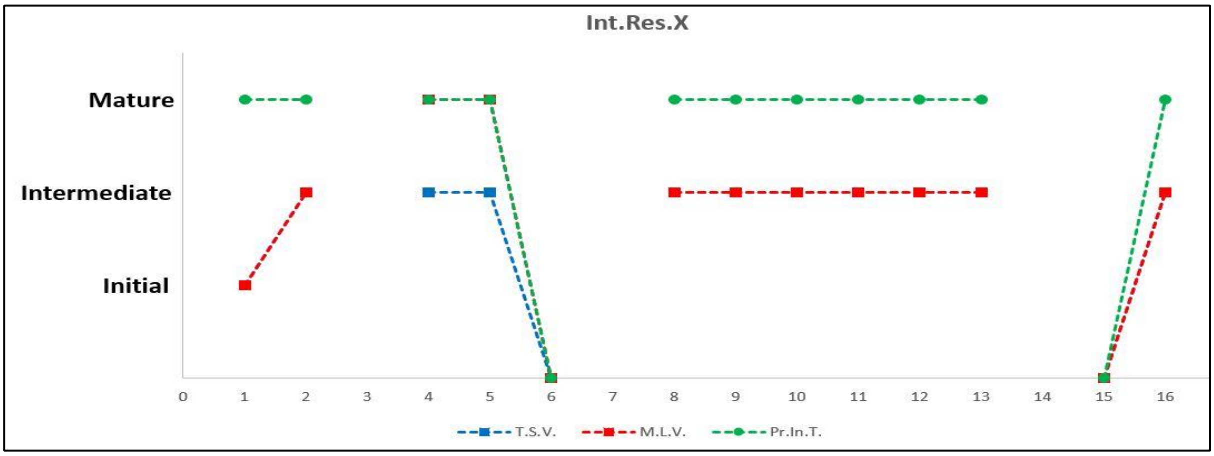


Diagram 10-3. Activation for competency aspect (Int.Res.X).

Competency aspect activations: location and frequencies

The competency activations by units were as follows: Periodic Functions [1-2], Exponential Growth & Regression [4-5], Population Dynamics [8-13], and Integrals & Modeling [16]. There was no activation of the competency aspect (Int.Res.X) during the last session of the Exponential Growth & Regression [6] and the first session [15] of the Integrals & Modeling units. The frequency of activation per units was [2, 2, 6, 1] and the total frequency of activation was, $n=11$.

Examples from the activations

Activation [1]: Evidence for the activation of the competency aspect (Int.Res.X) was provided by Erika when she noticed a particular part of the graph the teacher drew, “I see you started your graph [showing January on the x-axis] one point to the right of zero.” The task asked the students to graph the data from a table of monthly temperatures (warm up task on Periodic Functions) provided to the students and produce a periodic function out of it. She interpreted a mathematical result in an extra-mathematical context (monthly temperatures from an airport) and attempted to generalize the teacher’s solution for her task-situation because she had a different yet correct graph. She interpreted the result without any vision towards the solution (T.S.V., at Initial Activation Level, I) without knowing why the teacher asked this questions and she expressed herself using everyday expressions with no evidence of mathematical language (M.L.V., at Initial Activation Level, I). All that occurred with no prompting (Pr.In.T., at Mature Activation Level, III) from the teacher or anyone else in the class.

Activation [9]: Evidence for the activation of the competency aspect (Int.Res.X) was provided by Erika when she replied to a teacher’s question while working on the recycle task. After dividing the quantities (amount of recyclables in tones) from the table we found the value: 1.046; the teacher showing this value asked “but what is the rate?” and Erika replied “1.046 represents 4.26 in percentage” which was the correct answer. She made this interpretation in an extra-mathematical context (amount of recyclables in Minnesota) adapting to the task-situation context. She made this interpretation having a mid-range perspective towards the solution (T.S.V., at Intermediate Activation Level, II) being close to estimate when recycling will reach three million tons. She expressed herself making regular use of expressions with mathematical language (M.L.V., at Intermediate Activation Level, II) with minimal prompting from the teacher (Pr.In.T., at Mature Activation Level, III).

Summary of evidence

The summary of evidence regarding the changes within the competency activation levels and the consistency levels are displayed bellow.

Table 10-3. Summarized evidence for (Int.Res.X) activation.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	2	2	1	
Changes from higher to lower	1	1	1	
High consistency	1	1	1	
Moderate consistency	0	0	0	
Frequency of activation				11

To critique the model by reviewing or reflecting and questioning the results. (code: Cr.) (7)

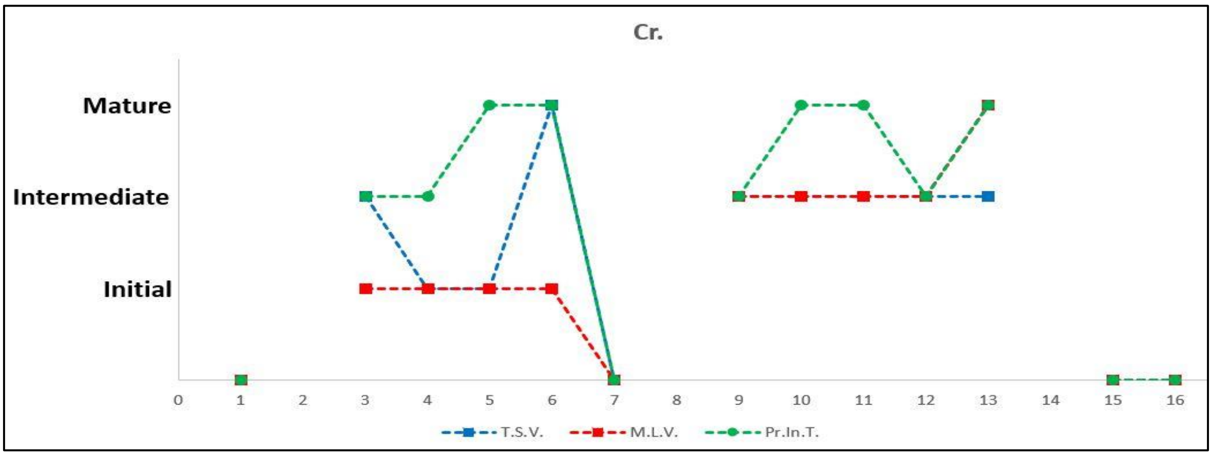


Diagram 10-4. Activation for competency aspect (Cr.).

Competency aspect activations: location and frequencies

The competency activations by units were: Exponential Growth & Regression [3-6] and Population Dynamics [9-13]. There was no activation of the (Cr.) competency aspect during the Periodic Functions, the last session of Exponential Growth & Regression [7], and the Integrals & Modeling units. The frequency of activation per units was [0, 4, 5, 0] and the total frequency of activation was, *n=9*.

<u>Examples from the activations</u>

Activation [6]: Evidence for the activation of the competency aspect (Cr.) was provided by Erika when she reflected on the result of the E coli task. She considered if she could use the "2^{3x}" expression (suggested by the teacher) to calculate time *t* from the equation: $2^t = 3.5 \cdot 10^{39}$. She said, "[I] suppose the '$3x$' part should be the time" having a long-range perspective towards the solution (T.S.V., at Mature Activation Level, III) while expressing herself making regular use of expressions with mathematical language (M.L.V., at Intermediate Activation Level, II). All that occurred with minimal prompting (Pr.In.T., at Mature Activation Level, III) with the teacher simply asking what they think about the aforementioned expression (i.e., 2^{3x}).

<u>Summary of evidence</u>

The summary of evidence regarding the changes within the competency activation levels and the consistency levels are displayed bellow.

Table 10-4. Summarized evidence for (Cr.) activation.

Type of evidence \ **Competency domain**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	1	1	3	
Changes from higher to lower	2	1	2	
High consistency	1	1(1)	0	
Moderate consistency	(1)	0	3	
Frequency of activation				9

<u>Look and search for available information and to differentiate between relevant and irrelevant information. (code: Info) (8)</u>

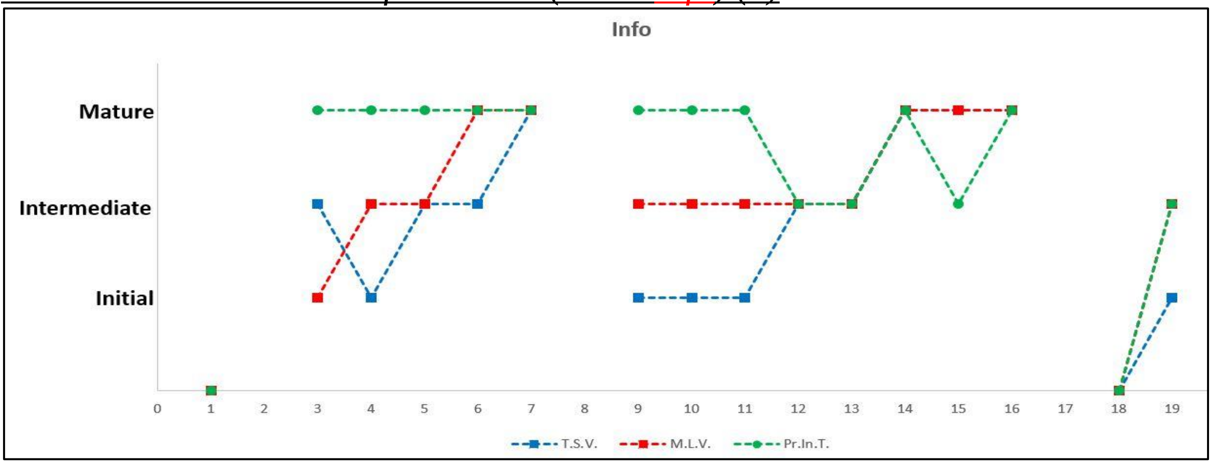


Diagram 10-5. Activation for competency aspect (Info).

Competency aspect activations: location and frequencies

The competency activations by units were: Exponential Growth & Regression [3-7], Population Dynamics [9-16], and Integrals & Modeling [19]. There was no activation of the (Info) aspect during the Periodic Functions and the first session [18] of the Integrals & Modeling units. The activation frequency per units was [0, 5, 8, 1] and the total activation frequency, $n=14$.

Examples from the activations

Activation [6]: Evidence for the activation of the competency aspect (Info) was provided by Erika when she claimed that "we need [the] generation time and the number of bacteria at the end" when asked to find the date when the world's population will be doubled using any kind of information. She proceeded to this claim having a mid-range perspective towards the solution (T.S.V., at Intermediate Activation Level, II) while expressing herself making regular and correct use of expressions with mathematical language (M.L.V., at Mature Activation Level, III). All that occurred with minimal prompting (Pr.In.T., at Mature Activation Level, III) because the teacher asked the students if they can "find a relationship between the number of the cells at the beginning and the number of the cells at the end."

Activation [19]: Evidence for the activation of the competency aspect (Info) was provided by Erika when asked if she remembered how the class solved the task of the previous week (consumption rate on earth's petroleum) by searching her available notes knowing where to look for them and saying "we were talking about the consumption rate using the increase at the rate of petroleum consumption" She had a short-range perspective towards the solution (T.S.V., at Initial Activation Level, I) while expressing herself making regular use of expressions with mathematical language (M.L.V., at Intermediate Activation Level, II). All that occurred with occasional prompting (Pr.In.T., at Intermediate Activation Level, II) because the teacher asked the students what they did last time so suggested that there is a connection to their latest notes.

Summary of evidence

The summary of evidence regarding the changes within the competency activation levels and the consistency levels are displayed bellow.

Table 10-5. Summarized evidence for (Info) activation.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	5	4	3	
Changes from higher to lower	1	0	2	
High consistency	0	0	1	
Moderate consistency	2(1)	4	1	
Frequency of activation				14

To choose appropriate mathematical graphical notations during the task solution process. (code: Graph) (9)

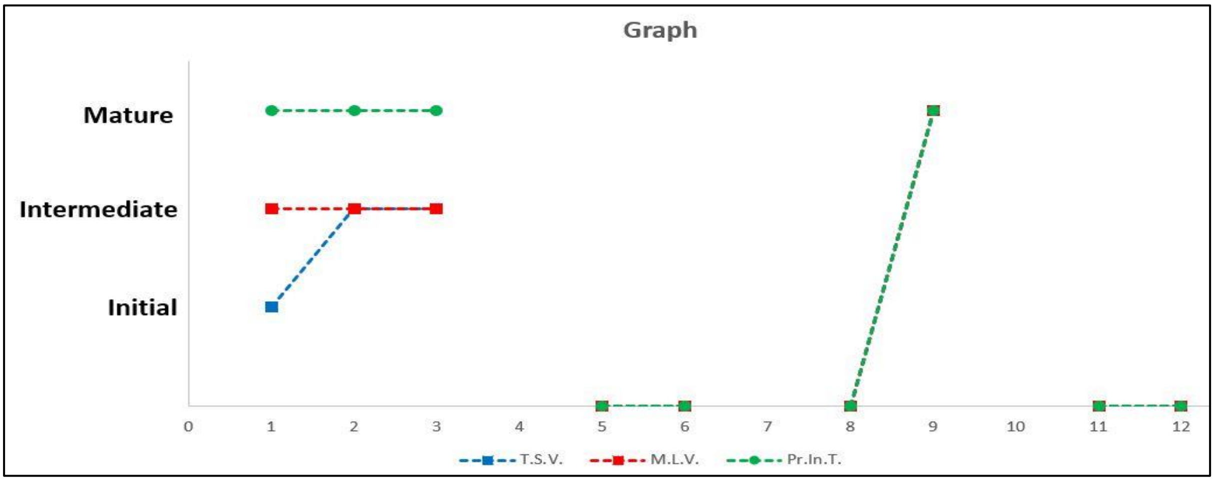


Diagram 10-6. Activation for competency aspect (Graph).

Competency aspect activations: location and frequencies

The competency activations by units were as follows: Periodic Functions [1-3] and Population Dynamics [9]. The activations during the Periodic Functions occurred during her written work only and not while discussing issues with her classmate of the teacher. There was no activation of the (Graph) competency aspect during the Exponential Growth & Regression, the first session [8] of the Population Dynamics, and the Integrals & Modeling units. The frequency of activation per units was [3, 0, 1, 0] and the total frequency of activation was, *n=4*.

Examples from the activations

Activation [3]: Evidence for the activation of the competency aspect (Graph) was provided by Erika when she correctly sketched the graph that modeled a creature's temperature. She sketched the graph having a mid-range perspective towards the solution (T.S.V., at Intermediate Activation Level, II) narrating while graphing, "it should focus on my starting point again, similarly to the previous problem" making regular use of expressions with mathematical language (M.L.V., at Intermediate Activation Level, II). All that occurred with no prompting (Pr.In.T., at Intermediate Activation Level, II) from the teacher until she produced the final graph.

Summary of evidence

Regarding the activation of the particular competency aspect of the modeling competency, the frequency of activation is rather low and almost all activations have been noticed during the Periodic Functions unit. Therefore, I do not provide a table of evidence for this case. That said, I observed that Erika worked independently and somehow comfortably while sketching by understanding what each part of the "$g(x) = a\cos[b(x-c)] + d$" function represented on the graph. She gradually had a clearer perspective towards the solution of both tasks.

Representing and Manipulating Symbolic Forms

Switch between representations. (code: Rep.↔Rep.) (11)

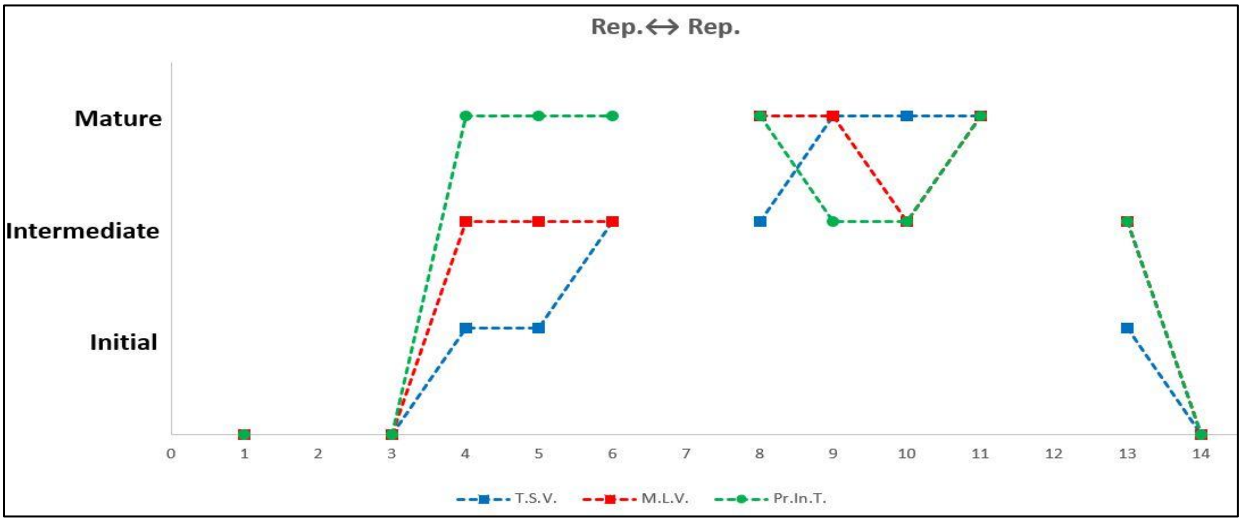


Diagram 10-7. Activation for competency aspect (Rep.↔Rep.)

Competency aspect activations: location and frequencies

The competency activations by units were as follows: Exponential Growth & Regression [3-6], Population Dynamics [8-11], and Integrals & Modeling [13]. There was no activation of the (Rep.↔Rep.) competency aspect during the Periodic Functions, the first session [3] of the Exponential Growth & Regression, and the last session [14] of the Integrals & Modeling units. The frequency of activation per units was [0, 3, 4, 1] and the total frequency of activation was, $n=8$.

Examples from the activations

Activation [13]: This activation is an example of a concurrent competency activation (two competency aspects activated simultaneously) where Erika provided evidence of competency aspect activation for (Rep.↔Rep.) at the same time with the (C.D.Rep.) aspect. Activation [13] here is activation [25] of the (C.D.Rep.) competency aspect that was presented in the previous result (Diagram 5-3). This occurred because apart from decoding the representation (as described in the previous aspect analysis) she also switched between different representations. When the teacher asked for help on "how do we write degrees per year" she initially wrote "$\frac{\Delta f}{\Delta t}$" but she switched to "$\frac{df}{dt}$" and eventually to "$f\,'(t)$" which is the translation from a natural language to a symbolic one that what was discussed in subsection 4.1.3. She switched between these representations having a short-range perspective towards the solution (T.S.V., at Initial Activation Level, I) being uncertain for the purpose of such an expression. She expressed herself making regular use of expressions with mathematical language and being familiar with the notation she used (M.L.V., at Intermediate Activation Level, II) with occasional prompting from the teacher (Pr.In.T., at Intermediate Activation Level, II).

Summary of evidence

The summary of evidence regarding the changes within the competency activation levels and the consistency levels are displayed bellow.

Table 10-6. Summarized evidence for (Rep.↔Rep.) activation.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	3	2	2	
Changes from higher to lower	1	2	2	
High consistency	1	1	1	
Moderate consistency	(1)	1	1	
Frequency of activation				8

Manipulate within a representation. (code: Man.Rep.) (12)

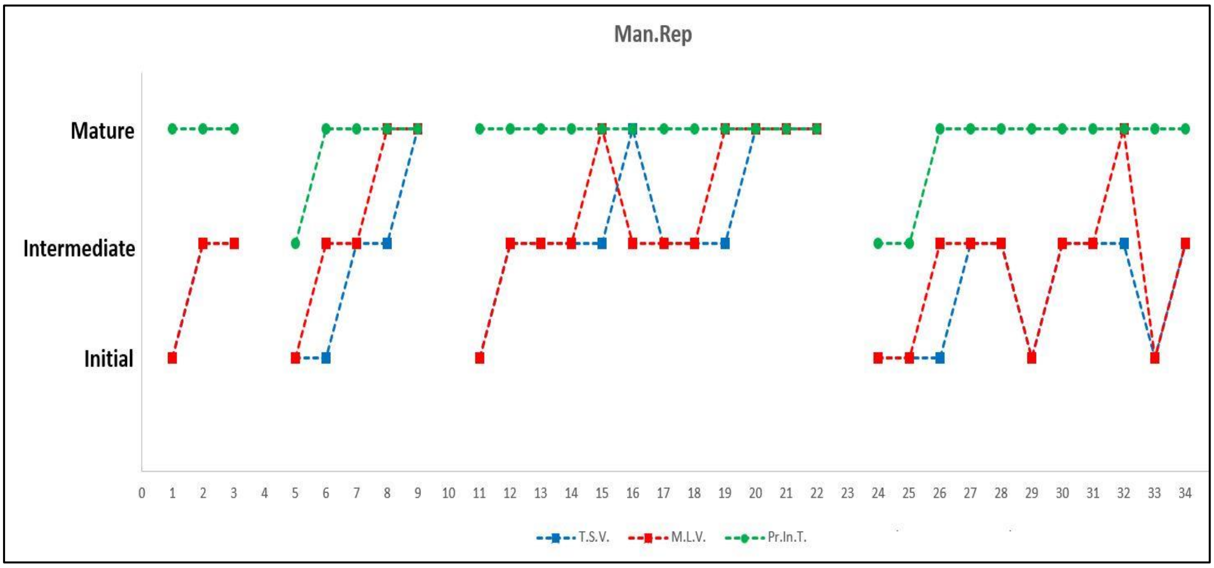


Diagram 10-8. Activation for competency aspect (Man.Rep.).

Competency aspect activations: location and frequencies

The competency activations by units were as follows: Periodic Functions [1-3], Exponential Growth & Regression [5-9], Population Dynamics [11-22], and Integrals & Modeling [24-34]. The frequency of activation per units was [3, 5, 12, 11] and the total frequency of activation was, *n=31*.

Examples from the activations

Activation [15]: Evidence for the activation of the competency aspect (Man.Rep.) was provided by Erika when she manipulated a representation from the E coli task. She worked on calculating time *t* from the equation: $2^t = 3.5 \cdot 10^{39}$, but from the description of the problem, time *t* was given in sets of 20 minutes. Erika, when calculated time *t*, said that, “131.9 represents the 20 minutes sets so we need to divide this by three.” She had a mid-range perspective towards the solution (T.S.V., at Intermediate Activation Level, II) while expressing herself making regular and correct use of expressions with mathematical language (M.L.V., at Mature Activation Level, III). All that occurred while she was working independently (Pr.In.T., at Mature Activation Level, III).

Activation [24]: Evidence for the activation of the competency aspect (Man.Rep.) was provided by Erika when during the discussion on the Fundamental Theorem of Calculus, she worked on the expression: ‘$F(x) = \int_a^x f(t)\,dt$’ where $F'(x) = f(x)$ and f is continuous on an interval I containing the point a. She successfully remembered to write down “$\int_a^x f(t)\,dt = F(x) = G(x) + c$” where $G'(x) = f(x)$ on I and c some constant. She had no perspective towards the solution (T.S.V., at Initial Activation Level, I), expressing herself making occasional use of everyday expressions with limited mathematical language (M.L.V., at Initial Activation Level, I) with occasional prompting from the teacher (Pr.In.T., at Intermediate Activation Level, II).

Summary of evidence

The summary of evidence regarding the changes within the competency activation levels and the consistency levels are displayed bellow.

Table 10-7. Summarized evidence for (Man.Rep.) activation.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	9	10	2	
Changes from higher to lower	3	3	0	
High consistency	0	0	4	
Moderate consistency	2(1)	3	0	
Frequency of activation				31

Mathematical Reasoning and Communicating competency

To understand others’ written, visual or oral ‘texts’, in a variety of linguistic registers, about matters having a mathematical content. To follow and assess chains of arguments, put forward by others. (code: Reason←) (13)

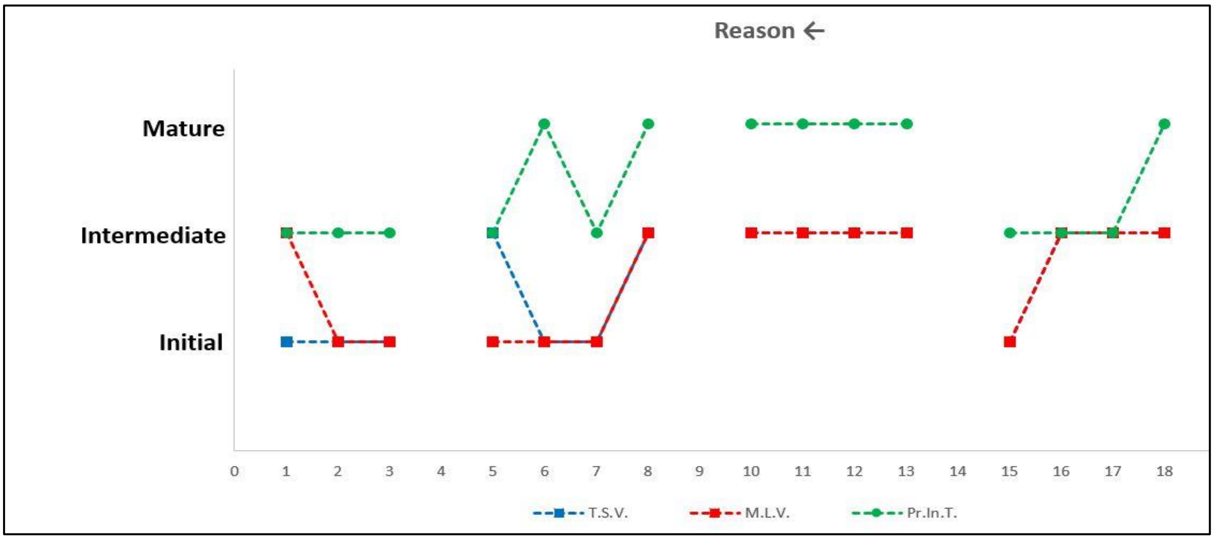


Diagram 10-9. Activation for competency aspect (Reason←).

Competency aspect activations: location and frequencies

The competency activations by units were as follows: Periodic Functions [1-3], Exponential Growth & Regression [5-8], Population Dynamics [10-13], and Integrals & Modeling [15-18]. The frequency of activation per units was [3, 4, 4, 4] and the total frequency of activation was, $n=15$.

Examples from the activations

Activation [2]: Evidence for the activation of the competency aspect (Reason←) was provided by Erika when she noticed the teacher's graph for the Airport Temperatures task. She said, "I see you started your graph on January one point to the right of the zero" and identified a difference in relation to her graph. She aimed to understand the teacher's graph and reacted by providing argumentation regarding her interpretation of the graph. She had a short-range perspective towards the solution (T.S.V., at Initial Activation Level, I) while expressing herself with everyday expressions with limited mathematical language (M.L.V., at Initial Activation Level, I). All that occurred with occasional prompting (Pr.In.T., at Intermediate Activation Level, II) from teacher who was leading the discussion at that point.

Activation [18]: Evidence for the activation of the competency aspect (Reason←) was provided by Erika while she was working on the petroleum consumption task and talking about the future amount of petroleum $M(t)$ which should be zero. The teacher said, "so we need $M(t) = 0$, but we the task gives us the $M'(t)$ and not the $M(t)$. Does this make sense to you?" To this, Erika replied by mentioning that "[this expression] makes sense because we have the rate not the final value, yes it makes sense." She had a mid-range perspective towards the solution (T.S.V., at Intermediate Activation Level, II) being close to work on calculating the relevant integral while expressing herself making regular use of expressions with mathematical language (M.L.V., at Intermediate Activation Level, II). She had minimal prompting from the teacher (Pr.In.T., at Mature Activation Level, III) while working on this part of the task.

Summary of evidence

The summary of evidence regarding the changes within the competency activation levels and the consistency levels are displayed bellow.

Table 10-8. Summarized evidence for (Reason←) activation.

Competency domain / Type of evidence	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	2	2	3	
Changes from higher to lower	1	1	1	
High consistency	2(1)	2(1)	3	
Moderate consistency	(1)	(1)	0	
Frequency of activation				15

Johannes

Mathematical Thinking and Acting competency

To pose questions that have a mathematical characteristic or to know the kinds of answers that mathematics may offer (entry). (code: P.Q.) (1)

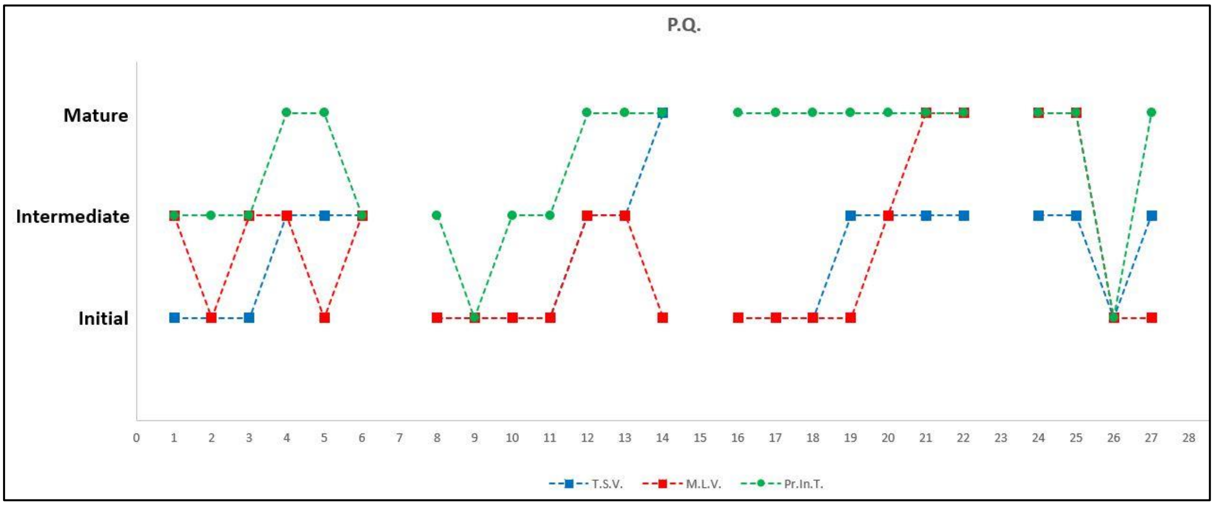


Diagram 10-10. Activation for competency aspect (P.Q.).

Competency aspect activations: location and frequencies

The competency activations by units were as follows: Periodic Function [1-6], Exponential Growth & Regression [8-14], Population Dynamics [16-22], and Integrals & Modeling [24-27]. The frequency of activation per units was [6, 7, 7, 4] and the total frequency of activation was, *n=24*.

Examples from the activations

Activation [1]: Evidence for the activation of the competency aspect (P.Q.) was provided by Johannes when he asked the teacher, "are we supposed to use 17.9 or a different number…is that the amplitude?" when Erika (his classmate) asked "what does *a* stands for? amplitude?" regarding the following equation $g(x) = a\cos[b(x-c)] + d$, while working on the Airport Temperatures task. He posed this question having a short-range perspective towards the solution (T.S.V., at Initial Activation Level, I) while expressing himself making regular use of expressions with mathematical language (M.L.V., at Intermediate Activation Level, II). All that occurred with occasional prompting (Pr.In.T., at Intermediate Activation Level, II) because his question originated from Erika's comment.

Activation [13]: Evidence for the activation of the competency aspect (P.Q.) was provided by Johannes when he asked one of his classmates, "how big should a cluster of cancer cells should be to be visible?" during the Exponential Growth & Regression unit and the Lung Cancer Cells task. He posed this question having a mid-range perspective towards the solution (T.S.V., at Intermediate Activation Level, II) being close to figure out the date when the cluster will be detectable by x-ray by asking about the optimal volume of the cells cluster. He expressed himself making regular use of expressions with mathematical language (M.L.V., at Intermediate Activation Level, II) with minimal prompting from the teacher (Pr.In.T., at Mature Activation Level, III).

Summary of evidence

The summary of evidence regarding the changes within the competency activation levels and the consistency levels are displayed bellow.

Table 10-9. Summarized evidence for (P.Q.) activation.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	5	5	4	
Changes from higher to lower	1	4	3	
High consistency	0	0	1	
Moderate consistency	3(2)	1(3)	3	
Frequency of activation				24

To understand and handle the scope and limitations of a given concept (entry). (code: Lim.) (2)

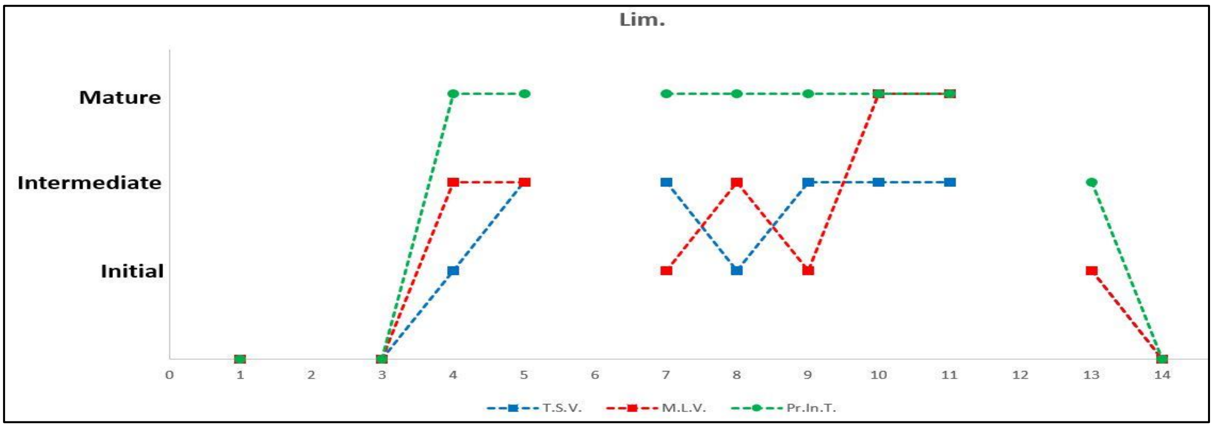


Diagram 10-11. Activation for competency aspect (Lim.).

Competency aspect activations: location and frequencies

The competency activations by units were as follows: Exponential Growth & Regression [3-5], Population Dynamics [7-11], and Integrals & Modeling [13]. There was no activation of the (Lim.) competency aspect during the Periodic Functions, the first session [3] of the Exponential Growth & Regression, and the last session [14] of the Integrals & Modeling units. The frequency of activation per units was [0, 2, 5, 1] and the total frequency of activation was, *n=8*.

Examples from the activations

Activation [9]: Evidence for the activation of the competency aspect (Lim.) was provided by Johannes when he indicated that he grasped the scope of the exponential growth concept during the E. Coli task. When begun to work on the task he said, "the starting point is a single cell…there will be 2, 4, 8… so we have to find what happens by the end of 24 hours." He expressed this limitation having a mid-range perspective towards the solution (T.S.V., at Intermediate Activation Level, II) while expressing himself making regular use of expressions with mathematical language (e.g., "the starting point is a single cell") (M.L.V., at Intermediate Activation Level, II). All that occurred with minimal prompting (Pr.In.T., at Intermediate Activation Level, III) because the teacher just repeated the question of the task "If a single cell of the bacterium E. coli divides every 20 minutes, how many E. coli would be there in 24 hours?"

Summary of evidence

The summary of evidence regarding the changes within the competency activation levels and the consistency levels are displayed bellow.

Table 10-10. Summarized evidence for (Lim.) activation.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	3	2	0	
Changes from higher to lower	2	2	1	
High consistency	0	0	1	
Moderate consistency	1	1	1	
Frequency of activation				8

Mathematical Modeling competency

To be able to assess the range and validity of existing models. (code: R.+V.) (4)

Competency aspect activations: note for location and frequencies

Johannes provided evidence for the activation of the competency aspect (R.+V.) only on three occasions. Twice during the Population Dynamics unit and once at the last session of the Integrals & Modeling unit. There was no evidence of competency activation during the Periodic Functions and the Exponential Growth & Regression units.

Examples from the activations

During the Population Doubled task, Johannes provided evidence for the activation of the competency aspect (R.+V.) when he searched for his notes from previous sessions to identify which model should be valid for the needs of the particular task, "I remember when we discussed about doublings in general that we used an expression with initial and final number of bacteria." He had a long-range perspective towards the solution (T.S.V., at Mature Activation Level, III) being close to estimate the date when the world's population will be doubled. He expressed himself making regular use of expressions with mathematical language (M.L.V., at Intermediate Activation Level, II) with minimal prompting from the teacher (Pr.In.T., at Mature Activation Level, III).

Summary of evidence

The low frequency of activation and the fact that there was no evidence of competency activation during the Periodic Functions and the Exponential Growth & Regression units resulted in considering that Johannes, regarding the activation of (R.+V.) competency aspect, provided limited evidence.

To interpret and translate the elements of a model during the mapping process. In this situation, the student should be able to understand the real problem and adapt a model based on reality. (code: Int.El.) (5)

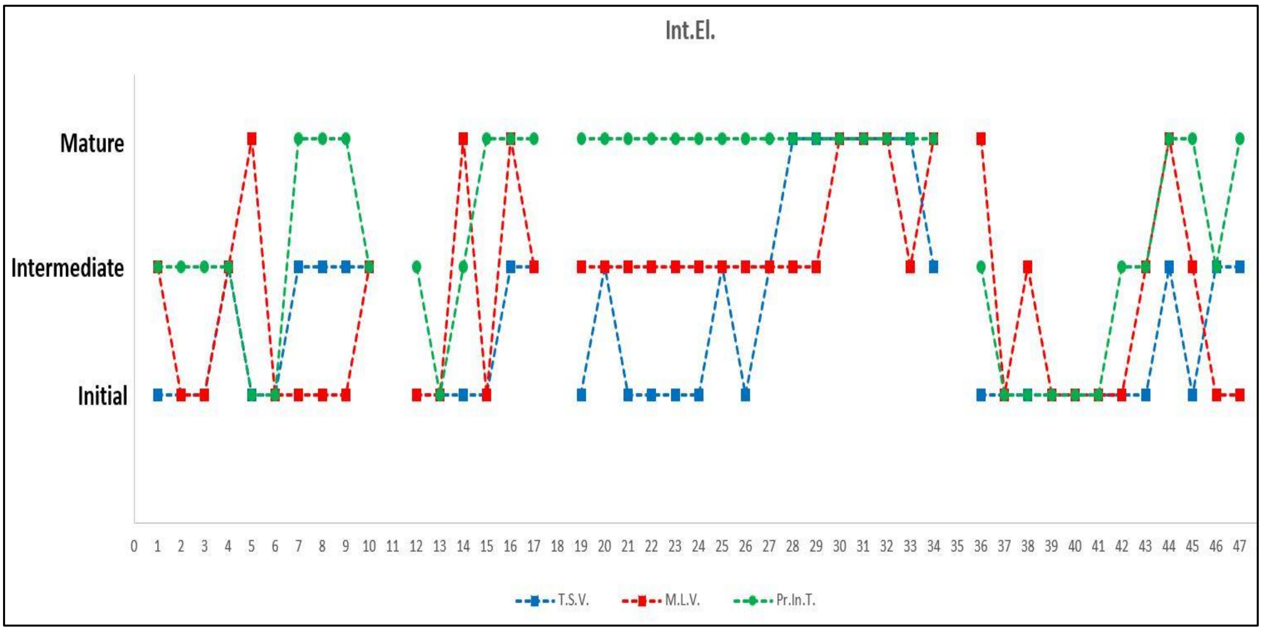


Diagram 10-12. Activation for competency aspect (Int.El.).

Competency aspect activations: location and frequencies

The competency activations by units were as follows: Periodic Functions [1-10], Exponential Growth & Regression [12-17], Population Dynamics [19-34], and Integrals & Modeling [36-47]. The frequency of activation per units was [10, 6, 16, 12] and the total frequency of activation was, *n=44*.

Examples from the activations

Activation [8]: Evidence for the activation of the competency aspect (Int.El.) was provided by Johannes when he interpreted the elements of the equation $g(x) = \mathrm{a}\cos[b(x - c)] + d$, while working on the Airport Temperatures task. He said, "So for *d* we take this point (showing the lowest point on the graph) lowest point plus the amplitude" He made this interpretation having a mid-range perspective towards the solution (T.S.V., at Intermediate Activation Level, II) while expressing himself with limited mathematical language and gestures (M.L.V., at Initial Activation Level, I). All that occurred with minimal prompting from the teacher (Pr.In.T., at Mature Activation Level, III). because the teacher let the students to begin working on the problem on their own with no initial help.

Activation [45]: Evidence for the activation of the competency aspect (Int.El.) was provided by Johannes while he was working on the Petroleum Consumption Rate task. In particular he replied to a teacher's question regarding the expression '$C(t) = C(0) \cdot e^{rt}$', having for $C(0)$, the value: 1.02. The teacher asked Johannes if he understood the expression and Johannes replied, "Yes, so we have the following expression: $1.02 = e^{r}$, and need to find the *r*." He interpreted the elements of the model having a short-range perspective towards the solution not sure what will follow (T.S.V., at Initial Activation Level, I) while expressing himself making regular use of expressions with mathematical language (M.L.V., at Intermediate Activation Level, II) with minimal prompting from the teacher (Pr.In.T., at Mature Activation Level, III).

Summary of evidence

The summary of evidence regarding the changes within the competency activation levels and the consistency levels are displayed bellow.

Table 10-11. Summarized evidence for (Int.El.) activation.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	9	10	6	
Changes from higher to lower	5	8	4	
High consistency	0	1	1	
Moderate consistency	2(2)	(2)	2(1)	
Frequency of activation				44

To interpret mathematical results in an extra-mathematical context and generalize the solutions developed for special task-situation. (code: Int.Res.X) (6)

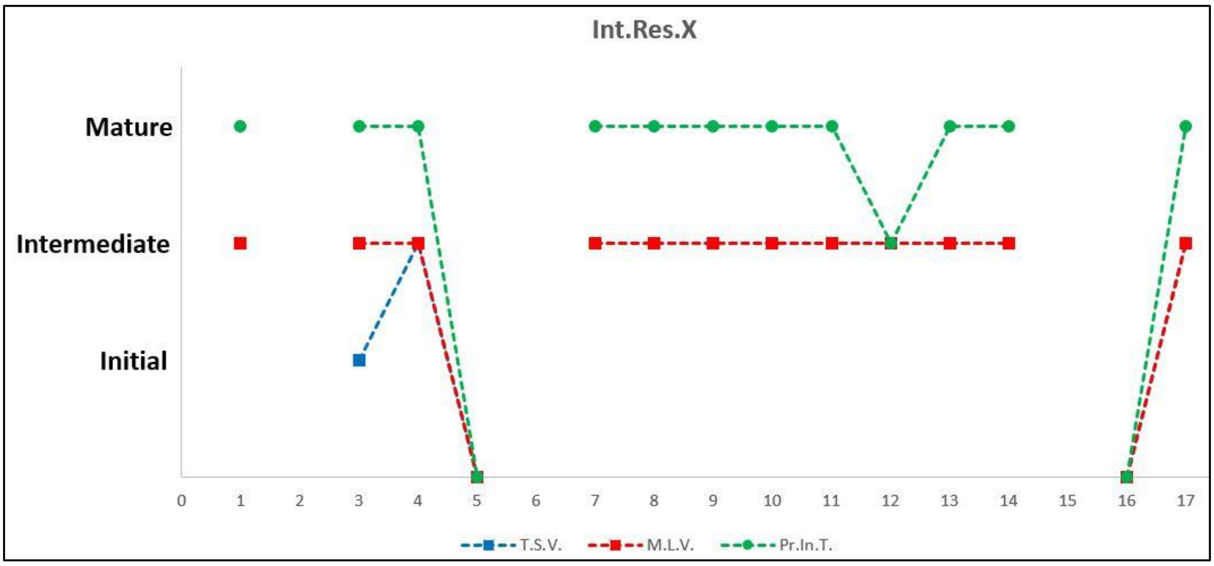


Diagram 10-13. Activation for competency aspect (Int.Res.X).

Competency aspect activations: location and frequencies

The competency activations by units were as follows: Periodic Functions [1], Exponential Growth & Regression [3-4], Population Dynamics [7-14], and Integrals & Modeling [17]. There was no activation of the (Int.Res.X) competency aspect during the second session [5] of the Exponential Growth & Regression, and the first session [16] of the Integrals & Modeling units. The frequency of activation per units was [1, 2, 8, 1] and the total frequency of activation was, *n=12*.

Examples from the activations

Activation [10]: Evidence for the activation of the competency aspect (Int.Res.X) was provided by Johannes while he was working on finding information on birth rates during the Population Dynamics unit. In particular he interpreted that the birth rate is "not negative [1.7% to 1.1%] but less than before" using also gestures and projecting this result to an extra-mathematical world when he reacted to the teacher's comment on declining population rates. He had a mid-range perspective towards the solution and the utility of such an approach (T.S.V., at Intermediate Activation Level, II) while expressing himself making regular use of expressions with mathematical language (M.L.V., at Intermediate Activation Level, II). All that occurred with occasional prompting (Pr.In.T., at Intermediate Activation Level, II) from the teacher.

Summary of evidence

The summary of evidence regarding the changes within the competency activation levels and the consistency levels are displayed bellow.

Table 10-12. Summarized evidence for (Int.Res.X) activation.

Type of evidence \ Competency domain	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	2	1	2	
Changes from higher to lower	1	1	2	
High consistency	1	1	0	
Moderate consistency	0	0	1	
Frequency of activation				12

To critique the model by reviewing or reflecting and questioning the results. (code: Cr.) (7)

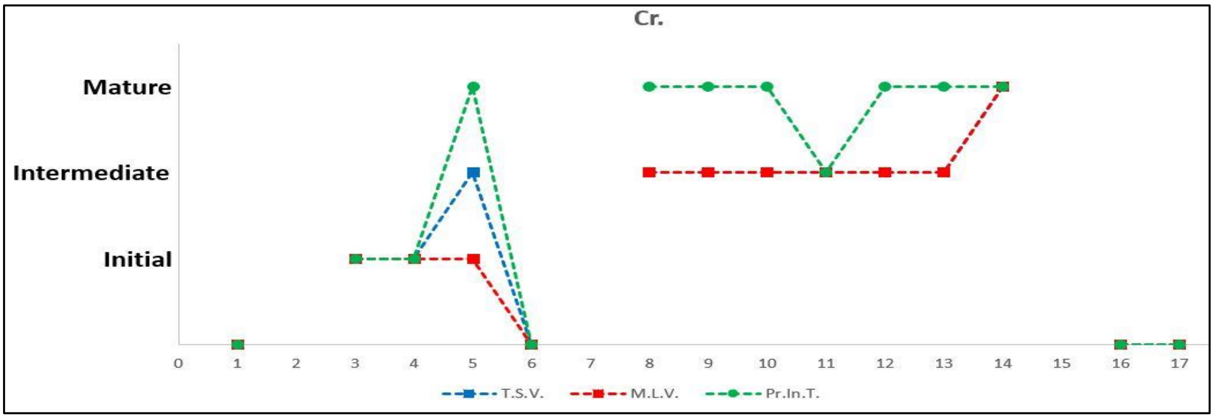


Diagram 10-14. Activation for competency aspect (Cr.).

Competency aspect activations: location and frequencies

The competency activations by units were: Exponential Growth & Regression [3-5], Population Dynamics [8-14]. There was no activation of the (Cr.) aspect during the Periodic Functions, the second session [6] of the Exponential Growth & Regression, and the Integrals & Modeling units. The activation frequency per units was [0, 3, 7, 0] and the total frequency, *n=10*.

Examples from the activations

Activation [14]: Johannes provided evidence for the (Cr.) activation when he considered that “it is really difficult to predict the future [world’s population] when human factor is involved” while discussing the models who predict populations. He reflected on this model having a long-range perspective towards the nature of such problems (T.S.V., at Mature Activation Level, III) while using regular and correct use of expressions with mathematical language (M.L.V., at Mature Activation Level, III). All that occurred with minimal prompting (Pr.In.T., at Mature Activation Level, III) from the teacher who just initiated the discussion about the models they encountered.

Summary of evidence

The summary of evidence regarding the changes within the competency activation levels and the consistency levels are displayed bellow.

Table 10-13. Summarized evidence for (Cr.) activation.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	2	1	2	
Changes from higher to lower	1	1	2	
High consistency	1	1(1)	0	
Moderate consistency	(1)	0	2(1)	
Frequency of activation				10

Representing and Manipulating symbolic forms competency

Choose (or decode in) a representation. (code: C.D.Rep.) (10)

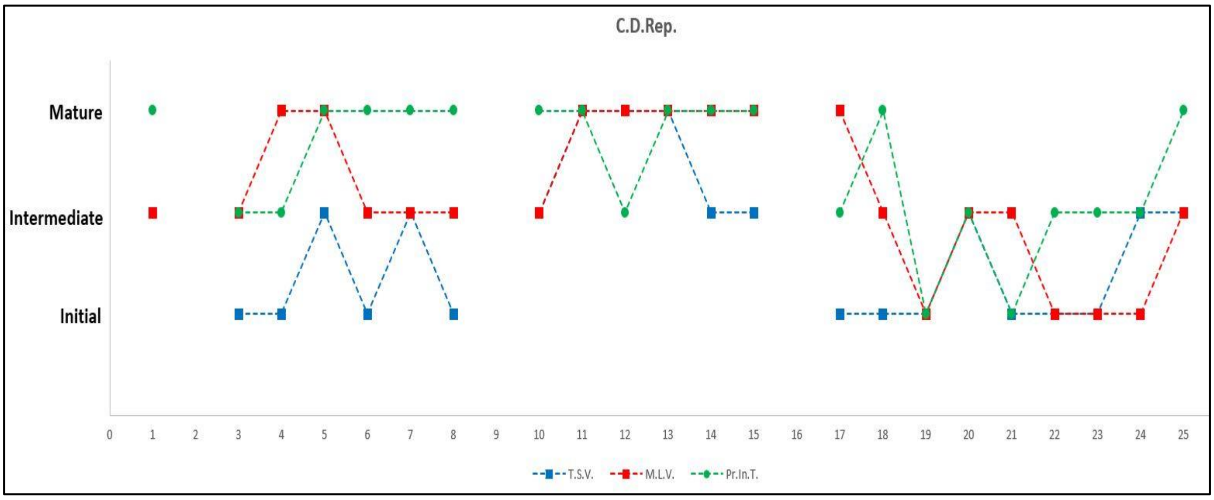


Diagram 10-15. Activation for competency aspect (C.D.Rep.).

Competency aspect activations: location and frequencies

The competency activations by units were as follows: Periodic Functions [1], Exponential Growth & Regression [3-8], Population Dynamics [10-15], and Integrals & Modeling [17-25]. The frequency of activation per units was [1, 6, 6, 9] and the total frequency of activation was, *n=22*.

Examples from the activations

Activation [7]: Evidence for the activation of the competency aspect (C.D.Rep.) was provided by Johannes when he suggested that "[they] can get the *t* down using the *ln* or the *log* [functions]" while working on a particular equation ($N_t = N_0e^{rt}$) during the Exponential Growth & Regression unit. He decoded this representation having a mid-range perspective towards the solution (T.S.V., at Intermediate Activation Level, II) while expressing himself making regular use of expressions with mathematical language (M.L.V., at Intermediate Activation Level, II). All that occurred with minimal prompting (Pr.In.T., at Intermediate Activation Level, III) because the teacher hinted that they need an equation that describes the temperature during the year and the graph, but he did not give any additional help.

Activation [24]: Evidence for the activation of the competency aspect (C.D.Rep.) was provided by Johannes while he was working on the Petroleum Consumption Rate task. He pointed out that "Basically, if it is not rate and just consumption the 2% every year is like year 1 is 2%, year 4 is 4% …" while working decoding the representation from the slides about the 2% increase at petroleum consumption rate. He had a mid-range perspective towards the solution (T.S.V., at Intermediate Activation Level, II) being close to realize the consumption rate. He expressed himself making use of everyday expressions with limited mathematical language (M.L.V., at Initial Activation Level, I) with occasional minimal prompting from the teacher (Pr.In.T., at Mature Activation Level, III) by asking Johannes to explore the "increases by 2% every year."

Summary of evidence

The summary of evidence regarding the changes within the competency activation levels and the consistency levels are displayed bellow.

Table 10-14. Summarized evidence for (C.D.Rep.) activation.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	5	4	6	
Changes from higher to lower	4	4	3	
High consistency	0	1	0	
Moderate consistency	1	1	2	
Frequency of activation				22

Mathematical Reasoning and Communicating competency

To express oneself, at various levels of theoretical and technical precision, in oral, visual or written form, about such matters. To provide explanations or justifications to support your result or idea. (code: Reason→) (14)

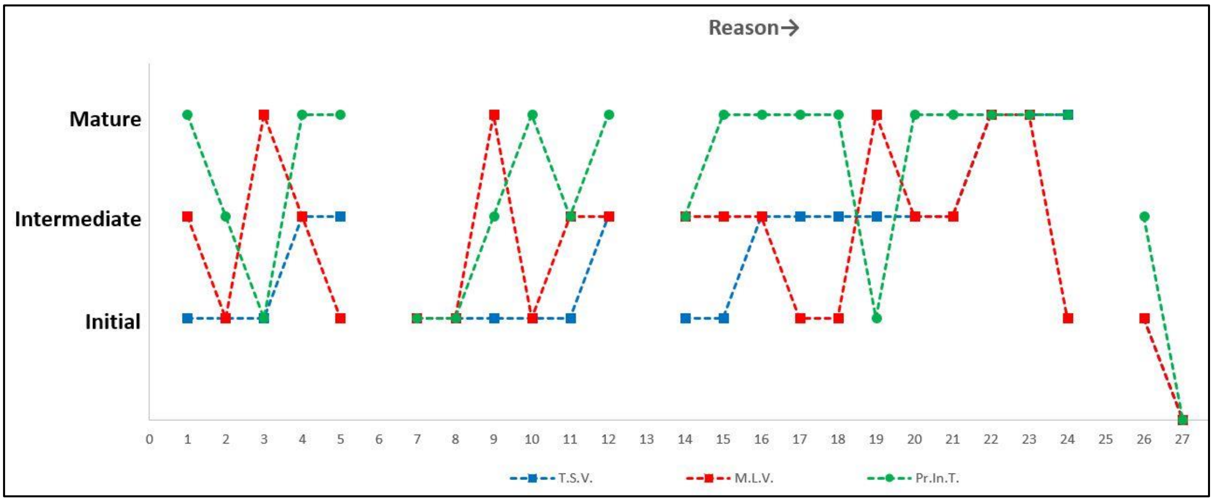


Diagram 10-16. Activation for competency aspect (Reason→).

Competency aspect activations: location and frequencies

The competency activations by units were: Periodic Functions [1-5], Exponential Growth & Regression [7-12], Population Dynamics [14-24], and Integrals & Modeling [27]. There was no activation of the (Reason→) aspect during the last session [27] of the Integrals & Modeling unit. The activation frequency per units was [5, 6, 11, 1] and the total frequency was, *n=23*.

Examples from the activations

Activation [4]: Evidence for the activation of the competency aspect (Reason→) was provided by Johannes during a discussion in class about an example with a closed environment with wolves and rabbits which was not part of their warm-up or main tasks. The teacher asked what could happen in such an environment and Johannes reasoned that "when the population of wolves is sinking then the population of rabbits is rising and vice versa." Johannes provided his reasoning having a mid-range clear perspective of what the hypothetical problem wanted them to infer (T.S.V., at Intermediate Activation Level, II) while expressing himself making regular use of expressions with mathematical language (M.L.V., at Intermediate Activation Level, II). All that occurred with limited prompting (Pr.In.T., at Mature Activation Level, III) because the teacher only posed the question by asking specifically what would happen if one of the two species for some reason will reduce its population.

Activation [19]: Evidence for the activation of the competency aspect (Reason→) was provided by Johannes when he described what he had to do to start working with the Population Doubled task. He reasoned that "first I have to find the current population which I did and then I need to find how much it's rising or decreasing." Johannes had a mid-range perspective towards the solution (T.S.V., at Intermediate Activation Level, II) showing awareness of the steps that he will need to follow. He expressed himself making regular and correct use of expressions with mathematical language (M.L.V., at Mature Activation Level, III) with occasional prompting from the teacher (Pr.In.T., at Intermediate Activation Level, II).

Summary of evidence

The summary of evidence regarding the changes within the competency activation levels and the consistency levels are displayed bellow.

Table 10-15. Summarized evidence for (Reason→) activation.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	4	5	6	
Changes from higher to lower	0	8	4	
High consistency	(1)	0	0	
Moderate consistency	2(1)	0	3	
Frequency of activation				23

APPENDIX E: Students' competency development results (remaining aspects)

In this appendix, I include all these competency development results from the competency aspects that were not included in the main body of the thesis in Section 6.1 and in particular in subsection 6.1.3 due to page restrictions and to avoid any unnecessary repetition.

Erika

Degree of Coverage

Mathematical Modeling competency

Overview for (Int.Res.X)

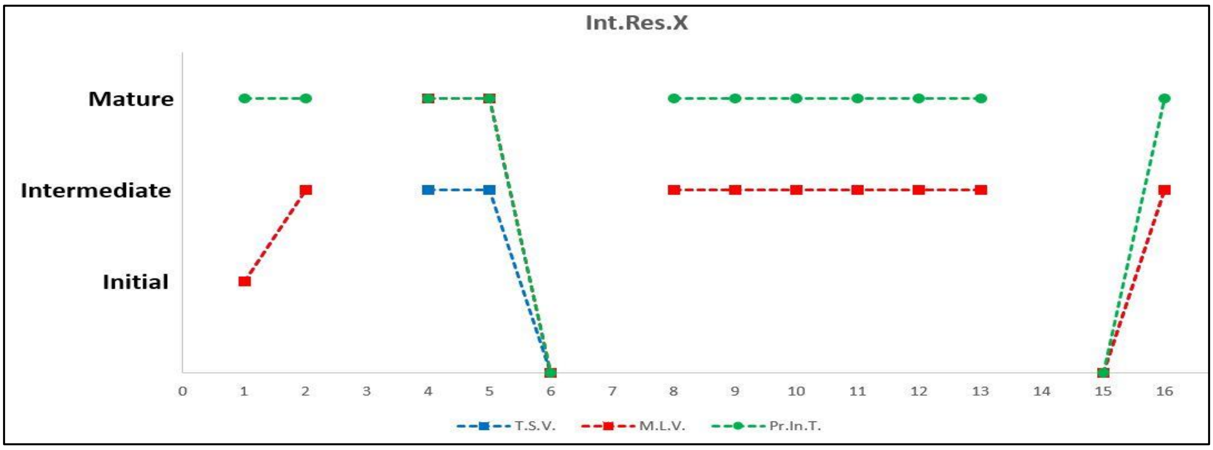


Diagram 10-17. Activation for competency aspect (Int.Res.X).

Table 10-16. Summarized evidence for (Int.Res.X) activation.

Competency domain / Type of evidence	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	2	2	1	
Changes from higher to lower	1	1	1	
High consistency	1	1	1	
Moderate consistency	0	0	0	
Frequency of activation				11

Frequency of competency activation

Regarding the first type of evidence, namely the *frequency of competency activation*, the more frequent activation located during the Population Dynamics unit. The rest of the sessions displayed less frequent activation. As shown in the diagram, the frequencies were [2, 2, 6, 1], and the total activation frequency was 11. There were two sessions where no evidence of competency activation was observed. All activations occurred above the Initial activation level except one that concerns the M.L.V. domain of the first activation.

Changes between activation levels

By incorporating the second type of evidence, *changes between activation levels*, the information from Table 6-2 informs us that the changes from lower to higher and higher to lower levels of competency activation by domain were: (2, 1) for T.S.V., (2, 1) for M.L.V., and (1, 1) for Pr.In.T. These observations suggest that the progress was more often from a lower to a higher level of activation, than the other way around. That said, the frequency of activation is rather low to make any strong inference regarding the development over time by examining through this type of evidence.

Levels of consistency

Regarding the third type of evidence, *levels of consistency*, Diagram 6-2 and Table 6-2 informs us that instances of consistency occurred only during the Population Dynamics unit because of the observed low activation frequency during the other units. These instances of high consistency occurred within the Mature or Intermediate level. This observations suggests evidence of competency development for the particular unit.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, I infer that there is a *moderate combination of evidence* for the development of the (Int.El.) aspect of the Mathematical Modeling competency.

Overview for (Cr.)

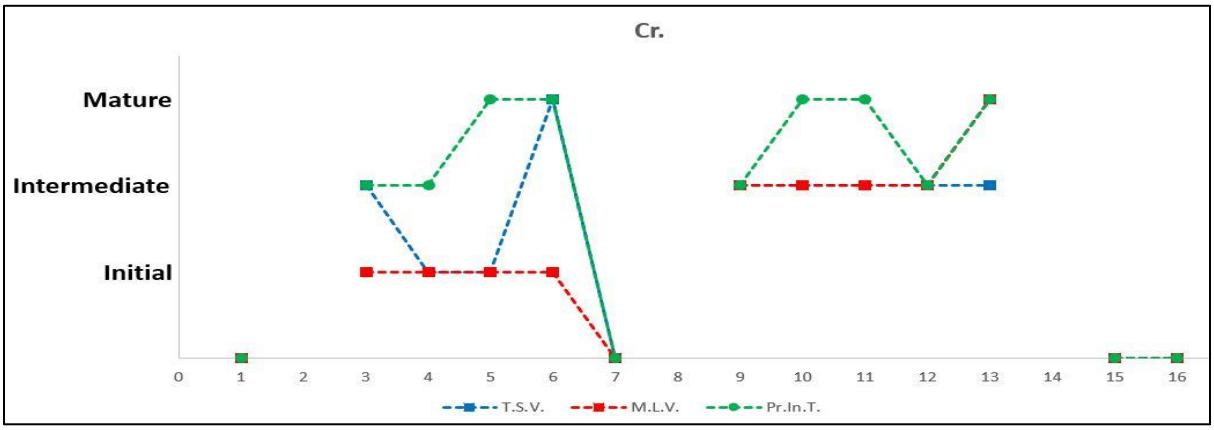


Diagram 10-18. Activation for competency aspect (Cr.)

Table 10-17. Summarized evidence for (Cr.) activation.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	1	1	3	
Changes from higher to lower	2	1	2	
High consistency	1	1(1)	0	
Moderate consistency	(1)	0	3	
Frequency of activation				9

Frequency of competency activation

Regarding the first type of evidence, namely the *frequency of competency activation*, the only observed activations have been located during the Exponential Growth & Regression (only during the first session of this unit) and the Population Dynamics units. No activation has been observed during the Periodic Functions and the Integrals & Modeling units. As shown in the diagram, the frequencies were [0, 4, 5, 0], and the total frequency of activation was 9.

Changes between activation levels

By incorporating the second type of evidence, *changes between activation levels*, the information from informs us that the changes from lower to higher and higher to lower levels of competency activation by domain were: (1, 2) for T.S.V., (1, 1) for M.L.V., and (3, 2) for Pr.In.T. These observations cannot provide reliable inferences regarding the progress of competency development because no direction (from lower to higher activation level or vice versa) was observed to be more often than the other.

Levels of consistency

Regarding the third type of evidence, *levels of consistency*, Diagram 6-2 and informs us that there was high or moderate consistency mainly at the Intermediate and Mature level of the scaling instrument. In particular, there were several occasions where Erika did not need any assistance or prompting to activate this competency aspect. The most noteworthy high consistency was observed mainly during the Population Dynamics unit, and this is also attributed to the fact that this was the only calculus unit where Erika activated the (Cr.) competency aspect in both sessions of the unit.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, I infer that there is a *moderate combination of evidence* for the development of the (Cr.) aspect of the Mathematical Modeling competency.

Overview for (Graph)

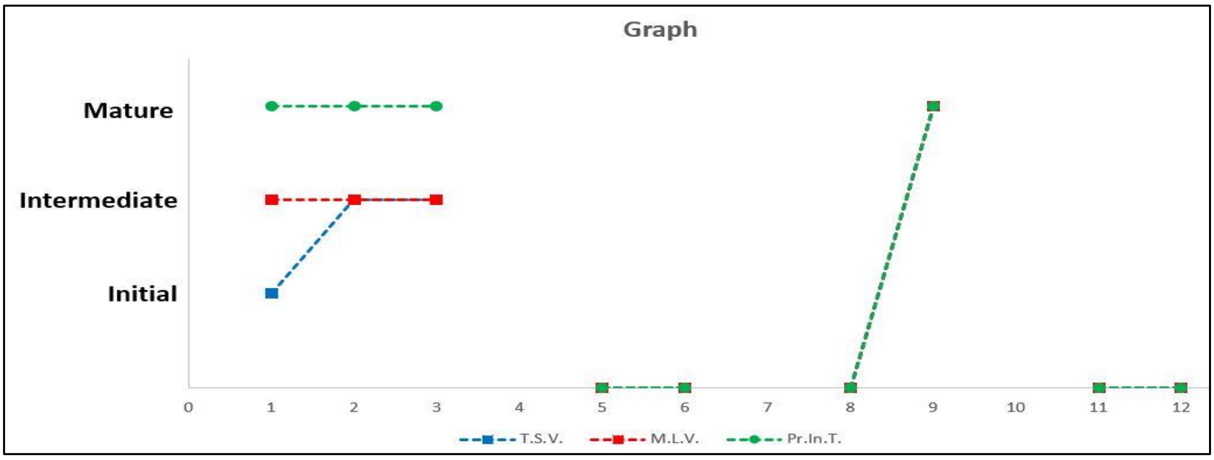


Diagram 10-19. Activation for competency aspect (Graph).

Frequency of competency activation

Regarding the first type of evidence, namely the *frequency of competency activation*, it is evident that activation has been observed mainly only during the Periodic Functions. This can be attributed to the fact that the tasks of the Periodic Unit specifically asked students to sketch several graphs. However, there were tasks in other units where students could possibly sketch a graph to assist themselves in understanding concepts discussed in these tasks. As shown in the diagram, the frequencies were [3, 0, 1, 0], and the total activation frequency was four.

Changes between activation levels

Due to the very low activation frequency, it is not feasible to make any inference for the development of this aspect through the analysis of this type of evidence. As mentioned above, activation was mainly observed during the Periodic Functions unit.

Levels of consistency

Regarding the third type of evidence, *levels of consistency*, Diagram 10-19 informs us that the only during the Periodic Functions, the activations displayed consistency in the Intermediate and Mature level of the scaling instrument.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, I infer that there is a *weak combination of evidence* for the development of the (Graph) aspect of the Mathematical Modeling competency.

Representing and Manipulating symbolic forms competency

Overview for (C.D.Rep)

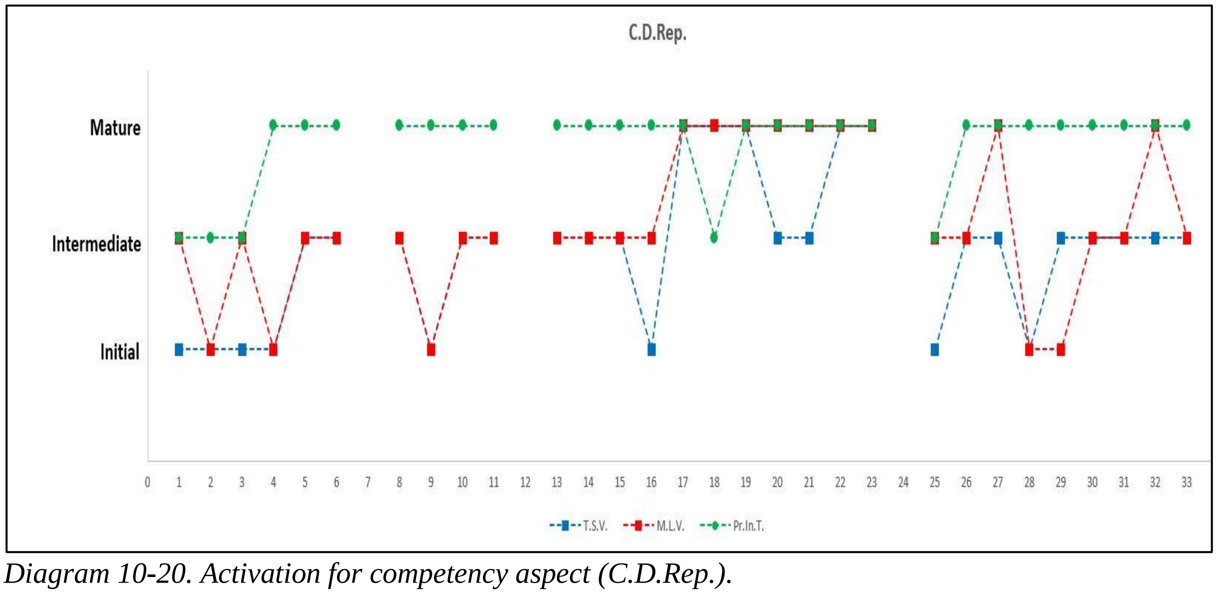


Diagram 10-20. Activation for competency aspect (C.D.Rep.).

Table 10-18. Summarized evidence for (C.D.Rep.) activation.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	6	7	3	
Changes from higher to lower	4	5	1	
High consistency	0	0	2	
Moderate consistency	2(1)	3	4	
Frequency of activation				30

Frequency of competency activation

Regarding the first type of evidence, namely the *frequency of competency activation*, frequent activation was located during the Population Dynamics and the Integrals & Modeling units, and the less frequent activation was located during the Exponential Growth & Regression unit. As shown in the diagram, the frequencies were [6, 4, 11, 9], and the total activation frequency was 30.

Changes between activation levels

By incorporating the second type of evidence, *changes between activation levels*, the information from Table 6-2Table 4-13 informs us that the changes from lower to higher and higher to lower levels of competency activation by domain were: (6, 4) for T.S.V., (7, 5) for M.L.V., and (3, 1) for Pr.In.T. These observations show that the progress was more often from a lower to a higher level of activation, than the other way around, suggesting a dynamic of development over time.

Levels of consistency

Regarding the third type of evidence, *levels of consistency*, Diagram 10-20 and Table 10-18 informs us that there were several instances of moderate consistency at the Mature or Intermediate level of the scaling instrument for all activation domains. This suggests that on many occasions, Erika activated the competency while working continuously independently with limited need for assistance, using expressions of mathematical language and having a clear perspective towards the steps of the solution process. Interestingly, there was only one occasion during the Periodic Functions session where Erika faced difficulties to obtain a clearer view towards the steps, she needed to take to reach a solution on the tasks she was assigned to work on during this session.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, I infer that there is a *strong combination of evidence* for the development of the (C.D.Rep.) aspect of the Representing and Manipulating symbolic forms competency.

Overview for (Rep↔Rep)

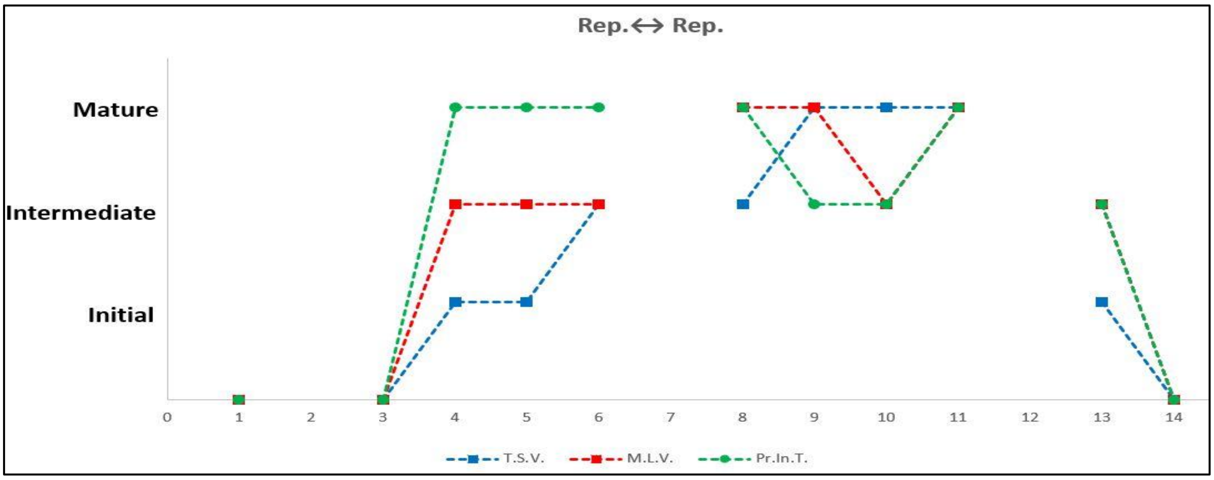


Diagram 10-21. Activation for competency aspect (Rep↔Rep.).

Table 10-19. Summarized evidence for(Rep↔Rep.) activation.

Competency domain / Type of evidence	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	3	2	2	
Changes from higher to lower	1	2	2	
High consistency	1	1	1	
Moderate consistency	(1)	1	1	
Frequency of activation				8

<u>Frequency of competency activation</u>

Regarding the first type of evidence, namely the *frequency of competency activation*, mainly the Exponential Growth & Regression and the Population Dynamics units provided evidence of competency activation. As shown in the diagram, the frequencies were [0, 3, 4, 1], and the total activation frequency was eight. There were three sessions where no evidence of competency activation was observed.

<u>Changes between activation levels</u>

By incorporating the second type of evidence, *changes between activation levels*, the information from Table 6-2Table 4-13 informs us that the changes from lower to higher and higher to lower levels of competency activation by domain were: (3, 1) for T.S.V., (2, 2) for M.L.V., and (2, 2) for Pr.In.T. These observations suggest that the progress was more often from a lower to a higher level of activation, only regarding Erika's perspective towards the solution of the tasks she faced.

<u>Levels of consistency</u>

Regarding the third type of evidence, *levels of consistency*, Diagram 10-21 and Table 10-19 informs us that there were instances of high consistency at the

Mature level for all domains of activation at the Intermediate or Mature level of the scaling instrument. However, the low frequency of activation renders difficult to make any conclusive inference regarding the competency development of the (Rep↔Rep) aspect.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, I infer that there is a *weak combination of evidence* for the development of the (Rep↔Rep) aspect of the Representing and Manipulating symbolic forms competency.

Mathematical Reasoning and Communicating competency

Overview for (Reason→)

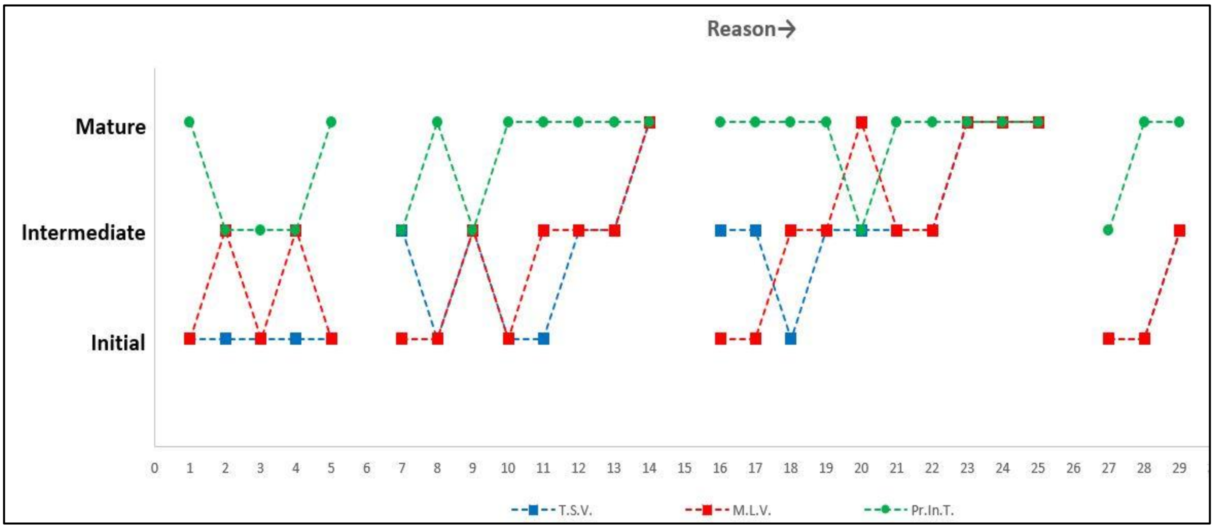


Diagram 10-22. Activation for competency aspect (Reason→).

Table 10-20. Summarized evidence for(Reason→) activation.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	6	9	5	
Changes from higher to lower	3	4	3	
High consistency	(1)	0	0	
Moderate consistency	1(1)	1(1)	5	
Frequency of activation				26

Frequency of competency activation

Regarding the first type of evidence, namely the *frequency of competency activation*, there was evidence of competency activation in all calculus units and every session of these units. As shown in the diagram, the frequencies were [5, 8, 10, 3], and the total frequency of activation was 26. This was the most frequently activated competency aspect of the Mathematical Reasoning and Communicating competency.

Changes between activation levels

By incorporating the second type of evidence, *changes between activation levels*, the information from Table 6-2Table 4-13 informs us that the changes from lower to higher and higher to lower levels of competency activation by domain were: (6, 3) for T.S.V., (9, 4) for M.L.V., and (5, 3) for Pr.In.T. These observations suggest that the progress was more often from a lower to a higher level of activation, for all the domains of activation, suggesting a dynamic of competency development over time.

Levels of consistency

Regarding the third type of evidence, *levels of consistency*, Diagram 10-22 and Diagram 10-20 displays that were few instances of consistency for the T.S.V. and M.L.V. domains with several of these instances in the Initial level of activation. This is attributed to the fact that there were many changes from a lower to a higher level of activation and therefore fewer instances of activation in a particular level of activation in a continuous fashion. However, inferences can be made for the Pr.In.T. domain where the consistency table suggests that Erika continuously worked independently with limited need for prompting or hints from the teacher.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, I infer that there is a *strong combination of evidence* for the development of the (Reason→) aspect of the Mathematical Reasoning and Communicating competency.

Use of Aids and Tools competency

Overview for (Ext.T.)

As mentioned in Section 5.6, no evidence was provided by Erika at any of the seven sessions regarding the activation of the (Ext.T) aspect. This finding suggests that it is possible the content of the sessions and the tasks did not provoke a discussion regarding the existence and properties of any tool or aid for mathematical activity. However, there were few instances from other students (e.g., Rene) who expressed their knowledge about the use of Desmos software for graphing activities. That said, it should also be considered that the description of this competency aspect may lead to difficulties in tracking instances of activation while engaging in a mathematical activity.

Radius of Action

Mathematical Modeling competency

Overview for (R.+V.)

As discussed in the Degree of Coverage part of this section, the low frequency of activation and the fact that there was no evidence of competency activation during the Periodic Functions and the Exponential Growth & Regression units resulted in considering that Erika provided limited evidence, regarding the activation of (R.+V.) competency aspect. I therefore infer that there is a *negligible combination of evidence* within the Radius of Action for the development of the (R.+V.) aspect of the Mathematical Modeling competency.

Overview for (Int.Res.X)

Table 10-21. Summarized evidence for (Int.Res.X) activation by unit.

Type of evidence / **Competency domain**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	(1,0,0,1)	(1,0,0,1)	(0,0,0,1)	
Changes from higher to lower	(0,1,0,0)	(0,1,0,0)	(0,1,0,0)	
High consistency	(0,0,1,0)	(0,0,1,0)	(0,0,1,0)	
Moderate consistency	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	
Frequency of activation				11

Frequency of competency activation

The activation frequencies were [2, 2, 6, 1]. A more frequent activation was noticed during the Periodic Functions unit followed by the Exponential Growth & Regression and the Population Dynamics units. There were few activations during the other calculus units.

Changes between activation levels

It is difficult to draw any inference by employing this type of evidence here because of the low frequency of activation that was observed for the Periodic Functions, Exponential Growth & Regression, and the Integrals & Modeling units. For the Population Dynamics, an important finding is identified in the next type of evidence.

Levels of consistency

Evidence of high consistency within the Intermediate or Mature level of activation was identified only during the Population dynamics unit for all domains of activation. The latter suggests a dynamic of development for the particular competency aspect. Erika was able to interpret results from the tasks she worked within a real-world context, consistently with limited prompting and help from the teacher, having a clear perspective towards the task solution and using expressions with formal mathematical language regularly.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, there is evidence of development mainly for one calculus unit: Population Dynamics. I, therefore, infer that there is a *weak combination of evidence* within the Radius of Action for the development of the (Int.Res.X) aspect of the Mathematical Thinking and Acting competency.

Overview for (Cr.)

Table 10-22. Summarized evidence for (Cr.) activation by unit.

Competency domain / Type of evidence	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	(0,1,0,0)	(0,0,0,0)	(0,1,2,0)	
Changes from higher to lower	(0,2,0,0)	(0,1,0,0)	(0,1,1,0)	
High consistency	(0,0,1,0)	(0,(1),1,0)	(0,0,0,0)	
Moderate consistency	(0,(1),0,0)	(0,0,0,0)	(0,2,1,0)	
Frequency of activation				9

Frequency of competency activation

The activation frequencies were [0, 4, 5, 0]. It is clear that Erika provided evidence of competency activation only in two calculus units: Population Dynamics and Exponential Growth & Regression.

Changes between activation levels

For these units in which activations were observed, there were both changes from a lower to a higher level of activation and vice versa for all domains of activation. A closer look at the table above indicates that even though Erika appeared to need less help to activate the (Cr.) competency aspect, her view towards the solutions of the tasks she faced and her use of mathematical language did not display a dynamic of development. That said, the next type of evidence (i.e., levels of consistency) can provide more solid indications.

Levels of consistency

Evidence of high consistency within the Intermediate or Mature level of activation was identified mainly during the Population dynamics unit. There were instances of moderate consistency during the Exponential Growth & Regression unit too but this consistency was also observed within the Initial level of activation of the scaling instrument. These two observations indicate that there is a dynamic of development for the calculus units of Population Dynamics and Exponential Growth & Regression.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, there is evidence of development mainly for two calculus units: Population Dynamics and Exponential Growth & Regression. I, therefore, infer that there is a *moderate combination of evidence* within the Radius of Action for the development of the (Cr.) aspect of the Mathematical Thinking and Acting competency.

Overview for (Graph)

As discussed in the Degree of Coverage part of this section, the fact that Erika provided evidence of activation mainly during the Periodic Functions unit and there was no evidence of activation for the Exponential Growth &

Regression and Integrals & Modeling units suggests that inferences can be drawn only for the Periodic Functions unit. Within this unit, Erika activated the competency consistently within the Moderate or Mature level of activation. This suggests that she worked continuously having a clear perspective towards the solution of the tasks she worked on, using expressions with mathematical language and with limited help from the teacher or a fellow student. The observations above suggest that there is evidence of development only for the Periodic Functions unit. I, therefore, infer that there is a *weak combination of evidence* within the Radius of Action for the development of the (Graph) aspect of the Mathematical Thinking and Acting competency.

Representing and Manipulating symbolic forms competency

Overview for (Rep↔ Rep)

Table 10-23. Summarized evidence for (Rep↔ Rep) activation by unit.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	(0,2,1,0)	(0,1,1,0)	(0,1,1,0)	
Changes from higher to lower	(0,0,0,1)	(0,0,1,1)	(0,0,1,1)	
High consistency	(0,0,1,0)	(0,1,0,0)	(0,1,0,0)	
Moderate consistency	(0,(1),0,0)	(0,0,1,0)	(0,0,1,0)	
Frequency of activation				8

<u>Frequency of competency activation</u>

The activation frequencies were [0, 3, 4, 1]. It is clear that Erika provided evidence of competency activation mainly for two calculus units: Population Dynamics and Exponential Growth & Regression.

<u>Changes between activation levels</u>

At these units where competency activation was observed, the changes from a lower to a higher level of activation were comparatively more than these changes to the opposite direction. This observation suggests a dynamic of development for the units of Population Dynamics and Exponential Growth & Regression.

<u>Levels of consistency</u>

Evidence of both high and moderate consistency within the Intermediate or Mature level of activation was identified for the Population dynamics and the Exponential Growth & Regression units. Furthermore, the evidence suggests that the consistency was observed with all domains of activation. This signifies that Erika worked completely independently and regularly using expressions with mathematical language and vocabulary while having awareness of the next steps of the solution process of the tasks she worked on for these calculus units.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, there is evidence of development for two calculus units: Population Dynamics and Exponential Growth & Regression. I, therefore, infer that there is a *moderate combination of evidence* within the Radius of Action for the development of the (Rep↔ Rep) aspect of the Mathematical Thinking and Acting competency.

Overview for (Man.Rep.)

Table 10-24. Summarized evidence for (Man.Rep.) activation by unit.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	(1,2,3,3)	(1,2,3,4)	(0,1,0,1)	
Changes from higher to lower	(0,0,1,2)	(0,0,1,2)	(0,0,0,0)	
High consistency	(0,0,0,0)	(0,0,0,0)	(1,1,1,1)	
Moderate consistency	(1,1(1),1,0)	(1,2,1,0)	(0,0,0,0)	
Frequency of activation				31

Frequency of competency activation

The activation frequencies were [3, 5, 12, 11]. A more frequent activation was noticed during the Population Dynamics and the Integrals & Modeling units. A less frequent activation was observed during Erika's work on the Exponential Growth & Regression and Periodic Functions units.

Changes between activation levels

Changes from a lower to a higher level of activation are identified at all four calculus units and are also comparatively more than those that occurred towards the other direction. This observation suggests a dynamic of development for all the sessions where activation occurred. At the same time the low frequency of activation of the Periodic Functions unit needs to be considered.

Levels of consistency

Evidence of high consistency within the Intermediate or Mature level of activation was identified at all four calculus units regarding the Pr.In.T. domain of activation. These observation suggests that Erika was consistently working independently with limited need for prompting while activating the particular competency. There were also several instances of moderate consistency during all calculus units. There was only one instance where consistency was located at the Initial level of activation regarding the T.S.V. domain of activation. This observation suggests that during the Exponential Growth & Regression unit, Erika had difficulties to obtain a clear perspective towards the solution steps she had to take to address the tasks she worked on. However, at the same sessions she regularly used expressions of mathematical language and vocabulary.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, there is evidence of development for all calculus units. I, therefore, infer that there is a *strong combination of evidence* within the Radius of Action for the development of the (Man.Rep.) aspect of the Mathematical Thinking and Acting competency.

Mathematical Reasoning and Communicating competency Overview for (Reason→)

Table 10-25. Summarized evidence for (Reason→) activation by unit.

Competency domain / Type of evidence	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	(0,3,2,1)	(2,3,3,1)	(1,2,1,1)	
Changes from higher to lower	(0,2,1,0)	(2,1,1,0)	(1,1,1,0)	
High consistency	((1),0,0,0)	(0,0,0,0)	(0,0,0,0)	
Moderate consistency	(0,0,1,(1))	(0,1,0,(1))	(1,1,2,1)	
Frequency of activation				26

<u>Frequency of competency activation</u>

The activation frequencies were [5, 8, 10, 3]. A more frequent activation was noticed during the Population Dynamics followed by the Exponential Growth & Regression unit. A less frequent activation was observed during Erika's work on the Periodic Functions and the Integrals & Modeling units.

<u>Changes between activation levels</u>

As seen in the table, there were more changes from a lower to a higher level of activation than the other way around for all calculus units except the Periodic Functions. This observation could signify a dynamic of competency development for the calculus units of Population Dynamics and Exponential Growth & Regression. It is difficult though to draw any inference, by employing this type of evidence, about the development of the (Reason→) aspect during the Integrals & Modeling unit because of the low frequency of activation that was observed.

<u>Levels of consistency</u>

Evidence of high consistency was only observed within the Initial level of activation during the Periodic Functions unit. This finding signifies the lack of evidence for the development of the (Reason→) competency aspect. There were instances of moderate consistency at the Intermediate or Mature level of activation during the Population Dynamics and the Exponential Growth & Regression units which signifies a dynamic of competency development during these two units. Erika was able to express herself and provide justification for her ideas and writings having a clear perspective towards the solution of the tasks or the issue that was discussed and she did that while using expressions with mathematical language and with limited prompting from the teacher.

<u>Strength of evidence combination: final evaluation</u>

Based on the summaries above for all three types of evidence, there is evidence of development for two calculus units: Population Dynamics and Exponential Growth & Regression. I, therefore, infer that there is a *moderate combination of evidence* within the Radius of Action for the development of the (Reason→) aspect of the Mathematical Thinking and Acting competency.

Use of Aids and Tools

Overview for (Ext.T.)

As it was discussed in the relevant overview regarding the Degree of Coverage no evidence was provided by Erika at any of the seven sessions regarding the activation of the (Ext.T) aspect and it is, therefore, impossible to draw any inferences regarding the development within the Radius of Action.

Technical Level

Representing and Manipulating symbolic forms competency

This competency consists of three aspects and to avoid repetition, I only mention here the relevant abbreviations (codes) that represent these aspects: (C.D.Rep.), (Rep.↔Rep.), and (Man.Rep.).

Overview for technical parts

Erika, on many occasions, activated the '*choose (or decode in) a representation*' competency aspect (C.D.Rep.) concurrently with the (Int.El.) competency aspect within the same turn (utterance). Therefore, the discussion about Erika's handling with the technical parts of the tasks presents similarities with the discussion undertaken regarding the (Int.El.) competency aspect. These similarities concern mainly the technical parts of the tasks during the Periodic Functions even though she activated this competency aspect in this unit more frequently than the (Int.El.) aspect. The same observation can be made for the Population Dynamics and Exponential Growth & Regression units. The major difference lies in the Integral & Modeling unit where she was able to work efficiently with the differentiation and integration processes along with a satisfying use of large decimal numbers.

The study of the '*switch between representations*' aspect (Rep.↔Rep.) manifested no activation during the Periodic Functions unit. Erika worked efficiently with logarithms and the exponential functions and she dealt resourcefully when she had to work with exponents and fractions with large decimal numbers. However, there was no evidence of efficient use of integrals and integration rules.

Finally, regarding the '*manipulate within a representation*' competency aspect (Man.Rep.), two important interrelations with two other competencies have been observed. These interrelations occurred with the (→Sol.) and (Int.El.) competency aspects. In conclusion, the analysis here is rather similar to the two competencies in question. The new addition here is the fact that Erika displayed intuition and determination to take initiative when dealing with the "E-coli bacterium task". She comprehended the "*If a single cell of the bacterium E. coli* *divides every 20 minutes, how many E. coli would be there in 24 hours*?" issue and worked efficiently with exponents in that situation.

Exploring the M.L.V. domain

In this part of the overview, I present the assembled diagrams (Figure 10.1) from the (C.D.Rep.), (Rep.↔Rep.), and (Man.Rep.) aspects of the Representing and Manipulating symbolic forms competency and provide the evidence focusing only for the M.L.V. domain of activation (red dotted line).

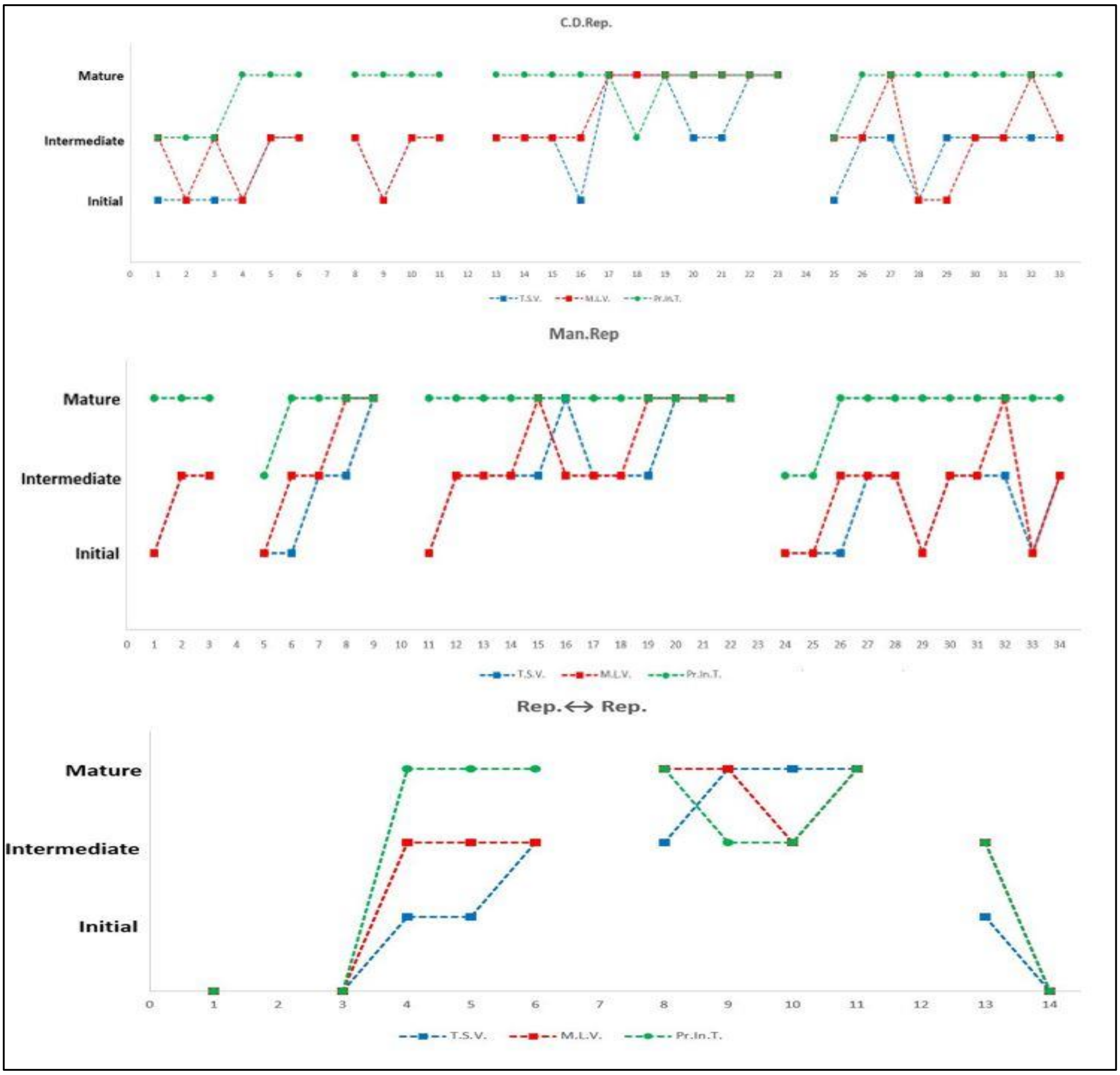


Figure 10.1. Assembled Diagrams: Representing and Manipulating symbolic forms competency (Erika).

Table 10-26 below, summarizes the two types of evidence (changes between activation levels and consistency of activation) within the M.L.V. domain for the three aspects of the Representing and Manipulating symbolic forms competency. For two aspects of this competency it is evident that the activations were observed to occur more often from a lower to a higher level of activation. It is also noticeable that all instances of moderate of high consistency were observed within the Intermediate of Mature level of the scaling instrument and there was no instance of consistency within the Initial level of activation, that is, a continuous activation at the higher levels of activation of the scaling instrument. This suggests a dynamic of development for all aspects of this competency.

Table 10-26. Summary of evidence for M.L.V. domain.

Competency aspect / Type of evidence	C.D.Rep	Rep.↔Rep.	Man.Rep.
Changes from lower to higher	7	2	10
Changes from higher to lower	5	2	3
High consistency	0	1	0
Moderate consistency	3	1	3

Final assessment for the Representing and Manipulating symbolic forms competency

Based on the overviews for the technical parts and the exploration of the M.L.V. domain regarding the combination of evidence for the progress of development of the three competency aspects that constitute the Representing and Manipulating symbolic forms competency, I infer that there is a *strong combination of evidence* that suggests development within the Technical Level of the competency.

Mathematical Reasoning and Communicating competency

This competency consists of two aspects and to avoid repetition, I only mention here the relevant abbreviations (codes) that represent these aspects: (Reason←) and (Reason→).

Overview for technical parts

Regarding the study of the 'to *express oneself, at various levels of theoretical and technical precision, in oral, visual or written form, about such matters. To provide explanations or justifications to support your result or idea.*' competency (code: Reason→), two fairly important interrelations with two other competency aspects have been observed. These interrelations (i.e., concurrent activations within the same turn) occurred with the (Int.El.) and (Man.Rep.) competencies. No important addition, regarding the technical parts of the tasks, can be made in this analysis apart from the fact that Erika exhibited efficient use of the formula in the "E-coli" task ($N_{today} = N_0 e^{rt}$) where she also helped the other member of the group (Johannes) when he encountered difficulties with a set of symbols he had to work on.

The analysis of the '*understand others' written, visual or oral 'texts', in a variety of linguistic registers, about matters having a mathematical content. To* *follow and assess chains of arguments, put forward by others*' competency aspect (Reason←) manifested interrelations with several other competency aspects: (Int.El.), (Reason→), (→Sol.), and (C.D.Rep.). No important addition, regarding the technical parts of the tasks, can be made in this analysis because the analysis has already been done with the aforementioned aspects. Therefore, to draw inferences for the development of this competency I referred to the previous overviews of the (Int.El.), (Reason→), (→Sol), and (C.D.Rep) aspects.

Exploring the M.L.V. domain

In this part of the overview, I present the assembled diagrams (Figure 10.2) from the (Reason←) and (Reason→) aspects of the Mathematical Reasoning and Communicating competency and provide the evidence focusing only for the M.L.V. domain of activation (red dotted line).

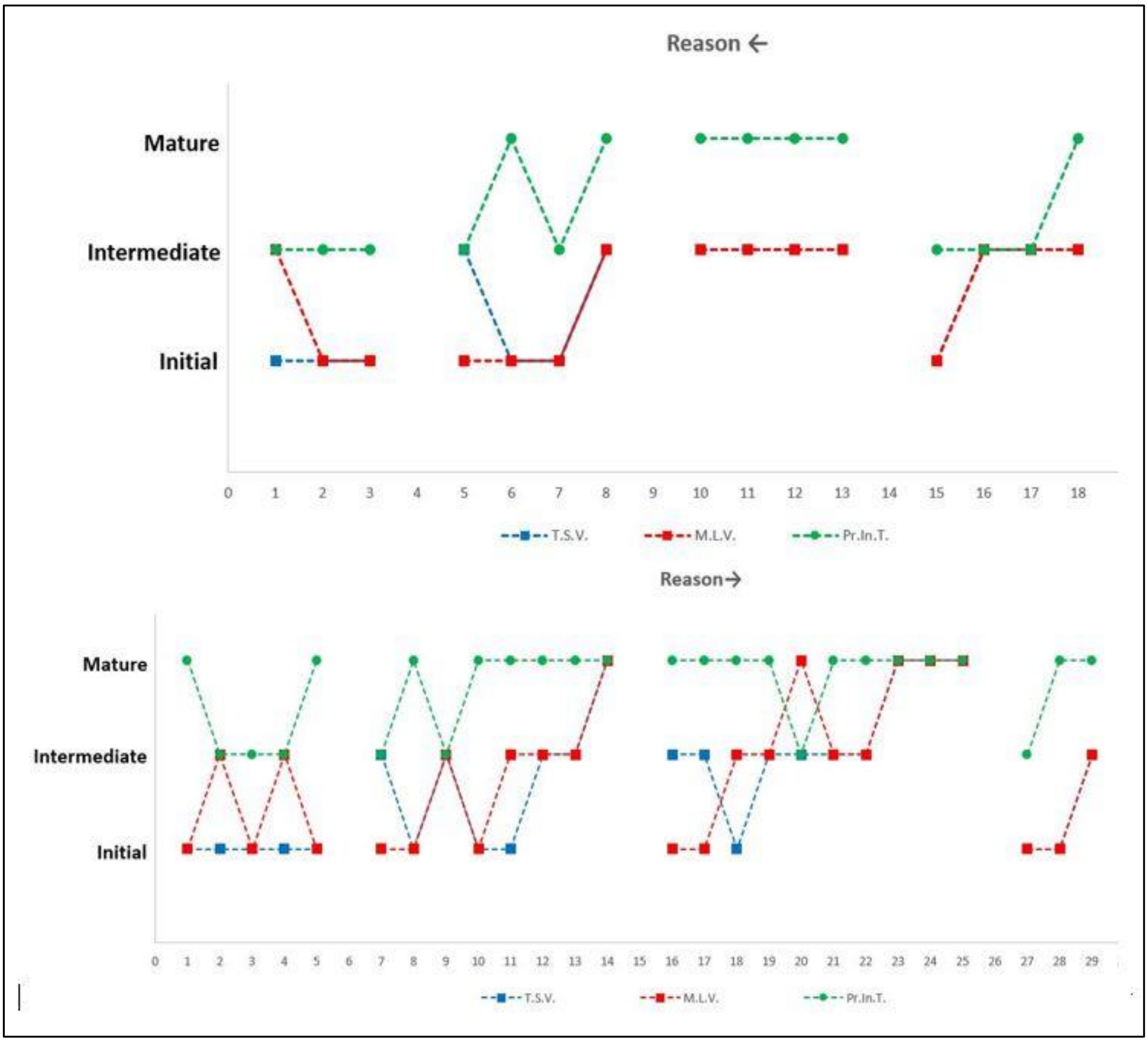


Figure 10.2. Assembled diagrams: Mathematical Reasoning and Communicating competency (Erika).

Table 10-27 below, summarizes the two types of evidence (changes between activation levels and consistency of activation) within the M.L.V. domain for the two aspects of the Mathematical Reasoning and Communicating competency. For these two aspects, it is observable that the activations were observed to occur more often from a lower to a higher level of activation. It is also noticeable that there were several instances of consistency in all levels of the scaling instrument. Therefore, even though there is a dynamic of development because of the previous observation she also displayed continuous activations at all levels of activation of the scaling instrument which includes the Initial level.

Table 10-27. Summary of evidence for M.L.V. domain.

Type of evidence	Reason←	Reason→
Changes from lower to higher	2	9
Changes from higher to lower	1	4
High consistency	2(1)	0
Moderate consistency	(1)	1(1)

Final assessment for the Mathematical Reasoning and Communicating competency

Based on the overviews for the technical parts and the exploration of the M.L.V. domain regarding the combination of evidence for the progress of development of the two competency aspects that constitute the Mathematical Reasoning and Communicating competency, I infer that there is a *moderate combination of evidence* that suggests development within the Technical Level of the competency.

Use of Aids and Tools competency

This competency consists of two aspects and to avoid repetition, I only mention here the relevant abbreviations (codes) that represent these aspects: (Ext.T.) and (U.T.).

Overview for technical parts

The unique nature of this competency addresses mainly issues of technical use of the calculator and the LiveScribe smartpen. Two interrelations with two other competencies took place during the activation of this competency: (→Sol.) and (Man.Rep.). Therefore, to draw inferences for the development of this competency I referred to the previous overviews of the aforementioned interrelated aspects.

Exploring the M.L.V. domain

In this part of the overview, I present the diagram from the (U.T.) aspect of the Use of Aids and Tools competency and provide the evidence focusing only for the M.L.V. domain of activation (red dotted line). No evidence was provided by Erika at any of the seven sessions regarding the activation of the (Ext.T) aspect.

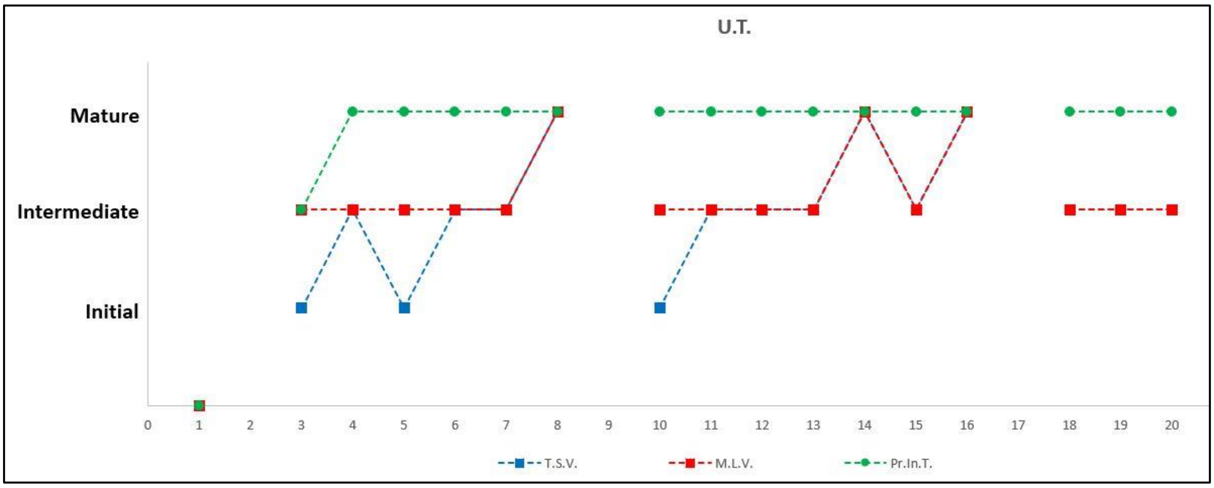


Diagram 10-23. Use of Aids and tools competency (Erika).

Table 10-28 below, summarizes the two types of evidence (changes between activation levels and consistency of activation) within the M.L.V. domain for the (U.T.) aspect of the Use of Aids and Tools competency. For this aspect, it is observable that the activations were observed to occur more often from a lower to a higher level of activation. It is also noticeable that all instances of moderate or high consistency were observed within the Intermediate or Mature level of the scaling instrument and there was no instance of consistency within the Initial level of activation, that is, a continuous activation at the higher levels of activation of the scaling instrument. This suggests a dynamic of development for the (U.T.) aspect of this competency.

Table 10-28. Summary of evidence for M.L.V. domain.

Competency aspect / Type of evidence	U.T.
Changes from lower to higher	3
Changes from higher to lower	1
High consistency	2
Moderate consistency	1

Final assessment for the Use of Aids and Tools competency

Based on the overview for the technical parts and the exploration of the M.L.V. domain regarding the combination of evidence for the progress of development of the (U.T.) aspect and due to the fact that there was no evidence of activation for the (Ext.T.) aspect, I infer that there is a *moderate combination of evidence* that suggests development within the Technical Level of the competency.

Johannes

Degree of Coverage

Mathematical Thinking and Acting competency

Overview for (P.Q.)

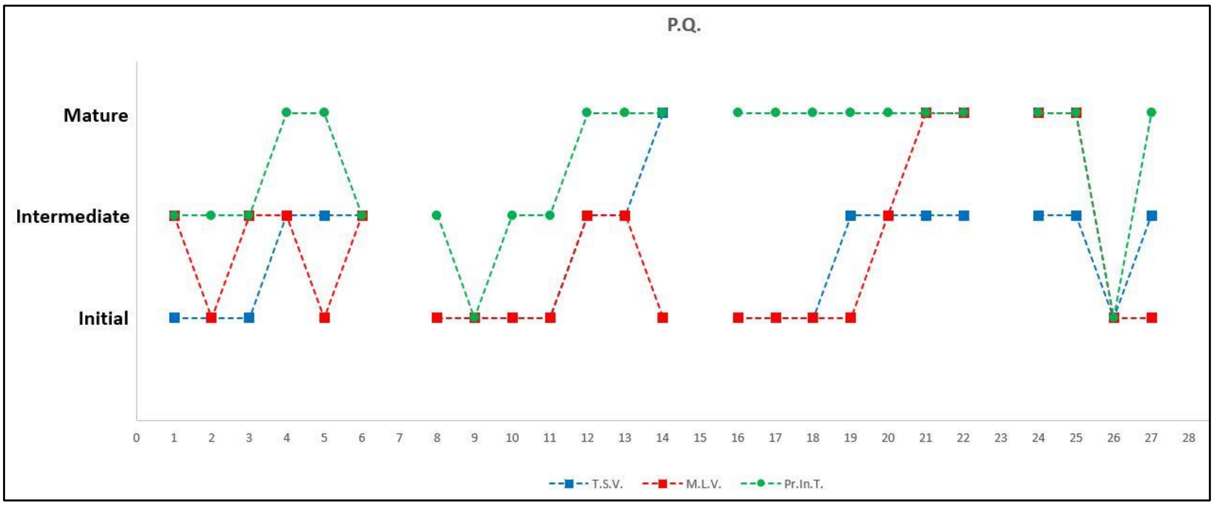


Diagram 10-24. Activation for competency aspect (P.Q.) (repetition).

Table 10-29. Summarized evidence for (P.Q.) activation.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	5	5	4	
Changes from higher to lower	1	4	3	
High consistency	0	0	1	
Moderate consistency	3(2)	1(3)	3	
Frequency of activation				24

<u>Frequency of competency activation</u>

As shown in the diagram, the frequencies were [6, 7, 7, 4] and the total frequency of activation was 24. Frequent activation was observed during the first three calculus units. A less frequent activation was observed during the Integrals & Modeling unit. The total frequency of activation was 24.

<u>Changes between activation levels</u>

The changes from lower to higher and higher to lower levels of competency activation by domain were: (5,1) for T.S.V., (5,4) for M.L.V., and (4,3) for Pr.In.T. These observations suggest that the progress was more often from a lower to a higher level of activation, than the other way around and shows a dynamic of development over time. Especially the changes within the T.S.V. domain suggest that Johannes developed his perspective towards the solution of the tasks or the challenges he faced during his engagement with the tasks he worked on.

Levels of consistency

There were several instances of moderate and high consistency for all the domains (i.e., T.S.V., M.L.V., and Pr.In.T.) at the Intermediate or Mature level. Two observations can be made within these type of evidence. First, Johannes faced difficulties to develop his use of mathematical language (M.L.V. domain) on several occasions. As the table informs us, there were three instances where he displayed consistent activation within the Initial level of activation. Second, the findings from Pr.In.T. column suggest that Johannes managed to work on many occasions continuously without needing assistance and that he received limited help from the teacher or a fellow classmate to activate the (P.Q.) aspect.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, I infer that there is a *moderate combination of evidence* for the development of the (P.Q.) aspect of the Mathematical Thinking and Acting competency.

Overview for (Lim.)

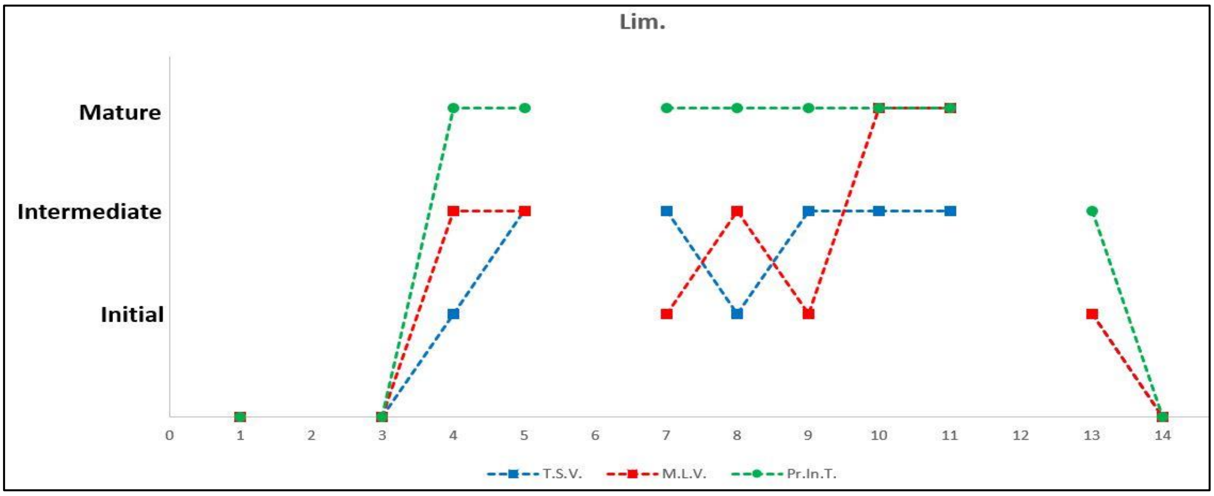


Diagram 10-25. Activation for competency aspect (Lim.) (repetition).

Table 10-30. Summarized evidence for (Lim.) activation. (repetition).

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	3	3	1	
Changes from higher to lower	2	1	1	
High consistency	0	0	1	
Moderate consistency	1	1	1	
Frequency of activation				8

Frequency of competency activation

As shown in the diagram, the frequencies were [0, 2, 5, 1] and the total frequency of activation was 8. Evidence of activation have been mainly tracked during the Population Dynamics unit. It appears that this was the least activated competency aspect among the three that constitute this competency. There were three sessions where no evidence of competency activation was observed.

Changes between activation levels

The changes from lower to higher and higher to lower levels of competency activation by domain were: (3, 2) for T.S.V., (3, 1) for M.L.V., and (1, 1) for Pr.In.T. These observations suggest that there is a dynamic of development over time even though the low frequency of activation must be always considered.

Levels of consistency

There were several instances of moderate consistency for all the domains (i.e., T.S.V., M.L.V., and Pr.In.T.) at the Intermediate or Mature level of the scaling instrument. Interestingly, there was no occasion of consistency within the Initial level of the scaling instrument. This suggests that even though the frequency of activation was low, there is a dynamic of development mainly during the Population Dynamics unit.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, I infer that there is a *moderate combination of evidence* for the development of the (Lim.) aspect of the Mathematical Thinking and Acting competency.

Mathematical Modeling competency

Overview for (R.+V.)

Johannes provided evidence for the activation of the competency aspect (R.+V.) only on three occasions. Twice during the Population Dynamics unit and once at the last session of the Integrals & Modeling unit. There was no evidence of competency activation during the Periodic Functions and the Exponential Growth & Regression units. Therefore, the low frequency of activation and the fact that there was no evidence of competency activation during the Periodic Functions and the Exponential Growth & Regression units resulted in considering that Johannes provided limited evidence, regarding the activation of (R.+V.) competency aspect. I therefore infer that there is a *negligible combination of evidence* for the development of the (R.+V.) aspect of the Mathematical Modeling competency.

Overview for (Int.El.)

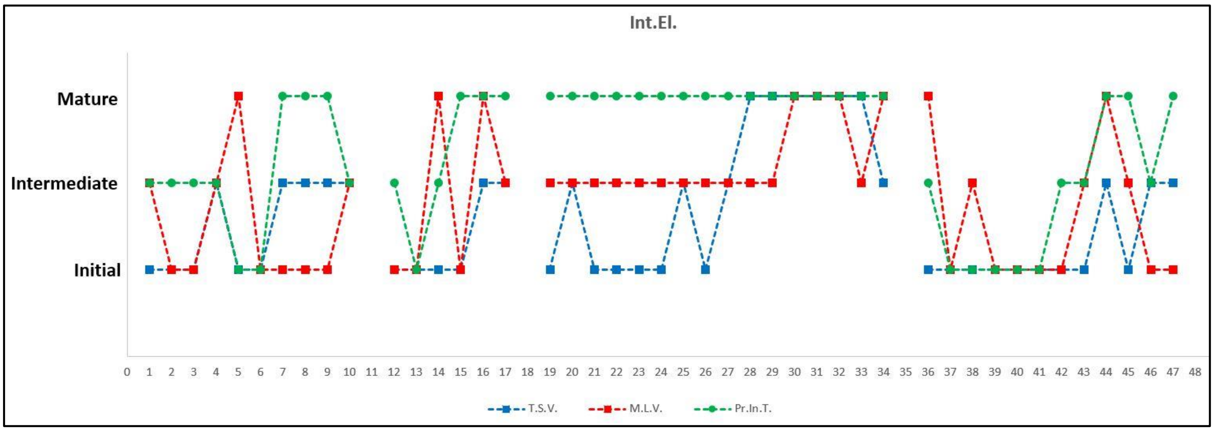


Diagram 10-26. Activation for competency aspect (Int.El.) (repetition).

Table 10-31. Summarized evidence for (Int.El.) activation.

Type of evidence \ Competency domain	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	9	10	6	
Changes from higher to lower	5	9	5	
High consistency	0	1	1	
Moderate consistency	2(2)	(1)	2(1)	
Frequency of activation				44

Frequency of competency activation

The frequencies were [10, 6, 16, 12], and the total frequency of activation was 44. Frequent activation located during the Population Dynamics, the Integrals & Modeling, and the Periodic Functions units. A less frequent activation was during the Exponential Growth & Regression unit. Importantly, this was the most frequently activated competency aspect of the Mathematical Modeling competency.

Changes between activation levels

The changes from lower to higher and higher to lower levels of competency activation by domain were: (9, 5) for T.S.V., (10, 9) for M.L.V., and (6, 5) for Pr.In.T. These observations suggest that the progress was more often from a lower to a higher level of activation, than the other way around for domains of activation which implies a dynamic of competency development over time.

Levels of consistency

There were several instances of moderate consistency at all levels of activation of the scaling instrument for all domains of activation. This suggests that Johannes, occasionally faced difficulties to interpret the elements of an equation or a model he was working on and even though he occasionally managed to activate the competency aspect in a higher activation level, he was not consistent in that level. However, there were also occasions where he had consistently a clear view towards the solution of the tasks he had to work on. It must be noted though that during the Population Dynamics unit, Johannes demonstrated high consistency at the Intermediate and Mature level of activation for the M.L.V. and the Pr.In.T. domains of activation respectively. It appears that during these two sessions of that unit he was continuously interpreting the elements of mathematical expressions using language with mathematical vocabulary and he was doing that with limited prompting and assistance from the teacher his fellow classmates.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, I infer that there is a *moderate combination of evidence* for the development of the (Int.El.) aspect of the Mathematical Modeling competency.

Overview for (Info)

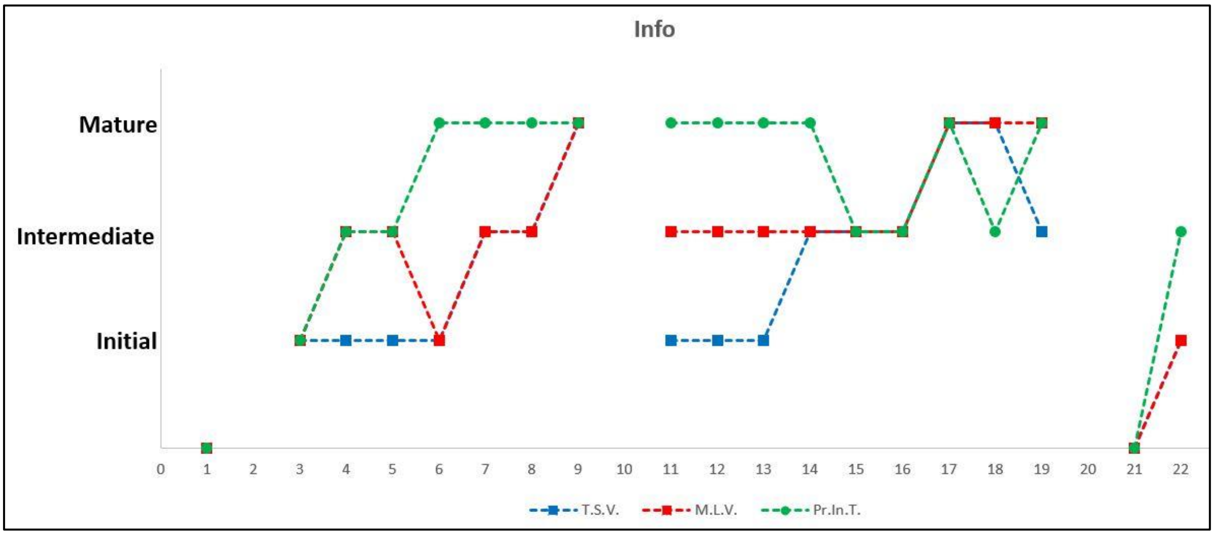


Diagram 10-27. Activation for competency aspect (Info) (repetition).

Table 10-32. Summarized evidence for (Info) activation.

Competency domain / Type of evidence	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	5	5	5	
Changes from higher to lower	1	1	2	
High consistency	0	0	0	
Moderate consistency	(1)	1	2	
Frequency of activation				17

Frequency of competency activation

Evidence of activations were located mainly during the Exponential Growth & Regression and the Population Dynamics units. No activation has been observed during the Periodic Functions. As shown in the diagram, the frequencies were [0, 7, 9, 1], and the total frequency of activation was 17. Interestingly again, the similarities of the above diagram with the diagram that described Erika's activation in this competency aspect are apparent.

Changes between activation levels

The changes from lower to higher and higher to lower levels of competency activation by domain were: (5, 1) for T.S.V., (5, 1) for M.L.V., and (5, 2) for Pr.In.T. These observations suggest that the progress was evidently more often from a lower to a higher level of activation, than the other way around and shows a dynamic of development over time.

Levels of consistency

There were instances of moderate consistency at all levels of activation but for the Pr.In.T. domain of activation there were occasions of moderate consistency only at the highest level of activation of the scaling instrument. This suggest that Johannes was able to find information, relevant to the task he was working on, continuously with limited prompting from the teacher or any fellow classmate.

<u>Strength of evidence combination: final evaluation</u>

Based on the summaries above for all three types of evidence, I infer that there is a *moderate combination of evidence* for the development of the (Info) aspect of the Mathematical Modeling competency.

Representing and Manipulating symbolic forms competency

Overview for (Rep.↔Rep.)

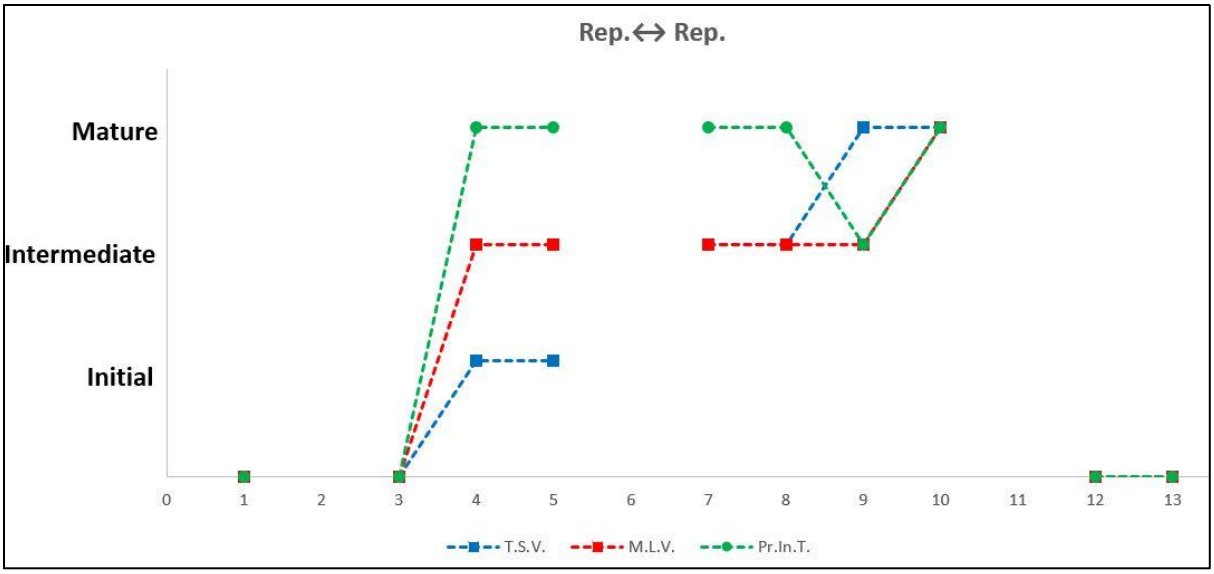


Diagram 10-28. Activation for competency aspect (Rep.↔Rep.) (repetition).

Table 10-33. Summarized evidence for (Rep.↔Rep.) activation.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	2	2	2	
Changes from higher to lower	0	0	1	
High consistency	0	1	0	
Moderate consistency	2	0	1	
Frequency of activation				6

<u>Frequency of competency activation</u>

As shown in the diagram, the frequencies were [0, 2, 4, 0], and the total frequency of activation was 6 which is rather low. There were two units, that of Periodic Functions and Integrals & Modeling, where no evidence of competency activation was identified.

<u>Changes between activation levels</u>

The changes from lower to higher and higher to lower levels of competency activation by domain were: (2, 0) for T.S.V., (2, 0) for M.L.V., and (2, 1) for

Pr.In.T. These observations suggest that the progress was more often from a lower to a higher level of activation, an observation that suggests a dynamic of development. However, this indication can be said mainly for the Population Dynamics unit as seen in the precious type of evidence (i.e., frequency of activation).

Levels of consistency

Even though the frequency of activation is low and evidence of such activation was only observed during the Exponential Growth & Regression and Population Dynamics units, Johannes displayed activated the (Rep↔Rep) aspect in a consistent manner for these units. This means Johannes was able to switch between representations continuously with limited prompting having a clear perspective towards the solution of the tasks while using expressions with mathematical language. However, the low frequency of activation renders difficult to make any conclusive inference regarding the competency development of the (Rep↔Rep) aspect.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, I infer that there is a *weak combination of evidence* for the development of the (Rep↔Rep) aspect of the Representing and Manipulating symbolic forms competency.

Overview for (Man.Rep.)

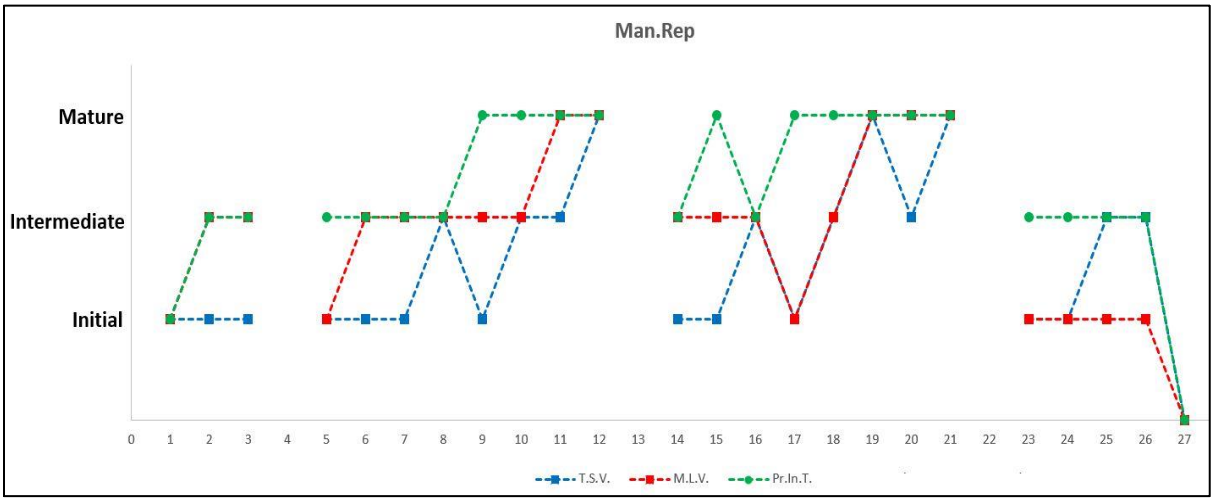


Diagram 10-29. Activation for competency aspect (Man.Rep.) (repetition).

Table 10-34. Summarized evidence for (Man.Rep.) activation.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	8	5	4	
Changes from higher to lower	4	2	2	
High consistency	(1)	(1)	1	
Moderate consistency	1(2)	4(1)	4	
Frequency of activation				23

Frequency of competency activation

As shown in the diagram, the frequencies were [3, 8, 8, 4], and the total frequency of activation was 23. The most frequent activation was observed during the Exponential Growth & Regression and the Population Dynamics units. A less frequent activation was observed for the Integrals & Modeling unit where evidence of activation was identified only for the first session of that unit, and during the Periodic Functions unit. This was the most frequently activated competency aspect of the Representing and Manipulating symbolic forms competency.

Changes between activation levels

The changes from lower to higher and higher to lower levels of competency activation by domain were: (8, 4) for T.S.V., (5, 2) for M.L.V., and (4, 2) for Pr.In.T. These observations suggest that the progress was more evidently often from a lower to a higher level of activation, for all the domains of activation and shows a dynamic of competency development over time.

Levels of consistency

There were several instances of moderate consistency at the higher levels of activation (i.e., Intermediate or Mature) of the scaling instrument for the M.L.V. and Pr.In.T. domains of activations. Importantly, for the Pr.In.T domain, there were many instances of moderate consistency only at the Mature or the Intermediate level suggesting that Johannes worked consistently independently while manipulating different kinds of mathematical representations (i.e., activation of the (Man.Rep.) aspect) and using expressions with mathematical language and vocabulary. Johannes occasionally faced difficulties to have a consistent clear view towards the solution of the tasks.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, I infer that there is a *strong combination of evidence* for the development of the (Man.Rep.) aspect of the Representing and Manipulating symbolic forms competency.

Mathematical Reasoning and Communicating competency

Overview for (Reason←)

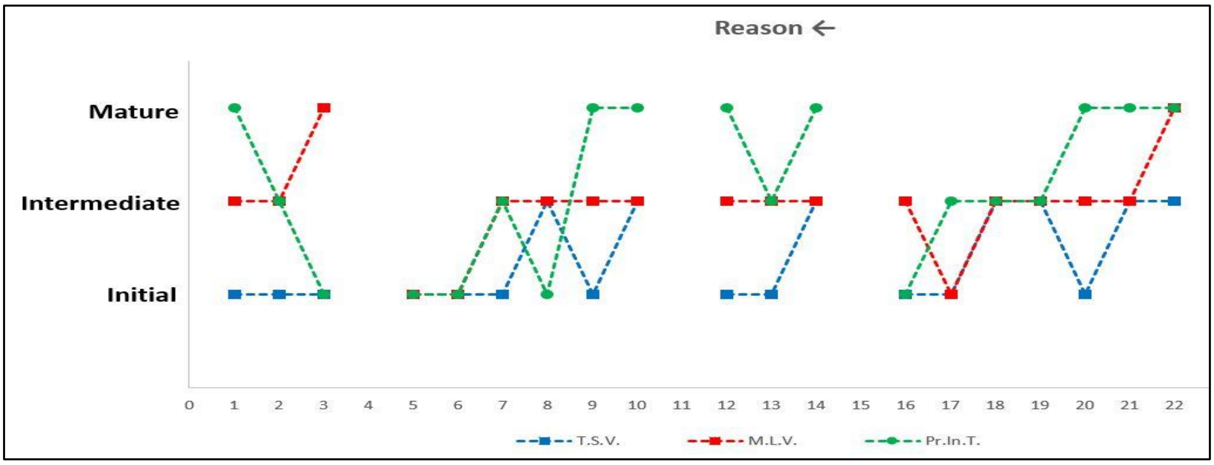


Diagram 10-30. Activation for competency aspect (Reason←) (repetition).

Table 10-35. Summarized evidence for (Reason←) activation.

Competency domain / Type of evidence	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	5	4	5	
Changes from higher to lower	2	1	4	
High consistency	(1)	1	0	
Moderate consistency	(2)	3	2	
Frequency of activation				19

Frequency of competency activation

As shown in the diagram, the frequencies were [3, 6, 3, 7], and the total frequency of activation was 19. The most frequent activation was observed during the Exponential Growth & Regression and the Integrals & Modeling units. Less frequent activation was observed for the units of Periodic Functions and Population Dynamics.

Changes between activation levels

The changes from lower to higher and higher to lower levels of competency activation by domain were: (5, 2) for T.S.V., (4, 1) for M.L.V., and (5, 4) for Pr.In.T. These observations suggest that the progress was more often from a lower to a higher level of activation, for all the domains of activation and shows a dynamic of competency development over time.

Levels of consistency

There were several instances of moderate consistency at the Mature or Intermediate level of activation for the M.L.V. and Pr.In.T. domains of activation and none within the Initial one. The latter suggests that Johannes used expressions with mathematical language consistently to activate the (Reason←) aspect while needing limited prompting. However, it must be noted that there were occasions of moderate and high consistency for the T.S.V. domain of activation which suggests that on several occasions, Johannes faced continuous difficulties to obtain a clear perspective towards the solution of the tasks he was working on.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, I infer that there is a *strong combination of evidence* for the development of the (Reason←) aspect of the Mathematical Reasoning and Communicating competency.

Use of Aids and Tools
Overview for (U.T.)

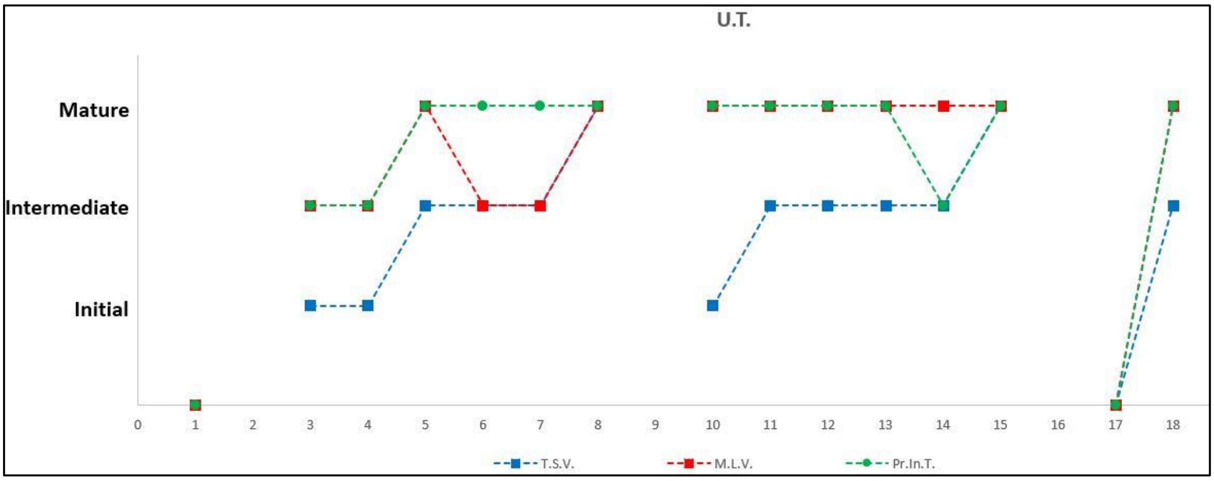


Diagram 10-31. Activation for competency aspect (U.T.) (repetition).

Table 10-36. Summarized evidence for (U.T.) activation.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	5	3	3	
Changes from higher to lower	0	1	1	
High consistency	0	1	0	
Moderate consistency	2	0	2	
Frequency of activation				13

Frequency of competency activation

As shown in the diagram, the frequencies were [0, 6, 6, 1], and the total frequency of activation was 13. Activation was mainly observed during the Population Dynamics and Exponential Growth & Regression units.

Changes between activation levels

The changes from lower to higher and higher to lower levels of competency activation by domain were: (5, 0) for T.S.V., (3, 1) for M.L.V., and (3, 1) for Pr.In.T. These observations suggest that the progress was more often from a lower to a higher level of activation, for all the domains of activation and shows a dynamic of competency development over time.

Levels of consistency

There were several instances of moderate or high consistency for all domains of activation with none of these instances in the Initial level of activation. Johannes (similarly to Erika) consistently used tools efficiently showing familiarity with the calculator and its features and he continuously worked independently with limited need for prompting or hints from the teacher.

Strength of evidence combination: final evaluation

Based on the summaries above for the three types of evidence, I infer that there is a *moderate combination of evidence* for the development of the (U.T.) aspect of the Use of Aids and Tools competency.

Radius of Action

Mathematical Thinking and Acting competency

Overview for (P.Q.)

Table 10-37. Summarized evidence for (P.Q.) activation by unit.

Type of evidence \ Competency domain	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	(1,2,1,1)	(2,1,2,0)	(1,2,0,1)	
Changes from higher to lower	(0,0,0,1)	(2,1,0,1)	(1,1,0,1)	
High consistency	(0,0,0,0)	(0,0,0,0)	(0,0,1,0)	
Moderate consistency	(1(1),(1),1(1),1)	(0,(1),(1),1(1))	(1,1,0,1)	
Frequency of activation				24

Frequency of competency activation

The activation frequencies were [6, 7, 7, 4], that is six instances of competency activation during the Periodic Functions unit, seven during the Exponential Growth & Regression and Population Dynamics unit, and four during the Integrals & Modeling unit. A more frequent activation was noticed during the Exponential Growth & Regression, the Population Dynamics, and the Periodic Functions units. Less frequent activation was observed during the Integrals & Modeling unit.

Changes between activation levels

More changes from a lower to a higher level of activation than the other way around were identified mostly at the first three calculus units. This suggests a dynamic of development over time. During the Integrals & Modeling unit and within the M.L.V. domain, only one change from a higher to a lower level of activation was identified. These findings could signify a dynamic of development for the first three calculus units and mainly for the Exponential Growth & Regression and the Population Dynamics units.

Levels of consistency

Evidence of moderate consistency within all activations levels was identified at all units. Evidence of high consistency was found only during Population Dynamics unit within the Pr.In.T. domain. This suggests that during this unit, Johannes consistently posed questions with limited prompting having a clear perspective towards the solution of the tasks of this unit. There were instances of moderate consistency at all levels for the first three calculus units. During the Integrals & Modeling unit there were more instances of moderate consistency within the Moderate of High activation level than those within the Initial one.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, there is evidence of development mainly for three calculus units: Population Dynamics and Exponential Growth & Regression. I, therefore, infer that there is a *moderate combination of evidence* within the Radius of Action for the development of the (P.Q.) aspect of the Mathematical Thinking and Acting competency.

Overview for (Lim.)

Table 10-38. Summarized evidence for (Lim.) activation by unit.

Type of evidence \ Competency domain	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	(0,2,1,0)	(0,1,2,0)	(0,1,0,0)	
Changes from higher to lower	(0,0,1,1)	(0,0,0,1)	(0,0,0,1)	
High consistency	(0,0,0,0)	(0,0,0,0)	(0,0,1,0)	
Moderate consistency	(0,0,1,0)	(0,0,1,0)	(0,0,0,0)	
Frequency of activation				8

Frequency of competency activation

The activation frequencies were [0, 2, 5, 1]. The only noticeable activation frequency was observed during the Population Dynamics unit. The low activation frequency during the other calculus units does not allow any strong inference to be made for the development within the Radius of Action using this type of evidence.

Changes between activation levels

More changes from a lower to a higher level of activation than the other way around were identified during the Population Dynamics and the Exponential Growth & Regression units. but none at the Integrals & Modeling unit. This suggests some form of development dynamic over time for these first three units but still it is vital to consider the low frequency of activation that has been observed for all units except the Population Dynamics.

Levels of consistency

Evidence of high consistency within the Intermediate or Mature level of activation was identified only during the Population dynamics unit. This suggests that Johannes was able to identify the limitations of the concept he encountered during this unit, continuously without any significant assistance from the teacher of a fellow classmate. However, it is difficult to make any inference for the consistency during the Exponential Growth & Regression and the Integrals & Modeling units because of the small frequency of activation (i.e., only one activation was identified.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, there is evidence of development mainly for one calculus unit: Population Dynamics. I, therefore, infer that there is a *weak combination of evidence* within the Radius of Action for the development of the (Lim.) aspect of the Mathematical Thinking and Acting competency.

Mathematical Modeling competency

Overview for (R.+V.)

As discussed in the Degree of Coverage part of this section, Johannes provided evidence for the activation of the competency aspect (R.+V.) only on three occasions. Twice during the Population Dynamics unit and once at the last session of the Integrals & Modeling unit. This low frequency of activation lead to the conclusion that there is a *negligible combination of evidence* within the Radius of Action for the development of the (R.+V.) aspect of the Mathematical Modeling competency.

Overview for (Int.El.)

Table 10-39. Summarized evidence for (Int.El.) activation by unit.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	(2,1,4,2)	(3,2,2,3)	(1,2,0,3)	
Changes from higher to lower	(1,0,3,1)	(2,2,1,4)	(2,1,0,2)	
High consistency	(0,0,0,0)	(0,0,1,0)	(0,0,1,0)	
Moderate consistency	(1,(1),1,(1))	((1),0,0,0)	(1,1,0,(1))	
Frequency of activation				44

Frequency of competency activation

The activation frequencies were: [10, 6, 16, 12]. Frequent activation was observed during the Population Dynamics, the Integrals & Modeling, and the Periodic Functions units. A less frequent activation was observed during the Exponential Growth & Regression unit.

Changes between activation levels

More changes from a lower to a higher level of activation than the other way around were identified at all four calculus units. This suggests that there is a dynamic of development for all units. These findings could signify a dynamic of development for all four calculus units but mainly for the Exponential Growth & Regression and the Population Dynamics units.

Levels of consistency

Evidence of high consistency within the Intermediate or Mature level of activation was identified only during the Population Dynamics unit and concerned the amount of help Johannes received to activate the (Int.El.) aspect. During this unit he worked independently while interpreting the elements of the models and any mathematical expression he encountered while using expressions of mathematical language in a continuous way. There were instances of moderate consistency at all levels of activation for all units except during the Population Dynamics sessions where Johannes activated the competency aspect only within the higher levels of activation (i.e., Intermediate or Mature). This suggests that during the Population Dynamics unit there was a dynamic of competency development over time.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, there is evidence of development for all calculus units and especially during the Population Dynamics sessions. I, therefore, infer that there is a *strong combination of evidence* within the Radius of Action for the development of the (Int.El.) aspect of the Mathematical Thinking and Acting competency.

Overview for (Cr.)

Table 10-40. Summarized evidence for (Cr.) activation by unit.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	(0,1,1,0)	(0,0,1,0)	(0,1,1,0)	
Changes from higher to lower	(0,1,0,0)	(0,1,0,0)	(0,1,1,0)	
High consistency	(0,0,1,0)	(0,(1),1,0)	(0,0,0,0)	
Moderate consistency	(0,(1),0,0)	(0,0,0,0)	(0,(1),2,0)	
Frequency of activation				10

Frequency of competency activation

The activation frequencies were: [0, 3, 7, 0] which suggests that Johannes provided evidence of competency activation only for two calculus units: Population Dynamics and Exponential Growth & Regression.

Changes between activation levels

For these units in which activations were observed, during the Population Dynamics sessions there were more changes from a lower to a higher level of activation than the other way around. However, this was not the case for the Exponential Growth & Regression unit. It is also important to remember that the activation frequency for this unit was low.

Levels of consistency

Evidence of high and moderate consistency within the Intermediate or Mature level of activation was identified only during the Population dynamics unit. There were instances of high and moderate consistency during the Exponential Growth & Regression unit too but this consistency was observed within the Initial level of activation of the scaling instrument. These two observations indicate that there is a dynamic of development mainly for the Population Dynamics unit.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, there is evidence of development mainly for one calculus units: Population Dynamics. I, therefore, infer that there is a *weak combination of evidence* within the Radius of Action for the development of the (Cr.) aspect of the Mathematical Thinking and Acting competency.

Overview for (Info)

Table 10-41. Summarized evidence for (Info) activation by unit.

Type of evidence \ **Competency domain**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	(0,2,2,1)	(0,3,1,1)	(0,2,2,1)	
Changes from higher to lower	(0,0,1,0)	(0,1,0,0)	(0,0,2,0)	
High consistency	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	
Moderate consistency	(0,(1),0,0)	(0,0,1,0)	(0,1,1,0)	
Frequency of activation				14

Frequency of competency activation

The activation frequencies were: [0, 7, 9, 1]. Again, it appears that Johannes provided evidence of competency activation mainly for two calculus units: Population Dynamics and Exponential Growth & Regression.

Changes between activation levels

For both units that were mentioned above, there were comparatively more changes from a lower to a higher level of activation than the other way around. This suggests that there is a dynamic of development over time for the calculus units of Population Dynamics and Exponential Growth & Regression.

Levels of consistency

There was evidence of moderate consistency within the Intermediate or Mature level of activation for the Population Dynamics and the Exponential Growth & Regression units. However, during the Exponential Growth & Regression sessions, Johannes faced difficulties (i.e., as seen from the (0,(1),0,0) symbolization in the table above) to obtain a clear perspective towards the solution of the tasks he needed to address and activated the competency continuously within the Initial level of activation for the T.S.V. perspective. During the Population Dynamics unit there were instances of moderate consistency only within the higher levels of activation. This suggest that Johannes was able to locate the relevant information that would assist him to address the tasks he was working on while using expressions with mathematical language on a regular basis.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, there is evidence of development mainly for two calculus units: Population Dynamics and Exponential Growth & Regression. I, therefore, infer that there is a *moderate combination of evidence* within the Radius of Action for the development of the (Info) aspect of the Mathematical Thinking and Acting competency.

Overview for (Graph)

As discussed in the Degree of Coverage part of this section Johannes provided evidence for the activation of the competency aspect (Graph) only on two occasions. Once during the first session of the Exponential Growth & Regression unit and once during the second session of the Population Dynamics unit. There was no evidence of competency activation during the Periodic Functions and the Exponential Growth & Regression units. Therefore, this competency aspect is considered *non-activated*.

Representing and Manipulating symbolic forms competency

Overview for (Rep↔Rep)

Table 10-42. Summarized evidence for (Rep.↔Rep.) activation by unit.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	(0,1,1,0)	(0,1,1,0)	(0,1,1,0)	
Changes from higher to lower	(0,0,0,0)	(0,0,0,0)	(0,0,1,0)	
High consistency	(0,0,0,0)	(0,0,1,0)	(0,0,0,0)	
Moderate consistency	(0,0,2,0)	(0,0,0,0)	(0,0,1,0)	
Frequency of activation				6

Frequency of competency activation

The activation frequencies were: [0, 2, 4, 0]. It is clear that Johannes provided evidence of competency activation only for two calculus units: Population Dynamics and Exponential Growth & Regression.

Changes between activation levels

At these units where competency activation was observed, the changes from a lower to a higher level of activation were comparatively more than these changes to the opposite direction. This observation suggests a dynamic of development for the units of Population Dynamics and Exponential Growth & Regression. Still, it must be noted that the frequency of activation, especially for the Exponential Growth & Regression unit, was evidently low.

Levels of consistency

Evidence of high consistency at the higher levels of activation was observed only for the Population Dynamics unit and within the M.L.V. domain. This suggests that Johannes while switching between mathematical representations, used expressions of mathematical language and vocabulary. At the same unit, there were several instances of moderate consistency within the Intermediate of Mature activation level.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, there is evidence of development for one calculus unit: Population Dynamics. I, therefore, infer that there is a *weak combination of evidence* within the Radius of Action for the development of the (Rep↔Rep) aspect of the Mathematical Thinking and Acting competency.

Overview for (Man.Rep.)

Table 10-43. Summarized evidence for (Man.Rep.) activation by unit.

Competency domain / **Type of evidence**	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	(0,3,4,1)	(1,2,2,0)	(1,1,2,0)	
Changes from higher to lower	(0,1,2,1)	(0,0,1,1)	(0,0,1,1)	
High consistency	((1),0,0,0)	(0,0,0,(1))	(0,0,0,1)	
Moderate consistency	(0,(1),0,1(1))	(1,1,2,0)	(1,2,1,0)	
Frequency of activation				23

Frequency of competency activation

The activation frequencies were: [3, 8, 8, 4]. A more frequent activation was noticed during the Population Dynamics and the Exponential Growth & Regression units. A less frequent activation was observed during Johannes's work on the Integrals & Modeling and the Periodic Functions units.

Changes between activation levels

For all units except the Integrals & Modeling sessions, the changes from a lower to a higher level of activation were identified to be comparatively more than those that occurred towards the other direction. This observation suggests a dynamic of development for all these three calculus units. At the same time the low frequency of activation of the Periodic Functions unit needs to be considered.

Levels of consistency

From all units, only during the Population Dynamics sessions there was no evidence of high or moderate consistency within the Initial level of activation. This suggests that during the sessions of this unit, Johannes was able to maintain a clear perspective towards the solution of the tasks he worked on while using expressions of mathematical language and vocabulary and while needing limited prompting or assistance to activate this competency aspect. For the sessions of the remaining units there were several instances of moderate consistency at all levels of activation. Importantly, there were several occasions during the Exponential Growth & Regression unit where Johannes was able to manipulate the mathematical representations he encountered while needing minimum assistance or prompting from a fellow classmate or the teacher.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, there is evidence of development for two calculus units: Periodic Functions and Exponential Growth & Regression. I, therefore, infer that there is a *moderate combination of evidence* within the Radius of Action for the development of the (Man.Rep.) aspect of the Mathematical Thinking and Acting competency.

Mathematical Reasoning and Communicating competency

Overview for (Reason←)

Table 10-44. Summarized evidence for (Reason←) activation by unit.

Competency domain / Type of evidence	T.S.V.	M.L.V.	Pr.In.T.	
Changes from lower to higher	(0,2,1,2)	(1,1,0,2)	(0,2,1,2)	
Changes from higher to lower	(0,1,0,1)	(0,0,0,1)	(2,1,1,0)	
High consistency	((1),0,0,0)	(0,0,1,0)	(0,0,0,0)	
Moderate consistency	(0,(1),(1),0)	(1,1,0,1)	(0,0,0,2)	
Frequency of activation				19

Frequency of competency activation

The activation frequencies were: [3, 6, 3, 7] which suggests that evidence of frequent competency activation frequency was observed mainly for the Integrals & Modeling and the Exponential Growth & Regression units.

Changes between activation levels

The changes from a lower to a higher level of activation were identified to be comparatively more than those that occurred towards the other direction during the Exponential Growth & Regression, the Integrals & Modeling, and the Population Dynamics units. This suggests a dynamic of competency development over time during the sessions of these units.

Levels of consistency

From all units, only during the Integrals & Modeling sessions there was no evidence of high or moderate consistency within the Initial level of activation. This suggests that during the sessions of this unit, Johannes was able to understand and follow the argumentations put by others while using expressions of mathematical language and vocabulary and while needing limited prompting or assistance to activate this competency aspect.

Strength of evidence combination: final evaluation

Based on the summaries above for all three types of evidence, there is evidence of development for two calculus units: Integrals & Modeling and Exponential Growth & Regression. I, therefore, infer that there is a *moderate combination of evidence* within the Radius of Action for the development of the (Reason←) aspect of the Mathematical Thinking and Acting competency.

Use of Aids and Tools

Overview for (Ext.T.)

As it was discussed in the relevant overview regarding the Degree of Coverage no evidence was provided by Johannes at any of the seven sessions regarding the activation of the (Ext.T) aspect. As with Erika's case, it is possible the content of the sessions did not provoke a discussion regarding the existence and use of tools or aids for mathematical activity.

Technical Level

Representing and Manipulating symbolic forms competency

This competency consists of three aspects and to avoid repetition, I only mention here the relevant abbreviations (codes) that represent these aspects: (C.D.Rep.), (Rep.↔Rep.), and (Man.Rep.).

Overview for technical parts

As discussed in Erika's case, there were several occasions where Johannes activated the '*choose (or decode in) a representation*' competency aspect (C.D.Rep.) concurrently with the (Int.El.) competency aspect. Therefore, the discussion about Johanne's handling with the technical parts of the tasks presents similarities with the discussion undertaken regarding the (Int.El.) competency aspect. These similarities, however, do not concern the technical parts of the tasks during the Periodic Functions unit (as in Erika's case) because Johannes provided evidence of activation of the (C.D.Rep.) only once during this unit. Similar observations (i.e., to the (Int.El.) aspect) can be made though, for the technical parts regarding his work on the Population Dynamics and Exponential Growth & Regression units. Johannes displayed familiarity with the concept of percentages and consumption rates while he was working on the Petroleum Consumption Rate task. He pointed out that "Basically, if it is not rate and just consumption the 2% every year is like year 1 is 2%, year 4 is 4% …" while working decoding the representation from the slides about the 2% increase at petroleum consumption rate.

Johannes provided evidence of activation for the '*switch between representations*' aspect (Rep.↔Rep.) mainly during the Population Dynamics where he was able to switch between representations (e.g., $P(t) = 2 \cdot P(0)$) with little assistance from the teacher or a fellow classmate. Finally, regarding the '*manipulate within a representation*' competency aspect (Man.Rep.), this aspect was observed to be activated often concurrently with the (→Sol.) and (Int.El.) competency aspects. Therefore the overview for the technical parts is similar to the analysis of the aforementioned aspects.

Exploring the M.L.V. domain

In this part of the overview, I present the assembled diagrams (Figure 10.3) from the (C.D.Rep.), (Rep.↔Rep.), and (Man.Rep.) aspects of the Representing and Manipulating symbolic forms competency and provide the evidence focusing only for the M.L.V. domain of activation (red dotted line).

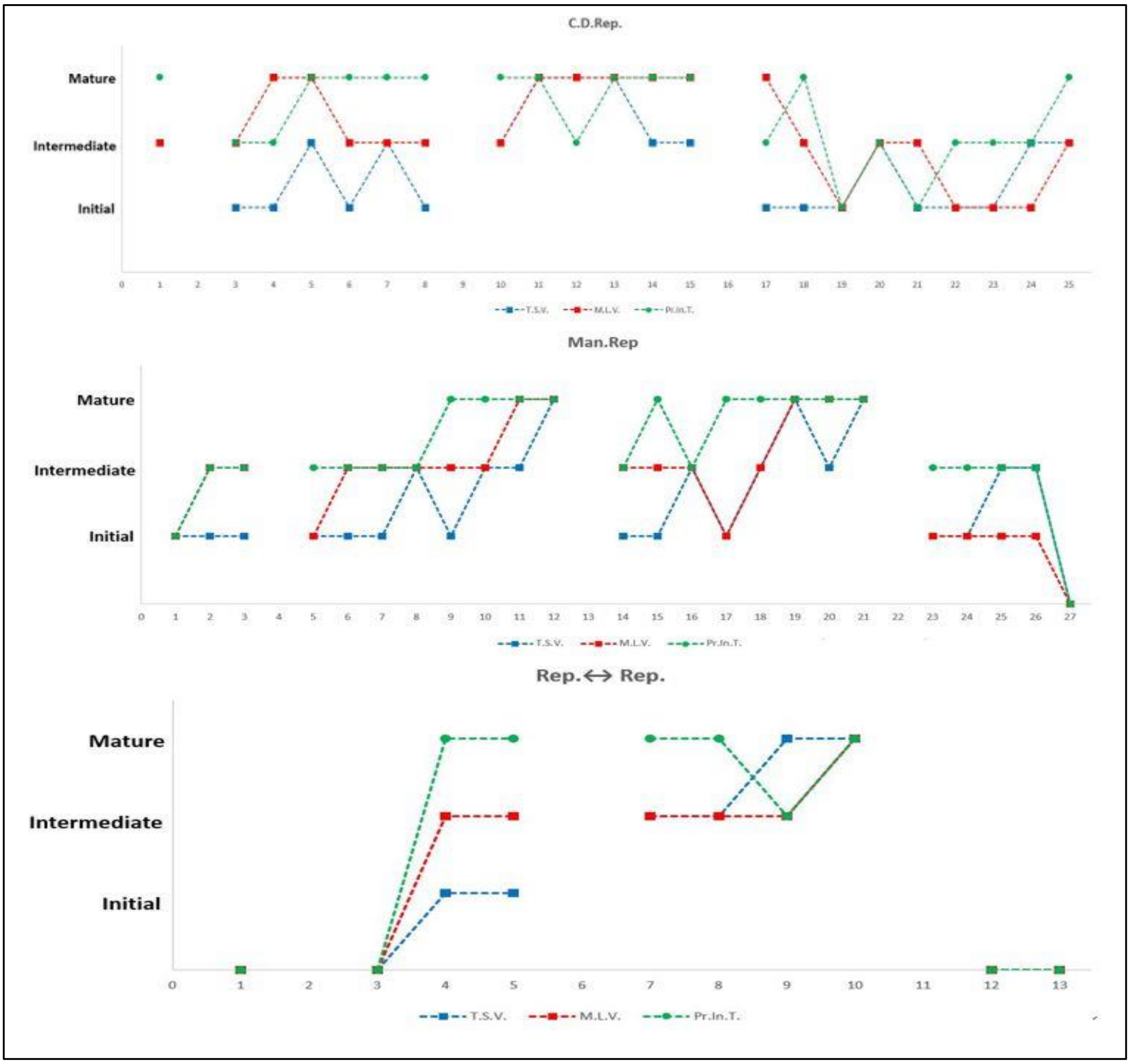


Figure 10.3. Assembled diagrams: Representing and Manipulating symbolic forms competency (Johannes).

Table 10-45 summarizes the two types of evidence (changes between activation levels and consistency of activation) within the M.L.V. domain for the three aspects of the Representing and Manipulating symbolic forms competency. For two aspects of this competency it is evident that the activations were observed to occur more often from a lower to a higher level of activation which suggests a dynamic of development. It is also noticeable that all instances of moderate consistency were observed within the Intermediate of Mature level of the scaling instrument and none within the Initial level of activation. Looking at the (Man.Rep.) column one can see that there was an instance of high consistency within the Initial level of activation which suggests that Johannes faced difficulties using expressions of mathematical language and vocabulary in a continuous manner. However, this incident occured only during the Integrals & Modeling sessions and furthermore, there were multiple occasions of moderate consistency within the higher levels of the scaling instrument. This suggests a dynamic of development for all aspects of this competency.

Table 10-45. Summary of evidence for M.L.V. domain.

Competency aspect / Type of evidence	C.D.Rep	Rep.↔Rep.	Man.Rep.
Changes from lower to higher	4	2	5
Changes from higher to lower	4	0	2
High consistency	1	1	(1)
Moderate consistency	1	0	4(1)

Final assessment for the Representing and Manipulating symbolic forms competency

Based on the overviews for the technical parts and the exploration of the M.L.V. domain regarding the combination of evidence for the progress of development of the three competency aspects that constitute the Representing and Manipulating symbolic forms competency, I infer that there is a *strong combination of evidence* that suggests development within the Technical Level of the competency.

Mathematical Reasoning and Communicating competency

This competency consists of two aspects and to avoid repetition, I only mention here the relevant abbreviations (codes) that represent these aspects: (Reason←) and (Reason→).

Overview for technical parts

Johannes provided evidence of activation of the (Reason→) competency aspect several times with a concurrent activation with the (Int.El.) and (Man.Rep.) competency aspects. Therefore, no important addition, regarding the technical parts of the tasks, can be made in this point of the analysis. Similarly, the activation of the (Reason←) observed on several occasions at the same time with the activation of the following competency aspects: (Int.El.), (Reason→), (→Sol.), and (C.D.Rep.). No important addition, regarding the technical parts of the tasks, can be made in this analysis because the analysis has already been done with the aforementioned aspects. Therefore, to draw inferences for the development of this competency I referred to the previous overviews of the (Int.El.), (Reason→), (→Sol), and (C.D.Rep) aspects.

Exploring the M.L.V. domain

In this part of the overview, I present the assembled diagrams (Figure 10.4) from the (Reason←) and (Reason→) aspects of the Mathematical Reasoning and Communicating competency and provide the evidence focusing only for the M.L.V. domain of activation (red dotted line).

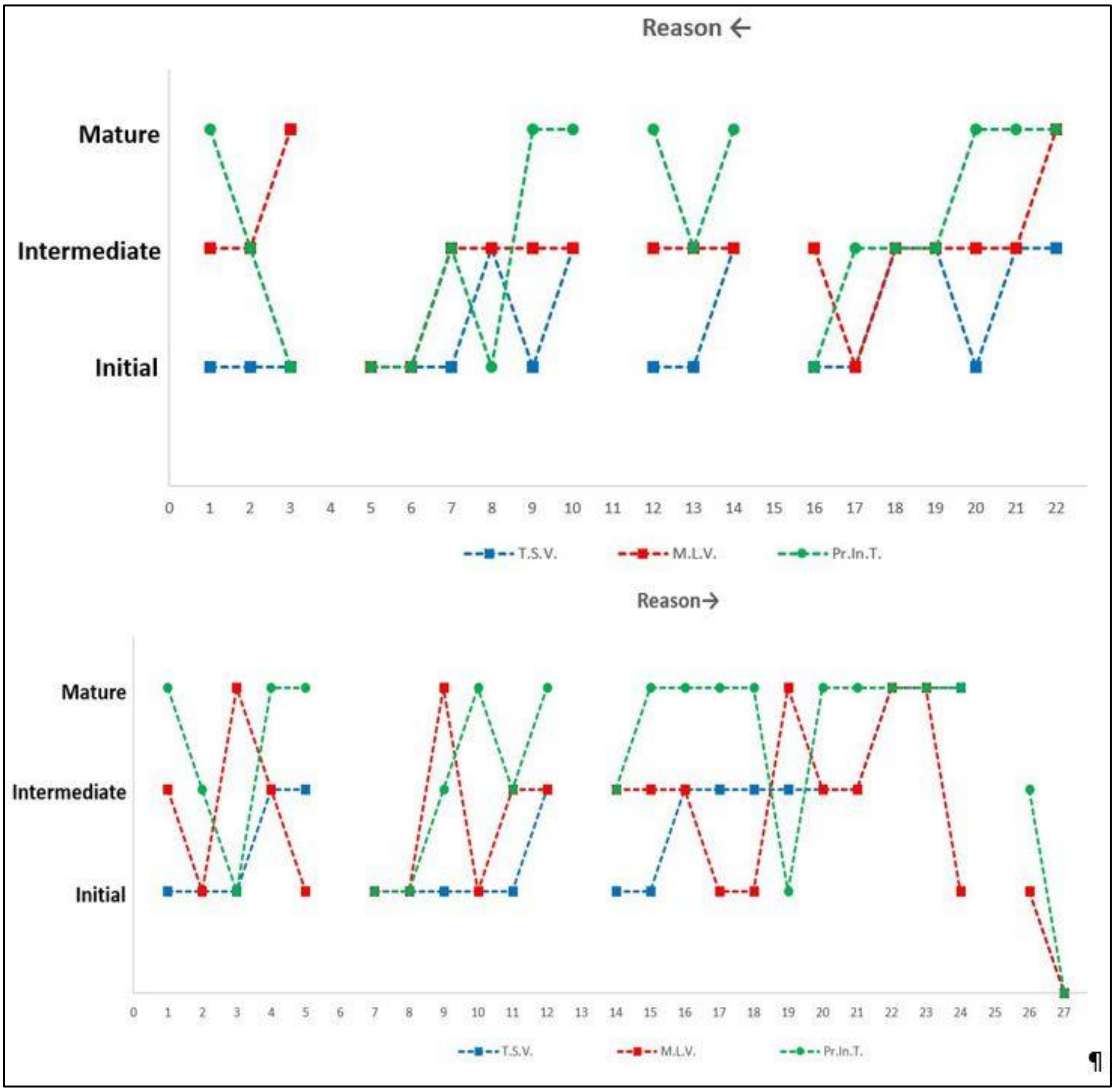


Figure 10.4. Assembled diagrams: Mathematical Reasoning and Communicating competency (Johannes).

Table 10-46 below, summarizes the two types of evidence (changes between activation levels and consistency of activation) within the M.L.V. domain for the two aspects of the Mathematical Reasoning and Communicating competency. For these two aspects, it is observable that the activations were observed to occur more often from a lower to a higher level of activation. It is also noticeable that there were several instances of consistency in all levels of the scaling instrument. Therefore, even though there is a dynamic of development because of the previous observation she also displayed continuous activations at all levels of activation of the scaling instrument which includes the Initial level.

Table 10-46. Summary of evidence for M.L.V. domain.

Competency aspect / Type of evidence	Reason←	Reason→
Changes from lower to higher	4	5
Changes from higher to lower	1	8
High consistency	1	0
Moderate consistency	3	0

Final assessment for the Mathematical Reasoning and Communicating competency

Based on the overviews for the technical parts and the exploration of the M.L.V. domain regarding the combination of evidence for the progress of development of the two competency aspects that constitute the Mathematical Reasoning and Communicating competency, I infer that there is a *moderate combination of evidence* that suggests development within the Technical Level of the competency.

Use of Aids and Tools competency

This competency consists of two aspects and to avoid repetition, I only mention here the relevant abbreviations (codes) that represent these aspects: (Ext.T.) and (U.T.).

Overview for technical parts

As noticed before, the nature of this competency aspect addresses mainly issues of technical use of the calculator and the LiveScribe smartpen. Two interrelations with two other competencies took place during the activation of this competency: (→Sol.) and (Man.Rep.). Therefore, to draw inferences for the development of this competency I referred to the previous overviews of the aforementioned interrelated aspects.

Exploring the M.L.V. domain

In this part of the overview, I present the diagram from the (U.T.) aspect of the Use of Aids and Tools competency (Diagram 10-32) and provide the evidence focusing only for the M.L.V. domain of activation (red dotted line). As mentioned before, no evidence was provided by Johannes at any of the seven sessions regarding the activation of the (Ext.T) aspect.

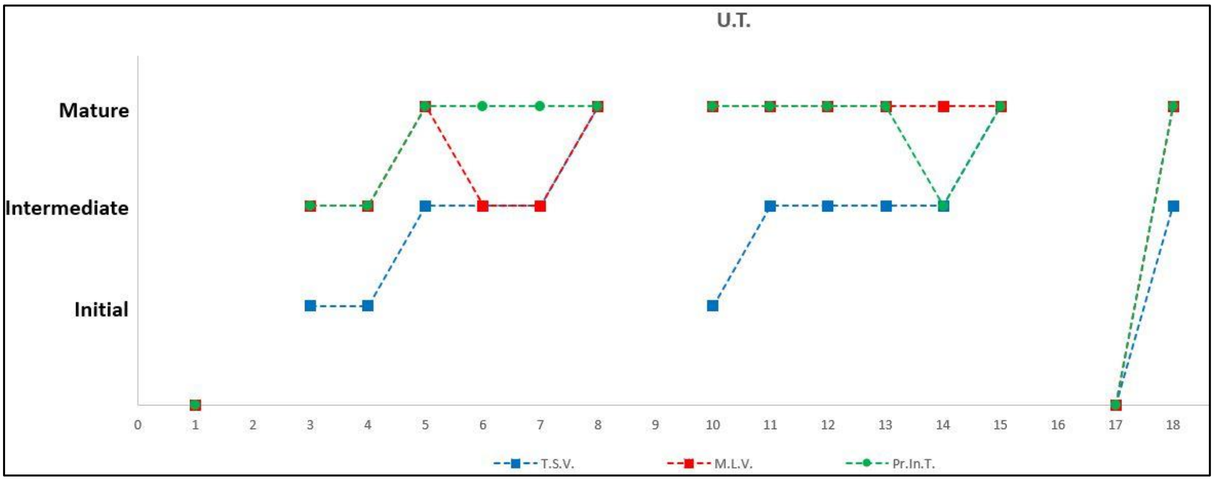


Diagram 10-32. Use of Aids and Tools competency (Johannes).

Table 10-47 below, summarizes the two types of evidence (changes between activation levels and consistency of activation) within the M.L.V. domain for the (U.T.) aspect of the Use of Aids and Tools competency. For this aspect, it is observable that the activations were observed to occur more often from a lower to a higher level of activation. It is also noticeable that there was only one instance of consistency (high) was observed within the Mature level of the scaling instrument, that is, a continuous activation at the higher level of activation of the scaling instrument. This suggests a dynamic of development for the (U.T.) aspect of this competency.

Table 10-47. Summary of evidence for M.L.V. domain.

Competency aspect / Type of evidence	U.T.
Changes from lower to higher	3
Changes from higher to lower	1
High consistency	1
Moderate consistency	0

Final assessment for the Use of Aids and Tools competency

Based on the overview for the technical parts and the exploration of the M.L.V. domain regarding the combination of evidence for the progress of development of the (U.T.) aspect and due to the fact that there was no evidence of activation for the (Ext.T.) aspect, I infer that there is a *weak combination of evidence* that suggests development within the Technical Level of the competency.

APPENDIX F: Intercoder reliability

In this appendix, I included a part of the transcriptions that have been coded and graded separately by two individuals (the author of this dissertation and the author's main supervisor: John Monaghan).

Intercoder reliability

Table 10-48.Code assignment.

Time	Who	Talk and description of actions	Coding: Y	Coding: J
29:19	K	Shall we see how many 20 minutes we have in 24 hours? There are 72 sets of 20 minutes in 24 hours... we will use the same model here because we have similar information	P.Q. Reason→ →Sol.	P.Q. Reason→ →Sol.
31:15	M	The starting point is one bacterium	Info Reason→	Reason→
31:30	K	The increase is 100% since it is double all the time	Reason→	Reason→
31:50	L	What then should be the right rate?	P.Q. Reason←	P.Q.
33:20	M	So, we have the time which is 24 hours, but we don't have the rate. At first, we have to find how many E. Coli will be in 24 hours... maybe it means that they will be divided 72 times, 72 doublings so maybe2^{72}?	Lim. Int.El. Reason→ →Sol. P.Q.	Lim. Int.El. Reason→ →Sol. P.Q.
34:00	K	Is it 72^2 or 2^{72} (*checks calculator*) seems more right the second-choice even though it is a big number?	P.Q. Not.+Graph U.T.	P.Q. Not.+Graph U.T.
	J	Why do we have + in front of Deaths? Is it maybe because the number of deaths count as negative?	P.Q. Reason →	P.Q. Reason →
	J	N_0 is the population at time 0, at the starting point	C.D.Rep.	C.D.Rep
	J	We can get t down using ln or log	→Sol C.D.Rep.	C.D.Rep.
12:00	E	Yes, the growth rate, this is what we are looking for	Reason←	Reason←
13:00	J	Yes, it is increasing and therefore the rate should be positive (looking at the	Reason→	Reason→

		table and indicating that the numbers are bigger every year)		

Table 10-49. Grade assignment.

Time	Who	Talk and description of actions	Coding: Y	Coding: J
29:19	K	Shall we see how many 20 minutes we have in 24 hours? There are 72 sets of 20 minutes in 24 hours... we will use the same model here because we have similar information	II, II, III II, II, III II, II, II	II, II, III II, II, III II, II, III
31:15	M	The starting point is one bacterium	II, II, II	I, II, II
31:30	K	The increase is 100% since it is double all the time	II, III, III	II, III, III
31:50	L	What then should be the right rate?	II, II, II	II, II, II
33:20	M	So, we have the time which is 24 hours, but we don't have the rate. At first, we have to find how many E. Coli will be in 24 hours... maybe it means that they will be divided 72 times, 72 doublings so maybe2^{72}?	II, II, II II, I, II II, II, II III, II, II II, II, III	II, II, II II, I, II II, II, III III, II, III II, II, II
34:00	K	Is it 72^2 or 2^{72} (*checks calculator*) seems more right the second-choice even though it is a big number?	III, II, II III, II, II III, III, II	III, II, II III, III, II III, III, II
	J	Why do we have + in front of Deaths? Is it maybe because the number of deaths count as negative?	II, II, III II, II, III	II, III, III II, II, III
	J	N_0 is the population at time 0, at the starting point	B, B, B	B, A, B
	J	We can get t down using ln or log	II, I, III	II, I, III
12:00	E	Yes, the growth rate, this is what we are looking for	II, II, I	II, I, I
13:00	J	Yes, it is increasing and therefore the rate should be positive (looking at the table and indicating that the numbers are bigger every year)	II, II, II	II, I, II

APPENDIX G: Students' invitation, letter of approval and figures

The appendix here includes the invitation that has been delivered to the students prior to their involvement to the project and the approval letter (Figure 10.5) for the fulfilment of this research by the Norwegian Centre for Research Data (NSD). It should be noted that within this invitation I refer to the sessions as 'Mathematical Modeling Units.' This term was eventually abandoned and the term 'Calculus Unit' has been adopted. Appendix G also includes figures from the initial hand-written draft for the data analysis planning and screenshots from the results when the Binomial Test was run.

Students' invitation

Dear Students,

I will try to describe in few words what we are planning to do during the next 4 months.

Who are we?

- the Department of Biology of UiB
- the Department of Mathematics of UiB
- the bioCEED – Centre of Excellence in Biology Education
- the MatRIC Centre for Research, Innovation and Coordination Mathematics Teaching of UiA
- and of course, … you and me, Yannis Liakos. I work and conducting my PhD research at the University of Agder

What is the plan?

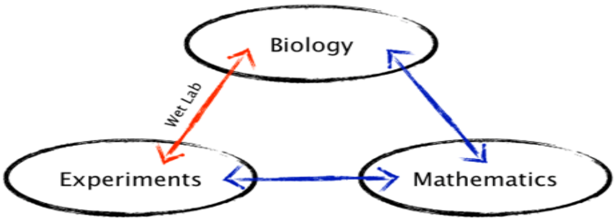
It is our belief that mathematical studies at university will be enriched by the inclusion of applications relevant to the main programme of study, in your case Biology. Our goal is to introduce and work on mathematical biology tasks and try to provide an alternative way to approach mathematics.

During your autumn semester we are planning to create a learning opportunity which we call «**Mathematical Modeling Unit**» that will be part of your regular **MAT101** course in elementary calculus. This opportunity should also help you with your assignments in MAT101. We will be organized in three groups (MC1, MC2 and MC3) and each group will attend a 45-50 minutes session on eight or nine occasions during the semester. In these sessions we will discover how mathematics can be applied to understand phenomena in the natural world. For research purposes, in every session, there will be a video camera at the back of the class facing mostly the whiteboard and not your faces. In these classes you will work in small groups of three or four to work on some tasks, which we hope you will find interesting because they will be focused on your own field of study: Biology. There will be times when I ask small groups to report to the whole class, on these occasions if you want the camera turned off you must say so. However, as it could make an important contribution to my PhD research, I hope you will allow it to continue running.

In addition to the above, I am seeking, in each class, one small group of three or four to be willing to be studied more closely. This will entail a video camera recording the group work and I will collect, digitally, copies of the groups' written work. This will be a very important part of the data I need for my project. The data I collect is for my personal use and will be shared only with my supervisors, if I use the video recordings in other circumstances; images will be pixelated to ensure no-one can be recognized.

What is mathematical modelling?

If we must answer this question with one picture this one could give a quite good description.



Remember

Mathematical Modelling may not give you the exact answer but it will tell you where you should look for it.

Population biologists are trying to predict future population of living species with various mathematical approaches. Charles Darwin (1872) in the 6th edition of the Origin of Species tried to illustrate the geometric process of population growth of elephants that could be produced beginning with a single pair. Biologists modeled a population genetic framework to explore the change in the frequency of genes controlling life history characters. We also know that some species of insects and fish produce thousands and millions of young per year per adult, while other organisms like elephants or humans produce only a handful of young over their lifetimes. Mathematical modeling may provide us some answers about what environmental features might make natural selection go in such opposite directions in terms of reproductive effort.

I could continue writing about the beautiful relationship between nature and mathematics, but I think that it would be a promising idea to discover it together in our future mathematical modeling sessions. Besides I heard that you are really engaged with the meteorite problem and that is good news.

With your participation we will try to build strong mathematical tools that will be helpful for your present and future studies.

Letter of approval from NSD

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Serviceboks 422
4604 KRISTIANSAND S

Vår dato: 31.03.2017 Vår ref: 53085 / 3 / BGH Deres dato: Deres ref:

TILBAKEMELDING PÅ MELDING OM BEHANDLING AV PERSONOPPLYSNINGER

Vi viser til melding om behandling av personopplysninger, mottatt 17.02.2017. Meldingen gjelder prosjektet:

53085	*Mathematical Modelling And Mathematical Competencies: The case of Biology students*
Behandlingsansvarlig	*Universitetet i Agder, ved institusjonens øverste leder*
Daglig ansvarlig	*Ioannis Liakos*

Personvernombudet har vurdert prosjektet og finner at behandlingen av personopplysninger er meldepliktig i henhold til personopplysningsloven § 31. Behandlingen tilfredsstiller kravene i personopplysningsloven.

Personvernombudets vurdering forutsetter at prosjektet gjennomføres i tråd med opplysningene gitt i meldeskjemaet, korrespondanse med ombudet, ombudets kommentarer samt personopplysningsloven og helseregisterloven med forskrifter. Behandlingen av personopplysninger kan settes i gang.

Det gjøres oppmerksom på at det skal gis ny melding dersom behandlingen endres i forhold til de opplysninger som ligger til grunn for personvernombudets vurdering. Endringsmeldinger gis via et eget skjema, http://www.nsd.uib.no/personvernombud/meld_prosjekt/meld_endringer.html. Det skal også gis melding etter tre år dersom prosjektet fortsatt pågår. Meldinger skal skje skriftlig til ombudet.

Personvernombudet har lagt ut opplysninger om prosjektet i en offentlig database, http://pvo.nsd.no/prosjekt.

Personvernombudet vil ved prosjektets avslutning, 01.09.2019, rette en henvendelse angående status for behandlingen av personopplysninger.

Vennlig hilsen

Kjersti Haugstvedt

Belinda Gloppen Helle

Kontaktperson: Belinda Gloppen Helle tlf: 55 58 28 74
Vedlegg: Prosjektvurdering

Dokumentet er elektronisk produsert og godkjent ved NSDs rutiner for elektronisk godkjenning.

NSD – Norsk senter for forskningsdata AS Harald Hårfagres gate 29 Tel: +47-55 58 21 17 nsd@nsd.no Org.nr. 985 321 884
NSD – Norwegian Centre for Research Data NO-5007 Bergen, NORWAY Faks: +47-55 58 96 50 www.nsd.no

Personvernombudet for forskning

Prosjektvurdering - Kommentar

Prosjektnr: 53085

COLLABORATION
The research project is a collaboration between UiA and UiB. UiA is the responsible data controller. The Data Protection Official for Research presupposes that the responsibility for processing personal data has been formally clarified between the institutions. We recommend that the division of responsibilities is formalized in a contract that includes structure of liabilities, who initiated the project, use of data and ownership.

INFORMATION AND CONSENT
The sample will receive written information about the project, and give their consent to participate. The letter of information is well formulated, however the end date of the project (01.09.2019) has to bed added, with information that within this date the data will be anonymized. Furthermore, if you wish access to the students' examination grades this has to be explicit in the information letter.

METHODS OF GATHERING DATA
According to the notification form data will be gathered through a questionnaire, observations and audio- and video recordings. Furthermore the researcher wishes to get access to the students' examination grades and written reports. The Data Protection Official presupposes that the students give their explicit consent to this.

Keep in mind that you need to give information to and get consent from all persons that you collect personal data about (also when other students/persons than the informants are recorded on video).

INFORMATION SECURITY
The Data Protection Official presupposes that the researcher follows internal routines of Universitetet i Agder regarding data security. If personal data is to be stored on portable storage devices, the information should be adequately encrypted.

THE END OF THE PROJECT
Estimated end date of the project is 01.09.2019. According to the notification form all collected data will be made anonymous by this date.
Making the data anonymous entails processing it in such a way that no individuals can be recognised. This is done by:
- deleting all direct personal data (such as names/lists of reference numbers)
- deleting/rewriting indirectly identifiable data (i.e. an identifying combination of background variables, such as residence/work place, age and gender)
- delete audio- and video recordings.

Figure 10.5. Approval letter from NSD.

Hand-written analysis drafts

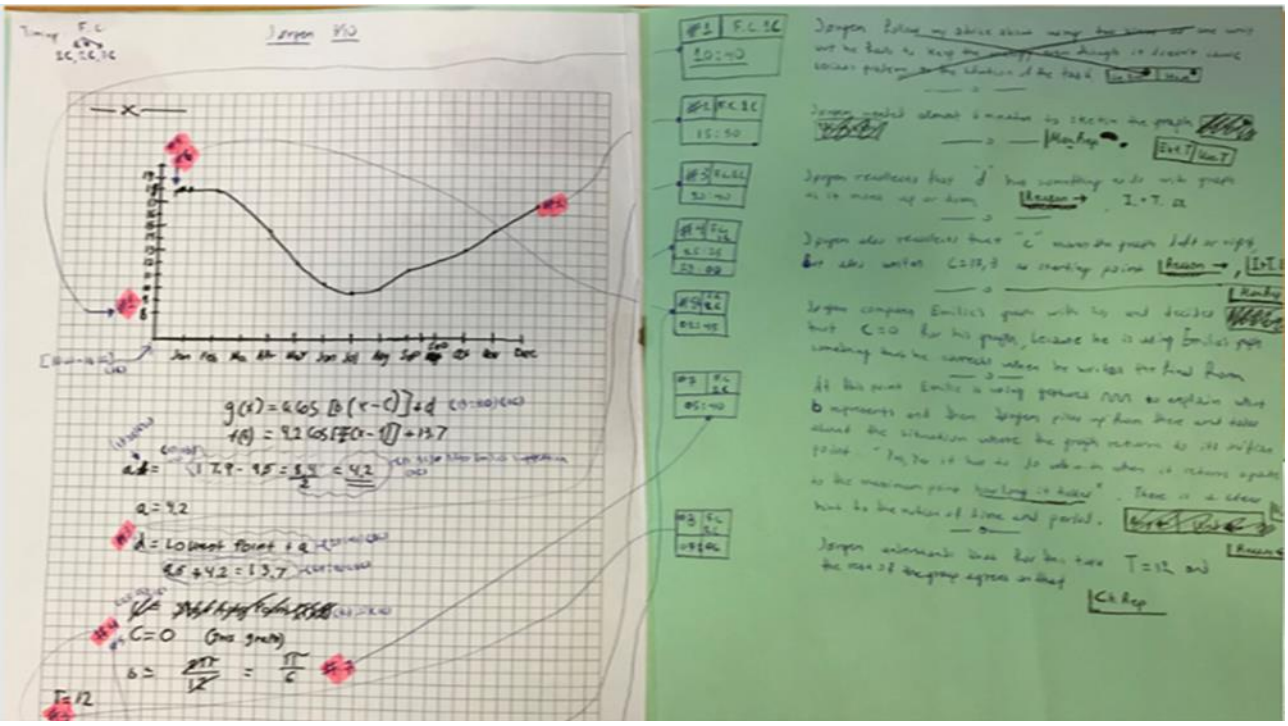

Figure 10.6. Initial hand-written draft for data analysis planning (a).

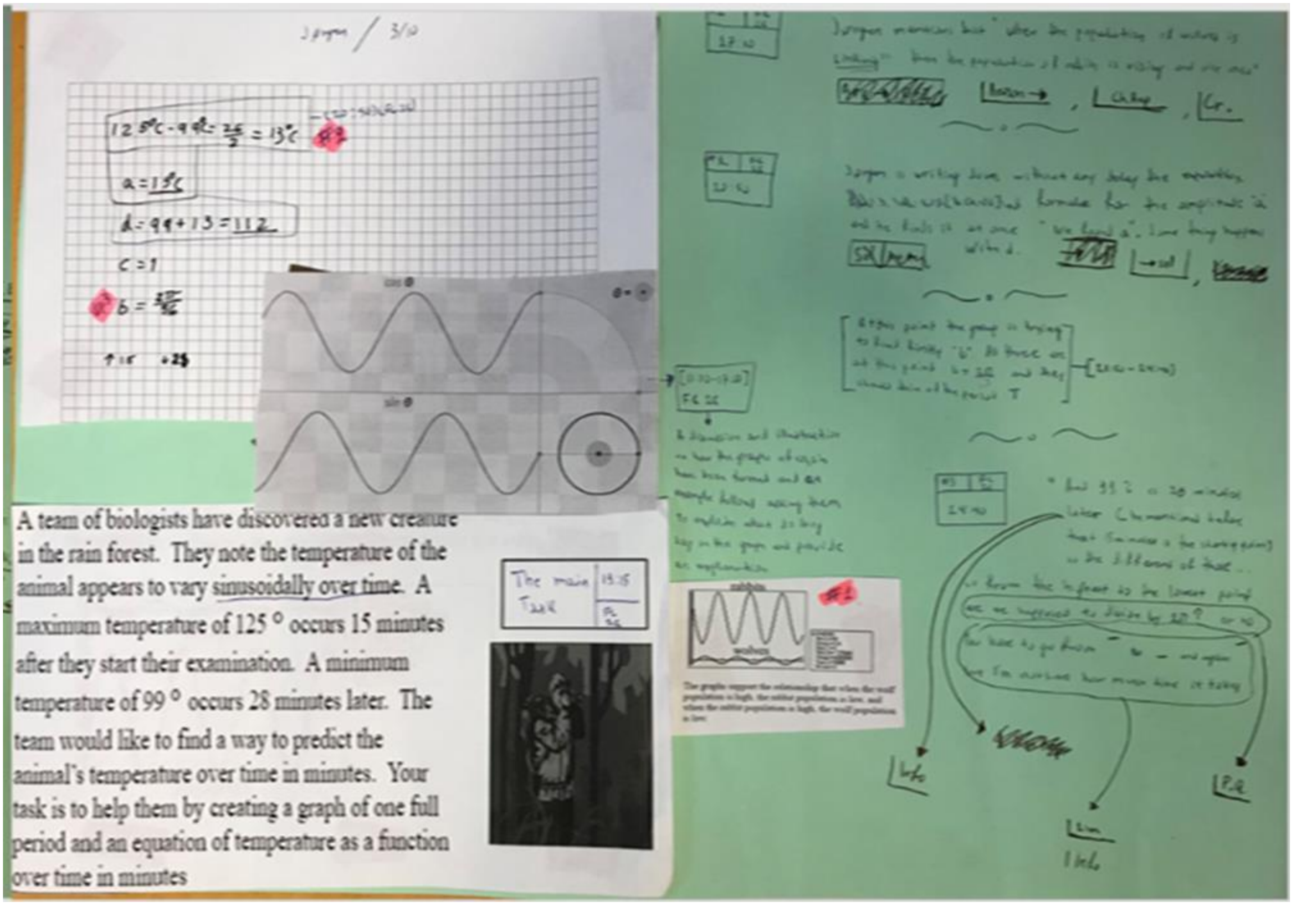

Figure 10.7. Initial hand-written draft for data analysis planning (b).

Binomial Test screenshots

Figure 10.8. Results from running the Binomial Test (i).

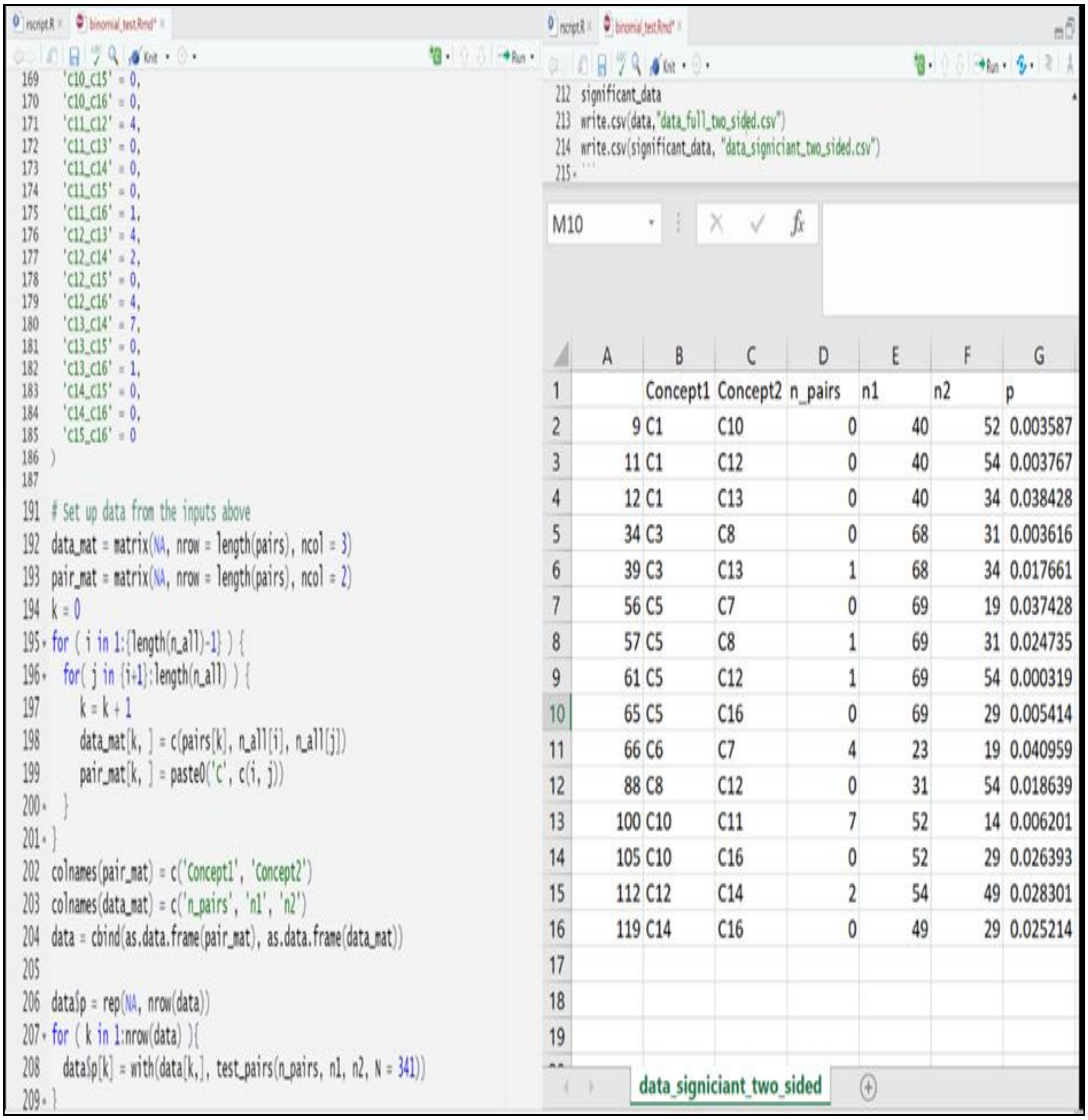

Figure 10.9. Results from running the Binomial Test (ii).