In-Service Teachers’ Perception of the Usefulness of Analyzing their Lower Secondary Students’ Written Argumentation

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Abstract

This study investigates how in-service teachers perceive analyzing their own lower secondary students’ written argumentation as useful. The participants were lower secondary school teachers enrolled in a university mathematics teaching program. The teachers planned and conducted a teaching lesson and wrote a report concerning the argumentation in their students' written work. The qualitative data consist of reports from in-service teachers and transcriptions of five follow-up interviews. Findings suggest that the teachers perceived the analysis useful for gaining insight into various parts of the students' prerequisites, establishing a basis for facilitating instruction, and acquiring knowledge related to different aspects of proof. The results imply that analyzing students' argumentation can be a meaningful and useful activity for in-service teachers to engage in.

Keywords: Mathematical reasoning, argumentation, reasoning-and-proving, in-service teachers, teacher education

Introduction

Reasoning, argumentation, and proving are recognized as important activities for students to engage in by researchers and educators worldwide, as they are fundamental to the field of mathematics, the work of mathematicians, and have the potential to promote mathematical understanding (e.g., Ball & Bass, 2003; Ball et al., 2003; G. Stylianides, 2009; Hanna & de Villers, 2012; National Council of Teachers of Mathematics, 2000; A. Stylianides, 2016; Stylianou et al., 2010).

At the same time, there is a lot of research suggesting that it is challenging to engage students in such activities, even for experienced teachers (Balacheff, 1988; Knuth, 2002; A. Stylianides 2016; G. Stylianides, 2008). This necessitates researchers' and teacher educators' attention; teacher education needs to provide teachers opportunities to develop necessary knowledge and dispositions, to be able to successfully engage students in reasoning, argumentation, and proving (A. Stylianides, 2007).

This preparation of mathematics teachers has been promoted as a challenging task, and there is a lack of research that teacher educators can rely on in handling this task (Ponte & Chapman, 2008; A. Stylianides, 2016; G. Stylianides et al., 2013; Wathne & Brodahl, 2019). If the universities are to succeed in preparing mathematics teachers to engage students in reasoning, argumentation, and proving, more research-based knowledge on how this can be achieved is necessary.

Current study

Aim of the study and research question

This study investigates how a specific analysis activity in a professional developmental program for in-service teachers potentially can be useful to engage in to develop knowledge related to proof identified as important. The goal was to gain insight into teachers' perception of the usefulness of analyzing their students' written argumentation, as a part of their university course. The research question guiding this investigation is therefore: "How do in-service teachers perceive analyzing their lower secondary students' written argumentation as useful?" For the sake of simplicity, I will refer to the in-service teachers also simply as teachers from here on.

The term *useful* in this setting refers to "the quality of having utility and especially practical worth of applicability" (Merriam-Webster, 2021). It is a question of whether the teachers perceived the analysis-task as beneficial or helpful to them as teachers in some way in their practice. The term *analyzing* in the research question refers to a systematic examination of the students' written work using a given analysis tool. In this case the analysis tool that they use is *Balacheff's theory*, which will be presented in the next part of this paper. This study uses the term *perception* to refer to how the teachers regard or 'see' the analysis as useful.

Answers to the posed research question can potentially be valuable for teacher educators aiming to give the necessary support and guidance to teachers or prospective teachers, as it gives some indications of the utility of engaging teachers in such activities in their professional development program. This provides information that teacher educators potentially can draw from when preparing teachers to be able to successfully engage students in reasoning, argumentation and proving.

Theoretical framework

The closely related terms *reasoning, argumentation, proof* and *proving* have been given diverse meanings and definitions in different research milieus in mathematics and it is therefore important to provide a clarification of what it is meant by the terms in the context of this study (Balacheff, 2002; Reid, 2005). The term *reasoning* is in this study referring to the thinking process that lies behind one's argumentation and proving. *Argumentation* is a term closely related to reasoning, as it is to be understood in this context as a way of communicating and sharing the reasoning. A *proof* is in this study defined as a special type of argumentation, namely one that demonstrates the truth or falsehood of a particular claim in an absolute sense (A. Stylianides, 2007). For mathematical argumentation to count as a proof, it must remove any doubt of the truth of an assertion by providing an explanation of why it (the strategy, method etc.) always work, for all cases within a certain class of objects. *Proving* is to be understood as the verb denoting the activity of constructing such a proof.

The development of a proof is typically the last stage of a longer process involving a range of other related mathematical activities, such as empirical exploration, identifying patterns, making conjectures based on the patterns, generalizing and testing conjectures, providing informal non-proof arguments to support mathematical claims (A. Stylianides, 2007; G. Stylianides, 2008; A. Stylianides & Ball, 2008). To emphasize how these activities are integrated in the proving process, and the development of proofs, G. Stylianides (2008) established the hyphenated term *reasoning-and-proving*. This term describes the overarching activity that encompasses the collection of activities often included in research mathematicians work related to proof.

Balacheff's theory

Balacheff (1988) developed a framework for conceptualizing the progression from inductive or empirical justifications toward being able to construct valid proofs. Based on an experimental study he classified students' work into four different types of proof, proposing a hierarchy reflecting higher levels of advanced thinking. The types of proof were suggested in the following order*: naive empiricism*, *the crucial experiment*, *the generic example,* and *the thought experiment.*

Naive empiricism involves verifying the truth of a claim by testing some selected examples. *The crucial experiment* involves testing a final selected example and then seeing this as absolute proof of the claim (Balacheff, 1988). These are what Balacheff calls *pragmatic proofs* meaning that they are based on direct showing through use of specific examples, i.e., *empirical arguments.* According to both Balacheff (1988) and A. Stylianides (2007), these forms of argumentation do not qualify as valid proofs as they use invalid forms of argumentation. Identifying them as 'levels of proof' is merely a way of recognizing that students themselves see them as proofs. This is what Harel and Sowder (1998, 2007) refer to as having *empirical proof schemes*, meaning that the individual is convinced by empirical arguments, or that they try to convince someone else of the truth of an assertion based on a few confirming examples.

The third level, *the generic example*, involves argumentation for something general, using a representative example. This could, for instance, be done by performing certain operations on the mathematical object, to show why this must apply to the class of cases that the object represents. This type of proof could be considered valid, as the particular case is then seen as a prototype, and therefore it demonstrates the truth for all objects, not just the example at hand (G. Stylianides & A. Stylianides, 2009). The last level, *the thought experiment*, is a fully conceptual form of proof as it involves argumentation without the use of any concrete examples. This means that one expresses the mathematical object in general, based on its properties, and performs operations on the object to demonstrate the truth of an assertion (Balacheff, 1988).

Mathematics teacher knowledge related to proof and proving

A. Stylianides (2016) points to the teacher's knowledge related to proof and proving as important to succeed in engaging students in reasoning-and-proving, and as an important factor as to whether such activities will be included in the classroom.

The teacher needs to be able to distinguish between valid and invalid forms of argumentation (Martin & Harel, 1989; Simon & Blume, 1996; A. Stylianides, 2007, A. Stylianides & Ball, 2008). It is therefore critical that previous research has shown that many in-service secondary school mathematics teachers perceive empirical arguments as valid proofs (Knuth, 2002). If the teacher perceives empirical argumentation as a valid form of proof, they will probably allow, and perhaps even encourage, their students to recognize this as proof (A. Stylianides, 2007; A. Stylianides & G. Stylianides, 2009). This would then lead to the students' misconceptions of proof not being challenged, and thus continue to perceive empirical arguments as valid proofs (A. Stylianides & G. Stylianides, 2009, p. 238).

Knowledge of typical developmental sequences from empirical reasoning towards deductive reasoning has also been highlighted as important for the teacher, as well as awareness of typical conceptions and misconceptions of proof students might have at different stages of their development (Lesseig, 2016). For instance, this involves teachers' awareness of the typical misconception students often have early in their development, where they perceive empirical arguments as valid proofs.

Teachers also need to know what types of arguments would be appropriate and accessible for their students at their stage of development and grade level (Lesseig, 2016). This involves knowing that generic examples are particularly useful in secondary school, as they can help students 'see' the general and prove mathematical claims through the use of a particular example (Mason & Pimm, 1984; A. Stylianides, 2016; G. Stylianides, 2009).

Another important aspect of knowledge on proof for teachers is understanding about existing classroom cultures, and how these could be changed over time (G. Stylianides et al., 2013). This involves being aware of the existing *didactical contract,* a term used to describe the invisible existing norms and expectations in the classroom (Brousseau et al., 2014). Balacheff (1999) pointed out that one most often becomes aware of the existing didactical contract when it is broken. An example of such a breach of contract could be if the students are not used to justifying their claims and are suddenly asked to do so (A. Stylianides & G. Stylianides, 2009). Teachers must be aware that there are different classroom cultures and existing didactical contracts and that they are aware that they probably will meet some form of resistance in their attempt to engage the students in new activities that can be unfamiliar to them (G. Stylianides et al., 2013).

Together with the teachers' knowledge, A. Stylianides (2016) also points to the teachers' beliefs about the role of proof and proving as another important factor in determining whether reasoning-and-proving will be included in the classroom or not. He points out that if a teacher believes that reasoning, argumentation, and proving are advanced mathematical topics and activities beyond the reach of most of their students, it is likely that reasoning-and-proving will have a marginal place in their classroom. If the teacher, on the other hand, knows what proving would mean in a school context and perceives it as a suitable activity for all their students, it is more likely that they will give opportunities for their class to engage in reasoning-and-proving (A. Stylianides, 2016).

Methodology

This study adopts a *case study design*, where the case is the group of in-service teachers (Bryman, 2012). The data were collected through teachers' project reports and in-depth interviews conducted by the researcher. In the following section, the setting behind the study and the participants will be presented. Also, the data collection process and research instruments will be described, followed by an explanation of the data analysis procedure.

Setting and participants

The participants were teachers enrolled in an online education program designed for in-service teachers who already have a general teaching credential, allowing them to further develop their mathematical and didactical competence. The course was divided into several modules, and one of them focuses on mathematical argumentation.

Prior to this module, the teachers received a request to participate in a research project. There were 43 teachers in total, who gave their consent to the report being a part of this research project with 25 of these teaching in lower secondary schools. Out of these twenty-six, 17 teachers also gave their consent to the possibility of being asked to participate in a follow-up interview. Based on the content in their reports, five of these 17 teachers were asked to participate in follow-up interviews.

Data collection

Data were collected as part of a two-week project assignment included in the above-described module. In this project, the teachers were challenged to engage their students in the mathematical problem "sum of consecutive odd numbers", which involves finding a general way to express the sum of consecutive odd numbers. The problem was presented in the form of a written "imaginary dialogue" (Wille, 2011; Wille & Boquet, 2009) between two made-up students, where they start exploring and discussing the mathematical problem. The students were challenged to explore this mathematical problem further and continue the written dialogue, taking the roles of the imaginary students. The teachers' task afterward was to analyze the written work using Balacheff's theory of levels of proof as framework and reflect on their experiences of doing this.

The reports were divided into three parts. The first part was to give a brief description of the planning and execution of the teaching session. The second part involved a presentation of the students' written work, identification of 'proof levels' (based on Balacheff's theory), and explanation of the identifications made. The teachers were then asked to reflect on their teaching experience and the analysis done, based on the three open questions:

1. What insights did you gain in students' argumentation, reasoning, and proof, in their written dialogues, in relation to Balacheff's theory of proof levels in school?

2. Which aspects in relation to Balacheff's theory of proof levels in school were helpful or useful to you in your identification and justification?

3. Which aspects in relation to Balacheff's theory of proof levels in school were challenging for you in your identification and justification?

The third part of the report consisted of a questionnaire where they were asked to rate on a scale from one (not useful) to ten (very useful) their level of agreement about 13 statements about experiences with the activity and the analysis-task.

After the module and the submission of the reports, five semi-structured in-depth interviews were conducted. The purpose of this was to gain supplementary information beyond what they had written in the reports, providing a better foundation to make inferences about the participants' views of usefulness. Only the second part of the reports, which included teachers' reflections on their own experiences, together with the data from the follow-up interviews, were used as data material for this study.

Data analysis

Data were analyzed using a *qualitative content analysis* approach (Bryman, 2012). Three main categories for usefulness were identified as a result of an iterative analysis process of coding and recoding. Through an inductive approach, dominant themes and codes in the second part of the teachers' reports were initially identified, listed, and organized in a codebook. These were then used to guide the coding of the transcripts. Initial themes and codes were adjusted, edited, and restructured in this process. The data material in total was at last coded, guided by the three identified main categories, and additionally by identified subcategories.

Results

Based on what the teachers wrote in their reports, and their responses from the interviews, three main categories for experienced usefulness were identified: (a) gaining insight into students' prerequisites; (b) enabling them to facilitate the teaching of mathematical argumentation; and (c) contributing to their professional growth. In the following section, I will describe in more depth what the teachers perceived as useful within these three main categories as well as give some samples of teachers' statements on the subjects.

Gaining insight into students' prerequisites

The teachers perceived the analysis as useful to gain insight into their students' seemingly little prior experience with reasoning, argumentation and proving. One of the teachers reported:

> It also became clear that the students are not trained in exploring, reasoning, or constructing mathematical arguments and proofs, especially not in putting mathematical thoughts and ideas

> down on paper. This is something that both secondary and primary schools may benefit from working on more. [all translations by the author].

Another student said: “It was clear that engaging in argumentation, reasoning, and proof is something they have not worked a lot on previously.”

Several teachers reported gaining insight into their students’ strategies for argumentation. One teacher put it the following way: “Seeing students in this light is new to me; Balacheff has provided me with a 'tool' for being more conscious of how students argument.” Another teacher reported: “Through this approach, I was able to acquire a new perspective on students’ relationship to argumentation, their capacity to justify their claims, and their relationship to proof as a concept.” A third teacher wrote: “What helped, or was valuable to me, in identifying and justifying according to Balacheff's theory of proof levels in school, was gaining insight into how students think.” This teacher further added: “The implementation results demonstrate that the students are unable to arrive at valid proofs; they conclude based on some individual examples.” This indicates that the analysis was perceived as useful to becoming aware of how their students go about trying to prove something mathematically and their perception of the concept of “proof.”

Some of the teachers also reported gaining insight into students’ lack of experience in using drawings, figures, and illustrations in the argumentation. An example statement from one of the teachers is the following:

> It seemed completely new to them, to go about using illustrations or drawings to justify something. The vast majority were just adding up and then they found out that it was true, and in a way stuck to the numbers. And those who started to illustrate didn't get very far. They just drew the same pyramids, and no one came to think of the idea of moving the bricks to create something else. It was, in a way, completely foreign to absolutely everyone.

The teacher went on to say: “So that was a bit of an eye-opener to me, realizing that this is something they are unfamiliar with. They aren't accustomed to thinking in this manner. To explain and argue for something general through drawings.”

Being enabled to facilitate the teaching and learning of mathematical argumentation

Some teachers stated that the analysis was useful to gain a foundation on which to guide the students. One of the teachers emphasized the benefits of knowledge of the different levels of proof: “It also works as a tool to ask better questions. Follow-up questions. If it turns out that the student is still playing within a small number-material, then I can challenge the student to raise the argument, and the series of arguments.” Another teacher stated it this way: “When they had to put it into words you could see how they reason.” She elaborated further on this: “And then it is somewhat easier to guide them, because then you understand what is going on in their head, and where they are, in a way. Mentally.” This indicates that they considered the analysis useful to see where the students are, and further, to be able to ask the students questions that challenge them to develop the argumentation further.

Several of the teachers seemed to find the analysis framework a useful tool to become aware of different types of arguments and to visualize what would be within the reach of students. One of the teachers suggested introducing the students to Balacheff's levels of proof: "to create a deeper understanding of what is needed to have a solid proof." This was explained further in the interview: "Because if they don't know exactly what is required or how a proof can be valid, it is very easy to just settle because then they think that they have 'done the job.' But by getting it a little more outlined, they may know which way they can go and what is expected. In a slightly more concrete way." She added the following:

> And not necessarily that you have to reach level four. This, I have come to understand, is quite rare in elementary and lower secondary school - but in a way, maybe reaching for level three, which I think many students could do if they only knew a little more about what is, in a way, expected, and what characterizes the different levels of proof.

The work with the analysis also inspired future teaching for many of the teachers. For example: "By using Balacheff's theory of proof levels in school, I gained a broader understanding of the importance of being able to argue in mathematics." Another wrote: "I think that mathematical argumentation is very important for students to learn, and I think that we should work more with it in the future so that they get a thorough understanding of mathematics and not just follow formulas uncritically."

Several teachers reported that the work with the analysis raised awareness of the styles of learning that they promote in their classrooms. In the interview, one of the teachers said the following: "This autumn, I have included reasoning and argumentation a lot more in the classroom." Another teacher expressed something similar in the interview: "I believe that that argumentation and proof have a lot of potential. Letting the students go a little further - triggering that curiosity." In the report, he also wrote the following: "Proof and argumentation for one's solutions have previously not been emphasized to a large extent at the lower secondary level, but I don't think it wouldn't hurt to include it more. And it is not too late to teach 'old' dogs to bark in a new way, but it is challenging to start in 9th grade if you do not prioritize it." This indicates that these teachers have opened their eyes to argumentation and proof and think that it would be wise to spend more time working on this.

Contribution to their professional growth

In terms of professional development, the teachers stated that working with the analysis was beneficial for learning about proof. One of the teachers stated in the interview, that working with the analysis was useful to become aware that there are different types of proof: "It made a lot of sense. It immediately helped me. I'd never considered the existence of different 'levels of proof'." Another teacher wrote the following in the report: "It is also good to know that levels 1 and 2 are not counted as valid proof (...)", indicating their increasing awareness of what does count as valid proof.

Some of the teachers even stated that they became aware of their limitations by working on this analysis project. One of them put it this way:

> When I read the dialogue (on the mathematical task) for the first time, I thought to myself, 'well, how in the world am I going to prove this?' It took me a long time to come up with anything reasonable. It could have something to do with the fact that I'm new to this way of thinking as well. But I have become more accustomed to and learned more about this way of thinking this fall.

Another teacher started becoming more aware of his limits of proving: "I don't think I'll be able to reach level four myself. Because I am so dependent on seeing things visually. Working with concretes."

Discussion

In this part of the paper, I will highlight and explain some of the key findings in light of the theoretical framework. Ending the discussion, I will provide some closing remarks, disclaimers and critical reflections upon the study and the findings. The posed research question for this study was: "How do in-service teachers perceive analyzing their lower secondary students' written argumentation as useful?".

The in-service teachers perceived the analysis as useful for providing insight into where students were in the process from pragmatic to conceptual proof (Balacheff, 1988) and more specifically visualizing students' empirical proof schemes (Harel & Sowder, 1998, 2007). This was perceived as useful to facilitate the right focus in the subsequent teaching. In this regard, Balacheff's theory was perceived as valuable as a framework to gain an understanding of their students' prerequisites and possibilities for further development (Lesseig, 2016). This was reported as useful for becoming aware that level three would be within the students' conceptual reach and something to aim for. For the facilitation of instruction, the analysis tool was in this sense useful for indicating the direction in which the students should be guided.

The analysis was seen as useful to become aware that reasoning-and-proving were not in line with the traditional teaching the students were used to, and hence teachers saw that the students were lacking experience in participating in proof-related activities. The analysis was in this way useful to make the existing didactical contract visible to the teachers by experiencing a breach of the didactical contract (Balacheff, 1999; Brousseau et al., 2014). This was perceived as useful to them as a way of highlighting the need to change the classroom culture to make their students more receptive to reasoning-and-proving. It was also perceived as useful for becoming aware that the students will need some more practice and experience in mathematical argumentation and that they might need to see some examples of how to use drawings, figures, and illustrations for argumentation.

The analysis was also regarded as useful for gaining inspiration to include reasoning-and-proving in their classroom. Some of the teachers stated that they have started to include more reasoning-and-proving in their teaching during the autumn and stated that they would prioritize setting aside more time to let students explore why things work the way they do. This is interesting due to the fact that teachers' beliefs were highlighted by A. Stylianides (2016) as an important factor that might determine the place of proof and proving at elementary school. It seems like these teachers perceive proving as an appropriate goal and suitable objective for their secondary students to engage in. It is then reasonable to believe that this might contribute to reasoning-and-proving gaining a larger role in their classrooms in the future.

The analysis was perceived as useful for becoming aware of different types of arguments and for learning what arguments could be considered valid proofs and which ones could not. These are, as we have seen, important aspects of a teacher's knowledge of proof (A. Stylianides, 2016). This awareness also becomes evident through the teachers' descriptions of the gained insight into students' prerequisites, and descriptions of ideas about facilitating teaching. It becomes clear that they do not perceive empirical argumentation as valid forms of proof, which supports the teachers own statement of becoming aware of what a valid proof is. This is an interesting finding in comparison to what Knuth (2002) reported about in-service teachers' knowledge of proof. In this study, the in-service teachers clearly discard empirical argumentation as proof, and thus have a seemingly correct perception of proof.

The findings of this study might suggest that the knowledge and beliefs of these practicing teachers have evolved or changed as a result of this analysis project, though it is not possible to make any such conclusions. It is beyond the scope of this research to evaluate and conclude what knowledge and beliefs were developed through this project. The aim of this research was to investigate how the teachers themself perceived the analysis as useful, as this indicates the utility of engaging teachers in such an analysis as a part of their professional developmental program.

This paper does not aim to suggest that teachers can learn all that they need to know about proof and proving through analyzing students' written argumentation. A single professional development activity alone cannot provide teachers with all they need to know, and there are obviously other aspects of mathematics teacher knowledge that is also needed to be able to successfully engage pupils in reasoning-and-proving. These results merely suggest that doing an analysis of students' written argumentation can be a meaningful and useful task for teachers to do, as a part of a larger professional development program aiming to prepare teachers.

Conclusion

The in-service teachers in this study found the analysis useful for acquiring insight into students' prerequisites for proving and conception of proof, and as a foundation for being able to facilitate instruction. The analysis was also perceived as useful as a vehicle to gain knowledge about various aspects of proof. These findings suggest that an analysis of students' written argumentation potentially could be a useful task for in-service teachers to engage in, as the results reveal that one could be benefitted in several ways from doing this task. This information might be useful for teacher educators aiming to prepare in-service teachers, and could perhaps give inspiration and ideas for tasks to include into professional development programs both for in-service teachers and pre-service teachers.

Engaging students in reasoning, argumentation, and proving is challenging and demands a lot from the teacher. Teacher educators need to provide opportunities for in-service teachers enrolled in a university program to develop knowledge and dispositions needed to facilitate such activities. Hopefully the findings in this study can inspire and support teacher educators aiming to prepare in-service teachers to handle this challenging task.

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