J. Math. Anal. Appl. 508 (2022) 125857

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Isometric factorization of vector measures and applications to spaces of integrable functions ☆

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A R T I C L E I N F O

Article history:
Received 4 September 2021
Available online 24 November 2021
Submitted by S. Hencl

Keywords:
Vector measure
Finite variation
Space of integrable functions
Integration operator
Davis-Figiel-Johnson-Pełczyński factorization

A B S T R A C T

Let X be a Banach space, Σ be a σ-algebra, and $m : \Sigma \to X$ be a (countably additive) vector measure. It is a well known consequence of the Davis-Figiel-Johnson-Pełczyński factorization procedure that there exist a reflexive Banach space Y, a vector measure $\tilde{m} : \Sigma \to Y$ and an injective operator $J : Y \to X$ such that m factors as $m = J \circ \tilde{m}$. We elaborate some theory of factoring vector measures and their integration operators with the help of the isometric version of the Davis-Figiel-Johnson-Pełczyński factorization procedure. Along this way, we sharpen a result of Okada and Ricker that if the integration operator on $L_1(m)$ is weakly compact, then $L_1(m)$ is equal, up to equivalence of norms, to some $L_1(\tilde{m})$ where Y is reflexive; here we prove that the above equality can be taken to be isometric.

1. Introduction

Let throughout (Ω, Σ) be a measurable space, X be a real Banach space, and $m : \Sigma \to X$ be a countably additive vector measure (not identically zero). Let us agree that m being a vector measure automatically means that m is countably additive and defined on some σ-algebra of subsets of some set.

The range of m, i.e., the set $\mathcal{R}(m) := \{m(A) : A \in \Sigma\}$ is relatively weakly compact by a classical result of Bartle, Dunford and Schwartz (see, e.g., [12, p. 14, Corollary 7]). So, the Davis-Figiel-Johnson-Pełczyński (DFJP) factorization method [10] applied to the closed absolute convex hull $\overline{\text{aco}}(\mathcal{R}(m))$ of $\mathcal{R}(m)$ ensures the existence of a reflexive Banach space Y and an injective operator $J : Y \to X$ such that $J(B_Y) \supseteq \mathcal{R}(m)$.

☆ The research of José Rodríguez was partially supported by *Agencia Estatal de Investigación* [MTM2017-86182-P, grant cofunded by ERDF, EU] and *Fundación Séneca* [20797/PI/18].

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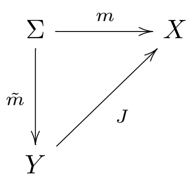
https://doi.org/10.1016/j.jmaa.2021.125857

Accordingly, m factors as $m = J \circ \tilde{m}$ for some map $\tilde{m} : \Sigma \to Y$ which turns out to be a vector measure as well (cf., [27, Theorem 2.1(i)]). In commutative diagram form:



Note that the vector measure $\tilde{m}$ need not have finite variation although m has finite variation. Indeed, if m does not have a Bochner derivative with respect to $|m|$, then neither does $\tilde{m}$ (since J is an operator) and so $\tilde{m}$ does not have finite variation, because Y has the Radon-Nikodým property and $\tilde{m}$ is $|m|$-continuous (by the injectivity of J). However, if m has finite variation and a Bochner derivative with respect to $|m|$, then m factors via a vector measure of finite variation taking values in a separable reflexive Banach space. This result is implicit in the proof of [28, Theorem 5.2] and we include it as Theorem 3.8 for the reader's convenience.

The previous way of factoring m is equivalent to applying the DFJP method to the (weakly compact) integration operator on the Banach lattice $L_\infty(m)$, i.e.,

$$I_m^{(\infty)} : L_\infty(m) \to X, \quad I_m^{(\infty)}(f) := \int_\Omega f \, dm,$$

because one has $\operatorname{aco}(\mathcal{R}(m)) \subseteq I_m^{(\infty)}(B_{L_\infty(m)}) \subseteq 2\overline{\operatorname{aco}}(\mathcal{R}(m))$ (see, e.g., [12, p. 263, Lemma 3(c)]). But there is still another approach which is based on factoring the integration operator on the (larger) Banach lattice $L_1(m)$ of all real-valued m-integrable functions defined on Ω, i.e.,

$$I_m : L_1(m) \to X, \quad I_m(f) := \int_\Omega f \, dm.$$

Note, however, that I_m need not be weakly compact and so in this case the DFJP method gives a factorization through a non-reflexive space. The DFJP factorization was already applied to I_m in [23] and [27]. Okada and Ricker showed that if I_m is weakly compact, then there exist a reflexive Banach space Y, a vector measure $\tilde{m} : \Sigma \to Y$ and an injective operator $J : Y \to X$ such that $m = J \circ \tilde{m}$ and $L_1(m) = L_1(\tilde{m})$ with *equivalent* norms (see [23, Proposition 2.1]).

In this paper we study the factorization of vector measures and their integration operators with the help of the *isometric* version of the DFJP procedure developed by Lima, Nygaard and Oja [19] (DFJP-LNO for short).

The paper is organized as follows. In Section 2 we include some preliminaries on spaces of integrable functions with respect to a vector measure and their integration operators, as well as on the DFJP-LNO method.

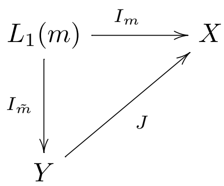
In Section 3 we obtain some results on factorization of integration operators that are a bit more general than applying the DFJP factorization procedure directly.

In Section 4 we present our main results. Theorem 4.1 collects some benefits of applying the DFJP-LNO factorization to $I_m^{(\infty)}$. Thus, one gets a reflexive Banach space Y, a vector measure $\tilde{m} : \Sigma \to Y$ with $\|m\|(\Omega) = \|\tilde{m}\|(\Omega)$ and an injective norm-one operator $J : Y \to X$ such that $I_m^{(\infty)} = J \circ I_{\tilde{m}}^{(\infty)}$. Moreover, the special features of the DFJP-LNO factorization also provide the following interpolation type inequality:

$$\|I_{\tilde{m}}^{(\infty)}(f)\|^2 \leq C \|m\|(\Omega) \|f\|_{L_\infty(m)} \|I_m^{(\infty)}(f)\| \quad \text{for all } f \in L_\infty(m), \tag{1.1}$$

where $C > 0$ is a universal constant. As a consequence, $I_{\tilde{m}}^{(\infty)}$ factors through the Lorentz space $L_{2,1}(\|m\|)$ associated to the semivariation of m (Proposition 4.3).

Theorem 4.5 gathers some consequences of the DFJP-LNO method when applied to I_m. In this case, one gets a factorization as follows:



where Y is a (not necessarily reflexive) Banach space, $\tilde{m} : \Sigma \to Y$ is a vector measure and J is an injective norm-one operator. Now, the equality

$$L_1(m) = L_1(\tilde{m})$$

holds with *equal* norms. Moreover, an inequality similar to (1.1) is the key to prove that $\tilde{m}$ has finite variation (resp., finite variation and a Bochner derivative with respect to it) whenever m does. As a particular case, we get the isometric version of the aforementioned result of Okada and Ricker (Corollary 4.7).

2. Preliminaries

By an *operator* we mean a continuous linear map between Banach spaces. The topological dual of a Banach space Z is denoted by Z^*. We write B_Z to denote the closed unit ball of Z, i.e., $B_Z = \{z \in Z : \|z\| \leq 1\}$. The absolute convex hull (resp., closed absolute convex hull) of a set $S \subseteq Z$ is denoted by $\mathrm{aco}(S)$ (resp., $\overline{\mathrm{aco}}(S)$).

Our source for basic information on vector measures is [12, Chapter I]. The symbol $|m|$ stands for the *variation* of m, while its *semivariation* is denoted by $\|m\|$. We write x^*m to denote the composition of $x^* \in X^*$ and m. A set $A \in \Sigma$ is said to be *m-null* if $\|m\|(A) = 0$ or, equivalently, $m(B) = 0$ for every $B \in \Sigma$ with $B \subseteq A$. The family of all m-null sets is denoted by $\mathcal{N}(m)$. A *control measure* of m is a non-negative finite measure μ on Σ such that m is μ-continuous, i.e., $\mathcal{N}(\mu) \subseteq \mathcal{N}(m)$; if μ is of the form $|x^*m|$ for some $x^* \in X^*$, then it is called a *Rybakov control measure*. Such control measures exist for any vector measure (see, e.g., [12, p. 268, Theorem 2]).

2.1. L_1-spaces of vector measures and integration operators

A suitable reference for basic information on L_1-spaces of vector measures is [24, Chapter 3]. A Σ-measurable function $f : \Omega \to \mathbb{R}$ is called *weakly m-integrable* if $\int_\Omega |f|\, d|x^*m| < \infty$ for every $x^* \in X^*$. In this case, for each $A \in \Sigma$ there is $\int_A f\, dm \in X^{**}$ such that

$$\Big(\int_A f\, dm\Big)(x^*) = \int_A f\, d(x^*m) \quad \text{for all } x^* \in X^*.$$

By identifying functions which coincide m-a.e., the set $L_1^w(m)$ of all weakly m-integrable functions forms a Banach lattice with the m-a.e. order and the norm

$$\|f\|_{L_1^w(m)} := \sup_{x^* \in B_{X^*}} \int_\Omega |f|\, d|x^*m|.$$

Given any Rybakov control measure μ of m, the space $L_1^w(m)$ *embeds continuously* into $L_1(\mu)$, i.e., the identity map $L_1^w(m) \to L_1(\mu)$ is an injective operator. From this embedding one gets the following well known property:

Fact 2.1. Let (f_n) be a sequence in $L_1^w(m)$ which converges in norm to some $f \in L_1^w(m)$. Then there is a subsequence (f_{n_k}) which converges to f m-a.e.

A Σ-measurable function $f : \Omega \to \mathbb{R}$ is said to be m-integrable if it is weakly m-integrable and $\int_A f\, dm \in X$ for all $A \in \Sigma$. The closed sublattice of $L_1^w(m)$ consisting of all m-integrable functions is denoted by $L_1(m)$. The Banach lattice $L_1(m)$ is order continuous and has a weak order unit (the function χ_Ω). The following result of Curbera (see [8, Theorem 8]) makes the class of $L_1(m)$-spaces extremely interesting: *if E is an order continuous Banach lattice with a weak order unit, then there exists an E-valued positive vector measure m such that $L_1(m)$ and E are lattice isometric.* (A vector measure taking values in a Banach lattice E is said to be *positive* if its range is contained in the positive cone of E.)

We write $\operatorname{sim}\Sigma$ to denote the set of all *simple functions* from Ω to $\mathbb{R}$. Just as for scalar L_1-spaces, $\operatorname{sim}\Sigma$ is a norm-dense linear subspace of $L_1(m)$. Note that $\int_\Omega \chi_A\, dm = m(A)$ and $\|\chi_A\|_{L_1(m)} = \|m\|(A)$ for all $A \in \Sigma$.

Any m-essentially bounded Σ-measurable function $f : \Omega \to \mathbb{R}$ is m-integrable. By identifying functions which coincide m-a.e., the set $L_\infty(m)$ of all m-essentially bounded Σ-measurable functions is a Banach lattice with the m-a.e. order and the m-essential sup-norm. Of course, $L_\infty(m)$ is equal to the usual space $L_\infty(\mu)$ for any Rybakov control measure μ of m (because $\mathcal{N}(\mu) = \mathcal{N}(m)$ in that case). It is known (see, e.g., [24, Proposition 3.31]) that:

(i) if $g \in L_\infty(m)$ and $f \in L_1(m)$, then $fg \in L_1(m)$ and
$$\|fg\|_{L_1(m)} \leq \|f\|_{L_1(m)}\|g\|_{L_\infty(m)};$$
(ii) the identity map $\alpha_\infty : L_\infty(m) \to L_1(m)$ is an (injective) weakly compact operator with $\|\alpha_\infty\| = \|m\|(\Omega)$.

The following formula for the norm on $L_1(m)$ will also be useful (see, e.g., [24, Lemma 3.11]):

Fact 2.2. For every $f \in L_1(m)$ we have
$$\|f\|_{L_1(m)} = \sup_{g \in B_{L_\infty(m)}} \Big\| \int_\Omega fg\, dm \Big\| = \sup_{g \in B_{L_\infty(m)} \cap \operatorname{sim}\Sigma} \Big\| \int_\Omega fg\, dm \Big\|.$$

A fundamental tool in the study of the space $L_1(m)$ is the *integration operator* $I_m : L^1(m) \to X$, which is the canonical map defined by
$$I_m(f) := \int_\Omega f\, dm \quad \text{for all } f \in L_1(m).$$
Note that $\|I_m\| = 1$ (see, e.g., [24, p. 152]). We may of course also look at the integration operator defined on $L_\infty(m)$, i.e., the composition
$$I_m^{(\infty)} := I_m \circ \alpha_\infty : L_\infty(m) \to X, \qquad I_m^{(\infty)}(f) = \int_\Omega f\, dm \quad \text{for all } f \in L_\infty(m).$$
The operator $I_m^{(\infty)}$ is thus weakly compact and $\|I_m^{(\infty)}\| = \|m\|(\Omega)$.

2.2. The isometric version of the Davis-Figiel-Johnson-Pełczyński procedure

Let us quickly recall the main construction and results from [10] together with the extra information obtained in [19].

Let $K \subseteq B_X$ be a closed absolutely convex set and fix $a \in (1, \infty)$. For each $n \in \mathbb{N}$, define the bounded absolutely convex set

$$K_n := a^n K + a^{-n} B_X$$

and denote by $\|\cdot\|_n$ the Minkowski functional defined by K_n, i.e.,

$$\|x\|_n := \inf\{t > 0 : x \in tK_n\} \quad \text{for all } x \in X.$$

Note that each $\|\cdot\|_n$ is an equivalent norm on X. The following statements now hold:

(i) $X_K := \{x \in X : \sum_{n=1}^{\infty} \|x\|_n^2 < \infty\}$ is a Banach space equipped with the norm

$$\|x\|_K := \sqrt{\sum_{n=1}^{\infty} \|x\|_n^2}.$$

(ii) The identity map $J_K : X_K \to X$ is an operator with $\|J_K\| \leq \frac{1}{f(a)}$ and $K \subseteq f(a) B_{X_K}$, where

$$f(a) := \sqrt{\sum_{n=1}^{\infty} \left(\frac{a^n}{a^{2n}+1}\right)^2}.$$

(iv) J_K^{**} is injective (equivalently, $J_K^*(X^*)$ is norm-dense in X_K^*).

(v) For each $x \in K$ we have

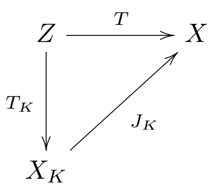
$$\|x\|_K^2 \leq \left(\frac{1}{4} + \frac{1}{2\ln a}\right)\|x\|. \qquad (K^2)$$

(vi) J_K is a norm-to-norm homeomorphism when restricted to K.

(vii) J_K is a weak-to-weak homeomorphism when restricted to B_{X_K}.

(viii) X_K is reflexive if and only if K is weakly compact.

Given a Banach space Z and a (non-zero) operator $T : Z \to X$, the previous procedure applied to $K := \frac{1}{\|T\|}\overline{T(B_Z)}$ gives a factorization



(2.1)

where T_K is an operator with $\|T_K\| \leq f(a)\|T\|$. The following statements hold:

(ix) X_K is reflexive if and only if T is weakly compact if and only if T_K is weakly compact if and only if J_K is weakly compact.

(x) T is compact if and only if T_K is compact if and only if J_K is compact. In this case, X_K is separable.

Let $\bar{a}$ be the unique element of $(1, \infty)$ such that $f(\bar{a}) = 1$. When the previous method is performed with $a = \bar{a}$, we have $\|T_K\| = \|T\|$ and $\|J_K\| = 1$, and (2.1) is called the *DFJP-LNO factorization* of T.

2.3. An observation on strong measurability

The next lemma will be needed in the proof of Theorem 4.5.

Lemma 2.3. *Let $K \subseteq B_X$ be a closed absolutely convex set and $a \in (1, \infty)$. Let $G : \Omega \to X$ be a function with $G(\Omega) \subseteq K$, $F : \Omega \to X_K$ be the function such that $J_K \circ F = G$, and μ be a non-negative finite measure on Σ. If G is strongly μ-measurable, then so is F.*

Proof. Since G is strongly μ-measurable, there is $E \in \Sigma$ with $\mu(\Omega \setminus E) = 0$ such that $G(E)$ is separable. Since $G(E) \subseteq K$, we have $F(E) \subseteq f(a)B_{X_K}$ (by (ii)). The separability of $G(E)$ and (vi) imply that $F(E)$ is separable. On the other hand, $y^* \circ F$ is μ-measurable for every $y^* \in J_K^*(X^*)$ (i.e., G is scalarly μ-measurable) and so the norm-density of $J_K^*(X^*)$ in X_K^* (property (iv)) implies that F is scalarly μ-measurable. An appeal to Pettis' measurability theorem (see, e.g., [12, p. 42, Theorem 2]) ensures that F is strongly μ-measurable. $\square$

Remark 2.4. The strong μ-measurability of a Banach space-valued function h defined on a finite measure space (Ω, Σ, μ) is characterized as follows: for each $\varepsilon > 0$ and each $A \in \Sigma$ with $\mu(A) > 0$ there is $B \subseteq A$, $B \in \Sigma$ with $\mu(B) > 0$, such that $\|h(\omega) - h(\omega')\| \leq \varepsilon$ for all $\omega, \omega' \in B$ (this is folklore, see [5, Proposition 2.2] for a sketch of proof). This characterization and the inequality

$$\|x - x'\|_K^2 \leq \left(\frac{1}{2} + \frac{1}{\ln a}\right)\|x - x'\| \quad \text{for all } x, x' \in K$$

(which follows from (K^2) and the absolute convexity of K) can be combined to give another proof of Lemma 2.3.

3. General factorization results

The following lemma is surely folklore to experts in vector measure theory, but as its proof requires some tools we provide a proof for the reader's convenience.

Lemma 3.1. *Let Y be a Banach space, $\Gamma \subseteq Y^*$ be a norm-dense set, and $\nu : \Sigma \to Y$ be a map such that $y^*\nu$ is countably additive for every $y^* \in \Gamma$. Then ν is a countably additive vector measure.*

Proof. By the Orlicz-Pettis theorem (see, e.g., [12, p. 22, Corollary 4]), in order to prove that ν is countably additive it suffices to show that $y^*\nu$ is countably additive for every $y^* \in Y^*$. Since Γ separates the points of Y, ν is finitely additive and has bounded range, by the Dieudonné-Grothendieck theorem (see, e.g., [12, p. 16, Corollary 3]). Fix $y^* \in Y^*$. Let (B_n) be a disjoint sequence in Σ and fix $\varepsilon > 0$. We can choose $y_0^* \in \Gamma$ such that $|y^*(\nu(A)) - y_0^*(\nu(A))| \leq \varepsilon$ for all $A \in \Sigma$. Since $y_0^*\nu$ is countably additive, there is $n_0 \in \mathbb{N}$ such that $|y_0^*(\nu(\bigcup_{n \geq n_1} B_n))| \leq \varepsilon$ for every $n_1 \geq n_0$. It follows that $|y^*(\nu(\bigcup_{n \geq n_1} B_n))| \leq 2\varepsilon$ for every $n_1 \geq n_0$. This shows that $y^*\nu$ is countably additive. $\square$

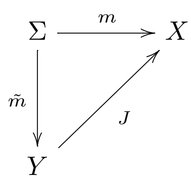
Remark 3.2.

(i) In Lemma 3.1 we do not really need that Γ is norm-dense. It suffices that Γ is what one could call a *Rainwater set*, i.e., a set with the following property: if (y_n) is a bounded sequence in Y and there is $y \in Y$ with $y^*(y_n) \to y^*(y)$ for every $y^* \in \Gamma$, then $y_n \to y$ weakly. See the proof of [15, Proposition 2.9]. The most general known Rainwater sets are (I)-generating sets, in particular James boundaries like the extreme points of B_{Y^*}; see [22] for the fact that (I)-generating sets are Rainwater (this was proved independently by Kalenda, private communication) and [17, Theorem 2.3] for the deep result that James boundaries are (I)-generating.

(ii) If Y contains no isomorphic copy of ℓ_∞, then the assertion of Lemma 3.1 holds for any set $\Gamma \subseteq Y^*$ which separates the points of Y, by a result of Diestel and Faires [11] (cf., [12, p. 23, Corollary 7]).

The following lemma is essentially known (see, e.g., [24, Lemma 3.27]). We add an estimate for the norm of the inclusion operator.

Lemma 3.3. *Suppose that m factors as*



where Y is a Banach space, $\tilde{m}$ is a countably additive vector measure and J is an injective operator. Then:

(i) $\mathcal{N}(m) = \mathcal{N}(\tilde{m})$.

(ii) $L_1(\tilde{m})$ *embeds continuously into* $L_1(m)$ *with norm* $\le \|J\|$, *i.e., the identity map* $L_1(\tilde{m}) \to L_1(m)$ *is an injective operator with norm* $\le \|J\|$.

(iii) $I_m^{(\infty)} = J \circ I_{\tilde{m}}^{(\infty)}$.

Proof. (i) The equality $\mathcal{N}(m) = \mathcal{N}(\tilde{m})$ follows at once from the injectivity of J.

(ii) If h is any $\tilde{m}$-integrable function, then h is m-integrable and the equality $J(\int_\Omega h\, d\tilde{m}) = \int_\Omega h\, dm$ holds (see, e.g., [24, Lemma 3.27]). Therefore, for each $f \in L_1(\tilde{m})$, we can apply Fact 2.2 twice to get

$$\|f\|_{L_1(m)} = \sup_{g \in B_{L_\infty(m)}} \Big\| \int_\Omega f g\, dm \Big\| = \sup_{g \in B_{L_\infty(m)}} \Big\| J\Big(\int_\Omega f g\, d\tilde{m} \Big) \Big\| \le \|J\| \sup_{g \in B_{L_\infty(m)}} \Big\| \int_\Omega f g\, d\tilde{m} \Big\| = \|J\| \|f\|_{L_1(\tilde{m})}.$$

(iii) follows from the density of $\operatorname{sim} \Sigma$ in $L_\infty(m)$ and the equality $m = J \circ \tilde{m}$. $\square$

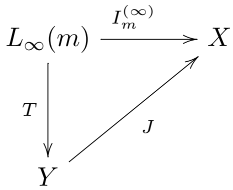
Remark 3.4. In the setting of the previous lemma:

$$|m|(A) \le \|J\| |\tilde{m}|(A) \quad \text{for all } A \in \Sigma.$$

In particular, m has finite variation whenever $\tilde{m}$ does. The converse fails in general, as we pointed out in the introduction.

Combining Lemmata 3.1 and 3.3 leads to a factorization result for $I_m^{(\infty)}$ that applies to the DFJP factorization procedure:

Corollary 3.5. *Suppose that* $I_m^{(\infty)}$ *factors as*



where Y *is a Banach space,* T *and* J *are operators and* $J^*(X^*)$ *is norm-dense in* Y^**. Define* $\tilde{m} : \Sigma \to Y$ *by* $\tilde{m}(A) := T(\chi_A)$ *for all* $A \in \Sigma$*. Then:*

(i) $\tilde{m}$ *is a countably additive vector measure and* $m = J \circ \tilde{m}$*.*
(ii) $\mathcal{N}(m) = \mathcal{N}(\tilde{m})$*.*
(iii) $L_1(\tilde{m})$ *embeds continuously into* $L_1(m)$ *with norm* $\leq \|J\|$*.*
(iv) $T = I_{\tilde{m}}^{(\infty)}$*.*

Proof. (i) follows from Lemma 3.1 applied to $\Gamma := J^*(X^*)$ and the countable additivity of m. (ii) and (iii) follow from Lemma 3.3, since J is injective. Finally, (iv) is a consequence of the continuity of both T and $I_{\tilde{m}}^{(\infty)}$, the density of $\operatorname{sim}\Sigma$ in $L_\infty(m)$, and the fact that $\int_\Omega h \, d\tilde{m} = T(h)$ for every $h \in \operatorname{sim}\Sigma$. $\square$

An isometric version of our next result was proved in [20, Lemma 6]. We include a similar proof for the sake of completeness.

Lemma 3.6. *Let* Y *be a Banach space and* $\tilde{m} : \Sigma \to Y$ *be a countably additive vector measure with* $\mathcal{N}(m) = \mathcal{N}(\tilde{m})$*. Suppose that there is a constant* $D > 0$ *such that* $\|f\|_{L_1(m)} \leq D\|f\|_{L_1(\tilde{m})}$ *for every* $f \in \operatorname{sim}\Sigma$*. Then* $L_1(\tilde{m})$ *embeds continuously into* $L_1(m)$ *with norm* $\leq D$*.*

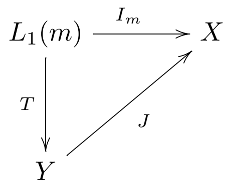
Proof. Consider $\operatorname{sim}\Sigma$ as a linear subspace of $L_1(\tilde{m})$. By the assumptions, the identity map $i : \operatorname{sim}\Sigma \to L_1(m)$ is well-defined, linear and continuous, with norm $\|i\| \leq D$. Since $\operatorname{sim}\Sigma$ is dense in $L_1(m)$, we can extend i to an operator

$$j : L_1(\tilde{m}) \to L_1(m)$$

with $\|j\| = \|i\| \leq D$. We claim that $j(f) = f$ for every $f \in L_1(\tilde{m})$. Indeed, choose a sequence (f_n) in $\operatorname{sim}\Sigma$ such that $\|f_n - f\|_{L_1(\tilde{m})} \to 0$. By passing to a subsequence, we can assume that $f_n \to f$ $\tilde{m}$-a.e. (Fact 2.1). Since j is an operator and $j(f_n) = f_n$ for all $n \in \mathbb{N}$, we have $\|f_n - j(f)\|_{L_1(m)} \to 0$. Another appeal to Fact 2.1 allows us to extract a further subsequence (f_{n_k}) such that $f_{n_k} \to j(f)$ m-a.e. It follows that $j(f) = f$. $\square$

The following result first appeared as [23, Lemma 2.2] (with a different and simpler proof in [27, Lemma 3.1]), but under the extra assumption that Y contains no isomorphic copy of ℓ_∞ (in order to prove the first part of (iii) below). In addition to removing this unnecessary condition, we also provide explicit estimates for the norm of the identity as a Banach lattice isomorphism between $L_1(m)$ and $L_1(\tilde{m})$.

Theorem 3.7. *Suppose that I_m factors as*



where Y is a Banach space, T and J are operators and J is injective. Define $\tilde{m} : \Sigma \to Y$ by $\tilde{m}(A) := T(\chi_A)$ for all $A \in \Sigma$. Then:

(i) *$\tilde{m}$ is a countably additive vector measure and $m = J \circ \tilde{m}$.*
(ii) *$\mathcal{N}(m) = \mathcal{N}(\tilde{m})$.*
(iii) *$L_1(\tilde{m}) = L_1(m)$ with equivalent norms. In fact, we have*

$$\|T\|^{-1}\|f\|_{L_1(\tilde{m})} \leq \|f\|_{L_1(m)} \leq \|J\|\|f\|_{L_1(\tilde{m})} \quad \text{for every } f \in L_1(m).$$

(iv) *$T = I_{\tilde{m}}$.*

Proof. (i) Clearly, $\tilde{m}$ is finitely additive. Now, its countable additivity follows from that of m and the inequality $\|\tilde{m}(A)\| \leq \|T\|\|\chi_A\|_{L_1(m)} = \|T\|\|m\|(A)$, for all $A \in \Sigma$. The equality $m = J \circ \tilde{m}$ is obvious.

Lemma 3.3 implies (ii) and the fact that any $f \in L_1(\tilde{m})$ belongs to $L_1(m)$, with $\|f\|_{L_1(m)} \leq \|J\|\|f\|_{L_1(\tilde{m})}$.

On the other hand, observe that for any $h \in \operatorname{sim}\Sigma$ we have $\int_\Omega h\, d\tilde{m} = T(h)$. Now, given any $f \in \operatorname{sim}\Sigma$, an appeal to Fact 2.2 yields

$$\begin{aligned} \|f\|_{L_1(\tilde{m})} &= \sup_{g \in B_{L_\infty(m)} \cap \operatorname{sim}\Sigma} \left\| \int_\Omega fg\, d\tilde{m} \right\| = \sup_{g \in B_{L_\infty(m)} \cap \operatorname{sim}\Sigma} \|T(fg)\| \\ &\leq \|T\| \sup_{g \in B_{L_\infty(m)} \cap \operatorname{sim}\Sigma} \|fg\|_{L_1(m)} \leq \|T\|\|f\|_{L_1(m)}. \end{aligned}$$

It follows that for every $f \in \operatorname{sim}\Sigma$ we have

$$\|T\|^{-1}\|f\|_{L_1(\tilde{m})} \leq \|f\|_{L_1(m)} \leq \|J\|\|f\|_{L_1(\tilde{m})}. \tag{3.1}$$

We can now apply Lemma 3.6 twice to conclude that $L_1(m) = L_1(\tilde{m})$ and that (3.1) holds for every $f \in L_1(m)$.

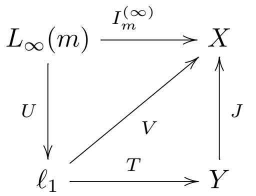
Finally, (iv) follows from the continuity of both T and $I_{\tilde{m}}$, the density of $\operatorname{sim}\Sigma$ in $L_1(\tilde{m})$, and the fact that $\int_\Omega h\, d\tilde{m} = T(h)$ for every $h \in \operatorname{sim}\Sigma$. □

3.1. An observation on vector measures with a Bochner derivative with respect to its variation

It is known that m has finite variation and a Bochner derivative with respect to $|m|$ if and only if $I_m^{(\infty)}$ is nuclear (see, e.g., [12, p. 173, Theorem 4]). Recall that an operator T from a Banach space Z to X is said to be *nuclear* if there exist sequences (z_n^*) in Z^* and (x_n) in X with $\sum_{n \in \mathbb{N}} \|z_n^*\|\|x_n\| < \infty$ such that $T(z) = \sum_{n \in \mathbb{N}} z_n^*(z) x_n$ for all $z \in Z$.

Theorem 3.8. *If m has finite variation and a Bochner derivative with respect to $|m|$, then there exist a separable reflexive Banach space Y, a countably additive vector measure $\tilde{m} : \Sigma \to Y$ having finite variation and an injective compact operator $J : Y \to X$ such that $m = J \circ \tilde{m}$.*

Proof. Since $I_m^{(\infty)}$ is a nuclear operator, it can be factored as $I_m^{(\infty)} = V \circ U$, where $U : L_\infty(m) \to \ell_1$ is a nuclear operator and $V : \ell_1 \to X$ is a compact operator (see, e.g., [13, Proposition 5.23]). Now, we can consider the DFJP factorization of V to obtain the commutative diagram

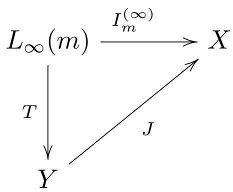


where Y is a separable reflexive Banach space, T and J are compact operators and $J^*(X^*)$ is norm-dense in Y^*. By Corollary 3.5, the map $\tilde{m} : \Sigma \to Y$ defined by $\tilde{m}(A) := (T \circ U)(\chi_A)$ for all $A \in \Sigma$ is a countably additive vector measure such that $\mathcal{N}(m) = \mathcal{N}(\tilde{m})$, $m = J \circ \tilde{m}$ and $I_{\tilde{m}}^{(\infty)} = T \circ U$. Since U is nuclear, the same holds for $I_{\tilde{m}}^{(\infty)}$, and so $\tilde{m}$ has finite variation. $\square$

4. DFJP-LNO factorization of integration operators

The following theorem collects some consequences of the DFJP-LNO factorization when applied to $I_m^{(\infty)}$.

Theorem 4.1. *Let us consider the DFJP-LNO factorization of $I_m^{(\infty)}$, as follows*



Let $\tilde{m} : \Sigma \to Y$ be the countably additive vector measure defined by $\tilde{m}(A) := T(\chi_A)$ for all $A \in \Sigma$ (see Corollary 3.5). Then:

(i) *Y is reflexive.*
(ii) *$L_1(\tilde{m})$ embeds continuously into $L_1(m)$ with norm ≤ 1.*
(iii) *$\|\tilde{m}\|(\Omega) = \|m\|(\Omega)$.*
(iv) *There is a universal constant $C > 0$ such that*

$$\|I_{\tilde{m}}^{(\infty)}(f)\|^2 \leq C\|m\|(\Omega)\|f\|_{L_\infty(m)}\|I_m^{(\infty)}(f)\| \quad \text{for every } f \in L_\infty(m). \tag{4.1}$$

In particular, $\|\tilde{m}(A)\|^2 \leq C\|m\|(\Omega)\|m(A)\|$ for all $A \in \Sigma$.

Proof. The DFJP-LNO factorization is done by using the set

$$K := \frac{1}{\|m\|(\Omega)} \overline{I_m^{(\infty)}(B_{L_\infty(m)})},$$

so that $T = T_K$, $J = J_K$, $\|T\| = \|I_m^{(\infty)}\| = \|m\|(\Omega)$ and $\|J\| = 1$. Since K is weakly compact, Y is reflexive. Statements (ii) and (iii) follow from Corollary 3.5, bearing in mind that $\|I_{\tilde{m}}^{(\infty)}\| = \|\tilde{m}\|(\Omega)$. Finally, (iv) follows from inequality (K^2), which implies $\|I_{\tilde{m}}^{(\infty)}(g)\|^2 \leq C\|m\|(\Omega)\|I_m^{(\infty)}(g)\|$ for every $g \in B_{L_\infty(m)}$. $\square$

Remark 4.2. In the setting of Theorem 4.1:

(i) $|m|(A) \leq |\tilde{m}|(A)$ for all $A \in \Sigma$ (because $\|J\| = 1$).

(ii) $\mathcal{R}(m)$ is relatively norm-compact if and only if $I_m^{(\infty)}$ is compact. In this case, $\mathcal{R}(\tilde{m})$ is relatively norm-compact (because the restriction $J|_K$ is a norm-to-norm homeomorphism and $\mathcal{R}(m) \subseteq \|m\|(\Omega)K$) and Y is separable.

(iii) $L_1(\tilde{m})$ is weakly sequentially complete because Y contains no isomorphic copy of c_0, see [8, Theorem 3] (cf., [3,25]). In general, $L_1(m)$ is not weakly sequentially complete, so the equality $L_1(\tilde{m}) = L_1(m)$ can fail.

An operator T from a Banach space Z to X is said to be $(2,1)$-*summing* if $\sum_{n\in\mathbb{N}} \|T(z_n)\|^2 < \infty$ for every unconditionally convergent series $\sum_{n\in\mathbb{N}} z_n$ in Z. A glance at inequality (4.1) reveals that $I_{\tilde{m}}^{(\infty)}$ is $(2,1)$-summing whenever $I_m^{(\infty)}$ is 1-summing, which in turn is equivalent to saying that m has finite variation (see, e.g., [12, Corollary 4, p. 164]).

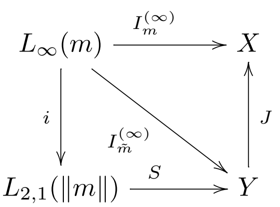
On the other hand, a result of Pisier [26] (cf., [13, Theorem 10.9]) characterizes $(2,1)$-summing operators from a $C(K)$ space (like $L_\infty(m)$) to a Banach space as those operators which factor through a Lorentz space $L_{2,1}(\mu)$ for some regular Borel probability on K, via the canonical map from $C(K)$ to $L_{2,1}(\mu)$; the hardest part of this result consists in obtaining an inequality similar to (4.1).

Inequality (4.1) and the easier part of Pisier's argument can be combined to obtain that $I_{\tilde{m}}^{(\infty)}$ can be extended to $L_{2,1}(|m|)$ whenever m has finite variation. In fact, $I_{\tilde{m}}^{(\infty)}$ can always be extended to the Lorentz type space $L_{2,1}(\|m\|)$ associated to the semivariation of m, no matter whether m has or not finite variation. We include this result in Proposition 4.3 below. Its proof is omitted since it can be done just by imitating some parts of the proof of [13, Theorem 10.9]. Let us recall that $L_{2,1}(\|m\|)$ is the set of all (m-a.e. equivalence classes of) Σ-measurable functions $f : \Omega \to \mathbb{R}$ for which

$$\|f\|_{L_{2,1}(\|m\|)} := 2 \int_0^\infty \sqrt{\|m\|_f(t)}\, dt < \infty,$$

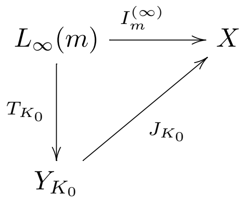
where $\|m\|_f$ is the distribution function of f with respect to $\|m\|$, defined by $\|m\|_f(t) := \|m\|(\{\omega \in \Omega : |f(\omega)| > t\})$ for all $t > 0$. The linear space $L_{2,1}(\|m\|)$ is a Banach lattice when equipped with a certain norm which is equivalent to the quasi-norm $\|\cdot\|_{L_{2,1}(\|m\|)}$. In general, $L_\infty(m) \subseteq L_{2,1}(\|m\|) \subseteq L_1(m)$ with continuous inclusions. Note that if m has finite variation, then the Lorentz space $L_{2,1}(|m|)$ embeds continuously into $L_{2,1}(\|m\|)$. The Lorentz spaces associated to the semivariation of a vector measure were introduced in [16] and studied further in [4].

Proposition 4.3. *In the setting of Theorem 4.1, $I_{\tilde{m}}^{(\infty)}$ factors as*



where i is the identity operator and S is an operator. In particular, if m has finite variation, then $I_{\tilde{m}}^{(\infty)}$ can be extended to the Lorentz space $L_{2,1}(|m|)$.

Remark 4.4. Suppose that X is a Banach lattice. If we apply the DFJP-LNO method to the closed convex solid hull K_0 of $\frac{1}{\|m\|(\Omega)} I_m^{(\infty)}(B_{L_\infty(m)})$, then we get a factorization as



where Y_{K_0} is a Banach lattice and both J_{K_0} and $J_{K_0}^*$ are interval preserving lattice homomorphisms (imitate the proof of [1, Theorem 5.41]). Moreover:

(i) The statements of Theorem 4.1, Remark 4.2(i)-(ii) and Proposition 4.3 also hold for this factorization, with the exception that Y_{K_0} need not be reflexive.

(ii) Y_{K_0} is reflexive whenever X has the property that the solid hull of any relatively weakly compact set is relatively weakly compact. This happens if either X contains no isomorphic copy of c_0 (see, e.g., [1, Theorems 4.39 and 4.60]) or X is order continuous and atomic, see [7, Theorem 2.4].

(iii) $\tilde{m}$ is positive and Y_{K_0} is reflexive whenever m is positive. Indeed, in this case an easy computation shows that $[-m(\Omega), m(\Omega)]$ is the solid hull of $\mathcal{R}(m)$ and that

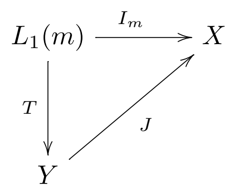
$$K_0 = \frac{1}{\|m\|(\Omega)}[-m(\Omega), m(\Omega)].$$

Now, from [14, Theorem 2.4] it follows that K_0 is L-weakly compact and so it is weakly compact (see, e.g., [21, Proposition 3.6.5]).

(iv) In general, $Y_{K_0}^*$ contains no isomorphic copy of c_0, because $I_m^{(\infty)}$ is weakly compact (imitate the proof of [1, Theorem 5.43]).

We next apply the DFJP-LNO factorization procedure to I_m. The following theorem gathers some consequences of it.

Theorem 4.5. *Let us consider the DFJP-LNO factorization of I_m, as follows*



Let $\tilde{m} : \Sigma \to Y$ be the countably additive vector measure defined by $\tilde{m}(A) := T(\chi_A)$ for all $A \in \Sigma$ (see Theorem 3.7). Then:

(i) *$L_1(\tilde{m}) = L_1(m)$ with equal norms, i.e.,*

$$\|f\|_{L_1(m)} = \|f\|_{L_1(\tilde{m})} \quad \textit{for all } f \in L_1(m).$$

In particular, $\|m\|(A) = \|\tilde{m}\|(A)$ for all $A \in \Sigma$.

(ii) *There is a universal constant $C > 0$ such that*

$$\|I_{\tilde{m}}(f)\|^2 \leq C\|f\|_{L_1(m)}\|I_m(f)\| \quad \textit{for every } f \in L_1(m). \tag{4.2}$$

In particular, $\|\tilde{m}(A)\|^2 \leq C\|m\|(A)\|m(A)\|$ *for all* $A \in \Sigma$.

(iii) $|m|(A) \leq |\tilde{m}|(A) \leq \sqrt{C}|m|(A)$ *for all* $A \in \Sigma$. *Therefore,* $\tilde{m}$ *has finite (resp.,* σ*-finite) variation whenever* m *does.*

(iv) *If* m *has finite variation and a Bochner derivative* G *with respect to* $|m|$*, then* $\tilde{m}$ *has a Bochner derivative* $\tilde{F}$ *with respect to* $|\tilde{m}|$ *and*

$$\int_\Omega \|\tilde{F}\|^2 \, d|\tilde{m}| \leq C \int_\Omega \|G\| \, d|m|.$$

Proof. The factorization is done by using the set $K := \overline{I_m(B_{L_1(m)})}$, so that $T = T_K$, $J = J_K$ and $\|T\| = \|J\| = 1$.

(i) follows from Theorem 3.7, while (ii) is consequence of inequality (K^2), which in this case reads as $\|I_{\tilde{m}}(g)\|^2 \leq C\|I_m(g)\|$ for every $g \in B_{L_1(m)}$, where we write $C := \frac{1}{4} + \frac{1}{2\ln \bar{a}}$ and $\bar{a}$ is as in Subsection 2.2.

(iii) Fix $A \in \Sigma$. The inequality $|m|(A) \leq |\tilde{m}|(A)$ follows at once from the fact that $\|J\| = 1$. On the other hand, the inequality $|\tilde{m}|(A) \leq \sqrt{C}|m|(A)$ is obvious if $|m|(A)$ is infinite, so we assume that $|m|(A) < \infty$. Now, given finitely many pairwise disjoint $A_1, \dots, A_n \in \Sigma$ with $A_i \subseteq A$, we have

$$\sum_{i=1}^n \frac{\|\tilde{m}(A_i)\|^2}{|m|(A_i)} \leq \sum_{i=1}^n \frac{\|\tilde{m}(A_i)\|^2}{\|m\|(A_i)} \overset{\text{(ii)}}{\leq} C\sum_{i=1}^n \|m(A_i)\| \leq C|m|(A) \tag{4.3}$$

(with the convention $\frac{0}{0} = 0$) and so the Cauchy-Schwarz inequality yields

$$\sum_{i=1}^n \|\tilde{m}(A_i)\| \leq \Big(\sum_{i=1}^n \frac{\|\tilde{m}(A_i)\|^2}{|m|(A_i)}\Big)^{1/2} \cdot \Big(\sum_{i=1}^n |m|(A_i)\Big)^{1/2} \overset{(4.3)}{\leq} \sqrt{C}|m|(A).$$

This shows that $|\tilde{m}|(A) \leq \sqrt{C}|m|(A)$.

(iv) Let $G : \Omega \to X$ be a Bochner derivative of m with respect to $|m|$. Then there is $E \in \Sigma$ with $|m|(\Omega \setminus E) = 0$ such that

$$G(E) \subseteq H := \overline{\Big\{\frac{m(A)}{|m|(A)} : A \in \Sigma, \, |m|(A) > 0\Big\}} \subseteq \overline{\text{aco}}\Big(\Big\{\frac{m(A)}{\|m\|(A)} : A \in \Sigma, \, \|m\|(A) > 0\Big\}\Big) \subseteq K$$

(see, e.g., [18, Lemma 2.3] or [6, Lemma 3.7]). We can assume without loss of generality that $E = \Omega$. Let $F : \Omega \to Y$ be the function satisfying $J \circ F = G$ (bear in mind that J is injective and that $G(\Omega) \subseteq K \subseteq J(Y)$). Then F is strongly $|m|$-measurable (by Lemma 2.3). Note that F is bounded (we have $F(\Omega) \subseteq K \subseteq B_Y$) and so F is Bochner integrable with respect to $|m|$. Since J is injective, we have $\int_A F \, d|m| = \tilde{m}(A)$ for all $A \in \Sigma$.

Note that $|m|$ and $|\tilde{m}|$ have the same null sets, hence F is strongly $|\tilde{m}|$-measurable as well. Since F is bounded, it is Bochner integrable with respect to $|\tilde{m}|$. On the other hand, let φ be the Radon-Nikodým derivative of $|m|$ with respect to $|\tilde{m}|$. Then $0 \leq \varphi \leq 1$ $|\tilde{m}|$-a.e. (because $|m|(A) \leq |\tilde{m}|(A)$ for all $A \in \Sigma$) and, therefore, the product $\tilde{F} := \varphi F : \Omega \to Y$ is Bochner integrable with respect to $|\tilde{m}|$, with integral $\int_A \tilde{F} \, d|\tilde{m}| = \int_A F \, d|m| = \tilde{m}(A)$ for all $A \in \Sigma$.

Finally, by (K^2) and the inclusion $G(\Omega) \subseteq K$, we have $\|F(\omega)\|^2 \leq C\|G(\omega)\|$ for every $\omega \in \Omega$, so that

$$\int_\Omega \|\tilde{F}\|^2 \, d|\tilde{m}| = \int_\Omega \varphi^2 \|F\|^2 \, d|\tilde{m}| = \int_\Omega \varphi \|F\|^2 \, d|m| \leq \int_\Omega \|F\|^2 \, d|m| \leq C \int_\Omega \|G\| \, d|m|,$$

as we wanted to prove. □

Remark 4.6. In the setting of Theorem 4.5, the following statements hold:

(i) If $\mathcal{R}(m)$ is relatively norm-compact, then so is $\mathcal{R}(\tilde{m})$ (because $J|_K$ is a norm-to-norm homeomorphism and $\mathcal{R}(m) \subseteq \|m\|(\Omega)K$).
(ii) If I_m is compact, then J and $I_{\tilde{m}}$ are compact as well.
(iii) If I_m is completely continuous, then so is $I_{\tilde{m}}$. This is also an immediate consequence of the fact that $J|_K$ is a norm-to-norm homeomorphism. It was proved in [27, Lemma 3.2(ii)] for the DFJP factorization, with a rather more complicated proof, under the additional assumption that Y contains no isomorphic copy of ℓ_∞.
(iv) If m has finite variation, then the 2-variation of $\tilde{m}$ with respect to $|m|$ is finite and less than or equal to $\sqrt{C|m|(\Omega)}$. This is a consequence of inequality (4.3).

We arrive at the isometric version of [23, Proposition 2.1] which we already mentioned in the introduction.

Corollary 4.7. *Suppose that I_m is weakly compact. Then there exist a reflexive Banach space Y, a countably additive vector measure $\tilde{m} : \Sigma \to Y$ and an injective operator $J : Y \to X$ such that $m = J \circ \tilde{m}$ and $L_1(m) = L_1(\tilde{m})$ with equal norms.*

Suppose that I_m is weakly compact. It is known that:

(i) If m has finite variation, then the composition of I_m and the continuous embedding of $L_1(|m|)$ into $L_1(m)$ (see, e.g., [24, Lemma 3.14]) is a weakly compact operator and so it is representable (see, e.g., [12, p. 75, Theorem 12]). Hence m admits a Bochner derivative with respect to $|m|$.
(ii) If m has σ-finite variation, then $\mathcal{R}(m)$ is relatively norm-compact, see [2, Corollary 3.11] (cf., [9, Claim 2]).

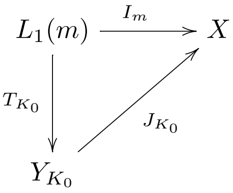
The previous statements can be improved as follows.

Corollary 4.8. *Suppose that I_m is weakly compact.*

(i) *If m has finite variation, then it admits a Bochner derivative with respect to any control measure.*
(ii) *If m has σ-finite variation, then it admits a strongly measurable Pettis integrable derivative with respect to any control measure.*

Proof. Let us consider a factorization of I_m as in Theorem 4.5. Since I_m is weakly compact, Y is reflexive. Observe that m and $\tilde{m}$ have the same control measures, because $\mathcal{N}(m) = \mathcal{N}(\tilde{m})$. Let μ be a control measure of both m and $\tilde{m}$. If m has finite (resp., σ-finite) variation, then the same holds for $\tilde{m}$. By the Radon-Nikodým property of Y, $\tilde{m}$ has a Bochner (resp., strongly measurable Pettis) derivative with respect to μ, say $F : \Omega \to Y$. The composition $J \circ F$ is then a Bochner (resp., strongly measurable Pettis) derivative of m with respect to μ. □

Remark 4.9. Suppose that X is a Banach lattice. As in Remark 4.4, we can apply the DFJP-LNO procedure to the closed convex solid hull K_0 of $I_m(B_{L_1(m)})$ to obtain a factorization



where Y_{K_0} is a Banach lattice and both J_{K_0} and $J^*_{K_0}$ are interval preserving lattice homomorphisms. The statements of Theorem 4.5 and Remark 4.6 also hold for this factorization. If I_m is weakly compact, then: (i) $Y^*_{K_0}$ contains no isomorphic copy of c_0 (imitate the proof of [1, Theorem 5.43]), and (ii) Y_{K_0} is reflexive whenever X has the property that the solid hull of any relatively weakly compact set is relatively weakly compact.

Acknowledgments

The research of José Rodríguez was partially supported by *Agencia Estatal de Investigación* [MTM2017-86182-P, grant cofunded by ERDF, EU] and *Fundación Séneca* [20797/PI/18].

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