Quotient Spaces and the Local Diameter 2 Property

With particular focus on ℓ_∞/c_0.

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Abstract

The goal of this thesis is to show that the quotient space ℓ_∞/c_0 has the local diameter 2 property. We will start by defining the quotient space X/Y when X is a vector space and Y is a subspace of X. We will see that when X is normed, then X/Y can be given a norm in a natural way, and that this norm is complete provided the norm in X is. In particular, we have that ℓ_∞/c_0 is a complete quotient space.

We will show that the dual of a quotient space X/Y is isometrically isomorphic to the annihilator of Y in X^*, and thus it follows that the dual of ℓ_∞/c_0 is isometrically isomorphic to a subspace of $(\ell_\infty)^*$.

We will realize the dual of ℓ_∞ as the space $ba(2^{\mathbb{N}})$ of finitely additive signed measures on $2^{\mathbb{N}}$ that are of bounded variation. Furthermore, we will show that the dual space action on ℓ_∞ is given by the integral of functions in ℓ_∞ with respect to such measures. Additionally, we will see that the dual space action on ℓ_∞/c_0 is also given by this integral.

Once the dual space action on ℓ_∞/c_0 is established, we can generate slices $S(\varphi, \varepsilon)$ of the unit ball B_{ℓ_∞/c_0} where $\varphi \in S_{(\ell_\infty/c_0)^*}$ and $\varepsilon > 0$. A slice is a set $S(\varphi, \varepsilon) := \{[x] \in B_{\ell_\infty/c_0} : \varphi([x]) > 1 - \varepsilon\}$. Furthermore, the diameter of a slice is the maximum distance between elements of the slice. By showing that any slice contains elements $[x], [y] \in \ell_\infty/c_0$ such that $x, y \in \ell_\infty$ have the values $1, -1$ respectively on a set $A_j \subset \mathbb{N}$ of infinite cardinality, we are able to show that all slices of B_{ℓ_∞/c_0} have diameter 2, i.e., ℓ_∞/c_0 has the local diameter 2 property.

Contents

Chapter 1

Introduction

The goal of this thesis is to show that the quotient space ℓ_∞/c_0 has the local diameter 2 property (Definition 5.1.4). Before we can discuss this property, we need some background that will be presented in the upcoming chapters.

In Chapter 2 we introduce the term quotient space X/Y, when X is a vector space and Y is a subspace of X. We will show that X/Y is a vector space, that X/Y is normed whenever X is normed and $Y \subset X$ is closed, and finally that X/Y is complete whenever X is complete.

Chapter 3 contains theory that will be needed when discussing the results presented in Chapter 4 and Chapter 5. We will begin by introducing duals of normed spaces along with the term dual space action. This will be needed when examining whether a space has the local diameter 2 property. Furthermore, the terms isomorphism and adjoint operator will be defined. Finally we will introduce the $ba(\mathcal{A})$-space of finitely additive signed measures on an algebra $\mathcal{A}$ that are of bounded variation. This will be needed when finding the dual of ℓ_∞/c_0.

We start Chapter 4 by showing that for any Banach space X and non-zero element $x^* \in X^*$, the quotient space $X/\ker x^*$ is isometrically isomorphic to $\mathbb{R}$. We will then move on to a more general result saying that when T is a bounded linear surjective operator between Banach spaces X and W, then the quotient space $X/\ker T$ is isomorphic to W. Furthermore we show that when Y is any closed subspace of X, then the dual $(X/Y)^*$ is isometrically isomorphic to the annihilator $Y^\perp$ in X^*. This shows that the dual of ℓ_∞/c_0 is isometrically isomorphic to a subset of $(\ell_\infty)^*$. Because of this, we will end Chapter 4 by establishing an isometric isomorphism between the dual space $(\ell_\infty)^*$ and $ba(2^{\mathbb{N}})$.

In Chapter 5 we present various examples of Banach spaces that have, and do not have, the local diameter 2 property. Finally, we show that ℓ_∞/c_0 has this property.

We will assume familiarity with real analysis and basic theory of measure and integration. We will only consider real vector spaces, so when discussing sequence spaces the elements will only be real valued sequences. Whenever we refer to a subspace Y of a vector space X, we mean that Y is a vector subspace.

Our main sources for the theory presented in chapters 2, 3 and 4 are [BK] and [FHHMZ]. We will use standard Banach space notation and terminology as used in these books.

Chapter 2

Quotient Spaces

In this chapter we will introduce the terms quotient space, normed space and Banach space. We will show that a quotient space can be given a norm in a natural way, and that the quotient space is a Banach space under certain conditions. Finally we will introduce the quotient space ℓ_∞/c_0.

2.1 Quotient Spaces and Normed Spaces

In this section we will define the terms quotient space and normed space. We will also present various examples of normed spaces which will be discussed later in the thesis. Finally we will define a norm on the quotient space.

Definition 2.1.1. Let X be a vector space and Y a subspace of X. A *coset* of Y in X is a set $[x] = x+Y = \{x+y : y \in Y\}$ for some $x \in X$. The *quotient space* X/Y is the set of all cosets of Y in X, i.e., $X/Y = \{x + Y : x \in X\}$.

The elements of the quotient space are cosets. It turns out that two elements v and w of X belong to the same coset if and only if their difference $v - w$ is an element of Y. Let us see why.

Proposition 2.1.2. *Let X/Y be a quotient space, where $[v]$ and $[w] \in X/Y$. Then, $[v] = [w] \iff v - w \in Y$.*

Proof. $\Longrightarrow$: Assume $[v] = [w]$. Then the two sets $v + Y$ and $w + Y$ are equal, and thus there exist $y \in Y$ such that $w + y = v + 0$ (note that $0 \in Y$ since Y is a vector space). Consequently, $v - w = y \in Y$.

$\Longleftarrow$: Assume $v - w \in Y$. Then we can write $v = w + y$ for some $y \in Y$. Hence $v + Y = w + y + Y = w + Y$. □

We can define *addition of cosets* and *multiplication of the cosets with scalars* to make X/Y a vector space. This is shown in the following proposition.

Proposition 2.1.3. *Let X be a vector space and Y be a subspace of X. Let $x_1, x_2 \in X$ and $\alpha \in \mathbb{R}$. The quotient space X/Y is a vector space with scalar multiplication given by $\alpha[x_1] = [\alpha x_1]$ and vector addition given by $[x_1] + [x_2] = [x_1 + x_2]$.*

Proof. Let us start by showing that the two vector space operations are well defined. Let us start with vector addition. Since each coset $[x_1], [x_2]$ can be expressed by several elements of X, we must ensure that the sum $[x_1] + [x_2]$ is unique. Let $x_1^{'} \in [x_1]$ and $x_2^{'} \in [x_2]$. We must show that $[x_1^{'} + x_2^{'}] = [x_1 + x_2]$. By the definition of coset, there exist $y_1, y_2 \in Y$ such that

$$x_1^{'} = x_1 + y_1 \quad \text{and} \quad x_2^{'} = x_2 + y_2.$$

Thus,

$$\begin{aligned}
(x_1^{'} + x_2^{'}) - (x_1 + x_2) &= \Big(x_1 + y_1 + x_2 + y_2\Big) - (x_1 + x_2) \\
&= x_1 - x_1 + x_2 - x_2 + y_1 + y_2 \\
&= y_1 + y_2 \in Y.
\end{aligned}$$

By Proposition 2.1.2, it follows that $[x_1^{'} + x_2^{'}] = [x_1 + x_2]$ which is what we needed. Since X is a vector space, we know that $(x_1 + x_2) \in X$. It follows that $[x_2 + x_2] \in X/Y$ by the definition of quotient space, and vector addition is well defined.

Now, let us look at scalar multiplication. Like before, we must ensure that, if $x_1^{'} \in [x_1]$, then $[\alpha x_1^{'}] = [\alpha x_1]$. Again, by the definition of coset, we have

$x_1^{'} = x_1 + y_1$ for some $y_1 \in Y$. Thus,

$$\begin{aligned} \alpha x_1^{'} - \alpha x_1 &= \alpha(x_1 + y_1) - \alpha x_1 \\ &= \alpha x_1 - \alpha x_1 + \alpha y_1 \\ &= \alpha y_1 \in Y, \end{aligned}$$

and by Proposition 2.1.2 we have the desired equality. Since X is a vector space we have that $\alpha x_1 \in X$, and consequently $[\alpha x_1] \in X/Y$. Thus, scalar multiplication is well defined.

For X/Y to be a vector space, it remains to show that eight axioms hold. To this end, let $[x_1], [x_2], [x_3] \in X/Y$ and $\alpha, \beta \in \mathbb{R}$. Then, since X is a vector space, we get

1. Associativity of vector addition:

$$\begin{aligned} \big([x_1] + [x_2]\big) + [x_3] &= [x_1 + x_2] + [x_3] \\ &= [(x_1 + x_2) + x_3] \\ &= [x_1 + (x_2 + x_3)] \\ &= [x_1] + [x_2 + x_3] = [x_1] + \big([x_2] + [x_3]\big). \end{aligned}$$

2. Commutativity of vector addition:

$$[x_1] + [x_2] = [x_1 + x_2] = [x_2 + x_1] = [x_2] + [x_1].$$

3. Identity element of vector addition:

$$[x_1] + [0] = [x_1 + 0] = [x_1].$$

4. Inverse element of vector addition:

$$[x_1] + [-x_1] = [x_1 + (-x_1)] = [0].$$

5. Compatibility of scalar multiplication with field multiplication:

$$\alpha\big(\beta[x_1]\big) = \alpha[\beta x_1] = [\alpha(\beta x_1)] = [(\alpha\beta)x_1] = (\alpha\beta)[x_1].$$

6. Identity element of scalar multiplication:

$$1[x_1] = [1x_1] = [x_1].$$

7. Distributivity of scalar multiplication with respect to vector addition:

$$\begin{aligned}\alpha([x_1] + [x_2]) = \alpha[x_1 + x_2] &= [\alpha(x_1 + x_2)] \\ &= [\alpha x_1 + \alpha x_2] \\ &= [\alpha x_1] + [\alpha x_2] = \alpha[x_1] + \alpha[x_2].\end{aligned}$$

8. Distributivity of scalar multiplication with respect to field addition:

$$\begin{aligned}(\alpha + \beta)[x_1] = [(\alpha + \beta)x_1] &= [\alpha x_1 + \beta x_1] \\ &= [\alpha x_1] + [\beta x_1] = \alpha[x_1] + \beta[x_1].\end{aligned}$$

□

Note that the zero vector in X/Y is Y because $[0] = 0 + Y = Y$.

The following example illustrates how cosets are generated.

Example 2.1.4. Let $Y := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2\}$. We will describe the quotient space $\mathbb{R}^2/Y$.

The set Y is a straight line through the origin with a slope of one, as shown on the left side of Figure 2.1. It can easily be checked that Y is a subspace of $\mathbb{R}^2$. By the definition of Y, each element of Y has the same real value in both coordinates. Thus, for $x = (x_1, x_2) \in \mathbb{R}^2$, we see that $[x] \in \mathbb{R}^2/Y$ is given by

$$\begin{aligned}[x] &= (x_1, x_2) + Y \\ &= \{(x_1, x_2) + (r, r) : r \in \mathbb{R}\} \\ &= \{(x_1 + r, x_2 + r) : r \in \mathbb{R}\}.\end{aligned}$$

The left side of Figure 2.1 demonstrates how we find a point in the coset $[x]$ by adding $y = (r, r) \in Y$ to x. The right side of the figure shows that the collection of all such points, letting r vary over all real numbers, form the coset

$[x]$, which is a straight line parallel to Y. Thus, no matter which coset we look at, a coset defines a straight line in $\mathbb{R}^2$ parallel to Y, and $\mathbb{R}^2/Y$ is the set of all lines parallel to Y.

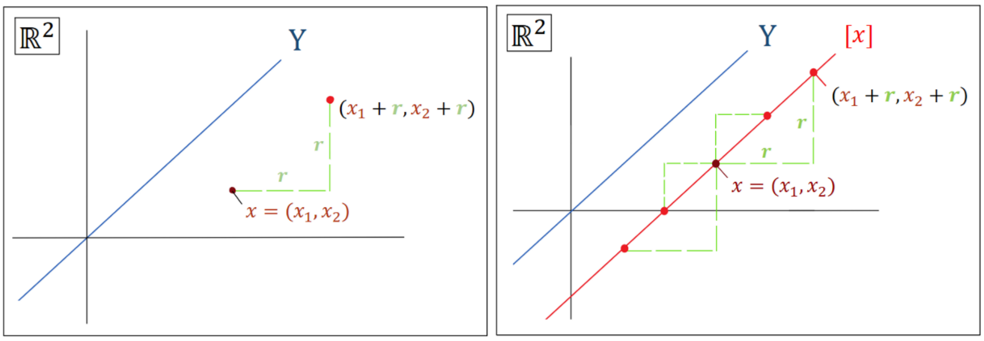

Figure 2.1: Constructing a coset $[x]$ in $\mathbb{R}^2/Y$.

We have seen how we generated the coset $[x]$ for some $x \in \mathbb{R}^2$. Let us see how we can find the coset $-[x]$. In Figure 2.2 the vectors x and $-x$ in $\mathbb{R}^2$ are shown as green arrows. By the definition of scalar multiplication in $\mathbb{R}^2/Y$ we have $-[x] = [-x]$, and this coset is shown as the yellow line in Figure 2.2.

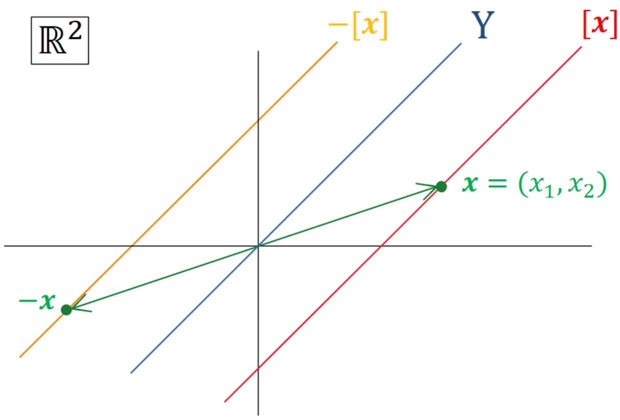

Figure 2.2: Finding $-[x]$.

Now, let $x, u \in \mathbb{R}^2$. Let us find the sum $[x] + [u]$. By the definition of vector addition in $\mathbb{R}^2/Y$ we have $[x] + [u] = [x + u]$. Figure 2.3 illustrates how we can find this coset by adding the vectors x and u in $\mathbb{R}^2$ and then generate the coset $[x + u]$ indicated by the pink line.

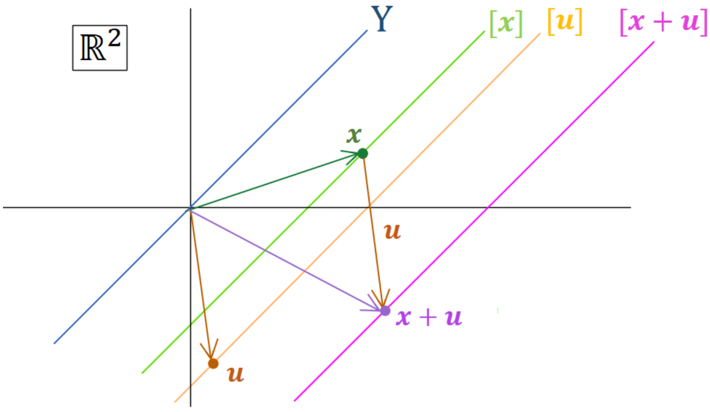

Figure 2.3: Finding $[x] + [u]$.

The Example above illustrates that the cosets $[x] \in X/Y$ are lines parallel to Y. More precisely, they are simply translations of Y.

If X is a normed space, it is possible to define a norm on the quotient space X/Y. Before we do this, let us recall what we mean by a normed space.

Definition 2.1.5. Let X be a vector space. A function $\|\cdot\| : X \to [0, \infty)$ defines a *norm* on X if it satisfies

(i) $\|x\| \geq 0$ for all $x \in X$, and $\|x\| = 0$ if and only if $x = 0$,

(ii) $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$ and all scalars $\alpha \in \mathbb{R}$, and

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

When X is endowed with a norm, X is said to be a *normed space.* It is possible to define different norms on the same vector space. Therefore, to be precise, it is the pair $(X, \|\cdot\|)$ which is a normed space, but we will simply refer to X as a normed space when it is obvious or irrelevant which norm we are using. Sometimes we will use the notation $\|\cdot\|_X$ to indicate that we are considering the norm on X.

It is natural to think of the norm as the distance between the element x and the origin. In a normed space X, the *open unit ball* is the set $U_X := \{x \in X :$

$\|x\| < 1\}$, and the *closed unit ball* is the set $B_X := \{x \in X : \|x\| \leq 1\}$. The *unit sphere* is the set $S_X := \{x \in X : \|x\| = 1\}$.

Now, let us look at some examples of normed spaces.

Example 2.1.6. The *Euclidean norm* on $\mathbb{R}^n$ is given by

$$\|(x_1, x_2, ..., x_n)\| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}. \tag{2.1}$$

The next examples deal with spaces where the elements are sequences. In this thesis, when we talk about a sequence $x = \{x_i\}_{i \in \mathbb{N}}$, we use the notation $x = \{x_i\}$. Now, let S denote the set of all scalar valued sequences, i.e.,

$$S := \{x = \{x_i\} : x_i \in \mathbb{R}\}.$$

Example 2.1.7. Let $\alpha \in \mathbb{R}$ and $\{x_i\}, \{y_i\} \in S$. The set S is a vector space where scalar multiplication is given by $\alpha\{x_i\} = \{\alpha x_i\}$ and vector addition is given by $\{x_i\} + \{y_i\} = \{x_i + y_i\}$. Clearly the two operations are well defined as $\{\alpha x_i\} \in S$ and $\{x_i + y_i\} \in S$. Furthermore, all eight axioms hold since $\mathbb{R}$ is a vector space.

To show that a subset X of a vector space V is itself a vector space, it suffices to show that X is closed under scalar multiplication and vector addition.

Example 2.1.8. The symbol ℓ_1 denotes the vector space of all scalar valued absolutely convergent sequences $x = \{x_i\}$ endowed with the norm

$$\|x\|_1 := \sum_{i=1}^{\infty} |x_i|. \tag{2.2}$$

That a scalar valued sequence $x = \{x_i\}$ is absolutely convergent means that $\sum\limits_{i=1}^{\infty} |x_i| < \infty$. Note that the set of all scalar valued absolutely convergent sequences is a subset of S from Example 2.1.7. Thus, to see that ℓ_1 is a vector space it suffices to show that it is closed under scalar multiplication and vector addition. Let $\alpha, \beta \in \mathbb{R}$ and $x = \{x_i\}, y = \{y_i\} \in \ell_1$. By definition,

$\alpha x + \beta y = \{\alpha x_i + \beta y_i\}$. Then, using the triangle inequality for the absolute value, we have

$$\sum_{i\in\mathbb{N}} |\alpha x_i + \beta y_i| \le \sum_{i\in\mathbb{N}} \Big(|\alpha||x_i| + |\beta||y_i|\Big)$$
$$= |\alpha| \sum_{i\in\mathbb{N}} |x_i| + |\beta| \sum_{i\in\mathbb{N}} |y_i| < \infty,$$

and ℓ_1 is a vector space. To see that (2.2) actually defines a norm on ℓ_1, we need to check the criteria of Definition 2.1.5. To this end, let $x = \{x_i\}, y = \{y_i\} \in \ell_1$ and $\alpha \in \mathbb{R}$.

(i) We see that $\|x\| = \sum_{i=1}^{\infty} |x_i| \ge 0$ for all $x \in \ell_1$. Now, let $x = 0$. Then, $\|x\| = \sum_{i=1}^{\infty} |0| = 0$. Next, let $\|x\| = 0$. Then, $|x_i| = 0$ for all i which means that $x = 0$.

(ii) $\|\alpha x\| = \sum_{i=1}^{\infty} |\alpha x_i| = |\alpha| \sum_{i=1}^{\infty} |x_i| = |\alpha|\|x\|$.

(iii) $\|x + y\| = \sum_{i=1}^{\infty} |x_i + y_i| \le \sum_{i=1}^{\infty} (|x_i| + |y_i|) = \sum_{i=1}^{\infty} |x_i| + \sum_{i=1}^{\infty} |y_i| = \|x\| + \|y\|$.

Example 2.1.9. The symbol ℓ_∞ denotes the vector space of all bounded scalar valued sequences endowed with the norm

$$\|x\|_\infty := \sup_{i\in\mathbb{N}} |x_i|. \tag{2.3}$$

That a scalar valued sequence $x = \{x_i\}$ is bounded means that $\sup_{i\in\mathbb{N}} |x_i| < \infty$. Note that the set of all bounded scalar valued sequences is a subset of S from Example 2.1.7. Thus, to show that ℓ_∞ is a vector space, it suffices to show that for $\alpha, \beta \in \mathbb{R}$ and $x = \{x_i\}, y = \{y_i\} \in \ell_\infty$, we have $\{\alpha x_i + \beta y_i\} \in \ell_\infty$. Using the triangle inequality for the absolute value and the properties of the supremum, we have

$$\sup_{i\in\mathbb{N}} |\alpha x_i + \beta y_i| \le \sup_{i\in\mathbb{N}} \Big(|\alpha||x_i| + |\beta||y_i|\Big) \le |\alpha| \sup_{i\in\mathbb{N}} |x_i| + |\beta| \sup_{i\in\mathbb{N}} |y_i| < \infty,$$

and we have the desired result. Furthermore, to see that (2.3) actually defines a norm on ℓ_∞, it must be shown that (2.3) meets the criteria of Definition 2.1.5. Let $\alpha \in \mathbb{R}$ and $x = \{x_i\}, y = \{y_i\} \in \ell_\infty$. Condition (i) and (ii) are met due to the properties of the supremum:

(i) We see that $\|x\| = \sup_{i\in\mathbb{N}} |x_i| \geq 0$ for all $x \in \ell_\infty$. Furthermore,
$x = 0 \iff \|x\| = \sup_{i\in\mathbb{N}} |x_i| = 0$,

(ii) $\|\alpha x\| = \sup_{i\in\mathbb{N}} |\alpha x_i| = |\alpha| \sup_{i\in\mathbb{N}} |x_i| = |\alpha|\|x\|$.

The third and last condition is also fulfilled, using the triangle inequality for the absolute value and the properties of the supremum:

$$\begin{aligned}\|x + y\| = \sup_{i\in\mathbb{N}} |x_i + y_i| &\leq \sup_{i\in\mathbb{N}} \Big(|x_i| + |y_i|\Big) \\ &\leq \sup_{i\in\mathbb{N}} |x_i| + \sup_{i\in\mathbb{N}} |y_i| = \|x\| + \|y\|.\end{aligned}$$

Example 2.1.10. The symbol c_0 denotes the subspace of ℓ_∞ consisting of all $x = \{x_i\}$ satisfying $\lim_{i\to\infty} x_i = 0$.

Clearly, c_0 is a subset of ℓ_∞ as all $x \in c_0$ are bounded. To show that c_0 is a subspace of ℓ_∞ and hence a normed space, it suffices to show that c_0 is closed under scalar multiplication and vector addition. Let $\alpha, \beta \in \mathbb{R}$ and $x, y \in c_0$. We have that $\alpha x + \beta y = \{\alpha x_i + \beta y_i\}$. Taking the limit of the sequence yields

$$\lim_{i\to\infty} (\alpha x_i + \beta y_i) = \alpha \lim_{i\to\infty} x_i + \beta \lim_{i\to\infty} y_i = \alpha \cdot 0 + \beta \cdot 0 = 0,$$

and $\alpha x + \beta y \in c_0$. The norm from (2.3) is defined for all elements of ℓ_∞ and is therefore also defined for all elements of c_0.

Example 2.1.11. The symbol c denotes the subspace of ℓ_∞ consisting of all $x = \{x_i\}$ such that $\lim_{i\to\infty} x_i$ exists in $\mathbb{R}$.

As all convergent sequences are bounded, c is a subset of ℓ_∞. To show that c is a normed subspace of ℓ_∞, it suffices to show that c is closed under scalar multiplication and vector addition. To this end, let $\alpha, \beta \in \mathbb{R}$ and $\{x_i\}, \{y_i\} \in c$. Let us denote the limits as $\lim_{i\to\infty} x_i = r_1$ and $\lim_{i\to\infty} y_i = r_2$. We have

$$\lim_{i\to\infty} (\alpha x_i + \beta y_i) = \alpha \lim_{i\to\infty} x_i + \beta \lim_{i\to\infty} y_i = \alpha r_1 + \beta r_2 \in \mathbb{R},$$

and we have the desired result.

We have now looked at various examples of normed spaces, all of which will be used later in this thesis. Another norm that will be frequently used throughout, is the operator norm (Example 2.1.17). As the operator norm is defined for bounded linear operators, let us now introduce these terms.

Definition 2.1.12. A function $T : X \to W$, where X and W are vector spaces, is called a *linear operator* if it satisfies

(i) $T(\alpha x) = \alpha Tx$ for all $\alpha \in \mathbb{R}$ and $x \in X$, and

(ii) $T(x_1 + x_2) = Tx_1 + Tx_2$ for all $x_1, x_2 \in X$.

Whenever $W = \mathbb{R}$ we call T a linear *functional*.

Definition 2.1.13. A linear operator $T : X \to W$, where X and W are normed spaces, is said to be *bounded* if there exists an $M \in \mathbb{R}$ such that $\|Tx\|_W \leq M\|x\|_X$ for all $x \in X$.

Definition 2.1.14. Let X be a normed space and let $r > 0$. The *open ball* centered at $x \in X$ with radius r is the set

$$B(x; r) := \{u \in X : \|x - u\| < r\}.$$

The corresponding closed ball is the set

$$\overline{B}(x; r) := \{u \in X : \|x - u\| \leq r\}.$$

Definition 2.1.15. Let $T : X \to W$ be an operator between two normed spaces. T is said to be *continuous at* $x \in X$ if there for every $\varepsilon > 0$ exists a $\delta > 0$ such that $T\big(B(x; \delta)\big) \subset B(Tx; \varepsilon)$. T is said to be *continuous* if T is continuous for every $x \in X$.

Note that whenever T is linear there is an equivalence between T being continuous at 0 and T being continuous everywhere. Additionally, it is not difficult to see that whenever an operator T is linear, there is an equivalence between T being bounded and T being continuous.

Definition 2.1.16. Let X and W be normed spaces. Then, $\mathcal{L}(X, W)$ denotes the vector space of all bounded linear operators from X into W where scalar multiplication is given by $(\alpha T)(x) = \alpha T x$ and vector addition is given by $(T + S)(x) = Tx + Sx$, for $\alpha \in \mathbb{R}$ and $T, S \in \mathcal{L}(X, W)$.

It is straightforward to prove that $\mathcal{L}(X, W)$ is indeed a vector space under the above mentioned operations. Now we can define a norm on $\mathcal{L}(X, W)$.

Example 2.1.17. The operator norm is a norm on $\mathcal{L}(X, W)$ and is defined for all T in $\mathcal{L}(X, W)$ by

$$\|T\| := \sup_{x \in B_X} \|Tx\|. \tag{2.4}$$

An equivalent definition is

$$\|T\| := \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}. \tag{2.5}$$

Note that an operator T is bounded exactly when $\|T\| < \infty$. The operator norm is the smallest M satisfying $\|Tx\| \leq M\|x\|$ for all $x \in X$.

Definition 2.1.18. Let X be a normed space. A sequence $\{x_i\} \subset X$ is said to *converge* to a point $x \in X$ if there for every $\varepsilon > 0$ exists an $N \in \mathbb{N}$ such that $\|x_n - x\| < \varepsilon$ whenever $n \geq N$.

Definition 2.1.19. Let X be a normed space. A set $U \subset X$ is called *open in* X if there for all $x \in U$ exists a ball $B(x; r)$ around x that is contained in U. A set $F \subset X$ is called *closed in* X if the complement F^c is open.

Remark 2.1.1. Note that an open ball $B(x; r)$ is an open set.

Remark 2.1.2. A subset Y of a normed space X is closed if and only if the limit of every sequence $\{y_i\} \subset Y$, convergent in X, also lies in Y, i.e., $\lim_{i \to \infty} y_i \in Y$.

Definition 2.1.20. Let X be a normed space with a closed subspace Y. For each $x \in X$, let $\|[x]\|_{X/Y} := \inf_{y \in Y} \|x + y\|_X$.

In the next proposition we will show that $\| \cdot \|_{X/Y}$ defines a norm on X/Y. In Figure 2.4 we see a $2D$ representation of how $\|[x]\|$ measures the "distance" from $[x]$ to Y.

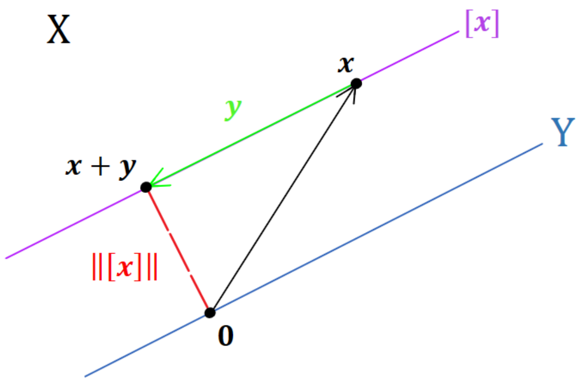

Figure 2.4: The coset $[x] \in X/Y$ generated by $x \in X$, and the norm $\|[x]\| = \inf_{y \in Y} \|x + y\|$.

Proposition 2.1.21. *Let X be a normed space with a closed subspace Y. Then $\|\cdot\|_{X/Y}$ from Definition 2.1.20 defines a norm on X/Y.*

Proof. Let $[x] \in X/Y$. Then $\|[x]\| = \inf_{y \in Y} \|x + y\|$. We must show that the defined norm satisfies the criteria in Definition 2.1.5.

(i) Since $\|x + y\| \geq 0$ for all $x + y \in X$, then $\|[x]\| = \inf_{y \in Y} \|x + y\| \geq 0$ for all $[x] \in X/Y$. Suppose $[x]$ is equal to the zero vector in X/Y, i.e., $[x] = Y$. Then $\|Y\| = \inf_{y \in Y} \|y\| = 0$ since $0 \in Y$.

Now suppose $\|[x]\| = 0$. It must be shown that $[x] = Y$. We have $0 = \|[x]\| = \inf_{y_n \in Y} \|x + y_n\|$. Because of this, for all $n \in \mathbb{N}$ there must exist some $y_n \in Y$ such that $\|x - (-y_n)\| < \frac{1}{n}$. Therefore $\{-y_n\}_{n=1}^{\infty}$ converges to x, and since Y is closed in X, it follows that $x \in Y$. Consequently, $[x] = x + Y = Y$.

(ii) Let α be a nonzero scalar. If $x \in X$, then

$$\begin{aligned} \|\alpha[x]\| = \inf_{y \in Y} \|\alpha x + y\| &= \inf_{z \in Y} \|\alpha x + \alpha z\| \\ &= \inf_{z \in Y} |\alpha| \|x + z\| = |\alpha| \inf_{z \in Y} \|x + z\| = |\alpha| \|[x]\|. \end{aligned}$$

(iii) Let $[x_1], [x_2] \in X/Y$ and $\varepsilon > 0$. By the definition of the norm, there exist elements $y_1, y_2 \in Y$ such that

$$\|x_1 + y_1\| < \|[x_1]\| + \varepsilon/2 \quad \text{and} \quad \|x_2 + y_2\| < \|[x_2]\| + \varepsilon/2.$$

Then, by the definition of vector addition in X/Y, and by using the triangle inequality for the norm on X, we get

$$\begin{aligned} \|[x_1] + [x_2]\| = \|[x_1 + x_2]\| &\leq \|x_1 + x_2 + (y_1 + y_2)\| \\ &\leq \|x_1 + y_1\| + \|x_2 + y_2\| \\ &< \|[x_1]\| + \|[x_2]\| + \varepsilon. \end{aligned}$$

Since ε was arbitrary, it follows that $\|[x_1] + [x_2]\| \leq \|[x_1]\| + \|[x_2]\|$. Hence, the triangle inequality is satisfied and $\|\cdot\|_{X/Y}$ is a norm on X/Y. □

Note that it is important that Y is closed in X for $\|\cdot\|_{X/Y}$ to be a norm. If Y is not closed and $\{y_i\} \subset Y$ such that $\lim_{i\to\infty} y_i = x \notin Y$, then $\|[x]\| = 0$ while $[x] \neq 0$.

2.2 Quotient Spaces and Banach Spaces

In this section we will define completeness and show that the quotient of a complete normed space with a closed subspace, is complete. We will also look at other examples of complete spaces.

Definition 2.2.1. A sequence $\{x_i\}$ of a normed space X is called a *Cauchy sequence* if there for every $\varepsilon > 0$ exists an $N \in \mathbb{N}$ such that $\|x_n - x_m\| < \varepsilon$ whenever $n, m \geq N$.

Note that all convergent sequences are Cauchy.

Definition 2.2.2. A *Banach space* is a *complete* normed space, i.e., a normed space X where all Cauchy sequences $\{x_i\} \subset X$ converge in X.

Proposition 2.2.3. *Let X be a Banach space and Y a subspace of X. Then,*

$$Y \text{ closed in } X \iff Y \text{ Banach.}$$

Proof. $\Longrightarrow$: Assume Y is closed and let $\{y_i\} \subset Y$ be Cauchy. Since X is complete, $\{y_i\}$ converges in X. Since Y is closed, the limit is in Y. Hence Y is complete.

$\Longleftarrow$: Assume Y is complete and let $\{y_i\} \subset Y$ be convergent. Since $\{y_i\}$ converges, it is Cauchy. Since Y is complete, the limit is in Y. Hence Y is closed. $\square$

Example 2.2.4. Let $P[0,1]$ denote the set of all real valued polynomials defined on the interval $[0,1]$. The symbol $C[0,1]$ denotes the vector space of all continuous functions on $[0,1]$ endowed with the supremum norm. The space $C[0,1]$ is complete, but $P[0,1]$ is not. Let us see why.

All polynomials $p : [0,1] \to \mathbb{R}$ where $p(x) = \sum_{i=0}^{n} a_i x^i$, $a_i \in \mathbb{R}$ are continuous, so $P[0,1]$ is a subset of the set of all continuous functions on $[0,1]$. The set $P[0,1]$ is closed under both scalar multiplication and vector addition and is therefore a subspace of $C[0,1]$. There is a well-known theorem in real analysis stating that $C[0,1]$ is complete: A uniformly convergent sequence of continuous functions converges to a continuous function. To show that $P[0,1]$ is not complete, let $p_n(x)$ be the truncated Maclaurin series of e^x, i.e.,

$$p_n(x) = \sum_{i=0}^{n} \frac{x^i}{i!}.$$

Then, p_n is a polynomial for all $n \in \mathbb{N}$, and p_n converges to $e^x \in C[0,1]$, but $e^x \notin P[0,1]$. Thus, $P[0,1]$ is not complete.

Example 2.2.5. ℓ_∞ is a Banach space.

In Example 2.1.9 it was shown that ℓ_∞ is a normed space when given the supremum norm defined in (2.3). It remains to show that ℓ_∞ is complete in this norm. Let $\{s^n\}_{n\in\mathbb{N}} \subset \ell_\infty$ be a Cauchy sequence where $s^n = \{x_i^n\}_{i\in\mathbb{N}} \in \ell_\infty$ for each $n \in \mathbb{N}$, i.e., $s^n = \{x_1^n, x_2^n, ..., x_i^n, ...\}$. Since $\{s^n\}$ is Cauchy, we have that for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that whenever $n, m \geq N$ we have

$$\|s^n - s^m\|_\infty = \sup_{i\in\mathbb{N}} |x_i^n - x_i^m| < \varepsilon/2.$$

This means that for all $n, m \geq N$ we have $|x_i^n - x_i^m| < \varepsilon/2$ for every $i \in \mathbb{N}$, so for each fixed i, the sequence $\{x_i^n\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, the sequence $\{x_i^n\}_{n\in\mathbb{N}}$ converges to some $x_i \in \mathbb{R}$ for each i. Moreover, note that for all $i \in \mathbb{N}$ and for $n \geq N$, we have $|x_i - x_i^n| < \varepsilon$. Now let $L = \{x_1, x_2, ..., x_i, ...\}$. We get

$$|x_i| = |x_i - x_i^N + x_i^N| \leq |x_i - x_i^N| + |x_i^N| < \varepsilon + \sup_{i\in\mathbb{N}} |x_i^N| = \varepsilon + \|s^N\|.$$

Since this holds for every $i \in \mathbb{N}$, each term of the sequence L is bounded by this value, and $L \in \ell_\infty$. Finally we see that

$$\|s^n - L\| = \sup_{i\in\mathbb{N}} |x_i^n - x_i| < \varepsilon \quad \text{for all} \quad n \geq N,$$

which shows that the Cauchy sequence $\{s^n\}$ converges to L in ℓ_∞, i.e., ℓ_∞ is complete.

Example 2.2.6. c_0 is a Banach space.

We already know from Example 2.1.10 that c_0 is a normed subspace of ℓ_∞. Showing that c_0 is complete is equivalent to showing that c_0 is closed in ℓ_∞, due to Proposition 2.2.3. Let $\{x^n\}$ be a sequence in c_0 converging to $a = \{a_i\}$. To prove that c_0 is closed in ℓ_∞, we must show that $a \in c_0$, i.e., that $a_i \to 0$ as $i \to \infty$. Because of convergence, for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that whenever $n \geq N$, we have

$$\|x^n - a\| = \sup_{i\in\mathbb{N}} |x_i^n - a_i| < \varepsilon/2.$$

In particular, the sequence x^N satisfies $|x_i^N - a_i| < \varepsilon/2$ for every $i \in \mathbb{N}$. Since x^N converges to zero we know that for our choice of ε there exists an $I \in \mathbb{N}$ such that $|x_i^N - 0| = |x_i^N| < \varepsilon/2$ whenever $i \geq I$. Combined, this yields

$$|a_i| = |a_i - x_i^N + x_i^N| \leq |a_i - x_i^N| + |x_i^N| < \varepsilon/2 + \varepsilon/2,$$

whenever $i \geq I$, and we have the desired result.

Example 2.2.7. c is a Banach space.

In Example 2.1.11 we showed that c is a normed subspace of ℓ_∞. To prove c is complete it suffices to show that c is closed in ℓ_∞ due to Proposition 2.2.3. Let $\{s^n\} \subset c$ be a sequence converging to $a = \{a_i\} \in \ell_\infty$ where $s^n = \{x_i^n\}_{i\in\mathbb{N}}$ for each n. It must be shown that $a \in c$, i.e., that $\lim_{i\to\infty} a_i$ exists in $\mathbb{R}$. Because of convergence, for all $\varepsilon > 0$ there exists an $N_1 \in \mathbb{N}$ such that

$$\|s^n - a\|_\infty = \sup_{i\in\mathbb{N}} |x_i^n - a_i| < \varepsilon/3 \quad \text{for all} \quad n \geq N_1. \tag{2.6}$$

Each sequence s^n is convergent, i.e., for each n there exists a $b^n \in \mathbb{R}$ such that $\lim_{i\to\infty} x_i^n = b^n$. Thus, for all $\varepsilon > 0$ and for each n there exists an $I_n \in \mathbb{N}$ such that

$$|x_i^n - b^n| < \varepsilon/3 \quad \text{for all} \quad i \geq I_n. \tag{2.7}$$

Since $\{s^n\} \subset c$ converges in ℓ_∞ the sequence is Cauchy. Thus, for all $\varepsilon > 0$ there exists $N_2 \in \mathbb{N}$ such that $\|s^n - s^m\|_\infty < \varepsilon/3$ whenever $n, m \geq N_2$. Thus, for all $i \in \mathbb{N}$,

$$|x_i^n - x_i^m| < \varepsilon/3 \quad \text{for all} \quad n \geq N_2. \tag{2.8}$$

Now, combining (2.7) and (2.8), we get that whenever $n \geq N_2$ and $i \geq \max\{I_n, I_m\}$, then

$$\begin{aligned} |b^n - b^m| &= |b^n - x_i^n + x_i^n - x_i^m + x_i^m - b^m| \\ &\leq |b^n - x_i^n| + |x_i^n - x_i^m| + |x_i^m - b^m| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3. \end{aligned}$$

Hence, the sequence $\{b^n\} \subset \mathbb{R}$ is Cauchy, and because of the completeness of $\mathbb{R}$ there exists an $N_3 \in \mathbb{N}$ and $b \in \mathbb{R}$ such that

$$|b^n - b| < \varepsilon/3 \quad \text{for all} \quad n \geq N_3. \tag{2.9}$$

Let $N = \max\{N_1, N_3\}$. Combining (2.6), (2.7) and (2.9) yields that for all $\varepsilon > 0$ and whenever $n \geq N$, $i \geq I_N$ we have

$$\begin{aligned} |a_i - b| &= |a_i - x_i^N + x_i^N - b^N + b^N - b| \\ &\leq |a_i - x_i^N| + |x_i^N - b^N| + |b^N - b| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3. \end{aligned}$$

Thus, $a_i \to b$ and the sequence a is in c.

Example 2.2.8. Let X and W be normed spaces. The space $\mathcal{L}(X, W)$ is complete whenever W is. We will skip the proof.

We are now ready to show that the norm $\|\cdot\|_{X/Y}$ from Definition 2.1.20 is complete whenever X is a complete normed space and Y a closed subspace of X.

Theorem 2.2.9. *Let X be a Banach space with a closed subspace Y and let $\|[x]\| = \inf_{y \in Y} \|x + y\|$ for all $[x] \in X/Y$. Then $(X/Y, \|\cdot\|)$ is a Banach space.*

Proof. Proposition 2.1.21 shows that this norm actually is a norm on X/Y. It remains to show that it is complete, i.e., that all Cauchy sequences in X/Y converge in this norm. Let $\{[x_k]\}_{k \in \mathbb{N}}$ be a Cauchy sequence in X/Y and let $\varepsilon > 0$. Then there exists an $N_1 \in \mathbb{N}$ such that whenever $n, m \geq N_1$ we have

$$\|[x_n] - [x_m]\| = \inf_{y \in Y} \|x_n - x_m + y\| < \varepsilon/2. \tag{2.10}$$

Since it is possible to get arbitrarily close to the infimum value, we can for each $n, m \in \mathbb{N}$ choose elements $y_n, y_m \in Y$ so that $x_n' = x_n + y_n$ and $x_m' = x_m + y_m$ fulfills

$$\|x_n' - x_m'\| < \|[x_n] - [x_m]\| + \frac{1}{n+m}. \tag{2.11}$$

The relation between (2.10) and (2.11) is illustrated in Figure 2.5.

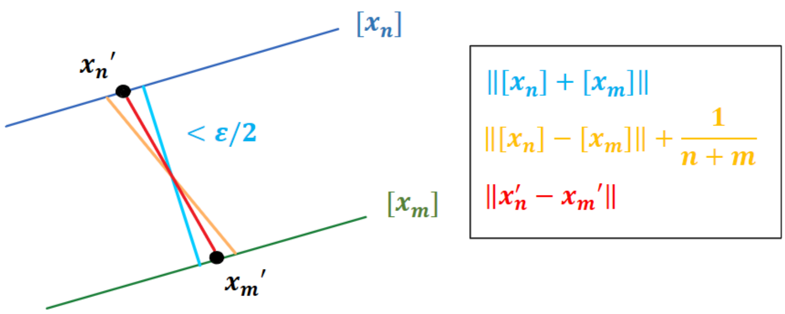

Figure 2.5: A 2D representation of the relation between (2.10) and (2.11).

Now, for our choice of ε, let $N_2 \in \mathbb{N}$ so that whenever $n, m \geq N_2$ we have

$$\frac{1}{n+m} < \varepsilon/2. \tag{2.12}$$

Let $N = \max\{N_1, N_2\}$. Thus, combining (2.10), (2.11) and (2.12), whenever $n, m \geq N$ we get

$$\|x_n^{'} - x_m^{'}\| < \|[x_n] - [x_m]\| + \frac{1}{n+m} < \varepsilon/2 + \varepsilon/2,$$

which means that $\{x_k^{'}\}$ is a Cauchy sequence in X. Since X is a Banach space, $x_k^{'}$ converges to some $x \in X$. Thus, for every $\varepsilon > 0$ there exists a $K \in \mathbb{N}$ so that whenever $k \geq K$ we have $\|x_k^{'} - x\| < \varepsilon$. Now, the coset generated by the limit x is $[x] = x + Y \in X/Y$. Thus, whenever $k \geq K$, we have

$$\|[x_k] - [x]\| = \|[x_k^{'}] - [x]\| = \inf_{y \in Y} \|x_k^{'} - x + y\| \leq \|x_k^{'} - x + 0\| < \varepsilon.$$

Note that, since we defined $x_k^{'}$ to be $x_k^{'} = x_k + y_k$ for some $y_k \in Y$, we have that $[x_k] = [x_k^{'}]$ due to Proposition 2.1.2. Thus, $\{[x_k]\}$ converges to $[x]$ in X/Y and X/Y is complete. □

We now have some general understanding of what a quotient space is, and we have shown that a quotient space X/Y is complete whenever X is.

Example 2.2.10. c/c_0 is a complete quotient space.

As shown in Example 2.2.7 c is complete. Clearly c_0 is a subset of c. Since c_0 is complete (Example 2.2.6), c_0 is a closed subspace of c due to Proposition 2.2.3. The quotient space c/c_0 meets the criteria of Theorem 2.2.9, and c/c_0 is complete.

Example 2.2.11. ℓ_∞/c_0 is a complete quotient space.

Example 2.2.5 shows that ℓ_∞ is complete. In Example 2.2.6 we show that c_0 is a closed subspace of ℓ_∞. It follows that ℓ_∞/c_0 is complete by Theorem 2.2.9.

Chapter 3

Toolbox

When we in Chapter 5 will discuss whether a space has the local diameter 2 property, it is important to know more about another Banach space, namely the dual space. This space will be discussed in the first section of this chapter. When discussing the dual of a quotient space X/Y in Section 4.3, we will use the terms isomorphism and adjoint operator, and these terms will be introduced in sections 3.2 and 3.3. Finally, to discuss the dual of ℓ_∞ in Section 4.4, we also need some measure theory which will be the topic of Section 3.4.

3.1 The Dual Space

In this section we will define dual space and dual space action and present some relevant examples.

Definition 3.1.1. Let X be a normed space. The vector space $\mathcal{L}(X, \mathbb{R})$ endowed with the operator norm (2.4) is the *dual space* of X and is denoted by X^*. The way the functionals $x^* \in X^*$ acts on elements $x \in X$ is called the *dual space action* on X.

Note that $X^* = \mathcal{L}(X, \mathbb{R})$ is complete because $\mathbb{R}$ is complete, as stated in Example 2.2.8.

For many spaces it is known how the dual space action works. Let us look at some examples of dual spaces X^* and their relation with X. In the following examples, $x^* = \{b_i\}$ will denote an element of the dual space in question and $x = \{a_i\} \in X$.

Example 3.1.2. The dual of $\mathbb{R}^n$ with the Euclidean norm (2.1) is $(\mathbb{R}^n)^* = \mathbb{R}^n$ with the Euclidean norm, and the dual space action is given by

$$x^*(x) = \sum_{i=1}^{n} b_i a_i.$$

Example 3.1.3. The dual of c_0 is $c_0^* = \ell_1$ with norm (2.2) and the dual of ℓ_1 is $\ell_1^* = \ell_\infty$ with the supremum norm (2.3). The dual space action in both cases is given by

$$x^*(x) = \sum_{i=1}^{\infty} b_i a_i. \tag{3.1}$$

Example 3.1.4. The dual of c is $c^* = \ell_1$ with norm (2.2) and the dual space action is given by

$$x^*(x) = b_1 \lim_{i\to\infty} a_i + \sum_{i=2}^{\infty} a_{i-1} b_i. \tag{3.2}$$

Note that the equalities in the previous examples, i.e. $(\mathbb{R}^n)^* = \mathbb{R}$, $c_0^* = \ell_1$, $c^* = \ell_1$ and $\ell_1^* = \ell_\infty$, are signifying that the respective spaces are isometrically isomorphic. The term isometrically isomorphic will be defined in the following section (Section 3.2), and instead of $=$ we will then write $\cong$.

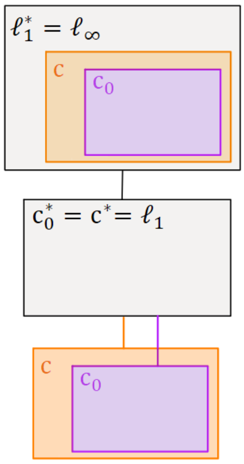

Figure 3.1: Some Banach spaces and their dual spaces.

Figure 3.1 illustrates the connection between the Banach spaces $c_0, c, \ell_1, \ell_\infty$ and their duals. Note that both c_0 and c are subspaces of ℓ_∞ as shown in Examples 2.1.10 and 2.1.11.

3.2 Isomorphisms and the Bounded Inverse Theorem

In this section we will introduce the notion called an isometric isomorphism between two normed spaces X and W. Then we will show that the quotient space $\mathbb{R}^2/Y$ (where Y is as in Example 2.1.4) is isometrically isomorphic to $\mathbb{R}$. Finally, we will introduce the Bounded Inverse Theorem. The theory presented in this section will be needed throughout Chapter 4.

Definition 3.2.1. A map $T : X \to W$ between two normed spaces X and W is called an *isomorphism* if the following conditions are met:

(i) T is bijective,

(ii) T is linear,

(iii) both T and T^{-1} are continuous.

When $T : X \to W$ is an isomorphism, we say that X and W are *isomorphic* and we write $X \simeq W$. If in addition to the above, $\|Tx\| = \|x\|$ for all $x \in X$, then T is called an *isometric isomorphism*. In this case we write $X \cong W$.

Note that, whenever X and W are normed spaces and $T : X \to W$ is a linear surjective and isometric operator, then T is automatically injective and bounded, and its inverse is continuous. Hence, $X \cong W$.

Definition 3.2.2. Let X and W be Banach spaces and let $T : X \to W$ be a bounded linear operator. The *kernel* of T is the set

$$\ker T := \{x \in X : Tx = 0\}.$$

Now let us show that $\mathbb{R}^2/Y$ (where Y is defined as in Example 2.1.4) is isometrically isomorphic to $\mathbb{R}$.

Example 3.2.3. Let $X = \mathbb{R}^2$ with the Euclidean norm (2.1) and recall that $Y := \{x = (x_1, x_2) \in X : x_1 = x_2\}$. The quotient space X/Y is isometrically isomorphic to $\mathbb{R}$. Let us see why.

As shown in Example 2.1.4, $X/Y = \mathbb{R}^2/Y$ is a quotient space where the cosets are lines parallel to Y. To show that X/Y is isometrically isomorphic to $\mathbb{R}$, it suffices to show that there exists a linear surjective and isometric operator $T : X/Y \to \mathbb{R}$.

To find the needed map $T : X/Y \to \mathbb{R}$, let us start with an observation. Consider $x^* = (1, -1)$ from the dual of X. Note that $X^* = X$, so x^* is indeed an element of X^*, and the dual space action is defined in Example 3.1.2. We have

$$x^*(x) = (1, -1)(x_1, x_2) = x_1 - x_2 \quad \text{for all} \;\; x \in X.$$

Since $x^*(x) = 0$ if and only if $x \in Y$, we have that $Y = \ker x^*$. Now, let $[z] \in X/Y$ and $w \in [z]$. By the definition of coset, $w = z + y_1$ for some $y_1 \in Y$, and $[w] = [z]$ due to Proposition 2.1.2. Choose $x \in X$ so that $x^*(x) = 1$. Note that $z \in X$ and $x^*(z) \in \mathbb{R}$. Thus, due to the linearity of x^*, we have

$$\begin{aligned} x^*(w - x^*(z)x) &= x^*(w) - x^*(z)x^*(x) \\ &= x^*(w) - x^*(z) \cdot 1 = x^*(w - z) = x^*(y_1) = 0. \end{aligned} \tag{3.3}$$

Consequently, $w - x^*(z)x \in \ker x^* = Y$. Thus, by Proposition 2.1.2,

$$[w] = [x^*(z)x].$$

Since $[w] = [z]$ and $x^*(z) = r \in \mathbb{R}$, we have $[z] = r[x]$. Thus,

$$\text{for all } [z] \in X/Y \;\; \text{there exists } r_{[z]} \in \mathbb{R} \;\; \text{such that} \;\; [z] = r_{[z]}[x]. \tag{3.4}$$

Let us now define an operator $T : X/Y \to \mathbb{R}$ by

$$T[u] = r_{[u]} \|[x]\|.$$

We see that $T[u]$ is unique for each $[u] \in X/Y$ due to (3.4), and T is well defined. To show that T is an isometric isomorphism we must show that T is linear, surjective, and that $\|T[x]\| = \|[x]\|$.

Let us see that T is linear. Let $\alpha \in \mathbb{R}$ and $[u] \in X/Y$. By (3.4) there exists $r_{[u]} \in \mathbb{R}$ such that $[u] = r_{[u]}[x]$. Furthermore, $\alpha[u] = \alpha r_{[u]}[x]$, which yields

$$T(\alpha[u]) = \alpha r_{[u]}\|[x]\| = \alpha T[u].$$

Now, let $[u], [v] \in X/Y$. By (3.4) there exist $r_{[u]}, r_{[v]} \in \mathbb{R}$ such that $[u] = r_{[u]}[x]$ and $[v] = r_{[v]}[x]$. Furthermore, $[u]+[v] = r_{[u]}[x]+r_{[v]}[x] = (r_{[u]}+r_{[v]})[x]$. Thus,

$$T([u]+[v]) = (r_{[u]} + r_{[v]})\|[x]\| = r_{[u]}\|[x]\| + r_{[v]}\|[x]\|| = T[u] + T[v],$$

and T is linear. Now, observe that for $[u] \in X/Y$ we have

$$\|[u]\| = \|r_{[u]}[x]\| = |r_{[u]}|\|[x]\| = \|T[u]\|,$$

so T an isometry. It remains to show that T is surjective. Since both X/Y and $\mathbb{R}$ are vector spaces and T is linear, we have that $T(X/Y) \subseteq \mathbb{R}$ is a vector space. Since $\mathbb{R}$ does not contain any proper subspaces, we have $T(X/Y) = \mathbb{R}$, i.e., T is surjective.

Example 3.2.3 shows that $X/Y = X/\ker x^* \cong \mathbb{R}$ where $X = \mathbb{R}^2$ and $x^* = (1, -1)$. In Section 4.1 we will see that we have this isometric isomorphism no matter which $x^* \in X^*$ we look at and for any Banach space X.

Let us now present some results that we will need when proving the Bounded inverse Theorem and the final result of this section.

Proposition 3.2.4. *Let $T : X \to W$ be an operator between two normed spaces. The following are equivalent:*

(i) T is continuous,

(ii) $U \subset W$ is open $\Longrightarrow T^{-1}(U)$ is open in X,

(iii) $F \subset W$ is closed $\Longrightarrow T^{-1}(F)$ is closed in X.

Proof. $(i) \implies (ii)$: Assume T is continuous. Let $x_0 \in T^{-1}(U)$. Since U is an open set, there exists $\varepsilon > 0$ such that the ball $B(Tx_0;\varepsilon)$ is contained in U. Since T is continuous we know that there for our choice of ε exists a $\delta > 0$ such that $T\big(B(x_0;\delta)\big) \subset B(Tx_0;\varepsilon) \subset U$. Consequently, $B(x_0;\delta) \subset T^{-1}(U)$. Since $x_0 \in T^{-1}(U)$ was arbitrary, $T^{-1}(U)$ is an open set.

$(ii) \implies (i)$: Assume that $U \subset W$ open $\implies T^{-1}(U)$ open in X. Let $x_0 \in T^{-1}(U)$. Since U is open, let $\varepsilon > 0$ such that $B(Tx_0;\varepsilon) \subset U$. Then, $B(Tx_0;\varepsilon)$ is an open set in U (see Remark 2.1.1). Let us denote this set by $S = B(Tx_0;\varepsilon)$. Thus, by our assumption, the set $T^{-1}(S) \subset T^{-1}(U)$ is open in X, i.e., for all $x \in T^{-1}(S)$ there exists $\delta > 0$ such that $B(x;\delta) \subset T^{-1}(S)$. Consequently, there exists $\delta > 0$ such that

$$T\big(B(x_0;\delta)\big) \subset S = B(Tx_0;\varepsilon),$$

and we are done.

$(ii) \implies (iii)$: Assume $U \subset W$ open $\implies T^{-1}(U)$ open in X. Let $F \subset W$ be a closed set. Thus, F^c is open, and by our assumption $T^{-1}(F^c)$ is open in X. Since inverse images commute with complements, the set $T^{-1}(F^c) = \big(T^{-1}(F)\big)^c$ is open in X. Consequently the set $T^{-1}(F)$ has an open complement, hence it is closed.

$(iii) \implies (ii)$: Assume $F \subset W$ closed $\implies T^{-1}(F)$ closed in X. Let $U \subset W$ be open. Then U^c is closed and $T^{-1}(U^c)$ is closed by assumption. Consequently, $T^{-1}(U^c) = \big(T^{-1}(U)\big)^c$ has an open complement, namely $T^{-1}(U)$. □

Definition 3.2.5. Let X and W be normed spaces. The map $T : X \to W$ is an *open map* if $T(U)$ is open in W whenever U is open in X.

Theorem 3.2.6. *(Open Mapping Theorem) If X and W are Banach spaces and $T : X \to W$ is a bounded linear surjective operator, then T is an open map.*

Proof. See [BK, Proposition 4.25]. □

Corollary 3.2.7. *(Bounded Inverse Theorem) If X and W are Banach spaces and $T : X \to W$ is a bounded linear bijection, then T^{-1} is a bounded linear operator. Consequently, any continuous linear bijection between Banach spaces is an isomorphism.*

Proof. Since T is bijective, the inverse $T^{-1} : W \to X$ is well defined. It can easily be checked that T^{-1} is linear whenever T is. By the Open Mapping Theorem (Theorem 3.2.6), T is an open map. Then, by Proposition 3.2.4, T^{-1} is continuous and thereby bounded. □

We will end this section with a final result.

Proposition 3.2.8. *Let X and W be normed spaces and let $T \in \mathcal{L}(X, W)$. Then $\ker T$ is a closed subspace of X.*

Proof. Let $\alpha \in \mathbb{R}$ and $u, v \in \ker T$, i.e., $Tu = Tv = 0$. Then, since T is linear,

$$T(\alpha u) = \alpha T u = 0 \implies \alpha u \in \ker T, \quad \text{and}$$
$$T(u + v) = Tu + Tv = 0 \implies u + v \in \ker T,$$

so $\ker T$ is a vector space. Since T is continuous and $\{0\}$ is closed in W, then $\ker T$ is closed in X by Proposition 3.2.4. □

Remark 3.2.1. Let X and W be normed spaces and let $T \in \mathcal{L}(X, W)$. Since $\ker T$ is a closed subspace of X by the previous proposition, $X/\ker T$ is a normed quotient space (Proposition 2.1.21). Furthermore, $X/\ker T$ is complete whenever X is complete due to Theorem 2.2.9.

3.3 Adjoints and the Hahn-Banach Theorem

In this section we will define the adjoint T^* of an operator T, and we will show that it is bounded and linear. First we will present the Hahn-Banach Theorem as this is needed to prove that $\|T^*\| = \|T\|$, which we do in the last result of this section. The adjoint is needed when establishing an isometric isomorphism between the dual of a quotient space X/Y and the set $Y^{\perp}$ (Definition 4.3.1), which we will do in Section 4.3.

Theorem 3.3.1. *(Hahn-Banach Theorem) Let X be a normed space and let $Y \subset X$ be a subspace. If $f \in Y^*$, then there exist a functional $\hat{f} \in X^*$ such that $\|\hat{f}\| = \|f\|$ and $\hat{f}(x) = f(x)$ for all $x \in Y$.*

Proof. See [BK, Theorem 3.9]. □

Proposition 3.3.2. *Let X be a normed space. If $x \in X$, then there exists $u^* \in X^*$ such that $\|u^*\| = 1$ and $u^*(x) = \|x\|$, and*

$$\|x\| = \sup_{x^* \in B_{X^*}} |x^*(x)|.$$

Proof. Let $x \in X$. We define a subspace Y_x to be

$$Y_x := \{rx : r \in \mathbb{R}\}.$$

Now we define a function $f : Y_x \to \mathbb{R}$ by $f(rx) = r\|x\|$ and let $\alpha \in \mathbb{R}$ and $r_1 x,\ r_2 x \in Y_x$. We have

$$f(\alpha(r_1 x)) = f((\alpha r_1)x) = (\alpha r_1)\|x\| = \alpha(r_1\|x\|) = \alpha f(r_1 x), \text{ and}$$
$$f(r_1 x + r_2 x) = f((r_1+r_2)x) = (r_1+r_2)\|x\| = r_1\|x\| + r_2\|x\| = f(r_1 x) + f(r_2 x),$$

and f is linear. Furthermore, by the definition of operator norm in (2.5), we have

$$\|f\| = \sup_{rx \in Y_x} \frac{|f(rx)|}{\|rx\|} = \sup_{rx \in Y_x} \frac{|r|\|x\|}{|r|\|x\|} = 1,$$

and f is bounded. Consequently f is an element of the dual space Y_x^*. Thus, by Theorem 3.3.1 there exists a functional $\hat{f} \in X^*$ with $\|\hat{f}\| = \|f\| = 1$ and $\hat{f}(u) = f(u)$ for all $u \in Y_x$. We see that $\hat{f}(x) = f(1 \cdot x) = \|x\|$. It remains to show the supremum-equality. Note that $\|\hat{f}\| = 1$, so $\hat{f} \in B_{X^*}$. Thus,

$$\|x\| = \hat{f}(x) \leq \sup_{x^* \in B_{X^*}} |x^*(x)|.$$

To show the other inequality, let $x^* \in B_{X^*}$ be arbitrary. We have

$$|x^*(x)| \leq \|x^*\|\|x\| \leq 1 \cdot \|x\| \quad \text{for all} \quad x^* \in B_{X^*}.$$

Taking the supremum over $x^* \in B_{X^*}$ on both sides gives the desired result. □

Definition 3.3.3. Let X and W be normed spaces and let $T \in \mathcal{L}(X, W)$. The *adjoint* of T is the map $T^* : W^* \to X^*$ defined by

$$(T^* w^*)(x) = w^*(Tx), \quad x \in X, \quad w^* \in W^*.$$

The relation between T and T^* is illustrated in Figure 3.2 where the purple lines illustrates how $(T^* w^*)(x) = w^*(Tx)$.

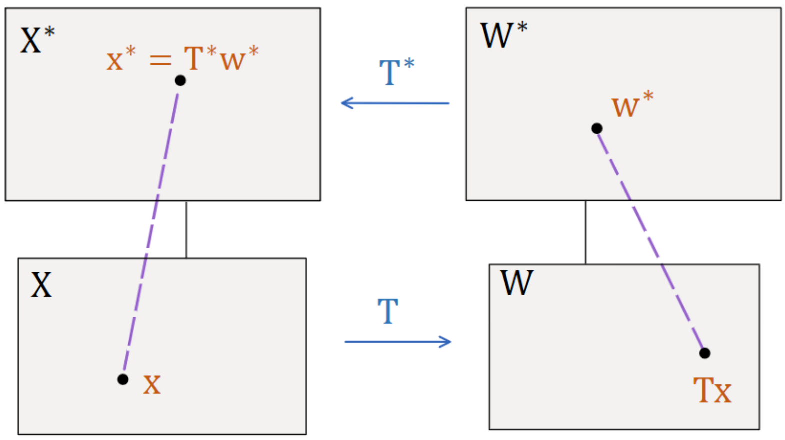

Figure 3.2: The adjoint of T defined by $(T^* w^*)(x) = w^*(Tx)$.

Proposition 3.3.4. *Let X and W be normed spaces and let $T \in \mathcal{L}(X, W)$. Then T^* is a bounded linear operator with $\|T^*\| = \|T\|$.*

Proof. First, let us see that T^* is well defined. Let $w^* \in W^*$. We can easily see that $T^* w^*$ is linear because T and w^* are linear. Furthermore, we have

$$\begin{aligned} \|T^* w^*\| = \sup_{x \in B_X} \|(T^* w^*)(x)\| &= \sup_{x \in B_X} |w^*(Tx)| \\ &\leq \sup_{x \in B_X} \|w^*\| \|(Tx)\| = \|w^*\| \|T\|, \end{aligned}$$

and $T^* w^*$ is bounded. It follows that $T^* w^* \in X^*$ and T^* is well defined. Now let us show that T^* is linear and bounded. Let $\alpha, \beta \in \mathbb{R}$ and $w_1^*, w_2^* \in W^*$. Then,

$$\begin{aligned} (T^*(\alpha w_1^* + \beta w_2^*))(x) &= (\alpha w_1^* + \beta w_2^*)(Tx) \\ &= (\alpha w_1^*)(Tx) + (\beta w_2^*)(Tx) \\ &= \alpha w_1^*(Tx) + \beta w_2^*(Tx) = \alpha (T^* w_1^*)(x) + \beta (T^* w_2^*)(x), \end{aligned}$$

which shows that T^* is linear. We have already established that $\|T^*w^*\| \leq \|w^*\|\|T\|$. Taking the supremum on both sides yields

$$\|T^*\| = \sup_{w^* \in B_{W^*}} \|T^*w^*\| \leq \sup_{w^* \in B_{W^*}} \|w^*\|\|T\| = \|T\|,$$

and T^* is bounded with $\|T^*\| \leq \|T\|$. It remains to show that $\|T^*\| \geq \|T\|$. Since $\|T\| = \sup_{x \in B_X} \|Tx\|$, there will for all $\varepsilon > 0$ exist $x \in B_X$ such that $\|T\| - \varepsilon < \|Tx\|$. By Proposition 3.3.2 there exists $w^* \in W^*$ with $\|w^*\| = 1$ such that $w^*(Tx) = \|Tx\|$. Combined with the fact that both $T^*w^* \in X^*$ and T^* are bounded operators, we get

$$\begin{aligned}
\|T\| - \varepsilon < \|Tx\| = w^*(Tx) &= (T^*w^*)(x) \\
&\leq |(T^*w^*)(x)| \\
&\leq \|T^*w^*\|\|x\| \leq \|T^*w^*\| \leq \|T^*\|\|w^*\| = \|T^*\|.
\end{aligned}$$

Thus, $\|T\| < \|T^*\| + \varepsilon$ for all $\varepsilon > 0$, and we have the desired result. □

Remark 3.3.1. It can be shown that if $T \in \mathcal{L}(X, W)$ is an isometric isomorphism, then $T^* \in \mathcal{L}(W^*, X^*)$ is an isometric isomorphism. In fact, the map $T \mapsto T^*$ is an isometric isomorphism from $\mathcal{L}(X, W)$ to $\mathcal{L}(W^*, X^*)$.

3.4 Measure Theory and the $ba(\mathcal{A})$-Space

This section will provide definitions and results that will be needed when identifying the dual of ℓ_∞ in Section 4.4. In Section 4.4 we will realize ℓ_∞ as the uniform closure of the linear span of the characteristic functions on the σ-algebra $2^{\mathbb{N}}$ presented in Example 3.4.3. Furthermore, the dual of ℓ_∞ will be realized as the set of some kind of "primitive" measures and the dual space action will be integration against these measures. Thus, in this section we will present a brief theory of such measures and the integration of uniform limits of simple functions on a σ-algebra. In fact, we will never use that the σ-algebra respects more than finite unions, and thus we will work on a more primitive family of sets.

3.4.1 Algebras and Finitely Additive Signed Measures

In this subsection we will define algebra, finitely additive measures and the total variation of such measures. Furthermore we will introduce the $ba(\mathcal{A})$-space and the Jordan decomposition of $\lambda \in ba(\mathcal{A})$.

Definition 3.4.1. Let X be a nonempty set. A collection $\mathcal{A}$ of subsets of X is called an *algebra* whenever the following conditions are met:

(i) $\emptyset \in \mathcal{A}$,

(ii) $E \in \mathcal{A} \Longrightarrow E^c \in \mathcal{A}$,

(iii) $\{E_i\}_{i=1}^n \subset \mathcal{A} \Longrightarrow \bigcup\limits_{i=1}^n E_i \in \mathcal{A}$.

This definition is the same as for σ-algebras except that for algebras we only assume closedness under finite unions. Note that $X \in \mathcal{A}$ because $X^c = \emptyset \in \mathcal{A}$.

Example 3.4.2. Let $X = \mathbb{N}$ and let $\mathcal{A} := \{E \subseteq X : \text{either } E \text{ or } E^c \text{ is finite}\}$. Then, $\mathcal{A}$ is an algebra, but not a σ-algebra.

Example 3.4.3. Let X be a nonempty set. The power set of X is the collection of all possible subsets of X, and is denoted 2^X. The power set is a σ-algebra.

Definition 3.4.4. Let $\mathcal{A}$ be an algebra of subsets of a set X. A *finitely additive signed measure* on $\mathcal{A}$ is a function $\lambda : \mathcal{A} \to \overline{\mathbb{R}}$ where

(i) $\lambda(\emptyset) = 0$,

(ii) Only $+\infty$ or $-\infty$ is included in the range (since $\infty - \infty$ is not defined),

(iii) Given a disjoint family $\{E_i\}_{i=1}^n \subset \mathcal{A}$, then $\lambda(\bigcup\limits_{i=1}^n E_i) = \sum\limits_{i=1}^n \lambda(E_i)$.

We say that λ is a *positive* finitely additive measure if λ only assumes non-negative values.

Definition 3.4.5. Let $\mathcal{A}$ be an algebra of subsets of a set X and let $\lambda : \mathcal{A} \to \overline{\mathbb{R}}$ be a finitely additive signed measure. For every $E \in \mathcal{A}$, the *total variation* of λ on E is defined as

$$|\lambda|(E) := \sup_{\pi} \sum_{i=1}^{n} |\lambda(E_i)|,$$

where $\pi = \{E_1, E_2, ..., E_n\}$ is a finite partition of E.

Remark 3.4.1. For a positive finitely additive measure λ, the total variation of λ on E is equal the measure of E, i.e.,

$$|\lambda|(E) = \sup_{\pi} \sum_{i=1}^{n} |\lambda(E_i)| = \sup_{\pi} \sum_{i=1}^{n} \lambda(E_i) = \lambda(E).$$

Now, let $\mathcal{A}$ be an algebra of subsets of a set X and let $A, B \in \mathcal{A}$ such that $A \subset B$. If $\{E_1, ..., E_n\}$ is a partition of A, then $\{E_1, ..., E_n, B \backslash A\}$ is a partition of B. Furthermore,

$$\sum_{i=1}^{n} |\lambda(E_i)| + |\lambda(B \backslash A)| \leq |\lambda|(A) + |\lambda(B \backslash A)| \leq |\lambda|(B),$$

and we see that $|\lambda|$ has maximum value $|\lambda|(X)$. When $|\lambda|(X)$ is finite, i.e., when $|\lambda|(X) < \infty$, we say that λ *is of bounded variation.*

Proposition 3.4.6. *Let λ be a finitely additive signed measure on the algebra $\mathcal{A}$. If λ is of bounded variation, then $|\lambda|$ is a positive finitely additive measure on $\mathcal{A}$.*

Proof. First of all, since $\emptyset$ is the only partition of $\emptyset$, we have $|\lambda|(\emptyset) = |\lambda(\emptyset)| = 0$. Furthermore, it is clear by the definition of $|\lambda|$ that $|\lambda|$ is a positive set function. For $|\lambda|$ to be a positive finitely additive measure, it remains to show that $|\lambda|(A \cup B) = |\lambda|(A) + |\lambda|(B)$ whenever $A, B \in \mathcal{A}$ are disjoint sets. To this end, let $\varepsilon > 0$ and let $\pi = \{E_1, E_2, ..., E_n\}$ be a partition of $A \cup B$ fulfilling

$$|\lambda|(A \cup B) - \varepsilon < \sum_{i=1}^{n} |\lambda(E_i)|. \tag{3.5}$$

Since A and B are disjoint, we can split each $E_i \in \pi$ into two disjoint sets $E_{i,A}$

and $E_{i,B}$. This yields

$$\begin{aligned}\sum_{i=1}^{n} |\lambda(E_i)| &= \sum_{i=1}^{n} |\lambda(E_{i,A} \cup E_{i,B})| \\ &= \sum_{i=1}^{n} |\lambda(E_{i,A}) + \lambda(E_{i,B})| \\ &\leq \sum_{i=1}^{n} |\lambda(E_{i,A})| + \sum_{i=1}^{n} |\lambda(E_{i,B})| \\ &\leq \sup_{\pi \text{of} A} \sum_{i=1}^{n} |\lambda(E_i)| + \sup_{\pi \text{of} B} \sum_{i=1}^{n} |\lambda(E_i)| \\ &= |\lambda|(A) + |\lambda|(B).\end{aligned}$$

Consequently, (3.5) yields $|\lambda|(A \cup B) \leq |\lambda|(A) + |\lambda|(B)$. It remains to show the other inequality. Assume $|\lambda|(A \cup B) < |\lambda|(A) + |\lambda(B)|$. Then, we can choose partitions π_A and π_B such that

$$|\lambda|(A \cup B) < \sum_{E \in \pi_A} |\lambda(E)| + \sum_{E \in \pi_B} |\lambda(E)|. \tag{3.6}$$

We see that $\pi = \pi_A \cup \pi_B$ is a partition of $A \cup B$, and thus (3.6) yields

$$|\lambda|(A \cup B) < \sum_{E \in \pi_A \cup \pi_B} |\lambda(E)|,$$

which is a contradiction. It follows that $|\lambda|(A \cup B) = |\lambda|(A) + |\lambda|(B)$, and $|\lambda|$ is a positive finitely additive measure. □

In the following we will only work on cases where λ is a finitely additive signed measures of bounded variation. Let us now introduce the $ba(\mathcal{A})$-space.

Definition 3.4.7. Let $\mathcal{A}$ be an algebra of subsets of a set X. The symbol $ba(\mathcal{A})$ denotes the vector space of all finitely additive signed measures $\lambda : \mathcal{A} \to \mathbb{R}$ of bounded variation. Scalar multiplication and vector addition is given by

$$\begin{aligned}&(\alpha\lambda)(E) = \alpha\lambda(E) \quad \text{for all} \quad \alpha \in \mathbb{R}, \lambda \in ba(\mathcal{A}), \\ &(\lambda + \eta)(E) = \lambda(E) + \eta(E) \quad \text{for all} \quad \lambda, \eta \in ba(\mathcal{A}).\end{aligned}$$

It is straightforward to prove that the two operations in Definition 3.4.7 are well defined. The following proposition shows that the total variation is a norm on $ba(\mathcal{A})$.

Proposition 3.4.8. *Let $\mathcal{A}$ be an algebra of subsets of a set X. The total variation $|\lambda|(X)$ defines a norm on $ba(\mathcal{A})$, i.e., $\|\lambda\| = |\lambda|(X)$ for all $\lambda \in ba(\mathcal{A})$.*

Proof. To see that $|\lambda|(X)$ is a norm on $ba(\mathcal{A})$, note that for all $\lambda \in ba(\mathcal{A})$ we have $|\lambda|(X) < \infty$. Thus, $\|\cdot\| : ba(\mathcal{A}) \to [0, \infty)$. Furthermore, it can easily be shown that the three criteria of Definition 2.1.5 hold partly due to the properties of supremum and partly due to the triangle inequality. $\square$

In Section 4.4 we will see that $ba(\mathcal{A})$ is a dual space and thereby a Banach space (Corollary 4.4.7).

Let us show that each finitely additive signed measure $\lambda \in ba(\mathcal{A})$ can be split into the difference of two positive finitely additive measures.

Proposition 3.4.9. *Let $\mathcal{A}$ be an algebra of subsets of a set X and let $\lambda \in ba(\mathcal{A})$. Define two set functions by*

$$\lambda^+ := \frac{1}{2}(|\lambda| + \lambda) \quad \textit{and} \quad \lambda^- := \frac{1}{2}(|\lambda| - \lambda).$$

Then, λ^+ and λ^- are both positive finitely additive measures of bounded variation.

Proof. Due to Proposition 3.4.6, $|\lambda|$ is a positive finitely additive measure. Thus, due to Remark 3.4.1, $|\lambda|$ is of bounded variation and $|\lambda| \in ba(\mathcal{A})$. Furthermore, since $ba(\mathcal{A})$ is a vector space, both λ^+ and λ^- are in $ba(\mathcal{A})$. It remains to show that λ^+ and λ^- only assume non-negative values. By the definition of $|\lambda|$ (Definition 3.4.5) we have that $|\lambda(E)| \leq |\lambda|(E) < \infty$ for all $E \in \mathcal{A}$ whenever $\lambda \in ba(\mathcal{A})$. Thus, both $\lambda^+(E)$ and $\lambda^-(E)$ are greater or equal to $\frac{1}{2}(|\lambda|(E) - |\lambda(E)|) \geq 0$, for all $E \in \mathcal{A}$. $\square$

Note that the definition of λ^+ and λ^- from the previous proposition yields

$$\lambda = \lambda^+ - \lambda^- \quad \text{and} \quad |\lambda| = \lambda^+ + \lambda^-, \tag{3.7}$$

for all $\lambda \in ba(\mathcal{A})$. The splitting $\lambda = \lambda^+ - \lambda^-$ is called the *Jordan decomposition* of λ. We refer to [BR, page 53] for more information on when a finitely additive signed measure admits a Jordan decomposition.

3.4.2 Integration Against $ba(\mathcal{A})$-Measures

In this subsection we will define characteristic function, simple function and the integral of such functions with respect to $\lambda \in ba(\mathcal{A})$. Furthermore, we will define the integral of a function $f : X \to \mathbb{R}$ when f is the uniform limit of a sequence of simple functions on X.

Definition 3.4.10. Let $\mathcal{A}$ be an algebra of subsets of a set X and let $E \in \mathcal{A}$. The *characteristic function* of E is the function $\mathbf{1}_E : X \to \mathbb{R}$ defined by

$$\mathbf{1}_E(x) = \begin{cases} 1, & x \in E \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3.4.11. Let $\mathcal{A}$ be an algebra of subsets of a set X. A function $h : X \to \mathbb{R}$ is called *simple* if it assumes only finitely many values. Let $c_1, ..., c_n$ be distinct nonzero values of h. Then h is of the form

$$h(x) = \sum_{i=1}^{n} c_i \mathbf{1}_{E_i}(x),$$

where $E_i = h^{-1}(c_i) \in \mathcal{A}$.

Definition 3.4.12. Let $\mathcal{A}$ be an algebra of subsets of a set X and let $\lambda \in ba(\mathcal{A})$. Let $h : X \to \mathbb{R}$ be a simple function defined by $h(x) = \sum\limits_{i=1}^{n} c_i \mathbf{1}_{E_i}(x)$ where $\{E_i\}_{i=1}^{n} \subset \mathcal{A}$. The integral of h over $E \in \mathcal{A}$ with respect to λ is

$$\int_E h \, d\lambda = \sum_{i=1}^{k} c_i \lambda(E \cap E_i). \tag{3.8}$$

Definition 3.4.13. Let $\mathcal{A}$ be an algebra of subsets of a set X and let $\lambda \in ba(\mathcal{A})$. If $f : X \to \mathbb{R}$ is the uniform limit of a sequence of simple functions $\{h_n\}$, then the integral of f over $E \in \mathcal{A}$ with respect to λ is defined by

$$\int_E f \, d\lambda = \lim_{n\to\infty} \int_E h_n \, d\lambda. \tag{3.9}$$

That f is the uniform limit of $\{h_n\}$ means that $\|f - h_n\|_\infty \to 0$.

It can be shown that the integral of f is well defined, i.e., that the limit in (3.9) is equal for all sequences $\{h_n\}$ converging uniformly to f.

Before we end this section, let us present some elementary results that hold when we are integrating with respect to $ba(\mathcal{A})$-measures.

Let $\mathcal{A}$ be an algebra of subsets of a set X and let $g, f : X \to \mathbb{R}$ be uniform limits of sequences of simple functions. Consider a positive finitely additive measure $\lambda \in ba(\mathcal{A})$. Whenever $g \leq f$, we have

$$\int g \; d\lambda \leq \int f d\lambda. \tag{3.10}$$

Now, let $\lambda \in ba(\mathcal{A})$. If $A, B \in \mathcal{A}$ are disjoint, then

$$\int_{A \cup B} f \; d\lambda = \int_A f \; d\lambda + \int_B f d\lambda. \tag{3.11}$$

Furthermore, for $\alpha, \beta \in \mathbb{R}$ and $E \in \mathcal{A}$, we have

$$\int_E (\alpha f + \beta g) \; d\lambda = \alpha \int_E f \; d\lambda + \beta \int_E g \; d\lambda. \tag{3.12}$$

(3.11) and (3.12) are referred to as additivity and linearity of the integral, respectively.

Chapter 4

Towards the Dual of ℓ_∞/c_0

The goal of this chapter is to identify the dual of a quotient space X/Y. First we will consider the case when Y is the kernel of a functional $x^* \in X^*$. Then we will discuss the case when Y is the kernel of an operator $T : X \to W$ between Banach spaces, and finally the most general case where Y is any closed subspace of X. In the final section we will establish an isometric isomorphism between $ba(2^{\mathbb{N}})$ and $(\ell_\infty)^*$, and this will be important for finding the dual of ℓ_∞/c_0.

4.1 The Quotient Space $X/\ker x^*$

The following proposition is more general than the result presented in Example 3.2.3 saying that $\mathbb{R}^2/Y$ is isometrically isomorphic to $\mathbb{R}$ when $Y = \ker x^*$ and $x^* = (1, -1)$.

Proposition 4.1.1. *Let X be a Banach space and let $x^* \in X^*$ ($x^* \neq 0$). Then the quotient space $X/\ker x^*$ is isometrically isomorphic to $\mathbb{R}$.*

Proof. To show that $X/\ker x^*$ is isometrically isomorphic to $\mathbb{R}$, it suffices to show that there exists $T : X/\ker x^* \to \mathbb{R}$ such that T is a linear surjective isometric operator. To this end, let $x \in X$ such that $x^*(x) = 1$, let $[z] \in X/\ker x^*$ and $w \in [z]$. By the definition of coset there exists $y_1 \in \ker x^*$ such that $w - z = y_1$. Like in (3.3) in Example 3.2.3, it can be shown that

$$x^*(w - x^*(z)x) = x^*(y_1) = 0,$$

which yields $w - x^*(z)x \in \ker x^*$. Thus, by Proposition 2.1.2, we have $[w] = [x^*(z)x]$. This means that for all $[z] \in X/\ker x^*$ there exists an $r_{[z]} = x^*(z) \in \mathbb{R}$ such that

$$[z] = [w] = x^*(z)[x] = r_{[z]}[x].$$

If we let $T : X/\ker x^* \to \mathbb{R}$ be defined by $T[z] = r_{[z]}\|[x]\|$, then T is a linear surjective isometric operator (proof identical to the proof in Example 3.2.3). Hence, $X/\ker x^* \cong \mathbb{R}$. □

The previous proposition states that $(X/\ker x^*) \cong \mathbb{R}$. Thus, the dual space $(X/\ker x^*)^*$ is isometrically isomorphic to $\mathbb{R}^* \cong \mathbb{R}$ due to Remark 3.3.1.

4.2 An Investigation of $X/\ker T$

In this section we will discuss the quotient space $X/\ker T$ where $T : X \to W$ is a bounded linear operator between the Banach spaces X and W. We will show that when T is surjective, then $X/\ker T$ is isomorphic to W. This extends Proposition 4.1.1 which shows that the latter holds for $W = \mathbb{R}$. Indeed, in this case every $T \in X^*$ ($T \neq 0$) is surjective. To establish the above mentioned isomorphism, we will start by defining the quotient map.

Definition 4.2.1. Let X be a Banach space with a closed subspace Y. The *quotient map* $Q : X \to X/Y$ is defined by $Q(x) = [x]$.

Proposition 4.2.2. *Let X be a Banach space with a closed subspace Y. The quotient map $Q : X \to X/Y$ has the property that it maps the open unit ball U_X in X onto the open unit ball $U_{X/Y}$ in X/Y, i.e., $Q(U_X) = U_{X/Y}$.*

Proof. Let $[x] \in U_{X/Y}$ and let $\{x + y_n\} \subset [x]$ such that

$$\|x + y_n\| \to \|[x]\| \quad \text{as} \quad n \to \infty.$$

(Note that $[x+y_n] = [x]$ for all n.) For all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|x + y_N\| < \|[x]\| + \varepsilon$. If we let ε be half of the distance between $\|[x]\|$ and one, i.e., $\varepsilon = \frac{1}{2}(1 - \|[x]\|)$, we have

$$\|x + y_N\| < \|[x]\| + \varepsilon < \|[x]\| + 2\varepsilon = 1.$$

Thus, for all $[x] \in U_{X/Y}$ there exists an element $(x + y_N) \in U_X$ such that $Q(x + y_N) = [x]$ which means that $U_{X/Y} \subseteq Q(U_X)$. Now, let $x \in U_X$ be arbitrary with $Q(x) = [x]$. We see that

$$\|[x]\| = \inf_{y \in Y} \|x + y\| \leq \|x + 0\| < 1,$$

which means that every point in U_X is mapped into $U_{X/Y}$, i.e., $Q(U_X) \subseteq U_{X/Y}$. We have the desired result. □

Proposition 4.2.3. *The quotient map from Definition 4.2.1 is linear and bounded with* $\|Q\| = 1$.

Proof. Let $\alpha \in \mathbb{R}$ and $x_1, x_2 \in X$. By Proposition 2.1.3 X/Y is a vector space, so we have

$$Q(\alpha x_1) = [\alpha x_1] = \alpha[x_1] = \alpha Q(x_1), \quad \text{and}$$

$$Q(x_1 + x_2) = [x_1 + x_2] = [x_1] + [x_2] = Q(x_1) + Q(x_2),$$

and Q is linear. Due to Proposition 4.2.2, we have the equality

$$\|Q\| = \sup_{x \in B_X} \|Q(x)\| = \sup_{x \in U_X} \|Q(x)\| = \sup_{[x] \in U_{X/Y}} \|[x]\| = 1. \tag{4.1}$$

□

Let X and W be Banach spaces and let $T \in \mathcal{L}(X, W)$. Then the quotient space $X/\ker T$ is complete due to Remark 3.2.1. Now let $\tilde{T}$ denote the map $\tilde{T} : X/\ker T \to W$ defined by $\tilde{T}[x] := Tx$. Then $\tilde{T}$ is well defined because T is well defined and linear (the linearity of T ensures that $\tilde{T}[x]$ is unique for each $[x]$). Furthermore, $\tilde{T}$ is surjective whenever T is.

Figure 4.1 illustrates the relation between the quotient map Q and the two operators $T \in \mathcal{L}(X, W)$ and $\tilde{T} : X/\ker T \to W$.

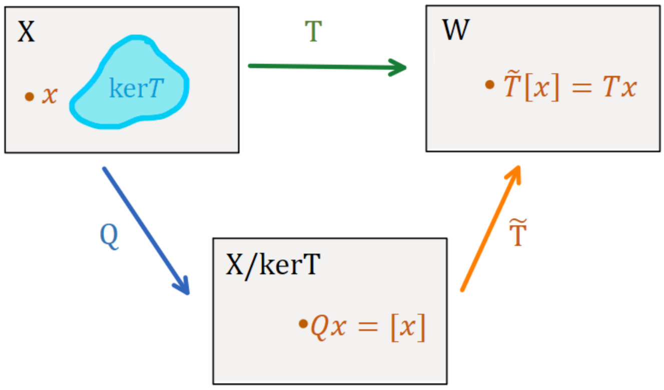

Figure 4.1: The relation between the three maps Q, T and $\tilde{T}$.

Proposition 4.2.4. *Let X and W be Banach spaces and let $T \in \mathcal{L}(X, W)$. Then $\tilde{T} : X/\ker T \to W$ defined by $\tilde{T}[x] = Tx$ is a bounded linear injective operator with $\|\tilde{T}\| = \|T\|$.*

Proof. Let $\alpha \in \mathbb{R}$ and $[x_1], [x_2] \in X/\ker T$. Then

$$\tilde{T}(\alpha[x]) = \tilde{T}[\alpha x] = T(\alpha x) = \alpha Tx = \alpha \tilde{T}[x], \quad \text{and}$$
$$\tilde{T}([x_1] + [x_2]) = \tilde{T}[x_1 + x_2] = T(x_1 + x_2) = Tx_1 + Tx_2 = \tilde{T}[x_1] + \tilde{T}[x_2],$$

and T is linear. To see that $\tilde{T}$ is injective, assume $\tilde{T}[x_1] = \tilde{T}[x_2]$. Then,

$$0 = \tilde{T}[x_1] - \tilde{T}[x_2] = \quad Tx_1 - Tx_2 = \quad T(x_1 - x_2),$$

which means that $(x_1 - x_2) \in \ker T$. It follows that $[x_1] = [x_2]$ due to Proposition 2.1.2. It only remains to show the equality of the operator norms. Since $Q(U_X) = U_{(X/\ker T)}$ due to Proposition 4.2.2, we have

$$\|\tilde{T}\| = \sup_{[x] \in U_{X/\ker T}} \|\tilde{T}[x]\| = \sup_{x \in U_X} \|\tilde{T}(Qx)\| = \sup_{x \in U_X} \|Tx\| = \|T\|.$$

□

Corollary 4.2.5. *Let X and W be Banach spaces. Whenever $T \in \mathcal{L}(X, W)$ is surjective, $\tilde{T}$ is an isomorphism between $X/\ker T$ and W.*

Proof. Recall that $X/\ker T$ is complete by Remark 3.2.1. We know from Proposition 4.2.4 that $\tilde{T}$ is a bounded linear injective operator. Furthermore, since T is surjective, $\tilde{T}$ is surjective. It follows by the Bounded Inverse Theorem (Corollary 3.2.7) that $\tilde{T}$ is an isomorphism. $\square$

Remark 4.2.1. It can be shown that $\tilde{T}$ is an isometric isomorphism whenever $T(B_X)$ is dense in B_W, but we will not go into detail on this. Therefore, whenever we have a bounded linear surjective operator T between Banach spaces X and W where $T(B_X)$ is dense in B_W, then $X/\ker T \cong W$.

Example 4.2.6. The quotient space c/c_0 is isometrically isomorphic to $\mathbb{R}$. To see this, let us define an operator $T : c \to \mathbb{R}$ by $T(\{a_i\}) = \lim_{i\to\infty} a_i$. It is easy to see that T is well defined, linear and surjective. T is also bounded:

$$\|T\| = \sup_{a\in B_c} \|Ta\| = \sup_{a\in B_c} |\lim_{i\to\infty} a_i| = 1.$$

Furthermore, we see that $\ker T = c_0$, and thus, due to Corollary 4.2.5 the quotient space c/c_0 is isomorphic to $\mathbb{R}$ through $\tilde{T}$. Since $T(B_c)$ is dense in $B_{\mathbb{R}}$, it follows from Remark 4.2.1 that $\tilde{T}$ is isometric.

The previous example shows that $c/c_0 \cong \mathbb{R}$ through $\tilde{T}$. By Remark 3.3.1, the dual $(c/c_0)^*$ is isometrically isomorphic to $\mathbb{R}^* \cong \mathbb{R}$ through the adjoint $(\tilde{T})^*$. (Note that T is a bounded linear functional on c, so $c/c_0 \cong \mathbb{R}$ through Proposition 4.1.1.)

4.3 The Dual of X/Y

In this section we will identify $(X/Y)^*$ when X is a Banach space and $Y \subset X$ is a closed subspace. We will do this by defining the annihilator of Y in X^* and show that the dual $(X/Y)^*$ is isometrically isomorphic to this set.

Definition 4.3.1. Let X be a normed space and Y a subset of X. The *annihilator* of Y is the set

$$Y^{\perp} := \{x^* \in X^* : x^*(x) = 0 \quad \forall x \in Y \} \subseteq X^*.$$

It can be shown that $Y^{\perp}$ is a closed subspace of X^* and thus a Banach space by Proposition 2.2.3.

Now, let X be a Banach space and $Y \subset X$ a closed subspace. Let us consider a functional $x^* \in Y^\perp$. Let $[x] \in X/Y$ and let $u \in [x]$, i.e., $u = x + y$ for some $y \in Y$. Observe that

$$x^*(u) = x^*(x+y) = x^*(x) + x^*(y) = x^*(x) \quad \text{for all} \quad u \in [x]. \tag{4.2}$$

Thus, it seems like each $x^* \in Y^\perp$ can be associated with a functional $\varphi \in (X/Y)^*$ through $\varphi[x] = x^*(x)$ as φ is uniquely defined for each $[x]$ through (4.2). Let us look closer at this connection in the following theorem.

Theorem 4.3.2. *Let X be a Banach space and Y a closed subspace. The dual of X/Y is isometrically isomorphic to the annihilator of Y in X^*, i.e., $(X/Y)^* \cong Y^\perp$.*

Proof. Let $Q : X \to X/Y$ be the quotient map from Definition 4.2.1. We will show that the adjoint $Q^* : (X/Y)^* \to X^*$ is an isometric isomorphism onto $Y^\perp$. To this end, recall that the quotient map is linear and bounded due to Proposition 4.2.3, i.e., $Q \in \mathcal{L}(X, X/Y)$. Thus, due to Proposition 3.3.4, the adjoint of Q is a bounded linear operator with $\|Q^*\| = \|Q\|$ where $Q^* : (X/Y)^* \to X^*$ is defined by

$$(Q^*\varphi)(x) = \varphi(Qx) = \varphi[x],$$

for all $x \in X$. Furthermore, due to Proposition 4.2.2 we see that

$$\|Q^*\varphi\|_{X^*} = \sup_{x \in U_X} |(Q^*\varphi)(x)| = \sup_{[x] \in U_{(X/Y)}} |\varphi[x]| = \|\varphi\|_{(X/Y)^*}, \tag{4.3}$$

and thus Q^* is isometric and thereby injective. For $(X/Y)^*$ to be isometrically isomorphic to $Y^\perp$, it remains to show that Q^* is onto $Y^\perp$. Let $y \in Y$. Recall that $[0] = Y$ is the zero element of X/Y, and that all $\varphi \in (X/Y)^*$ are linear. We have

$$(Q^*\varphi)(y) = \varphi[y] = \varphi(Y) = 0 \quad \text{for all} \quad y \in Y, \ \varphi \in (X/Y)^*,$$

which means that $Q^*\big((X/Y)^*\big) \subseteq Y^\perp$. Now, let $x^* \in Y^\perp$. Let $\varphi : X/Y \to \mathbb{R}$ be defined by $\varphi[x] = x^*(x)$. Due to (4.2), φ is well defined. Furthermore, φ is a bounded linear functional on X/Y since x^* is bounded and linear. It follows that Q^* is onto $Y^\perp$ and $(X/Y)^* \cong Y^\perp$. This relation is illustrated in Figure 4.2. $\square$

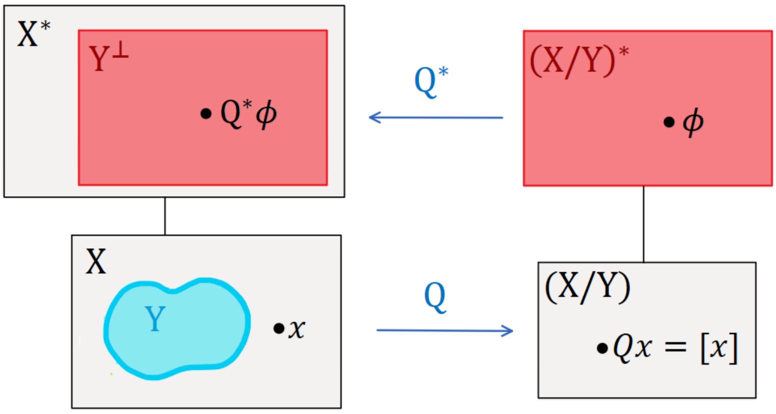

Figure 4.2: $(X/Y)^*$ isometrically isomorphic to $Y^\perp$ through Q^*.

Remark 4.3.1. Due to Theorem 4.3.2 it follows that the dual of c/c_0 is $c_0^\perp \subset c^* \cong \ell_1$, and the dual of ℓ_∞/c_0 is $c_0^\perp \subset (\ell_\infty)^*$.

4.4 The Dual of ℓ_∞

In the previous section we showed that $(\ell_\infty/c_0)^* \cong c_0^\perp \subset (\ell_\infty)^*$ (Remark 4.3.1). Thus, to understand the dual of ℓ_∞/c_0 we must understand the dual of ℓ_∞, and this is the goal of this section. We will start by defining generating subset of a normed space. Then we will show that all elements $x \in \ell_\infty$ are either characteristic functions, simple functions or uniform limits of sequences of simple functions. Finally we will use measure theory from Section 3.4 to establish an isometric isomorphism between $(\ell_\infty)^*$ and $ba(2^{\mathbb{N}})$, and we will see that the dual space action on ℓ_∞ is given by an integral with respect to $\lambda \in ba(2^{\mathbb{N}})$.

Definition 4.4.1. Let X be a normed space. A subset $M \subset X$ is *dense* in X if $\overline{M} = X$ where $\overline{M}$ is the closure of M in X.

Definition 4.4.2. Let X be a normed space. A subset $A \subset X$ is *generating* if the linear span of A is dense in X, i.e., if $\overline{M} = X$ where

$$M := \operatorname{span}(A) = \left\{ \sum_{i=1}^{n} c_i u_i : c_i \in \mathbb{R}, u_i \in A \right\}.$$

Note that we can reduce any generating set A to be a subset $\hat{A} \subset S_X$ by setting $\hat{A} = \{x/\|x\| : x \in A, x \neq 0\}$. For example, in ℓ_∞^2 the set $A = \{(2,0),(0,3)\}$ reduces to $\hat{A} = \{(1,0),(0,1)\}$.

Proposition 4.4.3. *Let A be the set of all characteristic functions of the σ-algebra $2^{\mathbb{N}}$, i.e., $A := \{\mathbf{1}_E : E \in 2^{\mathbb{N}}, E \neq \emptyset\}$. Then, A is a generating set in ℓ_∞.*

Proof. First of all, we see that all characteristic functions $\mathbf{1}_E \in A$ are sequences $x \in \ell_\infty$ since $x : \mathbb{N} \to \mathbb{R}$ is defined by $x(i) = x_i$. Furthermore, we see that the linear span of A is

$$M := \operatorname{span}(A) = \left\{ \sum_{i=1}^{n} c_i \mathbf{1}_{E_i} : c_i \in \mathbb{R}, E_i \in 2^{\mathbb{N}} \right\},$$

which is the set of all simple functions on $2^{\mathbb{N}}$. It remains to show that M is dense in ℓ_∞ and that the convergence of $\{h_n\} \subset M$ to $x \in \ell_\infty$ is uniform. To this end, let $x = \{x_i\} \in \ell_\infty$. Assume without loss of generality that $\|x\| = 1$. Split the interval $[-1,1] \subset \mathbb{R}$ into n different sets I_k for some $n \in \mathbb{N}$, $k \in \{1,2,..,n\}$. Let D_k be subsets of $\mathbb{N}$ containing those i's such that $x_i \in I_k$.

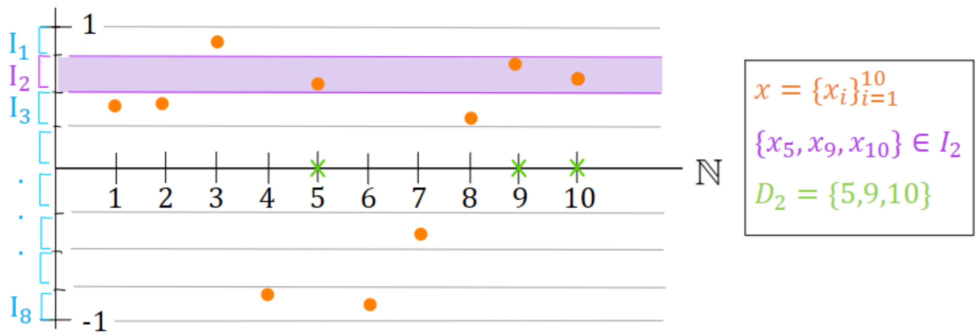

Figure 4.3: How we find the set $D_2 = \{n \in \mathbb{N} : x_i \in I_2\}$.

Figure 4.3 illustrates an example of how we construct D_k when looking at the ten first terms of a sequence $x \in B_{\ell_\infty}$. In the figure we split $[-1, 1]$ into eight sets and show that $D_2 = \{n \in \mathbb{N} : x_i \in I_2\}$.

The sets D_k are disjoint and $\bigcup_{k=1}^{n} D_k = \mathbb{N}$. Let a_k be the midpoint of each interval I_k and let $h_n = \sum_{k=1}^{n} a_k \mathbf{1}_{D_k}$. Then, h_n is a simple function such that $h_n \to x$ as $n \to \infty$, which shows that the simple functions are dense in ℓ_∞. Note that the convergence is uniform since $\sup_{i \in \mathbb{N}} |x_i - h_{ni}| \to 0$ as $n \to \infty$. $\square$

We will now define an operator $\Psi : ba(2^{\mathbb{N}}) \to (\ell_\infty)^*$ using an integral, and then we will show that Ψ is an isometric isomorphism. For the rest of this section we elaborate on the arguments given at pages 76–77 in [D].

Definition 4.4.4. Let $\Psi : ba(2^{\mathbb{N}}) \to (\ell_\infty)^*$ be defined by $\Psi(\lambda) = x^*_\lambda$ where

$$x^*_\lambda(x) := \int_{\mathbb{N}} x \, d\lambda, \quad \text{for all} \quad x \in \ell_\infty. \tag{4.4}$$

The integral (4.4) is well defined since by Proposition 4.4.3 every $x \in \ell_\infty$ is the uniform limit of a sequence of simple functions in ℓ_∞.

For Ψ to be well defined it only remains to show that Ψ maps to $(\ell_\infty)^*$, i.e., that x^*_λ is bounded and linear. Due to the linearity of the integral (3.12), x^*_λ is linear. Now, let $x \in B_{\ell_\infty} \cap M$ where M is the set of simple functions on $2^{\mathbb{N}}$. We have

$$\begin{aligned}
|x^*_\lambda(x)| &= \Big| \sum_{i=1}^{n} c_i \lambda(E_i) \Big| \\
&\leq \sum_{i=1}^{n} |c_i| |\lambda(E_i)| \\
&\leq \max_{i \in \{1,..,n\}} |c_i| \sum_{i=1}^{n} |\lambda(E_i)| \\
&\leq 1 \cdot \sup_{\pi} \sum_{i=1}^{n} |\lambda(E_i)| \\
&= |\lambda|(\mathbb{N}) = \|\lambda\|.
\end{aligned}$$

Furthermore, since M is dense in ℓ_∞ we have that $\sup_{x \in B_{\ell_\infty}} |x^*_\lambda(x)| = \sup_{x \in B_{\ell_\infty} \cap M} |x^*_\lambda(x)|$. Thus,

$$\|x^*_\lambda\| = \sup_{x \in B_{\ell_\infty} \cap M} |x^*_\lambda(x)| \leq \|\lambda\|. \tag{4.5}$$

It follows that $\Psi\lambda = x^*_\lambda$ is a bounded linear functional on ℓ_∞, and Ψ is well defined.

Proposition 4.4.5. *Ψ from Definition 4.4.4 is linear.*

Proof. Let $E \in 2^{\mathbb{N}}$, $\alpha \in \mathbb{R}$ and $\lambda, \mu \in ba(2^{\mathbb{N}})$. Due to the definition of Ψ combined with the definition of scalar multiplication and vector addition in $ba(2^{\mathbb{N}})$, we have

$$\begin{aligned} x^*_{\alpha\lambda}(\mathbf{1}_E) = (\alpha\lambda)(E) &= \alpha\lambda(E) = \alpha x^*_\lambda(\mathbf{1}_E), \quad \text{and} \\ x^*_{\lambda+\mu}(\mathbf{1}_E) = (\lambda+\mu)(E) &= \lambda(E) + \mu(E) = x^*_\lambda(\mathbf{1}_E) + x^*_\mu(\mathbf{1}_E), \end{aligned}$$

and Ψ is linear. □

Proposition 4.4.6. *Ψ from Definition 4.4.4 is a surjective isometry.*

Proof. Let $x^* \in (\ell_\infty)^*$. Let $\lambda_{x^*} : 2^{\mathbb{N}} \to \mathbb{R}$ be defined by $\lambda_{x^*}(E) := x^*(\mathbf{1}_E)$ for all $E \in 2^{\mathbb{N}}$ and let $E, F \in 2^{\mathbb{N}}$ be disjoint. Since x^* is linear, we have

$$\begin{aligned} \lambda_{x^*}(E \cup F) = x^*(\mathbf{1}_{E \cup F}) &= x^*(\mathbf{1}_E + \mathbf{1}_F) \\ &= x^*(\mathbf{1}_E) + x^*(\mathbf{1}_F) = \lambda_{x^*}(E) + \lambda_{x^*}(F), \end{aligned}$$

and λ_{x^*} is a finitely additive signed measure. For λ_{x^*} to be in $ba(2^{\mathbb{N}})$ it remains to show that λ_{x^*} is of bounded variation. To this end, let $\{E_i\}_{i=1}^n$ be a partition of $\mathbb{N}$. We have

$$\begin{aligned} \sum_{i=1}^n |\lambda_{x^*}(E_i)| &= \sum_{i=1}^n |x^*(\mathbf{1}_{E_i})| \\ &= \sum_{i=1}^n \Big(\text{sign } x^*(\mathbf{1}_{E_i}) \cdot x^*(\mathbf{1}_{E_i})\Big) \\ &= x^*\left[\sum_{i=1}^n \text{sign } x^*(\mathbf{1}_{E_i}) \cdot \mathbf{1}_{E_i}\right] \\ &\leq \|x^*\|, \end{aligned}$$

due to the linearity of x^* and since $\|\text{sign } x^*(\mathbf{1}_{E_i}) \cdot \mathbf{1}_{E_i}\| \leq 1$. Thus, taking the supremum over all partitions of $\mathbb{N}$, we get

$$|\lambda_{x^*}|(\mathbb{N}) = \|\lambda_{x^*}\| \leq \|x^*\|, \tag{4.6}$$

and λ_{x^*} is a finitely additive signed measure on $2^{\mathbb{N}}$ that is of bounded variation. Consequently, $\lambda_{x^*} \in ba(2^{\mathbb{N}})$ and Ψ is surjective. Finally, when combining (4.5) and (4.6) we see that

$$\|\Psi(\lambda)\| = \|x^*_\lambda\| = \|\lambda\|. \tag{4.7}$$

□

Corollary 4.4.7. *The operator Ψ from Definition 4.4.4 is an isometric isomorphism between $(\ell_\infty)^*$ and $ba(2^{\mathbb{N}})$. It follows that $ba(2^{\mathbb{N}})$ is a Banach space. The dual space action on ℓ_∞ is given by (4.4).*

Proof. Due to propositions 4.4.5 and 4.4.6, Ψ is a linear isometry onto $(\ell_\infty)^*$. It follows that Ψ is injective and bounded, and Ψ has a well defined bounded inverse. Consequently, Ψ is an isometric isomorphism and $ba(2^{\mathbb{N}})$ is complete.

□

The green area of Figure 4.4 illustrates the isometric isomorphism $ba(2^{\mathbb{N}}) \cong (\ell_\infty)^*$. The red arrow in the figure shows how the dual space action on ℓ_∞ is defined by an integral with respect to the finitely additive signed measure $\lambda \in ba(2^{\mathbb{N}})$.

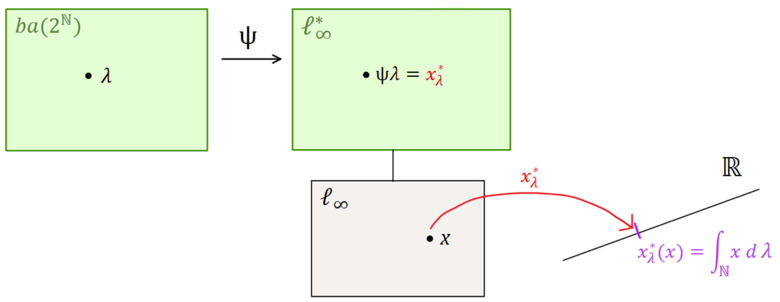

Figure 4.4: The isomorphism $ba(2^{\mathbb{N}}) \cong (\ell_\infty)^*$ and the dual space action $x^*_\lambda(x)$.

Chapter 5

The Local Diameter 2 Property

In this chapter we will introduce the local diameter 2 property, and we will look at various Banach spaces and see if they have this property. Finally, we will discuss the dual space $(\ell_\infty/c_0)^*$ and the dual space action on ℓ_∞/c_0 before we show that ℓ_∞/c_0 has the local diameter 2 property.

5.1 Introduction

In this section we will define a slice of the unit ball and look at two examples of slices of $B_{\ell_\infty^2}$. Then we will introduce the local diameter 2 property.

Definition 5.1.1. Let X be a normed space. The *diameter* of a set $Y \subset X$ is defined as the number

$$d := \sup_{x,y \in Y} \|x - y\|.$$

Definition 5.1.2. Let X be a Banach space with unit ball B_X and let $\varepsilon > 0$. A *slice* of B_X is a set of the form

$$S(x^*, \varepsilon) := \{x \in B_X : x^*(x) > 1 - \varepsilon\},$$

where $x^* \in S_{X^*}$.

Note that, since $1 = \|x^*\| = \sup_{x \in B_X} |x^*(x)|$, every slice is non-empty. Moreover, the diameter of a slice of B_X is at most 2.

Example 5.1.3. Let us look at an example of slices of $B_{\ell_\infty^2}$. Let $x = (x_1, x_2) \in \ell_\infty^2$. The norm on ℓ_∞^2 is $\|x\| = \max\{|x_1|, |x_2|\}$. The unit ball is the set $B_{\ell_\infty} = \{x \in \ell_\infty^2 : \max\{|x_1|, |x_2|\} \leq 1\}$ shown on the left side of Figure 5.1. It can be shown that the dual of ℓ_∞^2 is $(\ell_\infty^2)^* = \ell_1^2$ and that the dual space action is given by $x^*(x) = b_1x_1+b_2x_2$. The right side of Figure 5.1 shows the unit ball of ℓ_1^2 which is the set $B_{\ell_1^2} = \{x^* \in \ell_1^2 : \|x^*\| \leq 1\} = \{x^* \in \ell_1^2 : |b_1| + |b_2| \leq 1\}$.

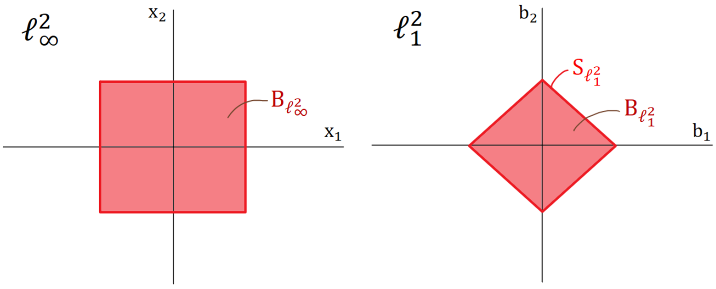

Figure 5.1: The unit ball of ℓ_∞^2 and ℓ_1^2.

Let $x_1^* \in \ell_1^2$ be $x_1^* = (1, 0)$ and let $\varepsilon = 1/2$. Then, the slice generated by x_1^* is

$$\begin{aligned} S_1(x_1^*, 1/2) &= \{x \in B_{\ell_\infty^2} : x_1^*(x) > 1 - 1/2\} \\ &= \{x \in B_{\ell_\infty^2} : (1, 0)(x_1, x_2) > 1/2\} \\ &= \{x \in B_{\ell_\infty^2} : x_1 > 1/2\}. \end{aligned}$$

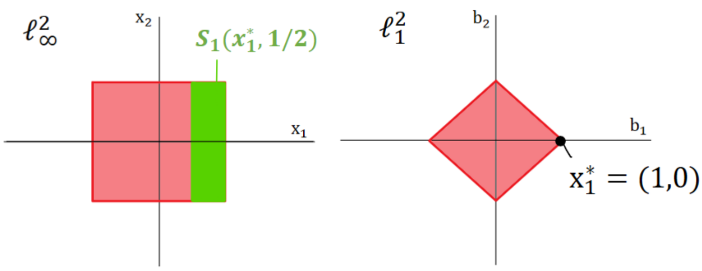

Figure 5.2: The slice $S_1(x_1^*, 1/2)$ for $x_1^* = (1, 0)$.

This slice is the green area in Figure 5.2. We see that the diameter of $S_1 = S_1(x_1^*, 1/2)$ is

$$\begin{aligned} d = \sup_{x,y \in S_1} \|x - y\| &\geq \|(1,1) - (1,-1))\| \\ &= \|(0,-2)\| = \max\{0, |-2|\} = 2, \end{aligned}$$

which yields $d = 2$.

Now let us look at another slice. Let $x_2^* \in \ell_1^2$ be $x_2^* = (1/2, 1/2)$ and let $\varepsilon = 1/2$. Then, the slice generated by x_2^* is

$$\begin{aligned} S_2(x_2^*, 1/2) &= \{x \in B_{\ell_\infty^2} : x_2^*(x) > 1 - 1/2\} \\ &= \{x \in B_{\ell_\infty^2} : (1/2, 1/2)(x_1, x_2) > 1/2\} \\ &= \{x \in B_{\ell_\infty^2} : x_1 + x_2 > 1\}. \end{aligned}$$

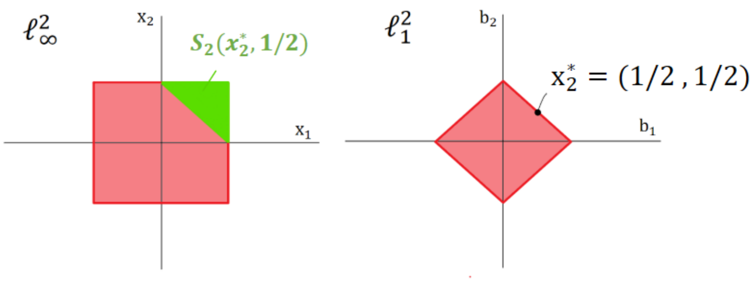

Figure 5.3: The slice $S_2(x_2^*, 1/2)$ for $x_2^* = (1/2, 1/2)$.

This slice is the green area in Figure 5.3. For all $x, y \in S_2(x_2^*, 1/2)$ we have

$$\|x - y\| = \|(x_1, x_2) - (y_1, y_2)\| \leq \max\{|x_i|, |y_i| : i = 1, 2\} \leq 1,$$

since $x_i, y_i \geq 0$. Since this holds for all elements of the slice, the slice has diameter $d \leq 1$. Now let $\delta \in (0, 1)$. Then, $(1, 1)$ and $(\delta, 1)$ are elements of $S_2(x_2^*, 1/2)$, and $d \geq \|(1, 1) - (\delta, 1)\| = 1 - \delta$. Consequently, the diameter of $S_2 = S_2(x_2^*, 1/2)$ is

$$d = \sup_{x,y \in S_2} \|x - y\| = \sup_{x,y \in S_2} \max\{|x_i - y_i| : i = 1, 2\} = 1. \tag{5.1}$$

Definition 5.1.4. ([ALN, Definition 1.1]) A Banach space X is said to have the *local diameter 2 property* (LD2P) if every slice of B_X has diameter 2.

In other words, that a Banach space X has the LD2P means that there for all $\delta > 0$ and every slice $S(x^*, \varepsilon)$ exist x, y in the slice such that $\|x - y\| > 2 - \delta$.

5.2 Spaces with and without the Local Diameter 2 Property

In this section we will see whether some well known Banach spaces have the LD2P. Recall that, for any slice $S(x^*, \varepsilon)$ of the unit ball B_X of a Banach space X, the maximum possible diameter of a slice is $d = 2$, as $x, y \in B_X$.

Proposition 5.2.1. *ℓ_∞^2 does not have the LD2P.*

Proof. To show that ℓ_∞^2 does not have the LD2P it suffices to show that there exists a slice of $B_{\ell_\infty^2}$ that does not have diameter 2. In Example 5.1.3 we saw that the slice $S_2(x_2^*, 1/2)$ of $B_{\ell_\infty^2}$, where $x_2^* = (1/2, 1/2)$, has diameter 1 (see (5.1)). Consequently, ℓ_∞^2 does not have the LD2P. $\square$

Proposition 5.2.2. *ℓ_1 does not have the LD2P.*

Proof. Like stated in Example 3.1.3 the dual of ℓ_1 is ℓ_∞ and the dual space action is given by (3.1). We must show that there exists a slice of B_{ℓ_1} that does not have diameter 2. Let $x^* \in \ell_\infty$ be $x^* = e_1^* = (1, 0, 0, ...)$ and let $\varepsilon = 1/2$. The slice generated by x^* is

$$\begin{aligned} S(x^*, 1/2) &= \{x \in B_{\ell_1} : x^*(x) > 1 - 1/2\} \\ &= \{\{x_i\} \in B_{\ell_1} : x_1 > 1/2\}. \end{aligned}$$

Thus, for all $x \in S(x^*, 1/2)$ we have $x_1 \in \left(\frac{1}{2}, 1\right]$. If we let $v, w \in S(x^*, 1/2)$, we have

$$|v_1 - w_1| < 1/2. \tag{5.2}$$

Furthermore, $\|v\| = \sum_{i=1}^{\infty} |v_i| \leq 1$ and $\|w\| = \sum_{i=1}^{\infty} |w_i| \leq 1$. Thus,

$$\sum_{i=2}^{\infty} |v_i| \leq 1 - |v_1| \ < \ 1/2, \quad \text{and} \quad \sum_{i=2}^{\infty} |w_i| \leq 1 - |w_1| \ < \ 1/2. \tag{5.3}$$

By the triangle inequality combined with (5.2) and (5.3), we have

$$\begin{aligned}\|v - w\| = \sum_{i=1}^{\infty} |v_i - w_i| &= |v_1 - w_1| + \sum_{i=2}^{\infty} |v_i - w_i| \\ &\leq |v_1 - w_1| + \sum_{i=2}^{\infty} |v_i| + \sum_{i=2}^{\infty} |w_i| \\ &< 1/2 + 1/2 + 1/2,\end{aligned}$$

for all $v, w \in S(x^*, 1/2)$. Since this holds for all elements of the slice, we have

$$d = \sup_{v,w} \|v - w\| \ < \ 3/2 \ < \ 2,$$

and the slice does not have diameter 2. □

In [ALN, page 440] we can read: "It is an interesting exercise to show that the classical spaces c_0, $C[0,1]$ and $L_1[0,1]$ have the diameter 2 properties.". Now, we will show that the spaces c_0, c, L_1 and ℓ_∞ have the local diameter 2 property.

Proposition 5.2.3. *c_0 has the LD2P.*

Proof. Let $\varepsilon > 0$. We know from Example 3.1.3 that the dual of c_0 is ℓ_1. Let $x^* \in S_{\ell_1}$ be $x^* = \{b_i\}$ and let $x = \{a_i\} \in c_0$. The dual space action on c_0 is given by (3.1) and the slice generated by x^* is

$$\begin{aligned}S(x^*, \varepsilon) &= \{x \in B_{c_0} : x^*(x) > 1 - \varepsilon\} \\ &= \{\{a_i\} \in B_{c_0} : \sum_{i=1}^{\infty} b_i a_i > 1 - \varepsilon\}.\end{aligned}$$

We must show that every slice of B_{c_0} has diameter 2. To this end, let $u_i = \text{sign } b_i$ and let $u = \sum_{i=1}^{N} u_i e_i$ for some $N \in \mathbb{N}$ (we will determine N later). Now let $v = u + e_{N+1}$ and $w = u - e_{N+1}$. Clearly $v, w \in B_{c_0}$ by definition. By the dual space action on c_0, we have

$$x^*(w) = \sum_{i=1}^{\infty} b_i w_i = \sum_{i=1}^{N} b_i \cdot \text{sign } b_i + b_{N+1} \cdot (-1) = \sum_{i=1}^{N} |b_i| - b_{N+1},$$

which yields

$$x^*(w) \geq \sum_{i=1}^{N} |b_i| - |b_{N+1}|. \tag{5.4}$$

Note that $1 = \|x^*\| = \sum_{i=1}^{\infty} |b_i|$. Thus, by adding $\sum_{i=N+1}^{\infty} |b_i| - \sum_{i=N+1}^{\infty} |b_i|$ to the right side of (5.4), we get

$$x^*(w) \geq \|x^*\| - \sum_{i=N+1}^{\infty} |b_i| - |b_{N+1}| = 1 - \Big(\sum_{i=N+1}^{\infty} |b_i| + |b_{N+1}| \Big). \tag{5.5}$$

Since $x^* \in \ell_1$ we know that $\sum_{i=N+1}^{\infty} |b_i| \to 0$ as $N \to \infty$. Thus, for each $\varepsilon > 0$ let $N \in \mathbb{N}$ be such that

$$\sum_{i=N+1}^{\infty} |b_i| + |b_{N+1}| < \varepsilon.$$

Then, (5.5) yields $x^*(w) > 1 - \varepsilon$ and $w \in S(x^*, \varepsilon)$. Similarly, we see that v is also in the slice:

$$\begin{aligned} x^*(v) = \sum_{i=1}^{N} |b_i| + b_{N+1} \geq \sum_{i=1}^{N} |b_i| - |b_{N+1}| \\ = 1 - \Big(\sum_{N+1}^{\infty} |b_i| + |b_{N+1}| \Big) > 1 - \varepsilon. \end{aligned}$$

Finally, for $S = S(x^*, \varepsilon)$, we have

$$\begin{aligned} d = \sup_{x,y \in S} \|x - y\| \geq \|v - w\| = \sup_{i \in \mathbb{N}} |v_i - w_i| \\ \geq |v_{N+1} - w_{N+1}| = |1 - (-1)| = 2, \end{aligned}$$

and $S(x^*, \varepsilon)$ has diameter 2. Since ε and x^* were arbitrary, c_0 has the LD2P.

□

Proposition 5.2.4. *c has the LD2P.*

Proof. We know from Example 3.1.4 that the dual of c is ℓ_1 and the dual space action is given by (3.2). Let $x^* = \{b_i\} \in \ell_1$ with $\|x^*\| = 1$ and let $\varepsilon > 0$. The slice generated by x^* is

$$\begin{aligned} S(x^*, \varepsilon) &= \{x \in B_c : x^*(x) > 1 - \varepsilon\} \\ &= \{\{a_i\} \in B_c : b_1 \lim_{i \to \infty} a_i + \sum_{i=2}^{\infty} a_{i-1} b_i > 1 - \varepsilon\}. \end{aligned}$$

To show that c has the LD2P, it suffices to show that the slice above has diameter 2. To this end, let $u_i = \text{sign } b_{i+1}$ and let $u = \sum_{i=1}^{N-1} u_i e_i$. Now, define v and w to be

$$v = u + e_N + \sum_{i=N+1}^{\infty} \text{sign } b_1 \cdot e_i \quad \text{and} \quad w = u - e_N + \sum_{i=N+1}^{\infty} \text{sign } b_1 \cdot e_i.$$

Then, we see that

$$\lim_{i\to\infty} v_i = \lim_{i\to\infty} w_i = \text{sign } b_1. \tag{5.6}$$

Clearly $v, w \in B_c$. By the dual space action on c combined with (5.6), we get

$$\begin{aligned} x^*(v) &= b_1 \lim_{i\to\infty} v_i + \sum_{i=2}^{\infty} v_{i-1} b_i \\ &= b_1 \text{sign } b_1 + \sum_{i=2}^{N} \text{sign } b_i \cdot b_i + 1 \cdot b_{N+1} + \text{sign } b_1 \sum_{i=N+2}^{\infty} b_i \\ &= \sum_{i=1}^{N} |b_i| + b_{N+1} + \text{sign } b_1 \sum_{i=N+2}^{\infty} b_i. \end{aligned} \tag{5.7}$$

By the triangle inequality, and by adding $\sum_{i=N+1}^{\infty} |b_i| - \sum_{i=N+1}^{\infty} |b_i|$ to the right side of (5.7), we get

$$\begin{aligned} x^*(v) &\geq \sum_{i=1}^{N} |b_i| - |b_{N+1}| - \sum_{i=N+2}^{\infty} |b_i| \\ &= \|x^*\| - 2 \sum_{i=N+1}^{\infty} |b_i|. \end{aligned} \tag{5.8}$$

Note that $\|x^*\| = 1$ and $\sum_{i=N+1}^{\infty} |b_i| \to 0$ as $N \to \infty$. Thus, for our choice of ε, let $N \in \mathbb{N}$ be such that

$$\sum_{i=N+1}^{\infty} |b_i| < \varepsilon/2.$$

Thus, (5.8) yields $x^*(v) > 1 - \varepsilon$, and $v \in S(x^*, \varepsilon)$. Similarly it can be shown that w is also in the slice. Finally, for $S = S(x^*, \varepsilon)$, we have

$$\begin{aligned} d = \sup_{x,y\in S} \|x - y\| \geq \|v - w\| &= \sup_{i\in\mathbb{N}} |v_i - w_i| \\ &\geq |v_N - w_N| = |1 - (-1)| = 2, \end{aligned}$$

and $S(x^*, \varepsilon)$ has diameter 2. $\square$

Proposition 5.2.5. *Let* $([0,1],\mathcal{A},\mu)$ *be a measure space where* $\mathcal{A}$ *is the Lebesgue* σ*-algebra on* $[0,1]$ *and where* μ *is the Lebesgue measure. Then, the space of integrable functions* $L_1 = L_1([0,1],\mathcal{A},\mu)$ *has the LD2P.*

Proof. First of all, it is well known that the dual of L_1 is L_∞, where L_∞ is the space of essentially bounded measurable functions, and that the dual space action on L_1 is given by

$$f(g) = \int_{[0,1]} fg \; d\mu,$$

for all $g \in L_1$. Let $\varepsilon > 0$ and $f \in S_{L_\infty}$. The slice generated by f is

$$\begin{aligned} S(f,\varepsilon) &= \{g \in B_{L_1} : f(g) > 1-\varepsilon\} \\ &= \{g \in B_{L_1} : \int_{[0,1]} fg \; d\mu > 1-\varepsilon\}. \end{aligned}$$

To show that L_1 has the LD2P, it suffices to show that the slice above has diameter 2. By the norm on L_∞, we have $1 = \|f\| = \operatorname*{ess\,sup}_{x\in[0,1]} |f(x)|$. Thus, for all $\varepsilon \in (0,1)$, there exists a measurable set $E \subset [0,1]$ such that

$$|f(x)| > 1-\varepsilon \quad \text{for all} \quad x \in E, \tag{5.9}$$

where the set $E \in \mathcal{A}$ has measure $\mu(E) > 0$. Let us split E into two disjoint sets $A_1 \cup A_2 = E$ so that $\mu(A_i) > 0$ for both sets. Now, define two functions g_1 and g_2 to be

$$\begin{aligned} g_1(x) &= \frac{1}{\mu(A_1)} \mathbf{1}_{A_1}(x) \cdot \operatorname{sign} f(x), \quad \text{and} \\ g_2(x) &= \frac{1}{\mu(A_2)} \mathbf{1}_{A_2}(x) \cdot \operatorname{sign} f(x). \end{aligned}$$

Then, by the norm on L_1, both g_1 and g_2 are in B_{L_1}:

$$\|g_i\| = \int_{[0,1]} |g_i| \; d\mu = \int_{A_i} \frac{1}{\mu(A_i)} \; d\mu = 1, \quad i \in \{1,2\}.$$

Furthermore, due to (5.9), both g_1 and g_2 are in the slice $S(f,\varepsilon)$:

$$\begin{aligned} f(g_i) = \int_{[0,1]} fg_i \; d\mu &= \int_{A_i} |f| \frac{1}{\mu(A_i)} \; d\mu \\ &> (1-\varepsilon) \int_{A_i} \frac{1}{\mu(A_i)} \; d\mu \;=\; 1-\varepsilon. \end{aligned}$$

Since g_1 and g_2 take nonzero values on different disjoint sets, we have that $|g_1 - g_2| = |g_1| + |g_2|$. Thus, for $S = S(f, \varepsilon)$, we have

$$\begin{aligned} d = \sup_{x,y \in S} \|x - y\| \geq \|g_1 - g_2\| &= \int_{[0,1]} |g_1 - g_2| \; d\mu \\ &= \int_E (|g_1| + |g_2|) \; d\mu \\ &= \int_{A_1} |g_1| \; d\mu + \int_{A_2} |g_2| \; d\mu = 2, \end{aligned}$$

and the slice has diameter 2. $\square$

Les us now present a lemma that will be needed when we show that ℓ_∞ has the LD2P in Proposition 5.2.7.

Lemma 5.2.6. *Let $\lambda \in ba(2^{\mathbb{N}})$. Then, for all $\eta > 0$ there exists an $n \in \mathbb{N}$ such that $|\lambda(\{n\})| < \eta$.*

Proof. We have $\|\lambda\| = |\lambda|(\mathbb{N})$ by Proposition 3.4.8. Let $\eta > 0$ and assume that $|\lambda(\{n\})| \geq \eta$ for all $n \in \mathbb{N}$. Choose a partition π of $\mathbb{N}$ to be

$$\pi = \{\{1\}, \{2\}, ..., \{k\}, \{A_m\}\} \quad \text{where} \quad A_m = \mathbb{N} \setminus \bigcup_{i=1}^{k} \{i\}.$$

Thus, using the definition of $|\lambda|(\mathbb{N})$ (Definition 3.4.5) combined with our assumption, we get

$$\|\lambda\| = |\lambda|(\mathbb{N}) \geq \sum_{i=1}^{k} |\lambda(\{i\})| + |\lambda(A_m)| \geq k \cdot \eta + |\lambda(A_m)|. \tag{5.10}$$

Let $k \in \mathbb{N}$ be such that $k \geq \|\lambda\|/\eta + 1$. Thus, (5.10) yields

$$\|\lambda\| \geq (\|\lambda\|/\eta + 1) \cdot \eta + |\lambda(A_m)| = \|\lambda\| + \eta + |\lambda(A_m)| > \|\lambda\|,$$

which is a contradiction. Consequently, for every $\eta > 0$ there exists at least one $n \in \mathbb{N}$ such that $|\lambda(\{n\})| < \eta$. $\square$

Proposition 5.2.7. *ℓ_∞ has the LD2P.*

Proof. Due to Corollary 4.4.7, we have $(\ell_\infty)^* \cong ba(2^{\mathbb{N}})$, and the dual space action is given by

$$x_\lambda^*(x) = \int_{\mathbb{N}} x \; d\lambda$$

for $x \in \ell_\infty$, $x^*_\lambda \in (\ell_\infty)^*$, $\lambda \in ba(2^{\mathbb{N}})$. Let $\varepsilon > 0$ and $x^*_\lambda \in S_{(\ell_\infty)^*}$. The slice generated by x^*_λ is

$$\begin{aligned} S(x^*_\lambda, \varepsilon) &= \{x \in B_{\ell_\infty} : x^*_\lambda(x) > 1 - \varepsilon\} \\ &= \{x \in B_{\ell_\infty} : \int_{\mathbb{N}} x \, d\lambda > 1 - \varepsilon\}. \end{aligned}$$

To show that ℓ_∞ has the LD2P, it suffices to show that the slice above has diameter 2. To this end, let u be an element of the slice $S(x^*_\lambda, \varepsilon/2)$, i.e.,

$$\int_{\mathbb{N}} u \, d\lambda > 1 - \varepsilon/2. \tag{5.11}$$

Let $\eta > 0$ such that $\eta = \varepsilon/4$. Due to Lemma 5.2.6 there exists an $n \in \mathbb{N}$ such that

$$|\lambda(\{n\})| < \varepsilon/4. \tag{5.12}$$

Now define v to be equal to u except for the term v_n, i.e.,

$$v = (u_1, u_2, ..., u_{n-1}, 1, u_{n+1}, ...).$$

Thus, u and v are equal on the set $\mathbb{N}\backslash\{n\}$, and due to the additive property of the integral (3.11), we get

$$\begin{aligned} x^*_\lambda(v) = \int_{\mathbb{N}} v \, d\lambda &= \int_{\mathbb{N}\backslash\{n\}} v \, d\lambda + \int_{\{n\}} v \, d\lambda \\ &= \int_{\mathbb{N}\backslash\{n\}} u \, d\lambda + 1 \cdot \lambda(\{n\}) + \int_{\{n\}} u \, d\lambda - \int_{\{n\}} u \, d\lambda \\ &= \int_{\mathbb{N}} u \, d\lambda + \lambda(\{n\})(1 - u_n). \end{aligned} \tag{5.13}$$

Note that $\int_{\{n\}} x \, d\lambda = 1 \cdot \lambda(\{n\})$ and $\int_{\{n\}} u \, d\lambda = u_n \lambda(\{n\})$ since x and u are simple functions on the set $\{n\}$. Furthermore, when combining (5.11) and (5.12) with the fact that $(1 - u_n) \in [0, 2]$, then (5.13) yields

$$\begin{aligned} x^*_\lambda(v) &\geq \int_{\mathbb{N}} u \, d\lambda - |\lambda(\{n\})| \cdot 2 \\ &> 1 - \varepsilon/2 - \varepsilon/2, \end{aligned}$$

which shows that $v \in S(x^*_\lambda, \varepsilon)$. Now define w to be

$$w = (u_1, u_2, ..., u_{n-1}, -1, u_{n+1}, ...)$$

where $w_n = -1$ for the n fulfilling (5.12). Performing similar computations as before, we get

$$x_\lambda^*(w) = \int_{\mathbb{N}} w \; d\lambda = \int_{\mathbb{N}} u \; d\lambda - \lambda(\{n\})(u_n + 1) \geq \int_{\mathbb{N}} u \; d\lambda - |\lambda(\{n\})| \cdot 2 > 1 - \varepsilon,$$

and $w \in S(x_\lambda^*, \varepsilon)$. Finally, for $S = S(x_\lambda^*, \varepsilon)$, we have

$$\begin{aligned} d = \sup_{x,y \in S} \|x - y\| \geq \|v - w\| &= \sup_{i \in \mathbb{N}} |v_i - w_i| \\ &\geq |v_n - w_n| = |1 - (-1)| = 2, \end{aligned}$$

and the slice has diameter 2. □

5.3 The Local Diameter 2 Property in ℓ_∞/c_0

It is time proceed to the goal of this thesis: We will show that the quotient space ℓ_∞/c_0 has the LD2P. For this we will need the dual space action on ℓ_∞/c_0 and an observation about the norm on the quotient space. We will begin with the latter.

By Definition 2.1.20 the norm on ℓ_∞/c_0 is given by $\|[x]\| = \inf_{a \in c_0} \|x + a\|$. Now, let $x = \{x_i\}$ and $y = \{y_i\}$ be elements of ℓ_∞ and let $a = \{a_i\} \in c_0$. Since $\{a_i\}$ converges to zero, then $|x_i - y_i + a_i| \to |x_i - y_i|$ as $i \to \infty$. Thus, if x and y are constant on an infinite set, i.e., if $x_i = r_1$ and $y_i = r_2$ for infinitely many i's, then we have

$$\|x - y + a\| = \sup_{i \in \mathbb{N}} |x_i - y_i + a_i| \geq |r_1 - r_2|,$$

for all $a \in c_0$, and so

$$\|[x] - [y]\| = \inf_{a \in c_0} \|x - y + a\| \geq \inf_{a \in c_0} |r_1 - r_2| = |r_1 - r_2|. \tag{5.14}$$

This observation will be helpful when showing that a slice of B_{ℓ_∞/c_0} has diameter 2.

Now, let us establish the dual space action on ℓ_∞/c_0. Due to Corollary 4.4.7, we know that $ba(2^{\mathbb{N}}) \cong (\ell_\infty)^*$ through the operator $\Psi : ba(2^{\mathbb{N}}) \to (\ell_\infty)^*$ defined by $(\Psi\lambda)(x) = x_\lambda^*(x) = \int_{\mathbb{N}} x \, d\lambda$ for all $x \in \ell_\infty$. This mapping is an isometric isomorphism due to (4.7), i.e., $\|\Psi(\lambda)\| = \|x_\lambda^*\| = \|\lambda\|$ for all $\lambda \in ba(2^{\mathbb{N}})$. The mapping is illustrated in green in Figure 5.4.

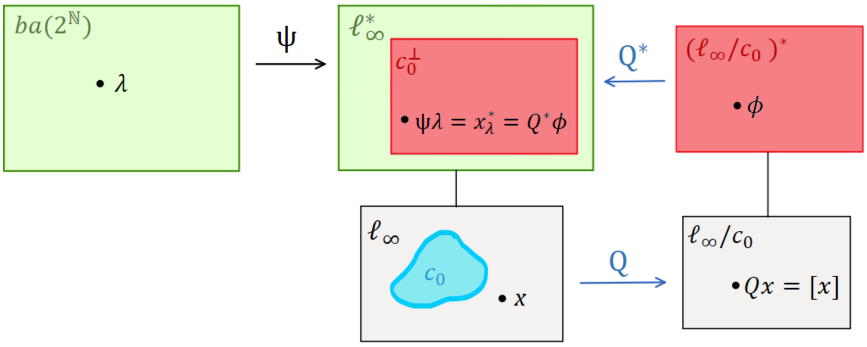

Figure 5.4: The two isometric isomorphisms $(\ell_\infty/c_0)^* \cong c_0^\perp \subset \ell_\infty^* \cong ba(2^{\mathbb{N}})$.

Furthermore, by Theorem 4.3.2 we know that the dual $(\ell_\infty/c_0)^*$ is isometrically isomorphic to $c_0^\perp \subset (\ell_\infty)^*$ through the adjoint Q^* of the quotient map Q, which maps $(\ell_\infty/c_0)^*$ onto $c_0^\perp$. This is illustrated in red in Figure 5.4. The adjoint Q^* is defined by $(Q^*\varphi)(x) = \varphi(Qx) = \varphi[x]$ for all $x \in \ell_\infty$ (Definition 3.3.3 combined with Definition 4.2.1). Recall that by (4.3) we have $\|Q^*\varphi\| = \|\varphi\|$ where $Q^*\varphi = x^* \in c_0^\perp \subset (\ell_\infty)^*$.

When combining the two previous paragraphs, we get that each functional $\varphi_\lambda \in (\ell_\infty/c_0)^*$ can be associated with a bounded finitely additive signed measure $\lambda \in ba(\mathcal{A})$ through

$$\varphi_\lambda[x] = \varphi_\lambda(Qx) = (Q^*\varphi_\lambda)(x) = x_\lambda^*(x) = (\Psi\lambda)(x), \tag{5.15}$$

where $x_\lambda^* \in c_0^\perp \subset \ell_\infty$. Due to Corollary 4.4.7, the dual space action on ℓ_∞ is given by $x_\lambda^*(x) = \int_{\mathbb{N}} x \, d\lambda$ for all $x \in \ell_\infty$. Thus, (5.15) yields that for all $\varphi_\lambda \in (\ell_\infty/c_0)^*$ and $[x] \in \ell_\infty/c_0$ the dual space action on ℓ_∞/c_0 is given by

$$\varphi_\lambda[x] = x_\lambda^*(x) = \int_{\mathbb{N}} x \, d\lambda, \tag{5.16}$$

where $\varphi_\lambda[x]$ is unique for each $[x] \in \ell_\infty/c_0$ since $x_\lambda^* \in c_0^\perp$. Furthermore, combining (5.15) with (4.3) and (4.7) we have the equality $\|\varphi_\lambda\| = \|x_\lambda^*\| = \|\lambda\|$.

Now, let us present a lemma.

Lemma 5.3.1. *Let $\eta > 0$ and $\lambda \in ba(2^{\mathbb{N}})$ with $\|\lambda\| = 1$. Let A_i be the set defined by $A_i := \{(n-1)k + i : n \in \mathbb{N}\}$ for some $k \in \mathbb{N}$, $i \in \{1, 2, ..., k\}$. Then there exists $k \in \mathbb{N}$ such that $|\lambda|(A_i) < \eta$ for some $i \in \{1, 2, ..., k\}$.*

Proof. First of all, let us visualise the sets A_i:

$$\begin{aligned} A_1 &= \{1, k+1, 2k+1, ...\} \\ A_2 &= \{2, k+2, 2k+2, ...\} \\ A_3 &= \{3, k+3, 2k+3, ...\} \\ &\vdots \\ A_k &= \{k, 2k, 3k, ...\}. \end{aligned}$$

If we let $\pi_1 = \{A_i\}_{i=1}^k$, we see that π_1 is a partition of $\mathbb{N}$. For $\eta > 0$ let $k \in \mathbb{N}$ be such that $k \geq (1/\eta + 1)$, and assume $|\lambda|(A_i) \geq \eta$ for all $A_i \in \pi_1$. Recall that $\|\lambda\| = |\lambda|(\mathbb{N})$ due to Proposition 3.4.8. Since $|\lambda|$ is a positive finitely additive measure by Proposition 3.4.6, we have

$$\begin{aligned} 1 = \|\lambda\| = |\lambda|(\mathbb{N}) &= \sum_{i=1}^{k} |\lambda|(A_i) \\ &\geq k \cdot \eta \geq (1/\eta + 1) \cdot \eta = 1 + \eta > 1, \end{aligned}$$

which is a contradiction. Consequently, there exists at least one set $A_i \in \pi_1$ such that $|\lambda|(A_i) < \eta$. □

Theorem 5.3.2. *ℓ_∞/c_0 has the LD2P.*

Proof. Let $\varepsilon > 0$ and $\varphi_\lambda \in S_{(\ell_\infty/c_0)^*}$. The dual space action on ℓ_∞/c_0 is given by (5.16), so the slice generated by φ_λ is

$$\begin{aligned} S(\varphi_\lambda, \varepsilon) &= \{[x] \in B_{\ell_\infty/c_0} : \varphi_\lambda[x] > 1 - \varepsilon\} \\ &= \{[x] \in B_{\ell_\infty/c_0} : \int_{\mathbb{N}} x \, d\lambda > 1 - \varepsilon\}. \end{aligned}$$

Let $[u] \in S(\varphi_\lambda, \varepsilon/2)$ with $\|[u]\| < 1$, i.e.,

$$\varphi_\lambda[u] = \int_{\mathbb{N}} u \, d\lambda > 1 - \varepsilon/2. \tag{5.17}$$

For $k \in \mathbb{N}$ and $i \in \{1, 2, ..., k\}$ let $A_i = \{(n-1)k + i : n \in \mathbb{N}\}$. Then $A_i \subset \mathbb{N}$ is of infinite cardinality. Note that $1 = \|\varphi_\lambda\| = \|\lambda\|$. Thus, due to Lemma 5.3.1, if we let $\eta = \varepsilon/4$ there exists $k \in \mathbb{N}$ and $j \in \{1, 2, ..., k\}$ such that the total variation of λ on the set $A_j = \{(n-1)k + j : n \in \mathbb{N}\}$ is

$$|\lambda|(A_j) < \varepsilon/4. \tag{5.18}$$

We know that $[u] \in U_{\ell_\infty/c_0}$. Thus, by Proposition 4.2.2, there exists an element $u \in [u]$ such that $u = \{u_i\} \in U_{\ell_\infty}$. For this u, define two elements $x, y \in \ell_\infty$ by

$$x = \{x_i\} = \begin{cases} 1, & i \in A_j \\ u_i & \text{otherwise}, \end{cases} \tag{5.19}$$

$$y = \{y_i\} = \begin{cases} -1, & i \in A_j \\ u_i & \text{otherwise}. \end{cases} \tag{5.20}$$

We see that $x, y \in B_{\ell_\infty}$. Since $\|[x]\| \leq \|x\|$, we have that $[x], [y] \in B_{\ell_\infty/c_0}$. We want to show that $[x]$ and $[y]$ are in the slice $S(\varphi_\lambda, \varepsilon)$. By the additive property of the integral (3.11), we have

$$\begin{aligned} \varphi_\lambda[x] &= \int_{\mathbb{N}} x \; d\lambda \\ &= \int_{\mathbb{N}\setminus A_j} u \; d\lambda + \int_{A_j} 1 \; d\lambda \\ &= \int_{\mathbb{N}} u \; d\lambda + \lambda(A_j) - \int_{A_j} u \; d\lambda. \end{aligned} \tag{5.21}$$

Note that $\lambda = \lambda^+ - \lambda^-$ by (3.7), where λ^+ and λ^- are positive finitely additive measures by Proposition 3.4.9. Thus, by (3.9) it can be shown that

$$\int x \; d\lambda = \int x \; d\lambda^+ - \int x \; d\lambda^-,$$

for all $x \in \ell_\infty$. Thus, (5.21) yields

$$\varphi_\lambda[x] = \int_{\mathbb{N}} u \; d\lambda + \lambda(A_j) - \int_{A_j} u \; d\lambda^+ + \int_{A_j} u \; d\lambda^-. \tag{5.22}$$

Now, let $w = \{\|u\|, \|u\|, \|u\|, ...\}$. Thus, w is a positive simple function in ℓ_∞ and $u \leq w$. Thus, $\int u \; d\lambda^\pm \leq \int w \; d\lambda^\pm$ by (3.10), and since $\int w \; d\lambda^\pm \geq 0$, then

(5.22) yields

$$\begin{aligned}\varphi_\lambda[x] &\geq \int_{\mathbb{N}} u\ d\lambda + \lambda(A_j) - \int_{A_j} w\ d\lambda^+ - \int_{A_j} w\ d\lambda^- \\ &= \int_{\mathbb{N}} u\ d\lambda + \lambda(A_j) - \|u\|\Big(\lambda^+(A_j) + \lambda^-(A_j)\Big) \\ &= \int_{\mathbb{N}} u\ d\lambda + \lambda(A_j) - \|u\||\lambda|(A_j).\end{aligned}$$

Furthermore, note that $|\lambda|(A_j)| \geq |\lambda(A_j)|$ which yields

$$\varphi_\lambda[x] \geq \int_{\mathbb{N}} u\ d\lambda - |\lambda|(A_j) - \|u\||\lambda|(A_j).$$

Finally, combining $\|u\| < 1$ with (5.17) and (5.18), we get

$$\begin{aligned}\varphi_\lambda[x] &\geq \int_{\mathbb{N}} u\ d\lambda - 2|\lambda|(A_j) \\ &> 1 - \varepsilon/2 - \varepsilon/2,\end{aligned}$$

which shows that $[x] \in S(\varphi_\lambda, \varepsilon)$. Similarly we can show that $[y]$ is in the slice:

$$\begin{aligned}\varphi_\lambda[y] &= \int_{\mathbb{N}\setminus A_j} u\ d\lambda - \int_{A_j} 1\ d\lambda \\ &= \int_{\mathbb{N}} u\ d\lambda - \lambda(A_j) - \int_{A_j} u\ d\lambda \\ &= \int_{\mathbb{N}} u\ d\lambda - \lambda(A_j) - \int_{A_j} u\ d\lambda^+ + \int_{A_j} u\ d\lambda^- \\ &\geq \int_{\mathbb{N}} u\ d\lambda - |\lambda|(A_j) - \int_{A_j} w\ d\lambda^+ - \int_{A_j} w\ d\lambda^- \\ &= \int_{\mathbb{N}} u\ d\lambda - |\lambda|(A_j) - \|u\|\Big(\lambda^+(A_j) + \lambda^-(A_j)\Big) \\ &\geq \int_{\mathbb{N}} u\ d\lambda - 2|\lambda|(A_j) \\ &> 1 - \varepsilon/2 - \varepsilon/2.\end{aligned}$$

Since ε and φ_λ were arbitrary, it only remains to show that $S = S(\varphi_\lambda, \varepsilon)$ has diameter 2. Since $x_i = 1$ and $y_i = -1$ on the infinite set A_j, then by (5.14), we have

$$d = \sup_{[v],[w]\in S} \|[v] - [w]\| \geq \|[x] - [y]\| \geq |1 - (-1)| = 2,$$

and the slice has diameter 2. □

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