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Random dynamical system generated by the 3D Navier-Stokes equation with rough transport noise*

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Abstract

We consider the Navier-Stokes system in three dimensions perturbed by a transport noise which is sufficiently smooth in space and rough in time. The existence of a weak solution was proved in [26], however, as in the deterministic setting the question of uniqueness remains a major open problem. An important feature of systems with uniqueness is the semigroup property satisfied by their solutions. Without uniqueness, this property cannot hold generally. We select a system of solutions satisfying the semigroup property with appropriately shifted rough path. In addition, the selected solutions respect the well accepted admissibility criterium for physical solutions, namely, maximization of the energy dissipation. Finally, under suitable assumptions on the driving rough path, we show that the Navier-Stokes system generates a measurable random dynamical system. To the best of our knowledge, this is the first construction of a measurable single-valued random dynamical system in the state space for an SPDE without uniqueness.

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1 Introduction

By now, the question of generating a random dynamical system by an Itô-type Stochastic Differential Equation (SDE) is well settled under natural assumptions on the coefficients implying in particular uniqueness, refer to [1, 15] for details. The crucial step consists in proving the stochastic flow property by using the Kolmogorov continuity theorem [30, Section 1.4] about the existence of a continuous random field with a finite-dimensional parameters range. Upon establishing the flow property, the existence of a random dynamical system induced by solution to considered SDE follows as derived in [35].

Unfortunately, it is in general not possible to extend the Kolmogorov continuity theorem to an infinite-dimensional parameter range and to obtain stochastic flows for Itô-type stochastic PDEs (SPDEs). The main issue here is that the solution to an Itô-type SPDE is defined almost surely where the exceptional set depends on the initial condition. Nevertheless, it is possible to obtain a stochastic flow, and in particular a random dynamical system, for Itô-type SPDEs driven by special noises provided uniqueness can be shown, refer to [16] for details. For instance, if the SPDE is driven by either additive noise or linear multiplicative noise, then it can be transformed into a random evolution equation. If additionally uniqueness can be proved for the transformed system then it is possible to construct the associated stochastic flow and to generate a random dynamical system. In the context of 2D stochastic Navier-Stokes equations, which has a unique global solution, this approach has been used successfully by Crauel and Flandoli [10] in bounded domain and Brzeźniak and Li [8] in an unbounded domain.

Due to non-uniqueness of global solutions to stochastic Navier-Stokes equations in three space dimensions, the above method fails. One way to overcome this difficulty, as given by Sell in [36], is to replace the state space by the phase space consisting of full trajectories of solutions of the equation. By modifying the Sell concept, Flandoli and Schmalfuss [19] were able to work with single-valued cocycles and proved the global existence of a random attractor for 3D stochastic Navier-Stokes equation with a multiplicative white noise. Another way to talk about a random dynamical system generated by an SPDE, which is not known to have a unique global solution, was established in [32], where the authors introduced a set-valued stochastic semiflow and a set-valued random dynamical system. In particular, by generalizing the idea of Ball [4] to the stochastic setting, they found a certain attracting set that depends on the random parameter and concluded the existence of a random attractor for the 3D Navier-Stokes equations driven by white noise.

As described above briefly, for some very specific structure of the noise driving an SPDE, a suitable transformation of the equation into a pathwise evolution equation allows to prove the existence of a random dynamical system. This forces us to rethink and consider a pathwise approach to construct solutions of an SPDE such that they satisfy the cocycle property. Recently, various techniques have been derived to give a pathwise meaning to the solutions of SPDEs by exploiting the ideas of Lyons' rough paths theory [31] in one way or other. However, there are very few results that explore the pathwise properties of the solutions to study the long time behavior of stochastic flows. For example, in [22, 23] the authors deal with random dynamical systems for SPDEs driven by a fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$ and $H \in (1/3, 1/2]$. The proof relies on the definition of a pathwise stochastic integral based on the Riemann-Liouville fractional derivatives. Using regularizing properties of analytic semigroups, the authors in [25] proved the existence of a unique global solution to infinite-dimensional parabolic rough evolution equations and investigated random dynamical systems for such equations. Existence of a unique random attractor applicable for a large class of SPDEs perturbed by Brownian motion, fractional Brownian motion and Lévy-type noise is shown in [24]. The results obtained in [24] are based on the variational approach to SPDE. Recently, the authors in [27], defined an intrinsic notion of solution based on ideas from the rough path theory and studied the well-posedness of 2D Navier-Stokes equations in an equivalent vorticity formulation. They derive rough path continuity of the equation and show that for a large class of driving signals, the system generates a continuous random dynamical system.

To the best of our knowledge, all the rough PDEs (RPDEs) considered so far to investigate the induced random dynamical system proved to have a unique global solution. However, from the point of view of applications there is a huge number of physically relevant systems where uniqueness is either unknown or not even valid. Such examples appear for instance in fluid dynamics and are represented by the iconic example of the Navier-Stokes system. Nevertheless, despite the theoretical difficulties, the Navier-Stokes system and related fluid dynamics equations are widely used in practice and count as a reliable basis for modeling and simulations. From the mathematics point of view, it is therefore essential to study such models, which are not known to have unique global solutions, and to develop methods to investigate their long time behavior.

Taking a step in this direction, we study the existence of a random dynamical system induced by solutions of a system of Navier-Stokes equations on the three-dimensional torus $\mathbf{T}^3$ subject to a rough transport noise. The noise arises from perturbing the transport advecting velocity field by a space-time dependent noise and is, at least formally, energy conservative. The rough path philosophy of splitting analysis from probability, as well as a Wong-Zakai stability result are the key ingredients of our construction. For the first time, we are able to construct a measurable single-valued random dynamical system in the state space for an SPDE without uniqueness. Our proof relies on a selection procedure and is direct, benefiting from the rough path theory rather than a transformation into a random PDE.

The system of interest governs the evolution of the velocity field $u : \mathbf{R}_+ \times \mathbf{T}^3 \to \mathbf{R}^3$ and the pressure $p : \mathbf{R}_+ \times \mathbf{T}^3 \to \mathbf{R}$ of an incompressible viscous fluid and reads as

$$\begin{aligned} \partial_t u + (u - \dot{a}) \cdot \nabla u + \nabla p &= \Delta u, \\ \nabla \cdot u &= 0, \\ u(0) &= u_0 \in L^2(\mathbf{T}^3; \mathbf{R}^3). \end{aligned} \tag{1.1}$$

Here $\dot{a}$ is the (formal) derivative in time of a function $a = a_t(x) : \mathbf{R}_+ \times \mathbf{T}^3 \to \mathbf{R}^3$ which can have the following factorization:

$$a_t(x) = \sigma_k(x) z_t^k = \sum_{k=1}^{K} \sigma_k(x) z_t^k, \tag{1.2}$$

where we adopt the convention of summation over repeated indices $k \in \{1, \dots, K\}$. We also assume that for all $k \in \{1, \dots, K\}$, the vector fields $\sigma_k : \mathbf{T}^3 \to \mathbf{R}^3$ are bounded, divergence-free, and twice-differentiable with uniformly bounded first and second derivatives. The driving signal z is assumed to be a $\mathbf{R}^K$-valued α-Hölder path for some $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$ which can be lifted to a geometric rough path $\mathbf{Z} = (Z, \mathbb{Z})$, see Section 2.4 for precise assumptions.

The system (1.1), (1.2) was studied recently in [26] within the framework of unbounded rough drivers as introduced in [2] and further developed in [14] and other works, e.g. [17]. The authors in [26] introduced a suitable notion of weak solution to (1.1) and proved existence, see [26, Theorem 2.13]. The proof relies on a Galerkin approximation in combination with uniform energy estimates of the solution as well as the corresponding remainder terms and a compactness argument. However, it turns out that this notion of solution is not suitable for the construction of a random dynamical system. More precisely, as the kinetic energy of weak solutions is only lower semicontinuous, the so-called energy sinks appear and cannot be avoided. Consequently, at the time of each energy sink, the actual energy of the system is not controlled by the energy of the solution. This is a problem since in a random dynamical system the future evolution after each time t has to be fully determined by the value of the solution at t.

Inspired by [5, 6, 7], we overcome this issue by including an auxiliary variable, i.e. the energy E, as a part of the solution. Accordingly, a weak solution in our context actually consists of a triple $[u, p, E]$ of the fluid velocity and the pressure together with the energy. In addition, the Navier-Stokes system (1.1), (1.2) needs to be supplemented by a suitable form of an energy inequality. As usual, see for e.g. [26, Section 4.1.2], the pressure can be recovered from the velocity so we do not specify it in our results or in the definition of a solution, see Definition 2.5. Including the energy as an additional datum may seem a bit superfluous at first sight, but it is indispensable in order to obtain the semigroup property for every time. Otherwise one would only obtain a statement for a.e. time as in [5, 18]. We refer to Remark 3.10 and to [28, Section 6] for a further discussion of this issue.

The first main result of the current paper can be stated as follows, see Theorem 3.9 for the precise formulation.

Theorem 1.1. *The Navier-Stokes equation* (1.1)-(1.2) *admits a semiflow selection in the class of weak solutions, that is, there is a measurable mapping*

$$U : [u_0, E_0, \boldsymbol{Z}] \mapsto [u, E],$$

which assigns to every initial condition $[u_0, E_0]$ *and a rough path* $\boldsymbol{Z}$ *one solution trajectory* $[u, E]$ *so that the following* ***semigroup property*** *holds true*

$$U\{u_0, E_0, \boldsymbol{Z}\}(t_1 + t_2) = U\{U\{u_0, E_0, \boldsymbol{Z}\}(t_1), \tilde{\boldsymbol{Z}}_{t_1}\}(t_2), \quad \text{for any } t_1, t_2 \geq 0,$$

where $\tilde{\boldsymbol{Z}}_{t_1}(\cdot)$ *denotes the shifted rough path* $\tilde{\boldsymbol{Z}}_{t_1}(\cdot) := \boldsymbol{Z}(t_1 + \cdot)$[1].

To prove this result we adapt to the rough path setting the selection procedure introduced in [9] and further generalized in [5, 6, 7]. The key property which then permits to deduce the existence of a measurable random dynamical system is the rough path stability in the spirit of a Wong-Zakai approximation result, see Theorem 3.3. Indeed, this permits to go back to probability and consider random driving rough paths which satisfy a suitable cocycle property as introduced in [3]. An example is a fractional Brownian motion with Hurst parameter $H > \frac{1}{3}$. This brings us to the second main result of this paper. See Theorem 4.4 for details and a precise formulation.

Theorem 1.2. *Under the assumption that the driving path* $\boldsymbol{Z}$ *is an* α*-Hölder rough path cocycle, see [3], the Borel measurable map*

$$\Phi : (t, \omega, [u_0, E_0]) \mapsto U\{u_0, E_0, \boldsymbol{Z}(\omega)\},$$

is a measurable random dynamical system over a measurable metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{T}})$*, that is, it satisfies the cocycle property, which roughly states that*

$$\Phi(t + s, \omega) = \Phi(t, \theta_s\omega) \circ \Phi(s, \omega) \quad \forall s, t \in \mathbb{T}, \omega \in \Omega.$$

We note that the possibility of changing measurability of the random dynamical system above into its continuity remains a major open question. Indeed, this corresponds to continuity with respect to the initial condition, which in the deterministic setting has

[1] With a slight abuse of notation we write $\mathbf{Z}(t_1 + \cdot)$ for the 2-index map $(Z, \mathbb{Z})_{t_1+\cdot, t_1+\cdot}$.

not been proved so far. In the stochastic setting, on the other hand, the probabilistic counterpart, i.e. the Feller property, was established only under the assumption that the noise is additive and sufficiently non-degenerate, see [12, 18].

The organization of the paper is as follows. In Section 2, we introduce our notation and provide the required definitions with the notion of weak solution used in the paper. In Section 3, we introduce the concept of a semiflow selection in terms of the two state variables: the velocity and the energy. We also analyze the properties (compactness, shift invariance and continuation) of the solution set and prove the existence of a semiflow selection, refer to Theorem 3.9. Section 4 is devoted to the existence of a random dynamical system, Theorem 4.4, which is a central result of this paper. We conclude the paper with Appendix A where we state all the required a priori estimates from [26, Section 3] and state a compact embedding lemma which is helpful in the proof of the crucial sequential stability result, see Theorem 3.3.

2 Notation

In this section we fix the notation which is used throughout the paper. Since we intend to construct a semiflow for weak solutions to (1.1) whose existence is proven in [26], to avoid any notational confusion, we closely follow [26, Section 2] and only state the required definitions and their properties.

We write $a \lesssim b$ if there exists a positive constant C such that $a \leq Cb$. If the constant C depends only on the parameters $p_1, \dots, p_n$, we write $C(p_1, \dots, p_n)$ and $\lesssim_{p_1,\dots,p_n}$, respectively, instead of C and $\lesssim$. Let $\mathbb{T} := [0, \infty)$. We let $\mathbf{N}$ to be the set of natural numbers. Let $\mathbf{T}^3$ denotes the three dimensional flat torus.

For given Banach spaces V_1 and V_2, the space of continuous linear operators from V_1 to V_2 will be denoted by $\mathcal{L}(V_1, V_2)$. Note that $\mathcal{L}(V_1, V_2)$ is endowed with the operator norm which we denote by $|\cdot|_{\mathcal{L}(V_1,V_2)}$. For a given σ-finite measure space $(X, \mathcal{X}, \mu)$, separable Banach space V with norm $|\cdot|_V$, and $p \in [1, \infty]$, we denote by $L^p(X; V)$ the Bochner space of strongly-measurable and L^p-integrable functions $f : X \to V$. For a given Hilbert space H and $T > 0$, we let $L^2_T H = L^2([0, T]; H)$ and $L^\infty_T H = L^\infty([0, T]; H)$. Moreover, we set $\mathbf{L}^2 = L^2(\mathbf{T}^3; \mathbf{R}^3)$. We use subscript "loc" to point out that the elements restricted to any bounded interval J belong to the space with domain J. For instance, by $L^\infty_{\text{loc}}(\mathbb{T}; \mathbf{R})$ we understand the set of all μ-equivalence-classes of strongly-measurable functions $f : \mathbb{T} \to \mathbf{R}$ such that $f|_J \in L^\infty(J; \mathbf{R})$ for every bounded $J \subset \mathbb{T}$. We write $C_T H = C([0, T]; H)$ to denote the Banach space of continuous functions from $[0, T]$ to H, endowed with the supremum norm in time. Moreover, if H is subject to weak topology, then we write $C([0, T]; H_w)$.

Let $\mathcal{S}$ be the Fréchet space of infinitely differentiable periodic complex-valued functions with the usual set of seminorms. Let $\mathcal{S}'$ be the continuous dual space of $\mathcal{S}$ endowed with the weak-star topology. In extension of these, we write $\mathbf{S}' = (\mathcal{S}')^d$ for the set of continuous linear functions from $\mathbf{S} = (\mathcal{S})^d$ to $\mathbf{C}$ endowed with the weak-star topology.

For a given $\beta \in \mathbf{R}$, the Hilbert space $\mathbf{W}^{\beta,2}$ is defined as

$$\mathbf{W}^{\beta,2} := (I - \Delta)^{-\frac{\beta}{2}} \mathbf{L}^2 = \{f \in \mathbf{S}' : (I - \Delta)^{\frac{\beta}{2}} f \in \mathbf{L}^2\},$$

with inner product

$$(f, g)_\beta := ((I - \Delta)^{\frac{\beta}{2}} f, (I - \Delta)^{\frac{\beta}{2}} g)_{\mathbf{L}^2}, \quad f, g \in \mathbf{W}^{\beta,2},$$

and induced norm $|\cdot|_\beta$. For notational simplicity, when $\beta = 0$ we omit the index in the inner product, i.e. $(\cdot, \cdot) := (\cdot, \cdot)_0$. Let

$$\mathbf{H}^0 := \left\{f \in \mathbf{W}^{0,2} : \nabla \cdot f = 0\right\}.$$

2.1 Helmholtz–Leray projection

We denote the Helmholtz–Leray projection by $P : \mathbf{S}' \to \mathbf{S}'$, which is well-known in the study of Navier-Stokes equation, see [38], and let $Q = I - P$. It follows that $P, Q \in \mathcal{L}(\mathbf{W}^{\beta,2}, \mathbf{W}^{\beta,2})$ and that the operator norms of P and Q are bounded by one for all $\beta \in \mathbf{R}$.

By setting

$$\mathbf{H}^{\beta} := P\mathbf{W}^{\beta,2} \qquad \text{and} \qquad \mathbf{H}^{\beta}_{\perp} := Q\mathbf{W}^{\beta,2},$$

it can be showed that for all $\beta \in \mathbf{R}$, see for e.g. [34, Lemma 3.7],

$$\mathbf{W}^{\beta,2} = \mathbf{H}^{\beta} \oplus \mathbf{H}^{\beta}_{\perp},$$

where

$$\mathbf{H}^{\beta} = \left\{ f \in \mathbf{W}^{\beta,2} : \ \nabla \cdot f = 0 \right\},$$

$$\mathbf{H}^{\beta}_{\perp} = \{ g \in \mathbf{W}^{\beta,2} : \langle f, g \rangle_{-\beta,\beta} = 0, \ \ \forall f \in \mathbf{H}^{-\beta} \}.$$

The following part of this subsection will shed light on the construction of the unbounded rough drivers associated with (1.1). Let $\sigma : \mathbf{T}^3 \to \mathbf{R}^3$ be twice differentiable and divergence-free. Moreover, assume that the derivatives of σ up to order two are bounded uniformly by a constant M_0.

Let $\mathcal{A}^1 := \sigma \cdot \nabla$ and $\mathcal{A}^2 := (\sigma \cdot \nabla)(\sigma \cdot \nabla)$. It follows that there is a constant M (depends on M_0, β) such that

$$|\mathcal{A}^1|_{\mathcal{L}(\mathbf{W}^{\beta+1,2}, \mathbf{W}^{\beta,2})} \leq M, \ \forall \beta \in [0,2], \qquad |\mathcal{A}^2 f|_{\mathcal{L}(\mathbf{W}^{\beta+2,2}, \mathbf{W}^{\beta,2})} \leq M, \ \forall \beta \in [0,1].$$

We ask the reader to see [33] for such estimates on the whole space, but they can easily be adapted to the periodic setting as required in the current paper.

Since $P \in \mathcal{L}(\mathbf{W}^{\beta,2}, \mathbf{H}^{\beta})$ and $Q \in \mathcal{L}(\mathbf{W}^{\beta,2}, \mathbf{H}^{\beta}_{\perp})$ for all $\beta \in \mathbf{R}$, both of which have operator norm bounded by 1, we have

$$|P\mathcal{A}^1|_{\mathcal{L}(\mathbf{H}^{\beta+1}, \mathbf{H}^{\beta})} \leq M, \forall \beta \in [0,2], \tag{2.1}$$

$$|P\mathcal{A}^2|_{\mathcal{L}(\mathbf{H}^{\beta+2}, \mathbf{H}^{\beta})} \leq M, \forall \beta \in [0,1], \tag{2.2}$$

and hence $(P\mathcal{A}^1)^* \in \mathcal{L}((\mathbf{H}^{\beta})^*, (\mathbf{H}^{\beta+1})^*)$ for $\beta \in [0,2]$ and $(P\mathcal{A}^2)^* \in \mathcal{L}((\mathbf{H}^{\beta})^*, (\mathbf{H}^{\beta+2})^*)$ for $\beta \in [0,1]$.

To analyze the convective term, we employ the following notation and bounds. Owing to [38, Lemma 2.1] adapted to fractional norms, the trilinear form defined by

$$b(u,v,w) := \int_{\mathbf{T}^3} ((u \cdot \nabla)v) \cdot w \ dx = \sum_{i,j=1}^{3} \int_{\mathbf{T}^3} u^i D_i v^j w^j \ dx,$$

satisfies

$$b(u,v,w) \lesssim_{\beta_1,\beta_2,\beta_3} |u|_{\beta_1} |v|_{\beta_2+1} |w|_{\beta_3}, \tag{2.3}$$

for every $\beta_1, \beta_2, \beta_3 \in \mathbf{R}_+$ such that

$$\beta_1 + \beta_2 + \beta_3 \geq \frac{3}{2}, \quad \text{if} \quad \beta_i \neq \frac{3}{2} \ \text{ for all } i \in \{1,2,3\},$$

$$\beta_1 + \beta_2 + \beta_3 > \frac{3}{2}, \quad \text{if } \beta_i = \frac{3}{2} \ \text{ for some } i \in \{1,2,3\}.$$

Furthermore, for all $u \in \mathbf{H}^{\beta_1}$ and $(v,w) \in \mathbf{W}^{\beta_2+1,2} \times \mathbf{W}^{\beta_3,2}$ such that $\beta_1, \beta_2, \beta_3$ satisfy (2.3), we have

$$b(u,v,w) = -b(u,w,v) \quad \text{and} \quad b(u,v,v) = 0. \tag{2.4}$$

For β_1, β_2, and β_3 that satisfy (2.3) and any given $(u,v) \in \mathbf{W}^{\beta_1,2} \times \mathbf{W}^{\beta_2+1,2}$, we define $B(u,v) \in \mathbf{W}^{-\beta_3,2}$ by

$$\langle B(u,v), w\rangle_{-\beta_3,\beta_3} = b(u,v,w), \quad \forall w \in \mathbf{W}^{\beta_3,2}.$$

In last, we define $B_P := PB$ and note that

$$B_P := PB : \mathbf{W}^{\beta_1,2} \times \mathbf{W}^{\beta_2+1,2} \to \mathbf{H}^{-\beta_3},$$

for β_1, β_2, and β_3 that satisfy (2.3). We set

$$B(u) = B(u,u), \quad \text{and} \quad B_P(u) := B_P(u,u).$$

2.2 Smoothing operators

Motivated by [2], we also need a family of self-adjoint smoothing operators $(J^\eta)_{\eta\in(0,1]}$ such that for all $\beta \in \mathbf{R}$ and $\gamma \in \mathbf{R}_+$,

$$|(I - J^\eta)f|_\beta \lesssim \eta^\gamma |f|_{\beta+\gamma} \qquad \text{and} \qquad |J^\eta f|_{\beta+\gamma} \lesssim \eta^{-\gamma}|f|_\beta. \tag{2.5}$$

For a construction of one such family, we refer [26, Section 2.2].

2.3 Rough path theory

For a given interval $I \subset \mathbf{R}$, we set

$$\Delta_I := \{(s,t) \in I^2 : s \le t\}, \qquad \Delta_I^{(2)} := \{(s,\theta,t) \in I^3 : s \le \theta \le t\}.$$

For a path $f : I \to \mathbf{R}^K$ we define its increment as $\delta f_{st} := f_t - f_s, \forall s,t \in I$ and for a two-index map $g : \Delta_I \to \mathbf{R}$, we define the second order increment operator

$$\delta g_{s\theta t} := g_{st} - g_{\theta t} - g_{s\theta}, \quad \forall (s,\theta,t) \in \Delta_I^{(2)}.$$

Let $\alpha > 0$ and J be a bounded interval in $\mathbf{R}$. We denote by $C_2^\alpha(J;\mathbf{R}^K)$ the closure of the set of smooth 2-index maps $g : \Delta_J \to \mathbf{R}^K$ with respect to the Hölder coefficient

$$[g]_{\alpha,J} := \sup_{s,t\in\Delta_J, s\neq t} \frac{|g_{st}|}{|t-s|^\alpha} < \infty.$$

Note that, since the zero element is $g_{st} = 0$ for all $(s,t) \in \Delta_J$, we infer that $[g]_{\alpha,J}$ is actually a norm on $C_2^\alpha(J;\mathbf{R}^K)$. Note that with this definition, the space $C_2^\alpha(J;\mathbf{R}^K)$ is Polish. By $C^\alpha(J;\mathbf{R}^K)$ we denote the closure of the set of smooth paths $f : J \to \mathbf{R}^K$ w.r.t. the semi-norm $[\delta f]_{\alpha,J}$. By $C^\alpha_{2,\mathrm{loc}}(\mathbb{T};\mathbf{R}^K)$ we denote the space of 2-index maps $g : \Delta_\mathbb{T} \to \mathbf{R}^K$ such that the restriction of g on every bounded interval $J \subset \mathbb{T}$, which we denote by $g|_{\Delta_J}$, belongs to $C_2^\alpha(J;\mathbf{R}^K)$.

Next, we present the definition of an α-Hölder rough path. A detailed exposition of rough path theory can be found in [20].

Definition 2.1. *Let $K \in \mathbf{N}$ and $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$. An α-Hölder rough path is a pair*

$$\boldsymbol{Z} = (Z, \mathbb{Z}) \in C^\alpha_{2,loc}(\mathbb{T};\mathbf{R}^K) \times C^{2\alpha}_{2,loc}(\mathbb{T};\mathbf{R}^{K\times K}), \tag{2.6}$$

such that for every bounded interval J in $\mathbb{T}$

$$(Z|_{\Delta_J}, \mathbb{Z}|_{\Delta_J}) \in C_2^\alpha(J;\mathbf{R}^K) \times C_2^{2\alpha}(J;\mathbf{R}^{K\times K}), \tag{2.7}$$

and satisfies the Chen's relation

$$\delta \mathbb{Z}_{s\theta t} = Z_{s\theta} \otimes Z_{\theta t}, \quad \forall (s,\theta,t) \in \Delta_J^{(2)}.$$

An α-Hölder rough path $\mathbf{Z} = (Z, \mathbb{Z})$ is said to be geometric if the restriction $\mathbf{Z}|_J$ can be obtained as the limit in the product topology of a sequence of rough paths $\{(Z^n, \mathbb{Z}^n)\}_{n\in\mathbf{N}} \subset C_2^{\alpha}(J;\mathbf{R}^K) \times C_2^{2\alpha}(J;\mathbf{R}^{K\times K})$ such that for each $n \in \mathbf{N}$,

$$Z^n_{st} := \delta z^n_{st} \quad \text{and} \quad \mathbb{Z}^n_{st} := \int_s^t \delta z^n_{s\theta} \otimes \mathrm{d}z^n_\theta,$$

for some smooth path $z^n : J \to \mathbf{R}^K$, where the iterated integral is understood in the Riemann sense.

We denote by $\mathcal{C}_g^{\alpha}(\mathbb{T};\mathbf{R}^K)$ the set of all geometric α-Hölder rough paths and endow it with the product topology.

For given bounded interval $J \subset \mathbf{R}$ and $(Z, \mathbb{Z}) \in C_2^{\alpha}(J;\mathbf{R}^K) \times C_2^{2\alpha}(J;\mathbf{R}^{K\times K})$, let us further set

$$||Z||_{\alpha,J} := \sup_{s,t\in\Delta_J, s\neq t} \frac{|Z_{st}|}{|t-s|^{\alpha}}, \qquad ||\mathbb{Z}||_{2\alpha,J} := \sup_{s,t\in\Delta_J, s\neq t} \frac{|\mathbb{Z}_{st}|}{|t-s|^{2\alpha}},$$

and

$$|||\mathbf{Z}|||_{\alpha,J} := ||Z||_{\alpha,J} + ||\mathbb{Z}||_{2\alpha,J}.$$

Note that $|||\cdot|||_{\alpha,J}$ is a norm on $C_2^{\alpha}(J;\mathbf{R}^K) \times C_2^{2\alpha}(J;\mathbf{R}^{K\times K})$.

We also need to deal with finite p-variation spaces. To introduce them let $\mathcal{P}(J)$ denote the set of all partitions of a bounded interval J and let V be a Banach space with norm $|\cdot|_V$. A function $g : \Delta_J \to V$ is said to have finite p-variation for some $p > 0$ on J if

$$|g|_{p-\mathrm{var};J;V} := \sup_{(t_i)\in\mathcal{P}(J)} \left(\sum_i |g_{t_i t_{i+1}}|_V^p \right)^{\frac{1}{p}} < \infty,$$

and we denote by $C_2^{p-\mathrm{var}}(J;V)$ the set of all continuous functions with finite p-variation on J equipped with the seminorm $|\cdot|_{p-\mathrm{var};J;V}$. We denote by $C^{p-\mathrm{var}}(J;V)$ the set of all paths $z : J \to V$ such that $\delta z \in C_2^{p-\mathrm{var}}(J;V)$.

A two-index map $\omega : \Delta_J \to [0,\infty)$ is called a control if

- it is continuous on Δ_J;
- it attains zero on diagonal i.e., for all $s \in J$, $\omega(s,s) = 0$;
- it is superadditive i.e., for all $(s,\theta,t) \in \Delta_J^{(2)}$, $\omega(s,\theta) + \omega(\theta,t) \leq \omega(s,t)$.

If for a given $p > 0$, $g \in C_2^{p-\mathrm{var}}(J;V)$, then it is well-known that the 2-index map $\omega_g : \Delta_J \to [0,\infty)$ defined by

$$\omega_g(s,t) := |g|^p_{p-\mathrm{var};[s,t]},$$

is a control, see [21, Proposition 5.8]. Moreover, in such situation, $|g_{st}|_V \leq \omega_g(s,t)^{\frac{1}{p}}$ for all $(s,t) \in \Delta_J$.

One can equivalently define a semi-norm on $C_2^{p-\mathrm{var}}(J;V)$ as, see [26, Section 2.3],

$$|g|_{p-var;[s,t]} = \inf\{\omega(s,t)^{\frac{1}{p}} : |g_{uv}|_V \leq \omega(u,v)^{\frac{1}{p}} \text{ for all } (u,v) \in \Delta_{[s,t]}\}. \tag{2.8}$$

Motivated by (2.8), we define a local version of the p-variation spaces.

Definition 2.2. *Given an interval $J = [a,b]$ for some $a, b \in \mathbb{T}$, a control ϖ on Δ_J, and a positive real number L, we denote by $C^{p-\mathrm{var}}_{2,\varpi,L}(J;V)$ the space of continuous two-index maps $g : \Delta_J \to V$ for which there exists at least one control ω such that $|g_{st}|_V \leq \omega(s,t)^{\frac{1}{p}}$ for every $(s,t) \in \Delta_J$ which gives $\varpi(s,t) \leq L$.*

We define a semi-norm on this space by

$$|g|_{p-\mathrm{var},\varpi,L;J} := \inf\left\{\omega(a,b)^{\frac{1}{p}} : \omega \text{ is a control s.t. } |g_{st}|_V \le \omega(s,t)^{\frac{1}{p}},\ \forall (s,t)\in\Delta_J \text{ with } \varpi(s,t)\le L\right\}.$$

By $C^{p-\mathrm{var}}_{2,\varpi,L,loc}(\mathbb{T};V)$ we mean the set of continuous two-index maps $g:\Delta_{\mathbb{T}}\to V$ such that for every bounded interval $J\subset\mathbb{T}$ the restriction $g|_{\Delta_J}$ belongs to $C^{p-\mathrm{var}}_{2,\varpi,L}(J;V)$.

Observe that, since the rough perturbation in (1.1) is (unbounded) operator valued, it is necessary to use the notion of unbounded rough drivers, which can be seen as operator valued rough paths with values in a suitable space of unbounded operators, see [2]. In what follows, by scale we mean a family $(E^\beta,|\cdot|_\beta)_{\beta\in\mathbf{R}_+}$ of Banach spaces such that $E^{\gamma+\beta}$ is continuously embedded into E^β for $\gamma\in\mathbf{R}_+$. For $\beta\in\mathbf{R}_+$, we denote by $E^{-\beta}$ the topological dual of E^β, and note that, in general, $E^{-0}\neq E^0$.

Definition 2.3. *Let $\alpha\in\left(\frac{1}{3},\frac{1}{2}\right]$ and a bounded interval $J\subset\mathbb{T}$ be given. A continuous unbounded α-rough driver with respect to the scale $(E^\beta,|\cdot|_\beta)_{\beta\in\mathbf{R}_+}$, is a pair $\mathbf{A}=(A^1,A^2)$ of 2-index maps such that there exists a control ω_A on J such that for every $(s,t)\in\Delta_J$,*

$$\begin{aligned}|A^1_{st}|_{\mathcal{L}(E^{-\beta},E^{-(\beta+1)})} &\le (\omega_A(s,t))^{\alpha} \quad \text{for } \beta\in[0,2],\\ |A^2_{st}|_{\mathcal{L}(E^{-\beta},E^{-(\beta+2)})} &\le (\omega_A(s,t))^{2\alpha} \quad \text{for } \beta\in[0,1],\end{aligned} \tag{2.9}$$

and the Chen relation holds true, that is,

$$\delta A^1_{s\theta t}=0,\qquad \delta A^2_{s\theta t}=A^1_{\theta t}A^1_{s\theta},\ \ \forall (s,\theta,t)\in\Delta^{(2)}_J. \tag{2.10}$$

2.4 Definition of weak solution

In this section, we define a notion of a weak solution to (1.1) and (1.2).

Let $z\in C^\alpha_{\mathrm{loc}}(\mathbb{T};\mathbf{R}^K)$ be such that it can be lifted to a continuous geometric α-Hölder rough path $\mathbf{Z}=(Z,\mathbb{Z})\in\mathcal{C}^\alpha_{g,\mathrm{loc}}(\mathbb{T};\mathbf{R}^K)$ for some $\alpha\in\left(\frac{1}{3},\frac{1}{2}\right]$. For each $k\in\{1,\dots,K\}$, assume that $\sigma_k:\mathbf{T}^3\to\mathbf{R}^3$ is twice differentiable and divergence-free. Moreover, assume that for all $k\in\{1,\dots,K\}$, σ_k and its derivatives up to order two are bounded uniformly.

Applying the Leray projection $P:\mathbf{W}^{\alpha,2}\to\mathbf{H}^\alpha$ and gradient projection $Q:\mathbf{W}^{\alpha,2}\to\mathbf{H}^\alpha_\perp$, defined in Section 2.1, separately to (1.1) with (1.2) yields

$$\partial_t u + P[(u\cdot\nabla)u] = \Delta u + P[(\sigma_k\cdot\nabla)u]\dot z^k_t, \tag{2.11}$$

$$\nabla p + Q[(u\cdot\nabla)u] = Q[(\sigma_k\cdot\nabla)u]\dot z^k_t. \tag{2.12}$$

By setting

$$\pi := \int_0^{\cdot} \nabla p_r\,dr,$$

and integrating the system (2.11)-(2.12) over $[s,t]$ and then iterating the equation into itself we obtain, see [26, Section 2.5] for a complete derivation,

$$\delta u_{st} + \int_s^t P[(u_r\cdot\nabla)u_r]\,dr = \int_s^t \Delta u_r dr + [A^{P,1}_{st}+A^{P,2}_{st}]u_s + u^{P,\natural}_{st}, \tag{2.13}$$

$$\delta \pi_{st} + \int_s^t Q[(u_r\cdot\nabla)u_r)]\,dr = [A^{Q,1}_{st}+A^{Q,2}_{st}]u_s + u^{Q,\natural}_{st}, \tag{2.14}$$

where

$$\begin{aligned} A^{P,1}_{st}\varphi &:= P[(\sigma_k\cdot\nabla)\varphi]\,Z^k_{st}, \qquad & A^{P,2}_{st}\varphi &:= P[(\sigma_k\cdot\nabla)P[(\sigma_i\cdot\nabla)\varphi]]\mathbb{Z}^{i,k}_{st},\\ A^{Q,1}_{st}\varphi &:= Q[(\sigma_k\cdot\nabla)\varphi]\,Z^k_{st}, \qquad & A^{Q,2}_{st}\varphi &:= Q[(\sigma_k\cdot\nabla)P[(\sigma_i\cdot\nabla)\varphi]]\,\mathbb{Z}^{i,k}_{st}.\end{aligned}$$

Remark 2.4. As shown in [26, Section 4.1.2], the pressure term π can be uniquely determined by the velocity u. Hence, we only concentrate on the construction of the random dynamical system associated to (2.11).

To define the considered notion of a weak solution which is suitable for semiflow selection, let us first define the set of admissible initial data

$$\mathbf{D} := \left\{ [x,e] \in \mathbf{H}^0 \times \mathbf{R}_+ : \frac{1}{2}|x|_0^2 \le e \right\}.$$

Note that $\mathbf{D}$ is a closed convex subset of $\mathbf{H}^0 \times \mathbf{R}_+$. Recall that $\mathbb{T} = [0,\infty)$.

Definition 2.5 (Weak solution). *Given $[u_0, E_0] \in \boldsymbol{D}$ and a geometric α-Hölder rough path*

$$\boldsymbol{Z} = (Z, \mathbb{Z}) \in C^{\alpha}_{2,loc}(\mathbb{T}; \mathbf{R}^K) \times C^{2\alpha}_{2,loc}(\mathbb{T}; \mathbf{R}^{K\times K}), \text{ for some } \alpha \in \left(\frac{1}{3}, \frac{1}{2}\right], \tag{2.15}$$

we say that a pair $[u, E]$ is a weak solution of (2.11) *if*

1. *$u : \mathbb{T} \to \boldsymbol{H}^0$ is a weakly continuous function and $u \in L^2_{loc}(\mathbb{T}; \boldsymbol{H}^1) \cap L^\infty_{loc}(\mathbb{T}; \boldsymbol{H}^0)$;*
2. *$E : \mathbb{T} \to \mathbf{R}_+$ satisfies $E(t) = \frac{1}{2}|u_t|_0^2$ a.e. $t \in \mathbb{T}$;*
3. *$E(t)$ is a non-increasing function of t. In the variational form we write this as $E(0-) = E(0)$ and*
$$[E\psi]_{t=\tau_1-}^{t=\tau_2+} - \int_{\tau_1}^{\tau_2} E\partial_t\psi\, dt + \int_{\tau_1}^{\tau_2} \psi \int_{\boldsymbol{T}^3} |\nabla u_t|^2\, dx\, dt \le 0, \tag{2.16}$$
for every $0 \le \tau_1 \le \tau_2$ and $\psi \in C^1_c(\mathbb{T})$ with $\psi \ge 0$;
4. *the remainder $u^{P,\natural} : \Delta_{\mathbb{T}} \to \boldsymbol{H}^{-3}$ which is defined, for all $\phi \in \boldsymbol{H}^3$, and $(s,t) \in \Delta_{\mathbb{T}}$ by*
$$u^{P,\natural}_{st}(\phi) := \delta u_{st}(\phi) + \int_s^t \left[(\nabla u_r, \nabla\phi) + B_P(u_r)(\phi)\right] dr - u_s([A^{P,1,*}_{st} + A^{P,2,*}_{st}]\phi), \tag{2.17}$$
satisfy
$$u^{P,\natural} \in C^{\frac{p}{3}-\text{var}}_{2,\varpi,L,loc}(\mathbb{T}; \boldsymbol{H}^{-3}), \tag{2.18}$$
for some control ϖ and $L > 0$.

The next result gives existence of a weak solution to (2.11) for any initial data and a rough transport perturbation. Even though the energy inequality (2.16) was not included in the corresponding definition of weak solution in [26], it can be verified that it is satisfied by the solutions constructed in [26, Theorem 2.13]. The necessary ideas are also discussed in the proof of stability in Theorem 3.3 below.

Theorem 2.6. *[26, Theorem 2.13] For a given initial data $[u_0, E_0] \in \boldsymbol{D}$, a geometric α-Hölder rough path $\boldsymbol{Z}$, for some $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$, there exists a weak solution to* (2.11)*, in the sense of Definition 2.5, which satisfies,*

$$\frac{1}{2}|u_t|_0^2 + \int_0^t |\nabla u_r|_0^2\, dr \le \frac{1}{2}|u_0|_0^2 \le E_0, \qquad \forall t \in \mathbb{T}. \tag{2.19}$$

Remark 2.7. Given $[u_0, E_0] \in \mathbf{D}$ and a geometric α-Hölder rough path $\mathbf{Z}$, if $[u, E]$ is a weak solution of (2.11), then we can always consider that $\frac{1}{2}|u_t|_0^2 \le E(t-)$ for all $t \in \mathbb{T}$. Moreover, we will write $\{[u,E](t); t \in \mathbb{T}\}$ and $\{[u_t, E(t-)]; t \in \mathbb{T}\}$ instead $[u, E]$ if we want to give information about the time scale.

3 Semiflow selection

Throughout this section, we continue our pathwise analysis and let

$$\mathbf{Z} = (Z, \mathbb{Z}) \in \mathcal{C}^{\alpha}_{g,\mathrm{loc}}(\mathbb{T}; \mathbf{R}^K), \quad \alpha \in \left(\frac{1}{3}, \frac{1}{2}\right],$$

be a continuous geometric α-Hölder rough path. Randomness is only going to reappear in Section 4 below. We consider the following separable metric space as the trajectory space

$$\mathbf{X} := C_{\mathrm{loc}}(\mathbb{T}; \mathbf{H}^{-1}) \times L^1_{\mathrm{loc}}(\mathbb{T}; \mathbf{R}). \tag{3.1}$$

For data $[x, e] \in \mathbf{D}$, we introduce the solution set

$$\mathcal{U}[x, e, \mathbf{Z}] := \left\{ [u, E] \in \mathbf{X} \;\middle|\; \begin{array}{c} [u, E] \text{ is a weak solution to (2.11) perturbed by } \mathbf{Z} \\ \text{with initial data } [x, e] \end{array} \right\}.$$

In order to fulfill the criterion of maximal energy dissipation, and following [6], for a fixed initial data and a rough path, we focus on a subclass of weak solutions consisting of the ones which minimize the total energy. To define this subclass we introduce a partial relation $\prec$ as follows: if $[u^i, E^i]$, $i = 1, 2$, are two weak solutions to (2.11) perturbed by the same rough path $\mathbf{Z}$ and starting from the same initial data $[u_0, E_0]$, we write $[u^1, E^1] \prec [u^2, E^2]$ iff

$$E^1(t\pm) \le E^2(t\pm) \text{ for every } t \in \mathbb{T} \setminus \{0\}.$$

Definition 3.1 (Admissible weak solution)**.** *We say that a weak solution $[u, E]$ to (2.11) perturbed by $\boldsymbol{Z}$ starting from the initial data $[u_0, E_0]$ is* admissible *if it is minimal with respect to the relation $\prec$. Specifically, if*

$$[\tilde{u}, \tilde{E}] \prec [u, E],$$

where $[\tilde{u}, \tilde{E}]$ is another weak solution to (2.11) driven by path $\boldsymbol{Z}$ and starting from $[u_0, E_0]$, then

$$E = \tilde{E} \text{ on } \mathbb{T}.$$

We can now define a semiflow selection to (2.11).

Definition 3.2 (Semiflow selection)**.** *A* semiflow selection *in the class of weak solutions for the problem (2.11) is a Borel measurable mapping*

$$U : \boldsymbol{D} \times \mathcal{C}^{\alpha}_{g,loc}(\mathbb{T}; \mathbf{R}^K) \to \mathbf{X},$$
$$U\{u_0, E_0, \boldsymbol{Z}\} \in \mathcal{U}[u_0, E_0, \boldsymbol{Z}] \text{ for any } [u_0, E_0, \boldsymbol{Z}] \in \boldsymbol{D} \times \mathcal{C}^{\alpha}_{g,loc}(\mathbb{T}; \mathbf{R}^K),$$

*which enjoys the following **semigroup property**:*

$$U\{u_0, E_0, \boldsymbol{Z}\}(t_1 + t_2) = U\left\{U\{u_0, E_0, \boldsymbol{Z}\}(t_1), \tilde{\boldsymbol{Z}}_{t_1}\right\}(t_2),$$

for any $[u_0, E_0] \in \boldsymbol{D}$ and any $t_1, t_2 \in \mathbb{T}$, where $\tilde{\boldsymbol{Z}}_{t_1}(\cdot) := \boldsymbol{Z}(t_1 + \cdot) = (Z, \mathbb{Z})_{t_1+\cdot, t_1+\cdot}$.

Observe that $\tilde{\mathbf{Z}}_{t_1}(\cdot)$ defined above is again a rough path, in particular, Chen's relation holds true.

3.1 Sequential stability

In this subsection we address the issue of sequential stability which will allow us to show the compactness of the set $\mathcal{U}[u_0, E_0, \mathbf{Z}]$ as well as the required measurability of the semiflow selection. It is also essential for proving the measurability of the random dynamical system constructed in Section 4.

Given $T > 0$, we let $\Delta_T := \Delta_{[0,T]}$ and $\Delta^{(2)}_T = \Delta^{(2)}_{[0,T]}$.

Theorem 3.3. *Let $\{\boldsymbol{Z}^N = (Z^N, \mathbb{Z}^N)\}_{N\in\mathbf{N}}$ be a sequence of geometric α-Hölder rough paths such that $\boldsymbol{Z}^N$ converges to some α-Hölder rough path $\boldsymbol{Z} = (Z, \mathbb{Z})$ in the product topology on $C^{\alpha}_{2,\mathrm{loc}}(\mathbb{T}; \mathbf{R}^K) \times C^{2\alpha}_{2,\mathrm{loc}}(\mathbb{T}; \mathbf{R}^{K\times K})$. Suppose that $\{[u_0^N, E_0^N]\}_{N\in\mathbf{N}} \subset \boldsymbol{D}$ is a sequence of initial data and that there exists a positive real number $\mathcal{E}$, such that*

$$E_0^N \le \mathcal{E}, \tag{3.2}$$

for every $N \in \mathbf{N}$. Let $[u^N, E^N] \in \mathcal{U}[u_0^N, E_0^N, \boldsymbol{Z}^N]$, $N \in \mathbf{N}$, be a family of associated weak solutions. Then

1. *there exist $[u_0, E_0] \in \boldsymbol{D}$ and a subsequence, indexed again by N, such that*

$$u_0^N \to u_0 \text{ weakly in } \boldsymbol{H}^0, \qquad E_0^N \to E_0. \tag{3.3}$$

2. *for the subsequence of solutions $\{[u^N, E^N]\}_{N\in\mathbf{N}}$, corresponding to the data $\{[u_0^N, E_0^N], \boldsymbol{Z}^N\}_{N\in\mathbf{N}}$ from part (a), there exists a weak solution $[u, E]$ such that the following hold*

$$\begin{aligned} u^N &\to u \text{ in } C_{\mathrm{loc}}(\mathbb{T}; \boldsymbol{H}^{-1}), \\ E^N(t) &\to E(t) \text{ for any } t \in \mathbb{T} \text{ and in } L^1_{\mathrm{loc}}(\mathbb{T}; \mathbf{R}). \end{aligned}$$

Proof of Theorem 3.3. First observe that the convergences in (3.3) follow immediately from the fact that E_0^N, in particular $|u_0^N|_0^2$, are uniformly bounded in N by $\mathcal{E}$. Let us fix an arbitrary time $T > 0$. Notice that to prove the Theorem 3.3, it is sufficient to prove all the required results on $[0, T]$.

Observe that due to convergence $\mathbf{Z}^N \to \mathbf{Z}$ in the mentioned product topology, for every $\varepsilon > 0$, there exists an $N_0 := N_0(\varepsilon) \in \mathbf{N}$ such that

$$|||\mathbf{Z}^N - \mathbf{Z}|||_{\alpha,[0,T]} < \varepsilon, \quad \text{for all} \quad N \ge N_0.$$

Consequently, the reverse triangle inequality yields, for all $N \ge N_0$,

$$|||\mathbf{Z}^N|||_{\alpha,[0,T]} < \varepsilon + |||\mathbf{Z}|||_{\alpha}.$$

Since the above holds for every $\varepsilon > 0$, we fix $\varepsilon = 1$ and get

$$|||\mathbf{Z}^N|||_{\alpha,[0,T]} \le \max\{|||\mathbf{Z}|||_{\alpha,[0,T]} + 1, |||\mathbf{Z}^1|||_{\alpha,[0,T]}, \cdots, |||\mathbf{Z}^{N_0}|||_{\alpha,[0,T]}\} =: R.$$

Let us set

$$\omega_Z(s,t) := (t-s)R^{1/\alpha}, \qquad (s,t) \in \Delta_T. \tag{3.4}$$

Then, it is easy to show that ω_Z is a control and we have

$$|Z^N_{st}| \le (\omega_Z(s,t))^{\alpha}, \qquad |\mathbb{Z}^N_{st}| \le (\omega_Z(s,t))^{2\alpha}, \quad \forall (s,t) \in \Delta_T. \tag{3.5}$$

To move further, let us define

$$\begin{aligned} A^{N,1}_{st}\phi &:= P\left[(\sigma_k \cdot \nabla)\phi\right] Z^{N,k}_{st}, \\ A^{N,2}_{st}\phi &:= P\left[(\sigma_k \cdot \nabla)P[(\sigma_j \cdot \nabla)\phi]\right] \mathbb{Z}^{N,j,k}_{st}. \end{aligned} \tag{3.6}$$

Next we claim that, for $\beta \in [0, 2]$,

$$|A^{N,1}_{st}|_{\mathcal{L}(\mathbf{H}^{\beta+1}, \mathbf{H}^{\beta})} \le M(\omega_Z(s,t))^{\alpha}, \quad \text{for } \beta \in [0,2], \tag{3.7}$$

$$|A^{N,2}_{st}|_{\mathcal{L}(\mathbf{H}^{\beta+2}, \mathbf{H}^{\beta})} \le M(\omega_Z(s,t))^{2\alpha}, \quad \text{for } \beta \in [0,1], \tag{3.8}$$

where M is introduced in Section 2.1. We will only prove (3.7), since the proof of (3.8) is similar. Observe that, for $\beta \in [0,2]$, estimates (3.5) and (2.1) give

$$|A^{N,1}_{st}|_{\mathcal{L}(\mathbf{H}^{\beta+1},\mathbf{H}^{\beta})} \leq |P\mathcal{A}^1|_{\mathcal{L}(\mathbf{H}^{\beta+1},\mathbf{H}^{\beta})}|Z^N_{st}| \leq M(\omega_Z(s,t))^{\alpha}.$$

Similarly we can get the inequality in (3.8). Hence, by Definition 2.3, the $\{(A^{N,1}_{st}, A^{N,2}_{st})\}_N$ is a family of unbounded rough drivers, with

$$\omega_{A^N}(s,t) := M^{1/\alpha}\omega_Z(s,t),$$

on the scale $(\mathbf{H}^{\beta})_{\beta\in\mathbf{R}_+}$ which is uniformly bounded in N.

Now, we prove the convergences in part (b). The case when the sequence of rough paths does not depend on N is proven [26, Theorem 4.1]. We still include the whole idea here with more details for the completion.

First observe that, without loss of generality, we may assume that the same control ϖ and constant $L > 0$ works for each element of sequence $\{u^N\}_{N\geq 1}$ in the Definition 2.5. Since, for every $N \in \mathbf{N}$, $\frac{1}{2}|u^N_0|^2_0 \leq \mathcal{E}$ and corresponding u^N satisfies the energy inequality (2.19), we get that the sequence $\{u^N\}_{N\geq 1}$ is uniformly bounded in $L^2_T\mathbf{H}^1 \cap L^\infty_T\mathbf{H}^0$, an application of Banach-Alaoglu theorem yields a subsequence, which we will index again as $\{u^N\}_{N\geq 1}$, that converges weakly in $L^2_T\mathbf{H}^1$ and weak-* in $L^\infty_T\mathbf{H}^0$.

To obtain a further subsequence that converges strongly in $L^2_T\mathbf{H}^0 \cap C_T\mathbf{H}^{-1}$, thanks to Lemma A.4, it is sufficient to show that there exist controls ω and $\bar{\omega}$ and $\bar{L}, \kappa > 0$, independent of N, such that $|\delta u^N_{st}|_{-1} \leq \omega(s,t)$ for all $(s,t) \in \Delta_T$ with $\bar{\omega}(s,t) \leq \bar{L}$.

Let $\phi \in \mathbf{H}^1$. Decomposition of δu^N_{st} into a smooth and non-smooth part using J^η (defined in Section 2.2) for some $\eta \in (0,1]$, yields

$$|\delta u^N_{st}(\phi)| \leq |\delta u^N_{st}(J^\eta\phi)| + |\delta u^N_{st}((I-J^\eta)\phi)|. \tag{3.9}$$

By applying (2.5) and (2.19) we estimate the second term in above as

$$|\delta u^N_{st}((I-J^\eta)\phi)| \lesssim |u^N|_{L^\infty_T\mathbf{H}^0}|(I-J^\eta)\phi|_0 \lesssim \eta|u^N_0|_0|\phi|_1 \leq \eta\sqrt{\mathcal{E}}|\phi|_1. \tag{3.10}$$

For the first term on the right hand side of (3.9), by letting

$$\mu^N_t(\phi) := -\int_0^t \left[(\nabla u^N_r, \nabla\phi) + B_P(u^N_r)(\phi)\right]\, dr, \quad \phi \in \mathbf{H}^1,$$

(2.17) gives that for all $(s,t) \in \Delta_T$,

$$\delta u^N_{st} = \delta\mu^N_{st} + A^{N,1}_{st}u_s + A^{N,2}_{st}u_s + u^{P,\natural,N}_{st}, \tag{3.11}$$

where the equality holds in $\mathbf{H}^{-3}$. Consequently, we get

$$|\delta u^N_{st}(J^\eta\phi)| \leq |u^{P,\natural,N}_{st}(J^\eta\phi)| + |\delta\mu^N_{st}(J^\eta\phi)| + |u^N_s(A^{N,1,*}_{st}J^\eta\phi)| + |u^N_s(A^{N,2,*}_{st}J^\eta\phi)|. \tag{3.12}$$

We estimate each term of (3.12) separately as follows:

1) By Lemma A.1 we infer that there is a positive constant $\tilde{L}$, depending only on p (i.e., independent of N), such that for all $(s,t) \in \Delta_T$ with $\varpi(s,t) \leq L$ and $M^{1/\alpha}\omega_Z(s,t) = \omega_{A^N}(s,t) \leq \tilde{L}$, the following inequality is true

$$\begin{aligned}\omega_{P,\natural,N}(s,t) &\lesssim_p |u^N|^{\frac{p}{3}}_{L^\infty_T\mathbf{H}^0}\omega_{A^N}(s,t) + (1+|u^N|_{L^\infty_T\mathbf{H}^0})^{\frac{2p}{3}}(t-s)^{\frac{p}{3}}\omega_{A^N}(s,t)^{\frac{1}{12}} \\ &\lesssim_p M^{1/\alpha}|u^N|^{\frac{p}{3}}_{L^\infty_T\mathbf{H}^0}\omega_Z(s,t) + M^{1/\alpha}(1+|u^N|_{L^\infty_T\mathbf{H}^0})^{\frac{2p}{3}}(t-s)^{\frac{p}{3}}\omega_Z(s,t)^{\frac{1}{12}},\end{aligned} \tag{3.13}$$

where $\omega_{P,\natural,N}(s,t) := |u^{P,\natural,N}|^{\frac{p}{3}}_{\frac{p}{3}-var;[s,t];\mathbf{H}^{-3}}$. Since $u^{P,\natural,N}$ is the remainder, (2.18) followed by (2.5) and (3.13) yield

$$\begin{aligned}|u^{P,\natural,\varepsilon}_{st}(J^\eta\phi)| &\le \omega_{P,\natural,N}(s,t)^{\frac{3}{p}}|J^\eta\phi|_3\\ &\lesssim_p \eta^{-2}M^{\frac{3}{p\alpha}}\left[|u^N|^{\frac{p}{3}}_{L^\infty_T\mathbf{H}^0}\omega_Z(s,t)+(1+|u^N|_{L^\infty_T\mathbf{H}^0})^{\frac{2p}{3}}(t-s)^{\frac{p}{3}}\omega_Z(s,t)^{\frac{1}{12}}\right]^{\frac{3}{p}}|\phi|_1\\ &\lesssim \eta^{-2}\left[\sqrt{\mathcal{E}}(\omega_Z(s,t))^{\frac{3}{p}}+(1+\mathcal{E})(t-s)(\omega_Z(s,t))^{\frac{1}{4p}}\right]|\phi|_1. \end{aligned}\tag{3.14}$$

2) Let us set $\omega_{\mu^N}(s,t) := \int_s^t(1+|u^N_r|_1)^2\,dr$. Since $u^N\in L^2_T\mathbf{H}^1$, we infer that ω_μ is a control. Using (2.3) and the the Cauchy-Schwartz inequality we deduce that

$$\begin{aligned}|\delta\mu_{st}|_{-3} &\le \sup_{|\phi|_3\le 1}\int_s^t\left[|(\nabla u_r,\nabla\phi)|+|B_P(u_r)(\phi)|\right]dr\\ &\le \sup_{|\phi|_3\le1}|\phi|_3\int_s^t|\nabla u_r|_0\,dr+\int_s^t|u_r|_1^2\,dr\lesssim\int_s^t\left(1+2|u_r|_1+|u_r|_1^2\right)dr=\omega_\mu(s,t).\end{aligned}$$

Then, by using (2.5) and (2.19) we obtain

$$|\delta\mu^N_{st}(J^\eta\phi)|\lesssim\eta^{-2}\omega_{\mu^N}(s,t)|\phi|_1=\eta^{-2}\left[\int_s^t(1+|u^N_r|_1)^2\,dr\right]|\phi|_1\lesssim\eta^{-2}(t-s)(1+\mathcal{E})|\phi|_1. \tag{3.15}$$

3) By applying (3.7) followed by (2.5) and (2.19) we yield, with $\alpha=\frac{1}{p}$,

$$\begin{aligned}|u^N_s(A^{N,1,*}_{st}J^\eta\phi)| &\lesssim M|u^N|_{L^\infty_T\mathbf{H}^0}(\omega_Z(s,t))^{\frac{1}{p}}|J^\eta\phi|_1\lesssim|u^N_0|_0(\omega_Z(s,t))^{\frac{1}{p}}|\phi|_1\\ &\le\sqrt{\mathcal{E}}(\omega_Z(s,t))^{\frac{1}{p}}|\phi|_1.\end{aligned}\tag{3.16}$$

4) Again by using (3.8) with (2.5) and (2.19) we get

$$\begin{aligned}|u^\varepsilon_s(A^{N,2,*}_{st}J^\eta\phi)| &\lesssim M|u^N|_{L^\infty_T\mathbf{H}^0}(\omega_Z(s,t))^{\frac{2}{p}}|J^\eta\phi|_2\lesssim\eta^{-1}|u^N_0|_0(\omega_Z(s,t))^{\frac{2}{p}}|\phi|_1\\ &\le\eta^{-1}\sqrt{\mathcal{E}}(\omega_Z(s,t))^{\frac{2}{p}}|\phi|_1.\end{aligned}\tag{3.17}$$

So by substituting (3.10), (3.14)-(3.17) into (3.12), for each $\phi\in\mathbf{H}^1$ we have

$$\begin{aligned}|\delta u^N_{st}(\phi)| &\lesssim\eta^{-2}\left[\sqrt{\mathcal{E}}(\omega_Z(s,t))^{\frac{3}{p}}+(1+\mathcal{E})(t-s)(\omega_Z(s,t))^{\frac{1}{4p}}\right]|\phi|_1+\eta^{-1}(t-s)(1+\mathcal{E})|\phi|_1\\ &\quad+\sqrt{\mathcal{E}}(\omega_Z(s,t))^{\frac{1}{p}}|\phi|_1+\eta^{-1}\sqrt{\mathcal{E}}(\omega_Z(s,t))^{\frac{2}{p}}|\phi|_1+\eta\sqrt{\mathcal{E}}|\phi|_1.\end{aligned}$$

Let us set $\eta := (\omega_Z(s,t))^{\frac{1}{p}}+(t-s)^{\frac{1}{p}}$. Observe that we can choose M such that $\eta\in[0,1)$. Indeed, since $\tilde{L}$ is fixed and $\omega_Z(s,t)$ defines as in (3.4), we can choose M large enough such that the inequalities (2.1)-(2.2), the relation $\omega_Z(s,t)\le\frac{\tilde{L}}{M^{1/\alpha}}<\frac{1}{2}$ and $(t-s)^{\frac{1}{p}}<\frac{1}{2}$ hold true for all $(s,t)\in\Delta_T$ with $\bar{\omega}(s,t)\le\bar{L}$.

Consequently, we infer that

$$|\delta u^N_{st}|_{-1}\lesssim_{M,\mathcal{E}}(1+|u_0|_0)^2(\omega_Z(s,t)^{\frac{1}{p}}+(t-s)^{1-\frac{2}{p}}). \tag{3.18}$$

Since for $p\ge2$, and $\kappa>0$

$$\omega_Z(s,t)^{\frac{1}{p}}+(t-s)^{1-\frac{2}{p}}\lesssim_{p,\kappa}\left(\omega_Z(s,t)^{\frac{\kappa}{p}}+(t-s)^{\kappa\left(1-\frac{2}{p}\right)}\right)^{\frac{1}{\kappa}},$$

by choosing κ which satisfy

$$\kappa\ge p \text{ and } \kappa\ge\frac{p}{p-2},$$

we deduce that

$$\tilde{\omega}(s,t) := \omega_Z(s,t)^{\frac{\kappa}{p}} + (t-s)^{\kappa\left(1-\frac{2}{p}\right)},$$

is a control.

Hence, due to Compactness Lemma A.4, there is a subsequence of $\{u^N\}_{N\in\mathbf{N}}$, which we continue to denote by $\{u^N\}_{N\in\mathbf{N}}$, converging strongly to an element u in $C_T\mathbf{H}^{-1}\cap L^2_T\mathbf{H}^0$.

Now recall that $\delta u^N_{st} := u^N_t - u^N_s$ and, from Definition 2.5,

$$\delta u^N_{st} + \int_s^t B_p(u^N_r)\,dr = \int_s^t \Delta u^N_r\,dr + [A^{N,1}_{st} + A^{N,2}_{st}]u^N_s + u^{P,\natural,N}_{st}, \tag{3.19}$$

where $A^{N,1}_{st}$ and $A^{N,2}_{st}$ are defined as in (3.6).

Our goal now is to prove that u is a weak solution to (2.11). The idea is to pass the limit in (3.19) tested against some $\phi\in\mathbf{H}^3$ as N tends to ∞.

For the terms with operators $A^{N,i}, i=1,2$, observe that

$$|(u^N_s, A^{N,i,*}_{st}\phi) - (u_s, A^{i,*}_{st}\phi)| \le |u^N_s - u_s|_{-1}|A^{N,i,*}_{st}\phi|_1 + |u^N_s - u_s|_0|(A^{N,i,*}_{st} - A^{i,*}_{st})\phi|_0. \tag{3.20}$$

The first term in the r.h.s of (3.20) goes to 0 as $N\to\infty$ because $u^N\to u$ in $C_T\mathbf{H}^{-1}$. To estimate the second term in (3.20) we proceed as follows: bound (2.1) yield

$$|(A^{N,1,*}_{st} - A^{1,*}_{st})\phi|_0 \le |P[\sigma_k\cdot\nabla]|_{\mathcal{L}(\mathbf{H}^1,\mathbf{H}^0)}|\phi|_1|Z^{N,k}_{st} - Z^k_{st}| \lesssim_M |\phi|_1|Z^N_{st} - Z_{st}|. \tag{3.21}$$

Similarly, the estimate (2.2) gives

$$|(A^{N,2,*}_{st} - A^{2,*}_{st})\phi|_0 \le |P[(\sigma_k\cdot\nabla)P[\sigma_j\cdot\nabla]]|_{\mathcal{L}(\mathbf{H}^2,\mathbf{H}^0)}|\phi|_2|\mathbb{Z}^{N,i,k}_{st} - \mathbb{Z}^{i,k}_{st}| \lesssim_M |\phi|_2|\mathbb{Z}^N_{st} - \mathbb{Z}_{st}|. \tag{3.22}$$

So, since $Z^N\to Z$ in $C^{\alpha}_{2,\mathrm{loc}}(\mathbb{T};\mathbf{R}^K)$ and $\mathbb{Z}^N\to\mathbb{Z}$ in $C^{2\alpha}_{2,\mathrm{loc}}(\mathbb{T};\mathbf{R}^{K\times K})$, from (3.21)-(3.22) we infer that, for $i=1,2$,

$$|(u^N_s, A^{N,i,*}_{st}\phi) - (u_s, A^{i,*}_{st}\phi)| \to 0 \text{ as } N\to\infty. \tag{3.23}$$

Further, using the Hölder inequality, the strong convergence in $L^2_T\mathbf{H}^0$ of $\{u^N\}_{\varepsilon>0}$ and trilinear estimate (2.3), we find

$$\begin{aligned}
&\left|\int_s^t \left[B_P(u_r)(\phi) - B_P(u^N_r)(\phi)\right]\,dr\right| \\
&\le \left|\int_s^t B_P(u_r - u^N_r, u_r)(\phi)\,dr\right| + \left|\int_s^t B_P(u^N_r, u_r - u^N_r)(\phi)\,dr\right| \\
&\lesssim \int_s^t |u_r - u^N_r|_0|u_r|_0\,dr|\phi|_3 + \int_s^t |u_r - u^N_r|_0|u^N_r|_0\,dr|\phi|_3 \\
&\le |\phi|_3\left(\int_s^t |u_r - u^N_r|_0^2\,dr\right)^{1/2}\left[\left(\int_s^t |u_r|_0^2\,dr\right)^{1/2} + \left(\int_s^t |u^N_r|_0^2\,dr\right)^{1/2}\right] \to 0,
\end{aligned}$$

as $N\to\infty$. Finally, using the Hölder inequality, the strong convergence in $L^2_T\mathbf{H}^0$ of $\{u^N\}_{\varepsilon>0}$ we have

$$\begin{aligned}
&\left|\int_s^t [\Delta u^N_r - \Delta u_r](\phi)\,dr\right| \\
&\quad\le |\Delta\phi|_0\int_s^t |u^N_r - u_r|_0\,dr \le |\Delta\phi|_0\left(\int_s^t |u_r - u^N_r|_0^2\,dr\right)^{1/2}|t-s|^{1/2}\to 0,
\end{aligned}$$

as $N \to \infty$.

Hence, since we have shown that all of the terms in equation (3.19) converge when applied to ϕ, the remainder $u_{st}^{P,\natural,N}(\phi)$ converges to some limit $u_{st}^{P,\natural}(\phi)$. Since $|u^{P,\natural}|_{\frac{p}{3}-var;[s,t];\mathbf{H}^{-3}}^{\frac{p}{3}}$ is equal to the infimum over all controls satisfying $|u_{st}^{P,\natural}|_{-3} \leq \omega_{P,\natural}(s,t)^{\frac{3}{p}}$, by above convergence results and (3.13) we obtain

$$|u_{st}^{P,\natural}|_{-3} \leq \sup_{|\phi|_3 \leq 1} |(u_{st}^{P,\natural} - u_{st}^{P,\natural,N})(\phi)| + \sup_{|\phi|_3 \leq 1} |u_{st}^{P,\natural,N}(\phi)|,$$

where, as in (3.14),

$$|u_{st}^{P,\natural,N}(\phi)| \lesssim \left[\mathcal{E}(\omega_Z(s,t))^{\frac{3}{p}} + (1+\mathcal{E})(t-s)(\omega_Z(s,t))^{\frac{1}{4p}}\right] |\phi|_3.$$

So, by taking the limit $N \to \infty$ we get that

$$|u_{st}^{P,\natural}|_{-3} \lesssim \left[\mathcal{E}(\omega_Z(s,t))^{\frac{3}{p}} + (1+\mathcal{E})(t-s)(\omega_Z(s,t))^{\frac{1}{4p}}\right].$$

Hence, $u^{P,\natural} \in C_{2,\varpi,L}^{\frac{p}{3}-\mathrm{var}}([0,T];\mathbf{H}^{-3})$ for some control ϖ depending only on ω_Z and $L > 0$ depending only on p.

Next, we prove that $u \in C_T\mathbf{H}_w^0$. Recall that $u \in L_T^\infty\mathbf{H}^0 \cap C_T\mathbf{H}^{-1}$. Let $\phi \in \mathbf{H}^0$. Since $\mathbf{H}^1$ is dense in $\mathbf{H}^0$, there exists a sequence $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathbf{H}^1$ such that $|\phi_n - \phi|_0 \to 0$ as $n \to \infty$. Then

$$|\langle u_t - u_s, \phi\rangle|_0 \leq 2|u|_{L_T^\infty\mathbf{H}^0}|\phi - \phi_n|_0 + |\langle u_t - u_s, \phi_n\rangle|_0.$$

Since $u \in C_T\mathbf{H}^{-1}$ and $\phi_n \in \mathbf{H}^1$, $|\langle u_t - u_s, \phi_n\rangle|_0 \to 0$ as $s \to t$. Consequently,

$$\lim_{s\to t} |\langle u_t - u_s, \phi\rangle|_0 \leq 2|u|_{L_T^\infty\mathbf{H}^0}\delta,$$

for any $\delta > 0$. So $\lim_{s\to t} u_s(\phi) = u_t(\phi), \forall \phi \in \mathbf{H}^0$.

Now note that, since E^N is non-increasing, for every $T > 0$ and any partition $0 = t_0 < t_1 < \cdots < t_n = T$ we have

$$\sum_{i=0}^{n-1} |E^N(t_{i+1}) - E^N(t_i)| \leq \mathcal{E}.$$

Consequently, we get that its total variation is uniformly bounded. Hence by the Helly selection theorem there exists a subsequence of $\{E^N\}_{N\in\mathbf{N}}$, which we index again by N, and a function $E : \mathbb{T} \to \mathbf{R}_+$ locally of bounded variation such that

$$E^N(t) \to E(t) \text{ for any } t \in \mathbb{T} \text{ and in } L^1_{\mathrm{loc}}(0,\infty).$$

But since $\{u^N\}_{N\in\mathbf{N}}$ converges strongly to u in $L_T^2\mathbf{H}^0$, we infer that $E(t) = \frac{1}{2}|u_t|_0^2$ for a.e. $t \in \mathbb{T}$.

Hence, $[u, E]$ is a solution of (2.11) in the sense of Definition 2.5 and the proof of Theorem 3.3 is complete. □

3.2 Shift invariance and continuation property

Here we prove the remaining two main ingredients for the construction of a semiflow, the shift invariance property and the continuation property of the set of solutions.

For $w \in \mathbf{X}$, we define the positive shift operator $S_T \circ w$ as

$$S_T \circ w(t) := w(T+t),\ t \geq 0.$$

Lemma 3.4 (Shift invariance property). *Let* $[u_0, E_0] \in \boldsymbol{D}$*,* $\boldsymbol{Z}$ *be a geometric* α*-Hölder rough path defined on* $\mathbb{T}$*, and* $[u, E] \in \mathcal{U}[u_0, E_0, \boldsymbol{Z}]$*. Then we have*

$$S_T \circ [u, E] \in \mathcal{U}[u(T), \mathcal{E}, \tilde{\boldsymbol{Z}}_T],$$

for any $T > 0$*, and any* $\mathcal{E} \geq E(T+)$*. Here we recall the notation* $\tilde{\boldsymbol{Z}}_T(t) := \boldsymbol{Z}(t+T)$ *for* $t \geq 0$*.*

Proof of Lemma 3.4. Let us fix $T > 0$. Based on the definition of S_T, we need to show that

$$(S_T \circ [u, E])(t) = \{[u_{t+T}, E(t+T)]; t \geq 0\} =: \{[\tilde{u}_t, \tilde{E}(t)]; t \geq 0\} \in \mathcal{U}[u(T), \mathcal{E}, \tilde{\mathbf{Z}}].$$

Firstly, we observe that since $u \in \mathcal{U}[u_0, E_0, \mathbf{Z}]$ it holds $\tilde{u} \in L^2_{\text{loc}}(\mathbb{T}; \mathbf{H}^1) \cap L^\infty_{\text{loc}}(\mathbb{T}; \mathbf{H}^0)$. Next, since $E(t)$ is a non-increasing function of t and satisfies (2.16), we have

$$\left[\tilde{E}(t)\psi(t)\right]_{t=\tau_1-}^{t=\tau_2+} - \int_{\tau_1}^{\tau_2} \tilde{E}(t)\partial_t\psi(t)\,dt + \int_{\tau_1}^{\tau_2} \psi \int_{\mathbf{T}^3} |\nabla \tilde{u}_t|^2\,dx\,dt \leq 0, \quad 0 \leq \tau_1 \leq \tau_2,$$

for every $\psi \in C^1_c(\mathbb{T})$ with $\psi \geq 0$.

Observe that, since $\mathcal{E} \geq E(T+)$, $[u(T), \mathcal{E}] \in \mathbf{D}$ due to (2.19). Moreover, (2.17) gives

$$\begin{aligned} u^{P,\natural}_{(s+T)(t+T)}(\phi) &= u_{t+T}(\phi) - u_{s+T}(\phi) + \int_{s+T}^{t+T} [(\nabla u_r, \nabla\phi) + B_P(u_r)(\phi)]\,dr \\ &\quad - u_{s+T}([A^{P,1,*}_{(s+T)(t+T)} + A^{P,2,*}_{(s+T)(t+T)}]\phi) \\ &= \tilde{u}_t(\phi) - \tilde{u}_s(\phi) + \int_s^t [(\nabla\tilde{u}_{\tilde{r}}, \nabla\phi) + B_P(\tilde{u}_{\tilde{r}})(\phi)]\,d\tilde{r} \\ &\quad - \tilde{u}_s([A^{P,1,*}_{(s+T)(t+T)} + A^{P,2,*}_{(s+T)(t+T)}]\phi). \end{aligned} \tag{3.24}$$

But, since under the notation $\tilde{Z}_{st} = Z_{(s+T)(t+T)}$ and $\tilde{\mathbb{Z}}_{st} = \mathbb{Z}_{(s+T)(t+T)}$, we have

$$A^{P,1}_{(s+T)(t+T)} = P[(\sigma_k \cdot \nabla)\varphi]\tilde{Z}^k_{st} = \tilde{A}^{P,1}_{st},$$

and

$$A^{P,2}_{(s+T)(t+T)} = P[(\sigma_k \cdot \nabla)P[(\sigma_l \cdot \nabla)\varphi]]\tilde{\mathbb{Z}}^{l,k}_{st} = \tilde{A}^{P,2}_{st}.$$

Hence, for $(s,t) \in \Delta_T$,

$$\begin{aligned} u^{P,\natural}_{(s+T)(t+T)}(\phi) &= \tilde{u}_t(\phi) - \tilde{u}_s(\phi) + \int_s^t [(\nabla\tilde{u}_{\tilde{r}}, \nabla\phi) + B_P(\tilde{u}_{\tilde{r}})(\phi)]\,d\tilde{r} \\ &\quad - \tilde{u}_s([\tilde{A}^{P,1,*}_{st} + \tilde{A}^{P,2,*}_{st}]\phi) =: \tilde{u}^{P,\natural}_{st}(\phi). \end{aligned} \tag{3.25}$$

To finish the proof of Lemma 3.4, it remains to show that for every $\tau > 0$, $\tilde{u}^{P,\natural} \in C^{\frac{p}{3}-\text{var}}_{2,\varpi,L}([0,\tau]; \mathbf{H}^{-3})$. But this we have since there exits a control $\tilde{w}_\natural$ such that

$$\|\tilde{u}^{P,\natural}_{st}\|_{-3} \leq c(\tilde{w}_\natural(s,t))^{\frac{3}{p}}, \quad \forall (s,t) \in \Delta_\tau.$$

Indeed, since $u^{P,\natural} \in C^{\frac{p}{3}-\text{var}}_{2,\varpi,L}([0,\tau]; \mathbf{H}^{-3})$, there exits a control $w_\natural$ such that, for every $\phi \in \mathbf{H}^3$, we have

$$|\tilde{u}^{P,\natural}_{st}(\phi)| = |u^{P,\natural}_{(s+T)(t+T)}(\phi)| \leq c\|\phi\|_{\mathbf{H}^3}(w_\natural(s+T, t+T))^{\frac{3}{p}}.$$

Hence, by setting $\tilde{w}_\natural(s,t) := w_\natural(s+T, t+T)$, we get $\|\tilde{u}^{P,\natural}_{st}\|_{-3} \leq c(\tilde{w}_\natural(s,t))^{\frac{3}{p}}, \forall (s,t) \in \Delta_\tau$ and finishes the proof of Lemma 3.4. □

For $w_1, w_2 \in \mathbf{X}$ and $T > 0$ we define the continuation operator $\omega_1 \cup_T \omega_2$ by

$$w_1 \cup_T w_2(\tau) := \begin{cases} w_1(\tau) \text{ for } 0 \le \tau \le T, \\ \\ w_2(\tau - T) \text{ for } \tau > T. \end{cases}$$

Lemma 3.5 (Continuation property)**.** *Let* $[u_0, E_0] \in \boldsymbol{D}$*,* $\boldsymbol{Z}$ *be an* α*-Hölder rough path, and*

$$[u, E] \in \mathcal{U}[u_0, E_0, \boldsymbol{Z}],\ [\tilde{u}, \tilde{E}] \in \mathcal{U}[u(T), \mathcal{E}, \tilde{\boldsymbol{Z}}] \quad \textit{for some} \quad \mathcal{E} \le E(T-).$$

Then

$$[u, E] \cup_T [\tilde{u}, \tilde{E}] \in \mathcal{U}[u_0, E_0, \boldsymbol{Z}].$$

Proof of Lemma 3.5*.* Since the initial energy for $[\tilde{u}, \tilde{E}]$ is less or equal to $E(T-)$, we have that the energy of the solution $[u, E] \cup_T [\tilde{u}, \tilde{E}]$ indeed remains non-increasing on $\mathbb{T}$ and bounded by E_0 from above.

Let us set $v := [u, E] \cup_T [\tilde{u}, \tilde{E}]$. It remains to show that, for every $\tau > 0$,

$$v^{P,\natural} \in C^{\frac{p}{3}-\mathrm{var}}_{2,\varpi,L,\mathrm{loc}}(\mathbb{T}; \mathbf{H}^{-3}),$$

where

$$\begin{aligned} v^{P,\natural}_{st}(\phi) := v_t(\phi) - v_s(\phi) + \int_s^t [(\nabla v_r, \nabla \phi) + B_P(v_r)(\phi)]\, dr \\ - v_s([A^{P,1,*}_{st} + A^{P,2,*}_{st}]\phi). \end{aligned}$$

For this we will prove that there exists a control $\omega_{P,A,u,\natural}$ such that

$$\|v^{P,\natural}_{st}\|_{\mathbf{H}^{-3}} \lesssim (\omega_{P,A,u,\natural}(s,t))^{\frac{3}{p}}, \quad \forall s < t. \tag{3.26}$$

Recall that, by definition of solution, for every $s < t$,

$$\begin{aligned} u^{P,\natural}_{st}(\phi) = u_t(\phi) - u_s(\phi) + \int_s^t [(\nabla v_r, \nabla \phi) + B_P(u_r)(\phi)]\, dr \\ - u_s([A^{P,1,*}_{st} + A^{P,2,*}_{st}]\phi), \end{aligned}$$

and

$$\begin{aligned} \tilde{u}^{P,\natural}_{st}(\phi) = \tilde{u}_t(\phi) - \tilde{u}_s(\phi) + \int_s^t [(\nabla \tilde{u}_r, \nabla \phi) + B_P(\tilde{u}_r)(\phi)]\, dr \\ - \tilde{u}_s([\tilde{A}^{P,1,*}_{st} + \tilde{A}^{P,2,*}_{st}]\phi), \end{aligned}$$

where

$$\begin{aligned} &A^{P,1}_{st} = P[(\sigma_k \cdot \nabla)] Z^k_{st} =: P_{1,k} Z^k_{st}, \quad A^{P,2}_{st} = P[(\sigma_k \cdot \nabla) P[(\sigma_l \cdot \nabla)]] \mathbb{Z}^{l,k}_{st} =: P_{2,l,k} \mathbb{Z}^{l,k}_{st} \\ &\tilde{A}^{P,1}_{st} = P_{1,k} \tilde{Z}^k_{st}, \quad \tilde{A}^{P,2}_{st} = P_{2,l,k} \tilde{\mathbb{Z}}^{l,k}_{st}. \end{aligned}$$

Note that the only interesting case is $s < T < t$ because if $s < t \le T$ or $s < t \in (T, \infty)$ then only one out of u or $\tilde{u}$ is active. Since $u_T(\phi) = \tilde{u}_0(\phi)$, for $s < T < t$ and $\phi \in \mathbf{H}^3$, we have

$$\begin{aligned} &\delta v_{st}(\phi) = \delta \tilde{u}_{0(t-T)}(\phi) + \delta u_{sT}(\phi) \\ &= \tilde{u}_0([\tilde{A}^{P,1,*}_{0(t-T)} + \tilde{A}^{P,2,*}_{0(t-T)}]\phi) - \int_0^{t-T} [(\nabla \tilde{u}_r, \nabla \phi) + B_P(\tilde{u}_r)(\phi)]\, dr + \tilde{u}^{P,\natural}_{0(t-T)}(\phi) \end{aligned}$$

$$
\begin{aligned}
&+ u_s([A^{P,1,*}_{sT} + A^{P,2,*}_{sT}]\phi) - \int_s^T [(\nabla u_r, \nabla\phi) + B_P(u_r)(\phi)]\, dr + u^{P,\natural}_{sT}(\phi)\\
&= u_T([A^{P,1,*}_{Tt} + A^{P,2,*}_{Tt}]\phi) - \int_s^t [(\nabla v_r, \nabla\phi) + B_P(v_r)(\phi)]\, dr + \tilde{u}^{P,\natural}_{0(t-T)}(\phi)\\
&+ u_s([A^{P,1,*}_{sT} + A^{P,2,*}_{sT}]\phi) + u^{P,\natural}_{sT}(\phi).
\end{aligned}
$$

Let us first observe that, since $u_s = v_s$, due to Chen's relation $\mathbb{Z}^{l,k}_{st} - \mathbb{Z}^{l,k}_{sT} - \mathbb{Z}^{l,k}_{Tt} = Z^l_{sT} \otimes Z^k_{Tt}$,

$$
\begin{aligned}
&[A^{P,1}_{Tt} + A^{P,2}_{Tt}]u_T + [A^{P,1}_{sT} + A^{P,2}_{sT}]u_s\\
&\quad = P_{1,k} u_T Z^k_{Tt} + P_{2,l,k} u_T \mathbb{Z}^{l,k}_{Tt} + P_{1,k} u_s Z^k_{sT} + P_{2,l,k} u_s \mathbb{Z}^{l,k}_{sT}\\
&\quad = P_{1,k} \delta u_{sT} Z^k_{Tt} + P_{2,l,k} \delta u_{sT} \mathbb{Z}^{l,k}_{Tt} + P_{1,k} u_s Z^k_{st} + P_{2,l,k} u_s (\mathbb{Z}^{l,k}_{st} - Z^l_{sT} \otimes Z^k_{Tt})\\
&\quad = [A^{P,1}_{st} + A^{P,2}_{st}] v_s + A^{P,1}_{Tt} \delta u_{sT} + A^{P,2}_{Tt} \delta u_{sT} - A^{P,1}_{Tt} A^{P,1}_{sT} u_s.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
v^{P,\natural}_{st}(\phi) &= \delta v_{st}(\phi) + \int_s^t [(\nabla v_r, \nabla\phi) + B_P(v_r)(\phi)]\, dr - v_s([A^{P,1,*}_{st} + A^{P,2,*}_{st}]\phi)\\
&= v_s([A^{P,1,*}_{st} + A^{P,2,*}_{st}]\phi) + \delta u_{sT} A^{P,1,*}_{Tt}\phi + \delta u_{sT} A^{P,2,*}_{Tt}\phi - u_s A^{P,1,*}_{Tt} A^{P,1,*}_{sT}\phi\\
&\quad + \tilde{u}^{P,\natural}_{0(t-T)}(\phi) + u^{P,\natural}_{sT}(\phi) - v_s([A^{P,1,*}_{st} + A^{P,2,*}_{st}]\phi)\\
&= \delta u_{sT} A^{P,2,*}_{Tt}\phi + A^{P,1,*}_{Tt}[\delta u_{sT} - u_s A^{P,1,*}_{sT}]\phi + \tilde{u}^{P,\natural}_{0(t-T)}(\phi) + u^{P,\natural}_{sT}(\phi).
\end{aligned}
$$

With $\alpha = \frac{1}{p}$, estimate (2.9) and Lemmata A.1-A.3 we obtain

$$
\begin{aligned}
\|v^{P,\natural}_{st}\|_{\mathbf{H}^{-3}} &\le \|\delta u_{sT}\|_{\mathbf{H}^{-1}} \|A^{P,2,*}_{Tt}\|_{\mathcal{L}(\mathbf{H}^{-1},\mathbf{H}^{-3})} + \|\delta u_{sT} - u_s A^{P,1,*}_{sT}\|_{\mathbf{H}^{-2}} \|A^{P,1,*}_{Tt}\|_{\mathcal{L}(\mathbf{H}^{-2},\mathbf{H}^{-3})}\\
&\quad + (\omega_{\tilde{u},\natural}(0, t-T))^{\frac{3}{p}} + (\omega_{u,\natural}(s,T))^{\frac{3}{p}}\\
&\le (\omega_A(s,t))^{\frac{2}{p}} (\omega_u(s,t))^{\frac{1}{p}} + (\omega_\natural(s,t))^{\frac{2}{p}} (\omega_A(s,t))^{\frac{1}{p}} + (\tilde{\omega}_\natural(s,t))^{\frac{3}{p}} + (\omega_{u,\natural}(s,t))^{\frac{3}{p}}\\
&\le (\omega_{A,u}(s,t))^{\frac{3}{p}} + (\omega_{A,\natural}(s,t))^{\frac{3}{p}} + (\tilde{\omega}_\natural(s,t))^{\frac{3}{p}} + (\omega_{u,\natural}(s,t))^{\frac{3}{p}},
\end{aligned}
$$

where we have used [21, Exercise 1.9 part (iii)] to conclude that

$$
\omega_{A,u} := (\omega_A(s,t))^{\frac{2}{3}} (\omega_u(s,t))^{\frac{1}{3}} \quad \text{and} \quad \omega_{A,\natural} := (\omega_\natural(s,t))^{\frac{2}{3}} (\omega_A(s,t))^{\frac{1}{3}},
$$

are controls. Hence, by setting $\omega_{P,A,u,\natural} := \omega_{A,u} + \omega_{A,\natural} + \tilde{\omega}_\natural + \omega_{u,\natural}$ we finish the proof of Lemma 3.5. □

3.3 General ansatz

Let us fix a rough path $\mathbf{Z}$. In summary, so far we have shown the existence of a set–valued mapping

$$
\mathbf{D} \times \mathcal{C}^\alpha_{g,\mathrm{loc}}(\mathbb{T}; \mathbf{R}^K) \to 2^{\mathbf{X}}, \quad [u_0, E_0, \mathbf{Z}] \mapsto \mathcal{U}[u_0, E_0, \mathbf{Z}], \tag{3.27}
$$

which enjoys the following properties:

(A1) **Compactness:** For any $[u_0, E_0, \mathbf{Z}] \in \mathbf{D} \times \mathcal{C}^\alpha_{g,\mathrm{loc}}(\mathbb{T}; \mathbf{R}^K)$, the set $\mathcal{U}[u_0, E_0, \mathbf{Z}]$ is a non–empty compact subset of $\mathbf{X}$. Indeed, the compactness is equivalent to the weak sequential stability of the solution set which we get from Theorem 3.3. Non-emptiness of $\mathcal{U}[u_0, E_0, \mathbf{Z}]$ follows from Theorem 2.6.

(A2) **Measurability:** The mapping (3.27) is Borel measurable, where the range of $\mathcal{U}$ is endowed with the Hausdorff metric. Indeed, since $\mathcal{U}[u_0, E_0, \mathbf{Z}]$ is a compact subset of the separable metric space $\mathbf{X}$, the Borel measurability of $\mathcal{U}$ is equivalent to the measurability with respect to the Hausdorff metric on the subspace of compact sets in $2^{\mathbf{X}}$. Whence, it is sufficient to apply the following Stroock and Varadhan Lemma with $Y = \mathbf{D}$ and $X = \mathbf{X}$.

Lemma 3.6. *[37, Lemma 12.1.8]*

Let Y be a metric space and $\mathcal{B}$ its Borel σ-field. Let $y \mapsto K_y$ be a map of Y into $Comp(X)$ for some separable metric space X, with $Comp(X)$ the set of all the compact subsets of X. Suppose for any sequence $y_n \mapsto y$ and $x_n \in K_{y_n}$, it is true that x_n has a limit point x in K_y. Then the map $y \mapsto K_y$ is a Borel map of Y into $Comp(X)$.

(A3) **Shift invariance:** For any $[u, E] \in \mathcal{U}[u_0, E_0, \mathbf{Z}]$, we have

$$S_T \circ [u, E] \in \mathcal{U}[u(T), E(T-), \tilde{\mathbf{Z}}_T] \text{ for any } T > 0,$$

where $\tilde{\mathbf{Z}}_T(t) := S_T \circ \mathbf{Z}(t)$ for all $t \geq 0$.

(A4) **Continuation:** If $T > 0$, and $[u, E] \in \mathcal{U}[u_0, E_0, \mathbf{Z}]$, $[\tilde{u}, \tilde{E}] \in \mathcal{U}[u(T), E(T-), \tilde{\mathbf{Z}}_T]$, then

$$[u, E] \cup_T [\tilde{u}, \tilde{E}] \in \mathcal{U}[u_0, E_0, \mathbf{Z}].$$

3.4 Selection sequence

Notice that, the idea for the construction of the selection is to make the set $\mathcal{U}[u_0, E_0, \mathbf{Z}]$ smaller and smaller by choosing the arguments of minima of particular functionals. More precisely, following the arguments presented in [5, 6, 9], we consider the following family of Krylov functionals, see [29],

$$I_{\lambda,F}[u, E] = \int_0^\infty e^{-\lambda t} F(u(t), E(t))dt, \quad \lambda > 0,$$

where $F : \mathbf{H}^{-1} \times \mathbf{R} \to \mathbf{R}$ is a bounded and continuous functional.

Given functional $I_{\lambda,F}$ and a set-valued mapping $\mathcal{U}$, we define a selection mapping $I_{\lambda,F} \circ \mathcal{U}$ by

$$\begin{aligned} &I_{\lambda,F} \circ \mathcal{U}[u_0, E_0, \mathbf{Z}] \\ &\quad = \{[u, E] \in \mathcal{U}[u_0, E_0, \mathbf{Z}] \mid I_{\lambda,F}[u, E] \leq I_{\lambda,F}[\tilde{u}, \tilde{E}] \text{ for all } [\tilde{u}, \tilde{E}] \in \mathcal{U}[u_0, E_0, \mathbf{Z}]\}. \end{aligned} \tag{3.28}$$

In other words, the selection is choosing arguments of minima of the functional $I_{\lambda,F}$. Observe that, since $I_{\lambda,F}$ is continuous on $\mathbf{X}$ and the set $\mathcal{U}[u_0, E_0, \mathbf{Z}]$ is compact in $\mathbf{X}$, set $I_{\lambda,F} \circ \mathcal{U}[u_0, E_0, \mathbf{Z}]$ is non-empty. Our next result says that the set $I_{\lambda,F} \circ \mathcal{U}$ enjoys the general ansatz if $\mathcal{U}$ does. Recall that the perturbation rough path $\mathbf{Z}$ is fixed.

Proposition 3.7. *Let $\lambda > 0$ and F be a bounded continuous functional on $\mathbf{H}^{-1} \times \mathbf{R}$. Let the multivalued mapping* (3.27) *have the properties **(A1)–(A4)**. Then the map $I_{\lambda,F} \circ \mathcal{U}$ enjoys **(A1)–(A4)** as well.*

Proof of Proposition 3.7. Since the analysis here is pathwise, we observe that compared to the proof of [6, Proposition 5.1] the existence of $\mathbf{Z}$ in the system does not create any extra difficulty. Consequently, the proof follows step by step the lines of [6, Proposition 5.1]. Let us only spell out the proof of the measurability **(A2)**, since this is of great importance for the measurability of the random dynamical system in Section 4.

(**A2**) Let $\mathcal{K} \subset 2^{\mathbf{X}}$ be the subspace of all the compact subsets of $\mathbf{X}$. Note that the map

$$\mathbf{D} \times \mathcal{C}^\alpha_{g,\mathrm{loc}}(\mathbb{T}; \mathbf{R}^K) \to 2^{\mathbf{X}}, \quad [u_0, E_0, \mathbf{Z}] \mapsto I_{\lambda,F} \circ \mathcal{U}[u_0, E_0, \mathbf{Z}], \tag{3.29}$$

takes values in $\mathcal{K}$. Moreover, it is the composition of the Borel measurable map (3.27) and the map defined using the continuous functional $\mathcal{I}_{\lambda,F}$

$$\mathcal{K} \to \mathcal{K}, \quad K \mapsto \mathcal{I}_{\lambda,F}[K] := \arg\min_K I_{\lambda,F}. \tag{3.30}$$

Hence, (3.29) is Borel measurable since (3.30) is Borel measurable due to [37, Lemma 12.1.7]. □

Next, we consider the functional $I_{1,\beta}$ where

$$\beta(u,E)=\beta(E),\ \beta:\mathbf{R}\to\mathbf{R} \text{ smooth, bounded, and strictly increasing.}$$

We also recall the following characterization of minimality w.r.t $\prec$, introduced in Definition 3.1.

Lemma 3.8. *[6, Lemma 5.2] Suppose that* $[u,E]\in\mathcal{U}[u_0,E_0,\boldsymbol{Z}]$ *satisfies*

$$\int_0^\infty \exp(-t)\beta(E(t))\,dt \le \int_0^\infty \exp(-t)\beta(\tilde{E}(t))\,dt,$$

for any $[\tilde{u},\tilde{E}]\in\mathcal{U}[u_0,E_0,\boldsymbol{Z}]$. *Then* $[u,E]$ *is* $\prec$ *minimal, meaning, admissible.*

Finally, we have all in hand to present the first main result of the present paper.

Theorem 3.9. *The Navier-Stokes equation* (2.11) *admits a semiflow selection* U *in the class of weak solutions in the sense of Definition 3.2. Moreover, we have that* $U\{u_0,E_0,\boldsymbol{Z}\}$ *is admissible in the sense of Definition 3.1, for any* $[u_0,E_0,\boldsymbol{Z}]\in\boldsymbol{D}\times\mathcal{C}^\alpha_{g,\mathrm{loc}}(\mathbb{T};\mathbf{R}^K)$.

Proof of Theorem 3.9. First note that by (3.28) it is clear that the new selection $I_{1,\beta}\circ\mathcal{U}$ from $\mathcal{U}$ contains only admissible solutions for any $[u_0,E_0,\mathbf{Z}]\in\mathbf{D}\times\mathcal{C}^\alpha_{g,\mathrm{loc}}(\mathbb{T};\mathbf{R}^K)$.

Next, we choose a countable basis $\{\mathbf{e}_n\}_{n\in\mathbf{N}}$ in $\mathbf{L}^2$, and a countable set $\{\lambda_k\}_{k\in\mathbf{N}}$ which is dense in $(0,\infty)$. We consider a countable family of functionals,

$$I_{k,0}[u,E]=\int_0^\infty e^{-\lambda_k t}\beta(E(t))dt,$$
$$I_{k,n}[u,E]=\int_0^\infty e^{-\lambda_k t}\beta\left(\int_{\mathbf{T}^3}u(t,\cdot)\cdot\mathbf{e}_n dx\right)dt.$$

The functionals are well defined since $u(t,\cdot)\in\mathbf{H}^{-1}(\mathbf{T}^3;\mathbf{R}^3)$ for all t. Let $\{(k(j),n(j))\}_{j=1}^\infty$ be an enumeration of the countable set

$$(\mathbf{N}\times\{0\})\cup(\mathbf{N}\times\mathbf{N}).$$

We define

$$\mathcal{U}^j := I_{k(j),n(j)}\circ\cdots\circ I_{k(1),n(1)}\circ I_{1,\beta}\circ\mathcal{U},\quad j=1,2,\ldots,$$

and

$$\mathcal{U}^\infty := \bigcap_{j=1}^\infty \mathcal{U}^j.$$

Next, we claim that the set-valued mapping

$$\mathbf{D}\times\mathcal{C}^\alpha_{g,\mathrm{loc}}(\mathbb{T};\mathbf{R}^K)\to 2^{\mathbf{X}},\quad [u_0,E_0,\mathbf{Z}]\mapsto\mathcal{U}^\infty[u_0,E_0,\mathbf{Z}], \tag{3.31}$$

enjoys the properties **(A1)**–**(A4)**. Indeed:

(A1) Let us take $[u_0,E_0,\mathbf{Z}]\in\mathbf{D}\times\mathcal{C}^\alpha_{g,\mathrm{loc}}(\mathbb{T};\mathbf{R}^K)$. Recall that, from Proposition 3.7, the set $I_{1,\beta}\circ\mathcal{U}[u_0,E_0,\mathbf{Z}]$ is compact. Since the sets $\mathcal{U}^j[u_0,E_0,\mathbf{Z}]$ are nested:

$$I_{1,\beta}\circ\mathcal{U}[u_0,E_0,\mathbf{Z}]\supseteq\mathcal{U}^1[u_0,E_0,\mathbf{Z}]\supseteq\cdots\supseteq\mathcal{U}^j[u_0,E_0,\mathbf{Z}]\supseteq\ldots.$$

by iterating the procedure of Proposition 3.7, we get that, for each $j\in\mathbf{N}$, $\mathcal{U}^j[u_0,E_0,\mathbf{Z}]$ is compact.

Since $\mathbf{X}$ is a Hausdorff space and $\mathcal{U}^\infty[u_0,E_0,\mathbf{Z}]$ is a closed subset of $I_{1,\beta}\circ\mathcal{U}[u_0,E_0,\mathbf{Z}]$, we infer that $\mathcal{U}^\infty[u_0,E_0,\mathbf{Z}]$ is compact. Moreover, by Proposition 3.7 we know that, for each $j\in\mathbf{N}$, $\mathcal{U}^j[u_0,E_0,\mathbf{Z}]$ is non-empty. Thus, due to the Cantor intersection theorem we have that $\mathcal{U}^\infty[u_0,E_0,\mathbf{Z}]\neq\emptyset$.

(A2) Since the intersection of measurable set–valued maps is measurable, the map (3.31) is measurable.

(A3) In order to prove the shift invariance property, let $[u_0, E_0, \mathbf{Z}] \in \mathbf{D} \times \mathcal{C}^{\alpha}_{g,\mathrm{loc}}(\mathbb{T}; \mathbf{R}^K)$ and $[u, E] \in \mathcal{U}^{\infty}[u_0, E_0, \mathbf{Z}]$. Thus, $[u, E] \in \mathcal{U}^j[u_0, E_0, \mathbf{Z}]$ for every $j \in \mathbf{N}$. Due to Proposition 3.7, we know that $I_{1,\beta} \circ \mathcal{U}$ satisfies the shift invariance property, that is, if $[u, E] \in I_{1,\beta} \circ \mathcal{U}[u_0, E_0, \mathbf{Z}]$, then

$$S_T \circ [u, E] \in I_{1,\beta} \circ \mathcal{U}[u(T), E(T-), \tilde{\mathbf{Z}}], \text{ for all } T > 0.$$

By iterating this procedure we obtain that the shift invariance property holds for every $\mathcal{U}^j$. This means that for

$$[u, E] \in \mathcal{U}^j[u_0, E_0, \mathbf{Z}] = I_{k(j),n(j)} \circ \cdots \circ I_{k(1),n(1)} \circ I_{1,\beta} \circ \mathcal{U}[u_0, E_0],$$

we have

$$S_T \circ [u, E] \in \mathcal{U}^j[u(T), E(T-), \tilde{\mathbf{Z}}], \text{ for all } j \text{ and all } T > 0.$$

Thus

$$S_T \circ [u, E] \in \mathcal{U}^{\infty}[u(T), E(T-), \tilde{\mathbf{Z}}], \text{ for all } T > 0.$$

(A4) In order to prove the continuation property, let $T > 0$, $[u, E] \in \mathcal{U}^{\infty}[u_0, E_0, \mathbf{Z}]$ and

$$[\tilde{u}, \tilde{E}] \in \mathcal{U}^{\infty}[u(T), E(T-), \tilde{\mathbf{Z}}].$$

Then, we have

$$[u, E] \in \mathcal{U}^j[u_0, E_0, \mathbf{Z}], \text{ and } [\tilde{u}, \tilde{E}] \in \mathcal{U}^j[u(T), E(T-), \tilde{\mathbf{Z}}], \qquad j \in \mathbf{N}.$$

By Proposition 3.7, we have that $I_{1,\beta} \circ \mathcal{U}$ satisfies the continuation property, and iterating this procedure we obtain that this property holds for every $\mathcal{U}^j$. This means that

$$[u, E] \cup_T [\tilde{u}, \tilde{E}] \in \mathcal{U}^j[u_0, E_0, \mathbf{Z}] \text{ for all } j \text{ and all } T > 0.$$

Thus

$$[u, E] \cup_T [\tilde{u}, \tilde{E}] \in \mathcal{U}^{\infty}[u_0, E_0, \mathbf{Z}] \text{ for all } T > 0.$$

Next, we claim that for every $[u_0, E_0, \mathbf{Z}] \in \mathbf{D} \times \mathcal{C}^{\alpha}_{g,\mathrm{loc}}(\mathbb{T}; \mathbf{R}^K)$ the set $\mathcal{U}^{\infty}$ is a singleton, i.e., there exists $U\{u_0, E_0, \mathbf{Z}\} \in \mathbf{X}$ such that

$$\mathcal{U}^{\infty}[u_0, E_0, \mathbf{Z}] = \big\{U\{u_0, E_0, \mathbf{Z}\}\big\}. \tag{3.32}$$

To prove this, first observe that by (3.28), for any $[u^1, E^1], [u^2, E^2] \in \mathcal{U}^{\infty}[u_0, E_0, \mathbf{Z}]$,

$$I_{k(j),n(j)}[u^1, E^1] = I_{k(j),n(j)}[u^2, E^2], \qquad j \in \mathbf{N}.$$

Since the integrals $I_{k(j),n(j)}$ can be seen as Laplace transforms

$$F(\lambda_k) = \int_0^{\infty} e^{-\lambda_k t} f(t) dt,$$

of the functions

$$f \in \left\{ \beta(E), \ \beta\left(\int_{\mathbf{T}^3} u \cdot \mathbf{e}_n dx \right) \right\},$$

the Lerch theorem [11, Theorem 2.1] implies that

$$\beta(E^1(t)) = \beta(E^2(t)),$$

$$\beta\left(\int_{\mathbf{T}^3} u^1(t,\cdot)\cdot \mathbf{e}_n dx\right) = \beta\left(\int_{\mathbf{T}^3} u^2(t,\cdot)\cdot \mathbf{e}_n dx\right),$$

for all $n \in \mathbf{N}$ and for a.e. $t \in (0,\infty)$. As β is strictly increasing, we must have

$$E^1(t-) = E^2(t-), \quad \langle u^1(t,\cdot), \mathbf{e}_n\rangle_{\mathbf{L}^2} = \langle u^2(t,\cdot), \mathbf{e}_n\rangle_{\mathbf{L}^2},$$

for all $n \in \mathbf{N}$ and for a.e. $t \in (0,\infty)$. Since $\{\mathbf{e}_n\}_{n\in\mathbf{N}}$ form a basis in $\mathbf{L}^2$, we deduce that

$$u^1 = u^2, \text{ and } E^1 = E^2 \text{ a.e. on } (0,\infty).$$

Due to (3.32), measurability of U follows from (**A2**) for $\mathcal{U}^\infty$. While the semigroup property follows from (**A3**). Indeed, for $t_1, t_2 \geq 0$ it holds

$$U\{u_0, E_0, \mathbf{Z}\}(t_1+t_2) = S_{t_1} \circ U\{u_0, E_0, \mathbf{Z}\}(t_2) = U\{U\{u_0, E_0, \mathbf{Z}\}(t_1), \tilde{\mathbf{Z}}_{t_1}\}(t_2),$$

where $\tilde{\mathbf{Z}}_{t_1}(t_2) := \mathbf{Z}(t_1+t_2)$. This completes the proof of Theorem 3.9. □

Remark 3.10. It is important to highlight that one can introduce a new selection, associated with the considered Navier-Stokes equation (2.11), defined only in terms of the initial velocity. However, in this case we can only achieve that, for each rough path $\mathbf{Z}$, the semigroup property holds almost everywhere in time. The proof of this argument in our framework is similar to [5, Section 5], where the author proves this claim for the compressible Navier-Stokes system without any perturbation.

4 Random dynamical system

Based on the semiflow selection from the previous section we investigate the existence of a random dynamical system for Navier-Stokes equation (2.11).

Let $(\Omega, \mathcal{F})$ be a measurable space. A family $\theta = (\theta_t)_{t\in\mathbb{T}}$ of maps from Ω to itself is called a **measurable dynamical system** provided

1. $(t,\omega) \mapsto \theta_t\omega$ is $\mathcal{B}(\mathbb{T}) \otimes \mathcal{F}/\mathcal{F}$-measurable, where $\mathcal{B}(\mathbb{T})$ is the Borel σ-algebra of $\mathbb{T}$,
2. $\theta_0 = \mathrm{Id}_\Omega$,
3. $\theta_{s+t} = \theta_t \circ \theta_s$ for all $s, t \in \mathbb{T}$.

If $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$ that is invariant under θ, i.e. $\mathbb{P} \circ \theta_t^{-1} = \mathbb{P}$ for all $t \in \mathbb{T}$, we call the quadruple $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ a measurable metric dynamical system.

The following is taken from L. Arnold's book, see [1, Definition 1.1.1].

Definition 4.1. *A* ***measurable random dynamical system*** *(MRDS) on a measurable space* $(X, \mathcal{X})$*, over a metric dynamical system* $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{T}})$ *with time* $\mathbb{T}$ *is a mapping*

$$\Phi : \mathbb{T} \times \Omega \times X \to X, \quad (t,\omega,x) \mapsto \Phi(t,\omega,x),$$

i) Measurability: Φ *is* $(\mathcal{B}(\mathbb{T}) \otimes \mathcal{F} \otimes \mathcal{X})/\mathcal{X}$ *measurable.*

ii) Cocycle property: The mappings $\Phi(t,\omega) := \Phi(t,\omega,\cdot) : X \to X$ *form a cocycle over* θ*, i.e. they satisfy*

$$\Phi(0,\omega) = \mathrm{id}_X \quad \forall \omega \in \Omega, \tag{4.1a}$$

$$\Phi(t+s,\omega) = \Phi(t,\theta_s\omega) \circ \Phi(s,\omega) \quad \forall s,t \in \mathbb{T}, \omega \in \Omega. \tag{4.1b}$$

Remark 4.2. If the mapping Φ in Definition 4.1 does not depend on ω, then the dynamics on X is independent of the underlying dynamical system on Ω, and $\Phi(t)$ satisfies the semigroup property.

Let us fix a measurable metric dynamical system $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{T}})$. As defined in [3, Section 2], for $\alpha\in\left(\frac{1}{3},\frac{1}{2}\right]$, we say that a measurable map

$$\mathbf{Z}=(Z,\mathbb{Z}):\Omega\to C^{\alpha}_{2,\mathrm{loc}}(\mathbb{T};\mathbf{R}^K)\times C^{2\alpha}_{2,\mathrm{loc}}(\mathbb{T};\mathbf{R}^{K\times K}),$$

is a geometric α-Hölder rough path cocycle provided $\mathbf{Z}(\omega)$ is a geometric α-Hölder rough path and the following cocycle property is satisfied

$$Z_{s,s+t}(\omega)=Z_{0,t}(\theta_s\omega),\qquad \mathbb{Z}_{s,s+t}(\omega)=\mathbb{Z}_{0,t}(\theta_s\omega),$$

holds true for every $s,t\in\mathbb{T}$ and $\omega\in\Omega$.

For the sake of completeness we include the following simple observation regarding the shift-property of an α-Hölder rough path.

Lemma 4.3. *Let $\mathbf{Z}=(Z,\mathbb{Z})$ be a geometric α-Hölder rough path cocycle for some $\alpha\in\left(\frac{1}{3},\frac{1}{2}\right]$. For every $0\le s\le t$, $h>0$ and $\omega\in\Omega$, we have $Z_{s+h,t+h}(\omega)=Z_{s,t}(\theta_h\omega)$ and $\mathbb{Z}_{s+h,t+h}(\omega)=\mathbb{Z}_{s,t}(\theta_h\omega)$.*

Finally, we have all in hand to formulate and prove the second main result of the present paper.

Theorem 4.4. *Assume that, for given measurable metric dynamical system $(\Omega,\mathcal{F},\mathbb{P},\theta)$, the driving rough path $\mathbf{Z}=(Z,\mathbb{Z})$ is a geometric α-Hölder rough path cocycle for some $\alpha\in\left(\frac{1}{3},\frac{1}{2}\right]$. Then the Navier-Stokes system* (2.11) *generates a measurable random dynamical system on **D**.*

***Proof of Theorem 4.4**.* Given the random rough path $\mathbf{Z}$ and the semiflow U constructed in Section 3 we define

$$\varphi:\Omega\times\mathbf{D}\to C_{\mathrm{loc}}\mathbf{H}^0_w\times L^1_{\mathrm{loc}}(\mathbb{T}),\quad (\omega,[u_0,E_0])\mapsto U\{u_0,E_0,\mathbf{Z}(\omega)\}.$$

By the definition of a rough path cocycle, this map factorizes as

$$(\omega,[u_0,E_0])\mapsto(u_0,E_0,\mathbf{Z}(\omega))\mapsto U\{u_0,E_0,\mathbf{Z}(\omega)\},$$

hence it is well-defined and measurable due to Theorem 3.9. This is the point where the Wong-Zakai stability in the rough path setting becomes essential. Notice that, since $\varphi(\omega,[u_0,E_0])\in\mathcal{U}^\infty[u_0,E_0,\mathbf{Z}(\omega)]$, we can evaluate it pointwise with respect to $t\in\mathbb{T}$.

We claim that

$$\Phi:\mathbb{T}\times\Omega\times\mathbf{D}\to\mathbf{D},\quad (t,\omega,[u_0,E_0])\mapsto\varphi(\omega,[u_0,E_0])(t),\tag{4.2}$$

is a measurable random dynamical system. To prove the claim, first observe that the map Φ is well-defined. Next, it is clear that the measurability of Φ can be deduced if we show that for given $\omega\in\Omega$ and $[u_0,E_0]\in\mathbf{D}$,

$$\mathbb{T}\to\mathbf{D},\quad t\mapsto\varphi(\omega,[u_0,E_0])(t)\quad\text{is measurable.}$$

But this is indeed the case, because if $[u,E]=\varphi(\omega,[u_0,E_0])$ then $\mathbb{T}\to\mathbf{H}^0$, $t\mapsto u_t$, is weakly continuous and the measurability of $\mathbb{T}\to\mathbf{R}_+$, $t\mapsto E(t-)$, is a consequence of the Lebesgue differentiation theorem, since

$$E(t-)=\lim_{h\to0}\frac{1}{h}\int_h^{t+h}E(s)\,ds.$$

It remains to verify the cocycle property of Φ. In view of the definition of Φ, the semiflow property of U as well as Lemma 4.3, we infer for all $t,s\in\mathbb{T}$, $\omega\in\Omega$

$$\begin{aligned}\Phi(t+s,\omega)([u_0,E_0])&=\varphi_{t+s}(\omega)([u_0,E_0])=U\{u_0,E_0,\mathbf{Z}(\omega)\}(t+s)\\&=U\{U\{u_0,E_0,\mathbf{Z}(\omega)\}(s),\tilde{\mathbf{Z}}_s(\omega)\}(t)\\&=U\{U\{u_0,E_0,\mathbf{Z}(\omega)\}(s),\mathbf{Z}(\theta_s\omega)\}(t)\\&=\varphi_t(\theta_s\omega)\circ\varphi_s(\omega)([u_0,E_0])=\Phi(t,\theta_s\omega)\circ\Phi(s,\omega)([u_0,E_0]),\end{aligned}$$

which completes the proof. □

A A priori estimate and compactness

Here we state, without proof, all the required a priori estimates from [26, Section 3]. Let us fix any $T>0$ and assume that u is the first component of a weak solution to (2.11) according to Definition 2.5.

Lemma A.1. *[26, Lemma 3.1] For $(s,t)\in\Delta_T$ such that $\varpi(s,t)\le L$, let $\omega_{P,\natural}(s,t):=|u^{P,\natural}|^{\frac{p}{3}}_{\frac{p}{3}-var;[s,t];\boldsymbol{H}^{-3}}$. Then there is a constant $\tilde L>0$, depending only on p and d, such that for all $(s,t)\in\Delta_T$ with $\varpi(s,t)\le L$ and $\omega_A(s,t)\le\tilde L$,*

$$\omega_{P,\natural}(s,t)\lesssim_p |u|^{\frac{p}{3}}_{L^\infty_T\boldsymbol{H}^0}\omega_A(s,t)+\omega_\mu(s,t)^{\frac{p}{3}}(\omega_A(s,t)^{\frac{1}{3}}+\omega_A(s,t)^{\frac{2}{3}}), \tag{A.1}$$

and

$$\omega_{P,\natural}(s,t)\lesssim_p |u|^{\frac{p}{3}}_{L^\infty_T\boldsymbol{H}^0}\omega_A(s,t)+(1+|u|_{L^\infty_T\boldsymbol{H}^0})^{\frac{2p}{3}}(t-s)^{\frac{p}{3}}\omega_A(s,t)^{\frac{1}{12}}. \tag{A.2}$$

Lemma A.2. *[26, Lemma 3.3] Solution u belongs to $C^{p-\mathrm{var}}([0,T];\boldsymbol{H}^{-1})$ and there is a constant $\tilde L>0$, depending only on p and d, such that for all $(s,t)\in\Delta_T$ with $\varpi(s,t)\le L$, $\omega_A(s,t)\le\tilde L$, and $\omega_{P,\natural}(s,t)\le\tilde L$, it holds that*

$$\omega_u(s,t)\lesssim_p (1+|u|_{L^\infty_T\boldsymbol{H}^0})^p(\omega_{P,\natural}(s,t)+\omega_\mu(s,t)^p+\omega_A(s,t)),$$

where $\omega_u(s,t):=|u|^p_{p-\mathrm{var};[s,t];\boldsymbol{H}^{-1}}$.

Lemma A.3. *[26, Lemma 3.4] The remainder $u^\sharp$ is in $C_2^{\frac{p}{2}-\mathrm{var}}([0,T];\boldsymbol{H}^{-2})$ and there is a constant $\tilde L>0$, depending only on p and d, such that for all $(s,t)\in\Delta_T$ with $\varpi(s,t)\le L$, $\omega_A(s,t)\le\tilde L$, and $\omega_{P,\natural}(s,t)\le\tilde L$, it holds that*

$$\omega_\sharp(s,t)\lesssim_p (1+|u|_{L^\infty_T\boldsymbol{H}^0})^{\frac{p}{2}}(\omega_{P,\natural}(s,t)+\omega_\mu(s,t)^{\frac{p}{2}}+\omega_A(s,t)),$$

where $\omega_\sharp(s,t):=|u^\sharp|^{\frac{p}{2}}_{\frac{p}{2}-\mathrm{var};[s,t];\boldsymbol{H}^{-2}}$.

The following compact embedding result is useful in the proof of Sequential Stability Theorem 3.3.

Lemma A.4. *[26, Lemma A.2] Let ω and ϖ be a controls on $[0,T]$ and $L,\kappa>0$. Let*

$$X=L^2_T\boldsymbol{H}^1\cap\left\{g\in C_T\boldsymbol{H}^{-1}:|\delta g_{st}|_{-1}\le\omega(s,t)^\kappa,\ \forall(s,t)\in\Delta_T\text{ with }\varpi(s,t)\le L\right\}$$

be endowed with the norm

$$|g|_X=|g|_{L^2_T\boldsymbol{H}^1}+\sup_{t\in[0,T]}|g_t|_{-1}+\sup\left\{\frac{|\delta g_{st}|_{-1}}{\omega(s,t)^\kappa}:(s,t)\in\Delta_T\ \text{ s.t. }\varpi(s,t)\le L\right\}.$$

Then X is compactly embedded into $C_T\boldsymbol{H}^{-1}$ and $L^2_T\boldsymbol{H}^0$.

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