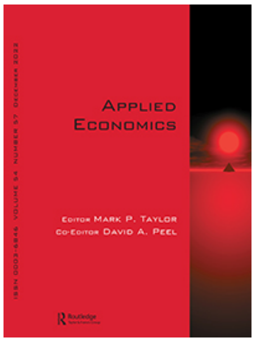

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ABSTRACT

There is a big controversy among both investment professionals and academics regarding how the termination probability of a market state depends on its age. Using more than two centuries of data on the broad US stock market index, we revisit the duration dependence in bull and bear markets. Our results suggest that the duration dependence for both bull and bear markets is a nonlinear function of the state age. It appears that the duration dependence in bear markets is strictly positive. For 93% of the bull markets, the duration dependence is also positive. Only about 7% of the bull markets, those with the longest durations, do not exhibit positive duration dependence. We also compare a few selected theoretical distributions on their ability to describe the duration dependence in bull and bear markets. Our results advocate that the gamma distribution most often provides the best fit for both the survivor and hazard functions of bull and bear markets. However, our results reveal that none of the selected distributions accurately describes the right tail of the hazard functions.

KEYWORDS

Stock market cycles; bull and bear markets; duration dependence; survivor function; hazard function

JEL CLASSIFICATION

C41; G10

I. Introduction

For more than a century, investment professionals have regularly referred to markets as being bullish and bearish. For example, the Dow Theory, developed at the end of the 19th century, postulates the existence of several types of trends in financial markets (see Brown, Goetzmann, and Kumar (1998)). The primary trend is the most dominant of all types of trends. Primary trends can be classified as bull and bear markets that tend to last for one year or more. Since bull and bear markets offer very different investment opportunities, investment professionals pay close attention to identifying and predicting bull and bear states in financial markets.

The majority of investment professionals believe that the probability that a bull or bear market terminates depends on the market's age. However, there are two opposite views regarding the duration dependence in bull and bear markets. Specifically, one group of investment professionals believes that the older the age of a bull (bear) market, the higher the probability that it terminates. This belief indicates a positive duration dependence. The other group thinks that the longer a bull (bear) market lasts, the lower the probability that it ends. This opinion suggests a negative duration dependence.

Not only investment professionals have contradictory opinions on duration dependence in bull and bear markets. Academics also present conflicting evidence regarding this dependence. For example, Cochran and Defina (1995), Ohn, Taylor, and Pagan (2004), and Harman and Zuehlke (2007) find positive duration dependence in both bull and bear markets in the US. In contrast, Maheu and McCurdy (2000) demonstrate negative duration dependence in both bull and bear markets in the US. Lunde and Timmermann (2004) document negative duration dependence in bull markets and positive duration dependence in bear markets. Zhou and Rigdon (2011) report a similar finding. Pagan and Sossounov (2003) find no conclusive evidence of duration dependence in the US stock market cycles.

This paper revisits the duration dependence in the US stock market cycles and sheds additional light on how a market state's termination probability depends on its age. The answer to this question is important not only to investment professionals but also to academics because of the following reason. There is rich literature on financial markets with regime-switching behaviour. This literature includes studies on identifying and predicting

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market states (some examples are Maheu and McCurdy (2000), Chen (2009), Maheu, McCurdy, and Song (2012), Nyberg (2013), De Angelis and Paas (2013), and Kole and van Dijk (2017)), optimal asset allocation decisions in regime-switching markets (see, among others, Sotomayor and Cadenillas (2009), Dai, Zhang, and Zhu (2010), Kong, Zhang, and Yin (2011), and Fei (2013)), and pricing of derivative securities in markets with regime-switching (a few examples are Edwards (2005), Boyarchenko and Levendorskii (2009), and van der Hoek and Elliott (2012)). In virtually all of these papers, academics assume that the market dynamics follow a standard time-homogeneous Markov process with two states: bull and bear. In such a model, the probability transition matrix between the states is constant, implying no duration dependence. That is, typically, academics take for granted that the duration of a bull or bear market does not depend on its age. If this assumption is not satisfied, academics must rework their models by incorporating a more accurate and realistic assumption about duration dependence.

This paper employs the most popular procedure to identify the bull and bear states of the financial market. This procedure is implemented using more than two centuries of monthly returns on the broad US stock market index. First, we present the descriptive statistics of bull and bear markets. Next, we examine the empirical survivor and hazard functions[1] of bull and bear markets. Since the shape of a hazard function reflects the duration dependence, we focus our attention on examining the empirical hazard functions. Our results suggest that the duration dependence is a nonlinear function of the state age for both bull and bear markets.

We study the nonlinear duration dependence in bull and bear markets utilizing a piecewise linear regression model with several unknown breakpoints. Our results suggest that the hazard functions for both bull and bear markets have three linear segments. For the bull markets, the duration dependence is as follows. The duration dependence is strictly positive in the first linear segment, which includes 93% of all observations. In contrast, the duration dependence is negative in the second linear segment. Finally, the duration dependence is again positive in the third linear segment. Only about 7% of the bull markets, those with the most prolonged durations, do not exhibit positive duration dependence. For the bear markets, our results on duration dependence are as follows. If a bear market's age is less than its average value, then the probability that it terminates increases relatively rapidly with age. Subsequently, when a bear market's age rises above its average value, the termination probability increases slowly. Eventually, for the bear markets with durations, which fall within 8% of the most prolonged durations, the hazard rate again increases rapidly with age.

Finally, we compare a few selected theoretical distributions on their ability to describe the empirical probability distributions of the bull and bear market durations and their hazard functions. These distributions are the most popular distributions for lifetime modelling and include the exponential, the Weibull, the log-logistic, the gamma, and the Burr distribution. We find that the gamma distribution most often provides the best fit for both the survivor and hazard functions of bull and bear markets. However, our results reveal that none of the selected distributions accurately describes the hazard functions' right tail.

The remainder of this paper is structured as follows. Section II presents the data and the descriptive statistics of bull and bear markets, while Section III presents the empirical survivor and hazard functions. Section IV examines the nonlinear behaviour of empirical hazard functions using a piecewise linear regression model. Section V compares the goodness-of-fit provided by a few selected theoretical distributions to the empirical survivor and hazard functions. Finally, Section VI concludes the paper.

II. Data and descriptive statistics of bull and bear markets

To ensure the best possible description of bull and bear stock markets, we use the longest possible time series of the US stock market returns from

[1]While the survivor function specifies the probability that a market state's duration will equal or exceed some specific length, the hazard function gives the probability that the market state terminates during the subsequent period under the condition that the termination has not occurred before.

January 1802 to December 2019. William Schwert[2] provides the data for the period from January 1802 to December 1925. The market returns for this period are constructed using a collection of early stock market indices for the US. The methodology of construction is described in all detail in Schwert (1990). From January 1926 to February 1957, the market returns are the returns on the Standard and Poor's 90 stock market index. From March 1957, the market returns are the returns on the Standard and Poor's 500 stock market index. Amit Goyal[3] provides the returns for the period from January 1926 to December 2019. All data come at a monthly frequency.

We follow the standard practice and use the nominal capital gain returns to identify bull and bear markets. There are several alternative methods to classify the stock market's states because of the lack of consensus on the formal definition of bull and bear markets in the finance literature. This study uses the most popular method proposed by Pagan and Sossounov (2003). In brief, Pagan and Sossounov (2003) adopt, with minor modifications, the dating algorithm proposed by Bry and Boschan (1971) and used to identify the US business cycle turning points. This method uses an elaborate set of rules and comprises two main stages: detecting initial turning points and censoring operations.

In the first stage, one identifies a peak (bottom) as a point higher (lower) than other points within an 8-month window around this date. Subsequently, one enforces the turning points' alternation by selecting the highest of multiple peaks and the lowest of multiple bottoms. In the second stage, one performs censoring operations to ensure that a stock market state lasts at least 4 months (unless the market move exceeds 20%), and a complete market cycle (a bull market and a subsequent bear market or vice versa) spans at least 16 months.

Table 1 presents the summary statistics of the bull and bear markets. From 1802 to 2019, there were 62 bear markets and 63 bull markets. A bull market tends to last longer than a bear market. The mean duration of a bull market is 25 months, whereas the mean bear market duration is 17 months. Consequently, the mean bull market duration is 1.5 times longer than the mean bear market duration. The median duration of a bull (bear) market is 23 (14) months. The median duration is smaller than the mean duration for both market states. Consequently, the duration distribution is right-skewed for each market state. In other words, the right tail of each distribution is longer than the left tail.

Table 1. Summary statistics of bull and bear market states. Duration is measured in months.

	Bull	Bear
Number of states	63	62
Minimum duration	4	3
Mean duration	25	17
Median duration	23	14
Maximum duration	74	44

We continue our analysis by computing the empirical density of bull and bear market durations. Figure 1 plots the histograms of the bull and bear market durations. Specifically, the left (right) panel in this plot shows the bull (bear) market duration histogram. Solid lines in each panel plot the estimated kernel densities of bull and bear market durations using a Gaussian smoother. It is worth noting that, for each market state, the shape of the empirical density function is a positively skewed bell curve.

III. Survivor and hazard functions

We denote a market state's duration by T and consider T as a random variable. It is worth emphasizing that each plot in Figure 1 shows the probability density function $f(t)$ of a market state duration. Alternatively, the probability distribution of durations can be specified by the cumulative distribution function

$$F(t) = Prob(T < t),$$

which is the probability that the random variable T is less than some certain value T. Given $F(t)$, the corresponding density function is $f(t) = dF(t)/dt$.

In duration analysis, instead of working with the cumulative distribution function, it is more convenient to work with the survivor function

[2]http://schwert.ssb.rochester.edu/data.htm.
[3]http://www.hec.unil.ch/agoyal/.

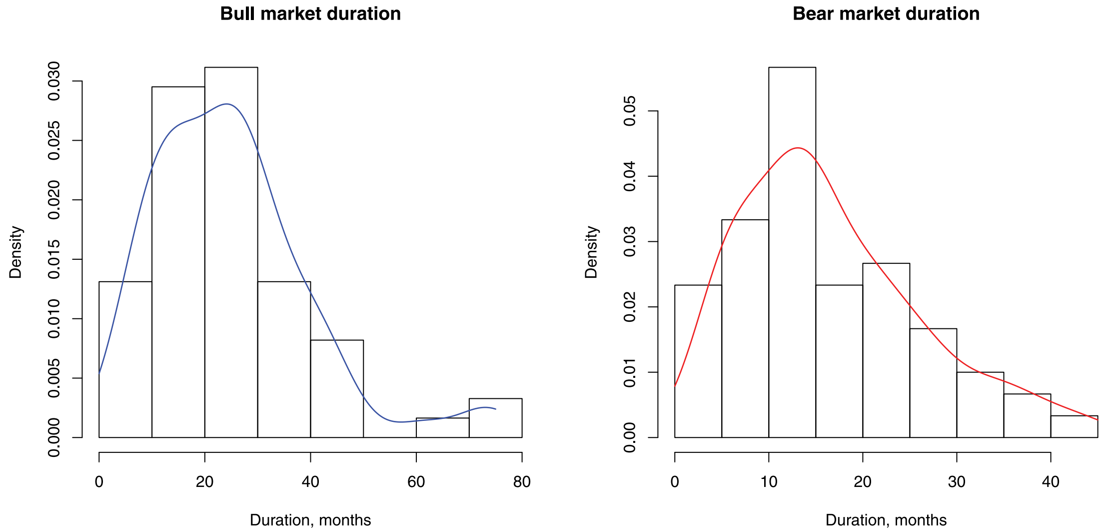

Figure 1. Empirical density of bull and bear market durations. Solid lines plot the estimated kernel densities of bull and bear market durations using a Gaussian smoother.

$$S(t) = 1 - F(t) = Prob(T \geq t).$$

The survivor function is the probability that the random variable T equals or exceeds some specific value t. Another very convenient function for analysing duration data is the hazard function

$$h(t) = \frac{f(t)}{S(t)}.$$

The hazard function gives a conditional failure rate. In our context, the hazard function is a probability that the market state ends during the period $[t, t + dt]$ under the condition that the market state lasted till t. In simple terms, the hazard function is a continuous-time version of a sequence of conditional probabilities.

If a hazard function is an increasing function of time, there is a positive duration dependence. In this case, the longer a market state lasts, the higher the probability that it ends. On the other hand, if a hazard function is a decreasing function of time, there is a negative duration dependence. In this circumstance, the longer a market state lasts, the lower the probability that it ends. When the hazard function is constant, there is no duration dependence. Provided the absence of duration dependence, at any time, the probability that a market state ends does not depend on how long the state has lasted. The hazard function can also take either a U-shaped or inverted U-shaped form. With a U-shape form common in demography, the hazard function first decreases and then increases.

The left panel in Figure 2 plots the empirical survivor functions for bull and bear markets. One can use the information in this plot to estimate the probability that a market state lasts longer than a specific number of months. It is worth noting that, for any given t, the probability of survival for a bull state is higher than that for a bear state. This relationship is a direct consequence of the fact that a bull state's mean duration is longer than a bear state's mean duration.

The right panel in Figure 2 plots the empirical hazard functions for bull and bear markets. Our first impression is that the hazard function is nonlinear for each market state. Whereas the duration dependence seems to be always positive for the bear state, the duration dependence is more complicated for the bull state. Specifically, for the bull state, the hazard function first increases, then decreases, and finally increases again. In the subsequent section, we take a closer look at the behaviour of the hazard functions. As a final remark to conclude this section, we note that for any given t, the bear state's hazard function is higher than the bull

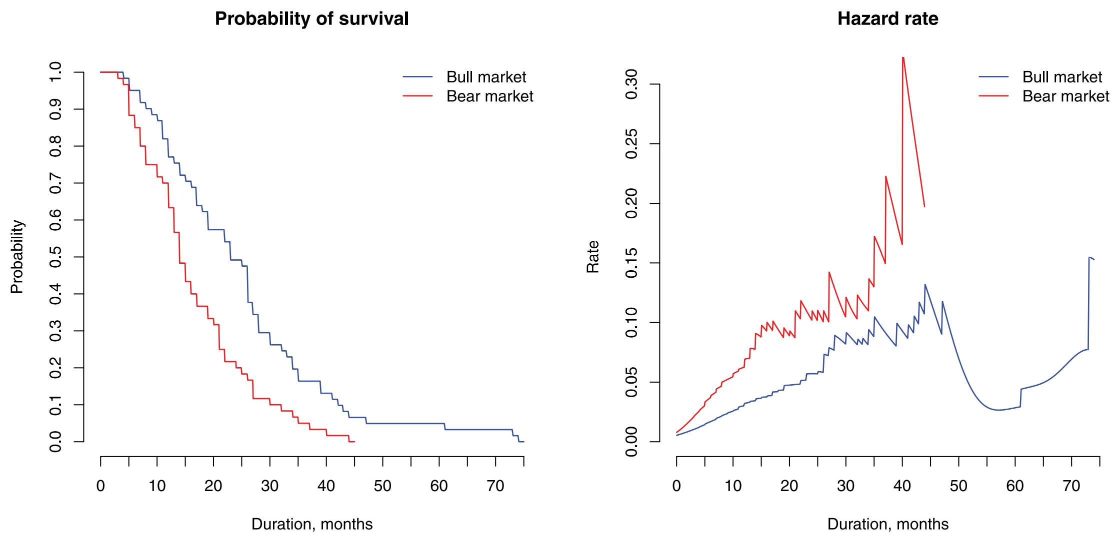

Figure 2. Left panel plots the empirical survivor functions for bull and bear markets. Right panel plots the empirical hazard functions for bull and bear markets.

state's hazard function. Again, this is a direct consequence of the fact that, on average, bull markets last longer than bear markets.

IV. Piecewise linear modelling of hazard functions

The plots of the bull and bear markets' empirical hazard functions motivate that the duration dependence in both market states is nonlinear. Namely, the hazard rate is a nonlinear function of the state age for each market state. The simplest way of modelling a nonlinear relationship is to assume that the relationship is piecewise linear. The idea is to split the nonlinear function into several linear pieces.

The methodology of estimating piecewise linear regression models with unknown breakpoints is developed by Muggeo (2003). The parametrization of the piecewise linear regression model for a hazard function with m unknown breakpoints is as follows

$$h(t) = \alpha + \beta_0 t + \sum_{i=1}^{m} \beta_i (t - \tau_i)^+ + \varepsilon(t), \qquad (1)$$

where t is the market state age, $h(t)$ is the empirical hazard function, α is the intercept, β_0 is the initial slope, β_i are the 'difference-in-slopes' parameters, τ_i are the breakpoints, and $\varepsilon(t)$ are the residuals. The notation $(t-\tau_i)^+$ means that the value $t-\tau_i > 0$ if it is positive and 0 otherwise.

As a simple illustration, consider the case where there is only one breakpoint in the relationship. In such a situation, the piecewise linear regression model (4.1) reduces to

$$h(t) = \alpha + \beta_0 t + \beta_1 (t - \tau_1)^+ + \varepsilon(t).$$

In this case, there are two linear segments in the model, and τ_1 is the breakpoint's location between the two segments. Note that on the left segment of the relationship, the line's slope is given by β_0. On the right segment of the relationship, the line's slope is given by $\beta_0 + \beta_1$.

Table 2. Breakpoints in the piecewise linear model for the hazard function for bull and bear markets. **Estimate** is the estimated location of the breakpoint, in months. **95% CI** is the 95% confidence interval for the location of the breakpoint. **Percentile** is the percentile of each estimated breakpoint within the corresponding distribution of durations.

	Bull markets			Bear markets		
Breakpoint	Estimate	95% CI	Percentile	Estimate	95% CI	Percentile
1st	45.40	(44.97, 45.83)	0.93	16.12	(14.00, 18.23)	0.60
2nd	56.44	(56.03, 56.84)	0.95	32.85	(32.11, 33.59)	0.92

Table 3. Slope estimates for each of the three linear segments between two breakpoints and the 95% confidence interval for each slope.

	Bull markets		Bear markets	
Segment	Slope	95% CI	Slope	95% CI
1st	0.0025	(0.0024, 0.0026)	0.0055	(0.0049, 0.0061)
2nd	−0.0091	(−0.0098, −0.0084)	0.0017	(0.0011, 0.0023)
3rd	0.0048	(0.0045, 0.0052)	0.0136	(0.0128, 0.0149)

We determine the breakpoints by the method of maximum likelihood. The application of the methodology developed by Muggeo (2003) requires specifying the number of unknown breakpoints and the initial guess for each breakpoint. We implement Muggeo's procedure of detecting the breakpoints for $m = 1, \ldots, 5$. As a model selection criterion, we use the adjusted R-squared. For both hazard functions, we find that the optimal number of breakpoints is $m = 2$. That is, our results suggest that the hazard function for both bull and bear markets has two breakpoints. Table 2 reports the location of the estimated breakpoints and the 95% confidence interval for each breakpoint. Besides, this table provides the percentile of each estimated breakpoint within the corresponding distribution of durations. Table 3 reports the slope estimates for each of the three linear segments between two breakpoints as well as the 95% confidence interval for each slope. Figure 3 plots the empirical hazard functions and fitted piecewise linear models with two breakpoints. Vertical dotted lines in this figure show the location of the breakpoints.

First, we discuss the results for the piecewise linear model of the hazard function for bear markets. Our main observation is that the bear markets' duration dependence is always positive. Yet, the hazard rate is nonlinear in the age of a bear market. Our results suggest that when a bear market is 'young', the hazard rate increases rapidly with its age. However, as a bear market 'matures' and its age exceeds 16.12 months, the hazard rate levels off. Specifically, while the hazard function's slope is 0.0055 for bear markets shorter than 16.12 months, the hazard function's slope decreases to 0.0017 when the bear market's age is between 16.12 and 32.85 months. That is, our results suggest a reduction of the slope coefficient by a factor of three. Eventually, when the bear market age surpasses 32.85 months, the hazard rate dramatically increases. The first breakpoint of 16.12 months corresponds to the 60th percentile, whereas the second breakpoint corresponds to the 92nd percentile within the distribution of bear market duration. Note that the estimate for the first breakpoint is roughly equal to the average bear market duration.

Consequently, our results imply that when a bear market's age is less than its average value, the probability that it terminates increases rapidly. However, when a bear market's age rises above its average value, the probability that it ends increases slowly. Only for bear markets

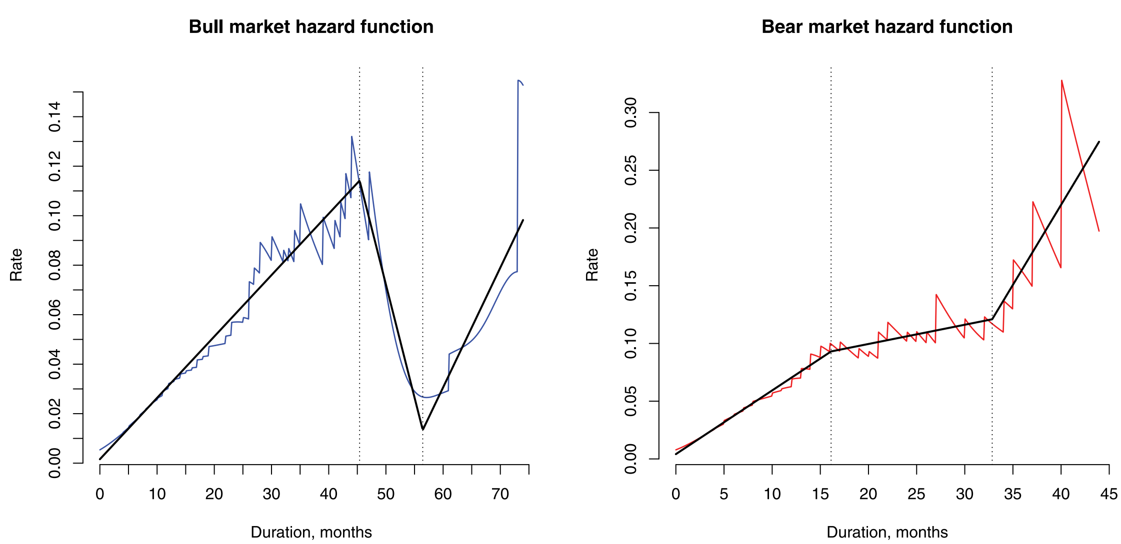

Figure 3. Empirical hazard functions and fitted piecewise linear models with 2 breakpoints. Vertical dotted lines show the location of the breakpoints.

with durations that fall within 8% of the longest durations, the hazard rate increases dramatically with age.

We turn to discuss the results for the piecewise linear model of the hazard function for bull markets. When the bull market's age is less than 45.40 months, the duration dependence is positive. This means that the longer a bull state lasts, the higher the probability that it ends. However, the duration dependence is negative when the bull market age lies between 45.40 and 56.44 months. This means that once the bull market age exceeds 45.40 months, the bull market gains 'momentum', and the probability that it ends decreases. Eventually, when the bull market age exceeds 56.44 months, the probability that it terminates again increases. However, it is worth emphasizing that the first breakpoint of 45.40 months corresponds to the 93rd percentile within the distribution of bull market duration. That is, for 93% of all bull markets, the duration dependence is strictly positive. Only 7% of all bull markets do not fit the positive duration dependence. Since the total number of bull market states is 63, only for 4 or 5 longest bull markets, the positive duration dependence breaks down.

In concluding this section, we want to briefly elaborate on why many bear markets are short and end quickly and why a few bull markets gain momentum and last long. To facilitate the discussion, Table 4 lists the five longest bull markets and shortest bear markets. First, we start with the short-lived bear markets. A review of these bear markets suggests the following story: An unexpected economic event triggers a stock market panic and sell-off. When investors realize that they have overreacted to this event, the stock market quickly recovers. For example, the plummeting gold price caused the stock market panic of 1869 (see Kindleberger (1978) and Morgan and Narron (2016)). The stock market crash in China precipitated the panic of 2015. In 1994, an unexpected rise in interest rates caused a debt crisis and a stock market turmoil. In 1990, Iraq invaded Kuwait, causing oil prices to increase substantially for a short period. The most often cited cause of the stock market crash in 1987 was 'portfolio insurance,' a popular hedging strategy executed by computers.

Table 4. The five longest bull markets and shortest bear markets. Duration is measured in months.

	Bull markets		Bear markets	
	Dates	Duration	Dates	Duration
1	Jul 1994 to Aug 2000	74	Sep 1987 to Nov 1987	3
2	Aug 1923 to Aug 1929	73	Sep 1869 to Dec 1869	4
3	Oct 2002 to Oct 2007	61	Jun 2015 to Sep 2015	4
4	Oct 2015 to Dec 2019	51	Jun 1990 to Oct 1990	5
5	Jul 1877 to May 1881	47	Feb 1994 to Jun 1994	5

Second, we continue with the longest bull markets. The two most prolonged bull markets ended with a speculative bubble in stocks and a subsequent stock market crash. Specifically, the Dot-com bubble of the 1990s burst in 2000, while the bubble of the 1920s (Roaring Twenties) led to the Great Crash of 1929. The third-longest bull market is associated with the US housing bubble, culminating in a bust of 2007 that led to the Global Financial Crisis. The fifth-longest bull market of 1877–1881 is characterized by a speculative mania in railroad stocks (Hughes 1955). It could be argued, therefore, that a bull market gains momentum and violates the positive duration dependence when a bull market develops into a full-fledged stock market bubble.

V. Fitting statistical distributions to bull and bear duration data

This section compares a few selected theoretical distributions in describing the empirical probability distributions of the bull and bear market durations and their hazard functions. These distributions include the most popular distributions for lifetime modelling. Each of these distributions is defined on the positive real line $t > 0$ and has from one to three parameters. These distributions are:

- Exponential with

$$f(t) = r\exp(-rt)$$

for $r > 0$,

- Weibull with

$$f(t) = \frac{k}{\lambda}\left(\frac{t}{\lambda}\right)^{k-1}\exp\left(-\left(\frac{t}{\lambda}\right)^{k}\right)$$

for $k > 0$ and $\lambda > 0$,

- Log-logistic with

$$f(t) = kr^k t^{k-1}\left(1 + (rt)^k\right)^{-2}$$

for $k > 0$ and $r > 0$,

- Gamma with

$$f(t) = \frac{r^k t^{k-1}}{\Gamma(k)} \exp(-rt)$$

for $k > 0$ and $r > 0$,

- Burr with

$$f(t) = \lambda kr^k t^{k-1}\left(1 + (rt)^k\right)^{-\lambda-1}$$

for $k > 0$, $\lambda > 0$ and $r > 0$.

The exponential distribution is widely used in lifetime modelling and reliability studies. This distribution is simple to work with and interpret. This distribution describes the state durations in a continuous-time Markov chain model. The hazard function in this distribution $h(t) = r$. That is, the hazard rate is constant, which means that there is no duration dependence. It is worth noting that the exponential is the only distribution with this property.

The Weibull distribution is described by two parameters and generalizes the exponential distribution. The hazard function in this distribution is given by

$$h(t) = \frac{k}{\lambda}\left(\frac{t}{\lambda}\right)^{k-1}.$$

When $k = 1$, the Weibull distribution becomes the exponential distribution with $r = 1/\lambda$. That is, there is no duration dependence if $k = 1$. The hazard function is increasing in duration if $k > 1$ and decreasing if $k < 1$. Consequently, the Weibull distribution can model both positive and negative duration dependence.

The hazard function in the log-logistic distribution is given by

$$h(t) = kr^k t^{k-1}\left(1 + (rt)^k\right)^{-1}.$$

For $0 < k \leq 1$, the hazard function decreases with duration. For $k > 1$, the hazard function first increases with duration, then decreases. Consequently, the log-logistic distribution can be used to model a hazard function that has an inverted U-shaped form.

The gamma distribution is described by two parameters and is very much like the Weibull distribution. If $k = 1$, the hazard function in the gamma distribution is constant. If $k < 1$ ($k > 1$), the hazard function is decreasing (increasing) in duration. Finally, the Burr distribution is described by three parameters. Therefore, it can generate a more intricate shape of the hazard function. For example, the hazard function in this distribution can first increase, then decrease, and then increase again.

Table 5 reports the maximum likelihood estimates of the selected distributions' parameters, along with their standard errors, fitted to the bull and bear markets duration data. It is worth observing that the parameter k in the Weibull and the gamma distribution is statistically significantly greater than 1, reflecting positive duration dependence.[4] Similarly, the parameter k in the log-logistic distribution is statistically significantly greater than 1, and this implies that the hazard function first increases with duration, then decreases. Note that for all but the Burr distribution, each parameter estimate is statistically significantly different from zero. Only for the Burr distribution, some parameters are not statistically significant. This result can be explained by the fact that the number of observations is relatively small, while the number of parameters in the Burr dis-

Table 5. Maximum likelihood parameter estimates (MLE) and their standard errors (SE).

	Parameter					
Distribution	r		k		λ	
	MLE	SE	MLE	SE	MLE	SE
Panel A: Bull markets						
Exponential	0.0389	0.0051				
Weibull			1.7381	0.1700	29.0390	2.3231
Log-logistic	0.0459	0.0038	2.7779	0.3027		
Gamma	0.1110	0.0213	2.8546	0.5021		
Burr	0.0310	0.0141	2.3302	0.4457	1.9552	1.3998
Panel B: Bear markets						
Exponential	0.0585	0.0077				
Weibull			1.9025	0.1932	19.3331	1.4228
Log-logistic	0.0672	0.0054	2.8636	0.3145		
Gamma	0.1844	0.0356	3.1505	0.5617		
Burr	0.0251	0.0223	2.1112	0.3607	5.3219	7.7218

[4]Statistical significance can be established using the estimated parameters and the standard errors of estimation of these parameters.

tribution is relatively large.

However, this section aims not to estimate the selected theoretical distributions' parameters but to evaluate how well these distributions fit the empirical data. For this purpose, we use three well-known goodness-of-fit test statistics (Stephens 1986): the Kolmogorov-Smirnov statistic, the Cramer-von Mises statistic, and the Anderson-Darling statistic. In each goodness-of-fit test statistic, the aim is to measure the distance between the fitted parametric distribution function $F(t)$ and the empirical distribution function $F_n(t)$. Table 6 provides the computational formulas for the three considered goodness-of-fit statistics. While the Kolmogorov-Smirnov statistic computes the maximum absolute difference between the theoretical and empirical distributions, the Cramer-von Mises statistic computes the integrated squared distance between the theoretical and empirical distributions. The Anderson-Darling statistic is also based on integrated squared distance, but it focuses on the distribution tails.

Table 7 displays the estimated goodness-of-fit statistics for the selected theoretical distributions. According to the Kolmogorov-Smirnov statistic, the bull market durations' empirical probability distribution is described best by the gamma distribution, while the Burr distribution comes second best. In contrast, both the Cramer-von Mises and Anderson-Darling statistics select the Burr distribution as the best one, whereas the gamma distribution comes second best. The log-logistic distribution is the third-best probability distribution to describe the bull duration data. Regarding the description of the bear market durations' empirical distribution, all three goodness-of-fit statistics are unanimous that the gamma distribution best fits the empirical data. The second-best theoretical distribution to describe the bear market duration is the Burr distribution, while the third-best is typically the Weibull distribution.

The Kolmogorov-Smirnov test statistic is, in fact, the statistic which is calculated under the null hypothesis that the theoretical probability distribution correctly describes the empirical distribution. Therefore, the Kolmogorov-Smirnov test is the test of whether two underlying probability distributions differ. The value of the Kolmogorov-Smirnov test statistic can be converted to a p-value of the test. For both the bull and bear markets, the Kolmogorov-Smirnov test rejects the null hypothesis only for the exponential distribution (these results are not reported to save space). In other words, this test cannot reject the null hypothesis that all but the exponential distribution correctly describe the empirical distributions of bull and bear market durations.

Figure 4 visualizes the goodness-of-fit for the three best theoretical distributions. In particular, the left panel in this plot shows the goodness-of-fit of the gamma, Burr, and log-logistic distributions to the survivor function for the bull markets. The right panel in this plot shows the goodness-of-fit of the gamma, Burr, and Weibull distributions to the survivor function for the bear markets.

The visual observation of the plots in Figure 4 suggests that all plotted theoretical distributions fit quite well the empirical survivor functions. However, since we are primarily interested in duration dependence, we want to know which theoretical distribution best fits the empirical hazard functions. To evaluate the goodness-of-fit to the empirical hazard functions, we employ two most common accuracy measures: the mean squared error (MSE) and the mean absolute error (MAE). These two measures of errors are computed according to

Table 6. Goodness-Of-Fit statistics as defined by Stephens (1986). n denotes the number of observations.

Statistic	Computational formula
Kolmogorov-Smirnov	$\sup\lvert F_n(t) - F(t)\rvert$
Cramer-von Mises	$n\int_{-\infty}^{\infty}(F_n(t) - F(t))^2 dt$
Anderson-Darling	$n\int_{-\infty}^{\infty}\frac{(F_n(t)-F(t))^2}{F(t)(1-F(t))} dt$

Table 7. Estimated goodness-of-fit statistics for the selected theoretical distributions. Each statistic evaluates the goodness-of-fit provided by the fitted theoretical distribution to the empirical cumulative distribution function.

	Probability distribution				
Statistic	Exp	Weibull	Loglogis	Gamma	Burr
Panel A: Bull markets					
Kolmogorov-Smirnov	0.2272	0.0867	0.0856	0.0621	0.0686
Cramer-von Mises	0.8506	0.0678	0.0497	0.0364	0.0331
Anderson-Darling	4.7006	0.5152	0.3045	0.2691	0.2198
Panel B: Bear markets					
Kolmogorov-Smirnov	0.2361	0.0834	0.0833	0.0688	0.0721
Cramer-von Mises	0.9389	0.0531	0.0562	0.0403	0.0428
Anderson-Darling	5.1498	0.3421	0.4520	0.2709	0.2864

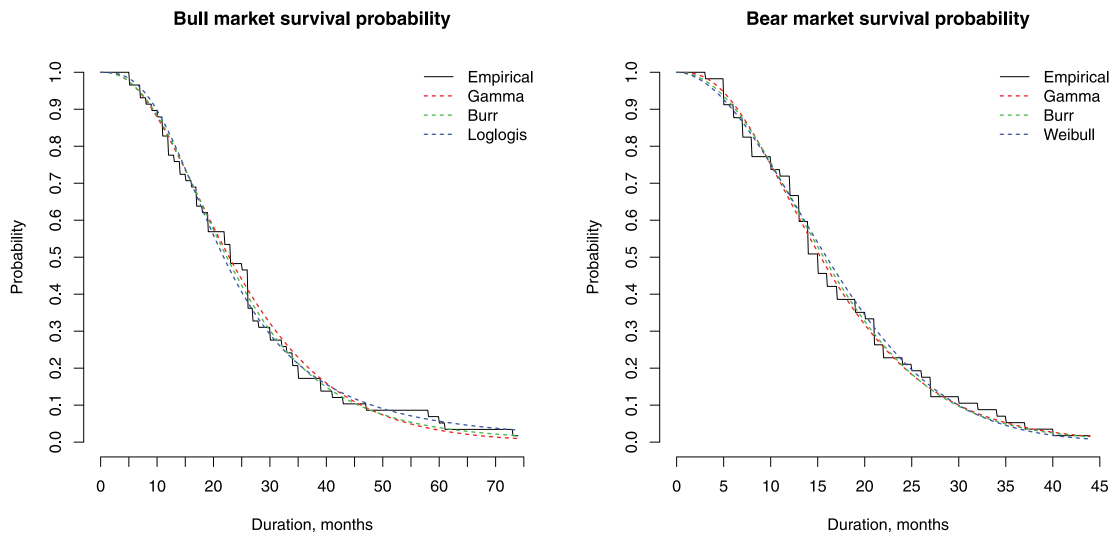

Figure 4. Illustration of the goodness-of-fit to the empirical survivor functions for the bull and bear markets provided by the three best theoretical distributions.

$$MSE = n^{-1}\sum_{i=1}^{n}(h_n(t) - h(t))^2,$$
$$MAE = n^{-1}\sum_{i=1}^{n}|h_n(t) - h(t)|,$$

where n is the number of observations, $h_n(t)$ is the empirical hazard function, and $h(t)$ is the theoretical hazard function.

Table 8 displays the estimated errors between the empirical and theoretical hazard functions for the selected distributions. Our main observation is that, regardless of the choice of the accuracy measure, the gamma distribution best fits the empirical hazard function for the bull markets. Therefore, the gamma distribution best fits both the empirical cumulative distribution function and the empirical hazard function. The second-best (third-best) fit is provided by the Burr (Weibull) distribution. In contrast, whereas the gamma distribution provides the best fit to the empirical cumulative distribution function of the bear market durations, the Weibull distribution provides the best fit to the empirical hazard function for the bear markets. The gamma (Burr) distribution gives the second-best (third-best) fit. Consequently, in describing bear market duration data, the ranking of probability distributions depends on whether one evaluates the fit to the cumulative distribution function or the hazard function.

Table 8. Estimated errors between the empirical and theoretical hazard functions for the selected distributions.

	Probability distribution				
Statistic	Exp	Weibull	Loglogis	Gamma	Burr
Panel A: Bull markets					
MSE	0.0109	0.0071	0.0167	0.0042	0.0044
MAE	0.0260	0.0185	0.0306	0.0138	0.0145
Panel B: Bear markets					
MSE	0.0706	0.0085	0.0904	0.0184	0.0179
MAE	0.0626	0.0171	0.0623	0.0224	0.0224

Figure 5 shows the goodness-of-fit for the three best theoretical distributions to the empirical hazard functions. Specifically, the left panel in this plot shows the goodness-of-fit of the gamma, Burr, and Weibull distributions to the empirical hazard function for the bull markets. The right panel in this plot shows the goodness-of-fit of the Weibull, gamma, and Burr distributions to the empirical hazard function for the bear markets. A visual inspection of the fitted curves suggests that all distributions correctly describe the duration dependence in the left tail of the hazard functions. Still, none of the fitted distributions accurately describes the right tail of the hazard functions. However, the right tail of each empirical hazard function is constructed using the data on the market states with the longest durations. Thus, it is computed using only a few observations. Therefore, we should be careful in interpreting the results based on a small number of observations.

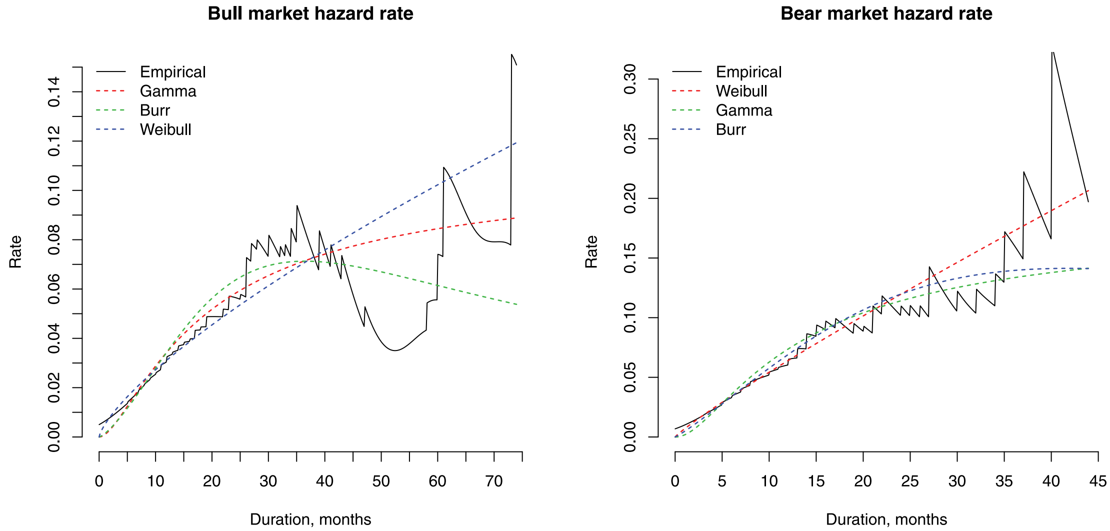

Figure 5. Illustration of the goodness-of-fit to the empirical hazard functions for the bull and bear markets provided by the three best theoretical distributions.

VI. Conclusions

There is a big controversy among both investment professionals and academics regarding how the termination probability of a market state depends on its age. Using more than two centuries of data on the broad US stock market index, we revisit the duration dependence in bull and bear markets. Our results suggest that the duration dependence for both bull and bear markets is a nonlinear function of the state age. It appears that the duration dependence in bear markets is strictly positive. When a bear market's age is less than its average value, the termination probability increases relatively rapidly with age. However, when a bear market's age rises above its average value, the probability that it terminates increases slowly. For 93% of the bull markets, the duration dependence is also positive. Only about 7% of the bull markets, those with the most prolonged durations, do not exhibit positive duration dependence. We also compare a few selected theoretical distributions in describing the duration dependence in bull and bear markets. Our results advocate that the gamma distribution most often provides the best fit for both the survivor and hazard functions of bull and bear markets. However, our results reveal that none of the selected distributions accurately describes the hazard functions' right tail.

Successful forecasting of the future market state can potentially deliver big profits to investors. Typically, the stock market states are predicted using either Markov or semi-Markov hidden models (Kole and van Dijk 2017). In this regard, our results are very relevant because they help to choose an appropriate probability distribution of a market state duration. However, our results also advocate that the duration of some bull and bear markets depends on investors' sentiment. Specifically, our review of the shortest bear and longest bull markets suggests that they are related to stock market panics and speculative manias. Therefore, in forecasting market states, there is a need to use investors' sentiment and advanced techniques such as machine learning, deep learning, and artificial intelligence. An example of such a forecasting technique is provided by Zhao and Chen (2021).

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No potential conflict of interest was reported by the author(s).

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