Advances in Mathematics 404 (2022) 108409

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Variational principles for fluid dynamics on rough paths

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ARTICLE INFO

Article history:
Received 25 February 2022
Accepted 6 April 2022
Available online 26 April 2022
Communicated by C. Fefferman

MSC:
primary 35Q35, 37K58, 60H15, 60L20, 60L50

Keywords:
Geometric rough paths
Fluid dynamics
Variational principles
Euler–Poincaré equations
Rough partial differential equations

ABSTRACT

In recent works, beginning with [76], several stochastic geophysical fluid dynamics (SGFD) models have been derived from variational principles. In this paper, we introduce a new framework for parametrization schemes (PS) in GFD. We derive a class of rough geophysical fluid dynamics (RGFD) models as critical points of rough action functionals using the theory of controlled rough paths. These RGFD models characterize Lagrangian trajectories in fluid dynamics as geometric rough paths (GRP) on the manifold of diffeomorphic maps. We formulate three constrained variational approaches for the derivation of these models. The first is the Clebsch formulation, in which the constraints are imposed as rough advection laws. The second is the Hamilton-Pontryagin formulation, in which the constraints are imposed as right-invariant rough vector fields. And the third is the Euler–Poincaré formulation, in which the variations are constrained. These constrained rough variational principles lead directly to the Lie–Poisson Hamiltonian formulation of fluid dynamics on GRP. The GRP framework preserves the geometric structure of fluid dynamics obtained by using Lie group reduction to pass from Lagrangian to Eulerian variational principles, yielding a rough formulation of the Kelvin circulation theorem. The rough formulation enhances its stochastic counterpart developed in [76], and extended

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https://doi.org/10.1016/j.aim.2022.108409

to semimartingales in [109]. For example, the rough-path variational approach includes non-Markovian perturbations of the Lagrangian fluid trajectories. In particular, memory effects can be introduced through a judicious choice of the rough path (e.g. a realization of a fractional Brownian motion). In the particular case when the rough path is a realization of a semimartingale, we recover the SGFD models in [76,109]. However, by eliminating the need for stochastic variational tools, we retain a *pathwise* interpretation of the Lagrangian trajectories. In contrast, the Lagrangian trajectories in the stochastic framework are described by stochastic integrals, which do not have a pathwise interpretation. Thus, the rough path formulation restores this property.

Contents

1. Introduction

The present work aims to transfer the fundamental properties of deterministic fluid dynamics derived by Hamilton's principle into their formulation on geometric rough paths. Recent work [44] concerning solution properties of Euler fluid dynamics on rough paths demonstrates the efficacy of this approach to produce previously unavailable results, such as the Beale-Kato-Majda blowup criterion for ideal fluid solutions on geometric rough paths.

To set the scene, we discuss some aspects of the Hamilton's principle variational approach to modelling fluid dynamics behaviour using its Lie group symmetry. Hamilton's principle states that critical points $\delta S = 0$ of a time integral $S = \int_0^T L\,dt$ with Lagrangian functional $L : TM \to \mathbb{R}$ determine dynamical equations on a manifold M. Since its inception, Hamilton's principle has provided a systematic mathematical framework for scientific investigation. For example, Lie symmetries of Hamilton's principle encode conservation laws (i.e., Noether's theorem [99,82,78]) on whose level sets the ensuing dynamics takes place. Lie symmetries of Hamilton's principle also reduce the number of dynamical degrees of freedom to equivalence classes of observables that transform under the corresponding Lie group.

The reduced Hamilton's principle leading to the Euler-Poincaré equations for ideal continuum mechanics was only developed recently in [77]. For the flow of ideal fluids in a fixed domain $M \subset \mathbb{R}^n$, Lie group symmetry reduces the number of degrees of freedom to the equivalence classes of observables that transform under pull-back by smooth invertible maps with smooth inverses ($\phi \in \mathrm{Diff}(M)$, diffeomorphisms) in which the composition of functions is understood as a Lie group operation. Euler fluid dynamics is then recast as a flow map ϕ_t which defines a *time-dependent* geodesic curve on the manifold of diffeomorphisms, cf., [6,77].

The present approach is based on the premise that Euler's fluid equations arise from Hamilton's variational principle for geodesic flow on the manifold of diffeomorphisms with respect to the metric defined by the kinetic energy of the fluid, [6,53]. The variations are constrained by the condition of right-invariance of the velocity vector field. Hamilton's principle for fluids is modified when advection by the fluid motion under the action of the diffeomorphisms carries fluid properties such as mass and heat, whose contribution to the thermodynamic equation of state affects the motion [77]. These advected fluid quantities are said to follow *Lagrangian trajectories* of fluid parcels in the flow. Since the Lagrangian trajectories for Euler's ideal fluid equations are *pushed forward* by time-dependent diffeomorphic maps, these trajectories may be regarded as curves parametrized by time on the manifold of smooth invertible maps (diffeomorphisms) [53].

Preserving the fundamental structure derived from Hamilton's principle in the course of more general fluid modelling is paramount. These theoretical considerations have helped in developing Hamilton's principle modelling for stochastic continuum mechanics in [76]. In turn, this new development has recently led to new methods for stochastic data assimilation using particle-filtering in geophysical fluid dynamics (GFD) [41].

The need for robust and computationally efficient Parametrization Schemes (PS) that model the effects of fast sub-grid scale physics and other unresolved processes is well understood in Geophysical Fluid Dynamics (GFD). See, for example, [64], for a recent overview. Stochastic Parameterization Schemes (SPS) have the additional ability to introduce model uncertainty [13] naturally. SPS have improved the probabilistic skill of the ensemble weather forecasts by increasing their reliability and reducing the error of the ensemble mean. The coming years are likely to see a further increase in the use of SPS in ensemble methods in forecasts and assimilation. This, however, will put increasing demands on the methods used to represent computational model uncertainty in the dynamical core and other components of the Earth system while maintaining overall computational efficiency [85].

The preservation of geometrical structure and physicality of fluid dynamics can serve as a guiding principle in designing robust PS for GFD. The PS are meant to preserve predictive power, accuracy, and computational efficiency in modelling the effects of both: (i) unresolved phenomena due to the known but unresolved rapid sub-grid scale physics, as well as (ii) uncertainty due to unknown bias in the data. Thus, in ensemble computations, PS face a daunting combination of tasks.

In this paper, we propose a structured approach for parametrization of the rapid scales of fluid motion by using a temporally rough vector field in the framework of geometric rough paths (GRP) [59], which we call Geometric Rough Path Parametrization Schemes (GRPPS). Namely, we will develop a new class of variational principles for fluids that model resolved and unresolved motions of fluid advection as GRPPS. Critical points of our rough-path constrained variational principles are rough partial differential equations (RPDEs), whose dynamics incorporate both the resolved-scale fluid velocity and the effects of the unresolved fluctuations.

In the particular case when the generating rough path is a straight line, or, more generally, a smooth curve, the GRPPS approach introduced here reduces to a PS approach obtained through classical/deterministic variational principles (see Section C). Similarly, when the generating rough path is a realization of a Brownian motion (or, more generally, of a semimartingale process), GRPPS specializes to a pathwise formulation of the SPS characterized through the stochastic variational principles first introduced in [76]. In other words, this work enhances the mathematical framework of *Stochastic Advection by Lie Transport* (SALT), in which the Lagrangian trajectories are treated as time-dependent Stratonovich stochastic processes [76].

Non-Markovian models include models with memory which are of interest in ocean dynamics, see, e.g., [119,51,98,19,120,84,102]. The variational treatment of fluid dynamics on rough paths enables the introduction of such models. This can be accomplished, for example, by realising the rough path as a fractional Brownian motion, or as a more general Gaussian process with suitably chosen time correlation. There is growing evidence that non-Markovianity improves models of the effects of fast sub-grid scales on the resolved scales (see, e.g., [5,61,33,86]). It stands to reason that such models could be useful in parametrising the sub-grid scales of real fluids. Our framework, in particular, includes spatially-local, non-Markovian SPS by using rough paths (rather than, for example, state-delay terms) to model the sub-grid scale terms. Since our framework retains the core geometric structure of fluid mechanics, one can expect the solution properties of our equations would track those of the deterministic unperturbed equations (e.g., stability up to blow-up time [44]). This sort of fidelity would be important in the calibration of our models, especially for those calibrations that use the modern 'solver-in-the loop' estimation procedures [115].

The contents of this paper. The overall goal of the present paper is to formulate rigorously in Theorems 2.6 and 3.12 variational principles for ideal fluid dynamics with advection of fluid quantities along Geometric Rough Paths. To achieve this goal, our first aim is to derive a rough version of the Lie chain rule in Theorem 3.3 leading to the GRP version of the classical Reynolds transport formula in Corollary 3.5. The Reynolds transport formula for momentum density encapsulates the force law which governs fluid motion. Mathematically, the formula describes the rate of change of the integral of the fluid momentum density over a moving control volume that is being transported by the rough (fluid) flow along a GRP. Thus, our key result is the Lie chain rule formula (3.4) in Theorem 3.3 for the rough differential (or increment in time) of the pull-back and push-forward of a tensor-field-valued GRP by a rough flow. The Lie chain rule formula (3.4) is intuitive and natural because it follows from the extension of ordinary calculus to GRP. The formula leads to a unified, stable, and flexible framework for modelling fluids whose Lagrangian parcels move along temporally rough paths. Theorem 3.3 for the Lie chain rule is the foundation on which the other contributions of the paper rest.

In order to derive the momentum and advection equations satisfied by the critical points of our variational principles, we required a rough version of the fundamental lemma of the calculus of variations. A version is formulated and proved in Section B.3. As far as we are aware, this is a new result.

Our work is an example of the rigorous content of the Malliavin transfer principle, which says that geometric constructions involving manifold-valued curves can be extended to manifold-valued stochastic paths by replacing classical calculus with geometric rough-path calculus (see, e.g. [104,54,32,21,50,3]). More specifically, we show that deterministic geometric continuum mechanics can be extended to rough-path geometric continuum mechanics. In the particular case when the rough path is a realization of a semimartingale, we recover the SGFD models in [76,109]. However, by eliminating the need for stochastic variational tools, we retain a *pathwise* interpretation of the Lagrangian trajectories. In contrast, the Lagrangian trajectories in the stochastic framework are described by stochastic integrals, which do not have a pathwise interpretation. In summary, the rough path formulation restores the pathwise property.

The paper is structured as follows:

- Section 2 formulates the first variational principle for fluid dynamics on geometric rough paths in Theorem 2.6, by imposing the *Clebsch constraint* for the advection fluid quantities along rough paths.
- Section 3 formulates the Lie Chain Rule Theorem 3.3 and the Reynolds Transport Corollary 3.5 for geometric rough paths. The Reynolds transport formula in the special case of one-forms yields the rough Kelvin–Noether Theorem 3.6. Section 3 also formulates the Hamilton-Pontryagin variational principle for rough paths in Theorem 3.12, which imposes the constraint that the vector fields which generate the Lagrangian trajectories are right-invariant under diffeomorphisms whose time dependence is rough. The Clebsch and Hamilton-Pontryagin variational principles for rough paths correspond to those derived in the SALT approach in [76] and [63], respectively. Next we formulate the Euler–Poincaré constrained variational principle for ideal fluid motion on GRP in Theorem 3.15. Here, we pose an open problem regarding the construction of variations used in this principle. Finally, we develop the Lie–Poisson Hamiltonian formulation of fluid dynamics on GRP in Corollary 3.19.
- Section 4 provides three examples of fluid equations on GRP. These are: i) the rough Euler equation for incompressible fluid flow; ii) the rough Camassa-Holm equation and its limiting case, the rough Burgers equation; and iii) the equations for ideal compressible adiabatic fluid dynamics on GRP.
- Finally, Section 5 contains the proofs of the main results formulated in Section 3.

In addition, the paper contains five Appendices which are meant to provide notation and background information, including proofs of key technical results invoked in the main text, a simple example of our procedure in the setting of smooth paths, and additional history and motivation. The first two Appendices are essential and contain together the key relationships and definitions needed in both geometric mechanics and the theory of rough paths for the present work, which as far as we know are found together nowhere else. The latter three Appendices provide additional information and motivation for the theory of rough paths.

Appendix A defines the notation we use and summarises the essential background and results for both rough paths and geometric mechanics that we use in the text. We choose to put this section in the appendix rather than in the main text since different classes of readers may be familiar with at least some of our notation, and might wish to see the statements of the main results presented first. The main text will refer to sections in Appendix A as needed if we think a notation is not standard. Appendix B contains proofs of selected technical results which facilitate the proofs in the main text. Appendix C illustrates the variational principles we use in the example of a homogeneous incompressible fluid flow perturbed by spatially and temporally smooth noise. This example serves as a guide for introducing rough perturbations into the variational principles for more general fluid theories. Appendix D provides a short history and motivation in the development of the theory of rough paths and Appendix E discusses the concrete example of Gaussian rough paths and provides additional references to this important class of rough paths.

Contributions of this paper.

This paper offers a variational framework that connects Geometric Rough Path Theory with Geophysical Fluid Dynamics, hopefully to the benefit of both fields. The geometric variational approach followed here may enhance the development of mathematical and numerical models in a range of investigations in Weather Prediction, Data Assimilation, Ocean Dynamics, Atmospheric Science, perhaps even Turbulence. For example, the model development may benefit from theoretical results (stability results, large deviation principles, splitting schemes) for random dynamical systems arising from rough partial differential equations [44]. In turn, the new connections between GFD and geometric rough paths may become a fruitful source of open problems in mathematics.

This paper introduces a GRPPS framework that transcends the scope of either deterministic or stochastic parametrization schemes by allowing GRP with Hölder index $\alpha \in \left(\frac{1}{3}, 1\right]$.[1] The case $\alpha = 1$ recovers deterministic fluid dynamics (see, e.g., Section C). The case $\alpha = 1/2 - \epsilon$ ($\epsilon \ll 1$) gives a pathwise characterization of SPS. Widening the choice of Hölder index provides a broader scope for modelling with PS. Indeed, one may also include models which are non-Markovian (for example, by choosing the rough path as a realization of a fractional Brownian motion, or of a more general Gaussian process with suitably chosen time dependence).

The GRPPS presented here possess the following fundamental properties:

- Being derived from Hamilton's variational principle, they preserve the geometric structure of fluid dynamics [77].
- They satisfy a Kelvin circulation theorem, which is the classical essence of fluid flow.
- They are consistent with the modern mathematical formulation of fluid flow as geodesic flows on the manifold of smooth invertible transformations, with respect to the metric associated with the fluid's kinetic energy [6].
- They accommodate Pontryagin's maximum principle for control in taking a dynamical system from one state to another, especially in the presence of constraints for the state or input controls [17].

Open problems Following this work, the following problems remain open:

- Completeness of the constrained velocity variations in formulating the RPDEs in the Euler-Poincaré Theorem 3.15.
- Well-posedness of RPDEs derived in Sec. 4.1 and Sec. 4.2.

[1] The analysis presented here can be extended to $\alpha \in (0, 1]$ at the expense of more elaborate computations.

- Estimation of the rough path properties and calibration of the GRPPS model from observed or simulated data.
- The development of pathwise data assimilation methods for the incorporation of data into GRPPS.
- Uncertainty quantification and forecast analysis using GRPPS.

Acknowledgments

All of the authors are grateful to our friends and colleagues who have generously offered their time, thoughts and encouragement in the course of this work during the time of COVID-19. We are particularly grateful to T.D. Drivas and S. Takao for thoughtful discussions. DC and DH are grateful for partial support from ERC Synergy Grant 856408 - STUOD (Stochastic Transport in Upper Ocean Dynamics). JML is grateful for partial support from US AFOSR Grant FA9550-19-1-7043 - FDGRP (Fluid Dynamics of Geometric Rough Paths) awarded to DH as PI. TN is grateful for partial support from the DFG via the Research Unit FOR 2402.

2. The Clebsch variational principle for geometric rough paths

To streamline the presentation of our results and to provide a ready reference for the reader, we have assembled the notation and background of geometric mechanics and rough paths theory needed for this paper into one place – Appendix A – rather than dispersing it sequentially in the main text.

Let $\mathbf{Z} = (Z, \mathbb{Z}) \in \mathcal{C}_T^{\alpha}(\mathbb{R}^K)$ be a given geometric rough path with Hölder index $\alpha \in (\frac{1}{3}, 1]$ defined in the time interval $t \in [0, T]$. Let $\mathfrak{X} = \mathfrak{X}_{\mathcal{F}_1}$ denote a function space of vector fields. Let $\mathfrak{X}^\vee = \mathfrak{X}^\vee_{\mathcal{F}_2}$ denote a function space of one-form densities such that the canonical pairing $\langle \cdot, \cdot \rangle_{\mathfrak{X}} : \mathfrak{X}^\vee_{C^\infty} \times \mathfrak{X}_{C^\infty} \to \mathbb{R}$ defined in (A.13) extends to a continuous pairing on $\mathfrak{X}^\vee \times \mathfrak{X}$. For incompressible fluids, we implicitly use the constructions of the divergence-free, or both divergence-free and harmonic-free vector fields and their canonical ‘duals’ (see Definition A.20). We write all spaces in brief notation without including Riemannian measure μ_g or other extraneous adornment for a unified treatment of the compressible and incompressible case. That is to say, for incompressible fluid flows, all vector fields (and variations of vector fields) and one-form densities are constrained. Using Cartan’s formula (A.12) and the Stokes theorem, one can show that for all $u \in \mathfrak{X}_{C^\infty}$, the adjoint (see (A.14)) of the vector-field operation

$$\mathrm{ad}_u = -\pounds_u : \mathfrak{X}_{C^\infty} \to \mathfrak{X}_{C^\infty}$$

relative to the canonical pairing $\langle \cdot, \cdot \rangle_{\mathfrak{X}}$ is given by $\mathrm{ad}^*_u = \pounds_u : \mathfrak{X}^\vee_{\mathcal{D}'} \to \mathfrak{X}^\vee_{\mathcal{D}'}$. Thus, for all $\alpha \otimes D \in \mathfrak{X}^\vee_{C^\infty}$, we have

$$\mathrm{ad}^*_u(\alpha \otimes D) = \pounds_u(\alpha \otimes D) = \pounds_u \alpha \otimes D + \alpha \otimes \pounds_u D = \pounds_u \alpha \otimes D + \alpha \otimes (\mathrm{div}_D\, uD).$$

Let A be a direct summand of alternating form bundles and tensor bundles such that the first component of A is the density bundle $\Lambda^d T^*M$. Let $A^\vee$ denote the canonical dual in Section A.2.2. Define $\langle \cdot, \cdot \rangle_{\mathfrak{A}} : \mathfrak{A}^\vee_{C^\infty} \times \mathfrak{A}_{C^\infty} \to \mathbb{R}$ via a sum as explained in (A.13). Let $\mathfrak{A} = \mathfrak{A}_{\mathcal{F}_3} = \Gamma_{\mathcal{F}_3}(A)$ and $\mathfrak{A}^\vee = \mathfrak{A}^\vee_{\mathcal{F}_4} = \Gamma_{\mathcal{F}_4}(A^\vee)$ be function spaces such that the pairing $\langle \cdot, \cdot \rangle_{\mathfrak{A}}$ extends to a continuous pairing on $\mathfrak{A}^\vee \times \mathfrak{A}$. For all $u \in \mathfrak{X}_{C^\infty}$, let $\pounds^*_u : \mathfrak{A}^\vee_{C^\infty} \to \mathfrak{A}^\vee_{C^\infty}$ denote the adjoint (see (A.14)) of the Lie derivative $\pounds_u : \mathfrak{A}^\vee_{C^\infty} \to \mathfrak{A}^\vee_{C^\infty}$ defined relative to the canonical pairing $\langle \cdot, \cdot \rangle_{\mathfrak{A}}$.

Definition 2.1 *(Diamond operator* $(\diamond)$*).* We define the bilinear *diamond operator* $\diamond : \mathfrak{A}^\vee_{C^\infty} \times \mathfrak{A}_{C^\infty} \to \mathfrak{X}^\vee_{C^\infty}$ via the relation

$$\langle \lambda \diamond a, u \rangle_{\mathfrak{X}} = - \langle \lambda, \pounds_u a \rangle_{\mathfrak{A}}, \quad \forall (\lambda, u, a) \in \mathfrak{A}^\vee_{C^\infty} \times \mathfrak{X}_{C^\infty} \times \mathfrak{A}_{C^\infty}.$$

We refer the reader to [77] and Section 4 (esp. Section 4.4) for explicit computations with the diamond operator in fluid dynamics. We assume that $\diamond$ extends to a continuous operator $\diamond : \mathfrak{A}^\vee \times \mathfrak{A} \to \mathfrak{X}^\vee$.

Assumption 2.2. Let $\ell : \mathfrak{X} \times \mathfrak{A} \to \mathbb{R}$. Assume there exist (functional derivatives) $\frac{\boldsymbol{\delta} \ell}{\boldsymbol{\delta} u} : \mathfrak{X} \times \mathfrak{A} \to \mathfrak{X}^\vee$ and $\frac{\boldsymbol{\delta} \ell}{\boldsymbol{\delta} a} : \mathfrak{X} \times \mathfrak{A} \to \mathfrak{A}^\vee$ such that for all $(u, a) \in \mathfrak{X} \times \mathfrak{A}$ and $(\boldsymbol{\delta} u, \boldsymbol{\delta} a) \in \mathfrak{X}_{C^\infty} \times \mathfrak{A}_{C^\infty}$:

(i)

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \ell(u + \epsilon \boldsymbol{\delta} u, a + \epsilon \boldsymbol{\delta} a) =: \left\langle \frac{\boldsymbol{\delta} \ell}{\boldsymbol{\delta} u}(u,a), \boldsymbol{\delta} u \right\rangle_{\mathfrak{X}} + \left\langle \frac{\boldsymbol{\delta} \ell}{\boldsymbol{\delta} a}(u,a), \boldsymbol{\delta} a \right\rangle_{\mathfrak{A}};$$

(ii) for any sequence $\{(u^n, a^n)\}_{n \in \mathbb{N}} \subset \mathfrak{X}_{C^\infty} \times \mathfrak{A}_{C^\infty}$ such that $(u^n, a^n) \to (u, a)$ as $n \to \infty$ in $\mathfrak{X} \times \mathfrak{A}$;

$$\lim_{n \to \infty} \left\langle \frac{\boldsymbol{\delta} \ell}{\boldsymbol{\delta} u}(u^n, a^n), \boldsymbol{\delta} u \right\rangle_{\mathfrak{X}} = \left\langle \frac{\boldsymbol{\delta} \ell}{\boldsymbol{\delta} u}(u,a), \boldsymbol{\delta} u \right\rangle_{\mathfrak{X}} \quad \text{and}$$
$$\lim_{n \to \infty} \left\langle \frac{\boldsymbol{\delta} \ell}{\boldsymbol{\delta} a}(u^n, a^n), \boldsymbol{\delta} a \right\rangle_{\mathfrak{A}} = \left\langle \frac{\boldsymbol{\delta} \ell}{\boldsymbol{\delta} a}(u,a), \boldsymbol{\delta} a \right\rangle_{\mathfrak{A}}.$$

(iii) the mapping $\frac{\boldsymbol{\delta} \ell}{\boldsymbol{\delta} u}(\cdot, a) : \mathfrak{X} \to \mathfrak{X}^\vee$ is an isomorphism.

Let $\xi \in \mathfrak{X}^K_{\mathcal{D}'}$ denote a fixed collection of vector fields.[2]

Definition 2.3. Let $Clb_{\mathbf{Z}}$ denote the space of all

$$(u, \mathbf{a}, \boldsymbol{\lambda}) \in C^\alpha_T(\mathfrak{X}) \times \boldsymbol{\mathcal{D}}_{Z,T}(\mathfrak{A}) \times \boldsymbol{\mathcal{D}}_{Z,T}(\mathfrak{A}^\vee)$$

such that

[2] It is worth noting that one can make ξ time dependent and depend on other quantities as in [63].

(i) for all $\phi \in \mathfrak{X}_{C^\infty}$, we have $\pounds_u a, \pounds_u \phi, \pounds_\phi a \in C_T^\alpha(\mathfrak{A})$, and there exists $(\pounds_\xi a)'$ such that $(\pounds_\xi a, (\pounds_\xi a)') \in \boldsymbol{\mathcal{D}}_{Z,T}(\mathfrak{A})$;

(ii) for all $\phi \in \mathfrak{X}_{C^\infty}$, we have $\pounds_u^* \lambda, \pounds_u^* \phi, \pounds_\phi^* \lambda \in C_T^\alpha(\mathfrak{A}^\vee)$, and there exists $(\pounds_\xi^* \lambda)'$ such that $(\pounds_\xi^* \lambda, (\pounds_\xi^* \lambda)') \in \boldsymbol{\mathcal{D}}_{Z,T}(\mathfrak{A}^\vee)$.

Remark 2.4. Let $\tilde{\mathfrak{X}}$, $\tilde{\mathfrak{A}}$, and $\tilde{\mathfrak{A}}^\vee$ be function spaces (see Section A.2.2) such that $\tilde{\mathfrak{X}} \hookrightarrow \mathfrak{X}$, $\tilde{\mathfrak{A}} \hookrightarrow \mathfrak{A}$, and $\tilde{\mathfrak{A}}^\vee \hookrightarrow \mathfrak{A}^\vee$, and $\pounds \in \boldsymbol{\mathcal{L}}(\tilde{\mathfrak{X}} \times \tilde{\mathfrak{A}}, \mathfrak{A})$ and $\pounds^* \in \boldsymbol{\mathcal{L}}(\tilde{\mathfrak{X}} \times \tilde{\mathfrak{A}}^\vee, \mathfrak{A}^\vee)$. If $(u, \mathbf{a}, \boldsymbol{\lambda}) \in C_T(\tilde{\mathfrak{X}}) \times \boldsymbol{\mathcal{D}}_{Z,T}(\tilde{\mathfrak{A}}) \times \boldsymbol{\mathcal{D}}_{Z,T}(\tilde{\mathfrak{A}^\vee})$ and $\xi \in \tilde{\mathfrak{X}}^K$, then (i) and (ii) hold.

Clebsch variational principle. The Clebsch action functional $S^{Clb_{\mathbf{Z}}} : Clb_{\mathbf{Z}} \to \mathbb{R}$ is defined by

$$S^{Clb_{\mathbf{Z}}}(u, \mathbf{a}, \boldsymbol{\lambda}) = \int_0^T \ell(u_t, a_t)\mathrm{d}t + \langle \lambda_t, \mathrm{d}\mathbf{a}_t + \pounds_{\mathrm{d}\mathbf{x}_t} a_t \rangle_{\mathfrak{A}}, \tag{2.1}$$

where

$$\langle \lambda_t, \pounds_{\mathrm{d}\mathbf{x}_t} a_t \rangle_{\mathfrak{A}} := \langle \lambda_t, \pounds_{u_t} a_t \rangle_{\mathfrak{A}} \mathrm{d}t + \langle \lambda_t, \pounds_\xi a_t \rangle_{\mathfrak{A}} \mathrm{d}\mathbf{Z}_t, \quad \mathrm{d}\mathbf{x}_t := u_t \mathrm{d}t + \xi \mathrm{d}\mathbf{Z}_t.$$

Remark 2.5. By Remark A.9, the integral $\int_0^T \langle \lambda_t, \mathrm{d}\mathbf{a}_t \rangle_{\mathfrak{A}}$ in the Clebsch action functional in (2.1) is well-defined. Indeed, the extra structure provided by the Gubinelli derivative in the controlled rough path space (whose elements are denoted with bold font, see Appendix subsection A.1.1) allows one to construct this integration.

A *variation* of $(u, \mathbf{a}, \boldsymbol{\lambda}) \in Clb_{\mathbf{Z}}$ is a curve $\{(u^\epsilon, \mathbf{a}^\epsilon, \boldsymbol{\lambda}^\epsilon)\}_{\epsilon \in (-1,1)} \subset Clb_{\mathbf{Z}}$ of the form

$$(u^\epsilon, \mathbf{a}^\epsilon, \boldsymbol{\lambda}^\epsilon) = (u + \epsilon \boldsymbol{\delta} u, \mathbf{a} + \epsilon \boldsymbol{\delta} a, \boldsymbol{\lambda} + \epsilon \boldsymbol{\delta} \lambda),$$

for arbitrarily chosen $(\boldsymbol{\delta} u, \boldsymbol{\delta} a, \boldsymbol{\delta} \lambda) \in C_T^\infty(\mathfrak{X}_{C^\infty} \times \mathfrak{A}_{C^\infty} \times \mathfrak{A}_{C^\infty}^\vee)$ such that $\boldsymbol{\delta} a$ vanishes at $t = 0$ and $t = T$. We say $(u, \mathbf{a}, \boldsymbol{\lambda}) \in Clb_{\mathbf{Z}}$ is a critical point of the action functional $S^{Clb_{\mathbf{Z}}}$, if for all variations one has

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} S^{Clb_{\mathbf{Z}}}(u^\epsilon, \mathbf{a}^\epsilon, \boldsymbol{\lambda}^\epsilon) = 0.$$

By virtue of the controlled rough path calculus and, in particular, Lemmas A.13 and B.4, we obtain the following Clebsch variational principle.[3]

Theorem 2.6 *(Clebsch variational principle on geometric rough paths). A curve* $(u, \mathbf{a}, \boldsymbol{\lambda}) \in Clb_{\mathbf{Z}}$ *is a critical point of* $S^{Clb_{\mathbf{Z}}}$ *in* (2.1) *iff for all* $t \in [0, T]$*, the following equations hold.*

[3] For more details about the history and applications of the Clebsch variational principle in fluid dynamics, see Appendix C.

$$\begin{aligned}
&m_t + \int_0^t \pounds_{u_s} m_s \mathrm{d}s + \int_0^t \pounds_\xi m_s \mathrm{d}\mathbf{Z}_s \overset{\mathfrak{X}^\vee}{=} m_0 + \int_0^t \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta}a}(u_s, a_s) \diamond a_s \mathrm{d}s, \quad m = \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta}u}(u,a) = \lambda \diamond a, \\
&a_t + \int_0^t \pounds_{u_s} a_s \mathrm{d}s + \int_0^t \pounds_\xi a_s \mathrm{d}\mathbf{Z}_s \overset{\mathfrak{A}}{=} a_0, \\
&\lambda_t \overset{\mathfrak{A}^\vee}{=} \lambda_0 + \int_0^t \left(\pounds_{u_t}^* \lambda_s + \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta}a}(u_s, a_s) \right) \mathrm{d}s + \int_0^t \pounds_\xi^* \lambda_s \mathrm{d}\mathbf{Z}_s.
\end{aligned} \tag{2.2}$$

Proof. See Section 5.1. □

Remark 2.7. The Lagrange multiplier λ enforces the constraint that 'a' satisfies

$$a_t + \int_0^t \pounds_{u_s} a_s \mathrm{d}s + \int_0^t \pounds_\xi a_s \mathrm{d}\mathbf{Z}_s \overset{\mathfrak{A}}{=} a_0, \quad \forall t \in [0, T].$$

That is, the quantity 'a' is (formally) advected by the integral curves of the vector field $\mathrm{d}x_t = u\mathrm{d}t + \xi \mathrm{d}\mathbf{Z}_t$. The Lie chain rule (Theorem 3.3) and Hamilton-Pontryagin variational principle in Section 3.3 explain the nature of this differential notation (see Remarks 3.4 and 3.9) which we will use freely. It follows that $\mathbf{a} = (a, -\pounds_\xi a) \in \boldsymbol{\mathcal{D}}_{Z,T}(\mathfrak{A})$ and $(\pounds_\xi a, (\pounds_\xi a)') \in \boldsymbol{\mathcal{D}}_{Z,T}(\mathfrak{A})$, where $(\pounds_\xi a)' = -(\pounds_{\xi_k} \pounds_{\xi_l} a)_{1 \le k,l \le K}$. For more information about rough partial differential equations (RPDEs) and their solutions, we refer the reader to [59,10,47,74,75]. We mention that to prove well-posedness and, in particular, to show that 'a' is controlled, one must obtain *a priori* estimates of a remainder term which contains third-order Lie derivatives, see equation (A.11).

Remark 2.8. Incorporating additional constraints into the action functional is straightforward. For example, it is possible to enforce incompressibility via Lagrange multipliers instead of through constraints on spaces as discussed at the beginning of this section and in Section A.2.3. Naturally, additional terms appear on the right-hand-side of the equation for momentum, m, corresponding to the pressure terms (rough and smooth in time). We will explain in the examples in Section 4 how one can impose the incompressibility constraint, either by using Lagrangian multipliers, or by constraining the space of vector fields and its dual.

The most commonly solved Euler equation for incompressible homogeneous flow of an ideal fluid with transport-type noise is, in addition, harmonic-free [43,28,27,42]. Indeed, in most papers, the authors prove well-posedness of a transport vorticity equation on the torus $\mathbb{T}^d$, $d = 2, 3$ with u recovered via the Biot-Savart law. By the Hodge decomposition theorem (see Section A.2.3), if the underlying equation for the fluid velocity u does not preserve mean-freeness (i.e., harmonic-freeness), then u cannot be recovered directly from the vorticity equation by the Biot-Savart law. As a result of the perturbative nature of our theory, our equations do not, in general, preserve harmonic-freeness at the level of velocity. By imposing constraints on spaces (i.e., projections), we can easily impose that u is both divergence and harmonic-free and derive the corresponding momentum equation with enough 'free-variables' to impose the divergence-free and harmonic-free constraints. In particular, we shall explain how the pressure and constant harmonic terms naturally decompose into a smooth and rough part, and how they can be recovered from u as was done in, for example, [94,95] and [74,75]).

3. The Lie chain rule for geometric rough paths and its applications

For an incompressible ideal fluid evolving on a compact oriented Riemannian manifold (M,g) with associated volume-form μ_g, the Lagrangian flow map $\eta : [0,T] \to \mathrm{Diff}_{\mu_g}$ may be regarded as a curve in the group $G := \mathrm{Diff}_{\mu_g}$ of volume-preserving diffeomorphisms on M endowed with some appropriate topology, initiated from the identity $\eta_0 = \mathrm{id}$ and parametrized by time, $t \in [0,T]$. In his seminal paper [6], V.I. Arnold showed that the configuration space for incompressible hydrodynamics is the space of volume-preserving diffeomorphisms and that Euler's equation for the Eulerian velocity field u (i.e., $\dot{\eta}_t = u_t \circ \eta_t$) is equivalent to the path η being a critical point of the kinetic energy action functional. That is, solutions of Euler's fluid equations are geodesic paths on the manifold of volume-preserving diffeomorphisms endowed with a right-invariant weak L^2-metric.

However, several geometric-analytic challenges arise if one wishes to make this viewpoint constructive and solve the geodesic equation as an ODE (and show there is no derivative loss). The crux of the matter is that composition from the right is not smooth if one wants to endow G with a Banach topology and work with a standard functional analytic tool-set [53]. The variational principles developed in this paper can be seen as extensions of the geodesic principle in [6] or, more generally, the overarching EPDiff theory [77].

3.1. Lie chain rule and Reynolds transport theorem

Theorem B.1 can be extended via a coordinate chart or approximate flow argument (see, e.g., [118,50,9]) to obtain the following theorem concerning smooth rough flows on the closed manifold M. We assume smoothness in the spatial variable and compactness of our manifolds for simplicity. More relaxed conditions can be found in, for example, in [118].

Theorem 3.1 *(Rough flow properties). There exists a unique continuous map*

$$\mathrm{Flow} : C_T^{\alpha}(\mathfrak{X}_{C^\infty}) \times C_T^{\infty}(\mathfrak{X}^K_{C^\infty}) \times \boldsymbol{\mathcal{C}}_{g,T}(\mathbb{R}^K) \to C^{\alpha}_{2,T}(\mathrm{Diff}_{C^\infty})$$

such that $\eta_{ts} = \mathrm{Flow}(u,\xi,\mathbf{Z})_{ts}$, $(s,t) \in [0,T]^2$, *satisfies the following properties:*

(i) for all $(s,\theta,t) \in [0,T]^3$, $\eta_{tt} = \mathrm{Id}$ *and*

$$\eta_{t\theta} \circ \eta_{\theta s} = \eta_{ts};$$

(ii) for all $(s,t) \in \Delta$ *and* $f \in C^\infty$,

$$\eta_{ts}^* f = f + \int_0^t \eta_{rs}^* u_r[f]\mathrm{d}r + \int_0^t \eta_{rs}^* \xi_r[f]\mathrm{d}\mathbf{Z}_r, \tag{3.1}$$

and

$$\eta_{ts*} f = f - \int_0^t u_r[\eta_{rs*} f]\mathrm{d}r - \int_0^t \xi_r[\eta_{rs*} f]\mathrm{d}\mathbf{Z}_r. \tag{3.2}$$

Remark 3.2. Let us recall that η_{ts}^* and η_{ts*} denote the pull-back and push-forward, respectively (see (A.10)). Item (ii) in (3.1) means that for all $X \in M$, the quantity $\eta_{\cdot s}X$ is the unique solution of the RDE

$$\mathrm{d}\eta_{ts}X = u_t(\eta_{ts}X)\mathrm{d}t + \xi_t(\eta_{ts}X)\mathrm{d}\mathbf{Z}_t, \;\; t \in (s,T], \;\; \eta_{ss}X = X, \tag{3.3}$$

for all $s \in [0,T]$.

We refer to the following theorem as the *Lie chain rule*. A stochastic version (i.e., Brownian case) of this theorem was proved in [46][Theorem 3.1].

Theorem 3.3 *(Rough Lie chain rule). For given* $\tau_0 \in \mathcal{T}^{lk}_{C^\infty}$, $\pi \in C_T(\mathcal{T}^{lk}_{C^\infty})$, *and* $\boldsymbol{\gamma} = (\gamma,\gamma') \in \boldsymbol{\mathcal{D}}_{Z,T}((\mathcal{T}^{lk}_{C^\infty})^K)$, *let*

$$\tau_t = \tau_0 + \int_0^t \pi_r \mathrm{d}r + \int_0^t \gamma_r \mathrm{d}\mathbf{Z}_r, \;\; t \in [0,T].$$

Then for all $(s,t) \in \Delta_T$,

$$\eta_{ts}^* \tau_t = \tau_s + \int_s^t \eta_{rs}^* \left(\pi_r + \pounds_{u_r}\tau_r\right)\mathrm{d}r + \int_s^t \eta_{rs}^* \left(\gamma_r + \pounds_{\xi_r}\tau_r\right)\mathrm{d}\mathbf{Z}_r, \tag{3.4}$$

and

$$\eta_{ts*}\tau_t = \tau_s + \int_s^t \left(\eta_{rs*}\pi_r - \pounds_{u_r}(\eta_{rs*}\tau_r)\right)\mathrm{d}r + \int_s^t \left(\eta_{rs*}\gamma_r - \pounds_{\xi_r}(\eta_{rs*}\tau_r)\right)\mathrm{d}\mathbf{Z}_r, \tag{3.5}$$

where the time-dependent vector fields u *and* ξ *are given in equation* (3.3).

Proof. See Section 5.2. □

Remark 3.4. By (3.5), for an arbitrary $\tau_0 \in \mathcal{T}^{r,s}_{C^\infty}$, it follows that $\tau_\cdot = \eta_{\cdot 0\,*}\tau_0$ is a classical solution of

$$\tau_t + \int_0^t \pounds_{u_r}\tau_r \mathrm{d}r + \int_0^t \pounds_{\xi_r}\tau_r \mathrm{d}\mathbf{Z}_r = \tau_0.$$

Notice that if we introduce the notation $\mathrm{d}x_t := \mathrm{d}\eta_{t0} \circ \eta_{t0}^{-1} := u_t \mathrm{d}t + \xi_t \mathrm{d}\mathbf{Z}_t$, then we may write

$$\tau_t + \int_0^t \pounds_{\mathrm{d}x_r}\tau_r = \tau_0,$$

which generalizes the dynamic definition of the Lie-derivative to the rough case.

The following corollary is an extension of the Reynolds transport theorem. It is a sequential application of the definition of the integral on manifolds (see, e.g., Sec. 8.1 and 8.2 of [1]), the global change of variables formula, the Lie chain rule (Theorem 3.3) and the rough Fubini theorem (Lemma B.3. We will use this formula next in the case $k = 1$ for the proof of the Kelvin circulation theorem (see Section 3.2).

Corollary 3.5 *(Rough Reynolds transport theorem). For given $\alpha_0 \in \Omega^k_{C^\infty}$, $\pi \in C_T(\Omega^k_{C^\infty})$, and $\boldsymbol{\gamma} = (\gamma, \gamma') \in \mathcal{D}_{Z,T}((\Omega^k_{C^\infty})^K)$, let*

$$\alpha_t = \alpha_0 + \int_0^t \pi_r \mathrm{d}r + \int_0^t \gamma_r \mathrm{d}\mathbf{Z}_r, \quad t \in [0, T].$$

Then for all k-dimensional smooth submanifolds Γ embedded in M and $(s,t) \in \Delta_T$, we have

$$\int_{\eta_{ts}(\Gamma)} \alpha_t = \int_\Gamma \alpha_s + \int_s^t \int_{\eta_{rs}(\Gamma)} (\pi_r + \pounds_{u_r}\tau_r)\, \mathrm{d}r + \int_s^t \int_{\eta_{rs}(\Gamma)} (\gamma_r + \pounds_{\xi_r}\tau_r)\, \mathrm{d}\mathbf{Z}_r,$$

where $\eta_{ts}(\Gamma)$ denotes the image of Γ under the action of the flow η.

3.2. Kelvin's circulation theorem

Assume that for all $t \in [0, T]$,

$$m_t \overset{\mathfrak{X}^\vee_{C^\infty}}{=} m_0 + \int_0^t \left(\frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta}a}(u_s, a_s) \diamond a_s - \pounds_{u_s} m_s\right) \mathrm{d}s - \int_0^t \pounds_\xi m_s \mathrm{d}\mathbf{Z}_s, \quad m = \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta}u}(u, a),$$

$$D_t + \int_0^t \pounds_{u_s} D_s \mathrm{d}r + \int_0^t \pounds_\xi D_s \mathrm{d}\mathbf{Z}_s \overset{\mathrm{Dens}_{C^\infty}}{=} D_0,$$

where all the paths and integrands are assumed to be smooth. By virtue of Theorem 3.1, there exists a flow of diffeomorphisms $\eta = \eta_{\cdot 0} \in C^\alpha_T(\mathrm{Diff}_{C^\infty})$ such that

$$\mathrm{d}\eta_t X = u_t(\eta_t X)\mathrm{d}t + \xi(\eta_t X)\mathrm{d}\mathbf{Z}_t, \;\; t \in (0,T], \quad \eta_0 X = X \in M.$$

We obtain the following rough version of the Kelvin-Noether theorem in [77] as an application of the Reynolds transport theorem in Corollary 3.5,

Theorem 3.6 *(Rough Kelvin-Noether Theorem). Let γ denote a compact embedded one-dimensional smooth submanifold of M and denote $\gamma_t = \eta_t(\gamma)$ for all $t \in [0,T]$. If D_0 is non-vanishing, then*

$$\oint_{\gamma_t} \frac{1}{D_t}\frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta}u}(u_t, a_t) = \oint_{\gamma_0} \frac{1}{D_0}\frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta}u}(u_0, a_0) + \int_0^t \oint_{\gamma_s} \frac{1}{D_s}\frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta}a}(u_s, a_s) \diamond a_s \mathrm{d}s.$$

Remark 3.7. Formula (A.15) explains that $\frac{1}{\mu} : \mathfrak{X}^\vee_{C^\infty} \to \Omega^1_{C^\infty}$ is defined by $m = \alpha \otimes \nu \mapsto \frac{m}{\mu} = \alpha \frac{d\nu}{d\mu}$.

Proof. See Section 5.3. □

3.3. The Hamilton-Pontryagin variational principle for geometric rough paths

In this section, in addition to the assumptions in Section 2, we require that $\mathfrak{A} = \mathfrak{A}_{C^\infty}$, $\mathfrak{X} = \mathfrak{X}_{C^\infty}$, $\xi \in \{\mathfrak{X}_{C^\infty}\}^K$, and $\mathbf{Z} \in \mathscr{C}^\alpha_{g,T}(\mathbb{R}^K)$, $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$, is *truly rough* as in Definition A.14. We define the space of rough diffeomorphisms by

$$\mathrm{Diff}_{\mathbf{Z},T,C^\infty} = \mathrm{Flow}(C^\alpha_T(\mathfrak{X}_{C^\infty}), C^\infty_T(\mathfrak{X}^K_{C^\infty}), \mathbf{Z})_{\cdot 0}.$$

For given $\eta = \mathrm{Flow}(v, \sigma, \mathbf{Z})_{\cdot 0} \in \mathrm{Diff}_{\mathbf{Z},T,C^\infty}$ and $\lambda \in \mathscr{D}_{Z,T}(\mathfrak{X}^\vee)$, we let

$$\int_0^T \langle \lambda_t, \mathrm{d}\eta_t \circ \eta_t^{-1} \rangle_{\mathfrak{X}} := \int_0^T \langle \lambda_t, v_t \rangle_{\mathfrak{X}} \mathrm{d}t + \int_0^T \langle \lambda_t, \sigma_t \rangle_{\mathfrak{X}} \mathrm{d}\mathbf{Z}_t. \tag{3.6}$$

Definition 3.8. Let $HP_{\mathbf{Z}}$ denote the space of

$$(u, \eta, \boldsymbol{\lambda}) \in C^\alpha_T(\mathfrak{X}_{C^\infty}) \times \mathrm{Diff}_{\mathbf{Z},T,C^\infty} \times \mathscr{D}_{Z,T}(\mathfrak{X}^\vee)$$

such that for all $\phi \in \mathfrak{X}_{C^\infty}$, $\pounds_u\lambda, \pounds_u\phi, \pounds_\phi\lambda \in C^\alpha_T(\mathfrak{A}^\vee)$, and there exists $(\pounds_\xi\lambda)'$ such that $(\pounds_\xi\lambda, (\pounds_\xi\lambda)') \in \mathscr{D}_{Z,T}(\mathfrak{X}^\vee)$.

For a given $a_0 \in \mathfrak{A}_{C^\infty}$, the Hamilton-Pontryagin action integral $S^{HP_\mathbf{Z}}_{a_0} : HP_\mathbf{Z} \to \mathbb{R}$ is defined by

$$S^{HP_\mathbf{Z}}_{a_0}(u, \eta, \boldsymbol{\lambda}) = \int_0^T \ell(u_t, \eta_{t*}a_0)\mathrm{d}t + \langle \lambda_t, \mathrm{d}\eta_t \circ \eta_t^{-1} - u_t \mathrm{d}t - \xi \mathrm{d}\mathbf{Z}_t \rangle_{\mathfrak{X}}. \tag{3.7}$$

Remark 3.9. By Theorem A.15, the Lagrange multiplier λ in (3.7) enforces

$$d\eta_t X = u_t(\eta_t X)\mathrm{d}t + \xi(\eta_t X)\mathrm{d}\mathbf{Z}_t, \;\; t \in (0, T], \quad \eta_0 X = X \in M.$$

The *true roughness* of the path $\mathbf{Z}$ defined in Definition A.14 and satisfying Theorem A.15 is required to ensure that (3.6) is well-specified and to conclude that $v \equiv u$ and $\sigma \equiv \xi$ in the proof of Theorem 3.12 (i.e., after taking variations). In contrast, we did not impose true roughness (see Remark 2.7) of the path for the Clebsch variational principle in Theorem 2.6 owing to the nature of the constraint and Lemma B.4.

By the Lie chain rule (Theorem 3.3), we find that $a_t = \eta_{t*}a_0$ satisfies

$$a_t + \int_0^t \pounds_{\mathrm{d}\mathbf{x}_s} a_s = a_0, \qquad \text{where } \mathrm{d}\mathbf{x}_t = u_t \mathrm{d}t + \xi \mathrm{d}\mathbf{Z}_t,$$

where the notation for $\mathrm{d}x_t$ is explained in Remark 3.4. That is, the quantity a is advected by the flow $\eta \in \mathrm{Diff}_{\mathbf{Z},T,C^\infty}$. This advection equation is used directly as the constraint in the Clebsch variational principal in Theorem 2.6.

Definition 3.10. A variation of $(u, \eta, \boldsymbol{\lambda}) \in HP_Z$ is a curve $\{(u^\epsilon, \eta^\epsilon, \boldsymbol{\lambda}^\epsilon)\}_{\epsilon \in (-1,1)} \subset HP_Z$ of the form

$$(u^\epsilon, \eta^\epsilon, \boldsymbol{\lambda}^\epsilon) = (u + \epsilon \boldsymbol{\delta} u, \psi^\epsilon \circ \eta, \boldsymbol{\lambda} + \epsilon \boldsymbol{\delta} \lambda),$$

where $\psi \in C^\infty([-1, 1] \times [0, T]; \mathrm{Diff}_{C^\infty})$ is defined to be the flow (in the t-variable) given by

$$\partial_t \psi^\epsilon_t X = \epsilon \partial_t \boldsymbol{\delta} w_t(\psi^\epsilon_t X), \quad \psi^\epsilon_0 X = X \in M,$$

for arbitrarily chosen $(\boldsymbol{\delta} u, \boldsymbol{\delta} w, \boldsymbol{\delta} \lambda) \in C^\infty_T(\mathfrak{X}_{C^\infty} \times \mathfrak{A}_{C^\infty} \times \mathfrak{A}^\vee_{C^\infty})$ such that $\boldsymbol{\delta} w$ vanishes at $t = 0$ and $t = T$.

Remark 3.11 *(Variation η^ϵ).* The type of variation we use for the rough diffeomorphism is common in the geometric mechanics community (see, e.g., Lemma 3.1 of [4]). Notice that for all $t \in [0, T]$ and $f \in C^\infty$,

$$\psi_t^{\epsilon *} f = f + \epsilon \int_0^t \psi_r^{\epsilon *}[\partial_t \boldsymbol{\delta} w_r f] \mathrm{d}r.$$

Applying Theorem 3.3 and using the natural property of the Lie derivative leads to

$$\eta_t^{\epsilon *} f = f + \int_0^t \eta_r^{\epsilon *} \left(\pounds_{v_r^\epsilon} f + \epsilon \pounds_{\partial_t \boldsymbol{\delta} w_r} f \right) \mathrm{d}r + \int_0^t \eta_r^{\epsilon *} \pounds_{\sigma_r^\epsilon} f \mathrm{d}\mathbf{Z}_r,$$

where $v_t^\epsilon = \psi_{t*}^\epsilon v$ and $\sigma_t^\epsilon = \psi_{t*}^\epsilon \sigma$. Thus, for a given $\eta = \mathrm{Flow}(v, \sigma, \mathbf{Z})_{\cdot 0}$, it follows that

$$d\eta_t^\epsilon X = \left(v_t^\epsilon(\eta_t^\epsilon X) + \epsilon \partial_t \boldsymbol{\delta} w_r(\eta_t^\epsilon X) \right) \mathrm{d}t + \sigma_t^\epsilon(\eta_t^\epsilon X) \mathrm{d}\mathbf{Z}_t, \quad \eta_0^\epsilon X = X \in M,$$

and hence

$$\eta^\epsilon = \mathrm{Flow}\left(v^\epsilon + \epsilon \partial_t \boldsymbol{\delta} w, \sigma^\epsilon, \mathbf{Z} \right)_{\cdot 0} \in \mathrm{Diff}_{\mathbf{Z}, T, C^\infty}.$$

The proof of the following theorem is given in Section 5.4.

Theorem 3.12 *(Hamilton-Pontryagin variational principle). A curve* $(u, \eta, \boldsymbol{\lambda}) \in HP_{\mathbf{Z}}$ *is a critical point of* $S^{HP_{\mathbf{Z}}}$ *if and only if for all* $[0, T]$,

$$m_t + \int_0^t \pounds_{u_s} m_s \mathrm{d}s + \int_0^t \pounds_\xi m_s \mathrm{d}\mathbf{Z}_s \overset{\mathfrak{X}^\vee}{=} m_0 + \int_0^t \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta}a}(u_s, a_s) \diamond a_s \mathrm{d}s, \quad m = \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta}u}(u, a) = \lambda,$$

$$a_t + \int_0^t \pounds_{u_s} a_s \mathrm{d}s + \int_0^t \pounds_\xi a_s \mathrm{d}\mathbf{Z}_s \overset{\mathfrak{A}_{C^\infty}}{=} a_0, \quad a_t = \eta_{t*} a_0,$$

$$\mathrm{d}\eta_t X = u_t(\eta_t X) \mathrm{d}t + \xi(\eta_t X) \mathrm{d}\mathbf{Z}_t, \;\; t \in (0, T], \;\; \eta_0 X = X \in M.$$

Remark 3.13. The corresponding Hamilton-Pontryagin principle was derived for SALT in [63].

Proof. See Section 5.4. □

Remark 3.14 *(Incompressible homogeneous Euler).* The rough incompressible homogeneous (unit density) Euler equations arise from the choice of the 'kinetic energy' Lagrangian $\ell : \dot{\mathfrak{X}}_{\mu_g} \to \mathbb{R}_+$ defined by

$$\ell(u) = \int_M g(u, u) \mu_g,$$

where (M, g) is an oriented Riemannian manifold with corresponding volume form μ_g. We refer to Sections 4.2 and 4.2 for more details. Letting $\dot{\mathfrak{X}}^{\vee}_{\mu_g}$ denote the space of one-form densities modulo exact and harmonic forms (see Definition A.20), we find

$$m = \lambda = \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta}u} = [u^\flat \otimes \mu_g] \in \dot{\mathfrak{X}}^{\vee}_{\mu_g},$$

and hence that $(u, \eta, \boldsymbol{\lambda})$ is a critical point of $HP_{\mathbf{Z}}$ iff

$$\mathrm{d}[u_t^\flat \otimes \mu_g] + \pounds_{u_t}[u_t^\flat \otimes \mu_g]\mathrm{d}t + \pounds_\xi[u_t^\flat \otimes \mu_g]\mathrm{d}\mathbf{Z}_t \overset{\dot{\mathfrak{X}}^{\vee}_{\mu_g}}{=} \frac{1}{2}[\mathbf{d}g(u_t, u_t) \otimes \mu_g]\mathrm{d}t,$$
$$\mathrm{d}\eta_t X = u_t(\eta_t X)\mathrm{d}t + \xi(\eta_t X)\mathrm{d}\mathbf{Z}_t, \quad t \in (0, T], \quad \eta_0 X = X \in M.$$

The first equation is equivalent to

$$\begin{cases} \mathrm{d}u_t^\flat + \pounds_{u_t}u_t^\flat \mathrm{d}t + \pounds_\xi u_t^\flat \mathrm{d}\mathbf{Z}_t = \frac{1}{2}\mathbf{d}g(u_t, u_t) - \mathbf{d}\mathrm{d}\mathbf{p}_t - \mathrm{d}\mathbf{c}_t, \\ \mathbf{d}^* u^\flat = \operatorname{div} u = 0, \\ H(u^\flat) = 0, \end{cases}$$

where $\mathrm{d}\mathbf{p}_t = p_t\mathrm{d}t + q_t\mathrm{d}\mathbf{Z}_t$ and $\mathrm{d}\mathbf{c}_t = c_t\mathrm{d}t + \tilde{c}_t\mathrm{d}\mathbf{Z}_t$ are the Lagrangian multipliers corresponding to the divergence and harmonic-free constraints. It follows that $\omega = \mathbf{d}u^\flat$ satisfies

$$\mathrm{d}\omega_t + \pounds_{u_t}\omega_t\mathrm{d}t + \pounds_\xi\omega_t\mathrm{d}\mathbf{Z}_t = 0.$$

In [44], we extend the work of [75], which studied the viscous case on the torus $M = \mathbb{T}^d$, to show that for an initial-velocity $u_0 \in \mathfrak{X}_{W_2^m}$ with $m > \frac{d}{2} + 1$, there exists a unique maximal Cauchy development $u \in C([0, T^*); \mathfrak{X}_{W_2^m}) \cap C^\alpha([0, T_{\max}; \mathfrak{X}_{W_2^{m-3}})$. Moreover, we show that if $T_{\max} < \infty$, then

$$\int_0^{T^*} |\omega_t|_{\Omega^2_{L^\infty}} \mu_g = +\infty.$$

That is, a Beale-Kato-Majda blowup criterion holds. In dimension two, identifying ω with a scalar $\tilde{\omega} = \star\omega \in \Omega^0$, we find that $|\tilde{\omega}_t|_{L^p} = |\tilde{\omega}_0|_{L^p}$ for all p so that $T^* = +\infty$.

Therefore, taking the initial data $u_0 \in \mathfrak{X}_{C^\infty}$ to be smooth, we obtain a solution $u \in C_T^\alpha(\mathfrak{X}_{C^\infty})$ on any interval $[0, T]$ with $T < T_{\max}$, and hence we may construct the flow $\eta = \mathrm{Flow}(u, \xi, \mathbf{Z}) \in \mathrm{Diff}_{\mathbf{Z},T,C^\infty}$. Consequently, we obtain a critical point $(u, \eta, \boldsymbol{\lambda})$ of $HP_{\mathbf{Z}}$ for any $T < T_{\max}$ with $\boldsymbol{\lambda} = (\lambda, \lambda') = ([u^\flat \otimes \mu_g], \pounds_\xi[u^\flat \otimes \mu_g])$.

3.4. An Euler–Poincaré variational principle for geometric rough paths

In this section, we assume that all stated quantities exist and are smooth; so, we can work formally (see Remark 3.17).

Theorem 3.15 *(Euler–Poincaré variational principle). Consider a path* $\eta = \mathrm{Flow}(u, \xi, \mathbf{Z}) \in \mathrm{Diff}_{\mathbf{Z},T,C^\infty}$*. The following are equivalent:*

(i) The constrained variational principle

$$\delta \int_0^T \ell(u_t, a_t) dt = 0$$

holds on $C_T^\alpha(\mathfrak{X}_{C^\infty}) \times C_T^\alpha(\mathfrak{A}_{C^\infty})$ *using variations of the form*

$$\boldsymbol{\delta} u \mathrm{d}t = \partial_t \boldsymbol{\delta} w \mathrm{d}t - \mathrm{ad}_{\mathrm{d}\mathbf{x}_t} \boldsymbol{\delta w} \qquad \text{and} \qquad \boldsymbol{\delta} a = - \pounds_{\boldsymbol{\delta w}} a, \tag{3.8}$$

for arbitrarily chosen $\boldsymbol{\delta} w \in C_T^\infty(\mathfrak{X}_{C^\infty})$ *which vanishes at* $t = 0$ *and* $t = T$*, where* $\mathrm{d}\mathbf{x}_t = u_t \mathrm{d}t + \xi \mathrm{d}\mathbf{Z}_t$*.*

(ii) The ***Euler–Poincaré*** *equations on geometric rough paths hold: that is, for all* $t \in [0, T]$*,*

$$m_t + \int_0^t \pounds_{u_s} m_s \mathrm{d}s + \int_0^t \pounds_\xi m_s \mathrm{d}\mathbf{Z}_s \overset{\mathfrak{X}^\vee_{C^\infty}}{=} m_0 + \int_0^t \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta} a}(u_s, a_s) \diamond a_s \mathrm{d}s, \quad m = \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta} u}(u, a),$$

$$a_t + \int_0^t \pounds_{u_s} a_s \mathrm{d}s + \int_0^t \pounds_\xi a_s \mathrm{d}\mathbf{Z}_s \overset{\mathfrak{A}_{C^\infty}}{=} a_0, \quad a_t = \eta_{t*} a_0.$$

Proof. See Section 5.5. □

Remark 3.16. Recalling that for all $u \in \mathfrak{X}^\infty$ the adjoint of $\mathrm{ad}_u = -\pounds_u : \mathfrak{X}_{C^\infty} \to \mathfrak{X}_{C^\infty}$ is $\mathrm{ad}_u^* = \pounds_u : \mathfrak{X}^\vee_{C^\infty} \to \mathfrak{X}^\vee_{C^\infty}$, we have

$$\begin{aligned} \mathrm{d}m_t + \mathrm{ad}^*_{\mathrm{d}\mathbf{x}_t} m_t &= \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta} a_t} \diamond a_t \\ \mathrm{d}a_t + \pounds_{\mathrm{d}\mathbf{x}_t} a_t &= 0. \end{aligned} \tag{3.9}$$

Remark 3.17. This is not strictly a variational principle in the same sense as the standard Hamilton's principle. It is akin to the classic Lagrange d'Alembert principle for dynamics with nonholonomic constraints, because the variations of $\boldsymbol{\delta} u$ and $\boldsymbol{\delta} a$ in (3.8) are restricted in terms of $\boldsymbol{\delta} w$. These restrictions are discussed next.

Let $\eta = \mathrm{Flow}(u, \xi, \mathbf{Z})_{\cdot,0}$. Assume that for all $\boldsymbol{\delta} w \in \mathfrak{X}_{C^\infty}$ and $\boldsymbol{\delta} w_0 = \boldsymbol{\delta} w_T = 0$ we can construct a variation $\{\eta^\epsilon\}_{\epsilon \in [-1,1]}$ such that

$$\mathrm{d}\eta^\epsilon_t X = u^\epsilon_t(\eta^\epsilon_t X)\mathrm{d}t + \xi(\eta^\epsilon_t X)\mathrm{d}\mathbf{Z}_t, \quad t \in (0, T], \quad \eta^\epsilon_0 X = X \in M,$$

and for all $t \in [0, T]$,

(i)

$$\frac{\partial}{\partial\epsilon}\mathrm{d}\eta_t^\epsilon\big|_{\epsilon=0} = \mathrm{d}\frac{\partial}{\partial\epsilon}\eta_t^\epsilon\big|_{\epsilon=0};$$

(ii)

$$\boldsymbol{\delta}w_t = \left(\frac{\partial}{\partial\epsilon}\big|_{\epsilon=0}\eta_t^\epsilon\right)\circ\eta_t^{-1} \quad\Leftrightarrow\quad \frac{\partial}{\partial\epsilon}\big|_{\epsilon=0}\eta_t^\epsilon X = \boldsymbol{\delta}w_t(\eta_t X).$$

Define $\boldsymbol{\delta}u_t := \frac{\partial}{\partial\epsilon}\big|_{\epsilon=0}u_t^\epsilon$. Then

$$\mathrm{d}\frac{\partial}{\partial\epsilon}\eta_t^\epsilon\big|_{\epsilon=0} = \mathrm{d}(\boldsymbol{\delta}w_t\circ\eta_t) = (\partial_t\boldsymbol{\delta}w_t)\circ\eta_t\mathrm{d}t + (T\boldsymbol{\delta}w_t\circ\eta_t)(u_t\circ\eta_t\mathrm{d}t + \xi\circ\eta_t\mathrm{d}\mathbf{Z}_t)$$

and

$$\begin{aligned}\frac{\partial}{\partial\epsilon}\mathrm{d}\eta_t^\epsilon\big|_{\epsilon=0} &= \frac{\partial}{\partial\epsilon}(u^\epsilon\circ\eta_t^\epsilon\mathrm{d}t + \xi\circ\eta_t^\epsilon\mathrm{d}\mathbf{Z}_t)\big|_{\epsilon=0}\\ &= (\boldsymbol{\delta}u_t\circ\eta_t + (Tu_t\circ\eta_t)(\boldsymbol{\delta}w_t\circ\eta_t))\,\mathrm{d}t + (T\xi\circ\eta_t)(\boldsymbol{\delta}w_t\circ\eta_t)\mathrm{d}\mathbf{Z}_t\end{aligned}$$

Using the equality of mixed derivatives, we find

$$\begin{aligned}\boldsymbol{\delta}u_t\circ\eta_t\mathrm{d}t &= ((\partial_t\boldsymbol{\delta}w_t)\circ\eta_t + [\boldsymbol{\delta}w_t, u_t]\circ\eta_t)\,\mathrm{d}t + [\boldsymbol{\delta}w_t,\xi]\circ\eta_t\mathbf{Z}_t\\ &= ((\partial_t\boldsymbol{\delta}w_t)\circ\eta_t - \mathrm{ad}_{u_t}\boldsymbol{\delta}w_t\circ\eta_t)\,\mathrm{d}t - \mathrm{ad}_\xi\boldsymbol{\delta}w_t\circ\eta_t\mathbf{Z}_t.\end{aligned}$$

It follows from $a_t^\epsilon = \eta_{t*}^\epsilon a_0$ that $\boldsymbol{\delta}a_t = \frac{\partial}{\partial\epsilon}\big|_{\epsilon=0}\eta_{t*}^\epsilon a_0 = -\pounds_{\boldsymbol{\delta}w}a_t$. Two issues now arise: i) we do not have a proof that such variations exist as we did for the Hamilton-Pontryagin variational principle; ii) it is not clear how to deduce

$$\boldsymbol{\delta}u_t\mathrm{d}t = (\partial_t\boldsymbol{\delta}w_t - \mathrm{ad}_{u_t}\boldsymbol{\delta}w_t)\,\mathrm{d}t - \mathrm{ad}_\xi\boldsymbol{\delta}w_t\mathrm{d}\mathbf{Z}_t.$$

We shall leave the clarification of these issues about the Euler-Poincaré variations as an open problem.

3.5. A Lie–Poisson bracket for Hamiltonian dynamics on geometric rough paths

Definition 3.18. We define $h : \boldsymbol{\mathcal{D}}_{Z,T}(\mathfrak{X}^\vee)\oplus\boldsymbol{\mathcal{D}}_{Z,T}(\mathfrak{A})\to\boldsymbol{\mathcal{D}}_{Z,T}(\mathbb{R})$ by

$$h_t(m,a) := \int_0^t (\langle m_s, u_s\rangle_{\mathfrak{X}} - \ell(u_s,a_s))\,\mathrm{d}s + \int_0^t \langle m_s,\xi\rangle_{\mathfrak{X}}\mathrm{d}\mathbf{Z}_s,$$
$$(m,a)\in\boldsymbol{\mathcal{D}}_{Z,T}(\mathfrak{X}^\vee)\oplus\boldsymbol{\mathcal{D}}_{Z,T}(\mathfrak{A}),\quad t\in[0,T],$$

where u denotes the inverse of $\frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta}u}(\cdot,a):\mathfrak{X}\to\mathfrak{X}^\vee$ applied to m; that is, $m = \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta}u}(u,a)$.

The rough Hamiltonian $h_t(m,a)$ is the sum of the deterministic Hamiltonian defined by Legendre transformation associated with the Lagrangian ℓ and given by $H(m,a) = \langle m, u\rangle_{\mathfrak{X}} - \ell(u,a)$, plus $G(m) = \langle m, \xi_k\rangle_{\mathfrak{X}}$, so that

$$h_t(m,a) = \int_0^t H(m_s,a_s)\mathrm{d}s + \int_0^t G(m_s)\mathrm{d}\mathbf{Z}_s.$$

Let us take variations of $h(m,a)$ in m and a. For arbitrary $\boldsymbol{\delta} m \in \mathfrak{X}^\vee_{C^\infty}$ and $\boldsymbol{\delta} a \in \mathfrak{A}_{C^\infty}$, we find

$$\begin{aligned}\boldsymbol{\delta} h_t &= \int_0^t \left(\langle m_s, \frac{\boldsymbol{\delta} u}{\boldsymbol{\delta} m}\rangle_{\mathfrak{X}} + \langle \boldsymbol{\delta} m_s, u_s\rangle_{\mathfrak{X}} - \langle \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta} u}(u_s,a_s), \frac{\boldsymbol{\delta} u}{\boldsymbol{\delta} m}\rangle_{\mathfrak{X}} - \langle \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta} a}(u_s,a_s), \boldsymbol{\delta} a\rangle_{\mathfrak{A}}\right)\mathrm{d}s \\ &\quad + \int_0^t \langle \boldsymbol{\delta} m_s, \xi\rangle_{\mathfrak{X}}\mathrm{d}\mathbf{Z}_s \\ &= \int_0^t \langle \boldsymbol{\delta} m_s, \mathrm{d}x_s\rangle_{\mathfrak{X}} - \int_0^t \langle \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta} a}(u_s,a_s), \boldsymbol{\delta} a\rangle_{\mathfrak{A}}\mathrm{d}s, \quad \forall t \in [0,T],\end{aligned}$$

where in the second equality we have used $m := \frac{\boldsymbol{\delta}\ell}{\delta u}$ and set $\mathrm{d}x_t := u_t\mathrm{d}t + \xi\mathrm{d}\mathbf{Z}_t$. Thus,

$$\mathrm{d}\frac{\boldsymbol{\delta} h_t}{\boldsymbol{\delta} m}(m,a) = \mathrm{d}x_t \quad \text{and} \quad \mathrm{d}\frac{\boldsymbol{\delta} h}{\boldsymbol{\delta} a}(m,a) = -\frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta} a}(u,a)\mathrm{d}t,$$

which is to say

$$\begin{gathered}\frac{\boldsymbol{\delta} h_t}{\boldsymbol{\delta} m}(m,a) = \int_0^t u_s\mathrm{d}s + \int_0^t \xi\mathrm{d}\mathbf{Z}_s = \int_0^t \frac{\boldsymbol{\delta} H}{\boldsymbol{\delta} m}(m_s,a_s)\mathrm{d}s + \int_0^t \frac{\boldsymbol{\delta} G}{\boldsymbol{\delta} m}(m_s)\mathrm{d}s \\ \frac{\boldsymbol{\delta} h_t}{\boldsymbol{\delta} a}(m,a) = -\frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta} a}(u_t,a_t) = \frac{\boldsymbol{\delta} H}{\boldsymbol{\delta} a}(m_t,a_t).\end{gathered}$$

Corollary 3.19 *(Lie–Poisson Hamiltonian form). The Euler–Poincaré equations in* (3.9) *can be written in Lie–Poisson bracket form as*

$$\begin{bmatrix} m_t \\ a_t \end{bmatrix} \overset{\mathfrak{X}^\vee \oplus \mathfrak{A}}{=} \begin{bmatrix} m_0 \\ a_0 \end{bmatrix} - \int_0^t \begin{bmatrix} \mathrm{ad}^*_\square m_s & \square \diamond a_s \\ \pounds_\square a_s & 0 \end{bmatrix} \begin{bmatrix} \mathrm{d}\boldsymbol{\delta} h_s/\boldsymbol{\delta} m(m,a) \\ \mathrm{d}\boldsymbol{\delta} h_s/\boldsymbol{\delta} a(m,a) \end{bmatrix},$$

in which the boxes ($\square$) represent substitution in the operator. For arbitrary $f : \mathfrak{X}^\vee \times \mathfrak{A} \to \mathbb{R}$ such that $\frac{\boldsymbol{\delta} f}{\boldsymbol{\delta} m}(m,a) \in C_T(\mathfrak{X})$ and $\frac{\boldsymbol{\delta} f}{\boldsymbol{\delta} a}(m,a) \in \boldsymbol{\mathcal{D}}_{Z,T}(\mathfrak{A}^\vee)$ exist (see Lemma A.13), we have

$$f(m_t,a_t) = f(m_0,a_0) + \int_0^t \langle \mathrm{d}m_s, \frac{\boldsymbol{\delta} f}{\boldsymbol{\delta} m}(m_s,a_s)\rangle_{\mathfrak{X}} + \int_0^t \langle \mathrm{d}a_s, \frac{\boldsymbol{\delta} f}{\boldsymbol{\delta} a}(m_s,a_s)\rangle_{\mathfrak{A}^\vee}$$
$$= -\int_0^t \left\langle \begin{bmatrix} \mathrm{ad}^*_\square m_s & \square \diamond a_s \\ \pounds_\square a_s & 0 \end{bmatrix} \begin{bmatrix} \mathrm{d}\delta h/\boldsymbol{\delta} m(m_s,a_s) \\ \mathrm{d}\delta h/\boldsymbol{\delta} a(m_s,a_s) \end{bmatrix}, \begin{bmatrix} \boldsymbol{\delta} f/\boldsymbol{\delta} m(m_s,a_s) \\ \boldsymbol{\delta} f/\boldsymbol{\delta} a(m_s,a_s) \end{bmatrix} \right\rangle_{\mathfrak{X}\oplus\mathfrak{A}^\vee}$$
$$=: \int_0^t \Big\{ f, \mathrm{d}h_s \Big\}(m_s,a_s),$$

in which the last equality adopts the notation for the semidirect-product Lie–Poisson bracket given in [77]. In differential notation, we find

$$\mathrm{d}f(m_t,a_t) = \{f, \mathrm{d}h_t\}(m_t,a_t) = \{f,H\}(m_t,a_t)\mathrm{d}t + \{f,G\}(m_t,a_t)\mathrm{d}\mathbf{Z}_t.$$

Remark 3.20. Stochastic Hamilton equations were introduced along parallel lines with the deterministic canonical theory in [15]. These results were later extended to include reduction by symmetry in [83]. Reduction by symmetry of expected-value stochastic variational principles for Euler–Poincaré equations was developed in [4] and [36]. Stochastic variational principles were also used in constructing stochastic variational integrators in Bou-Rabee and Owhadi [20].

4. Examples

4.1. Rough incompressible Euler equation via Lagrange multipliers

Let (M,g) denote a smooth, compact, connected, oriented d-dimensional Riemannian manifold without boundary. Denote by $\mu_g \in \mathrm{Dens}_{C^\infty}$ the associated volume form, which is given in local coordinates by

$$\mu_g = \sqrt{|\det[g_{ij}]|}\, dx^1 \wedge \cdots \wedge dx^d.$$

Let $A = \Lambda^d T^*M \oplus \Lambda^0 T^*M$ and $A^\vee = \Lambda^0 T^*M \oplus \Lambda^d T^*M$. Denote the advected variables by $\mathbf{a} = (\mathbf{D}, \boldsymbol{\rho}) \in \mathfrak{A} = \mathrm{Dens}_{\mathcal{F}_3} \oplus \Omega^0_{\mathcal{F}_3}$ and the associated Lagrangian multipliers by $\boldsymbol{\lambda} = (\boldsymbol{f}, \boldsymbol{\beta}) \in \mathfrak{A}^\vee = \Omega^0_{\mathcal{F}_4} \oplus \mathrm{Dens}_{\mathcal{F}_4}$. In the following example, we will explain how to impose incompressibility through projections and spatial constraints. Toward this end, we introduce an additional Lagrangian multiplier $\boldsymbol{\pi} \in \boldsymbol{\mathcal{D}}_{Z,T}(\mathcal{F}_3 \cap \mathcal{F}_4)$ to enforce incompressibility. We consider the Clebsch action functional

$$S^{Clb_{\mathbf{Z}}}(u,\mathbf{a},\boldsymbol{\lambda},\boldsymbol{\pi}) = \int_0^T \ell(u_t,a_t)\mathrm{d}t + \big\langle \mathrm{d}\boldsymbol{\pi}_t\,,\, D_t - \rho_t\mu_g \big\rangle_{\Omega^d} + \big\langle f_t\,,\, \mathrm{d}D_t + \pounds_{dx_t} D_t \big\rangle_{\Omega^d}$$
$$+ \big\langle \beta_t\,,\, \mathrm{d}\rho_t + \pounds_{dx_t}\rho_t \big\rangle_{\Omega^0},$$

where $dx_t = u_t dt + \xi d\mathbf{Z}_t$ and the Lagrangian $\ell : \mathfrak{X} \times \mathfrak{A} \to \mathbb{R}$ is defined by

$$\ell(u,a) = \frac{1}{2}\int_M g(u,u)D = \frac{1}{2}\langle u^\flat \otimes D, u\rangle_{\mathfrak{X}},$$

where the $\flat$ operation is defined in Section A.2.3. We take variations of $(u, \mathbf{a}, \boldsymbol{\lambda})$ as defined in Section 2. A variation of $\boldsymbol{\pi}$ is defined to be $\boldsymbol{\pi}^\epsilon = \boldsymbol{\pi} + \epsilon\boldsymbol{\delta}\pi$ for $\boldsymbol{\delta}\pi \in C_T^\infty(C^\infty)$ such that $\boldsymbol{\delta}\pi_0 = \boldsymbol{\delta}\pi_T = 0$.

It follows that all $(u,a) \in \mathfrak{X} \times \mathfrak{A}$,

$$m = \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta}u}(u,a) = u^\flat \otimes D \in \mathfrak{X}^\vee, \qquad u = \frac{\sharp}{D}m, \qquad \text{and} \quad \frac{\boldsymbol{\delta}\ell}{\delta a}(u,a) = \left(\frac{1}{2}g(u,u),\ \ 0\right) \in \mathfrak{A}^\vee,$$

where the diffeomorphism $\frac{\sharp}{D}$ is defined in Section A.2.2. Let us now compute the relevant diamond terms (see Definition 2.1). For all $(h,\nu) \in C^\infty \times \mathrm{Dens}_{C^\infty}$ and $u \in \mathfrak{X}_{C^\infty}$, we have

$$-\langle h, \pounds_u \nu\rangle_{\Omega^d} = -\int_M h\pounds_u\nu = -\int_M h\mathbf{d}\mathbf{i}_u\nu = \int_M \nu\mathbf{i}_u\mathbf{d}h = \langle \pounds_u h, \nu\rangle_{\Omega^d} = \langle \mathbf{d}h\otimes\nu, u\rangle_{\mathfrak{X}},$$

which implies $h \diamond \nu = \mathbf{d}g \otimes \nu$. Moreover, since

$$-\langle \nu, \pounds_u h\rangle_{\Omega^0} = -\int_M \nu i_u \mathbf{d}h = -\langle \mathbf{d}h \otimes \nu, u\rangle_{\mathfrak{X}} \quad \Rightarrow \quad \nu \diamond h = -\mathbf{d}h \otimes \nu.$$

With minor modifications of the Proof of Theorem 2.6, we find that $(u, \mathbf{a}, \boldsymbol{\lambda}, \boldsymbol{\pi})$ is a critical point of $S^{Clb\mathbf{z}} = 0$, if and only if for all $t \in [0,T]$:

$$m_t + \int_0^t \pounds_{dx_s} m_s \overset{\mathfrak{X}^\vee}{=} m_0 + \int_0^t \frac{1}{2}g(u_s,u_s)\diamond D_s ds + \int_0^t d\boldsymbol{\pi}_s \diamond D_s - \int_0^t d\boldsymbol{\pi}_s\mu_g \diamond \rho_s,$$

$$m_t = f_t \diamond D_t + \beta_t \diamond \rho_t,$$

$$D_t + \int_0^t \pounds_{dx_s} D_s \overset{\mathrm{Dens}}{=} D_0, \quad \rho_t + \int_0^t \pounds_{dx_s}\rho_s \overset{\Omega^0}{=} \rho_0, \quad D_t = \rho_t\mu_g,$$

$$f_t + \int_0^t \pounds_{dx_s} f_s \overset{\Omega^0}{=} f_0 + \int_0^t \boldsymbol{\pi}_s ds + \frac{1}{2}\int_0^t g(u_s,u_s)ds \qquad \beta_t + \int_0^t \pounds_{dx_s}\beta_t \overset{\mathrm{Dens}}{=} \beta_0 - \int_0^t \boldsymbol{\pi}_s\mu_g ds.$$

Substituting $D = \rho\mu_g$ into the equation for D and applying the diffeomorphism $\frac{1}{\mu_g}$ (see (A.15)), we find

$$\rho_t + \int_0^t \left(\pounds_{dx_s} \rho_s + \operatorname{div}_{\mu_g} dx_s \right) = \rho_0.$$

Since ρ is advected, we obtain for all $t \in [0, T]$,

$$\int_0^t \operatorname{div}_{\mu_g} u_s \mathrm{d}s + \int_0^t \operatorname{div}_{\mu_g} \xi \mathrm{d}\mathbf{Z}_s = 0.$$

In order to conclude that $\operatorname{div}_{\mu_g} u \equiv 0$, we need to assume either $\operatorname{div}_{\mu_g} \xi \equiv 0$, or true roughness of $\mathbf{Z}$. In the next example, we do not require additional assumptions to conclude that u is divergence-free, since we will impose this directly.

Substituting $m = u^\flat \otimes D$ into the momentum equation and recalling that D is advected, that the Lie derivative is a derivation, and that the product rule (Lemma A.13) holds, we find

$$\mathrm{d}u_t^\flat \otimes \rho_t \mu_g + \pounds_{dx_t} u_t^\flat \otimes \rho_t \mu_g + = \frac{1}{2} \mathbf{d} g(u_t, u_t) \otimes \rho_t \mu_g + \mathbf{d}\mathrm{d}\boldsymbol{\pi}_t \otimes \rho_t \mu_g + \mathbf{d}\rho_t \otimes \mathrm{d}\boldsymbol{\pi}_t \mu_g.$$

Assuming ρ is non-vanishing and applying the diffeomorphism $\frac{1}{D}$ (see (A.15)) yields

$$\begin{aligned} \mathrm{d}u_t^\flat + \pounds_{dx_t} u_t^\flat &\overset{\Omega^1}{=} \frac{1}{2} \mathbf{d} g(u_t, u_t) + \mathbf{d}\mathrm{d}\boldsymbol{\pi}_t + \frac{1}{\rho_t} \mathbf{d}\rho_t \mathrm{d}\boldsymbol{\pi}_t = \frac{1}{2} \mathbf{d} g(u_t, u_t) + \frac{1}{\rho_t} \mathbf{d}(\rho_t \mathrm{d}\boldsymbol{\pi}_t) \\ &= \frac{1}{2} \mathbf{d} g(u_t, u_t) - \frac{1}{\rho_t} \mathbf{d}\mathrm{d}\mathbf{p}_t, \end{aligned} \tag{4.1}$$

in which the pressure is identified in terms of the Lagrange multiplier $\mathrm{d}\boldsymbol{\pi}_t$ as $\mathrm{d}\mathbf{p}_t := -\rho_t \mathrm{d}\boldsymbol{\pi}_t$. We will elaborate more on this equation in the following example.

4.2. Rough incompressible Euler equation via constraint on spaces

Let (M, g) and μ_g be as in the previous example. Let $A = \Lambda^d T^* M$ and $A^\vee = \Lambda^0 T^* M$ in this example, and notice that for all $D \in \mathfrak{A} = \mathrm{Dens}_{\mathcal{F}^3}$, there exists $\rho \in \Omega^0_{\mathcal{F}^3}$ such that $D = \rho \mu_g$.

Let $\mathfrak{X}_{\mu_g} = \mathfrak{X}_{\mu_g, \mathcal{F}_1}$ denote the space of incompressible vector fields and $\mathfrak{X}^\vee_{\mu_g} = \mathfrak{X}_{\mu_g, \mathcal{F}_2}$ denote the dual space of one-form densities modulo the kernel of the divergence-free projection as defined in Definition A.20. Denote by $\langle \cdot, \cdot \rangle_{\mathfrak{X}_{\mu_g}} : \mathfrak{X}^\vee_{\mu_g} \times \mathfrak{X}_{\mu_g} \to \mathbb{R}$ the canonical pairing defined in (A.18) in Definition A.20. Define the Lagrangian $\ell : \mathfrak{X}_{\mu_g} \times \mathfrak{A} \to \mathbb{R}$ by

$$\ell(u, D) = \frac{1}{2} \int_M \rho g(u, u) \mu_g = \frac{1}{2} \int_M g(u, u) D = \frac{1}{2} \langle u^\flat \otimes D, u \rangle_{\mathfrak{X}} = \frac{1}{2} \langle [u^\flat \otimes D], u \rangle_{\mathfrak{X}_{\mu_g}}.$$

The square brackets denote an equivalence class, whose elements satisfy $[\mathbf{d} f \otimes \mu_g]. = [0]$. It follows that for all $(u, D) \in \mathfrak{X}_{\mu_g} \times \mathfrak{A}$,

$$m = \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta}u}(u, D) = [u^\flat \otimes D] \in \mathfrak{X}^\vee_{\mu_g}, \quad u = \frac{\sharp}{D}m, \quad \text{and} \quad \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta}D}(u, D) = \frac{1}{2}g(u, u),$$

where the diffeomorphism $\frac{\sharp}{D}$ is defined in Section A.2.2. Using the diamond operation computed in the previous example, we find

$$\frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta}D}(u, D) \diamond D = \left[\frac{1}{2}\mathbf{d}g(u, u) \otimes D\right] \in \mathfrak{X}^\vee_{\mu_g}.$$

Clebsch critical points (Theorem 2.6) thus satisfy

$$\begin{aligned}
&\mathrm{d}m_t + \pounds_{u_t} m_t \mathrm{d}t + \pounds_\xi m_t \mathrm{d}\mathbf{Z}_t \overset{\mathfrak{X}^\vee_{\mu_g}}{=} [\frac{1}{2}\mathbf{d}g(u_t, u_t) \otimes D_t]\mathrm{d}t, \quad m = [u^\flat \otimes D],\\
&\mathrm{d}D_t + \pounds_{u_t} D_t \mathrm{d}t + \pounds_\xi D_t \mathrm{d}\mathbf{Z}_t \overset{\mathfrak{A}}{=} 0,\\
&\mathrm{d}\lambda_t \overset{\mathfrak{A}^\vee}{=} \left(\pounds_{u_t}\lambda_t + \frac{1}{2}g(u_t, u_t)\right)\mathrm{d}t + \pounds_\xi \lambda_t \mathrm{d}\mathbf{Z}_t.
\end{aligned}$$

Critical points of the Hamilton-Pontryagin action functional (Theorem 3.12) also satisfy the first two equations. Since D is Lie-advected and the Lie derivative is a derivation, using the product rule (Lemma A.13), we find

$$[\mathrm{d}u^\flat_t \otimes D_t] + [\pounds_{u_t} u^\flat_t \otimes D_t]\mathrm{d}t + [\pounds_\xi u^\flat_t \otimes D_t]\mathrm{d}\mathbf{Z}_t \overset{\mathfrak{X}^\vee_{\mu_g}}{=} [\frac{1}{2}\mathbf{d}g(u_t, u_t) \otimes D_t]\mathrm{d}t,$$

or equivalently

$$\mathrm{d}u^\flat_t \otimes D_t + P(\pounds_{u_t} u^\flat_t \otimes D_t)\mathrm{d}t + P(\pounds_\xi u^\flat_t \otimes D_t)\mathrm{d}\mathbf{Z}_t \overset{\mathfrak{X}^\vee_{\mu_g}}{=} P(\frac{1}{2}\mathbf{d}g(u_t, u_t) \otimes D_t)\mathrm{d}t.$$

Upon invoking the definition of $\mathfrak{X}^\vee_{\mu_g}$ in Definition A.20, we find

$$\mathrm{d}u^\flat_t \otimes D_t + \pounds_{u_t} u^\flat_t \otimes D_t \mathrm{d}t + \pounds_\xi u^\flat_t \otimes D_t \mathrm{d}\mathbf{Z}_t \overset{\mathfrak{X}^\vee}{=} \frac{1}{2}\mathbf{d}g(u_t, u_t) \otimes D_t \mathrm{d}t - \mathbf{d}p \otimes \mu_g \mathrm{d}t - \mathbf{d}q \otimes \mu_g \mathrm{d}\mathbf{Z}_t,$$

where $\mathbf{d}p \in C^\alpha_T(\Omega^0)$ and $\mathbf{d}q \in \boldsymbol{\mathcal{D}}_{Z,T}((\Omega^0)^K)$.

Applying $\frac{1}{D}$ (as defined in (A.15)) and recalling that $\operatorname{div}_{\mu_g} u_t \equiv 0$ yields

$$\begin{aligned}
&\mathrm{d}u^\flat_t + \pounds_{u_t} u^\flat_t \mathrm{d}t + \pounds_\xi u^\flat_t \mathrm{d}\mathbf{Z}_t \overset{\Omega^1}{=} \frac{1}{2}\mathbf{d}g(u_t, u_t)\mathrm{d}t - \frac{1}{\rho_t}\mathbf{d}p_t \mathrm{d}t - \frac{1}{\rho_t}\mathbf{d}q_t \mathrm{d}\mathbf{Z}_t,\\
&\mathbf{d}^* u^\flat = 0 = \operatorname{div}_{\mu_g} u_t,\\
&\mathrm{d}\rho_t + \pounds_{u_t}\rho_t \mathrm{d}t + (\pounds_\xi \rho_t + \operatorname{div}_{\mu_g}\xi)\mathrm{d}\mathbf{Z}_t \overset{\Omega^0}{=} 0.
\end{aligned}$$

Remark 4.1. Thus, one sees that the Hodge decomposition necessitates introducing a 'rough' Lagrangian multiplier (i.e., pressure term) $\mathrm{d}\mathbf{p}_t = -\rho_t \mathrm{d}\boldsymbol{\pi}_t$ in (4.1). That is,

$$\mathrm{d}\mathbf{p}_t = \mathbf{d}p \otimes \mu_g \mathrm{d}t - \mathbf{d}q \otimes \mu_g \mathrm{d}\mathbf{Z}_t,$$

where

$$\mathbf{d}p_t = -Q\left(\rho_t\left(\pounds_{u_t} u_t^\flat - \frac{1}{2}\mathbf{d}g(u_t, u_t)\right)\right) \quad \text{and} \quad \mathbf{d}q_t = -Q\left(\rho_t \pounds_\xi u_t^\flat\right),$$

and $Q : \Omega^1 \to \mathbf{d}\Omega^0$ denotes the projection (A.16) onto flat one-forms.

The following identity is well-known:

$$\pounds_v v^\flat - \frac{1}{2}\mathbf{d}g(v,v) = (\nabla_v v)^\flat, \quad \forall v \in \mathfrak{X}_{C^\infty},$$

where $\nabla : \mathfrak{X}_{C^\infty} \times \mathfrak{X}_{C^\infty} \to \mathfrak{X}_{C^\infty}$ is Levi-Civita connection (see, e.g., [52][Section 3]).[4] Thus, applying the $\sharp$ operator to the equation for $u^\flat$ yields

$$\mathrm{d}u_t + \nabla_{u_t} u_t \mathrm{d}t + \left(\pounds_\xi u_t^\flat\right)^\sharp \mathrm{d}\mathbf{Z}_t = \frac{1}{\rho_t}\nabla p_t \mathrm{d}t + \frac{1}{\rho_t}\nabla q_t \mathrm{d}\mathbf{Z}_t,$$

where in a local coordinate chart (see (A.11)),

$$(\pounds_\xi u^\flat)^\sharp = \left(\xi^j \partial_{x^j} u^k + g^{ik}\xi^j u^l \partial_{x^j} g_{li} + g^{ik} g_{lj} u^l \partial_{x^i}\xi^j\right)\partial_{x^k}.$$

It is worth noting that for all $u \in \mathfrak{X}_{\mu_g, C^\infty}$ and $v, w \in \mathfrak{X}_{C^\infty}$,

$$\begin{aligned}(w, \mathrm{ad}_u v)_{\mathfrak{X}_{L^2}} &= \langle w^\flat \otimes \mu_g, \mathrm{ad}_u v\rangle_{\mathfrak{X}} = \langle \pounds_u(w^\flat \otimes \mu_g), v\rangle_{\mathfrak{X}} = \langle \pounds_u w^\flat \otimes \mu_g, v\rangle_{\mathfrak{X}} \\ &= ((\pounds_u w^\flat)^\sharp, v)_{\mathfrak{X}_{L^2}}.\end{aligned}$$

Remark 4.2. When $\xi \equiv 0$, the corresponding equation is the usual deterministic incompressible non-homogeneous Euler fluid equation (see, e.g., [24][Ch. VI] or [92]).

In case of a homogeneous fluid ($\rho \equiv 1$) we find

$$\mathrm{d}u_t^\flat + \pounds_{u_t} u_t^\flat \mathrm{d}t + \pounds_\xi u_t^\flat \mathrm{d}\mathbf{Z}_t = \frac{1}{2}\mathbf{d}g(u_t, u_t)\mathrm{d}t - \mathbf{d}p_t \mathrm{d}t - \mathbf{d}q_t \mathrm{d}\mathbf{Z}_t,$$

---

[4] For the convenience of the reader, we repeat the proof. For a given $u \in \mathfrak{X}_{C^\infty}$, define the tensor derivation $A_u = \pounds_u - \nabla_u$. It follows that $A_u f \equiv 0$ for all $f \in C^\infty$ and that $A_u v = -\nabla_v u$ by the torsion-free property of the connection. Using these properties and that A_u is a derivation, for a given $\alpha \in \Omega^1_{C^\infty}$, we have $\mathbf{i}_v(A_u \alpha) = \mathbf{i}_{\nabla_v u}\alpha$, and hence $\mathbf{i}_w(A_u v^\flat) = \mathbf{i}_{\nabla_w u} v^\flat = g(v, \nabla_w u)$ for all $w \in \mathfrak{X}_{C^\infty}$. Therefore,

$$\mathbf{i}_w(\mathbf{d}g(u,v)) = \nabla_w[g(u,v)] = g(v, \nabla_w u) + g(u, \nabla_w v) = \mathbf{i}_w(A_u v^\flat + A_v u^\flat), \quad \forall u, v, w \in \mathfrak{X}_{C^\infty},$$

where we have also used $\nabla\eta = 0$. Thus, $\pounds_u v^\flat - \nabla_u v^\flat + \pounds_v u^\flat - \nabla_v u^\flat = \mathbf{d}g(u,v)$, which gives the formula upon setting $u = v$.

or equivalently,

$$\mathrm{d}u_t + \nabla_{u_t} u_t \mathrm{d}t + \left(\pounds_\xi u_t^\flat\right)^\sharp \mathrm{d}\mathbf{Z}_t = -\nabla p_t \mathrm{d}t - \nabla q_t \mathrm{d}\mathbf{Z}_t.$$

In this case, another advected quantity $a \in \oplus_{i=1}^d \Lambda^0 T^*M$ and its Lagrange multiplier $\lambda \in \oplus_{i=1}^d \Lambda^d T^*M$ should be introduced into the Clebsch constraint to avoid reduction to potential flow (see, e.g., Section C).

Vorticity dynamics We will now discuss vorticity dynamics in both the inhomogeneous and homogeneous case. We assume $\mathrm{div}_{\mu_g} \xi \equiv 0$ and that all quantities are regular enough subsequently to perform each calculation. First, notice that for every $f \in C^\infty$,

$$\mathrm{d}f(\rho_t) + \pounds_{u_t} f(\rho_t)\mathrm{d}t + \pounds_\xi f(\rho_t)\mathrm{d}\mathbf{Z}_t = 0.$$

That is, $f(\rho_t)$ is advected by $\mathrm{d}x_t$ (see Remark 3.4). Let $\omega = \mathbf{d}u^\flat \in \Omega^2$ be the vorticity two-form. Since the exterior derivative $\mathbf{d}$ commutes with the Lie derivative, we obtain

$$\mathrm{d}\omega_t + \pounds_{u_t}\omega_t \mathrm{d}t + \pounds_\xi \omega_t \mathrm{d}\mathbf{Z}_t \overset{\Omega^2}{=} -\mathbf{d}\rho_t^{-1} \wedge \mathbf{d}p_t \mathrm{d}t - \mathbf{d}\rho_t^{-1} \wedge \mathbf{d}q_t \mathrm{d}\mathbf{Z}_t \quad \text{and}$$

$$\mathrm{d}\mathbf{d}f(\rho_t) + \pounds_{u_t}(\mathbf{d}f(\rho_t))\mathrm{d}t + \pounds_\xi(\mathbf{d}f(\rho_t))\mathrm{d}\mathbf{Z}_t \overset{\Omega^1}{=} 0.$$

Thus, by the product rule (Lemma A.13), we get

$$\begin{aligned}\mathrm{d}(\omega_t \wedge \mathbf{d}f(\rho_t)) + \pounds_{u_t}((\omega_t \wedge \mathbf{d}f(\rho_t))\mathrm{d}t + \pounds_\xi((\omega_t \wedge \mathbf{d}f(\rho_t))\mathrm{d}\mathbf{Z}_t \\ = -\mathbf{d}\rho_t^{-1} \wedge \mathbf{d}p_t \wedge \mathbf{d}f(\rho_t)\mathrm{d}t - \mathbf{d}\rho_t^{-1} \wedge \mathbf{d}q_t \wedge \mathbf{d}f(\rho_t)\mathrm{d}\mathbf{Z}_t.\end{aligned}$$

In particular, in dimension three, using that $\mathbf{d}\rho_t \wedge \mathbf{d}p_t \wedge \mathbf{d}\rho_t \equiv \mathbf{d}\rho_t \wedge \mathbf{d}q_t \wedge \mathbf{d}\rho_t \equiv 0$, we find

$$\mathrm{d}(\omega_t \wedge \mathbf{d}\rho_t) + \pounds_{u_t}(\omega_t \wedge \mathbf{d}\rho_t)\mathrm{d}t + \pounds_\xi(\omega_t \wedge \mathbf{d}\rho_t)\mathrm{d}\mathbf{Z}_t \overset{\Omega^3}{=} 0.$$

Moreover, in dimension three, applying Stokes theorem, we get

$$\int_M \omega_t \wedge \mathbf{d}f(\rho_t) = \int_M \omega_0 \wedge \mathbf{d}f(\rho_0).$$

We also have that

$$\mathrm{d}(\omega_t f(\rho_t)) + \pounds_{u_t}(\omega_t f(\rho_t)\mathrm{d}t + \pounds_\xi(\omega)t f(\rho_t))\mathrm{d}\mathbf{Z}_t = -\mathbf{d}G(\rho_t) \wedge \mathbf{d}p_t \mathrm{d}t - \mathbf{d}G(\rho_t) \wedge \mathbf{d}q_t \mathrm{d}\mathbf{Z}_t,$$

where G is the anti-derivative of $g(\rho) = \frac{f(\rho)}{\rho^2}$. Thus, in dimension two, we obtain

$$\int_M \omega_t f(\rho_t) = \int_M \omega_0 f(\rho_0).$$

In the homogeneous setting, the vorticity equation is given by

$$\mathrm{d}\omega_t + \pounds_{u_t}\omega_t \mathrm{d}t + \pounds_\xi \omega_t \mathrm{d}\mathbf{Z}_t = 0.$$

Using the Hodge-star operator $\star : \Omega^2 \to \Omega^{d-2}$ and setting $\tilde{\omega} = \star\omega \in \Omega^0$ in dimension two and $\tilde{\omega} = \sharp \star \omega \in \dot{\mathfrak{X}}_{\mu_g}$ in dimension three, we find

$$\partial_t \tilde{\omega}_t + ((u_t \cdot \nabla)\tilde{\omega}_t - \mathbf{1}_{d=3}(\tilde{\omega}_t \cdot \nabla)u_t)\,\mathrm{d}t + ((\xi \cdot \nabla)\tilde{\omega}_t - \mathbf{1}_{d=3}(\tilde{\omega}_t \cdot \nabla)\xi)\,\mathrm{d}t = 0.$$

Here, we have used that $\sharp\star$ and the Lie derivative commute (see, e.g., Section A.6 of [14]).

In three dimensions, the helicity, defined as

$$\Lambda(\tilde{\omega}_t) = \int_M u_t^\flat \wedge \omega_t$$

measures the linkage of field lines of the divergence-free vector field $\tilde{\omega}$ [7]. It follows that

$$\mathrm{d}(u^\flat \wedge \omega) + \pounds_{u_t}(u^\flat \wedge \omega)\mathrm{d}t + \pounds_\xi(u^\flat \wedge \omega)\mathrm{d}\mathbf{Z}_t = -\mathbf{d}\tilde{p}_t \wedge \omega_t \mathrm{d}t - \mathbf{d}q \wedge \omega_t \mathrm{d}\mathbf{Z}_t,$$

where $\tilde{p} = p - \frac{1}{2}\mathbf{d}g(u,u)$. Thus, we find

$$\Lambda(\tilde{\omega}_t) = \int_M u_t^\flat \wedge \omega_t = \int_M u_0^\flat \wedge \omega_0 = \Lambda(\tilde{\omega}_0).$$

Therefore, the linkage number of the vorticity vector field $\Lambda(\tilde{\omega})$ is preserved by the 3D Euler fluid equations.

In two dimensions, for any smooth $f \in C^\infty$,

$$\mathrm{d}f(\tilde{\omega}_t) + \pounds_{u_t} f(\tilde{\omega}_t)\mathrm{d}t + \pounds_\xi f(\tilde{\omega}_t)\mathrm{d}\mathbf{Z}_t = 0,$$

and hence

$$\int_M f(\tilde{\omega}_t)\mu_g = \int_M f(\tilde{\omega}_0)\mu_g.$$

Letting $f(x) = x^2$, we obtain

$$\int_M \tilde{\omega}_t^2 \mu_g = \int_M \tilde{\omega}_0^2 \mu_g,$$

which implies that in dimension two enstrophy is conserved.

Divergence and harmonic-free It is worth noting that in the homogeneous setting, u can only be recovered directly from ω if $\mathcal{H}^1_\Delta = \emptyset$ via the Biot-Savart law (see Section A.2.3). Otherwise one needs to keep track of the harmonic constant. Nevertheless, one can repeat the above analysis with the harmonic and divergence-free spaces $\dot{\mathfrak{X}}_{\mu_g}$ and $\dot{\mathfrak{X}}^\vee_{\mu_g}$ (see Definition A.20) to derive

$$\mathrm{d}u_t^\flat + \dot{P}\pounds_{u_t} u_t^\flat \mathrm{d}t + \dot{P}\pounds_\xi u_t^\flat \mathrm{d}\mathbf{Z}_t = \frac{1}{2}\dot{P}\mathbf{d}g(u_t, u_t)\mathrm{d}t,$$

and hence

$$\mathrm{d}u_t^\flat + \pounds_{u_t} u_t^\flat \mathrm{d}t + \pounds_\xi u_t^\flat \mathrm{d}\mathbf{Z}_t = \frac{1}{2}\mathbf{d}g(u_t, u_t)\mathrm{d}t + (c_t - \mathbf{d}p_t)\mathrm{d}t + (\tilde{c}_t - \mathbf{d}q_t)\mathrm{d}\mathbf{Z}_t,$$

where $c = H(\pounds_u u^\flat) \in C^\alpha_T(\mathcal{H}^1_\Delta)$ and $\tilde{c} = H(\pounds_\xi u^\flat) \in \boldsymbol{\mathcal{D}}_{Z,T}((\mathcal{H}^1_\Delta)^K)$ and H is the projection onto Harmonic one-forms. Here, u is constrained to be both divergence free and harmonic free. For this equation, $u^\flat$ (and u) can be recovered directly from ω via the Biot-Savart operator. This equation has been studied in [43,28,27,42]. See, also, the discussion in [55].

4.3. Rough Camassa-Holm equation and Burgers equation

Let $M = \mathbb{S}$ be the flat one-dimensional torus (i.e., the circle). Denote the standard normalized volume form by $\mu \in \mathrm{Dens}_{C^\infty}$ and coordinates by x. In this example, we take $A = \emptyset = A^\vee$. We define the Lagrangian $\ell : \mathfrak{X} \to \mathbb{R}$ by

$$\ell(u) = \frac{1}{2}\int_{\mathbb{S}} (|u|^2 + \alpha^2 |\nabla_x u|^2)\mu = \frac{1}{2}\langle (\Lambda^2 u)^\flat \otimes \mu, u\rangle_{\mathfrak{X}}, \quad \text{where} \quad \Lambda^2 := 1 - \alpha^2 \nabla_x^2.$$

It follows that

$$m = \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta}u}(u) = (\Lambda^2 u)^\flat \otimes \mu \in \mathfrak{X}^\vee \quad \text{and} \quad u = \Lambda^{-2}\left(\frac{\sharp}{\mu} m\right).$$

If (u, η) is a critical point of the Hamilton-Pontryagin action functional (Theorem 3.12), then

$$\mathrm{d}m_t + \pounds_{u_t} m_t \mathrm{d}t + \pounds_\xi m_t \mathrm{d}\mathbf{Z}_t = 0,$$

which we may interpret as Lie transport of the momentum 1-form density in the Camassa-Holm equation along rough paths.

Since the Lie derivative is a derivation and we have the explicit formula (A.11) for one-forms, we find

$$\pounds_v m = \Big(\pounds_v (\Lambda^2 u)^\flat + (\Lambda^2 u)^\flat \operatorname{div}_\mu v\Big) \otimes \mu = \big(v\nabla_x(\Lambda^2 u) + 2(\Lambda^2 u)\nabla_x v\big)^\flat \otimes \mu, \quad \forall v \in \mathfrak{X}.$$

Thus, identifying u with a scalar-valued function, we get

$$\mathrm{d}u_t + \Lambda^{-2}\left(u_t\partial_x(\Lambda^2 u_t) + 2(\Lambda^2 u_t)\partial_x u_t\right)\mathrm{d}t + \Lambda^{-2}\left(\xi\partial_x(\Lambda^2 u_t) + 2(\Lambda^2 u_t)\partial_x\xi\right)\mathrm{d}\mathbf{Z}_t = 0.$$

After some simplification, we find

$$\begin{aligned}\mathrm{d}u_t &+ \left(u_t\partial_x u_t + \partial_x\Lambda^{-2}\left(\left(|u_t|^2 + \frac{\alpha^2}{2}|\partial_x u_t|^2\right)\right)\right)\mathrm{d}t \\ &+ \left(\xi\partial_x u_t + \Lambda^{-2}\left(2u_t\partial_x\xi + \alpha^2\partial_x^2\xi\partial_x u_t\right)\right)\mathrm{d}\mathbf{Z}_t = 0,\end{aligned}$$

written as a nonlocal Cauchy problem with the pseudo-differential operator Λ^{-2}. Indeed, notice that if one substitutes ξ with u in the $\mathrm{d}\mathbf{Z}_t$-term, then one obtains the same operator in $\mathrm{d}t$-term.

Now, if $\alpha = 0$, we obtain the Burgers equation on rough paths,

$$\mathrm{d}u_t + 3u_t\partial_x u_t\mathrm{d}t + (\xi\partial_x u_t + 2u_t\partial_x\xi)\,\mathrm{d}\mathbf{Z}_t = 0.$$

The properties of the stochastic Burgers equation with stochastic transport noise have been investigated, e.g., in [2], and the properties of the Burgers equation with rough transport noise have been studied in [73].

4.4. Rough Euler equations for adiabatic compressible flows

Let $A = \Lambda^d T^*M \oplus \Lambda^0 T^*M$ and $A^\vee = \Lambda^0 T^*M \oplus \Lambda^d T^*M$. Denote the advected variables by $\mathbf{a} = (\mathbf{D}, \mathbf{s}) \in \mathfrak{A}_{\mathcal{F}_3} = \mathrm{Dens}_{\mathcal{F}_3} \oplus \Omega^0_{\mathcal{F}_3}$ and the associated Lagrangian multipliers by $\boldsymbol{\lambda} = (\boldsymbol{f}, \boldsymbol{\beta}) \in \mathfrak{A}^\vee_{\mathcal{F}_4} = \Omega^0_{\mathcal{F}_4} \oplus \mathrm{Dens}_{\mathcal{F}_4}$. Let $\rho \in \Omega^0_{\mathcal{F}_3}$ be such that $D = \rho\mu_g$. The advected variables comprise the thermodynamic evolution variables mass/volume, ρ and the entropy/mass, s. The internal energy/mass, $e(\rho, s)$, obeys the First Law of Thermodynamics, given by

$$de(\rho, s) = \frac{p}{\rho^2}d\rho + Tds,$$

with pressure $p(\rho, s)$ and temperature $T(\rho, s)$.

Define the Lagrangian $\ell : \mathfrak{X}_{\mu_g} \times \mathfrak{A} \to \mathbb{R}$ by

$$\ell(u, a) = \int_M \left(\frac{1}{2}g(u, u) - e(\rho, s)\right) D.$$

It follows that

$$m = \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta}u}(u, a) = u^\flat \otimes D \in \mathfrak{X}^\vee \quad \text{and} \quad \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta}a}(u, a) = \left(\frac{1}{2}g(u, u) - h(p, s),\ -TD\right),$$

where $h(p, s) = e(\rho, s) + p/\rho$ is the specific enthalpy/mass, which satisfies

$$dh(p,s) = \frac{1}{\rho}dp + Tds.$$

Applying the calculations with the diamond operation ($\diamond$) in Section 4.1 yields

$$\frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta}a}(u,a)\diamond a = \left(\left(\frac{1}{2}\mathbf{d}g(u,u) - \mathbf{d}h(p,s)\right)\otimes D,\; T\mathbf{d}s\otimes D\right).$$

Critical points of the Clebsch and Hamilton-Pontryagin action functionals satisfy

$$\mathrm{d}m_t + \pounds_{dx_t}m_t\mathrm{d}t \overset{\mathfrak{X}^\vee}{=} \left(\frac{1}{2}\mathbf{d}g(u_t,u_t) - \mathbf{d}h(p_t,s_t)\right)\otimes D_t\mathrm{d}t + T_t\mathbf{d}s_t\otimes D_t\mathrm{d}t,$$

$$\mathrm{d}D_t + \pounds_{dx_t}D_t\mathrm{d}t \overset{\mathrm{Dens}}{=} 0, \quad \& \quad \mathrm{d}s_t + \pounds_{dx_t}s_t \overset{\Omega^0}{=} 0.$$

Since D is Lie-advected and the Lie derivative is a derivation, using the product rule (Lemma A.13) and applying the diffeomorphism $\frac{1}{D}$ yields

$$\mathrm{d}u_t^\flat + \pounds_{\mathrm{d}x_t}u_t^\flat = \left(\mathbf{d}\left(\frac{1}{2}g(u_t,u_t) - h(p_t,s_t)\right) + T_t\mathbf{d}s_t\right)\mathrm{d}t =: \left(\frac{1}{2}\mathbf{d}g(u_t,u_t) - \frac{1}{\rho_t}\mathbf{d}p_t\right)\mathrm{d}t\,.$$

Restricting to dimension three and working with enough regularity to perform the subsequent calculations implies three advected quantities,

$$(\mathrm{d}+\pounds_{\mathrm{d}x_t})D_t = 0\,, \qquad (\mathrm{d}+\pounds_{\mathrm{d}x_t})s_t = 0\,, \qquad (\mathrm{d}+\pounds_{\mathrm{d}x_t})(\mathbf{d}u_t^\flat\wedge\mathbf{d}s_t) \overset{\Omega^3}{=} 0\,. \quad (4.2)$$

Let $\omega = \mathbf{d}u^\flat$ denote the vorticity two-form and let $\tilde{\omega}$ denote the corresponding divergence-free vector field. From the three quantities in (4.2), one may construct the following advected scalar quantity known as the *potential vorticity*

$$(\mathrm{d}+\pounds_{\mathrm{d}x_t})\Omega_t \overset{\Omega^0}{=} 0, \quad \text{where} \quad \Omega_t := D_t^{-1}\boldsymbol{\omega}_t\wedge\mathbf{d}s_t = \rho_t^{-1}\tilde{\omega}\cdot\nabla s_t.$$

Consequently, the following functional is conserved for the adiabatic compressible Euler equations on GRPs

$$C_\Phi := \int_M \Phi(\Omega_t, s_t)D,$$

for any smooth function $\Phi : \mathbb{R}^2 \to \mathbb{R}$.

5. Proof of main results

5.1. Proof of the Clebsch variational principle Theorem 2.6

Proof. It is worth noting that this proof closely mirrors the proof in [76] for stochastic variational principles. Nonetheless, we repeat the proof for the convenience of the reader.

If $(u, a, \lambda) \in Clb_{\mathbf{Z}}$ is a critical point of the action functional, then it satisfies

$$0 = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} S^{Clb_{\mathbf{Z}}}(u^\epsilon, \mathbf{a}^\epsilon, \boldsymbol{\lambda}^\epsilon) = I(\boldsymbol{\delta} u) + II(\boldsymbol{\delta} a) + III(\boldsymbol{\delta}\lambda),$$

where

$$I(\boldsymbol{\delta} u) = \int_0^T \left\langle \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta} u}(u_t, a_t) - \lambda_t \diamond a_t, \boldsymbol{\delta} u_t \right\rangle_{\mathfrak{X}} \mathrm{d}t$$

$$II(\boldsymbol{\delta} a) = \int_0^T \langle \lambda_t, \partial_t \boldsymbol{\delta} a_t\rangle_{\mathfrak{A}} \mathrm{d}t + \int_0^T \langle \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta} a}(u_t, a_t) + \pounds_{u_t}^T \lambda_t, \boldsymbol{\delta} a_t\rangle_{\mathfrak{A}} \mathrm{d}t + \int_0^T \langle \pounds_\xi^* \lambda_t, \boldsymbol{\delta} a_t\rangle_{\mathfrak{A}} \mathrm{d}\mathbf{Z}_t$$

$$III(\boldsymbol{\delta}\lambda) = \int_0^T \langle \boldsymbol{\delta}\lambda_t, \mathrm{d}\mathbf{a}_t\rangle_{\mathfrak{A}} + \int_0^T \langle \boldsymbol{\delta}\lambda_t, \pounds_{u_t} a_t\rangle_{\mathfrak{A}} \mathrm{d}t + \int_0^T \langle \boldsymbol{\delta}\lambda_t, \pounds_\xi a_t\rangle_{\mathfrak{A}} \mathrm{d}\mathbf{Z}_t.$$

Here, we have used the definition of the diamond operator and (2.2) to exchange the order of derivative in ϵ and the time-integral for the Lagrangian terms. Since we may always take $\boldsymbol{\delta} u \equiv 0$, $\boldsymbol{\delta} a \equiv 0$, and $\boldsymbol{\delta}\lambda \equiv 0$, we conclude that $I(\boldsymbol{\delta} u) = 0$, $II(\boldsymbol{\delta} a) = 0$, and $III(\boldsymbol{\delta}\lambda) = 0$ for all smooth $(\boldsymbol{\delta} u, \boldsymbol{\delta}\lambda, \boldsymbol{\delta} a)$ that vanish at $t = 0$ and $t = T$. Splitting the variations in time and space and applying the fundamental lemma of calculus of variations in Lemmas B.4, and A.13, we find $m = \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta} u}(u, a) \overset{\mathfrak{X}^\vee}{=} \lambda \diamond a$ and that a, λ solve the equations given in (2.2). Upon applying Lemma A.13 with the continuous bilinear pairing $\diamond : \mathfrak{X} \times \mathfrak{A} \to \mathfrak{X}^\vee$, we obtain

$$m_t = \lambda_0 \diamond a_0 + \int_0^t \mathrm{d}\lambda_r \diamond a_r + \int_0^t \lambda_r \diamond \mathrm{d}a_r.$$

Subtracting $\int_0^t \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta} a}(u_r, a_r)\mathrm{d}r$ from both sides of the above, testing against a smooth $\phi \in \mathfrak{X}_{C^\infty}$ and working in differential notation yields

$$\begin{aligned}
\langle \mathrm{d}m - \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta} a} \diamond a\mathrm{d}t, \phi\rangle_{\mathfrak{X}} &= \langle (\mathrm{d}\lambda - \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta} a}\mathrm{d}t) \diamond a + \lambda \diamond \mathrm{d}a, \phi\rangle_{\mathfrak{X}} \\
&= \langle \pounds_u^* \lambda \diamond a, \phi\rangle_{\mathfrak{X}} \mathrm{d}t + \langle \pounds_\xi^* \lambda \diamond a, \phi\rangle_{\mathfrak{X}} \mathrm{d}\mathbf{Z} - \langle \lambda \diamond \pounds_u a, \phi\rangle_{\mathfrak{X}} \mathrm{d}t - \langle \lambda \diamond \pounds_\xi a, \phi\rangle_{\mathfrak{X}} \mathrm{d}\mathbf{Z} \\
&\qquad \text{(by eqn. for } \lambda \ \& \ a) \\
&= -\langle \pounds_u^* \lambda, \pounds_\phi a\rangle_{\mathfrak{A}} \mathrm{d}t - \langle \pounds_\xi^* \lambda, \pounds_\phi a\rangle_{\mathfrak{A}} \mathrm{d}\mathbf{Z} + \langle \lambda, \pounds_\phi \pounds_u a\rangle_{\mathfrak{A}} \mathrm{d}t + \langle \lambda, \pounds_\phi \pounds_\xi a\rangle_{\mathfrak{A}} \mathrm{d}\mathbf{Z} \\
&\qquad \text{(by def. of } \diamond) \\
&= -\langle \lambda, (\pounds_u \pounds_\phi - \pounds_\phi \pounds_u) a\rangle_{\mathfrak{A}} \mathrm{d}t - \langle \lambda, (\pounds_\xi \pounds_\phi - \pounds_\phi \pounds_\xi) a\rangle_{\mathfrak{A}} \mathrm{d}\mathbf{Z} \\
&= -\langle \lambda, \pounds_{[u,\phi]} a\rangle_{\mathfrak{A}} \mathrm{d}t - \langle \lambda, \pounds_{[\xi,\phi]} a\rangle_{\mathfrak{A}} \mathrm{d}\mathbf{Z} \qquad \text{(by prop. of Lie derivative)}
\end{aligned}$$

$$= \langle \lambda \diamond a, [u, \phi] \rangle_{\mathfrak{X}} \mathrm{d}t + \langle \lambda \diamond a, [\xi, \phi] \rangle_{\mathfrak{X}} \mathrm{d}\mathbf{Z} \qquad \text{(by def. of } \diamond)$$
$$= -\langle \lambda \diamond a, \mathrm{ad}_u \, \phi \rangle_{\mathfrak{X}} \mathrm{d}t - \langle \lambda \diamond a, \mathrm{ad}_\xi \, \phi \rangle_{\mathfrak{X}} \mathrm{d}\mathbf{Z} \qquad \text{(by def. of } \mathrm{ad}_u)$$
$$= -\langle \pounds_u m, \phi \rangle_{\mathfrak{X}} \mathrm{d}t - \langle \pounds_\xi m, \phi \rangle_{\mathfrak{X}} \mathrm{d}\mathbf{Z} \qquad \text{(b/c } \mathrm{ad}^*_v \, m = \pounds_v m).$$

Consequently,

$$\mathrm{d}m_t + \pounds_{u_t} m_t \mathrm{d}t + \pounds_\xi m_t \mathrm{d}\mathbf{Z}_t \overset{\mathfrak{X}_\vee}{=} \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta}a} \diamond a_t \mathrm{d}t.$$

The converse can be obtained by reversing the above proof. □

5.2. Proof of the rough Lie chain rule Theorem 3.3

Proof. We will prove the statement using the following steps.

1. We prove the formula for scalar functions by working in a local chart;
2. We prove the formula for vectors by reducing to step 1 and using the product formula;
3. We prove the formula for one-forms from steps 1 and 2 and the product formula;
4. Using steps 3 and 4, we apply an induction argument to prove the general formula.

For simplicity, we will drop the dt-terms and time-dependence on ξ, and assume $s = 0$. That is, we consider the C^∞-flow $\eta \in C^\alpha_T(\mathrm{Diff}_{C^\infty})$ satisfying

$$d\eta_t X = \xi(\eta_t X) \mathrm{d}\mathbf{Z}_t, \quad \eta_0 X = X \in M.$$

Since we are working in C^∞, we will simply write $\boldsymbol{\mathcal{D}}_{\mathbf{Z}}$ for the controlled spaces.

Step 1. Assume that $f \in C^\alpha_T(C^\infty)$ has the decomposition

$$f_t = f_0 + \int_0^t \pi_r \mathrm{d}\mathbf{Z}_r, \quad t \in [0, T].$$

We aim to show that

$$\eta_t^* f_t = f_0 + \int_0^t \eta_r^* \left(\pi_r + \xi[f_r] \right) \mathrm{d}\mathbf{Z}_r \tag{5.1}$$

and

$$\eta_{t*} f_t = f_0 + \int_0^t \left(\eta_{r*} \pi_r - \xi[\eta_{r*} f_r] \right) \mathrm{d}\mathbf{Z}_r,$$

where both integrals are understood in the sense of controlled calculus. We focus only on the pull-back formula (5.1) as the equation for the push-forward can be shown in a similar way.

Towards this end, let us fix a coordinate chart (U, x). Let $\rho := \phi \circ \eta$, $\Xi^i := \xi[\phi^i]$, $F_t = f_t \circ \phi^{-1}$ and $b(t, \cdot) = \pi_t(\phi^{-1}(\cdot))$. We will now show that

$$F_t(\rho_t) = F_0 + \int_0^t \left(\Xi^i(\rho_r)\partial_{x^i} F_r(\rho_r) + b(r, \rho_r)\right) \mathrm{d}\mathbf{Z}_r, \tag{5.2}$$

which is (5.1) written in local coordinates. Since (U, ϕ) was arbitrary, proving (5.2) completes step 1.

To see this, it will be convenient to spell out the expansion in terms of scalars. We identify $\Xi(\cdot)$ as an operator on $\mathcal{L}(\mathbb{R}^K, \mathbb{R}^d)$ acting on Z with $\Xi_k(\cdot)\delta Z^k_{st}$ and write the Davie's expansions of ρ and F:

$$\delta\rho_{st} = \Xi_k(\rho_s)\delta Z^k_{st} + \partial_{x^i}\Xi_k(\rho_s)\Xi^i_l(\rho_s)\mathbb{Z}^{lk}_{st} + \rho^\natural_{st} \tag{5.3}$$

and

$$\delta F_{st}(\cdot) = b_k(s, \cdot)\delta Z^k_{st} + b'_{k,l}(s, \cdot)\mathbb{Z}^{lk}_{st} + b^\natural_{st}(\cdot), \tag{5.4}$$

where $|\rho^\natural_{st}| \lesssim |t-s|^{3\alpha}$ and $|b^\natural_{st}|_{C^\infty} \lesssim |t-s|^{3\alpha}$.

To prove (5.2), we Taylor expand F_s, $b_k(s, \cdot)$, and $b'_{k,l}(s, \cdot)$, and use (5.4) to write

$$\begin{aligned}
F_t(\rho_t) - F_s(\rho_s) &= F_t(\rho_t) - F_s(\rho_t) + F_s(\rho_t) - F_s(\rho_s) \\
&= b_k(s, \rho_t)\delta Z^k_{st} + b'_{k,l}(s, \rho_t)\mathbb{Z}^{lk}_{st} + b^\natural_{st}(\rho_t) + \partial_{x^i}F_s(\rho_s)\delta\rho^i_{st} \\
&\quad + \frac{1}{2}\partial_{x^j}\partial_{x^i}F_s(\rho_s)\delta\rho^i_{st}\delta\rho^j_{st} + o(|\delta\rho_{st}|^3) \\
&= (b_k(s, \rho_s) + \partial_{x^i}b_k(s, \rho_s)\delta\rho^i_{st} + o(|\delta\rho_{st}|^2))\delta Z^k_{st} \\
&\quad + (b'_{k,l}(s, \rho_s) + o(|\delta\rho_{st}|))\mathbb{Z}^{lk}_{st} + b^\natural_{st}(\rho_t) \\
&+ \partial_{x^i}F_s(\rho_s)\Xi^i_k(\rho_s)\delta Z^k_{st} + \partial_{x^i}F_s(\rho_s)\partial_{x^j}\Xi^i_k(\rho_s)\Xi^j_l(\rho_s)\mathbb{Z}^{lk}_{st} + \partial_{x^i}F_s(\rho_s)\rho^{i,\natural}_{st} \\
&+ \frac{1}{2}\partial_{x^j}\partial_{x^i}F_s(\rho_s)\Xi^i_l\Xi^j_k\delta Z^k_{st}\delta Z^l_{st} + o(|\delta\rho_{st}|^3).
\end{aligned}$$

Since $\mathbf{Z}$ is geometric (i.e., $\mathbb{Z}^{lk}_{st} + \mathbb{Z}^{kl}_{st} = Z^l_{st}Z^k_{st}$), plugging in the expansion (5.3), we find

$$\begin{aligned}
F_t(\rho_t) - F_s(\rho_s) &= \left(b_k(s, \rho_s) + \partial_{x^i}F_s(\rho_s)\Xi^i_k(\rho_s)\right)\delta Z^k_{st} \\
&\quad + \left(\partial_{x^i}b_k(s, \rho_s)\Xi^i_l(\rho_s) + \partial_{x^i}b_l(s, \rho_s)\Xi^i_k(\rho_s)\right)\mathbb{Z}^{lk}_{st} \\
&\quad + \left(\partial_{x^j}\partial_{x^i}F_s(\rho_s)\Xi^i_l\Xi^j_k + \partial_{x^i}F_s(\rho_s)\partial_{x^j}\Xi^i_k(\rho_s)\Xi^j_l(\rho_s)\right)\mathbb{Z}^{lk}_{st} + o(|t-s|^{3\alpha}).
\end{aligned}$$

Straightforward, but tedious, computations show that the local expansion

$$\begin{aligned}\Psi_{st} := &\left(b_k(s,\rho_s) + \partial_{x^i} F_s(\rho_s)\Xi^i_k(\rho_s)\right)\delta Z^k_{st} \\ &+ \Big(\partial_{x^i} b_k(s,\rho_s)\Xi^i_l(\rho_s) + \partial_{x^i} b_l(s,\rho_s)\Xi^i_k(\rho_s) + \partial_{x^j}\partial_{x^i} F_s(\rho_s)\Xi^i_l\Xi^j_k \\ &+ \partial_{x^i} F_s(\rho_s)\partial_{x^j}\Xi^i_k(\rho_s)\Xi^j_l(\rho_s)\Big)\mathbb{Z}^{lk}_{st}\end{aligned}$$

satisfies $|\delta_2\Psi_{s\theta t}| \lesssim |t-s|^{3\alpha}$. By the uniqueness in Lemma A.1, we get (5.2).

Step 2. Let $V \in C^\alpha_T(\mathfrak{X}_{C^\infty})$ be such that

$$V_t = V_0 + \int_0^t \pi_r \mathrm{d}\mathbf{Z}_r, \quad t \in [0,T],$$

for some $(\pi,\pi') \in \mathcal{D}_Z$. For any $f \in C^\infty$, we have

$$(\eta^*_t V_t)[f] = (\eta^*_t V_t)[\eta^*_t \eta_{t*} f] = \eta^*_t (V_t[\eta_{t*}]).$$

Recall that $\eta_t = \eta_{t*} \in C^\alpha_T(C^\infty)$ satisfies (3.2) with $s=0$. Applying Lemma A.13 with the continuous bilinear map $B : \mathfrak{X}_{C^\infty} \times C^\infty \to C^\infty$, we find

$$V_t[\eta_t] = V_0[f] + \int_0^t (\pi_r[g_r] - V_r[\xi[g_r]])\,\mathrm{d}\mathbf{Z}_r.$$

Moreover, making use of step 1 with $f_t = V_t[\eta_t] \in C^\infty$, we obtain

$$\begin{aligned}(\eta^*_t V_t)[f] = \eta^*_t(V_t[\eta_t]) &= V_0[f] + \int_0^t \eta^*_r\left(\pi_r[g_r] + \xi[V_r[g_r]] - V_r[\xi[g_r]]\right)\mathrm{d}\mathbf{Z}_r \\ &= V_0[f] + \int_0^t \eta^*_r\left(\pi_r[g_r] + [\xi,V_r][g_r]\right)\mathrm{d}\mathbf{Z}_r \\ &= V_0[f] + \int_0^t \left(\eta^*_r\left(\pi_r + [\xi,V_r]\right)\right)[f]\mathrm{d}\mathbf{Z}_r.\end{aligned}$$

Because f was arbitrarily chosen, we conclude that $(\eta^* V, \eta^* \pounds_\xi V + \eta^*\pi) \in \boldsymbol{\mathcal{D}}_{\mathbf{Z}}$ and

$$\eta^*_t V_t = V_0 + \int_0^t \eta^*_r\left(\pi_r + \pounds_\xi V_r\right)\mathrm{d}\mathbf{Z}_r.$$

Noting that

$$(\eta_{t*}V_t)[f] = (\eta_{t*}V_t)[\eta_{t*}\eta_t^* f] = \eta_{t*}(V_t[\eta_t^* f]),$$

and following a similar proof, we find that $(\eta_* V, -\pounds_\xi \eta_* V + \eta_* \pi) \in \boldsymbol{\mathcal{D}}_{\mathbf{Z}}$ and

$$\eta_{t*}V_t = V_0 + \int_0^t (\eta_{r*}\pi_r - \pounds_\xi(\eta_{r*}V_r))\, \mathrm{d}\mathbf{Z}_r. \tag{5.5}$$

Step 3. Assume that $\alpha \in C_T^\alpha(\Omega^1_{C^\infty})$ has the decomposition

$$\alpha_t = \alpha_0 + \int_0^t \pi_r \mathrm{d}\mathbf{Z}_r.$$

Fix an arbitrary vector $V \in \mathfrak{X}_{C^\infty}$ independent of t. Using (5.5) and Lemma A.13, we get

$$\alpha_t(\eta_{t*}V) = \alpha_0(V) + \int_0^t (\pi_r(\eta_{r*}V) - \alpha_r([\xi, \eta_{r*}V]))\, \mathrm{d}\mathbf{Z}_r.$$

Applying (5.1), we obtain

$$\begin{aligned}(\eta_t^*\alpha_t)(V) &= \eta_t^*\left(\alpha_t(\eta_{t*}V)\right) \\ &= \alpha_0(V) + \int_0^t \eta_r^*\left(\pi_r(\eta_{r*}V) - \alpha_r([\xi, \eta_{r*}V]) + [\xi(\alpha_r(\eta_{r*}V)]\right) \mathrm{d}\mathbf{Z}_r.\end{aligned}$$

The derivation property of the Lie derivative implies

$$\eta_t^*\xi[\alpha_t(\eta_{t*}V)] = (\eta_t^*(\pounds_\xi \alpha_t))(V) + \eta_t^*\left(\alpha_t([\xi, \eta_{t*}V])\right).$$

Noting that $\eta_t^*\left(\pi_t(\eta_{t*}V)\right) = (\eta_t^*\pi_t)(V)$ and that V was arbitrary, we obtain

$$\eta_t^*\alpha_t = \alpha_0 + \int_0^t \eta_r^*\left(\pi_r + \pounds_\xi \alpha_r\right) \mathrm{d}\mathbf{Z}_r.$$

Following a similar argument, we get

$$\eta_{t*}\alpha_t = \alpha_0 + \int_0^t \eta_{r*}\pi_r - \pounds_\xi\left(\eta_{r*}\alpha_r\right) \mathrm{d}\mathbf{Z}_r,$$

which completes step 3.

Step 4. Let us show how to extend to $\mathcal{T}^{lk}_{C^\infty}$. Let $V_1, \ldots, V_k \in \mathfrak{X}_{C^\infty}$, and $\alpha_1, \ldots, \alpha_l \in \Omega^1_{C^\infty}$. We recall that

$$(\eta_t^* \tau_t)(\alpha_1, \ldots, \alpha_l, X_1, \ldots, X_k) = \eta_t^* \big(\tau_t(\eta_{t*}\alpha_1, \ldots, \eta_{t*}\alpha_l, \eta_{t*}X_1, \ldots, \eta_{t*}X_k)\big).$$

Using induction in the product formula, we obtain

$$\begin{aligned}
\tau_t(\eta_{t*}\alpha_1, \ldots, \eta_{t*}\alpha_l, \eta_{t*}V_1, \ldots, \eta_{t*}V_k) &= \tau_0(\alpha_1, \ldots, \alpha_l, V_1, \ldots, V_k) \\
&+ \int_0^t \big[\gamma_r(\eta_{r*}\alpha_1, \ldots, \eta_{r*}\alpha_l, \eta_{r*}V_1, \ldots, \eta_{r*}V_k) \\
&- \sum_{j=1}^{l} \tau_r(\eta_{r*}\alpha_1, \ldots, \pounds_\xi \eta_{r*}\alpha_j, \ldots, \eta_{r*}\alpha_l, \eta_{r*}V_1, \ldots, \eta_{r*}V_k) \\
&- \sum_{j=1}^{k} \tau_r(\eta_{r*}\alpha_1, \ldots, \eta_{r*}\alpha_l, \eta_{r*}V_1, \ldots, \pounds_\xi \eta_{r*}V_j, \ldots, \eta_{r*}V_k)\big] d\mathbf{Z}_r,
\end{aligned}$$

which from (5.1) yields

$$\begin{aligned}
\eta_t^* \tau_t(\alpha_1, \ldots, \alpha_l, V_1, \ldots, V_k) &= \tau_0(\alpha_1, \ldots, \alpha_l, V_1, \ldots, V_k) \\
&+ \int_0^t \big[\eta_r^*(\gamma_r(\eta_{r*}\alpha_1, \ldots, \eta_{r*}\alpha_l, \eta_{r*}V_1, \ldots, \eta_{r*}V_k)) \\
&- \sum_{j=1}^{k} \tau_r(\eta_{r*}\alpha_1, \ldots, \pounds_\xi \eta_{r*}\alpha_j, \ldots, \eta_{r*}\alpha_l, \eta_{r*}V_1, \ldots, \eta_{r*}V_k) \\
&- \sum_{j=1}^{k} \tau_r(\eta_{r*}\alpha_1, \ldots, \eta_{r*}\alpha_l, \eta_{r*}V_1, \ldots, \pounds_\xi \eta_{r*}V_j, \ldots, \eta_{r*}V_k)\big] d\mathbf{Z}_r \\
&+ \int_0^t \eta_r^*(\xi[\tau_r(\eta_{r*}V_1, \ldots, \eta_{r*}V_k)]) \mathrm{d}\mathbf{Z}_r.
\end{aligned}$$

By the derivation property of the Lie derivative, we get

$$\begin{aligned}
&\xi[\tau_r(\eta_{r*}\alpha_1, \ldots, \eta_{r*}\alpha_l, \eta_{r*}V_1, \ldots, \eta_{r*}V_k)] \\
&\quad = (\pounds_\xi \tau_r)(\eta_{r*}\alpha_1, \ldots, \eta_{r*}\alpha_l, \eta_{r*}V_1, \ldots, \eta_{r*}V_k) \\
&\qquad + \sum_{j=1}^{l} \tau_r(\eta_{r*}\alpha_1, \ldots, \pounds_\xi \eta_{r*}\alpha_j, \ldots, \eta_{r*}\alpha_l, \eta_{r*}V_1, \ldots, \eta_{r*}V_k) \\
&\qquad + \sum_{j=1}^{k} \tau_r(\eta_{r*}\alpha_1, \ldots, \eta_{r*}\alpha_l, \eta_{r*}V_1, \ldots, \pounds_\xi \eta_{r*}V_j, \ldots, \eta_{r*}V_k),
\end{aligned}$$

and hence

$$\begin{aligned}
&\eta_t^*\tau_t(\alpha_1,\dots,\alpha_l,V_1,\dots,V_k)\\
&\quad=\tau_0(V_1,\dots,V_k)+\int_0^t\big[\eta_r^*\gamma_r(\eta_{r*}\alpha_1,\dots,\eta_{r*}\alpha_l,\eta_{r*}V_1,\dots,\eta_{r*}V_k)\\
&\qquad+\eta_r^*(\pounds_\xi\tau_r)(\eta_{r*}\alpha_1,\dots,\eta_{r*}\alpha_l,\eta_{r*}V_1,\dots,\eta_{r*}V_k)\big]d\mathbf{Z}_r\\
&\quad=\tau_0(\alpha_1,\dots,\alpha_l,V_1,\dots,V_k)\\
&\qquad+\int_0^t(\eta_r^*\gamma_r)(\alpha_1,\dots,\alpha_l,V_1,\dots,V_k)+(\eta_r^*\pounds_\xi\tau_r)(\alpha_1,\dots,\alpha_l,V_1,\dots,V_k)d\mathbf{Z}_r.
\end{aligned}$$

Since $\alpha_1,\dots,\alpha_l$ and $V_1,\dots,V_k$ were arbitrary, the result follows. □

5.3. Proof of the rough Kelvin–Noether Theorem 3.6

Proof. Let $\mu\in\mathrm{Dens}_{C^\infty}$ be an arbitrary non-vanishing density and set $\rho=\frac{dD}{d\mu}\in C^\infty$ so that $D=\rho\mu$. Recall that for all $w\in\mathfrak{X}_{C^\infty}$, $\pounds_w D=(\pounds_w\rho+\mathrm{div}_\mu w)\mu$. It follows that for all $t\in[0,T]$,

$$\rho_t=\rho_0-\int_0^t(\pounds_{u_r}\rho_r+\rho_r\,\mathrm{div}_\mu u_r)\mathrm{d}r-\int_0^t(\pounds_\xi\rho_r+\rho_r\,\mathrm{div}_\mu\xi)\mathrm{d}\mathbf{Z}_r.$$

Using the Lemma A.13 and the identity $\pounds_w\frac{1}{\rho}=-\frac{1}{\rho^2}\pounds_w\rho$, $w\in\mathfrak{X}_{C^\infty}$, we find

$$\frac{1}{\rho_t}=\frac{1}{\rho_0}+\int_0^t\left(-\pounds_{u_r}\frac{1}{\rho_r}+\frac{1}{\rho_r}\mathrm{div}_\mu u_r\right)\mathrm{d}r+\int_0^t\left(-\pounds_\xi\frac{1}{\rho_r}+\frac{1}{\rho_r}\mathrm{div}_\mu\xi\right)\mathrm{d}\mathbf{Z}_r.$$

For all $m=\alpha\otimes\nu\in\mathfrak{X}^\vee_{C^\infty}$ and $w\in\mathfrak{X}_{C^\infty}$, we have

$$\begin{aligned}
\pounds_w m=\pounds_w\left(\alpha\frac{d\nu}{d\mu}\otimes\mu\right)&=\left(\pounds_w\left(\alpha_i\frac{d\nu}{d\mu}\right)+(\mathrm{div}_\mu w)\alpha_i\frac{d\nu}{d\mu}\right)\otimes\mu\\
&=\pounds_w\left(\frac{m}{\mu}\right)+(\mathrm{div}_\mu w)\frac{m}{\mu}.
\end{aligned}$$

Therefore,

$$\frac{m_t}{\mu}=\frac{m_0}{\mu}+\int_0^t\left(\frac{1}{\mu}\frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta}a}(u_r,a_r)\diamond a_t-\pounds_{u_r}\left(\frac{m_r}{\mu}\right)-(\mathrm{div}_\mu u_r)\frac{m_r}{\mu}\right)\mathrm{d}r$$

$$
- \int_s^t \left(\pounds_\xi \left(\frac{m_r}{\mu} \right) + (\operatorname{div}_\mu \xi) \frac{m_r}{\mu} \right) \mathrm{d}\mathbf{Z}_r.
$$

Applying the Lemma A.13 and the identity $\frac{m}{D_t} = \frac{1}{\frac{dD_t}{d\nu}} \frac{m}{\mu} = \frac{1}{\rho_t} \frac{m}{\mu}$, we arrive at

$$
\frac{m_t}{D_t} = \frac{m_s}{D_s} + \int_s^t \frac{1}{D_r} \left(\frac{\boldsymbol{\delta} \ell}{\boldsymbol{\delta} a}(u_t, a_t) \diamond a_t - \pounds_{u_r} \left(\frac{m_r}{D_r} \right) \right) \mathrm{d}r - \int_s^t \frac{1}{D_r} \pounds_\xi \left(\frac{m_r}{D_r} \right) \mathrm{d}\mathbf{Z}_r.
$$

We then complete the proof by applying Corollary 3.5 with $\alpha = m/D$. □

5.4. Proof of the rough Hamilton–Pontryagin Theorem 3.12

Proof. If $(u, g, \boldsymbol{\lambda}) \in HP_{\mathbf{Z}}$ is a critical point of the action functional, then

$$
0 = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} S_{a_0}^{HP_{\mathbf{Z}}}(u^\epsilon, \eta^\epsilon, \boldsymbol{\lambda}^\epsilon) = I(\boldsymbol{\delta} u) + II(\boldsymbol{\delta} w) + III(\boldsymbol{\delta} \lambda),
$$

where

$$
\begin{aligned}
I(\boldsymbol{\delta} u) &= \int_0^T \left\langle \frac{\boldsymbol{\delta} \ell}{\boldsymbol{\delta} u}(u_t, a_t) - \lambda_t, \boldsymbol{\delta} u_t \right\rangle_{\mathfrak{X}} \mathrm{d}t \\
II(\boldsymbol{\delta} w) &= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \int_0^T \ell(u_t, (\eta_t^\epsilon)_* a_0) \mathrm{d}t + \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \int_0^T \langle \lambda_t, \mathrm{d}\eta_t^{\epsilon,-1} \eta_t^\epsilon \rangle_{\mathfrak{X}} \\
III(\boldsymbol{\delta} \lambda) &= \int_0^T \langle \boldsymbol{\delta} \lambda_t, \mathrm{d}\eta_t \circ \eta_t^{-1} \rangle_{\mathfrak{X}} - \int_0^T \langle \boldsymbol{\delta} \lambda_t, u_t \rangle_{\mathfrak{X}} \mathrm{d}t - \int_0^T \langle \boldsymbol{\delta} \lambda_t, \xi \rangle_{\mathfrak{X}} \mathrm{d}\mathbf{Z}_t.
\end{aligned}
$$

By virtue of the fundamental lemma of calculus of variations, $I(\boldsymbol{\delta} u) = 0$ implies $m = \frac{\boldsymbol{\delta} \ell}{\boldsymbol{\delta} u}(u, a) \overset{\mathfrak{X}^\vee}{=} \lambda$. Separating variations in time and space and applying Theorem A.15, from $III(\boldsymbol{\delta} \lambda) = 0$, deduce that $v \equiv u$ and $\sigma \equiv \xi$.

We now focus on $II(\boldsymbol{\delta} w) = 0$. By the equality of mixed derivatives (see, also, Lemma 3.1 in [4]), we have

$$
\frac{\partial^2 \psi_t^\epsilon}{\partial t \partial \epsilon} = \frac{\partial^2 \psi_t^\epsilon}{\partial \epsilon \partial t} = \partial_t \boldsymbol{\delta} w_t \circ \psi_t^\epsilon + \epsilon \frac{\partial}{\partial \epsilon} \left[\partial_t \boldsymbol{\delta} w_t \circ \psi_t^\epsilon \right], \quad \forall (\epsilon, t) \in [-1, 1] \times [0, T].
$$

Using the above relation and that $\psi_t^0 X = X$, we find $\frac{\partial \psi^\epsilon}{\partial \epsilon}\big|_{\epsilon=0} = \boldsymbol{\delta} w$, and hence

$$
\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} v_t^\epsilon = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} (\psi_t^\epsilon)_* v = -[\boldsymbol{\delta} w_t, v_t] = \operatorname{ad}_{\boldsymbol{\delta} w_t} v_t.
$$

Therefore,

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0}\int_0^T \langle \lambda_t, \mathrm{d}\eta_t^{\epsilon,-1}\eta_t^\epsilon\rangle_{\mathfrak{X}} = \int_0^T \langle \lambda_t, \partial_t \boldsymbol{\delta} w_t + \mathrm{ad}_{\boldsymbol{\delta} w_t}\, v_t\rangle_{\mathfrak{X}}\mathrm{d}t + \int_0^T \langle \lambda_t, \mathrm{ad}_{\boldsymbol{\delta} w_t}\, \sigma_t\rangle_{\mathfrak{X}}\mathrm{d}\mathbf{Z}_t,$$

where we have exchanged the order of $\frac{d}{d\epsilon}$ and the rough integral using Theorem A.8. Moreover,

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0}(\eta_t^\epsilon)_* a_0 = \frac{d}{d\epsilon}\bigg|_{\epsilon=0}(\psi_t^\epsilon)_* a_t = -\pounds_{\boldsymbol{\delta} w_t} a_t, \quad \forall t \in [0,T],$$

which implies that

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0}\int_0^T \ell(u_t, (\eta_t^\epsilon)_* a_0)\mathrm{d}t = \int_0^T \langle \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta} a}(u_t,a_t), -\pounds_{\boldsymbol{\delta} w_t} a_t\rangle_{\mathfrak{A}}\mathrm{d}t = \int_0^T \langle \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta} a}(u_t,a_t)\diamond a_t, \boldsymbol{\delta} w_t\rangle_{\mathfrak{A}}\mathrm{d}t.$$

The proof is completed by splitting the variations of $\boldsymbol{\delta} w$ in space and time and applying Lemma B.4. □

5.5. *Proof of the rough Euler–Poincaré Theorem 3.15*

Proof. Using the definitions of $\boldsymbol{\delta} u$ and $\boldsymbol{\delta} a$ in (3.8), integrating by parts, and taking the endpoint conditions $w_0 = w_T = 0$ into account, we find

$$\begin{aligned}
\delta S^{EP\mathbf{z}} &= \int_0^T \langle \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta} u_t}(u_t,a_t)\,, \boldsymbol{\delta} u_t\rangle_{\mathfrak{X}} dt + \langle \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta} a_t}(u_t,a_t), \boldsymbol{\delta} a_t\rangle_{\mathfrak{A}} \\
&= \int_0^T \langle \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta} u_t}(u_t,a_t)\,, \partial_t \boldsymbol{\delta} w\rangle_{\mathfrak{X}}\mathrm{d}t + \langle \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta} u_t}(u_t,a_t)\,, \mathrm{ad}_{\mathrm{d}x_t}\, \boldsymbol{\delta} w\rangle_{\mathfrak{X}} \\
&\quad - \langle \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta} a_t}(u_t,a_t)\diamond a_t\,, \boldsymbol{\delta} w\rangle_{\mathfrak{X}}\mathrm{d}t \\
&= \int_0^T \langle -\mathrm{d}\left(\frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta} u_t}\right) - \mathrm{ad}^*_{\mathrm{d}x_t}\frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta} u_t}\,, \boldsymbol{\delta} w\rangle_{\mathfrak{X}} + \langle \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta} a_t}\diamond a_t\,, \boldsymbol{\delta} w\rangle_{\mathfrak{X}}\, dt \\
&= \int_0^T \langle -\mathrm{d}\left(\frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta} u_t}\right) - \pounds^*_{\mathrm{d}x_t}\frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta} u_t} + \frac{\boldsymbol{\delta}\ell}{\boldsymbol{\delta} a_t}\diamond a_t\, dt\,, \boldsymbol{\delta} w\rangle_{\mathfrak{X}}.
\end{aligned}$$

We conclude with the corresponding momentum equation by splitting up variations in space and time and applying Lemma B.4. In addition, the advection equation $\mathrm{d}a_t +$

$\pounds_{\mathrm{d}x_t} a_t = 0$ follows from the push-forward relation $a_t = (\eta_t)_* a_0$ by the Lie chain rule in Theorem 3.3. These two results complete the proof of Theorem 3.15. □

Appendix A. Notation and required background

A.1. Geometric rough paths

In this section, we will provide an overview of the theory of geometric rough paths. We invite the reader to consult Appendix D for a historical account motivating the use of rough paths and [90,60,59,8] for more thorough expositions.

Let $T > 0$, $\Delta_T^2 = \{(s,t) \in [0,T]^2 : s \leq t\}$ and $\Delta_T^3 = \{(s,\theta,t) \in [0,T]^3 : s \leq \theta \leq t\}$. Let E denote an arbitrary Fréchet space E with family of seminorms $\boldsymbol{\mathcal{P}}$. Elements of family of seminorms $\boldsymbol{\mathcal{P}}$ will be denoted by p. For a given $\alpha \in (0,1]$, let $C_T^\alpha(E)$ denote the space of Hölder continuous paths; in particular, $C_T^1(E)$ is the space of Lipschitz paths. Moreover, for a given $m \in \{2,3\}$[5] and $\alpha \in \mathbb{R}_+$, denote by $C_{m,T}^\alpha(E)$ the space of functions that satisfy

$$[\Xi]_{\alpha,p} = \sup_{\substack{(t_1,\cdots,t_m)\in\Delta_T^m \\ t_1\neq t_m}} \frac{p\left(\Xi_{t_1,\ldots,t_m}\right)}{|t_m - t_1|^\alpha} < \infty, \quad p \in \boldsymbol{\mathcal{P}}.$$

Define $\delta : C_T^\alpha(E) \to C_{2,T}^\alpha(E)$ by $\delta f =_{st} := f_t - f_s$ for $f \in C_T^\alpha(E)$ and $\delta_2 : C_{2,T}^\alpha(E) \to C_{3,T}^\alpha(E)$ by

$$\delta_2 \Xi_{s\theta t} := \Xi_{st} - \Xi_{s\theta} - \Xi_{\theta t}, \quad (s,\theta,t) \in \Delta_T^3, \quad \Xi \in C_{2,T}^\alpha(E).$$

It follows that $\delta_2 \circ \delta : C_T(E) \to C_{3,T}(E)$ is the zero operator.

For a given $\Xi \in C_{2,T}^\alpha(E)$, $\beta \in \mathbb{R}_+$, and $p \in \boldsymbol{\mathcal{P}}$, the quantity $[\delta_2\Xi]_{\beta,p}$, defined above, may be regarded as a measure of the extent to which Ξ is an increment δf for some $f \in C_T^\alpha(E)$. The following lemma, proved in [72][Proposition A.1], is referred to as the *sewing lemma*. The lemma says that if $\beta > 1$, one can construct a "unique" $f \in C_T^\alpha(E)$ such that Ξ is close to δf in $C_{2,T}^\beta(E)$ by (A.1).

Lemma A.1 *(Sewing Lemma). There exists a unique continuous linear map $\boldsymbol{\mathcal{I}}$: $C_{2,T}^{\alpha,\beta}(E) \to C_T^\alpha(E)$ satisfying $\boldsymbol{\mathcal{I}}\Xi_0 = 0$ and $[\delta\boldsymbol{\mathcal{I}}\Xi - \Xi]_\beta \lesssim_\beta [\delta_2\Xi]_{\beta,p}$ for all $\Xi \in C_{2,T}^{\alpha,\beta}(E)$ and $p \in \boldsymbol{\mathcal{P}}$. More explicitly, for a given $(s,t) \in \Delta_T^2$,*

$$\delta(\boldsymbol{\mathcal{I}}\Xi)_{st} = \lim_{|\mathcal{P}([s,t])|\to 0} \sum_{[t_i,t_{i+1}]\in\mathcal{P}([s,t])} \Xi_{t_i t_{i+1}}, \tag{A.1}$$

[5] We only need $m = 2,3$ because we consider only rough paths with Hölder regularity $\alpha \in (\frac{1}{3}, \frac{1}{2}]$.

where $\mathcal{P}([s,t])$ denotes a finite partition of the interval $[s,t]$, $|\mathcal{P}([s,t])|$ denotes its mesh size, and the limit is understood as a limit of nets (with the directed set of partitions partially ordered by inclusion).

Remark A.2. Notice that if $\tilde{\Xi} \in C^{\alpha,\beta}_{2,T}(E)$ and $\Xi - \tilde{\Xi} \in C^{\beta}_{2,T}(E)$, then $\mathcal{I}(\Xi) = \mathcal{I}(\tilde{\Xi})$.

For a given Fréchet space E and $K \in \mathbb{N}$, let E^K denote the direct sum of E with itself K-times. By virtue of the Sewing Lemma, one can construct an integral of $Y \in C^{\beta}_T(E^K)$ against $Z \in C^{\alpha}_T(\mathbb{R}^K)$ if $\alpha + \beta > 1$ by letting $\Xi_{st} = Y_s \delta Z_{st} = \sum_{k=1}^K Y^k_s \delta Z^k_{st}$ for all $(s,t) \in \Delta^2_T$ and defining

$$\int_0^t Y_r \mathrm{d}Z_r = \mathcal{I}(\Xi)_t, \quad t \in [0,T].$$

This integral construction coincides with the integral that L.C. Young [121] constructed. In particular, for $Z \in C^{\alpha}_T(\mathbb{R}^K)$ with $\alpha \in \left(\frac{1}{2}, \infty\right)$, we may define $\mathbb{Z} \in C^2_{2,T}(\mathbb{R}^{K\times K})$ by

$$\mathbb{Z}_{st} = \int_s^t \int_s^{t_2} \mathrm{d}Z_{t_1} \otimes \mathrm{d}Z_{t_2} = \int_s^t \delta Z_{st_2} \otimes \mathrm{d}Z_{t_2}, \quad (s,t) \in \Delta_T,$$

where we have used the δ notation defined above in the second equality. One can easily verify that $(Z, \mathbb{Z}) \in C^{\alpha}_T(\mathbb{R}^K) \times C^{2\alpha}_T(\mathbb{R}^{K\times K})$ satisfies

$$\delta_2 \mathbb{Z}_{st} = \delta Z_{s\theta} \otimes \delta Z_{\theta t}, \quad \forall (s,\theta,t) \in \Delta^3_T, \tag{A.2}$$

and

$$\mathrm{Sym}(\mathbb{Z}_{st}) = \frac{1}{2} \delta Z_{st} \otimes \delta Z_{st} \quad \forall (s,t) \in \Delta^2_T. \tag{A.3}$$

The condition (A.3) is a geometric property which encodes the usual chain and product rules, upon which our variational theory is based. Paths $Z \in C^{\alpha}_T(\mathbb{R}^K)$ with $\alpha \in (\frac{1}{2}, 1]$ are referred to as Young paths. Young paths are distinguished from rough paths $\mathbf{Z} = (Z, \mathbb{Z}) \in C^{\alpha}_T(\mathbb{R}^K) \times C^{\alpha}_T(\mathbb{R}^{K\times K})$, $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, which are defined to be paths such that an a priori postulated two-parameter path $\mathbb{Z} \in C^{2\alpha}_T(\mathbb{R}^{K\times K})$ satisfies (A.2). A subclass of rough paths are the geometric rough paths, for which a classical calculus can be developed. In particular, (A.3) holds. For a few more words of motivation about rough paths see Appendix D.

Definition A.3. For a given $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$, define the set $\mathcal{C}^{\alpha}_{g,T}(\mathbb{R}^K)$ of *geometric K-dimensional α-Hölder rough paths on the interval* $[0, T]$ to be the closure of

$$\left\{(Z, \mathbb{Z}) \in C^1_T(\mathbb{R}^K) \oplus C^1_{2,T}(\mathbb{R}^{K\times K}) : \mathbb{Z} = \iint \mathrm{d}Z \otimes \mathrm{d}Z\right\}$$

in $C^{\alpha}_T(\mathbb{R}^K) \oplus C^{\alpha}_{2,T}(\mathbb{R}^{K\times K})$ with respect to the metric

$$\rho(\mathbf{Z}^{(1)}, \mathbf{Z}^{(2)}) = [Z^{(1)} - Z^{(2)}]_{\alpha} + [\mathbb{Z}^{(1)} - \mathbb{Z}^{(2)}]_{2\alpha}.$$

It follows that both (A.2) and (A.3) hold for all $\mathbf{Z} = (Z, \mathbb{Z}) \in \mathcal{C}^{\alpha}_{g,T}(\mathbb{R}^K)$ by a limiting argument. For a given $\alpha \in \left(\frac{1}{2}, 1\right]$, we denote $\mathcal{C}^{\alpha}_{g,T}(\mathbb{R}^K) = C^{\alpha}_T(\mathbb{R}^K)$.

Remark A.4. To have a uniform notation for all $\alpha \in \left(\frac{1}{3}, 1\right]$, we write $\mathbf{Z} = Z \in \mathcal{C}^{\alpha}_{g,T}(\mathbb{R}^K)$ if $\alpha \in \left(\frac{1}{2}, 1\right]$.

It is possible to consider infinite-dimensional geometric rough paths, but for simplicity we restrict ourselves to finite-dimensional paths. However, we consider controlled rough paths (defined in the next section) in Fréchet spaces. We also remark that our theory can be extended to more irregular paths $\alpha < \frac{1}{3}$, but this requires higher-order iterated integrals and more cumbersome notation.

There is a large class of Gaussian processes that belong to $\mathcal{C}^{\alpha}_{g,T}(\mathbb{R}^K)$ for $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$. We refer the reader to Appendix E for a slightly more in-depth discussion of Gaussian rough paths. The present discussion will be brief.

Example A.5 *(Stratonovich Brownian motion).* Consider a Brownian motion $B : \Omega \times [0, T] \to \mathbb{R}^K$ on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathbb{B}$ be the stochastic iterated integral constructed from Stratonovich integration theory. By virtue of the Kolmogorov continuity theorem, one can find a event $\tilde{\Omega} \in \mathcal{F}$ with $\mathbb{P}(\tilde{\Omega}) = 1$ such that $\mathbf{B}(\omega) = (B(\omega), \mathbb{B}(\omega) \in \mathcal{C}^{\alpha}_{g,T}(\mathbb{R}^K)$ for any $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$. This the Stratonovich lift of Brownian motion. Indeed, the Stratonovich integral is a limit of integrals of piecewise-linear approximations of Brownian motion.

Example A.6 *(Gaussian rough paths).* More broadly, a Gaussian process $Z : \Omega \times [0, T] \to \mathbb{R}^K$ can be lifted to a geometric rough path $\mathbf{Z}(\omega) = (Z(\omega), \mathbb{Z}(\omega) \in \mathcal{C}^{\alpha}_{g,T}(\mathbb{R}^K)$ for $\alpha \in \left(\frac{1}{3}, \tilde{\alpha}\right]$, provided the correlation in time of the process is fast enough depending on $\tilde{\alpha}$ (see Appendix E). In particular, fractional Brownian motion $B^H : \Omega \times [0, T] \to \mathbb{R}^K$ can be lifted to a strong geometric rough path $\mathbf{B}^H(\omega) = (B^H(\omega), \mathbb{B}^H(\omega) \in \mathcal{C}^{\alpha}_{g,T}(\mathbb{R}^K)$ for all $\alpha \in \left(\frac{1}{3}, \frac{1}{4H}\right)$ for all ω in some event of probability one.

A.1.1. Controlled rough paths and integration

Let us first describe the integration theory for paths $\mathbf{Z} = (Z, \mathbb{Z}) \in \mathcal{C}^{\alpha}_{g,T}(\mathbb{R}^K)$ such that $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$.

Definition A.7 *(Controlled rough path).* We say a path $Y \in C^{\alpha}_T(E)$ is *controlled* by Z, if there exists a $Y' \in C^{\alpha}_T(E^K)$ such that $R^Y : \Delta^2_T \to W$ defined by

$$R^Y_{st} = \delta Y_{st} - Y'_s \delta Z_{st} = \delta Y_{st} - \sum_{k=1}^{K} Y'^{\,k}_s \delta Z^k_{st}, \quad (s,t) \in \Delta^2_T, \tag{A.4}$$

satisfies $R^Y \in C^{2\alpha}_{2,T}(E)$. For $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$, we define the linear space $\mathcal{D}_{Z,T}(E)$ of controlled rough paths to be those pairs $\mathbf{Y} = (Y, Y') \in C^{\alpha}_T(E) \oplus C^{\alpha}_T(E^K)$ such that $R^Y \in C^{2\alpha}_{2,T}(E)$. The function Y' is referred to as the *Gubinelli derivative* [65]. The space $\mathcal{D}_{Z,T}(E)$ is a Fréchet space with seminorms

$$|\mathbf{Y}|_{\mathbf{Z},p} = |Y_0|_p + |Y'_0|_p + [Y']_{\alpha,p} + [R^Y]_{2\alpha,p}, \quad p \in \mathcal{P}. \tag{A.5}$$

We note that any $Y \in C^{2\alpha}_T(E)$ satisfies (A.4) with $Y' \equiv 0$. Moreover, $\mathbf{Z}$ itself is controlled with $Z' \equiv \mathrm{id}$. However, the additional structure provided by Y' is natural in the context of rough differential equations (see Remark A.17). It is worth mentioning that Y' is not uniquely specified unless the path is truly rough (see Definition A.14 and Theorem A.15 below). The integration of controlled rough paths is an immediate consequence of Lemma A.1

Theorem A.8. *Let $\mathbf{Z} = (Z, \mathbb{Z}) \in \mathcal{C}^{\alpha}_{g,T}(\mathbb{R}^K)$ for a given $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$. There exists a linear continuous map $\mathbf{I}_{\mathbf{Z}} : \mathcal{D}_{Z,T}(E^K) \to \mathcal{D}_{Z,T}(E)$ defined by $\mathbf{I}_{\mathbf{Z}}(\mathbf{Y}) = (\mathcal{I}(\Xi), Y)$, where*

$$\Xi_{st} = Y_s \delta Z_{st} + Y'_s \mathbb{Z}_{st} = \sum_{k=1}^{K} Y^k_s \delta Z^k_{st} + \sum_{k=1}^{K} Y'^{\,kl}_s \mathbb{Z}^{lk}, \quad (s,t) \in \Delta^2_T,$$

and $\mathcal{I}$ is as in Lemma A.1. We write

$$\int_0^t Y_r \mathrm{d}\mathbf{Z}_r = \mathcal{I}(\Xi) \in C^{\alpha}_T(E), \quad t \in [0,T].$$

Remark A.9 *(Integral of controlled path against a controlled path).* Let F, G denote a Fréchet space and $B : F \times E \to G$ be continuous and bilinear. For $\mathbf{Y} = (Y, Y') \in \mathcal{D}_{Z,T}(E)$ and $\mathbf{X} = (X, X') \in \mathcal{D}_{Z,T}(F)$, we define

$$\int_s^t B(X_r, \mathrm{d}\mathbf{Y}_r) = (\delta \mathcal{I}\Xi)_{st}, \quad \text{where} \quad \Xi_{st} = B(X_s, \delta Y_{st}) + B(X'_s, Y'_s)\mathbb{Z}_{st}, \quad (s,t) \in \Delta^2_T. \tag{A.6}$$

Indeed, for all $(s,\theta,t) \in \Delta_T^3$,

$$\delta_2\Xi_{s\theta t} = -B(R^X_{s\theta}, R^Y_{\theta t}) - B(R^X_{s\theta}, Y'_\theta)\delta Z_{\theta t} - B(X'_s, R^Y_{\theta t})\delta Z_{s\theta} - B(X'_s, \delta Y'_{s\theta})\delta Z_{s\theta} \otimes \delta Z_{\theta t} \\ + (B(X'_s, Y'_s) - B(X'_\theta, Y'_\theta))\mathbb{Z}_{\theta t},$$

which implies in $\Xi \in C^{3\alpha}_{2,T}(G)$, so that we may apply Lemma A.1. Notice that if $Y \in C^{2\alpha}_T(E)$ and $Y' \equiv 0$, then (A.6) agrees with the Young integral. This definition is used in the Clebsch variational principle in order to define the integral of the Lagrange multiplier against an advected quantity (see Remark 2.7).

For Young paths, the extra structure provided by the Gubinelli derivatives is not needed.

Definition A.10 *(Controlled paths in the Young case).* We define $\boldsymbol{\mathcal{D}}_{Z,T}(E) = C^\alpha_T(E)$ if $\alpha \in \left(\frac{1}{2}, 1\right]$.

Remark A.11. To have a uniform notation for all $\alpha \in \left(\frac{1}{3}, 1\right]$, we write $\mathbf{Y} = Y \in \boldsymbol{\mathcal{D}}_{Z,T}(E)$ if $\alpha \in \left(\frac{1}{2}, 1\right]$. We also remark that obviously the controlled space does not depend on Z in this case.

A.1.2. The rough chain and product rule

Let E and F be Fréchet spaces and $C(E;F)$ denote the space of continuous maps. Let $C^1_b(E;F)$ denote the space of bounded functions $\Phi : E \to F$ such that the limit

$$D\Phi(e)h = \lim_{\epsilon\to 0} \frac{\Phi(e+th) - \Phi(e)}{t}$$

exists for all $e, h \in E$ and $D\Phi : E \times E \to F$ is continuous and bounded. We define $C^m_b(E;F)$ for $m \geq 2$ analogously (see [69][Def. 3.1.1 & Section I.3.6]). Let $N_\alpha = 0$ if $\alpha = 1$, $N_\alpha = 1$ if $\alpha \in \left(\frac{1}{2}, 1\right)$ and $N_\alpha = 2$ if $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$. The following lemma says that controlled rough paths are stable under composition and products. Their proof can be found in Lemma 7.3 and Corollary 7.4 of [59].

Lemma A.12.

(i) *If* $\mathbf{Y} = (Y, Y') \in \boldsymbol{\mathcal{D}}_{Z,T}(E)$ *and* $\Phi \in C^1_T(C^{N_\alpha}_b(E;F))$, *then* $\Phi(\mathbf{Y}) = (\phi(Y), D\phi(Y)Y') \in \boldsymbol{\mathcal{D}}_{Z,T}(F)$.

(ii) *Let* $B : F \times E \to G$ *be continuous and bilinear. If* $\mathbf{X} = (X, X') \in \boldsymbol{\mathcal{D}}_{Z,T}(F)$ *and* $\mathbf{Y} = (Y, Y') \in \boldsymbol{\mathcal{D}}_{Z,T}(E)$, *then* $B(\mathbf{X}, \mathbf{Y}) = (B(X,Y), B(X',Y) + B(X,Y')) \in \boldsymbol{\mathcal{D}}_{Z,T}(G)$.

In order to construct the integration theory given above, we have actually not needed the geometric nature of the path (i.e., (A.3)). However, to obtain an extension of the ordinary chain and product rule, we require (A.3) to hold (see [59][Section 7.5]).

Lemma A.13.

(i) For a given, $Y_0 \in E$, $\beta \in C_T(E)$, *and* $(\sigma, \sigma') \in \boldsymbol{\mathcal{D}}_{Z,T}(E^K)$, *let*

$$Y_t = Y_0 + \int_0^t \beta_r \mathrm{d}r + \int_0^t \sigma_r \mathrm{d}\mathbf{Z}_r, \quad t \in [0, T]. \tag{A.7}$$

If $\Phi \in C_T^1(C_b^{N_\alpha+1}(E; F))$, *then for all* $t \in [0, T]$

$$\Phi_t(Y_t) = \Phi_0(Y_0) + \int_0^t (\partial_t \Phi_r(Y_r) + D\Phi_r(Y_r))\, \beta_r \mathrm{d}r + \int_0^t D\Phi_r(Y_r)\sigma_r \mathrm{d}\mathbf{Z}_r.$$

(ii) For a given, $X_0 \in F$, $\tilde{\beta} \in C_T(F)$, *and* $(\tilde{\sigma}, \tilde{\sigma}') \in \boldsymbol{\mathcal{D}}_{Z,T}(F^K)$, *let*

$$X_t = X_0 + \int_0^t \tilde{\beta}_r \mathrm{d}r + \int_0^t \tilde{\sigma}_r \mathrm{d}\mathbf{Z}_r, \quad t \in [0, T],$$

and Y *be as specified in (i). Let* $B : F \times E \to G$ *be continuous and bilinear. Then for all* $t \in [0, T]$,

$$B(X_t, Y_t) = B(X_0, Y_0) + \int_0^t \big(B(\tilde{\beta}_r, Y_r) + B(X_r, \beta_r)\big)\, \mathrm{d}r$$

$$+ \int_0^t (B(\tilde{\sigma}_r, Y_r) + B(X_r, \sigma_r))\, \mathrm{d}\mathbf{Z}_r.$$

In Section 3.3, we need the decomposition of paths Y satisfying the relation (A.7) to be unique. A decomposition of a path Y satisfying (A.7) is unique if the rough path $\mathbf{Z}$ is *truly rough* (Theorem 6.5 of [59]). Examples of truly rough paths include fractional Brownian motion B^H with $H \in \left(\frac{1}{3}, \frac{1}{2}\right]$.

Definition A.14 *(Truly rough path).* Let $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$ and $\mathbf{Z} \in \boldsymbol{\mathcal{C}}^\alpha_{g,T}(\mathbb{R}^K)$. We say $\mathbf{Z}$ is *truly rough* if for all s in a dense set in $[0, T]$,

$$\limsup_{t \downarrow s} \frac{|\delta Z_{st}|}{|t - s|^{2\alpha}} = \infty.$$

Theorem A.15. *If* $\mathbf{Z}$ *is truly rough and*

$$Y_t = Y_0 + \int_0^t \beta_r \mathrm{d}r + \int_0^t \sigma_r \mathrm{d}\mathbf{Z}_r = \tilde{Y}_0 + \int_0^t \tilde{\beta}_r \mathrm{d}r + \int_0^t \tilde{\sigma}_r \mathrm{d}\mathbf{Z}_r, \quad \forall t \in [0, T],$$

where $Y_0, \tilde{Y}_0 \in E$, $\beta, \tilde{\beta} \in C_T(E)$, *and* $(\sigma, \sigma'), (\tilde{\sigma}, \tilde{\sigma}') \in \boldsymbol{\mathcal{D}}_{Z,T}(E^K)$, *then* $\beta \equiv \tilde{\beta}$ *and* $(\sigma, \sigma') \equiv (\tilde{\sigma}, \tilde{\sigma}')$.

A.1.3. Solutions of rough differential equations (RDEs)

We will now introduce the definition of solution to an RDE. Let V denote a Banach space.

Definition A.16. Let $u \in C_T(C_b(V;V))$ and $\xi \in C^1_T(C_b^{N_\alpha}(V;V)^K)$. We say Y is a solution of

$$\mathrm{d}Y_t = u_t(Y_t)\mathrm{d}t + \xi_t(Y_t)\mathrm{d}\mathbf{Z}_t, \quad Y_0 = v \in V, \tag{A.8}$$

on the interval $[0,T]$, if $\mathbf{Y} = (Y, \xi(Y)) \in \boldsymbol{\mathcal{D}}_{Z,T}(V)$ and

$$Y_t = v + \int_0^t u_r(Y_r)\mathrm{d}r + \int_0^t \xi_r(Y_r)\mathrm{d}\mathbf{Z}_r, \quad \forall t \in [0,T]. \tag{A.9}$$

Remark A.17. The rough integral in (A.9) is well-defined by virtue of Lemma A.12.

The following lemma concerns equivalent notions of solutions. Its proof is a direct application of Theorem A.8 and A.13. The first formulation is referred to as the Davie's formulation [45] and the second naturally extends to the manifold setting.

Lemma A.18. *Y is a solution of* (A.8) *on the interval* $[0,T]$ *if and only if*

(i) $Y^\natural_{st} := \delta Y_{st} - \int_s^t u_r(Y_r)\mathrm{d}r - \xi_s(Y_s)(\delta Z_{st}) - D\xi_s(Y_s)\xi_s(Y_s)(\mathbb{Z}_{st})$, $(s,t) \in \Delta^2_T$, *satisfies* $Y^\natural \in C^{3\alpha}_{2,T}$;
(ii) $f(Y_t) = f(v) + \int_0^t Df(Y_r)u_r(Y_r)\mathrm{d}r + \int_0^t Df(Y_r)\xi_r(Y_r)\mathrm{d}\mathbf{Z}_r$, $\forall t \in [0,T]$, $\forall f \in C_b^\infty(V;\mathbb{R})$.

The proof of existence and uniqueness for RDEs uses a Picard iteration argument in the controlled rough path topology (i.e., (A.5)). We refer the reader to, e.g., [59][Section 8.5] for a proof. Moreover, in Section B.1 we give more details about flows on Euclidean spaces.

Theorem A.19. *There exists a unique continuous solution map*

$$S : V \times C_T(C^1_b(V;V)) \times C^1_T(C_b^{N_\alpha+1}(V;V)^K) \times \boldsymbol{\mathcal{C}}_{g,T}(\mathbb{R}^K) \to \boldsymbol{\mathcal{D}}_{Z,T}(V)$$
$$(v, u, \xi, \mathbf{Z}) \mapsto (Y, \xi(Y)).$$

A.2. Differential geometry

A.2.1. Basic setting and the Lie derivative

Let M denote a smooth compact, connected, oriented d-dimensional manifold without boundary. For an arbitrarily given rank p vector bundle E over M, denote by $\Gamma_{C^\infty}(E)$ the space of smooth sections endowed with the Fréchet topology defined through a cover of total trivializations of E. Denote by $C^\infty = \Gamma_{C^\infty}(\mathbb{R})$ the space of smooth functions on M, $\mathfrak{X}_{C^\infty} = \Gamma_{C^\infty}(TM)$ the space of smooth vector fields on M, $\mathcal{T}^{lk}_{C^\infty} = \Gamma_{C^\infty}(T^{lk}TM)$ the space of smooth (l-contravariant, k-covariant) tensor fields on M. Denote $\Omega^k_{C^\infty} = \Gamma_{C^\infty}(\Lambda^k T^*M)$ as the space of smooth alternating k-forms on M. We let $\mathrm{Dens}_{C^\infty} := \Omega^d_{C^\infty}$. It is worth remarking that non-orientability and tensor-densities can be easily accommodated by introducing weighted densities (see, e.g., [108,1,116]). However, we avoid this extension for brevity in the presentation here.

Denote the wedge and tensor product by $\wedge$ and $\otimes$, respectively. Let $\mathbf{d} : \Omega^k_{C^\infty} \to \Omega^{k+1}_{C^\infty}$ denote the exterior derivative operator and $\mathbf{i}_u : \Omega^k_{C^\infty} \to \Omega^{k-1}_{C^\infty}$ denote the interior product operator for an arbitrarily given $u \in \mathfrak{X}_{C^\infty}$. For given $F \in \mathrm{Diff}_{C^\infty}$ and $\tau \in \mathcal{T}^{lk}_{C^\infty}$, the push-forward and pull-back are defined by

$$F_*\tau = (TF)_* \circ \tau \circ F^{-1} \quad \text{and} \quad F^*\tau = (F^{-1})_*\tau, \tag{A.10}$$

respectively, where $TF \in C^\infty(TM; TM)$ is the tangent map of F, which extends to an isomorphism (on fibers) $(TF)_* \in C^\infty(T^{lk}TM; T^{lk}TM)$.

For a given time-dependent vector field $u \in C^\infty(\mathbb{R} \times M; TM)$, let $\eta : \mathbb{R}^2 \times M \to M$ denote the two-parameter smooth flow of diffeomorphisms generated by u[6]; that is, for all $(s,t) \in \mathbb{R}^2$, $\eta_{t\theta} \circ \eta_{\theta s} = \eta_{ts}$, and $\eta_{\cdot s}X$ is the unique integral curve

$$\dot{\eta}_{ts}X = u_t(\eta_{ts}X), \quad \eta_{ss}X = X \in M.$$

For given $u \in C^\infty(\mathbb{R} \times M; TM)$ and $t \in \mathbb{R}$, the Lie derivative $\pounds_{u_t} : \mathcal{T}^{lk}_{C^\infty} \to \mathcal{T}^{lk}_{C^\infty}$ is defined by

$$\pounds_{u_t}\tau = \frac{d}{d\theta}\bigg|_{\theta=t} (\eta_{\theta t})^*\tau \quad \Leftrightarrow \quad \frac{d}{dt}\eta_{ts}\tau = \eta_{ts}\pounds_{u_t}\tau.$$

If u is independent of time, we define $\pounds_u\tau = \frac{d}{dt}\big|_{t=0} \eta_t^*\tau$, where $\eta_t = \eta_{t0}$ is the corresponding one-parameter flow map. It is well-known that the Lie derivative (see, e.g.,

[6] The time-dependent vector field u may be associated with a time-independent vector field $\bar{u} \in C^\infty(\mathbb{R} \times M, T\mathbb{R} \times TM)$ on the manifold $T\mathbb{R} \times TM$ via $\bar{u}_t(p) = \{u_t(p), 1_t\} \in T_t\mathbb{R} \times T_pM$ for all $(t,p) \in \mathbb{R} \times M$. Thus, the two-parameter flow may be defined in terms of the one-parameter flow of $\bar{u}$ by $\eta_{t-s}(x,s) = \{\eta_{ts}, t\}$. It follows from $\eta_{st} \circ \eta_{ts} = \mathrm{id}$ that for all $s \in \mathbb{R}$

$$\frac{d}{dt}\eta_{st} = -T\eta_{st} \circ u_t = -(\eta_{st})_* u_t \circ \eta_{st}.$$

Equivalently, for all $f \in C^\infty$, $h_\cdot = \eta_{\cdot s*}f = (\eta_{s\cdot})^*f \in C^m$ solves the PDE $\partial_t h_t + \pounds_{u_t} h_t = 0$.

[78,1]) is the unique operator on the tensor algebra $\oplus_{l,k}\mathcal{T}^{lk}_{C^\infty}$ that i) commutes with tensor contractions, ii) is natural with respect to restrictions, and iii) satisfies for a given local chart (U,ϕ) and all $f\in C^\infty|_U$, $u,v\in\mathfrak{X}_{C^\infty}|_U$,

$$\pounds_u f = u[f] = u^i\partial_{x^i}f = \mathbf{i}_u\mathbf{d}f \quad \text{and} \quad (\pounds_u v)^i = u^j\partial_{x^j}v^i - v^j\partial_{x^j}u^i.$$

It follows that for all $u\in\mathfrak{X}_{C^\infty}|_U$, $\alpha\in\Omega^k_{C^\infty}|_U$, and $\tau\in\mathcal{T}^{lk}_{C^\infty}|_U$,

$$\begin{aligned}(\pounds_u\alpha)_{i_1\cdots i_k} &= u^j\partial_{x^j}\alpha_{i_1\cdots i_k} + \alpha_{j\cdots i_k}\partial_{x^{i_1}}u^j + \cdots + \alpha_{i_1\cdots j}\partial_{x^{i_k}}u^j\\ (\pounds_u\tau)^{j_1\cdots j_l}_{i_1\cdots i_k} &= u^j_t\partial_{x^j}\tau^{j_1\cdots j_l}_{i_1\cdots i_k} - \tau^{j_1\cdots j_l}_{i_1\cdots i_k}\partial_{x^{j_1}}u^j - \cdots - \tau^{j_1\cdots j}_{i_1\cdots i_k}\partial_{x^{j_l}}u^j + \tau^{j_1\cdots j_k}_{j\cdots i_k}\partial_{x^{i_1}}u^j\\ &\quad + \cdots + \tau^{j_1\cdots j_l}_{i_1\cdots j}\partial_{x^{i_k}}u^j.\end{aligned} \tag{A.11}$$

Thus, for given $u\in C^\infty(\mathbb{R}\times M, TM)$, the Lie derivative $\pounds_{u_t}$ is a first-order differential operator on the bundles Λ^kT^*M and $T^{lk}TM$. For non-vanishing $\mu\in\Omega^d_{C^\infty}$, the operator $\operatorname{div}_\mu:\mathfrak{X}_{C^\infty}\to C^\infty$ is defined by the relation

$$\pounds_u\mu = (\operatorname{div}_\mu u)\mu.$$

For $u,v\in\mathfrak{X}_{C^\infty}$ we let $[u,v]=\pounds_u v$ and $\operatorname{ad}_u v := -\pounds_u v$ and note that

$$(\pounds_u\pounds_v - \pounds_v\pounds_u)\,\tau = \pounds_{[u,v]}\tau = -\pounds_{\operatorname{ad}_u v}\tau, \quad \forall\tau\in\mathcal{T}^{lk}_{C^\infty}.$$

Moreover, for all $u\in\mathfrak{X}_{C^\infty}$ and $\alpha\in\ \Omega^k_{C^\infty}$, we have

$$\pounds_u\alpha = \mathbf{d}(\mathbf{i}_u\alpha) + \mathbf{i}_u\mathbf{d}\alpha\,, \tag{A.12}$$

which is referred to as *Cartan's formula.*

A.2.2. Vector bundles: canonical pairings, adjoints, and function spaces

For a given vector bundle E, denote by E^* the dual bundle. Let $E^\vee = E^*\otimes\Lambda^dT^*M$ and we may extend the dual pairing between E and E^* to a mapping $\langle\cdot,\cdot\rangle_E : E^\vee\times E\to\Lambda^dT^*M$. The bundle $E^\vee$ is often called the functional dual bundle. We may then define the canonical pairing $\langle\cdot,\cdot\rangle_{\Gamma(E)}:\Gamma_{C^\infty}(E^\vee)\times\Gamma_{C^\infty}(E)\to\mathbb{R}$ by

$$\langle s',s\rangle_{\Gamma(E)} = \int_M \langle s',s\rangle_E, \qquad (s',s)\in\Gamma_{C^\infty}(E^\vee)\times\Gamma_{C^\infty}(E). \tag{A.13}$$

The quantity $\langle s',s\rangle_E\in\operatorname{Dens}_{C^\infty}$ in the integrand is a volume form and it is being integrated over the manifold M. The distributional sections of E and $E^\vee$ are defined by $\Gamma_{\mathcal{D}'}(E):=\Gamma_{C^\infty}(E^\vee)^*$ and $\Gamma_{\mathcal{D}'}(E^\vee):=\Gamma_{C^\infty}(E)^*$, respectively. The canonical pairing (A.13) induces the following dense embeddings:

$$\Gamma_{C^\infty}(E) \hookrightarrow \Gamma_{\mathcal{D}'}(E) \text{ via } s \mapsto l_s = \langle \cdot, s \rangle_{\Gamma(E)} \quad \text{and}$$
$$\Gamma_{C^\infty}(E^\vee) \hookrightarrow \Gamma_{\mathcal{D}'}(E^\vee) \text{ via } \tilde{s} \mapsto l_{s'} = \langle s', \cdot \rangle_{\Gamma(E)}.$$

The pairing and definitions of distributions are canonical in the sense that no metric or volume form is needed to define them. We extend the pairing $\langle \cdot, \cdot \rangle_{\Gamma(E)}$ to $\Gamma_{\mathcal{D}'}(E^\vee) \times \Gamma_{C^\infty}(E)$ and $\Gamma_{C^\infty}(E^\vee) \times \Gamma_{\mathcal{D}'}(E)$ in the usual way.

The adjoint of a linear differential operator $L : \Gamma_{C^\infty}(E) \to \Gamma_{C^\infty}(E)$, denoted by $L^* : \Gamma_{\mathcal{D}'}(E^\vee) \to \Gamma_{\mathcal{D}'}(E^\vee)$, is defined by

$$\langle L^* s', s \rangle_{\Gamma(E)} = \langle s', Ls \rangle_{\Gamma(E)}, \quad \forall (s', s) \in \Gamma_{\mathcal{D}'}(E^\vee) \times \Gamma_{C^\infty}(E). \tag{A.14}$$

It follows that L^* restricts to $L^* : \Gamma_{C^\infty}(E^\vee) \to \Gamma_{C^\infty}(E^\vee)$, and we write $L = L^{**} : \Gamma_{\mathcal{D}'}(E) \to \Gamma_{\mathcal{D}'}(E)$.

For a normal, local, and invariant Fréchet, Banach, or Hilbert function space $\mathcal{F}$ on $\mathbb{R}^d$,[7] we define a Fréchet, Banach, or Hilbert, respectively, of sections $\Gamma_{\mathcal{F}}(E)$ via a cover of total trivializations.[8] It follows that

$$\Gamma_{C^\infty}(E) \hookrightarrow \Gamma_{\mathcal{F}}(E) \hookrightarrow \Gamma_{\mathcal{D}'}(E),$$

where the embedding $\Gamma_{C^\infty}(E) \hookrightarrow \Gamma_{\mathcal{F}}(E)$ is dense. We refer the reader to [116][Ch. 3] for more details. Exactly the same construction applies to obtain a function space $\Gamma_{\mathcal{F}}(E^\vee)$:

$$\Gamma_{C^\infty}(E^\vee) \hookrightarrow \Gamma_{\mathcal{F}}(E^\vee) \hookrightarrow \Gamma_{\mathcal{D}'}(E^\vee).$$

In the present work, we assume that all function spaces $\mathcal{F}$ are normal, local, and invariant. In particular, we let

$$\mathcal{F} = \Gamma_{\mathcal{F}}(\mathbb{R}), \;\; \mathfrak{X}_{\mathcal{F}} = \Gamma_{\mathcal{F}}(TM), \;\; \mathrm{Dens}_{\mathcal{F}}^\vee = \mathcal{F}, \;\; \mathcal{T}_{\mathcal{F}}^{lk} = \Gamma_{\mathcal{F}}(T^{lk}TM), \;\; \Omega_{\mathcal{F}}^k = \Gamma_{\mathcal{F}}(\Lambda^k T^* M),$$
$$\mathcal{F}^\vee = \Gamma_{\mathcal{F}}(\Lambda^d T^* M) = \mathrm{Dens}_{\mathcal{F}}, \;\; \mathfrak{X}_{\mathcal{F}}^\vee = \Gamma_{\mathcal{F}}(TM^\vee), \quad (\mathcal{T}_{\mathcal{F}}^{lk})^\vee = \Gamma_{\mathcal{F}}((T^{lk}TM)^\vee),$$
$$(\Omega_{\mathcal{F}}^k)^\vee = \Gamma_{\mathcal{F}}(\Lambda^k T^* M^\vee).$$

Any strong bundle pseudo-metric $(\cdot, \cdot)_E : E \times E \to \mathbb{R}$ induces an isomorphism $\flat : \Gamma_{\mathcal{F}}(E) \to \Gamma_{\mathcal{F}}(E^*)$ with inverse denoted $\sharp : \Gamma_{\mathcal{F}}(E^*) \to \Gamma_{\mathcal{F}}(E)$ for an arbitrarily given function space $\mathcal{F}$. Moreover, a non-vanishing volume form $\mu \in \mathrm{Dens}_{C^\infty}$ induces an

[7] Let $\mathcal{F}$ denote a locally convex topological vector space of functions $f : \mathbb{R}^d \to \mathbb{R}$ such that $C_c^\infty(\mathbb{R}^d) \hookrightarrow \mathcal{F} \hookrightarrow \mathcal{D}'(\mathbb{R}^d) := C_c^\infty(\mathbb{R}^d)^*$ and such that pointwise multiplication of functions in $\mathcal{F}$ by functions in $C_c^\infty(\mathbb{R}^d)$ is a continuous operation. We say that a function space $\mathcal{F}$ is normal if the embedding $C_c^\infty(\mathbb{R}^d) \hookrightarrow \mathcal{F}$ is dense, local if $\mathcal{F} = \{u \in \mathcal{D}'(\mathbb{R}^d) : \phi u \in \mathcal{F}, \;\; \forall \phi \in C_c^\infty(\mathbb{R}^d)\}$, and invariant if any smooth diffeomorphism $\chi \in \mathrm{Diff}_{C^\infty}$ induces an topological isomorphism on $\mathcal{F}$ via push-forward.

[8] A total trivialization is a triple (U, ϕ, ψ) such that (U, ϕ) is a local chart of M and $\psi : E_U \to U \times \mathbb{R}^{\mathrm{rank}(E)}$ is a trivialization of E over U. Any such trivialization induces an isomorphism $h_{\phi,\psi} : \Gamma_{\mathcal{D}'}(E|_U) \to \Gamma_{\mathcal{D}'}(\phi(U))$. A section $s \in \Gamma_{\mathcal{D}'}(E)$ belongs to $\Gamma_{\mathcal{F}}(E)$ if for every total trivialization (U, ϕ, ψ), $h_{\phi,\psi}(s|_U) \in \mathcal{F}(\phi(U))^{\mathrm{rank}(E)}$.

isomorphism $\mathrm{id} \otimes \mu : \Gamma_{\mathcal{F}}(E^*) \to \Gamma_{\mathcal{F}}(E^\vee)$ with inverse $\frac{1}{\mu} : \Gamma_{\mathcal{F}}(E^\vee) \to \Gamma_{\mathcal{F}}(E^*)$. For every $s \in \Gamma_{\mathcal{F}}(E^*)$,

$$\mathrm{id} \otimes \mu(s) = s \otimes \mu.$$

To describe the inverse, note that for all densities $\nu \in \mathrm{Dens}_{C^\infty}$, there exists $\frac{d\nu}{d\mu} \in C^\infty$ such that $\nu = \frac{d\nu}{d\mu}\mu$. The inverse is induced by

$$\frac{1}{\mu}(s \otimes \nu) = s\frac{d\nu}{d\mu}, \quad s \otimes \mu \in \Gamma_{\mathcal{F}}(E^\vee). \tag{A.15}$$

Composing these isomorphisms, we obtain an isomorphism $\flat \otimes \mu : \Gamma_{\mathcal{F}}(E) \to \Gamma_{\mathcal{F}}(E^\vee)$ with inverse $\frac{\sharp}{\mu} : \Gamma_{\mathcal{F}}(E^\vee) \to \Gamma_{\mathcal{F}}(E)$. In particular, we may define a pairing $(\cdot, \cdot)_{\Gamma_{L^2}(E)} : \Gamma_{L^2}(E) \times \Gamma_{L^2}(E) \to \mathbb{R}$ by

$$(s_1, s_2)_{\Gamma_{L^2}(E)} = \langle \flat \otimes \mu(s_1), s_2\rangle_{\Gamma(E)} = \int_M \langle s_1^\flat, s_2\rangle_E \mu = \int_M (s_1, s_2)_E \mu, \quad s_1, s_2 \in \Gamma_{C^\infty}(E),$$

which may be extended to $\Gamma_{\mathcal{D}'}(E) \times \Gamma_{C^\infty}(E)$ via the isomorphisms $(\flat \otimes \mu)^* : \Gamma_{\mathcal{D}'}(E) \to \Gamma_{\mathcal{D}'}(E^\vee)$.

If $(\cdot, \cdot)_E$ is a metric, we obtain a Hilbert structure on $\Gamma_{L^2}(E)$, the space of square-integrable equivalence classes of measurable sections. Furthermore, for every $s \in \mathbb{R}$, there exists an order s elliptic operator A satisfying $A : \Gamma_{W_2^s}(E) \to \Gamma_{W_2^{s-1}}(E)$ and $\Gamma_{W_2^s}(E) \cong A^{-1}\Gamma_{L^2}(E)$, where $W_2^s = (I - \Delta)^{-\frac{s}{2}} L^2$ denotes the Bessel-potential spaces, which provides a Hilbert structure to $\Gamma_{W_2^s}(E)$ [71,93]. Moreover, if $L : \Gamma_{C^\infty}(E) \to \Gamma_{C^\infty}(E)$, then $L^T_{\flat\otimes\mu} := \frac{\sharp}{\mu} \circ L^T \circ \flat \otimes \mu : \Gamma_{C^\infty}(E) \to \Gamma_{C^\infty}(E)$ is the adjoint of L relative to the pairing $(\cdot, \cdot)_{\Gamma_{L^2}(E)}$.

A.2.3. Riemannian manifolds and the Hodge decomposition

Any Riemannian metric g on M gives rise to a volume form μ_g defined in a local coordinate chart (U, ϕ) by

$$\mu_g = \sqrt{\det[g_{ij}]}\, dx^1 \wedge \cdots \wedge dx^d.$$

Furthermore, the metric g extends to bundle metrics on $T^{lk}TM$ and $\Lambda^k T^*M$ in the usual way. In particular, we obtain the diffeomorphisms $\flat, \mathrm{id} \otimes \mu_g$ and $\flat \otimes \mu_g$ discussed in the previous section. For every $k \in \{0, 1, \ldots, d\}$, the metric and orientation gives rise to the inner product on $\Omega^k_{L^2}$ defined by

$$(\alpha, \beta)_{\Omega^k_{L^2}} = \int_M g(\alpha, \beta)\mu_g = \int_M \alpha \wedge \star\beta, \quad \alpha, \beta \in \Omega^k_{L^2},$$

where we have used the Hodge-star diffeomorphism $\star : \Omega^k_{\mathcal{F}} \to \Omega^{d-k}_{\mathcal{F}}$ defined by

$$\alpha \wedge \star\beta = g(\alpha, \beta)\mu_g, \quad \forall \alpha, \beta \in \Omega^k_{\mathcal{F}}.$$

The adjoint of $\mathbf{d} : \Omega^k_{C^\infty} \to \Omega^{k+1}_{C^\infty}$ with respect to the L^2-pairing is given by $\mathbf{d}^* := (-1)^{dk+1} \star \mathbf{d} \star : \Omega^{k+1}_{\mathcal{D}'} \to \Omega^k_{\mathcal{D}'}$.

The Hodge decomposition plays an essential role in incompressible fluids on manifolds. We now briefly describe the decomposition and the canonical pairing we use in the incompressible case.

Let $\Delta_H = \mathbf{d}\mathbf{d}^* + \mathbf{d}^*\mathbf{d} : \oplus_{k=0}^d \Omega^k_{\mathcal{D}'} \to \oplus_{k=0}^d \Omega^k_{\mathcal{D}'}$ denote the Hodge Laplacian, which is formally self-adjoint and non-negative with respect to the inner product $\sum_{k=0}^d (\cdot,\cdot)_{\Omega^k}$. Let

$$\mathcal{H}^k_\Delta = \left\{\alpha \in \Omega^k_{C^\infty} : \Delta_H \alpha = 0\right\} = \left\{\alpha \in \Omega^k_{C^\infty} : \mathbf{d}\alpha = \delta\alpha = 0\right\}$$

denote the finite-dimensional space of harmonic k-forms. It follows that harmonic 0-forms are constant.

Let $\mathcal{F}$ denote either the smooth $\mathcal{F} = C^\infty$, the Bessel-potential $\mathcal{F} = W^s_p$, $s \geq 0$, $p \in [1, \infty)$, or the Hölder functions $\mathcal{F} = C^{m,\alpha}$, $m \geq 0$, $\alpha \in (0,1)$. The Hodge decomposition of $\Omega^k_{\mathcal{F}}$ is given by

$$\Omega^k_{\mathcal{F}} = \mathcal{H}^k_\Delta \oplus \Delta_H G \Omega^k_{\mathcal{F}} = \mathcal{H}^k_\Delta \oplus \mathbf{d}^* \Omega^{k+1}_{\mathcal{F}^{+1}} \oplus \mathbf{d}\Omega^{k-1}_{\mathcal{F}^{+1}}, \tag{A.16}$$

where $G : \Omega^k_{\mathcal{F}} \to \Omega^k_{\mathcal{F}^{+2}}$ satisfies $\Delta_H G\alpha = \alpha - H\alpha$, $H : \Omega^k_{\mathcal{F}} \to \mathcal{H}^k_\Delta$ is the harmonic projection [100,117,97,105,94], and $\mathcal{F}^{+1}$ and $\mathcal{F}^{+2}$ are the one and two-more regular spaces (in the non-smooth case). That is, $\mathcal{F}^{+1} = W^{s+1}_p$ and $\mathcal{F}^{+2} = W^{s+2}_p$, and similarly for Hölder spaces.

Letting $k = 1$ in (A.16), applying the diffeomorphism $\sharp : \Omega^1_{\mathcal{F}} \to \mathfrak{X}_{\mathcal{F}}$, and defining

$$\nabla \mathcal{F}^{+1} := \sharp \mathbf{d}\mathcal{F}^{+1}, \quad \mathfrak{X}_{\mathcal{F},\mu_g} := \sharp \mathcal{H}^k_\Delta \oplus \sharp \mathbf{d}^* \Omega^2_{\mathcal{F}^{+1}}, \quad \& \quad \dot{\mathfrak{X}}_{\mathcal{F},\mu_g} := \sharp \mathbf{d}^* \Omega^2_{\mathcal{F}^{+1}},$$

we obtain an extension of the Helmholtz decomposition of (possibly non-smooth) vector fields to manifolds:

$$\mathfrak{X}_{\mathcal{F}} = \mathfrak{X}_{\mathcal{F},\mu_g} \oplus \nabla \mathcal{F}^{+1} = (\mathcal{H}^1_\Delta)^\sharp \oplus \dot{\mathfrak{X}}_{\mathcal{F},\mu_g} \oplus \nabla \mathcal{F}^{+1}, \tag{A.17}$$

which is an orthogonal decomposition with respect to the inner product $(\cdot,\cdot)_{\mathfrak{X}_{L^2}} : \mathfrak{X}_{L^2} \times \mathfrak{X}_{L^2} \to \mathbb{R}$ defined by

$$(u, v)_{\mathfrak{X}_{L^2}} = \int_M g(u, v)\mu_g, \quad u, v \in \mathfrak{X}_{L^2}.$$

Using $\mathbf{i}_u \mu_g = \star u^\flat$ and Cartan's formula, we find $\operatorname{div}_{\mu_g} u = -\mathbf{d}^* u^\flat = 0$ for all $u \in \mathfrak{X}_{\mathcal{F},\mu_g}$. Thus, $\mathfrak{X}_{\mathcal{F},\mu_g}$ consists of divergence-free vector fields and $\dot{\mathfrak{X}}_{\mathcal{F},\mu_g}$ consists of harmonic-free and divergence-free vector fields.

Let us recall the canonical pairing (A.13) $\langle\cdot,\cdot\rangle_{\mathfrak{X}} : \mathfrak{X}^\vee_{C^\infty} \times \mathfrak{X}_{C^\infty} \to \mathbb{R}$:

$$\langle \alpha\otimes\mu, u\rangle_{\mathfrak{X}} = \int_M \alpha(u)\mu,$$

and diffeomorphism $\flat\otimes\mu_g : \mathfrak{X}_{C^\infty} \to \mathfrak{X}^\vee_{C^\infty}$, which satisfies $\langle \flat\otimes\mu_g(v), u\rangle_{\mathfrak{X}} = (v,u)_{\mathfrak{X}_{L^2}}$ for all $u, v \in \mathfrak{X}_{C^\infty}$. Applying the diffeomorphism $\flat\otimes\mu_g$ to (A.17), we get

$$\mathfrak{X}^\vee_{C^\infty} = (\mathrm{id}\otimes\mu_g)\Omega^1_{C^\infty} = (\mathrm{id}\otimes\mu_g)\mathcal{H}^1_\Delta \oplus (\mathrm{id}\otimes\mu_g)\delta\Omega^2_{C^\infty} \oplus (\mathrm{id}\otimes\mu_g)\mathbf{d}C^\infty.$$

Define the 'projection' operators $P : \mathfrak{X}^\vee_{C^\infty} \to (\mathrm{id}\otimes\mu_g)\mathcal{H}^1_\Delta \oplus (\mathrm{id}\otimes\mu_g)\mathbf{d}^*\Omega^2_{C^\infty}$ and $\dot{P} : \mathfrak{X}^\vee_{C^\infty} \to (\mathrm{id}\otimes\mu_g)\mathbf{d}^*\Omega^2_{C^\infty}$, which act only on the one-form component. Clearly, if we restrict the canonical pairing $\langle\cdot,\cdot\rangle_{\mathfrak{X}}$ to $\mathfrak{X}^\vee_{C^\infty} \times \mathfrak{X}_{C^\infty,\mu_g}$ and $\mathfrak{X}^\vee_{C^\infty} \times \dot{\mathfrak{X}}_{C^\infty,\mu_g}$, then the pairing is degenerate; indeed,

$$\begin{aligned} \langle \alpha\otimes\mu, u\rangle_{\mathfrak{X}} = 0, \quad \forall u \in \mathfrak{X}_{C^\infty,\mu_g} &\quad\Longrightarrow\quad P(\alpha\otimes\mu) = 0,\\ \langle \alpha\otimes\mu, u\rangle_{\mathfrak{X}} = 0, \quad \forall u \in \dot{\mathfrak{X}}_{C^\infty,\mu_g} &\quad\Longrightarrow\quad \dot{P}(\alpha\otimes\mu) = 0. \end{aligned}$$

Notice that the kernel of P is $(\mathrm{id}\otimes\mu_g)\mathbf{d}C^\infty$ and the kernel of $\dot{P}$ is $(\mathrm{id}\otimes\mu_g)\mathcal{H}^1_\Delta \oplus (\mathrm{id}\otimes\mu_g)\mathbf{d}C^\infty$. To restore non-degeneracy, we mod out by the kernel; the following definition is standard [7,80,81].

Definition A.20. Let $\mathfrak{X}^\vee_{\mathcal{F},\mu_g} := \mathfrak{X}^\vee_{\mathcal{F}}/(\mathrm{id}\otimes\mu_g)\mathbf{d}\mathcal{F}^{+1}$ and $\dot{\mathfrak{X}}^\vee_{\mathcal{F},\mu_g} = \mathfrak{X}^\vee_{\mathcal{F}}/(\mathrm{id}\otimes\mu_g)\mathcal{H}^1_\Delta \oplus (\mathrm{id}\otimes\mu_g)\mathbf{d}\mathcal{F}$. Moreover, we define the canonical pairings $\langle\cdot,\cdot\rangle_{\mathfrak{X}_{\mu_g}} : \mathfrak{X}^\vee_{C^\infty,\mu_g} \times \mathfrak{X}_{C^\infty,\mu_g} \to \mathbb{R}$ and $\langle\cdot,\cdot\rangle_{\dot{\mathfrak{X}}_{\mu_g}} : \dot{\mathfrak{X}}^\vee_{C^\infty,\mu_g} \otimes \dot{\mathfrak{X}}_{C^\infty,\mu_g} \to \mathbb{R}$ by

$$\begin{aligned} \langle [\alpha\otimes\mu], u\rangle_{\mathfrak{X}_{\mu_g}} &= \langle \alpha\otimes\mu, u\rangle_{\mathfrak{X}}, \ \ \forall\, ([\alpha\otimes\mu], u) \in \mathfrak{X}^\vee_{C^\infty,\mu_g} \times \mathfrak{X}_{C^\infty,\mu_g}\\ \langle [\alpha\otimes\mu], v\rangle_{\dot{\mathfrak{X}}_{\mu_g}} &= \langle \alpha\otimes\mu, v\rangle_{\mathfrak{X}}, \ \ \forall\, ([\beta\otimes\nu], v) \in \dot{\mathfrak{X}}^\vee_{C^\infty,\mu_g} \times \dot{\mathfrak{X}}_{C^\infty,\mu_g}, \end{aligned} \tag{A.18}$$

where the $[\alpha\otimes\mu]$ denotes an equivalence class with representative $\alpha\otimes\mu$. It follows that $\flat\otimes\mu_g : \mathfrak{X}_{\mathcal{F},\mu_g} \to \mathfrak{X}^\vee_{\mathcal{F},\mu_g}$ and $\flat\otimes\mu_g : \dot{\mathfrak{X}}_{\mathcal{F},\mu_g} \to \dot{\mathfrak{X}}^\vee_{\mathcal{F},\mu_g}$ are diffeomorphisms.

It can easily be checked the definition is well-defined in the sense that the right-hand-sides of (A.18) are independent of the representative. Indeed, for any two given representatives $\alpha\otimes\mu$ and $\beta\otimes\nu$ of an equivalence class of $\mathfrak{X}^\vee_{\mathcal{F},\mu_g}$, we have

$$P(\alpha\otimes\mu) = P(\beta\otimes\nu) \quad\Longleftrightarrow\quad \alpha\otimes\mu = \beta\otimes\mu + \mathbf{d}f\otimes\mu_g \ \text{ for some } f\in\mathcal{F}^{+1},$$

and for any two given representatives $\alpha\otimes\mu$ and $\beta\otimes\nu$ of an equivalence class of $\dot{\mathfrak{X}}^\vee_{\mathcal{F},\mu_g}$

$$\dot{P}(\alpha\otimes\mu) = P(\beta\otimes\nu) \quad\Longleftrightarrow\quad \alpha\otimes\mu = \beta\otimes\mu + (\mathbf{d}f + c)\otimes\mu_g \ \text{ for some } f\in\mathcal{F}^{+1} \ \&\ c\in\mathcal{H}^1_\Delta.$$

Appendix B. Auxiliary results

B.1. Rough flows on Euclidean space

Theorem B.1. *There exists a continuous map*

$$\text{Flow} : C_T^\alpha\left(\mathfrak{X}_{C_b^\infty}(\mathbb{R}^d)\right) \times C_T^\infty\left(\mathfrak{X}_{C_b^\infty}(\mathbb{R}^d)^K\right) \times \boldsymbol{\mathcal{C}}_{g,T}(\mathbb{R}^K) \to C_{2,T}^\alpha(\text{Diff}_{C^\infty}(\mathbb{R}^d))$$

such that the flow $\eta_{ts} = \text{Flow}(u, \xi, \mathbf{Z})_{ts}$, $(s,t) \in [0,T]^2$ *satisfies the following properties:*

(i) *for all* $(s,\theta,t) \in [0,T]^3$, $\eta_{tt} = \text{Id}$ *and* $\eta_{t\theta} \circ \eta_{\theta s} = \eta_{ts}$*;*

(ii) $Y_\cdot = \eta_{\cdot s}(X) \in C^\alpha([s,T];\mathbb{R}^d)$ *is the unique solution of the equation*

$$\mathrm{d}Y_t = u_t(Y_t)\mathrm{d}t + \xi_t(Y_t)\mathrm{d}\mathbf{Z}_t, \quad Y_s = X \in \mathbb{R}^d;$$

(iii) η *is the unique two-parameter flow satisfying (i) and*

$$|\eta_{ts} - \mu_{ts}|_\infty \le C|t-s|^{3\alpha}, \quad \forall (s,t) \in [0,T]^2,$$

for a constant C*, where* $\mu \in C_{2,T}^\alpha(\text{Diff}_{C^\infty}(\mathbb{R}^d))$ *is the* C^∞*-approximate flow given by*

$$\mu_{ts} := \exp\left(u_s(t-s) + \sum_{k=1}^K \xi_k(s)\mathbb{Z}_{st}^k + \sum_{1\le k<l\le K} [\xi_k(s), \xi_l(s)]\mathbb{A}_{st}^{kl}\right),$$

or equivalently by $\mu_{ts}(X) := Y_1$ *such that*

$$\dot{Y}_\theta = u_s(Y_\theta)(t-s) + \sum_{k=1}^K \xi_k(s)(Y_\theta)\delta Z_{st}^k + \sum_{1\le k<l\le K} [\xi_k(s), \xi_l(s)](Y_\theta)\mathbb{A}_{st}^{kl},$$
$$\theta \le 1, \quad Y_0 = X \in \mathbb{R}^d;$$

(iv) *for all* $f \in C_b^\infty(\mathbb{R}^d;\mathbb{R})$ *and* $s \in [0,T]$, $\eta = f(\eta_{\cdot s}^{-1}) \in C^\alpha([s,T]; C^\infty(\mathbb{R}^d;\mathbb{R}))$ *satisfies*

$$\eta_t + \int_s^t \pounds_{u_r} g_r \mathrm{d}r + \int_s^t \pounds_{\xi_r} g_r \mathrm{d}\mathbf{Z}_r = f;$$

in $C_b^\infty(\mathbb{R}^d)$*; that is,* $(\xi[g], -\xi[\xi[g]]) \in \boldsymbol{\mathcal{D}}_{\mathbf{Z}}([s,T]; C_b^\infty(\mathbb{R}^d))$.

Remark B.2. Claims (i-iii) are a direct extension of Corollary 11.14 of [60]; one can easily verify the Davie's estimates (Corollary 11.14 of [60]). We do not impose that our drift coefficient is Lipschitz in time because it is the solution of a rough partial differential equation driven by the path $\mathbf{Z}$ in our framework, and hence it can only be expected to be α-Hölder continuous. We also allow for time dependence in the vector field ξ since this is used in Section 3.3 to take variations. Claim (iv) is a minor extension of Theorem 16 of [8,11] (or Theorem 1.27 of [50]), which uses the method of approximate flows. It is possible to weaken the required regularity in space and time of the coefficients, but for simplicity, we do not pursue this.

Claim (iv) is the initial-value first-order linear transport rough partial differential equation (RPDE) for the inverse flow. We understand g as the classical solution in the spatial variable and in the sense of controlled rough paths in time. In [30][Corollary 8], the method of characteristics solution theory for initial-value RPDEs (in the case $u \equiv 0$) is established and the solutions are characterized as being a limit point of $g^n = f(X_t^n)$, where X^n is the solution of the time-reversal along a sequence of smooth paths $\mathbf{Z}^n = (Z^n, \mathbb{Z}^n)$ converging to $\mathbf{Z}$ in the rough path topology. It is not clear that one can deduce a stronger notion of solution (in the sense of controlled rough paths) from this result in a simple manner (see, also, Remark 2.10 of [48]). The works [48] and [12] prove the well-posedness of the final-value transport equation and its adjoint, the initial-value continuity equation, in the sense of controlled rough paths. We were not able to find the exact result in the literature.

Nevertheless, the solution of the RPDE can be derived using theory of unbounded rough drivers ([10,47]), which is analogous to the energy method in deterministic PDE. Indeed, one may first derive a solution $g \in C([s,T]; W_2^n(\mathbb{R}^d))$ under the assumption $u \in C_T(C^m(\mathbb{R}^d;\mathbb{R}^d))$, $\xi \in C^{m+3}(\mathbb{R}^d;\mathbb{R}^d)$, and $f \in W_2^m(\mathbb{R}^d)$ for any $m \in \mathbb{N}_0$ by adapting Theorem 2 of [72] and Section 5.2 of [39]. Then one may obtain a solution $u \in C_T^{\alpha}(C^{\infty}(\mathbb{R}^d;\mathbb{R}))$ by applying the Sobolev embedding. Finally, one can apply the pull-back version of the Lie chain rule Theorem 3.3 to show that $g(\eta_{\cdot s}) = f$.

B.2. Rough Fubini's theorem

Let $T > 0$, $\alpha \in \left(\frac{1}{3}, 1\right]$, and $\mathbf{Z} \in \mathcal{C}_{g,T}^{\alpha}(\mathbb{R}^K)$. By virtue of the fact that rough integration is a linear continuous map, we can easily obtain a version of Fubini's theorem. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measured space and W be a Banach space. Denote by $L^1(X;W)$ the Banach space of equivalent classes of Bochner integrable functions $f : X \to W$ endowed with the norm

$$|f|_{L^1(X;W)} = \int_X |f|_V d\mu, \quad f \in L^1(X;W).$$

Recall that for an arbitrary Banach space V and linear map $L \in \mathcal{L}(W, V)$,

$$L \int_X f d\mu = \int_X L f d\mu, \quad \forall f \in L^1(X;W). \tag{B.1}$$

The following lemma is then a straightforward application of (B.1), Theorem A.8, and

$$L^1(X;\mathcal{D}^{\alpha}_{Z,T}(V^K)) \subset \mathcal{D}_{Z,T}(L^1(X;V)^K),$$

which itself follows from Fatou's lemma.

Lemma B.3 *(Rough Fubini). If* $\mathbf{F} = (F, F') \in L^1(X;\mathcal{D}^{\alpha}_{Z,T}(V^K))$*, then for all* $(s,t) \in \Delta^2_T$*,*

$$\int_X \int_s^t F_r \mathrm{d}\mathbf{Z}_r d\mu = \int_s^t \int_X F_r d\mu \mathrm{d}\mathbf{Z}_r.$$

B.3. Fundamental lemma of the calculus of rough variations

Let $T > 0$, $\alpha \in \left(\frac{1}{3}, 1\right]$, and $\mathbf{Z} \in \mathcal{C}^{\alpha}_{g,T}(\mathbb{R}^K)$.

Lemma B.4. *Assume that* $\mathbf{Y} = (Y, Y') \in \mathcal{D}_{Z,T}(\mathbb{R}^K)$ *and* $\lambda \in C_T(\mathbb{R})$ *satisfy*

$$\int_a^b \lambda_t \dot{\phi}_t \mathrm{d}t = \int_a^b \phi_t Y_t \mathrm{d}\mathbf{Z}_t \tag{B.2}$$

for all $\phi \in C^{\infty}_T(\mathbb{R})$ *such that* $\phi_0 = \phi_T = 0$*. Then for all* $(s,t) \in \Delta^2_T$*,*

$$\delta\lambda_{st} = \int_s^t Y_r \mathrm{d}\mathbf{Z}_r. \tag{B.3}$$

Remark B.5. On the right-hand-side of (B.2), we have used that $(\phi, 0) \in \mathcal{D}_{Z,T}(\mathbb{R})$, and thus that $\phi\mathbf{Y} = (\phi Y, \phi Y') \in \mathcal{D}_{Z,T}(\mathbb{R}^K)$ by Lemma A.12.

Proof. Step 1. We will begin by showing that equality (B.2) must hold for any Lipschitz $\phi \in C^1_T(\mathbb{R})$ such that $\phi_0 = \phi_T = 0$, where $\dot{\phi}$ on the left hand side is the bounded weak derivative (which exists by Rademacher's theorem [70][Theorem 6.15]). Consider a mollifier on $\mathbb{R}$ defined by $\rho_n(\theta) := n\rho(n\theta)$, $n \in \mathbb{N}$, where $\int_{\mathbb{R}} \rho(\theta)\mathrm{d}\theta = 1$ and $\operatorname{supp}\rho \subset [0,T]$. Because ϕ vanishes at the end points, we can extend ϕ by zero

$$\tilde{\phi}_t = \begin{cases} \phi_t & t \in [0,T] \\ 0 & t \notin [0,T] \end{cases}$$

and note that $\tilde{\phi} \in C^1_T(\mathbb{R})$ has the same Lipschitz constant as ϕ. For a given $n \in \mathbb{N}$, define

$$\phi^n_t := \tilde{\phi} * \rho_n(t) = \int_{\mathbb{R}} \tilde{\phi}_{t-\theta}\rho_n(\theta)d\theta = \int_a^b \phi_\theta \rho_n(t-\theta)d\theta, \quad t \in \mathbb{R},$$

which is clearly bounded in $C_T(\mathbb{R})$. For all $n \in \mathbb{N}$ and $s, t \in [0, T]$, we find

$$|\phi_t^n - \tilde{\phi}_t| = \left| \int_{\mathbb{R}} (\tilde{\phi}_{t-\theta} - \tilde{\phi}_t)\rho_n(\theta)d\theta \right| \lesssim \int_{\mathbb{R}} |\theta|\rho_n(\theta)d\theta = n^{-1} \int_{\mathbb{R}} |\theta|\rho(\theta)d\theta$$

and

$$|\delta\phi_{st}^n| = \int_{\mathbb{R}} (\tilde{\phi}_{t-\theta} - \tilde{\phi}_{s-\theta})\rho_n(\theta)d\theta \lesssim |t-s| \int_{\mathbb{R}} \rho_n(\theta)d\theta = |t-s|.$$

By Arzela-Ascoli's theorem, $\phi^n \to \phi$ uniformly, and, in fact, in $C_T^\beta(\mathbb{R})$ for all $\beta < 1$. A classical argument shows that

$$\lim_{n\to\infty} \int_0^T \dot{\phi}_t^n \lambda_t \mathrm{d}t = \int_0^T \dot{\phi}_t \lambda_t \mathrm{d}t.$$

For fixed $\mathbf{Y} \in \boldsymbol{\mathcal{D}}_{Z,T}(\mathbb{R}^K)$, the mapping $\psi \mapsto \psi\mathbf{Y} := (\psi Y, \psi Y')$ is a linear and continuous operation from $C_T^\beta(\mathbb{R})$ to $\boldsymbol{\mathcal{D}}_{Z,T}(\mathbb{R}^K)$ for all $\beta \geq 2\alpha$. Moreover,

$$|\psi\mathbf{Y}|_{\mathbf{Z}} \leq |\psi|_\beta |\mathbf{Y}|_{\mathbf{Z}}(|Y|_\infty + |Y'|_\infty). \tag{B.4}$$

Thus, by the continuity of the rough path integral (Theorem A.8), we obtain

$$\lim_{n\to\infty} \int_a^b \phi_t^n Y_t \mathrm{d}\mathbf{Z}_t = \int_a^b \phi_t Y_t \mathrm{d}\mathbf{Z}_t,$$

which completes step 1.

Step 2. We will now construct a sequence of Lipschitz functions $\{\phi^n\}_{n\in\mathbb{N}} \subset C_T^1(\mathbb{R})$ converging to the characteristic function $\mathbf{1}_{[s,t]}$, for $s, t \in \mathbb{R}$ such that $0 < s < t < T$, and then pass to the limit on both sides of (B.2) to obtain (B.3). We then extend the equality to $(s, t) \in \Delta_T^2$ by continuity. Toward this end, for large enough $n \in \mathbb{N}$ and $r \in [0, T]$, define

$$\phi_r^n = \begin{cases} 1 & r \in [s,t] \\ n(r-s)+1 & s \in [s-n^{-1}, s] \\ n(t-r)+1 & s \in [t, t+n^{-1}] \\ 0 & \text{otherwise} \end{cases},$$

so that $|\phi^n|_\infty = 1$ and $|\dot{\phi}^n|_\infty = n$ where $\dot{\phi}^n$ is the weak derivative defined by

$$\dot{\phi}^n_r = \begin{cases} n & r \in [s-n^{-1}, s] \\ -n & r \in [t, t+n^{-1}] \\ 0 & \text{otherwise} \end{cases} .$$

A classical argument shows that

$$\lim_{n\to\infty} \int_0^T \lambda_r \dot{\phi}^n_r \mathrm{d}r = \delta\lambda_{st}.$$

Since the rough integral is an increment, we have

$$\int_0^T \phi^n_r Y_r \mathrm{d}\mathbf{Z}_r = \int_{s-n^{-1}}^{s} \phi^n_r Y_r \mathrm{d}\mathbf{Z}_r + \int_s^t Y_r \mathrm{d}\mathbf{Z}_r + \int_t^{t+n^{-1}} \phi^n_r Y_r \mathrm{d}\mathbf{Z}_r.$$

If we can show that the first and last integrals converge to zero as $n \to \infty$, then we are finished. We will only show that the last term converges to zero, as the argument for the first integral is easier. Let C denote a constant that is independent of n and may vary from line to line. By Theorem A.8 and the fact that $|\phi^n \mathbf{Y}|_{\mathbf{Z}} \leq Cn$ by (B.4), we find

$$\int_t^{t+n^{-1}} \phi^n_r \mathbf{Y}_r \mathrm{d}\mathbf{Z}_r = \phi^n_t Y_t \delta Z_{t,t+n^{-1}} + \phi^n_t Y'_t \mathbb{Z}_{t,t+n^{-1}} + R^n\left(t, t+n^{-1}\right),$$

where

$$|R^n(t,t+n^{-1})| \leq C(([Z]_\alpha + [\mathbb{Z}]_{2\alpha})|\phi^n \mathbf{Y}||t+n^{-1}-t|^{3\alpha} \leq Cn^{1-3\alpha} \to 0,$$

as $n \to \infty$. Moreover,

$$\left|\phi^n_t Y_t \delta Z_{t,t+n^{-1}} + \phi^n_t Y'_t \mathbb{Z}_{t,t+n^{-1}}\right| \leq |Y|_\infty [Z]_\alpha n^{-\alpha} + |Y'|_\infty [\mathbb{Z}]_{2\alpha} n^{-2\alpha} \to 0,$$

as $n \to \infty$, which completes the proof. □

Appendix C. The variational principle for incompressible fluids on smooth paths

The purpose of this section is to explain the variational principles we formulate in this paper in the simplified setting of an incompressible homogeneous ideal fluid evolving on the torus with a smooth perturbation. The beginning of the section can be read with no knowledge of differential geometry. The rest of the section assumes some basic knowledge of differential geometry (see Section A.2).

We also explain the presence of the so-called line-element stretching term in our main equation. The presence of this term distinguishes our equations from a pure transport perturbation of the deterministic Euler equation on flat space in velocity form. In particular, we show that the stretching term arises as a direct consequence of the variational principle and not by how momentum is characterized; that is to say, our variational principle indirectly enforces a covariant formulation, which naturally leads to a Kelvin's circulation theorem, helicity conservation in dimension three, and enstrophy conservation in dimension two.

By appealing to the Helmholtz decomposition (Hodge decomposition), we explicitly show the decomposition of the pressure terms into the unperturbed and perturbed part, which motivates the corresponding decomposition in the rough case. As a result of the presence of the stretching term, our equations do not preserve mean-freeness (i.e., harmonic-freeness) unless we impose an additional constraint in the variational principle. By imposing this constraint, the velocity u can be recovered directly from the vorticity $\tilde{\omega} = \nabla \times u$ via the Biot-Savart law. In vorticity form, our equations are a pure transport perturbation of the deterministic Euler equation in dimension two. The associated vorticity equation in the Brownian setting has been studied in the literature with u recovered directly from the vorticity ω via Biot-Savart [28,43,42,27].

We consider an incompressible homogeneous fluid moving on the flat d-dimensional torus $\mathbb{T}^d$ with the standard volume form dV. Denote by $\mathfrak{X}$ the space of smooth vector fields, $\mathfrak{X}_{dV}$ the space of smooth divergence-free vector fields and $\dot{\mathfrak{X}}_{dV}$ the space of smooth divergence and mean-free vector fields. It follows that

$$\mathfrak{X} = \mathfrak{X}_{dV} \oplus \nabla C^\infty = \dot{\mathfrak{X}}_{dV} \oplus \mathbb{R}^d \oplus \nabla C^\infty,$$

where the decomposition is orthogonal with respect to the L^2-inner product. Let $P : \mathfrak{X} \to \mathfrak{X}_{dV}$, $Q : \mathfrak{X} \to \nabla C^\infty$, $\dot{P} : \mathfrak{X} \to \dot{\mathfrak{X}}_{dV}$, and $H : \mathfrak{X} \to \mathbb{R}^d$ denote the corresponding projections (see Section A.2.3 and (A.17)). We recall that in dimension three, $\text{curl} : \dot{\mathfrak{X}}_{dV} \to \dot{\mathfrak{X}}_{dV}$ is an isomorphism, and in dimension two, $\text{curl} : \dot{\mathfrak{X}}_{dV} \to C^\infty$ is an isomorphism. Denote the inverse of curl by BS (for Biot-Savart).

We assume that the Eulerian velocity field $v : [0,T] \to \mathfrak{X}_{dV}$ of the fluid admits a decomposition into a sum of a dynamical velocity variable $u : [0,T] \to \dot{\mathfrak{X}}_{dV}$ and a *known* model vector field $\zeta : [0,T] \to \mathfrak{X}_{dV}$:

$$v_t = u_t + \zeta_t, \tag{C.1}$$

The vector field ζt in (C.1) admits the specified decomposition

$$\zeta_t(x) = \xi(x)\dot{Z}_t = \sum_{k=1}^{K} \xi_k(x)\dot{Z}_t^k, \quad (t,x) \in [0,T] \times \mathbb{T}^d,$$

where $\xi \in \dot{\mathfrak{X}}_{dV}^K$ and $Z : [0,T] \to \mathbb{R}^K$ in this appendix is a differentiable path, as opposed to the rough paths in the main text.

Review of geometric ideal incompressible fluid dynamics. In ideal incompressible fluid dynamics, the fluid flow is obtained as a smooth, time-dependent volume-preserving diffeomorphism $\eta : [0,T] \times \mathbb{T}^d \to \mathbb{T}^d$ by integrating the velocity vector field

$$\dot{\eta}_t = v_t \circ \eta_t = u_t \circ \eta_t + \xi_t \circ \eta_t, \quad \eta_0 = \mathrm{id}\,.$$

In fact, η may be regarded as a curve in the group of volume-preserving diffeomorphisms on M, denoted by $G = \mathrm{Diff}_{dV}(\mathbb{T}^d)$ and endowed with some appropriate topology. The Lagrangian, or material, velocity, is the velocity of the particle labelled by $X \in \mathbb{T}^d$ at time t. The Lagrangian velocity is given by $U_t(X) = \dot{\eta}_t X = v_t(\eta_t X)$; that is, $U = v \circ \eta$. The Eulerian velocity, which is the velocity of the particle currently in position $x \in \mathbb{T}^d$ at time t (i.e., $x = x(X,t) = \eta_t X$), can be expressed as

$$v_t(x) = U_t(X) = U_t(\eta_t^{-1} x) \quad \text{or} \quad v_t = \dot{\eta}_t \eta_t^{-1} = T_{\eta_t} R_{\eta_t^{-1}} \dot{\eta}_t,$$

where the notation in the right-most expression is the right action (technically the tangent lift of the action) of the inverse map η_t^{-1} on the tangent vector $\dot{\eta}_t \in T_{\eta_t} G$ by the inverse map η_t^{-1}. The action by the inverse map translates the tangent vector $\dot{\eta}_t$ at η_t back to the identity $\mathfrak{g} = T_{\mathrm{id}} G \cong \mathfrak{X}_R(G) \cong \mathfrak{X}_{dV}$ (the space of divergence-free vector fields). It follows that $v_t = \dot{\eta}_t \eta_t^{-1}$ is invariant under the action of the diffeomorphisms from the right given by $\eta_t \to \eta_t h$ for any fixed diffeomorphism $h \in \mathrm{Diff}_{dV}$. This symmetry corresponds to the well-known invariance of the Eulerian fluid velocity vector field v_t under relabelling of the Lagrangian coordinates as $X \to hX$. As discussed in Section 3.2, right-invariance is the key to understanding the Kelvin circulation theorem from the viewpoint of Noether's theorem.

Clebsch constrained variational principle. In order to derive an equation for u, we will apply a Clebsch constrained variational principle. For arbitrary $u : [0,T] \to \dot{\mathfrak{X}}_{dV}$ and $\lambda, a : [0,T] \times \mathbb{T}^d \to \mathbb{R}^d$, we define

$$S(u,a,\lambda) = \int_0^T \int_{\mathbb{T}^d} \left[\frac{1}{2}|u_t|^2 + \sum_{q=1}^d \lambda_t^q \left(\partial_t a_t^a + (v_t \cdot \nabla) a_t^q \right) \right] dV \mathrm{d}t, \tag{C.2}$$

where until otherwise specified we will work in the Euclidean coordinate system. Henceforth, we will also drop the summation over $q \in \{1, \ldots, d\}$.

The history of the Clebsch constrained variational principle $\boldsymbol{\delta} S(u,a,\lambda) = 0$ goes back to [38], as reviewed for fluid dynamics, e.g., in [106]. The first term in the Clebsch action integrand in (C.2) corresponds to the kinetic energy of the unperturbed velocity u in the decomposition (C.1), rather than that of the total velocity, v. The second term indirectly imposes the constraint $\dot{\eta} = v \circ \eta$ through the advection relation. Indeed, the method of characteristics shows for a given $a_0 : \mathbb{T}^d \to \mathbb{R}$ that the path $a_t = a_0(\eta_t^{-1}) = \eta_{t*} a_0$ (the push-forward of a_0 by η_t) satisfies the advection equation

$$\partial_t a_t + (v_t \cdot \nabla) a_t = \partial_t a_t + (u_t \cdot \nabla) a_t + (\xi \cdot \nabla) a_t \dot{Z}_t = 0.$$

To continue, we consider variations of the form

$$u^\epsilon = u + \epsilon \boldsymbol{\delta} u, \quad a^\epsilon = a + \epsilon \boldsymbol{\delta} a, \quad \lambda^\epsilon = \lambda + \epsilon \boldsymbol{\delta} \lambda, \quad \epsilon \in (-1, 1),$$

for arbitrarily given $\boldsymbol{\delta} u : [0, T] \to \dot{\mathfrak{X}}_{dV}$ and $\boldsymbol{\delta} a, \boldsymbol{\delta} \lambda : [0, T] \times \mathbb{T}^d \to \mathbb{R}^d$ such that $\boldsymbol{\delta} u, \boldsymbol{\delta} a, \boldsymbol{\delta} \lambda|_{t=0,T} \equiv 0$. Upon taking these variations of the action functional, one finds

$$0 = \boldsymbol{\delta} S(u, a, \lambda) = \int_0^T \int_{\mathbb{T}^d} \big[(u + \lambda \nabla a) \cdot \boldsymbol{\delta} u + \lambda \left(\partial_t \boldsymbol{\delta} a + (v \cdot \nabla) \boldsymbol{\delta} a \right) + \boldsymbol{\delta} \lambda \left(\partial_t a + (v \cdot \nabla) a \right) \big] dV \mathrm{d}t. \tag{C.3}$$

Here, the dot product with '$\cdot$' denotes the inner product on $\mathbb{R}^d$ relative to the Euclidean coordinate system (i.e., the flat metric δ_{ij}). We note also that since u and $\boldsymbol{\delta} u$ are constrained to be mean-free and divergence-free, we have

$$\int_{\mathbb{T}^d} (u + \lambda \nabla a) \cdot \boldsymbol{\delta} u dV = \int_{\mathbb{T}^d} \left(u + \dot{P} \lambda \nabla a \right) \cdot \boldsymbol{\delta} u dV.$$

By using integration by parts in space and time, we find that (u, a, λ) is a critical point of S if and only if

$$u = -\dot{P}(\lambda \cdot \nabla a), \qquad \partial_t \lambda + (v \cdot \nabla) \lambda = 0, \qquad \partial_t a + (v \cdot \nabla) a = 0.$$

It follows that

$$\begin{aligned} \partial_t \dot{P} u &= -\dot{P} \partial_t \lambda \nabla a - \dot{P} \lambda \nabla \partial_t a = \dot{P}((v \cdot \nabla) \lambda) \nabla a + \dot{P} \lambda \nabla ((v \cdot \nabla) a) \\ &= \left(\dot{P}((v \cdot \nabla) \lambda) \nabla a + \dot{P} \lambda (v \cdot \nabla) \nabla a \right) + \dot{P} \lambda \partial_{x^j} a \nabla v^j \\ &= -\dot{P}(v \cdot \nabla) u - \dot{P}(\nabla v)^T \cdot u. \end{aligned} \tag{C.4}$$

Here $((\nabla v)^T \cdot u)^i := \delta^{ij} u^k \partial_{x^j} v^k$, and we have used the δ^{ij} in order to maintain the geometric index convention even though we are working on flat space. Therefore,

$$\partial_t u_t + \dot{P}(v_t \cdot \nabla) u_t + \dot{P}(\nabla v_t)^T \cdot u_t = 0 \quad \Leftrightarrow \quad \partial_t u_t + (v_t \cdot \nabla) u_t + (\nabla v_t)^T \cdot u_t = -\nabla p_t + c_t. \tag{C.5}$$

In terms of the projections Q and H, we find

$$-\nabla p = Q(v_t \cdot \nabla) u_t + Q(\nabla v_t)^T u_t = Q(u_t \cdot \nabla) u_t + \left(Q(\xi \cdot \nabla) u_t + Q(\nabla \xi)^T u_t \right) \dot{Z}_t$$

$$c_t = H(v_t \cdot \nabla) u_t + H(\nabla v_t)^T \cdot u_t = H(\nabla v_t)^T u_t = H(\nabla \xi)^T \cdot u_t \dot{Z}_t = \int_{\mathbb{T}^d} (\nabla \xi)^T \cdot u_t dV \dot{Z}_t.$$

We note that the pressure p enables us to enforce the constraint that u is incompressible and the constant (in space) c enables us to enforce that u is mean-free. Substituting in $v = u + \xi \dot{Z}$, we find

$$\partial_t u_t + (u_t \cdot \nabla) u_t + \left((\xi \cdot \nabla) u_t + (\nabla \xi)^T \cdot u_t \right) \dot{Z}_t = -\nabla \tilde{p}_t + c_t, \quad \tilde{p}_t = p_t + \frac{1}{2} |u_t|^2.$$

In dimension two and three, one can readily check an equivalent formulation in terms of the vorticity $\tilde{\omega} = \operatorname{Curl} u$:

$$\partial_t \tilde{\omega}_t + (v_t \cdot \nabla) \tilde{\omega}_t - \mathbf{1}_{d=3} (\tilde{\omega}_t \cdot \nabla) v_t = 0, \qquad u = \mathrm{BS}(\omega). \tag{C.6}$$

From this point on, we assume the reader is familiar with basic differential geometry (see Section A.2). Let us introduce an arbitrary coordinate system on a Riemannian manifold with metric g. Denote by $\{dx^i\}_{i=1}^d$ a global frame of Ω^1. Moreover, let the musical notation $\flat : \mathfrak{X} \to \Omega^1$ denote the isomorphism between vector fields and one-forms. Equation (C.5) can be expressed covariantly as

$$\partial_t u_t^\flat + \pounds_{v_t} u_t^\flat = -\mathbf{d} \tilde{p} + c_t^\flat, \tag{C.7}$$

where the Lie-derivative operator $\pounds_{v_t}$ acts on the one-form $u^\flat$ to produce the one-form $\pounds_{v_t} u^\flat$, given by

$$\begin{aligned} \pounds_{v_t} u^\flat = \pounds_{v_t} (g_{ki} u^k dx^i) &= \left(v_t^j \partial_{x^j} (g_{ki} u^k) + g_{kj} u^k \partial_{x^i} v_t^j \right) dx^i \\ &= \left(v_t^j u^k \partial_{x^j} g_{ki} + g_{ki} v_t^j \partial_{x^j} u^k + g_{kj} u^k \partial_{x^i} v_t^j \right) dx^i. \end{aligned}$$

Here $\mathbf{d} \tilde{p}$ is exterior derivative of the scalar-field $\tilde{p}$.

Let $\omega = \mathbf{d} u^\flat \in \Omega^2$ denote the vorticity two-form obtained by applying the exterior derivative operator $\mathbf{d}$. Since the exterior derivative commutes with the Lie derivative, one finds

$$\partial_t \omega_t + \pounds_{v_t} \omega_t = 0. \tag{C.8}$$

The two characterizations of the vorticity ω and $\tilde{\omega}$ satisfying, (C.6) and (C.8), respectively, are related by the Hodge-star operator $\star : \Omega^2 \to \Omega^{d-2}$. In dimension two, $\tilde{\omega} = \star \omega \in \Omega^0$, and in dimension three, $\tilde{\omega} = \sharp \star \omega \in \dot{\mathfrak{X}}_{dV}$. In order to obtain equation (C.6) directly from (C.8), one uses that $\sharp\star$ and the Lie derivative commute (see, e.g., [14][Section A.6]).

Kelvin circulation theorem. The covariant formulation immediately implies a Kelvin circulation theorem. Let γ denote a closed loop in $\mathbb{T}^d$. Then using Reynolds transport theorem,

$$\frac{d}{dt}\oint\limits_{\eta_t(\gamma)} u_t^\flat = \oint\limits_{\eta_t(\gamma)} (\partial_t u_t^\flat + \pounds_{v_t} u_t^\flat) = \oint\limits_{\eta_t(\gamma)} \mathbf{d}\tilde{p} = 0,$$

where one transforms the integration around the moving loop $\eta_t(\gamma)$ to the loop γ in the material frame by applying the pull back η_t^* to the integrand, then takes the time derivative, applies the dynamic definition of the Lie-derivative, transforms back and substitutes the covariant equation of fluid motion (C.7).

Helicity conservation. In three dimensions, the helicity, defined as

$$\Lambda(\tilde{\omega}) = \int\limits_{\mathbb{T}^3} u^\flat \wedge \omega = \int\limits_{\mathbb{T}^3} u^\flat \wedge \mathbf{d}u^\flat$$

measures the linkage of field lines of the divergence-free vector field $\tilde{\omega}$ [7]. Owing to (C.7) and (C.8), we have

$$\partial_t(u^\flat \wedge \omega) = -\pounds_{v_t}(u^\flat) \wedge \omega - u^\flat \wedge \pounds_{v_t}\omega - \mathbf{d}\tilde{p} \wedge \omega,$$

and hence

$$\frac{d\Lambda}{dt}(\tilde{\omega}) = \frac{d}{dt}\int\limits_{\mathbb{T}^3} u^\flat \wedge \omega = \int\limits_{\mathbb{T}^3} \mathbf{d}\tilde{p} \wedge \mathbf{d}u^\flat = \int\limits_{\mathbb{T}^3} \mathbf{d}(\tilde{p}\,\mathbf{d}u^\flat) = 0\,.$$

Thus, the linkage number of the vorticity vector field $\Lambda(\tilde{\omega})$ is preserved by the 3D Euler fluid equations (C.7).

Enstrophy conservation in two-dimensions. In two dimensions, for any $f \in C^\infty$, we find

$$\partial_t f(\tilde{\omega}_t) + (v_t \cdot \nabla f)(\tilde{\omega}_t) = 0,$$

and hence

$$\int\limits_{\mathbb{T}^2} f(\tilde{\omega}_t) dV = \int\limits_{\mathbb{T}^2} f(\tilde{\omega}_0) dV.$$

In particular, taking $f(x) = x^2$, we find

$$\int\limits_{\mathbb{T}^2} |\tilde{\omega}_t|^2 dV = \int\limits_{\mathbb{T}^2} |\tilde{\omega}_0|^2 dV,$$

which implies that enstrophy is conserved in two-dimensions.

Momentum representation. The Lie derivative of the volume form dV along v is zero since $\pounds_v dV = (\operatorname{div} v)\, dV = 0$. Thus, we can also write equation (C.5) as

$$\partial_t m_t + \pounds_{v_t} m_t = \mathbf{d}\tilde{p} \otimes dV + c_t \otimes dV, \tag{C.9}$$

where $m_t = u^\flat \otimes dV \in \mathfrak{X}^\vee := \Omega^1 \otimes \mathrm{Dens}$ denotes the space of smooth one-form densities. In Sections 2, 3.3, and 3.4, the momentum is regarded as a one-form density in order to conveniently incorporate both the inhomogeneous and compressible settings and work canonically. One can always transform between equivalent formulations once a metric and volume form have been fixed. We will now explain how one can derive the various equivalent formulations directly from the Clebsch action functional.

Clebsch constrained variational principle revisited. Let us now explain how we can directly derive (C.7) and (C.9) from the Clebsch action functional. The first term on the RHS of (C.3) can be expressed in the following three equivalent coordinate-free ways:

(i)

$$(u + \lambda\nabla a, \boldsymbol{\delta} u)_{\mathfrak{X}_{L^2}} = \int_{\mathbb{T}^d} g(u + \lambda\nabla a, \boldsymbol{\delta} u) dV, \quad \text{where } (\cdot,\cdot)_{\mathfrak{X}_{L^2}} : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R};$$

(ii)

$$\langle u^\flat + \lambda \mathbf{d}a, \boldsymbol{\delta} u\rangle = \int_{\mathbb{T}^d} \mathbf{i}_{\boldsymbol{\delta} u}(u^\flat + \lambda \mathbf{d}a) dV, \quad \text{where } \langle\cdot,\cdot\rangle : \Omega^1 \times \mathfrak{X} \to \mathbb{R};$$

(iii)

$$\langle u^\flat \otimes dV + \mathbf{d}a \otimes \lambda dV, \boldsymbol{\delta} u\rangle_{\mathfrak{X}} = \int_{\mathbb{T}^d} \mathbf{i}_v \left[(u^\flat \otimes dV + \mathbf{d}a \otimes \lambda dV) \right],$$

$$\text{where } \langle\cdot,\cdot\rangle_{\mathfrak{X}} : \mathfrak{X}^\vee \times \mathfrak{X} \to \mathbb{R}.$$

Let us denote

$$(i)\ \ m = u \in \mathfrak{X}, \quad (ii)\ \ m = u^\flat \in \Omega^1, \quad (iii)\ \ m = u^\flat \otimes dV \in \mathfrak{X}^\vee.$$

Let

$$(\lambda, a)_{L^2} = \int_{\mathbb{T}^d} \lambda a dV, \quad \text{where } (\cdot,\cdot) : \Omega^0 \times \Omega^0 \to \mathbb{R}.$$

It follows that

$$(i)\ \ (\lambda, \pounds_v a)_{L^2} = -(\lambda \diamond a, v)_{\mathfrak{X}_{L^2}}, \quad (ii)\ \ (\lambda, \pounds_v a)_{L^2} = -\langle \lambda \diamond a, v\rangle, \quad \text{or}$$
$$(iii)\ \ (\lambda, \pounds_v a)_{L^2} = -\langle \lambda \diamond a, v\rangle_{\mathfrak{X}},$$

where

$$(i)\ \ \lambda \diamond a = -\lambda\nabla a, \quad (ii)\ \ \lambda \diamond a = -\lambda \mathbf{d}a, \quad \text{or} \quad (iii)\ \ \lambda \diamond a = -\mathbf{d}a \otimes \lambda dV,$$

respectively.

A critical point of the Clebsch action S in (C.2) then satisfies

$$\dot{P}m = \dot{P}(\lambda \diamond a),$$

where we use the same notation $\dot{P}$ for the corresponding projection onto 'divergence and harmonic-free' parts (see Section A.2.3) in all three cases. In all three cases, following a similar calculation to the one given in (C.4), we obtain

$$\partial_t m_t + \dot{P}\pounds_{v_t} m_t = 0.$$

The first case (i) agrees with the direct calculus computation given above. In general, the main ingredients of this computation (see Section 5.1) are 1) the definition of $\diamond$, 2) the relation for all $v, w \in \mathfrak{X}$ and $a \in \Omega^0$ (i.e., for all tensor fields, a) that

$$\pounds_v \pounds_w a - \pounds_w \pounds_v a = \pounds_{[v,w]} a,$$

and 3) that

$$\langle m, \operatorname{ad}_v w\rangle = \langle \pounds_v m, w\rangle,$$

for all of the above pairings. That is, $\operatorname{ad}^*_v m = \pounds_v m$. If v is not divergence-free, then $\operatorname{ad}^*_v m = \pounds_v m$ is only true for the pairing $\langle\cdot,\cdot\rangle_{\mathfrak{X}}$.

Thus, one may characterize the 'momentum' m in various ways if a metric and volume form are fixed. However, the pairing $\langle\cdot,\cdot\rangle_{\mathfrak{X}}$ is canonical in that it does not require a metric or volume form to be fixed (see the discussion in Section A.2.2), and we use this pairing above.

As a consequence of this discussion, we see that the line-element stretching term $(\nabla v_t)^T \cdot u_t$ in equation (C.5) does not arise because we have characterized momentum in a certain way. This term appears even if we treat m as a vector and work in a fixed standard coordinate system. As derived here, the stretching term tells us that the Clebsch variational principle has produced covariant coordinate-free equations. This is simply the generalized-coordinate theorem for the covariance of variational principles, the first being the Euler-Lagrange equations in classical mechanics, which are valued for precisely this reason.

Hamilton-Pontryagin variational principle.

Another way to impose the constraint on the deterministic flow decomposition is through the *Hamilton-Pontryagin variational principle*. The Hamilton-Pontryagin action integral on $[0, T]$ is given by

$$S(u, \eta, \lambda) = \int_0^T \int_{\mathbb{T}^d} \left[\frac{1}{2}|u_t|^2 + \lambda_t \cdot \left(\dot{\eta}_t \eta_t^{-1} - v_t\right)\right] dV \mathrm{d}t.$$

Here, $\eta : [0,T] \to \mathrm{Diff}_{dV}$ is an arbitrary time-dependent map. The second-term corresponds to the Lagrangian dynamical constraint $\dot{\eta} = v \circ \eta$. A variation of η is simply a two-parameter curve $\eta : [-1,1] \times [0,T] \to \mathrm{Diff}_{dV}$ with equality of mixed-derivatives.

One refers to the stationary principle $\boldsymbol{\delta} S = 0$ for the action integral above as the *Hamilton-Pontryagin* variational principle since the Lagrangian constraint variable λ is the symmetry-reduced version of the adjoint variable in the Pontryagin maximum principle, as first discussed for fluids in [17]. To explain this analogue further, the cost may be regarded as the L^2-norm of the (control) u, the path is constrained to satisfy $\dot{\eta}_t = v_t \circ \eta_t$, the endpoints of η are treated as fixed (i.e., $\eta_0 = \mathrm{id}$ and $\eta_T = \psi$), and one seeks to find a path that minimizes the cost. However, in general, critical points are not global minimizers [25,26].

Appendix D. A few words of motivation for the theory of rough paths

Let $\{\xi_k\}_{k=1}^K \subset \mathfrak{X}_{C^\infty}$ be a family of smooth vector fields on a closed manifold M. Let $\alpha \in (0,1]$ and $Z \in C_T^\alpha(\mathbb{R}^K)$. Consider the ordinary differential equation

$$\mathrm{d}Y_t = \sum_{k=1}^K \xi_k(Y_t)\mathrm{d}Z_t^k, \quad Y_t|_{t=0} = Y_0. \tag{D.1}$$

If we can solve (D.1), then we expect for any $f \in C^\infty(M)$ that $f(Y) \in C_T^\alpha(\mathbb{R})$, and hence

$$\xi_k[f](Y) = \xi_k^i(Y)\partial_{x^i} f(Y) \in C_T^\alpha(\mathbb{R}^K).$$

If we require $2\alpha > 1$, then the integral $\int_0^t \xi_k[f](Y_s)\mathrm{d}Z_s^k$ in

$$f(Y_t) = f(Y_0) + \sum_{k=1}^K \int_0^t \xi_k[f](Y_s)\mathrm{d}Z_t^k \tag{D.2}$$

may be defined as a Young integral [121] (see Lemma A.1), and we expect to have stability properties of the solution in terms of the path Z; that is, the mapping $Z \in C_T^\alpha(\mathbb{R}^K) \mapsto f(Y) \in C_T^\alpha(\mathbb{R})$ is continuous for all $f \in C^\infty(M)$. However, if $2\alpha \le 1$, then Young integration is inadequate to develop a pathwise solution theory with a stability property.

A prime example of such a path is a realization of a K-dimensional real Brownian motion $Z_t^k = B_t^k(\omega)$, $\omega \in \Omega$, for which it is known that on a set of probability one, $B(\omega) \in C_T^\alpha(\mathbb{R}^K)$ for $\alpha < \frac{1}{2}$. Indeed, T. Lyons showed [89] (see, also, Prop. 1.1. in [59]) that there exists no separable Banach space $\mathcal{B} \subset C_T(\mathbb{R}^K)$ in which the sample paths of Brownian motion lie and for which the integral $\int_0^\cdot f_t dg_t : C_T^\infty(\mathbb{R}) \times C_T^\infty(\mathbb{R}) \to C_T^\infty$ extends in a continuous way to $\mathcal{B} \times \mathcal{B} \to C_T(\mathbb{R}^K)$. Since the integral $\int_0^t B_s^1(\omega)\mathrm{d}B_s^2(\omega)$ is expected to be the solution of the simplest differential equation driven by a two-dimensional

Brownian motion $B(\omega) = (B^1(\omega), B^2(\omega))$, the result of T. Lyons indicates that the development of a pathwise theory must take into account the additional structure of the solution Y and the path Z.

If, however, $K = 1$ or the vector fields commute (i.e. $[\xi_{k_1}, \xi_{k_2}] \equiv 0$ for all k_1, k_2), then a solution theory can be developed for continuous paths $Z \in C_T(\mathbb{R}^K)$. Indeed, H. Doss and H. Sussman [49,110] showed that the solution can be defined by

$$Y_t = \exp\left(\sum_{k=1}^{K} \xi_k Z_t^k\right) Y_0,$$

where $\exp\left(\sum_{k=1}^{K} \xi_k Z_t^k\right)$ is the flow of the vector field $\sum_{k=1}^{K} \xi_k Z_t^k$ with t-fixed at time $t = 1$ (i.e., the time-one map). This is clearly a continuous function of Z and satisfies the equation exactly if Z is differentiable. In [110][pg. 21], H. Sussman discussed the connection of pathwise solutions with so-called Wong-Zakai results/anomalies (see, e.g., [114]) and clearly indicated that: (i) extending this result to $K > 1$ in the non-commutative case would require substantially new methods; and (ii) finding such an extension would lead to significant progress in our understanding of the anomalies.

The key idea for extending the pathwise theory came from T. Lyons [107,88,87] as a *tour de force* which combined iterated integrals [101,18,91,34], control theory [35,56,111, 57,112], system identification and filtering [96,23,22], numerical schemes [29,37,113,62, 67], and renormalization [66,56,40,16].

To describe this idea, let us assume for the moment that third-order brackets vanish (i.e., $[\xi_{k_1}, [\xi_{k_2}, \xi_{k_3}]] = 0$ for all k_1, k_2, k_3) and that $Z_t^k = B_t^k(\omega)$ is a realization of a K-dimensional Brownian motion. Consider for all $(s,t) \in \Delta_T^2$, the time-one map

$$\begin{aligned}\mu_{st}(\omega) &= \exp\left(\sum_{k=1}^{K} \xi_k \delta B_{st}^k(\omega) + \frac{1}{2}\sum_{k,l=1}^{K} [\xi_l, \xi_k]\mathbb{B}_{st}^{lk}(\omega)\right)\\ &= \exp\left(\sum_{k=1}^{K} \xi_k \delta B_{st}^k(\omega) + \sum_{k<l} [\xi_l, \xi_k]\mathbb{A}_{st}^{lk}(\omega)\right),\end{aligned}$$

where the quantity

$$\mathbb{B}_{st}^{lk}(\omega) := \left(\int_s^t \int_s^{t_1} \mathrm{d}B_{t_2}^l \circ \mathrm{d}B_{t_1}^k\right)(\omega) \tag{D.3}$$

is the 2α-Hölder modification of the Stratonovich integral evaluated at ω and $\mathbb{A}_{st}^{lk}(\omega) = \frac{1}{2}\left(\mathbb{B}_{st}^{lk}(\omega) - \mathbb{B}_{st}^{kl}(\omega)\right)$. Then $Y_t(\omega) := \mu_{0t}(\omega)Y_0$ is the pathwise solution of the SDE. Thus, the notion of path is enhanced to include the addition of the iterated-integral

$$\mathbf{B}(\omega) = (B(\omega), \mathbb{B}(\omega)) \in C_T^{\alpha}(\mathbb{R}^K) \times C_{2,T}^{2\alpha}(\mathbb{R}^{K\times K}), \quad \alpha < \frac{1}{2},$$

where ω belongs to a set $\Omega' \in \mathcal{F}$ of probability one. Of course, we are able to construct a pathwise solution because probability theory (specifically, the $L^2(\Omega)$-closure of time-approximating discretizations of the quantity in (D.3)) enabled us to construct the iterated integral of the path $Z_t^k = B_t^k(\omega)$, and the Kolmogorov continuity theorem allowed us to obtain a 2α-Hölder version of the iterated integral. Furthermore, the map is stable in the sense that for any $\{B^n(\omega)\}_{n\in\mathbb{N}}$ such that $\mathbf{B}^n(\omega) = (B^n(\omega), \mathbb{B}^n(\omega)) \to \mathbf{B}(\omega)$, one has $\mu_{st}^n(\omega) \to \mu_{st}(\omega)$ as $n \to \infty$. It is in this sense that the $Y_t(\omega)$ is a pathwise solution. The reader familiar with Magnus expansions will notice that μ is essentially the second-order Magnus expansion and the expansion is exact because of the third-order bracket condition.

Use of the relation $\mathbb{B}_{st}^{lk}(\omega) + \mathbb{B}_{st}^{kl}(\omega) = \delta B_{st}^l(\omega)\delta B_{st}^k(\omega)$ shows that for all $(s,t) \in \Delta_T$ and $f \in C^\infty(M)$,

$$f(Y_t(\omega)) = f(Y_s(\omega)) + \sum_{k=1}^K \xi_k[f](Y_s(\omega))\delta B_{st}^k(\omega) + \sum_{k,l=1}^K \xi_l[\xi_k[f]](Y_s(\omega))\mathbb{B}_{st}^{lk}(\omega) + f_{st}^\sharp(\omega), \tag{D.4}$$

where $f^\sharp : \Delta_T^2 \to \mathbb{R}$ satisfies for a constant $C > 0$

$$|f_{st}^\sharp(\omega)| \le C|\xi|_{C^3}([B(\omega)]_\alpha + [\mathbb{B}(\omega)]_{2\alpha})^2|t-s|^{3\alpha}.$$

Upon defining for all $(s,t) \in \Delta_T$ and $\omega \in \Omega'$,

$$\Xi_{st} = \xi_k[f](Y_s(\omega))\delta B_{st}^k(\omega) + \xi_l[\xi_k[f]](Y_s)\mathbb{B}_{st}^{lk}(\omega) + f_{st}^\sharp(\omega),$$

and invoking $|f_{st}^\sharp(Y_s)(\omega)| \le C(\omega)|t-s|^{3\alpha}$ and $\delta_2\mathbb{B}_{s\theta t}^{lk}(\omega) = \delta B_{s\theta}^l(\omega)\delta B_{\theta t}^k(\omega)$, one can directly check that $\Xi \in C_{2,T}^{\alpha,3\alpha}(\mathbb{R})$. Hence, one may apply Lemma A.1 to construct the integral $\mathcal{I}\Xi = (\int \xi[f]\mathrm{d}\mathbf{B})(\omega)$. This integral agrees with the Stratonovich integral $\left(\int_s^t \xi[f](Y_s) \circ \mathrm{d}B_s\right)$ on a set of probability one (see Theorem A.8).

The expansion (D.4) is called the second-order Chen-Fleiss expansion in the system-identification and control literature. Here, B can be interpreted as a control. Such expansions illustrate that all information of the controls impact on the system is contained in the iterated integrals of the control. The Chen-Fleiss expansion can be obtained directly from (D.2) by formally iterating the integral (Taylor series) with $Z = B$ and then evaluating at ω. One immediately recognizes the advantage of the Magnus expansion over the Chen-Fleiss series. Namely, the Magnus expansion is an exact solution of an approximating system, while the Chen-Fleiss series is not [79]. Nevertheless, such expansions are of great utility in the study of controllability and analysis of control systems [112].

From the above discussion, we understand that for all $\omega \in \Omega'$, $(B(\omega), \mathbb{B}(\omega))$ belongs to the class of $(Z, \mathbb{Z}) \in C_T^\alpha(\mathbb{R}^K) \times C_{2,T}^{2\alpha}(\mathbb{R}^{K\times K})$ such that for all $(s,t) \in \Delta_T$

$$\delta_2\mathbb{Z}_{s\theta t}^{lk} = \delta Z_{s\theta}^l \delta Z_{\theta t}^k, \quad \mathbb{Z}_{st}^{lk} + \mathbb{Z}_{st}^{kl} = \delta Z_{st}^l \delta Z_{st}^k. \tag{D.5}$$

For $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$, the closure of the set of Lipschitz paths in $C^{\alpha}_T(\mathbb{R}^K) \times C^{2\alpha}_{2,T}(\mathbb{R}^{K\times K})$ that satisfy the above properties is called the space of geometric rough paths.

The fundamental idea of T. Lyons is that (in the general case that the Lie brackets among the ξ do not vanish) the notion exists of the solution of equations driven by geometric rough paths $\mathbf{Z} = (Z, \mathbb{Z})$ and an accompanying well-posedness theory can be developed. There are many equivalent notions of the solutions of such equations (see Lemma A.18). For example, the Chen-Fleiss expansion up to level two can be used to define an intrinsic notion of the solutions by additionally specifying that the remainder $f^\sharp$ belongs $C^{3\alpha}_{2,T}(\mathbb{R})$ for any $f \in C^\infty(M)$ [45]. Higher-order iterated integrals are needed if $\alpha < \frac{1}{3}$. However, one still needs a means of constructing $\mathbb{Z}$, and probability is the main tool used to do so. Effectively, then, the technical ingredient necessary to develop the basic theory of rough paths is the sewing lemma (Lemma A.1) [65]. To wit, the sewing lemma is used to establish the existence of integrals against $\mathbf{Z}$ and to obtain bounds on 'remainder' $f^\sharp_{st}$. It is also possible to prove that there exists a unique two-parameter flow associated with the time-one map

$$\mu_{st}(\omega) = \exp\left(\sum_{k=1}^K \xi_k \delta Z^k_{st}(\omega) + \frac{1}{2}\sum_{k,l=1}^K [\xi_l, \xi_k]\mathbb{Z}^{lk}_{st}(\omega)\right), \quad \forall (s,t) \in \Delta^2_T,$$

even if the third-order Lie-brackets of ξ do not vanish. The main ingredient in this approach is the multiplicative sewing lemma, developed by I. Bailleul [8].

A prophethetical quote of M. Fleiss [56][pg. 33] translated into English reads,

> We know (cf. Schwartz [103]) that it is generally impossible to multiply the distributions and, in particular, that the powers $\delta^2, \delta^3, \cdots$, of the Dirac impulse are not distributions. Similarly here, we cannot represent the square of a Dirac impulse by a series of Chen. However, it is possible to propose what is called in physics a renormalization (et. Güttinger [66]) based on natural combinatorial considerations.

T. Lyons showed that by postulating the existence of objects $\mathbb{Z}$ which satisfy (D.5), a solution theory can be developed for differential equations driven by rough paths. As explained above, probability is used to construct $\mathbb{Z}$. Thus, probability can be understood as a tool to renormalize through its construction of otherwise analytically ill-defined quantities $\mathbb{Z}$ – and it is only this quantity that needs to be defined to construct a solution. M. Hairer extended the T. Lyons program by developing the theory of *regularity structures* as the basis of a solution theory for stochastic partial differential equations driven by white noise [68] (see, also, [59]). One of the key theorems in M. Hairer's theory is the Reconstruction Theorem, which is a substantial generalization of the sewing lemma. As predicted by M. Fleiss [56] and H. Sussman [110], this theory has had a transformative impact on renormalization in statistical physics, and of our understanding of stochastic differential equations (in finite and infinite dimensions) and the so-called Wong-Zakai anomalies.

Appendix E. Gaussian rough paths

A broad class of geometric rough paths are given by the Gaussian rough paths. Fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a K-dimensional Gaussian process $\{Z_t\}_{t\leq T}$ with independent components and zero mean. Let $R_k(s,t) = \mathbb{E}[Z_s^k Z_t^k]$ denote the corresponding covariance functions and

$$R_{k,uv}^{st} = \mathbb{E}[\delta Z_{st}^k \delta Z_{uv}^k] = R_k(s,u) + R_k(t,v) - R_k(s,v) - R_k(t,u).$$

The existence of a rough path lift for X is contingent upon sufficient rate of decay of the correlation of the increments. If for a given $q \in [1, \frac{3}{2})$, there exist a constant $C > 0$ such that for all $k \in \{1, \dots, K\}$ and $(s,t) \in \Delta_T$,

$$\sup_{\mathcal{P}([s,t]^2)} \sum_{[t_i,t_{i+1}]\times[s_i,s_{i+1}]\in\mathcal{P}([s,t]^2)} |R_{k,s_i,s_{i+1}}^{t_i,t_{i+1}}|^q \leq C|t-s|, \tag{E.1}$$

where the supremum is taken over all finite partitions $\mathcal{P}([s,t]^2)$ of the interval $[s,t]^2$, then there is a random variable $\mathbb{Z}$ and set $\bar{\Omega} \in \mathcal{F}$ for which $\mathbb{P}(\bar{\Omega}) = 1$ and such that for all $\omega \in \bar{\Omega}$, $\mathbf{Z}(\omega) = (Z(\omega), \mathbb{Z}(\omega))) \in \mathcal{C}_{g,T}^{\alpha}(\mathbb{R}^K)$ for $\alpha \in (\frac{1}{3}, \frac{1}{2q})$. Furthermore, the lift is canonical in the sense that for all $(s,t) \in \Delta_T^2$,

$$\lim_{|\mathcal{P}([s,t])|\to 0} \mathbb{E}\left| \sum_{[t_i,t_{i+1}]\in\mathcal{P}([s,t])} \delta Z_{st_i} \otimes \delta Z_{t_i t_{i+1}} - \mathbb{Z}_{st} \right|^2 = 0,$$

where $\mathcal{P}([s,t])$ denotes a finite partition of the interval $[s,t]$ and $|\mathcal{P}([s,t])|$ denotes its mesh size and the integral is understood in the sense of a limit of nets.

If X is stationary and

$$\sigma_k^2(\tau) := R_{k,t(t+\tau)}^{t(t+\tau)} \tag{E.2}$$

is concave and non-decreasing as a function of τ on an interval $[0,h]$ for some $h > 0$ and there is a constant $C > 0$ such that for all $k \in \{1, \dots, K\}$ and $\tau \in [0,h]$,

$$|\sigma_k^2(\tau)| \leq C|\tau|^{\frac{1}{q}},$$

then (E.1) holds. We refer the reader to [59][Ch. 10] and [58][Ch. 15] for a more thorough expositions.

Example E.1 *(Fractional Brownian motion).* The prototypical Gaussian process satisfying these assumptions is a K-dimensional fractional Brownian motion B^H, $H \in (\frac{1}{3}, 1]$, which has the covariance function

$$R^H(s,t) = \frac{1}{2}\left[s^{2H} + t^{2H} - |t-s|^{2H}\right] \times I_K \quad \Rightarrow \quad \sigma_k^2(\tau) = \tau^{2H},$$

where I_K is the $K \times K$-identity matrix and $\sigma_k^2(\tau)$ is defined in (E.2). Thus, B^H lifts to a geometric rough path $\mathbf{B}^H(\omega) = (B^H(\omega), \mathbb{B}^H(\omega)) \in \mathcal{C}^\alpha_{g,T}(\mathbb{R}^K)$, $\alpha \in (\frac{1}{3}, \frac{1}{4H})$ for all ω in a set of probability one. In particular, for $H = \frac{1}{2}$, $B := B^{\frac{1}{2}}$ is a standard Brownian motion, $\mathbf{B}(\omega) = (B(\omega), \mathbb{B}(\omega)) \in \mathcal{C}^\alpha_{g,T}(\mathbb{R}^K)$, $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$, and

$$\mathbb{B}_{st}(\omega) = \left(\int_s^t \delta B_{st_2} \otimes \circ \mathrm{d}B_{t_1} \right)(\omega), \quad (s,t) \in \Delta_T^2.$$

We note that

$$\mathbb{B}^{lk}_{st}(\omega) \neq \int_s^t \delta B^l_{st_2}(\omega) \circ \mathrm{d}B^k_{t_1}(\omega),$$

because stochastic integrals are defined for non-simple processes via an $L^2(\Omega)$-closure and there is no pathwise way (i.e., in the sense that it is robust under smooth approximations of the path) to make sense of the right-hand-side other than by simply defining via the left-hand-side.

Example E.2 *(Volterra Gaussian processes).* A Volterra kernel $K : [0,T]^2 \to \mathbb{R}$ is a square integrable function such that $K(s,t) = 0$ for $s \geq t$. One can find conditions on the kernels $K : [0,T]^2 \to \mathbb{R}$ such that the corresponding Volterra Gaussian processes

$$Z_t = \int_0^T K(t,s)\mathrm{d}B_s, \qquad R(s,t) = \int_0^{t\wedge s} K(t,r)K(s,r)\mathrm{d}r,$$

can be lifted to a geometric rough path. We refer the reader to [31] for a more in depth discussion of Volterra Gaussian processes and even how to extend the setup to more irregular paths. Fractional Brownian motion, Riemann-Liouville, and more simply, Ornstein-Uhlenbeck processes are all examples of Volterra Gaussian rough paths.

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