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ORIGINAL PAPER

Asymptotic geometry and Delta-points

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Abstract

We study Daugavet- and Δ-points in Banach spaces. A norm one element x is a Daugavet-point (respectively, a Δ-point) if in every slice of the unit ball (respectively, in every slice of the unit ball containing x) you can find another element of distance as close to 2 from x as desired. In this paper, we look for criteria and properties ensuring that a norm one element is not a Daugavet- or Δ-point. We show that asymptotically uniformly smooth spaces and reflexive asymptotically uniformly convex spaces do not contain Δ-points. We also show that the same conclusion holds true for the James tree space as well as for its predual. Finally, we prove that there exists a superreflexive Banach space with a Daugavet- or Δ-point provided there exists such a space satisfying a weaker condition.

Keywords Delta-point · Daugavet-point · Asymptotic uniform smoothness · Asymptotic uniform convexity · Uniformly non-square norm

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1 Introduction

Daugavet- and Δ-points first appeared in [1] as natural pointwise versions of geometric characterizations of the Daugavet property [44, Lemma 2] and of the so-called spaces with bad projections [27, Theorem 1.4] (also known as spaces with the diametral local diameter two property (DLD2P) [8]). We refer to Sect. 2 for precise definitions and equivalent reformulations.

From their introduction on, Daugavet- and Δ-points attracted a lot of attention and were intensively studied in classical Banach spaces [1–3, 23]. In particular, a strong emphasis was put on finding linear or geometric properties that would prevent norm one elements in a space to be Daugavet- or Δ-points. It soon appeared that even nice properties which on a global level prevent the space to have the Daugavet property or the DLD2P do not provide an obstruction to the existence of Daugavet- or Δ-points in the space. For example, there exists a Banach space with a 1-unconditional basis such that the set of Daugavet-points are weakly dense in the unit ball [3, Theorem 4.7].

Another striking example illustrating this was obtained in the context of Lipschitz-free spaces. The study of Daugavet- and Δ-points in this context started in [30] where a characterization of Daugavet-points in free-spaces over compact metric spaces was discovered. It was observed that Daugavet-points have to be at distance 2 from every denting point of the unit ball in any given Banach space [30, Proposition 3.1], and it was proved that the converse holds in every free-space in the compact setting. Extending this result to the general setting, Veeorg was then able to provide in [46] a surprising example of a metric space whose free space has the Radon–Nikodým property (RNP) and admits a Daugavet-point.

In the present paper, we continue the investigation of Daugavet- and Δ-points in general Banach spaces by focusing on the interactions between those points and the asymptotic geometry of the space. We provide new examples of Banach spaces failing to contain Δ-points and we introduce weaker notions which can be viewed as a step forward in the direction of constructing an example of a superreflexive space with a Daugavet- or a Δ-point.

Let us now describe the content of the paper and expose our main results. In Sect. 2 we recall the notion of slices, give the definition of Daugavet- and Δ-points, and state a few simple geometric lemmata. As a warm up, we give a simple proof that uniformly non-square spaces do not admit Δ-points and explain why simple considerations on the diameter of slices cannot rule out Δ-points outside of this setting. We end the section with the necessary background on asymptotic uniform properties of Banach spaces.

In Sect. 3, we focus on asymptotic smoothness. Our main result there is a condition on the modulus of asymptotic smoothness $\overline{\rho}_X(t,x)$ at some point x which prevents the considered point to be a Δ-point. We will show in particular that no asymptotically smooth point can be a Δ-point. As a consequence, we obtain that asymptotically uniformly smooth spaces fail to contain Δ-points. We then apply this theorem to obtain new examples of classical spaces failing to contain Δ

-points, specifically spaces with Kalton's property (M^*) and the predual JT_* of the James tree space.

In Sect. 4, we obtain partial pointwise results for asymptotic convexity. As a consequence, we show that spaces with the property (α) of Rolewicz and in particular reflexive asymptotically uniformly convex spaces do not admit Δ-points. As a consequence we obtain that the Baernstein space B as well as its dual do not admit Δ-points.

Motivated by the result from the preceding section, we look in Sect. 5 at the existence of Daugavet- and Δ-points in the James tree space JT. Our main result there is that JT does not admit Δ-points. We also have some partial results for the dual JT^*.

In the final section, we introduce a natural weaker notion of $(2-\varepsilon)$ Daugavet- and Δ-points and we look at Banach spaces in which such points exits for every $\varepsilon > 0$. We show that if there exists a superreflexive space X which has a $(2-\varepsilon)$ Daugavet-point (respectively, a $(2-\varepsilon)$ Δ-point) for every $\varepsilon > 0$, then there exists a superreflexive space (an ultrapower of X) with a Daugavet-point (respectively, a Δ-point).

2 Preliminaries

Let X be a Banach space, B_X its unit ball, S_X its unit sphere, and X^* its dual space. We only consider real Banach spaces.

For any $x^* \in S_{X^*}$ and $\delta > 0$ we define a *slice* of B_X by

$$S(x^*, \delta) := \{y \in B_X : x^*(y) > 1 - \delta\}.$$

The corresponding closed slice is denoted

$$\overline{S}(x^*, \delta) := \left\{y \in B_X : x^*(y) \geq 1 - \delta\right\}.$$

Slices and closed slices of B_X can also be defined using non-zero functionals in X^* by replacing the 1 above with the norm of the corresponding functional. We will also informally say that a Banach space has *thin slices of arbitrary large diameter*, when the diameter of $S(x^*, \delta)$ is as close as we want to 2 for small δ.

For every $x \in X$, let us write $D(x) := \{x^* \in S_{X^*} : x^*(x) = \|x\|\}$. We will make extensive use of the following lemma.

Lemma 2.1 *Let $n \geq 1$, $x_1^*, \ldots, x_n^* \in S_{X^*}$, and $\delta > 0$. Define $\delta' := \frac{\delta}{2n}$. If $\left\|\frac{1}{n}\sum_{i=1}^n x_i^*\right\| > 1 - \delta'$, then*

$$S\left(\frac{1}{n}\sum_{i=1}^n x_i^*, \delta'\right) \subset \bigcap_{i=1}^n S(x_i^*, \delta).$$

Moreover, if $x \in S_X$ and $x_1^*, \ldots, x_n^* \in D(x)$, then the inclusion holds with $\delta' = \frac{\delta}{n}$ for all $\delta > 0$.

Proof Let $x_1^*, \dots, x_n^* \in S_{X^*}$ and $\delta > 0$. Assume that $x^* := \frac{1}{n}\sum_{i=1}^n x_i^*$ satisfies $\|x^*\| > 1 - \delta'$ where $\delta' := \frac{\delta}{2n}$. For every $y \in S\big(x^*, \frac{\delta}{2n}\big)$ we have $x^*(y) > \|x^*\| - \delta' > 1 - 2\delta'$. Thus for every $1 \le j \le n$, we have

$$\frac{1}{n}x_j^*(y) > 1 - \frac{\delta}{n} - \frac{1}{n}\sum_{\substack{i=1 \\ i\neq j}}^{n} x_i^*(y)$$

so that

$$x_j^*(y) > n - \delta - (n-1) = 1 - \delta$$

and $y \in S(x_j^*, \delta)$. The conclusion follows.

Now if we fix $x \in S_X$ and $x_1^*, \dots, x_n^* \in D(x)$, then $x^*(x) = 1$ and $\|x^*\| = 1$. Similar computations show that $S\big(x^*, \frac{\delta}{n}\big) \subset \bigcap_{i=1}^n S(x_i^*, \delta)$ for all $\delta > 0$. □

Let X be a Banach space and $x \in S_X$. For $\varepsilon > 0$ we define

$$\Delta_\varepsilon(x) := \{y \in B_X : \|x - y\| \ge 2 - \varepsilon\}.$$

Following [1], we say that x is a *Daugavet-point* if $B_X = \overline{\text{conv}}\,\Delta_\varepsilon(x)$ for all $\varepsilon > 0$ and we say that x is a Δ-*point* if $x \in \overline{\text{conv}}\,\Delta_\varepsilon(x)$ for all $\varepsilon > 0$.

Using Hahn–Banach separation, we get the following well-known lemma that we will use without reference.

Lemma 2.2 *Let X be a Banach space and let $x \in S_X$.*

(a) *x is a Daugavet-point if and only if for all $\varepsilon > 0$, all $\delta > 0$, and all $x^* \in S_{X^*}$, there exists $y \in S(x^*, \delta)$ such that $\|x - y\| > 2 - \varepsilon$.*
(b) *x is a Δ-point if and only if for all $\varepsilon > 0$, all $\delta > 0$, and all $x^* \in S_{X^*}$ such that $x \in S(x^*, \delta)$, there exists $y \in S(x^*, \delta)$ such that $\|x - y\| > 2 - \varepsilon$.*

The above lemma appears in [1, Lemma 2.1], but note that there is a misprint in the statement of (2) in [1, Lemma 2.1]. Following [30] we say that an element $x^* \in S_{X^*}$ is a *weak* Daugavet-point* if for all $\varepsilon > 0$, all $\delta > 0$, and all $x \in S_X$ (naturally identified with an element of X^{**}) there exists $y^* \in S(x, \delta)$ such that $\|x^* - y^*\| > 2 - \varepsilon$. Similarly we say that $x^* \in S_{X^*}$ is a *weak* Δ-point* if for all $\varepsilon > 0$, all $\delta > 0$, and all $x \in S_X$ such that $x^* \in S(x, \delta)$, there exists $y^* \in S(x, \delta)$ such that $\|x^* - y^*\| > 2 - \varepsilon$.

A Banach space is said to be *uniformly non-square* if there exists $\varepsilon > 0$ such that for all $x, y \in B_X$ we have either $\|\frac{1}{2}(x + y)\| \le 1 - \varepsilon$ or $\|\frac{1}{2}(x - y)\| \le 1 - \varepsilon$.

Uniformly non-square spaces were introduced by James [28]. It is well known that uniformly non-square spaces are superreflexive and that uniformly convex and uniformly smooth spaces are uniformly non-square (see, e.g., [32, Proposition 1]). The following simple observations characterizes uniformly non-square spaces in terms of slices.

Proposition 2.3 *Let X be a Banach space. The following are equivalent.*

(a) *X is uniformly non-square.*
(b) *There exists $\varepsilon > 0$ such that for all $x^* \in S_{X^*}$ the diameter of $S(x^*, \varepsilon)$ is less than $2 - 2\varepsilon$.*

Proof (a) $\Rightarrow$ (b). Let $\varepsilon > 0$ be such that for all $x, y \in B_X$ we have either $\|\frac{1}{2}(x+y)\| \leq 1 - \varepsilon$ or $\|\frac{1}{2}(x-y)\| \leq 1 - \varepsilon$. If $x^* \in S_{X^*}$ and $x, y \in S(x^*, \varepsilon)$, then $\|x+y\| \geq x^*(x+y) \geq 2 - 2\varepsilon$, so $\|x-y\| \leq 2 - 2\varepsilon$.

(b) $\Rightarrow$ (a). If X is not uniformly non-square, then for every $\varepsilon > 0$ there exists $x, y \in B_X$ such that $\|x+y\| > 2 - \varepsilon$ and $\|x-y\| > 2 - \varepsilon$.

Find $x^* \in S_{X^*}$ such that $x^*(x+y) > 2 - \varepsilon$ and observe that $x, y \in S(x^*, \varepsilon)$ with $\|x-y\| > 2 - \varepsilon$. □

If x is a Δ-point then for all $\delta > 0$ and $x^* \in S_{X^*}$ with $x^*(x) > 1 - \delta$ we have $\operatorname{diam} S(x^*, \delta) = 2$ so we immediately get the following.

Corollary 2.4 *Let X be a uniformly non-square Banach space. Then X does not admit Δ-points.*

If a Banach space X is *not* uniformly non-square, then for every $\varepsilon > 0$, there is a slice $S(x^*, \varepsilon)$, $x^* \in S_{X^*}$, with diameter strictly greater than $2 - 2\varepsilon$, so X admits thin slices of arbitrarily large diameter. Observe that if X is a dual space, then we may assume that the functional x^* in the proof of (b) $\Rightarrow$ (a) in Proposition 2.3 is weak* continuous. Hence a dual space which is not uniformly non-square has a unit ball with thin weak* slices of diameter arbitrary close to 2. These simple observations show that for spaces that are not uniformly non-square, there is no hope of ruling out Δ-points by some upper bound on the diameter of slices of the unit ball. Note that any Banach space X of dimension greater or equal to 2 has an equivalent norm $|\cdot|$ such that $(X, |\cdot|)$ is *not* uniformly non-square [32, Corollary 1]. In particular, Corollary 2.4 does not rule out Δ-points in superreflexive or finite dimensional spaces.

Let X be a Banach space. Let $\operatorname{cof}(X)$ denote the set of all subspaces of finite co-dimension of X. For $t > 0$ and $x \in S_X$ consider

$$\bar{\rho}_X(t, x) = \inf_{Y \in \operatorname{cof}(X)} \sup_{y \in S_Y} \{ \|x + ty\| - 1 \}$$

and

$$\bar{\delta}_X(t, x) = \sup_{Y \in \operatorname{cof}(X)} \inf_{y \in S_Y} \{ \|x + ty\| - 1 \}.$$

The *modulus of asymptotic convexity* of X is given by

$$\bar{\delta}_X(t) = \inf_{x \in S_X} \bar{\delta}_X(t, x).$$

and the *modulus of asymptotic smoothness* of X is given by

$$\bar{\rho}_X(t) = \sup_{x\in S_X} \bar{\rho}_X(t,x).$$

The space X is said to be *asymptotically uniformly smooth* (AUS for short) if $\lim_{t\to 0} t^{-1}\bar{\rho}_X(t) = 0$ and it is *asymptotically uniformly convex* (AUC) if $\bar{\delta}_X(t) > 0$ for all $t > 0$. Similarly in X^*, there is a weak*-modulus of asymptotic uniform convexity defined by

$$\bar{\delta}^*_X(t) = \inf_{x^*\in S_{X^*}} \sup_{E} \inf_{y^*\in S_E} \{\|x^* + ty^*\| - 1\},$$

where E runs through all weak*-closed subspaces of X^* of finite codimension. We say that X^* is *AUC** if $\bar{\delta}^*_X(t) > 0$ for all $t > 0$. Clearly, a dual space which is AUC* is also AUC. The converse does not hold true since the James tree space *JT* is known to have an AUC dual space (see [20]) but to admit no equivalent norm whose dual norm is AUC*.

It is well known (see, e.g., [11, Corollary 2.4]) that X is AUS if and only if X^* is AUC*, and that if X is reflexive then X is AUS if and only if X^* is AUC. The space ℓ_1 is an example of a non-reflexive AUC space.

A Banach space X is said to have the *uniform Kadec–Klee* (UKK) property if for every $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $(x_n) \subset B_X$ with $\|x_n - x_m\| \geq \varepsilon$ for all $n \neq m$ and $x_n \to x$ weakly for some $x \in X$, then it follows that $\|x\| \leq 1 - \delta$. This property was introduced by Huff [26]. The UKK property is a close relative of asymptotic uniform convexity. It is not too difficult too show that X is AUC if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in B_X$ and all weak neighborhoods V of x satisfy $\operatorname{diam}(V \cap B_X) > \varepsilon$, then $\|x\| \leq 1 - \delta$. The latter property is a generalization of UKK and it shows X AUC implies X UKK. (This generalization was used in, e.g., [33], and Lancien actually used it as a definition of UKK.) It is known that if X does not contain a copy of ℓ_1, then the reverse implication holds and X is AUC if and only if X is UKK. In particular, AUC and UKK are equivalent for reflexive spaces. The idea is that a Banach space X failing this property contains a point $x \in S_X$ and a net $(x_V) \subset B_X$ converging weakly to x such that $\|x - x_V\| \geq \varepsilon$ for every weak neighborhood V of x for some fixed $\varepsilon > 0$. Now, if X does not contain ℓ_1, the non-separable version of Rosenthal's result [42, Theorem 3] proved in [21, Theorem 2.6] tells us that the set $\{x_V\}$ is weakly sequentially dense in its weak-closure and we can, thus, extract a sequence converging weakly to x and providing an obstruction to the UKK property (up to further extractions in order to obtain an $\frac{\varepsilon}{2}$-separated sequence).

Let X be a Banach space and A a bounded subset of X. By $\alpha(A)$ we denote the *Kuratowski measure of non-compactness* of A which is defined as the infimum of $\varepsilon > 0$ such that A can be covered by a finite number of sets with diameters less than ε, that is,

$$\alpha(A) := \inf\left\{\varepsilon > 0 : A \subset \bigcup_{i=1}^{n} A_i, A_i \subset X,\ \operatorname{diam}(A_i) < \varepsilon, i = 1, 2, \dots, n\right\}.$$

A Banach space X has *Rolewicz' property* (α) if for every $x^* \in S_{X^*}$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $\alpha(S(x^*, \delta)) \leq \varepsilon$. We say that X has *uniform property* (α) if the same δ works for all $x^* \in S_{X^*}$. These properties were introduced by Rolewicz in [41]. If X has Rolewicz' property (α), then it is reflexive (see, e.g., [36]).

Implicit in Rolewicz [41, Theorem 3] is the result that X is AUC and reflexive if and only if X has uniform property (α), Rolewicz uses the term "X is Δ-uniformly convex" instead of X is AUC and reflexive. It is known that X is AUC and reflexive if and only if X is nearly uniformly convex (NUC) [26] if and only if X is Δ-uniformly convex. The difference between these two types of uniform convexity is that they use different (but equivalent) measures of non-compactness.

We also note that for dual spaces we have that if X^* is AUC* then X^* is has *weak** uniform property (α), that is, for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\alpha(S(x, \delta)) \leq \varepsilon$ for all $x \in S_X$. Our Corollary 3.4 is a pointwise version of this result. Lennard [34, Proposition 1.3] states that X^* has the weak* version of the uniform Kadec–Klee property if and only if X^* has weak* uniform property (α). Note that Lennard credits this result to Sims and he uses a different measure of non-compactness.

3 Asymptotic uniform smoothness

The goal of this section is to show that Banach spaces with an asymptotic uniformly smooth norm do not admit Δ-points.

Our first result connects the pointwise modulus of asymptotic smoothness $\overline{\rho}_X(t, x)$ at $x \in S_X$ with the measure of non-compactness of the slices defined by x. In his thesis, Dutrieux gave a proof that a separable Banach space is AUS if and only if its dual is weak* uniformly Kadec–Klee [12, Proposition 36]. Our proof of the following proposition follows closely part of the proof given by Dutrieux, but we do not assume separability.

Proposition 3.1 *Let X be a Banach space, and fix $x \in S_X$ and $\varepsilon > 0$. If there exists $t > 0$ such that $\bar{\rho}_X(t, x)/t < \varepsilon$, then we can find $\delta > 0$ such that $\alpha(S(x, \delta)) < 4\varepsilon$.*

To prove this proposition, we will need two well-known lemmas. The first lemma is from [29, Lemma 2.13]. A nice different proof can be found in the thesis of Dutrieux [12, Lemma 38].

Lemma 3.2 *Let X be a Banach space. For all $Y \in \operatorname{cof}(X)$ and all $\varepsilon > 0$, there exists a compact set K_ε such that $B_X \subset K_\varepsilon + (2 + \varepsilon)B_Y$.*

The proof of the next lemma is simple using contradiction, so we skip it.

Lemma 3.3 *Let X be a Banach space. Let $x^* \in B_{X^*}$ and $\varepsilon > 0$.*

If $\limsup_\alpha \|x^* - x^*_\alpha\| < \varepsilon$ whenever $(x^*_\alpha) \subseteq B_{X^*}$ and $x^*_\alpha \xrightarrow{w^*} x^*$, then there exists a weak-neighborhood V of x^* with $V \cap B_{X^*} \subseteq B(x^*, \varepsilon)$.*

Proof Let us assume that $\bar{\rho}_X(t,x)/t < \varepsilon$ and let us write $\delta_x = \varepsilon t - \bar{\rho}_X(t,x)$. We then have the following.

Claim. Take any $\delta < \delta_x$. For every $x^* \in S(x,\delta)$ and every net $(x^*_\alpha) \subset B_{X^*}$ such that $x^*_\alpha \to x^*$ weak* we have $\limsup_\alpha \|x^* - x^*_\alpha\| < 2\varepsilon$.

Before proving the claim let us see how to finish the proof. Pick any $\delta < \delta_x$ and take δ' such that $\delta < \delta' < \delta_x$. By the claim and by Lemma 3.3 we can find for every $x^* \in S(x,\delta')$ a weak*-neighborhood V_{x^*} of x^* with $V_{x^*} \cap B_{X^*} \subseteq B(x^*, 2\varepsilon)$. Then, $(V_{x^*})_{x^* \in S(x,\delta')}$ is an open cover of $S(x,\delta')$ and therefore, an open cover of the weak* compact set $\bar{S}(x,\delta)$. By compactness there is a finite subcover.

Proof of the Claim. Fix some $\delta < \delta_x$ and let $x^* \in S(x,\delta)$ and $(x^*_\alpha) \subset B_{X^*}$ such that $x^*_\alpha \to x^*$ weak*. We will show that $L = \limsup_\alpha \|x^* - x^*_\alpha\| < 2\varepsilon$. Again pick any δ' such that $\delta < \delta' < \delta_x$. By definition of δ_x, we then have $\bar{\rho}_X(t,x) < \varepsilon t - \delta'$ so there exists $Z \in \operatorname{cof}(X)$ such that

$$\sup_{z \in S_Z} \|x + tz\| \leq 1 + \varepsilon t - \delta'.$$

Since for any $s \in [0,1]$

$$\|x + tsz\| = \|(1-s)x + s(x+tz)\| \leq 1 + \varepsilon t - \delta',$$

we get

$$\sup_{z \in B_Z} \|x + tz\| \leq 1 + \varepsilon t - \delta'.$$

Now let $Y = Z \cap \ker(x^*)$ and let $\eta > 0$. By Lemma 3.2 there exists a compact set K in X such that $B_X \subseteq K + (2+\eta)B_Y$.

By compactness of K and boundedness of (x^*_α), we have that $x^*_\alpha \to x^*$ uniformly on K. We may, therefore, choose β such that

- $|\langle x^*_\beta - x^*, k\rangle| < \eta$ for all $k \in K$;
- $x^*_\beta(x) > 1 - \delta$;
- $|\|x^*_\beta - x^*\| - L| < \eta$.

Choose $x_\beta \in S_X$ such that $\langle x^*_\beta - x^*, x_\beta\rangle > L - \eta$. Now write $x_\beta = k_\beta + (2+\eta)y_\beta$ with $k_\beta \in K$ and $y_\beta \in B_Y$. We get

$$x^*_\beta(y_\beta) = \langle x^*_\beta - x^*, y_\beta\rangle = \frac{\langle x^*_\beta - x^*, x_\beta\rangle - \langle x^*_\beta - x^*, k_\beta\rangle}{2+\eta} > \frac{L - 2\eta}{2+\eta}$$

since $y_\beta \in \ker(x^*)$. Therefore,

$$1 - \delta + \frac{t(L-2\eta)}{2+\eta} < \langle x^*_\beta, x\rangle + \frac{t(L-2\eta)}{2+\eta} < \langle x^*_\beta, x + ty_\beta\rangle \leq \|x + ty_\beta\| \leq 1 + \varepsilon t - \delta'.$$

Finally,

$$L - 2\eta < (2+\eta)(\varepsilon - \theta)$$

with $\theta = \frac{\delta'-\delta}{t} > 0$. Since $\eta > 0$ was arbitrary, we get $L \leq 2(\varepsilon - \theta) < 2\varepsilon$ as desired.

As an immediate corollary, we get.

Corollary 3.4 *Let X be a Banach space and let $x \in S_X$ be an asymptotically smooth point, that is a point for which $\lim_{t\to 0} \frac{\bar{\rho}_X(t,x)}{t} = 0$. Then $\lim_{\delta\to 0} \alpha(S(x,\delta)) = 0$.*

We will now show that this condition on the Kuratowski index of the slices $S(x,\delta)$ prevents the point x to be a Δ-point. In fact it appears that this is the case as soon as the slice $S(x,\delta)$ admits a non-trivial covering by finitely many balls.

Theorem 3.5 *Let X be a Banach space and let $x \in S_X$. If there exists a $\delta > 0$ such that $\alpha(S(x,\delta)) < 2$, then x is not a Δ-point.*

Proof Take $\delta_0 > 0$ and $\varepsilon \in (0,2)$ such that $\alpha\big(S(x,\delta_0)\big) < \varepsilon$, pick any $\delta \leq \delta_0$ such that $\varepsilon + \delta < 2$, and let $\eta = 2 - (\varepsilon + \delta)$. We have $\alpha(S(x,\delta)) \leq \alpha\big(S(x,\delta_0)\big) < \varepsilon$, so we can find $n \geq 1$ and non-empty subsets $A_1, \dots, A_n$ of X^* with diameter smaller than ε such that $S(x,\delta) \subset \bigcup_{i=1}^n A_i$. By an easy refinement we obtain the following.

Claim. For every $\delta' \leq \delta$, we can find $1 \leq m \leq n$ and $y_1^*, \dots, y_m^* \in X^*$ such that

(a) $y_j^* \in S_{X^*} \cap S(x,\delta')$ for every $1 \leq j \leq m$;
(b) $S_{X^*} \cap S(x,\delta') \subset \bigcup_{j=1}^m B(y_j^*, \varepsilon)$.

Proof of the Claim For any $\delta' \leq \delta$, we have $S_{X^*} \cap S(x,\delta') \subset \bigcup_{i=1}^n A_i$. Now this set is not empty since it contains $D(x)$, and the set $J = \{1 \leq i \leq n : (S_{X^*} \cap S(x,\delta')) \cap A_i \neq \emptyset\}$, thus, has cardinality $|J| = m$ for some $1 \leq m \leq n$.

Picking for every $j \in J$ an element $y_j^* \in (S_{X^*} \cap S(x,\delta')) \cap A_j$, we obtain $S_{X^*} \cap S(x,\delta') \subset \bigcup_{j\in J} A_j \subset \bigcup_{j\in J} B(y_j^*, \varepsilon)$ since A_j has diameter smaller than ε, and the conclusion follows (relabeling the y_j^*'s if necessary). ☐

Now let us find $1 \leq m \leq n$ and $y_1^*, \dots y_m^* \in S_{X^*}$ as in the Claim for $\delta' = \frac{\delta}{2n}$, and let us define $y^* = \frac{1}{m}\sum_{j=1}^m y_j^*$. Since $y_j^*(x) > 1 - \delta'$ for every $1 \leq j \leq m$, we have $y^*(x) > 1 - \delta' \geq \|y^*\| - \delta'$ so that $x \in S\big(y^*, \delta'\big)$ and $\|y^*\| > 1 - \delta' \geq 1 - \frac{\delta}{2m}$. It, thus, follows from Lemma 2.1 that

$$S\Big(y^*, \frac{\delta}{2m}\Big) \subset \bigcap_{j=1}^m S(y_j^*, \delta)$$

and as a consequence, we have

$$x \in S\big(y^*, \delta'\big) \subset \bigcap_{j=1}^m S(y_j^*, \delta).$$

To conclude, let us take $y \in S(y^*, \delta')$, and let us take $z^* \in S_{X^*}$ such that $\|x - y\| = z^*(x - y)$. We distinguish two cases.

Case 1: If $z^*(x) > 1 - \delta'$, then

$$z^* \in S_{X^*} \cap S\big(x, \delta'\big) \subset \bigcup_{j=1}^{m} B(y_j^*, \varepsilon)$$

so there exists $j_0 \in \{1, \dots, m\}$ such that $\|z^* - y_{j_0}^*\| \le \varepsilon$. Now, we have $y \in S(y_{j_0}^*, \delta)$ by the above inclusion so that

$$\|x - y\| = z^*(x) + (y_{j_0}^* - z^*)(y) - y_{j_0}^*(y) \le 1 + \varepsilon - (1 - \delta) = \varepsilon + \delta = 2 - \eta.$$

Case 2: If $z^*(x) \le 1 - \delta'$, we have

$$\|x - y\| = z^*(x) - z^*(y) \le 1 - \delta' + 1 = 2 - \delta'.$$

Combining the two cases, we obtain $\|x - y\| \le \max\{2 - \delta', 2 - \eta\}$ and x cannot be a Δ-point. □

Combining Theorem 3.5 and Proposition 3.1, we then get

Proposition 3.6 *Let X be a Banach space and let $x \in S_X$ satisfy $\overline{\rho}_X(t, x) < \frac{t}{2}$ for some $t > 0$. Then, x is not a Δ-point.*

In particular, no asymptotically smooth point can be a Δ-point and we obtain

Theorem 3.7 *Let X be an AUS Banach space. Then X does not admit a Δ-point.*

Next, let us collect some examples where the above corollary applies.

Recall that a Banach space has Kalton's property (M) if whenever $x, y \in X$ with $\|x\| = \|y\|$ and (x_α) is a bounded weakly null net in X, then

$$\limsup_\alpha \|x + x_\alpha\| = \limsup_\alpha \|y + x_\alpha\|.$$

Similarly X has property (M^*) if whenever $x^*, y^* \in X^*$ with $\|x^*\| = \|y^*\|$ and (x_α^*) is a bounded weak* null net in X^*, then

$$\limsup_\alpha \|x^* + x_\alpha^*\| = \limsup_\alpha \|y^* + x_\alpha^*\|.$$

If X has property (M^*), then X has property (M) and X is an M-ideal in X^{**} (see, e.g., [24, Proposition VI.4.15]). In particular, X is an Asplund space (see, e.g., [24, Theorem III.3.1]). It is well known that property (M^*) is inherited by both subspaces and quotients (see, e.g., [38]).

By chasing references, we find that the following proposition holds.

Proposition 3.8 *Assume a Banach space X has property (M). The following are equivalent:*

(a) *X is AUS;*
(b) *X contains no copy of ℓ_1;*

(c) *X has property* (M^*).

Proof (a) ⇒ (b). If X is AUS, then X is Asplund (see, e.g., [29, Proposition 2.4]). Hence, X contains no copy of ℓ_1.

(b) ⇒ (c). If X contains no copy of ℓ_1, then no separable subspace of X can contain ℓ_1. Clearly every separable closed subspace of X has property (M) (both net and sequential version, see [37, Proposition 1]) and then they all have property (M^*) (both net and sequential version) by Theorem 2.6 in [31]. Finally, X has property (M^*) if every separable closed subspace does [38, Proposition 3.1].

(c) ⇒ (a). Dutta and Godard [13] proved that if X is a separable Banach space with property (M^*), then X is AUS. However, using property (M^*) and Proposition 2.2 in [16] one finds $\bar{\rho}_X(t,x) = \bar{\rho}_X(t)$ for all $x \in S_X$ and their proof also works in the non-separable case. □

Since c_0 has property (M^*), we have that all subspaces and quotients of c_0 are AUS and they all fail to contain Δ-points. All these examples are M-ideals in their bidual, that is, they are M-embedded [24, Chapter 3].

Note that there are M-embedded spaces which are not AUS. For example the Schreier space $\mathcal{S}$ is not AUS since it does not have property (M^*). Indeed, if a Banach space X has property (M^*), then the relative norm and weak* topologies on S_{X^*} coincide (see e.g. [24, Proposition VI.4.15]). But if (e_i) is the unit vector basis in $\mathcal{S}$ and (e_i^*) the biorthogonal functionals in the dual, then $e_2^* + e_i^* \in S_{\mathcal{S}^*}$ and converges weak* to e_2^*, but not in norm. Note however that $\mathcal{S}$ does not admit Δ-points by Proposition 2.15 in [3].

Let X be a Banach space with a normalized basis (e_i) (or more generally an FDD (E_i)). We say that (e_i) *admits block upper* ℓ_p *estimates* for some $p \in (1,\infty)$ if there is a constant $C > 0$ such that for every finite blocks $x_1, \dots, x_N$ of (e_i) with consecutive disjoint supports we have $\left\|\sum_{n=1}^N x_n\right\|^p \leq C \sum_{n=1}^N \|x_n\|^p$. We say that (e_i) admits block lower ℓ_q estimates for some $q \in (1,\infty)$ if there is a constant $c > 0$ such that for every finite blocks $x_1, \dots, x_N$ of (e_i) with consecutive disjoint supports we have $\left\|\sum_{n=1}^N x_n\right\|^q \geq c \sum_{n=1}^N \|x_n\|^q$. It is well known that a basis admitting upper ℓ_p estimates is shrinking while a basis admitting lower ℓ_q estimates is boundedly complete. The latter can be proved using the following criterion, which is left as an exercise in [4, Exercise 3.8] and whose proof can be found in [10, Proposition 3.1]: a basis is boundedly complete if and only if $\sup_N \left\|\sum_{n=1}^N x_n\right\| = \infty$ for every block sequence (x_n) of (e_i) that is bounded away from 0. The former is then obtained by duality. Applying [16, Corollary 2.4] we then have that any space with a basis admitting block upper ℓ_p estimates is AUS (with power type p) and that any space admitting a basis with block lower ℓ_q estimates is AUC* (with power type q) as the dual of the space of $Y = [e_i^*]$. As a consequence, a Banach space with a basis admitting block upper ℓ_p estimates does not admit Δ-points and the predual of a Banach space with a basis admitting lower ℓ_q estimates does not admit Δ-points. This applies in particular to the predual of the James tree space (see Sect. 5) and to the Baernstein space B (see at the end of Sect. 4).

4 Asymptotic uniform convexity

The main result of this section is that reflexive AUC spaces do not have Δ-points. The proof uses the characterization of reflexive AUC spaces in terms of the measure of non-compactness of slices and relies on the following result of Kuratowski (see e.g. [7, p. 151])

Lemma 4.1 *Let (M, d) be a complete metric space. If $(F_n)_{n=1}^{\infty}$ is a decreasing sequence of non-empty, closed, and bounded subsets of M such that* $\lim_n \alpha(F_n) = 0$, *then the intersection $F_\infty = \bigcap_{n=1}^{\infty} F_n$ is a non-empty compact subset of M.*

Let us first prove a pointwise version of the main result.

Theorem 4.2 *Let X be a Banach space and $x \in S_X$. If there exists $x_0^* \in D(x)$ such that* $\lim_{\delta\to 0} \alpha(S(x_0^*, \delta)) = 0$, *then x is not a Δ-point.*

Proof Let $x \in S_X$. Assume that $x_0^* \in D(x)$ such that $\lim_{\delta\to 0} \alpha(S(x_0^*, \delta)) = 0$ and let us assume for contradiction that x is a Δ-point.

Define a set of functionals norming x by

$$D_0(x) := \left\{ \frac{x^* + x_0^*}{2} : x^* \in D(x) \right\}.$$

If $\delta > 0$ and $f = \frac{x^*+x_0^*}{2} \in D_0(x)$, then by Lemma 2.1

$$S\left(f, \frac{\delta}{2}\right) \subset S(x^*, \delta) \cap S(x_0^*, \delta)$$

and therefore, $\lim_{\delta\to 0} \alpha(\overline{S}(f, \delta)) = \lim_{\delta\to 0} \alpha(S(f, \delta)) = 0$ for all $f \in D_0(x)$.

Given $f \in D_0(x)$ and a sequence $(\delta_n) \subseteq (0, 1)$ decreasing to 0 we define for each n

$$F_n^f := \overline{S}(f, \delta_n) \cap \Delta_{\delta_n}(x).$$

Since $x \in S(f, \delta)$ for every $\delta > 0$ and since x is a Δ-point, each F_n^f is a closed non-empty subset of B_X. Thus, since $\alpha(F_n^f) \to 0$, $F^f = \cap_n F_n^f$ is non-empty and compact for every $f \in D_0(x)$ by Lemma 4.1. Using a compactness argument, we can then prove the following.

Claim.

$$F := \bigcap_{f\in D_0(x)} F^f \neq \emptyset.$$

Before proving this claim, let us see that it will give us the desired conclusion. Indeed, if $z \in F$, then $\|x - z\| = 2$ and for all $f \in D_0(x)$ we must have $f(z) = 1$, in particular $x_0^*(z) = 1$. But this is nonsense since for any $x^* \in S_{X^*}$ with $x^*(x - z) = 2$, we must have $x^*(x) = 1$ and $x^*(z) = -1$, so

$$1 = \frac{x^* + x_0^*}{2}(z) = \frac{-1+1}{2} = 0$$

since $\frac{x^*+x_0^*}{2} \in D_0(x)$. This contradiction shows that x is not a Δ-point.

To finish the proof, we only need to prove the claim that $F \neq \emptyset$. It is enough to show that $(F^f)_{f\in D_0(x)}$ has the finite intersection property since all the sets F^f are compact and non-empty.

Let $f_1, \dots, f_k \in D_0(x)$, which means $f_j = \frac{x_j^*+x_0^*}{2}$ for $x_j^* \in D(x)$. Define

$$f = \frac{1}{k}\sum_{j=1}^{k} f_j = \frac{x_0^* + \frac{1}{k}\sum_{j=1}^{k} x_j^*}{2}.$$

Clearly $f \in D_0(x)$. By Lemma 2.1, we have for all $\delta > 0$

$$S\left(f, \frac{\delta}{k}\right) \subset \bigcap_{j=1}^{k} S(f_j, \delta)$$

and hence

$$S\left(f, \frac{\delta}{k}\right) \cap \Delta_{\frac{\delta}{k}}(x) \subset \bigcap_{j=1}^{k} S(f_j, \delta) \cap \Delta_\delta(x). \tag{1}$$

Since (δ_n) is going to 0, there must for any n exist m with $\delta_m < \delta_n/k$. By (1) we get

$$F_m^f \subset \bigcap_{j=1}^{k} F_n^{f_j}$$

and hence by commutativity of intersections

$$\emptyset \neq \bigcap_n F_n^f \subseteq \bigcap_{j=1}^{k}\bigcap_n F_n^{f_j} = \bigcap_{j=1}^{k} F^{f_j}$$

and the claim is proved. □

From Theorem 4.2, we immediately get

Theorem 4.3 *If X has Rolewicz' property (α), then X does not have Δ-points.*

As we noted in Sect. 2, a Banach space X is reflexive and AUC if and only if it has uniform property (α). Also finite-dimensional spaces are trivially AUC since for example $\alpha(S(x^*, \delta)) = 0$ for slices of B_X in finite-dimensional spaces.

Theorem 4.4 *If X is reflexive and AUC, then X does not have Δ-points.*
In particular, if X is finite-dimensional then X does not have Δ-points.

***Remark* 4.5** Let X be a Banach space such that for every $x \in S_X$ there exists $x^* \in D(x)$ with $\lim_{\delta\to 0} \alpha(S(x^*, \delta)) = 0$. Then by Theorem 4.2, X does not admit a Δ-point. Note that unlike Rolewicz' property (α) (see [40]) this property does not imply reflexivity.

Indeed, every separable Banach space has an equivalent locally uniformly rotund renorming and if (the norm of) X is locally uniformly rotund, then every $x \in S_X$ is strongly exposed by $x^* \in D(x)$ so that for all $\varepsilon > 0$ there exists $\delta > 0$ such that $S(x^*, \delta)$ has diameter less than ε. In particular, $\alpha(S(x^*, \delta)) < \varepsilon$.

Using the duality AUS/AUC in reflexive spaces, we can in fact combine Theorem 3.7 and 4.4.

Corollary 4.6 *If X is reflexive and AUC, then neither X nor X^* admit Δ-points.*

In particular, we can apply this result to the Baernstein's space B whose construction and basic properties are given in the introductory Chapter 0 of [9, Construction 0.9]. This space was originally introduced in [5] as an example of a reflexive space failing the Banach–Saks property. It is known to have a normalized (unconditional) basis with block lower ℓ_2 estimates and thus to be 2-AUC (see [39, Theorem 3] with the NUC terminology). Also the optimal modulus of near convexity of B has been estimated in [6]. From our preceding results the space B and its dual space B^* both fail to have Δ-points.

Here is a pointwise application of Theorem 4.2.

Corollary 4.7 *Let X be a Banach space and let $x^* \in S_{X^*}$ be a norm one functional which attains its norm at some $x \in S_X$. If $\lim_{t\to 0} \frac{\overline{\rho}(t,x)}{t} = 0$, then x^* is not a Δ-point.*

Proof By Corollary 3.4 we have $\lim_{\delta\to 0} \alpha(S(x, \delta)) = 0$ and the conclusion follows directly from Theorem 4.2 since $x \in D(x^*)$. □

Let X be a Banach space. An element $x \in B_X$ is said to be a *quasi-denting point* if given $\varepsilon > 0$, there exists $x^* \in S_{X^*}$ and $\delta > 0$ with $x \in S(x^*, \delta)$ such that $\alpha(S(x^*, \delta)) < \varepsilon$. Recall that x is a *denting-point* if the above can be strengthened to $S(x^*, \delta) \subset B(x, \varepsilon)$. In dual spaces, one can similarly define weak* (quasi-)denting points by requiring x^* to be weak* continuous. Giles and Moors [19] introduced quasi-denting points (under the name α-denting points) see, e.g., [35] or [45].

Let X be a Banach space such that the dual is AUC*, then every $x^* \in S_{X^*}$ is a quasi-denting point. This is essentially contained in e.g. [22, Proposition 4.8], but we include the straightforward argument. The AUC* is equivalent to the following version of UKK*: For every $\varepsilon > 0$, there exists $\delta > 0$ such that if $x^* \in B_{X^*}$ and all weak* neighborhoods V of x^* satisfy $\operatorname{diam}(V \cap B_{X^*}) > \varepsilon$, then $\|x^*\| \le 1 - \delta$. Now fix $\varepsilon > 0$ and choose $\delta > 0$ as above. If $x^* \in S_{X^*}$, then we can find $x \in S_X$ with $x^*(x) > 1 - \delta$. Let $0 < \delta' < \delta$. If $y^* \in S(x, \delta)$, then $\|y^*\| > 1 - \delta$ and the exits a weak* open neighborhood V_{y^*} of y^* with $\operatorname{diam}(V_{y^*} \cap B_{X^*}) \le \varepsilon$. We, therefore, have an open cover of the weak* compact set $\bar{S}(x, \delta')$ and by compactness we have a finite cover and hence x^* is a weak* quasi-denting point.

If the dual X^* is AUC*, then X is AUS and by Corollary 4.7 no norm-attaining $x^* \in S_{X^*}$ can be a Δ-point, but we do not know if a weak*-quasi-denting point, or more generally a quasi-denting point, can be a Daugavet-point or a Δ-point. For non-reflexive AUC spaces, we do not even know if every element of the unit sphere is quasi-denting.

5 The James tree space

Let $T = \{\emptyset\} \cup \bigcup_{n\geq 1}\{0,1\}^n$ be the infinite binary tree and let us denote by $\leq$ the natural ordering on T. A totally ordered subset S of T is called a segment if it satisfies:

$$\forall s, t \in S,\ [s,t] = \{u \in T\ :\ s \leq u \leq t\} \subset S.$$

An infinite segment of T is also called a branch of T. Let us denote by $\mathcal{F}$ the set of all finite families of disjoint segments in T. The James tree space JT is defined as follows:

$$JT = \left\{ x = (x_s)_{s\in T} \subset \mathbb{R}\ :\ \|x\|_{JT}^2 = \sup_{F\in\mathcal{F}} \sum_{S\in F} x_S^2 < \infty \right\},$$

where $x_S = \sum_{s\in S} x_s$ for every non-empty segment S and $x_\emptyset = 0$.

It is well known that JT is a Banach space and that the set of canonical unit vectors $\{e_t\}_{t\in T}$ of $c_{00}(T)$ forms, for the lexicographic order, a normalized monotone boundedly complete basis of JT. Moreover, it is also known that the closed linear span $JT_* = [e_t^*]_{t\in T}$ of the set of biorthogonal functionals in JT^* is a unique isometric predual of JT. If S is a segment of T or if β is a branch of T, we will sometimes refer to the set $\{e_s\}_{s\in S}$ as a segment of JT and the set $\{e_t\}_{t\in\beta}$ as a branch of JT. From the definition of the norm, it is easy to show that $\|x+y\|_{JT}^2 \geq \|x\|_{JT}^2 + \|y\|_{JT}^2$ whenever $x, y \in JT$ have totally disconnected supports (that is if $\operatorname{conv}(\operatorname{supp}_T x) \cap \operatorname{conv}(\operatorname{supp}_T y) = \emptyset$). Since we are working with the lexicographic order on T, this applies whenever $\operatorname{supp} x < \operatorname{supp} y$ with respect to the ordering of the basis, and $\{e_t\}_{t\in T}$, thus, satisfies block lower ℓ_2 estimates. It follows that JT is 2-AUC* and Theorem 3.7 applied to the 2-AUS JT_* yields the following result.

Theorem 5.1 *The predual JT_* of the James tree space does not admit Δ-points.*

Let us also emphasize that $\|x\|_{JT} \geq \|x\|_{\ell_2}$ for every $x \in JT$. It is also worth mentioning that if one consider the equivalent norm $\|x\|^2 = \sup_{S_1<\cdots<S_n} \sum_{i=1}^n x_{S_i}^2$ on the James space J, where the S_i are segments of $\mathbb{N}$, then one obtains a Banach space isometric to the closed linear span $[e_t]_{t\in\beta}$ of any branch of JT. In particular all the results we will obtain for the space JT will also apply to $(J, \|.\|)$.

As we see from Proposition 2.3, the unit ball of spaces that are not uniformly non-square contain thin slices of diameter arbitrary close to 2. Let us start by illustrating this by exhibiting slices of diameter 2 in JT.

Proposition 5.2 *For every $\delta > 0$ we can find some $x^* \in S_{JT^*}$ such that* $\operatorname{diam} S(x^*, \delta) = 2$ *and the diameter is attained.*

Proof For convenience, let us work in the space $(J, \|.\|)$ introduced above. Doing the same construction on any branch $(x_t)_{t\in\beta}$ of JT would do the work for JT.

Fix $\delta > 0$, fix $n \geq 1$, and let

$$x = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (-1)^{i+1} e_{2i-1} = \frac{1}{\sqrt{n}}(1, 0, -1, 0, \dots, -1, 0, 0, 0, \dots)$$

and

$$y = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (-1)^{i} e_{2i} = \frac{1}{\sqrt{n}}(0, -1, 0, 1, \dots, 0, 1, 0, 0, \dots)$$

so that

$$x - y = \frac{1}{\sqrt{n}}(1, 1, -1, -1, \dots, -1, -1, 0, 0, \dots)$$

and

$$x + y = \frac{1}{\sqrt{n}}(1, -1, -1, 1, 1, \dots, -1, -1, 1, 0, 0, \dots).$$

It is easy to check that $\|x\| = \|y\| = 1, \|x - y\| = 2$, and $\|x + y\| = \sqrt{\frac{1+(n-1)\times 2^2+1}{n}} = \sqrt{4 - \frac{2}{n}}$.

By assuming that n was chosen large enough so that $\sqrt{4 - \frac{2}{n}} > 2 - \delta$ and by choosing a norming functional $x^* \in S_{J^*}$ for $x + y$ we then have $x^*(x + y) = \|x + y\| > 2 - \delta$ and this implies that $x, y \in S(x^*, \delta)$. Since $\|x - y\| = 2$, the conclusion follows. □

We start by showing that the elements of the basis of JT are weak* denting (and even strongly exposed by an element of the predual) and then use this to prove that JT does not admit weak* Daugavet-points.

Lemma 5.3 *For each $t \in T$, $\lim_{\delta\to 0} \operatorname{diam} S(e_t^*, \delta) = 0$.*

Proof Let us fix $t \in T$ and $\delta > 0$, and let us take $y \in S(e_t^*, \delta)$. By the triangle inequality, we have

$$\|y - e_t\|_{JT} \leq \|y - y_t e_t\|_{JT} + (1 - y_t) \leq \|y - y_t e_t\|_{JT} + \delta$$

so we only need to estimate $\|y - y_t e_t\|_{JT}$.

If t is not in the support of a family $F \in \mathcal{F}$, that is if $t \notin \bigcup_{S\in F} S$, then the segment $\{t\}$ is disjoint from all the segments in F and we have

$$(1-\delta)^2 + \sum_{S\in F}(y - y_t e_t)_S^2 < y_t^2 + \sum_{S\in F}(y - y_t e_t)_S^2 = y_t^2 + \sum_{S\in F} y_S^2 \le \|y\|_{JT}^2 \le 1$$

so that

$$\sum_{S\in F}(y - y_t e_t)_S^2 \le 1 - (1-\delta)^2 \le 2\delta.$$

Now if we take any segment S of T containing t, we can write $S = S^- \cup \{t\} \cup S^+$ with $S^- < \{t\} < S^+$ segments of T, and by the preceding computations we have

$$(y - y_t e_t)_S^2 \le y_{S^-}^2 + y_{S^+}^2 + 2\sqrt{y_{S^-}^2 y_{S^+}^2} \le 6\delta.$$

By combining the two observations we get $\sum_{S\in F}(y - y_t e_t)_S^2 \le 8\delta$ for every $F \in \mathcal{F}$, that is $\|y - y_t e_t\|_{JT}^2 \le 8\delta$. The conclusion follows. □

Corollary 5.4 *The space JT does not admit weak* Daugavet-points.*

Proof Let us fix $x \in S_{JT}$. If $x = e_t$ for some $t \in T$ then x is a weak* denting point by the preceding lemma and x cannot be a weak* Daugavet-point. So let us assume that x is not of that form and let us take $t \in T$ such that $x_t \neq 0$ and $x_s = 0$ for every $s < t$.

Now let us write $x_t = \theta\alpha$ with $\alpha \in (0,1)$ and $\theta \in \{-1, 1\}$. Because of the choice of t we clearly have

$$\|x - x_t e_t\|_{JT}^2 = \sup_{F\in\mathcal{F}_t} \sum_{S\in F}^{n} x_S^2,$$

where $\mathcal{F}_t$ is the set of finite families of disjoint segments of T not intersecting $[\emptyset, t]$. Since

$$x_t^2 + \sum_{S\in F} x_S^2 \le \|x\|_{JT}^2$$

for every such family, we obtain

$$\|x - x_t e_t\|_{JT}^2 \le 1 - \alpha^2 < 1.$$

Thus,

$$\|x - \theta e_t\|_{JT} \le \|x - x_t e_t\|_{JT} + (1-\alpha) \le 2 - \alpha$$

so that x is at distance strictly less than 2 from a (weak*) denting point of JT. The conclusion follows from a weak* version of [30, Proposition 3.1]. □

Now, we will show that JT does not admit Δ-points. For this purpose let us introduce some more notations. Let us write $\mathcal{F}_\infty$ for the set of (finite or infinite) families of disjoint segments in T. Note that every infinite $F \in \mathcal{F}_\infty$ has to be countable. For any segment S of T, let us write $\mathbb{1}_S = \sum_{s\in S} e_s^*$. A molecule in JT^* is an element of the form $m := \sum_{i\geq 1} \lambda_i \mathbb{1}_{S_i}$, where $\{S_i\} \in \mathcal{F}_\infty$ and $\lambda := (\lambda_i) \in B_{\ell_2}$. Observe that $m(x) = \langle(\lambda_i), (x_{S_i})\rangle_{\ell_2} \leq \|\lambda\|_{\ell_2}\|x\|_{JT}$ for every $x \in JT$ by the Cauchy–Schwartz inequality and by definition of the norm of JT so that $\|m\|_{JT^*} \leq \|\lambda\|_{\ell_2} \leq 1$. The molecule is finite if the family $\{S_i\}$ contains only finitely many non-empty segments, and we will write $\mathcal{M}$ (resp. $\mathcal{M}_\infty$) for the set of finite (resp. finite or infinite) molecules of JT^*.

Molecules play an important role in the study of the space JT because they turn computations in JT into computations in ℓ_2 and because of the following result due to Schachermayer [43, Proposition 2.2].

Theorem 5.5 *The unit ball B_{JT^*} of the dual of JT is the norm closed convex hull of $\mathcal{M}$.*

Remark **5.6** It is also known that $\mathcal{M}_\infty$ is the weak* closure of $\mathcal{M}$ in B_{JT^*}.

We will use some specific molecules to provide norming functionals for elements of S_{JT}. For $x \in S_{JT}$ and $F \in \mathcal{F}_\infty$, let us write $m_{x,F} = \sum_{S\in F} x_S \mathbb{1}_S$. Since $\sum_{S\in F} x_S^2 \leq \|x\|_{JT} = 1$, those elements are molecules in JT^*. Moreover, we have the following result.

Lemma 5.7 *Let us assume that $\sum_{S\in F} x_S^2 = \|x\|_{JT} = 1$. Then the molecule $m_{x,F}$ belongs to $D(x)$ (that is, it is a norm one functional and it norms x). Moreover, if $y \in S(m_{x,F}, \delta)$ for some $\delta > 0$, then we have*

$$\sum_{S\in F}(x_S - y_S)^2 \leq 2\delta.$$

Proof By assumption $m_{x,F}(x) = \|(x_S)_{S\in F}\|_{\ell_2}^2 = 1$ so $m_{x,F}$ has norm 1 and is norming for x. Now $y \in B_{JT}$ belongs to $S(m_{x,F}, \delta)$ if and only if $\langle (x_S)_{S\in F}, (y_S)_{S\in F}\rangle_{\ell_2} > 1 - \delta$ and thus every element y of $S(m_{x,F}, \delta)$ satisfies

$$\begin{aligned}\|(x_S - y_S)_{S\in F}\|_{\ell_2}^2 &= \|(x_S)_{S\in F}\|_{\ell_2}^2 + \|(y_S)_{S\in F}\|_{\ell_2}^2 - 2\langle (x_S)_{S\in F}, (y_S)_{S\in F}\rangle_{\ell_2} \\ &\leq 2 - 2(1-\delta) = 2\delta.\end{aligned}$$

□

Note that it is not obvious at first that such a norm attaining family exists. We will first do a warm up with finitely supported elements in JT for which this is obvious, and then prove it in a lemma.

Proposition 5.8 *Let $x \in S_{JT}$ be an element of finite support. Then, x is not a weak* Δ-point.*

Proof Let $x \in S_{JT}$ be an element of finite support Σ and let $\mathcal{F}_\Sigma$ be the set of families of disjoint segments of the convex hull of Σ in T (that is the smaller subset of T containing $[s, t]$ for every $s \leq t$ in Σ). Then, $\mathcal{F}_\Sigma$ is finite and we have

$$\|x\|_{JT}^2 = \max_{F \in \mathcal{F}_\Sigma} \sum_{S \in F} x_S^2.$$

Now let $\mathcal{D} = \{F \in \mathcal{F}_\Sigma : \sum_{S \in F} x_S^2 = 1\}$. The set $\mathcal{D}$ is a non-empty subset of $\mathcal{F}_\Sigma$ and since this set is finite we can find a constant $\eta_x > 0$ such that for every $F \in \mathcal{F}_\Sigma \setminus \mathcal{D}$ we have

$$\sum_{S \in F} x_S^2 < (1 - \eta_x)^2.$$

Let us introduce $x^* = \frac{1}{|\mathcal{D}|} \sum_{F \in \mathcal{D}} m_{x,F}$ where $m_{x,F}$ is the molecule associated to x and F. For every $F \in \mathcal{D}$, the molecule $m_{x,F}$ is in $D(x)$ by the preceding lemma, and by Lemma 2.1, we know that x^* is in $D(x)$ and that for every choice of a $\delta > 0$ we have

$$S\left(x^*, \frac{\delta}{|\mathcal{D}|}\right) \subset \bigcap_{F \in \mathcal{D}} S(m_{x,F}, \delta).$$

To conclude choose any $\delta \in (0, 1)$ and let us take $y \in S\left(x^*, \frac{\delta}{|\mathcal{D}|}\right)$. If $F \in \mathcal{F}$ does not belong to $\mathcal{D}$, then by Minkowski's inequality we have

$$\sum_{S \in F} (x_S - y_S)^2 \leq \left(\sqrt{\sum_{S \in F} x_S^2} + \sqrt{\sum_{S \in F} y_S^2}\right)^2 \leq (2 - \eta_x)^2.$$

Now if $F \in \mathcal{D}$ we have $y \in S(m_{x,F}, \delta)$ and by the preceding lemma we get

$$\sum_{S \in F} (x_S - y_S)^2 \leq 2\delta.$$

Finally $\|x - y\|_{JT} \leq \max\{2 - \eta_x, \sqrt{2\delta}\}$ and this quantity is strictly less than 2 since $\delta < 1$. To conclude, note that $x^* \in JT_*$ since all the segments involved in the construction are finite. □

To tackle the other elements of JT, we first need to ensure the existence of norm attaining (possibly infinite) families.

Lemma 5.9 *Let $x \in S_{JT}$. Then there is an $F \in \mathcal{F}_\infty$ such that $\sum_{S \in F} x_S^2 = 1$.*

Proof Let $x \in S_{JT}$. By Krein–Milman x attains its norm on an extreme point $x^* \in B_{JT*}$. By Milman's converse, Lemma 5.5 and Remark 5.6, $x^* \in M_\infty$, so we can write $x^* = \sum_{i \geq 1} \lambda_i \mathbb{1}_{S_i}$ for some $\lambda = (\lambda_i)$ in B_{ℓ_2} and $\{S_i\}$ in $\mathcal{F}_\infty$. To conclude, consider $\mu = (x_{S_i})$ in B_{ℓ_2} and observe that

$$\langle \lambda, \mu \rangle_{\ell_2} = x^*(x) = 1,$$

so λ strongly exposes μ in B_{ℓ_2}. Thus $\lambda = \mu$ and $\|\mu\|_{\ell_2} = 1$ which is precisely the desired result. □

For every $x \in S_{JT}$, let us introduce $\mathcal{D}(x) = \{F \in \mathcal{F}_\infty : \sum_{S \in F} x_S^2 = 1\}$. By the preceding lemma $\mathcal{D}(x)$ is non-empty, but it does not need to be finite. To get around this problem we will consider the restriction of families in $\mathcal{D}(x)$ to a subtree of finite level. So for every $N \geq 1$ let us write $T_N = \text{level}[0, N] = \{\emptyset\} \cup \bigcup_{n=1}^{N} \{0,1\}^n$ the binary tree of height N and let us write $\mathcal{D}_N(x) = \{F \cap T_N : F \in \mathcal{D}(x)\}$, where $F \cap T_N = \{S \cap T_N : S \in F\}$. Then $\mathcal{D}_N(x)$ is a finite non-empty set and we have, similarly to the finite support case, the following lemma.

Lemma 5.10 *For every $x \in S_{JT}$ and for every $N \geq 1$, there is a constant $\eta_{x,N} > 0$ such that $\sum_{S \in F} x_S^2 < (1 - \eta_{x,N})^2$ for every F in $\mathcal{F}$ for which $F \cap T_N$ does not belong to $\mathcal{D}_N(x)$.*

Proof If this was not true, then since $\mathcal{F} \cap T_N$ is also finite we could find a family $F \in \mathcal{F}$ of segments of T_N not belonging to $\mathcal{D}_N(x)$ and a sequence $(F_i)_{i \geq 1} \subset \mathcal{F}$ such that $F_i \cap T_N = F$ and $\sum_{S \in F_i} x_S^2 > \left(1 - \frac{1}{i}\right)^2$ for every $i \geq 1$. Using a compactness argument (weak* extractions for the sequence of corresponding molecules in B_{JT^*} and metrizability of the weak* topology on B_{JT^*}) we would then obtain a family $G \in \mathcal{F}_\infty$ for which $\sum_{S \in G} x_S^2 = 1$ and such that $G \cap T_N = F$. But then this would mean that $F \in \mathcal{D}_N(x)$ and we would get a contradiction. □

We will need a last easy fact which states that any $x \in S_{JT}$ has its norm almost concentrated on a subtree of finite level. The proof is elementary and comes from the definition of the norm of JT.

Lemma 5.11 *Let $x \in S_{JT}$ and let $\varepsilon > 0$. There is an $N \geq 1$ such that for every family $F \in \mathcal{F}_\infty$ of segments of T which do not intersect T_N one has $\sum_{S \in F} x_S^2 \leq \varepsilon^2$.*

With those tools in hand, we can now prove the main result of this section.

Theorem 5.12 *Let $x \in S_{JT}$. Then x is not a Δ-point.*

Proof Let us fix some $x \in S_{JT}$ and let us assume that x has infinite support. Let us also fix some $\varepsilon > 0$ and let us take some $N \geq 1$ for which x is almost concentrated on T_N in the sense of the previous lemma.

For every F in $\mathcal{D}_N(x)$ we pick a representative family $F_\mathcal{R}$ in $\mathcal{D}(x)$ which satisfy $F_\mathcal{R} \cap T_N = F$. This means that for every $S \in F$ there is (a unique) $S_\mathcal{R} \in F_\mathcal{R}$ such that $S_\mathcal{R} \cap T_N = S$. For every such F, we define $x_F^* = m_{x,F_\mathcal{R}}$ to be the molecule associated with x and $F_\mathcal{R}$ and we let x^* be the average of the x_F^*'s with $F \in \mathcal{D}_N(x)$. Since each x_F^* is in $D(x)$, Lemma 2.1 tells us that $x^* \in D(x)$ and that

$$S\left(x^*, \frac{\varepsilon^2}{2n}\right) \subset \bigcap_{F \in \mathcal{D}_N(x)} S\left(x_F^*, \frac{\varepsilon^2}{2}\right)$$

for $n = |\mathcal{D}_N(x)|$. We now want to get a uniform bound for $\|x - y\|_{JT}$ for any $y \in S\left(x^*, \frac{\varepsilon^2}{2n}\right)$.

So let us take $y \in S\left(x^*, \frac{\varepsilon^2}{2n}\right)$ and let us fix some $G \in \mathcal{F}$. First, observe that Lemma 5.10 allows us to get rid of the case $G \cap T_N \notin \mathcal{D}_N(x)$ exactly as in the finite support proof because it yields

$$\sum_{S \in G} (x_S - y_S)^2 \leq (2 - \eta_{x,N})^2.$$

So let us assume that $F = G \cap T_N$ belongs to $\mathcal{D}_N(x)$. We will split the family G into 4 disjoint subfamilies G_i, $1 \leq i \leq 4$, and we will estimate separately the sums $\sum_{S \in G_i} (x_S - y_S)^2$. For this, we will use repeatedly the two following inequalities.

Claim A. Let H be a subfamily of $F_{\mathcal{R}}$. Then

$$\sum_{S \in H} (x_S - y_S)^2 \leq \varepsilon^2.$$

Proof of Claim A Since segments in H are in $F_{\mathcal{R}}$ and since $y \in S\left(x_F^*, \frac{\varepsilon^2}{2}\right)$ with $x_F^* = m_{x,F_{\mathcal{R}}}$ Lemma 5.7 yields

$$\sum_{S \in H} (x_S - y_S)^2 \leq \sum_{S \in F_{\mathcal{R}}} (x_S - y_S)^2 \leq \frac{2\varepsilon^2}{2} = \varepsilon^2.$$

□

Claim B. Let H be a family of disjoint segments of T which do not intersect T_N. Then,

$$\sum_{S \in H} (x_S - y_S)^2 \leq \sum_{S \in H} y_S^2 + \varepsilon^2 + 2\varepsilon.$$

Proof of Claim B Since segments in H do not intersect T_N we have, by our initial choice of N, $\sum_{S \in H} x_S^2 \leq \varepsilon^2$ and so using Minkowski's inequality

$$\begin{aligned}\sum_{S \in H} (x_S - y_S)^2 &\leq \sum_{S \in H} x_S^2 + \sum_{S \in H} y_S^2 + 2\sqrt{\sum_{S \in H} x_S^2}\sqrt{\sum_{S \in H} y_S^2} \\ &\leq \sum_{S \in H} y_S^2 + \varepsilon^2 + 2\varepsilon.\end{aligned}$$

□

Now, let us split the family G and let us start the computations of the corresponding sums.

Claim 1. Let $G_1 = \{S \in G : S \subset T_{N-1}\}$. There is a $\gamma_1 = \gamma_1(\varepsilon)$ such that

$$\sum_{S \in G_1} (x_S - y_S)^2 \leq \gamma_1.$$

Proof of Claim 1 Since segments in G_1 are contained in T_{N-1}, every segment S in G_1 has to be equal to its representative $S_{\mathcal{R}}$ because $S_{\mathcal{R}} \cap T_N = S$. This means that G_1 is a subfamily of $F_{\mathcal{R}}$ and the result follows directly from Claim A with $\gamma_1 := \varepsilon^2$. □

Claim 2. Let $G_2 = \{S \in G : S \subset T \backslash T_N\}$. There is a $\gamma_2 = \gamma_2(\varepsilon)$ such that

$$\sum_{S \in G_2} (x_S - y_S)^2 \leq \sum_{S \in G_2} y_S^2 + \gamma_2.$$

Proof of Claim 2 Since segments in G_2 do not intersect T_N, the result follows directly from Claim B with $\gamma_2 := \varepsilon^2 + 2\varepsilon$. □

The remaining segments of G are those that intersect the N^{th} level of T. For those, we will distinguish between segments S such that S and its representative $S_{\mathcal{R}}$ are on the same branch of T and those for which S and $S_{\mathcal{R}}$ split at some higher level. So, let G_3 be the subset of the remaining segments of G satisfying the first condition and let G_4 be the subset of those satisfying the second condition.

Note that if S is a segment in G_3, then since S and $S_{\mathcal{R}}$ are on the same branch and are equal in T_N, we have either $S \subset S_{\mathcal{R}}$ or $S_{\mathcal{R}} \subset S$ and the complements are in either case segments which do not intersect T_N. Thus,

Claim 3. Let $G_{3,1} = \{S \in G_3 : S = S_{\mathcal{R}}\}$, let $G_{3,2} = \{S \in G_3 : S \subsetneq S_{\mathcal{R}}\}$ and let $G_{3,3} = \{S \in G_3 : S_{\mathcal{R}} \subsetneq S\}$. There is a $\gamma_3 = \gamma_3(\varepsilon)$ such that

$$\sum_{S \in G_3} (x_S - y_S)^2 \leq \sum_{S \in G_{3,2}} y_{S_{\mathcal{R}} \backslash S}^2 + \sum_{S \in G_{3,3}} y_{S \backslash S_{\mathcal{R}}}^2 + \gamma_3.$$

Proof of Claim 3 By definition $G_{3,1}$ is a subfamily of $F_{\mathcal{R}}$ so Claim A yields

$$\sum_{S \in G_{3,1}} (x_S - y_S)^2 \leq \varepsilon^2.$$

Now let us deal with $G_{3,2}$. By Claim A we know that

$$\sum_{S \in G_{3,2}} (x_{S_{\mathcal{R}}} - y_{S_{\mathcal{R}}})^2 \leq \varepsilon^2.$$

Moreover, for every $S \in G_{3,2}$, we know that the segment $S_{\mathcal{R}} \backslash S$ does not intersect T_N and thus Claim B yields

$$\sum_{S\in G_{3,2}} (x_{S_{\mathcal{R}}\setminus S} - y_{S_{\mathcal{R}}\setminus S})^2 \leq \sum_{S\in G_{3,2}} y^2_{S_{\mathcal{R}}\setminus S} + \varepsilon^2 + 2\varepsilon.$$

So, finally we get

$$\begin{aligned}
\sum_{S\in G_{3,2}} (x_S - y_S)^2 &= \sum_{S\in G_{3,2}} [(x_{S_{\mathcal{R}}} - y_{S_{\mathcal{R}}}) - (x_{S_{\mathcal{R}}\setminus S} - y_{S_{\mathcal{R}}\setminus S})]^2 \\
&\leq \sum_{S\in G_{3,2}} (x_{S_{\mathcal{R}}} - y_{S_{\mathcal{R}}})^2 + \sum_{S\in G_{3,2}} (x_{S_{\mathcal{R}}\setminus S} - y_{S_{\mathcal{R}}\setminus S})^2 \\
&\quad + 2\sqrt{\sum_{S\in G_{3,2}} (x_{S_{\mathcal{R}}} - y_{S_{\mathcal{R}}})^2}\sqrt{\sum_{S\in G_{3,2}} (x_{S_{\mathcal{R}}\setminus S} - y_{S_{\mathcal{R}}\setminus S})^2} \\
&\leq \sum_{S\in G_{3,2}} y^2_{S_{\mathcal{R}}\setminus S} + \varepsilon^2 + (\varepsilon^2 + 2\varepsilon) + 4\varepsilon.
\end{aligned}$$

Similar computations allow us to deal with $G_{3,3}$ and the conclusion follows by combining the 3 inequalities with $\gamma_3 := \varepsilon^2 + 2(2\varepsilon^2 + 6\varepsilon)$. □

Finally, let us take $S \in G_4$. Since S and its representative $S_{\mathcal{R}}$ are equal on T_N the segments $S^- = S\setminus(S \cap S_{\mathcal{R}})$ and $S^-_{\mathcal{R}} = S_{\mathcal{R}}\setminus(S \cap S_{\mathcal{R}})$ do not intersect T_N (and are non-empty). Thus,

Claim 4. There is a $\gamma_4 = \gamma_4(\varepsilon)$ such that

$$\sum_{S\in G_4} (x_S - y_S)^2 \leq 1 + \sum_{S\in G_4} y^2_{S^-} + \sum_{S\in G_4} y^2_{S^-_{\mathcal{R}}} + \gamma_4.$$

Proof of Claim 4 Using again Claim A and Claim B, we have

$$a = \sum_{S\in G_4} (x_{S_{\mathcal{R}}} - y_{S_{\mathcal{R}}})^2 \leq \varepsilon^2,$$

$$b = \sum_{S\in G_4} (x_{S^-} - y_{S^-})^2 \leq \sum_{S\in G_4} y^2_{S^-} + \varepsilon^2 + 2\varepsilon,$$

and

$$c = \sum_{S\in G_4} (x_{S^-_{\mathcal{R}}} - y_{S^-_{\mathcal{R}}})^2 \leq \sum_{S\in G_4} y^2_{S^-_{\mathcal{R}}} + \varepsilon^2 + 2\varepsilon.$$

Thus we get, using the $\sqrt{u+v} \leq \sqrt{u} + \sqrt{v}$ inequality for the last term:

$$\begin{aligned}\sum_{S\in G_4}(x_S-y_S)^2 &= \sum_{S\in G_4}[(x_{S_{\mathcal{R}}}-y_{S_{\mathcal{R}}})-(x_{S^-_{\mathcal{R}}}-y_{S^-_{\mathcal{R}}})+(x_{S^-}-y_{S^-})]^2\\ &\le a+b+c+2\sqrt{ab}+2\sqrt{ac}+2\sqrt{bc}\\ &\le \varepsilon^2+\sum_{S\in G_4}y^2_{S^-}+\sum_{S\in G_4}y^2_{S^-_{\mathcal{R}}}+2(\varepsilon^2+2\varepsilon)\\ &\quad+4\sqrt{\varepsilon^2(1+\varepsilon^2+2\varepsilon)}+2\sqrt{\sum_{S\in G_4}y^2_{S^-}}\sqrt{\sum_{S\in G_4}y^2_{S^-_{\mathcal{R}}}}\\ &\quad+2\sqrt{2(\varepsilon^2+2\varepsilon)+(\varepsilon^2+2\varepsilon)^2}\end{aligned}$$

so it only remains to show that

$$2\sqrt{\sum_{S\in G_4}y^2_{S^-}}\sqrt{\sum_{S\in G_4}y^2_{S^-_{\mathcal{R}}}}\le 1.$$

To show this, observe that if we take two different segments S and T in G_4, then we obviously have $S\cap T=\emptyset$ and $S_{\mathcal{R}}\cap T_{\mathcal{R}}=\emptyset$ since we work with families of disjoint segments, but we also have $S\cap T_{\mathcal{R}}=\emptyset$ and $T\cap S_{\mathcal{R}}=\emptyset$ since S and T have disjoint starting parts in T_N. Consequently:

$$\sum_{S\in G_4}y^2_{S^-}+\sum_{S\in G_4}y^2_{S^-_{\mathcal{R}}}\le \|y\|^2_{JT}=1.$$

The conclusion follows since the function $f(x)=x\sqrt{1-x^2}$ has maximum $\frac{1}{2}$ on $[0,1]$ (attained at $\frac{1}{\sqrt{2}}$), and we have

$$\gamma_4 := \varepsilon^2+2(\varepsilon^2+2\varepsilon)+4\sqrt{\varepsilon^2(1+\varepsilon^2+2\varepsilon)}+2\sqrt{2(\varepsilon^2+2\varepsilon)+(\varepsilon^2+2\varepsilon)^2}.$$

□

Now letting $\gamma := \sum_{i=1}^4\gamma_i$ and combining the results from the 4 claims, we obtain:

$$\begin{aligned}\sum_{S\in G}(x_S-y_S)^2\le 1&+\sum_{S\in G_2}y_S^2+\sum_{S\in G_{3,3}}y^2_{S\setminus S_{\mathcal{R}}}+\sum_{S\in G_4}y^2_{S^-}\\ &+\sum_{S\in G_{3,2}}y^2_{S_{\mathcal{R}}\setminus S}+\sum_{S\in G_4}y^2_{S^-_{\mathcal{R}}}+\gamma,\end{aligned}$$

and since we work with families of disjoint segments we get

$$\sum_{S\in G}(x_S-y_S)^2\le 3+\gamma.$$

Finally, $\|x-y\|_{JT}\le\max\{2-\eta_{x,N},\sqrt{3+\gamma}\}$ and this quantity is strictly less than 2 if we take a small enough ε since γ goes to 0 as ε goes to 0. □

***Remark* 5.13** Here, the segments involved can be infinite so the functional x^* we slice with does not have to be in JT_*. This raises the following question.

Question 5.14 *Does JT admit weak** Δ*-points?*

From the work of [20], we know that the dual of the James tree space JT^* is AUC. Now JT does not admit an equivalent norm whose dual norm is AUC*. Indeed such norm would be AUS by the asymptotic duality. This is impossible because JT is not Asplund (it is separable while JT^* is not), see [29, Proposition 2.4]. In view of those observations, JT^* would be a natural candidate for our study and the question of the existence of Δ- or Daugavet-points there, as well as the question of the existence of non quasi-denting points would be particularly relevant. Unfortunately, it seems that computations are still out of reach in this context, although the use of molecules facilitates them, and we were only able to obtain a few partial results even while trying to restrict ourselves to J^*.

Lemma 5.15 *Let* $s, t \in T$ *be two distinct points. For any* $\varepsilon > 0$ *and* $z^* \in JT^*$ *with* $\{s, t\} \cap \operatorname{supp}(z^*) = \emptyset$ *we have*

$$\|e_t^* - \varepsilon e_s^* + z^*\| > 1.$$

Proof Let $x^* = e_t^* - \varepsilon e_s^* + z^*$ for some $\varepsilon > 0$ and z^* supported in $T \backslash \{s, t\}$. Assume first that $\varepsilon \geq 1$, and let $\alpha = 1/\sqrt{2}$ and $x = \alpha e_t - \alpha e_s$. We have $\|x\| = 1$ and

$$x^*(x) = \alpha + \varepsilon\alpha \geq 2/\sqrt{2} > 1.$$

Next if $\varepsilon < 1$ we let $\alpha = \sqrt{1 - \varepsilon^2}$ and $x = \alpha e_t - \varepsilon e_s$. We have $\|x\| = 1$ since $\alpha^2 + \varepsilon^2 = 1$ and $(\alpha - \varepsilon)^2 \leq 1 - 2\alpha\varepsilon < 1$. Now

$$x^*(x) = \alpha + \varepsilon^2 > \alpha^2 + \varepsilon^2 = 1.$$

In both cases, $\|x^*\| > 1$. □

Lemma 5.16 *In* JT^* *every basis vector* e_t^* *is an extreme point of* B_{JT^*}*, and therefore a weak* denting point.*

Proof Assume that x^* and y^* are norm one elements such that $e_t^* = \frac{x^*+y^*}{2}$. Then, we have $x^*(e_t) = y^*(e_t) = 1$ and $x^*(e_s) = -y^*(e_s)$ for all $s \neq t$. If we have $x^*(e_s) < 0$ for some $s \neq t$, then with $z^* = x^* - (e_t^* + x^*(e_s)e_s^*)$ we get

$$\|x^*\| = \|e_t^* + x^*(e_s)e_s^* + z^*\| > 1$$

by Lemma 5.15. Hence $x^* = y^* = e_t^*$.

By Proposition 3.d.19 in [15] we know that e_t^* is a point of weak* to norm continuity on the unit ball of JT^* so the conclusion follows by applying Choquet's lemma (see for example [14, Lemma 3.69]) which tells that the weak* slices form a neighborhood basis of e_t^*. Indeed, the continuity of the identity map at e_t^* then ensures that any ball around e_t^* contains a weak* slice containing e_t^*. □

Corollary 5.17 *No molecule in JT^* is a weak* Daugavet-point.*

Proof Let $x^* = \sum_{i\geq 1} \lambda_i \mathbb{1}_{S_i}$ where $\sum_{i\geq 1} \lambda_i^2 \leq 1$ and S_i are disjoint segments of T. If there is an $i_0 \geq 1$ such that $\lambda_{i_0} = 1$, then $x^* = \mathbb{1}_{S_{i_0}}$. Since the biorthogonal functionals are (weak*) denting points, we may assume that S_{i_0} contains at least two points of T. Now if we let s_{i_0} be the starting point of this segment, and if we let $T_{i_0} = S_{i_0} \setminus \{s_{i_0}\}$, we have $\left\| x^* - e^*_{s_{i_0}} \right\| = \left\| \mathbb{1}_{T_{i_0}} \right\| = 1$ so x^* is at distance strictly less than 2 from a (weak*) denting point in JT^* and thus cannot be a weak* Daugavet-point.

Next let us assume that $\lambda_i \in (0,1)$ for every $i \geq 1$ and let us fix any $i_0 \geq 1$. We define as above s_{i_0} and $T_{s_{i_0}}$ (which might eventually be empty) and we let $T_i = S_i$ for any $i \neq i_0$. Since the T_i are disjoint segments of T we have $\left\| x^* - e^*_{s_{i_0}} \right\| \leq 1 - \left|\lambda_{i_0}\right| + \left\| x^* - \lambda_{s_{i_0}} e^*_{s_{i_0}} \right\| = 1 - \left|\lambda_{i_0}\right| + \left\| \sum_{i\geq 1} \lambda_i \mathbb{1}_{T_i} \right\| \leq 2 - \lambda_{s_{i_0}} < 2$ and the conclusion follows as above. □

In light of the above we ask:

Question 5.18 *Can molecules in JT^* be Δ-points?*

Note that the observation from [30, Remark 2.4] applied to the set of molecules in JT^* which satisfies as mentioned $\overline{\text{conv}}\, \mathcal{M} = B_{X^*}$ gives the following simplification.

Lemma 5.19 *Let $x^* \in S_{JT^*}$. Then x^* is a Δ-point if and only if we can find, for every $\varepsilon > 0$ and for every slice S of B_{JT^*}, some $m \in \mathcal{M} \cap S$ such that $\|x - m\| \geq 2 - \varepsilon$.*

Although it is possible to do some computations in very specific cases (for example when x^* is a molecule supported on two segments) it seems to be difficult to estimate the distance between two molecules in general even when they are supported on the same branch of T. Moreover, it is not completely trivial to distinguish norm one molecules in $\mathcal{M}$ and to find suitable norming elements in JT.

Let us mention that with techniques similar to those for JT, it is possible to prove that also J with the equivalent norms $\|\cdot\|_{\mathcal{J}}$ and $\|\cdot\|_0$ as given on p. 62 in [4], fail to contain Δ-points.

However, if we replace the binary tree T by the countably branching tree $T_\infty = \{\emptyset\} \cup \bigcup_{n\geq 1} \mathbb{N}^n$ in the definition of the James tree space we obtain the Banach space JT_∞ originally introduced in [17] which shares some of the basic properties of JT, but presents a few striking dissimilarities. Indeed, it is proved in [17] that the predual of JT_∞ fails the PCP and as a consequence admits no equivalent AUC norm (in fact $\overline{\delta}_{|.|}(\frac{1}{2}) = 0$ for any equivalent norm $|.|$ on $(JT_\infty)_*$, see [20]) while it satisfy the so called convex PCP, see [18]. For our study, we have as before that JT_∞ is 2-AUC* and admits no Daugavet-points, but our proof of non-existence of Δ-points fails for infinitely supported elements since restricting the support of such element to a finite level of the tree does not necessarily provide finitely many nodes of T_∞ anymore. It is thus natural to ask the following.

Question 5.20 *Does JT_∞ admit Δ-points?*

6 Almost Daugavet- and Delta-points

Looking at Corollary 2.4, it is natural to ask:

Question 6.1 *Do all superreflexive Banach spaces fail to contain Δ-points?*

This question is not trivial since, as we have seen, even superreflexive spaces whose norm is not uniformly non-square have thin slices of the unit ball with diameter arbitrary close to 2. However, for a superreflexive space X we know that the dual of an ultrapower $X^{\mathcal{U}}$ is isometric to $(X^*)^{\mathcal{U}}$, and this opens the door to using ultrafilter limits. Since $X^{\mathcal{U}}$ is also superreflexive, we do not leave the context of superreflexive spaces when passing to ultrapowers. Motivated by this, we introduce the following definitions which are a weakening of the notions of Daugavet- and Δ-points, and we will see how those can help simplify the problem from Question 6.1.

Let X be a Banach space and let $\varepsilon > 0$. We say that $x \in B_X$ is a $(2-\varepsilon)$ *Daugavet-point* if $B_X \subset \overline{\text{conv}}\, \Delta_\varepsilon(x)$ or equivalently if we can find for every $\delta > 0$ and for every $x^* \in S_{X^*}$ an element $y \in S(x^*, \delta)$ satisfying $\|x - y\| \geq 2 - \varepsilon$. We say that $x \in B_X$ is a $(2-\varepsilon)$ Δ-*point* if $x \in \overline{\text{conv}}\, \Delta_\varepsilon(x)$ or equivalently if we can find for every $\delta > 0$ and for every $x^* \in S_{X^*}$ such that $x \in S(x^*, \delta)$ an element $y \in S(x^*, \delta)$ satisfying $\|x - y\| \geq 2 - \varepsilon$.

We say that X *admits almost Daugavet-points* if it admits a $(2-\varepsilon)$ Daugavet-point for every $\varepsilon > 0$, and we say that X *admits almost Δ-points* if it admits a $(2-\varepsilon)$ Δ-point for every $\varepsilon > 0$.

We say that a Banach space X *contains ℓ_p^n's uniformly* $(1 \leq p \leq \infty)$ if for all $\varepsilon > 0$ and $n \in \mathbb{N}$ there exist $x_1, \ldots, x_n \in B_X$ such that for all sequences of scalars (a_k)

$$(1+\varepsilon)^{-1}\|(a_n)\|_p \leq \left\| \sum_{k=1}^{n} a_k x_k \right\| \leq \|(a_n)\|_p.$$

We will start by studying the existence of almost Daugavet- and Δ-points in some classical spaces. We highlight the following observation, which although obvious from the definition will provide an easy way of constructing almost Δ-points.

Observation 6.2 *Let $\varepsilon > 0$. If $x \in$ conv $\Delta_\varepsilon(x)$, then x is a $(2-\varepsilon)$ Δ-point.*

Using this, we can prove the two following lemmas.

Lemma 6.3 *If a Banach space X contains ℓ_1^n's uniformly, then X admits almost Δ-points.*

Proof Let $\varepsilon > 0$ and $n \in \mathbb{N}$ and find $x_1, \dots, x_n \in B_X$ from the definition of containing ℓ_1^n's uniformly. Let $x = \frac{1}{n}\sum_{k=1}^n x_k$. Then $\|x\| \le 1$ and for each j

$$\|x - x_j\| = \left\| \sum_{k \neq j} \frac{1}{n} x_k + \left(\frac{1}{n} - 1\right) x_j \right\| \ge (1+\varepsilon)^{-1}\left(2 - \frac{2}{n}\right)$$

which can be made as close to 2 as we like. □

Lemma 6.4 *If a Banach space X contains ℓ_∞^n's uniformly, then X admits almost Δ-points.*

Proof It is well known that ℓ_1^n can be isometrically embedded into $\ell_\infty^{2^n}$ for every $n \ge 1$ so that this result can be seen as an immediate corollary of Lemma 6.3. We present a direct approach here. Let $\varepsilon > 0$ and $n \in \mathbb{N}$ and find $x_1, \dots, x_n \in B_X$ from the definition of containing ℓ_∞^n's uniformly. For $i = 1, \dots, n$ we define

$$y_i = -2x_i + \sum_{k=1}^n x_k$$

and $x = \frac{1}{n}\sum_{i=1}^n y_i$, that is

$$x = \left(1 - \frac{2}{n}\right) \sum_{k=1}^n x_k.$$

Then $\|x\| \le 1$ and for each j

$$\|y_j - x\| = \left\| \sum_{k \neq j} \frac{2}{n} x_k + \left(\frac{2}{n} - 2\right) x_j \right\| \ge (1+\varepsilon)^{-1}\left(2 - \frac{2}{n}\right)$$

which can be made as close to 2 as we like. □

The two lemmas above can be formulated as: If X does not have finite co-type or does not have non-trivial type, then X admits almost Δ-points. In particular the spaces c_0 and ℓ_1 both admits almost Δ-points. Let us recall that they do not admit Δ-points (by, e.g., [3, Theorem 2.17]). For those spaces we can say more.

Lemma 6.5 *The space c_0 does not admit almost Daugavet-points.*

Proof Let $x = (x_i) \in S_{c_0}$. Given $\delta \in (0, 1)$, there exists a finite non-empty set $J \subset \mathbb{N}$ of cardinality $n \ge 1$ such that $|x_j| \ge 1 - \delta$ for every $j \in J$ and $|x_i| < 1 - \delta$ for every $i \in \mathbb{N} \setminus J$. Now let $x^* = \frac{1}{n}\sum_{j \in J} \operatorname{sign}(x_j) e_j^* \in S_{\ell_1}$. By Lemma 2.1, we have $S(x^*, \frac{\delta}{n}) \subset \bigcap_{j \in J} S(\operatorname{sign}(x_j) e_j^*, \delta)$ so if $y = (y_i)_{i \ge 1}$ is in $S(x^*, \frac{\delta}{n})$, then it satisfies

$\left|x_j - y_j\right| \leq \delta$ for every $j \in J$, and thus $\|x - y\| \leq \max\{\delta,\ 2 - \delta\}$. The conclusion follows. □

Lemma 6.6 *The space ℓ_1 admits almost Daugavet-points.*

Proof Fix $n \geq 1$ and let $x = \frac{1}{n}\sum_{i=1}^{n} e_i \in S_{\ell_1}$ where (e_i) is the unit vector basis of ℓ_1. Then x is a $(2 - \frac{2}{n})$ Δ-point by construction and we will show that it is in fact a $(2 - \frac{2}{n})$ Daugavet-point. Indeed fix $x^* \in S_{\ell_\infty}$ and $\delta > 0$. Since $\sup_i \left|x^*(e_i)\right| = 1$ we can find some i_0 such that $\left|x^*(e_{i_0})\right| > 1 - \delta$ that is $e_{i_0} \in S(x^*, \delta)$ or $-e_{i_0} \in S(x^*, \delta)$. Now, it is easy to check that $\left\|x \pm e_{i_0}\right\| \geq 2 - \frac{2}{n}$ so we are done. □

Remark **6.7** We have seen in Theorem 4.3 that if a Banach space X has Rolewicz' property (α), then X has no Δ-points. But X can contain almost Δ-points.

Indeed, let T be the Tsirelson space. Even though T fails Rolewicz' property (α), there exists an equivalent norm $|\cdot|$, so that $(T, |\cdot|)$ has Rolewicz' property (α) by [36, Theorems 3 and 4].

Since T contains ℓ_1^n's uniformly the same holds for $(T, |\cdot|)$ by James' ℓ_1-distortion theorem. Thus, $(T, |\cdot|)$ admits almost Δ-points by Lemma 6.3.

The following result, whose proof is clear from Proposition 2.3, covers a lot of classical norms, and in particular uniformly smooth and uniformly convex ones.

Proposition 6.8 *If X is uniformly non-square, then X does not admit almost Δ -points.*

It is clear from Proposition 2.3 that the unit ball of every non uniformly non-square norm admits thin slices of diameter arbitrarily close to 2, so ruling out almost Daugavet- and Δ-points is non-trivial even in superreflexive spaces.

The following result is the main reason for the introduction of the notions almost Daugavet- and almost Δ-points.

Proposition 6.9 *Let X be a superreflexive Banach space and let $\mathcal{U}$ be some free ultrafilter on $\mathbb{N}$.*

If X admits almost Δ- (resp. Daugavet-)points, then there exists $x \in X^{\mathcal{U}}$ with $\|x\| = 1$ such that for any slice S of $B_{X^{\mathcal{U}}}$ containing x (resp. any slice S of $B_{X^{\mathcal{U}}}$) there exists $y \in S$ with $\|y\| = 1$ and $\|x - y\| = 2$.

In particular, if there exists a superreflexive Banach space which admits an almost Δ- (resp. Daugavet-)point, then there exists a superreflexive Banach space with a Δ- (resp. Daugavet-)point.

Proof Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Since X is superreflexive we have $(X^{\mathcal{U}})^* = (X^*)^{\mathcal{U}}$ (see, e.g., [25, Proposition 7.1]). For each $n \in \mathbb{N}$ choose a $(2 - \frac{1}{n})$ Δ -point $x_n \in B_X$. Let $x = (x_n)_{\mathcal{U}}$. We have $\|x_n\| \geq 1 - \frac{1}{n}$ hence $\|x\| = \lim_{\mathcal{U}} \|x_n\| = 1$.

Let $x^* = (x_n^*)_{\mathcal{U}}$ with $\|x^*\| = 1$ and $\delta > 0$. Assume that $x \in S(x^*, \delta)$. This means that there is an $\eta > 0$ such that $x^*(x) = \lim_{\mathcal{U}} x_n^*(x_n) > 1 - \delta + \eta$ and thus, there is a set $A \in \mathcal{U}$ such that

$$x_n^*(x_n) > 1 - \delta + \eta = \|x_n^*\| - (\|x_n^*\| - (1 - \delta + \eta)),$$

for every $n \in A$. In particular, we have $\|x_n^*\| > 1 - \delta + \eta$ for these n and

$$x_n \in S_n := \{y \in B_X : x_n^*(y) > \|x_n^*\| - (\|x_n^*\| - (1 - \delta + \eta))\}.$$

We can then use the definition to find $y_n \in S_n$ with $\|x_n - y_n\| \geq 2 - \frac{1}{n}$. For $n \notin A$, we just let $y_n = 0$.

Now $y = (y_n) \in X^{\mathcal{U}}$ and

$$\|y\| = \lim_{\mathcal{U}} \|y_n\| \geq \lim_{\mathcal{U}}(\|x_n - y_n\| - \|x_n\|) \geq \lim_{\mathcal{U}} \left(2 - \frac{1}{n} - 1\right) = 1.$$

It is also clear that $\|x - y\| = \lim_{\mathcal{U}} \|x_n - y_n\| = 2$. Finally we check that y is in the slice $S(x^*, \delta)$

$$x^*(y) = \lim_{\mathcal{U}} x_n^*(y_n) = \lim_{\mathcal{U}} \|x_n^*\| \lim_{\mathcal{U}} \frac{x_n^*}{\|x_n^*\|}(y_n) \geq 1 - \delta + \eta > 1 - \delta.$$

For the in particular part, we just note that ultrapowers of superreflexive spaces are superreflexive by finite representability of $X^{\mathcal{U}}$ in X. □

Corollary 6.10 *If X is finite dimensional, then it does not admit almost Δ-points.*

Proof This is an immediate consequence of the previous proposition since finite-dimensional spaces do not admit Δ-points and since any ultrapower of a finite-dimensional space is trivially identified with the space itself (under the diagonal map). □

By Proposition 6.9, we can, thus, restate Question 6.1 in the following way.

Question 6.11 *Is it possible to find a superreflexive space admitting almost Δ-points or almost Daugavet-points?*

In particular, it would be interesting to investigate the following question.

Question 6.12 *Is it possible to find a renorming of ℓ_2 with almost Δ-points or almost Daugavet-points?*

Also note that proving that superreflexive spaces do not admit Δ- (or Daugavet-) points in general would immediately imply by the preceding proposition that superreflexive spaces do not admit almost Δ- (or Daugavet-) points.

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References

1. Abrahamsen, T.A., Haller, R., Lima, V., Pirk, K.: Delta- and Daugavet points in Banach spaces. Proc. Edinb. Math. Soc.(2) **63**(2), 475–496 (2020)
2. Abrahamsen, T.A., Lima, V., Martiny, A.: Delta-points in Banach spaces generated by adequate families. Ill. J. Math. (to appear). arXiv:2012.00406
3. Abrahamsen, T.A., Lima, V., Martiny, A., Troyanski, S.: Daugavet- and delta-points in Banach spaces with unconditional bases. Trans. Am. Math. Soc. Ser. B **8**, 379–398 (2021)
4. Albiac, F., Kalton, N.J.: Topics in Banach Space Theory. Graduate Texts in Mathematics, vol. 233. Springer, New York (2006)
5. Baernstein, A.: On reflexivity and summability. Stud. Math. **42**, 91–94 (1972)
6. Banaś, J., Olszowy, L., Sadarangani, K.: Moduli of near convexity of the Baernstein space. Proc. Am. Math. Soc. **123**(12), 3693–3699 (1995)
7. Banaś, J., Mursaleen, M.: Sequence Spaces and Measures of Noncompactness with Applications to Differential and Integral Equations. Springer, New Delhi (2014)
8. Becerra Guerrero, J., López-Pérez, G., Rueda Zoca, A.: Diametral diameter two properties in Banach spaces. J. Convex Anal. **25**(3), 817–840 (2018)
9. Casazza, P.G., Shura, T.J.: Tsirelson's spacE. Lecture Notes in Mathematics, vol. 1363. Springer, Berlin (1989)
10. Causey, R.: Shrinking and Boundedly Complete Bases. Lecture Notes. https://people.math.sc.edu/causeyrm/lecture6.5.pdf
11. Dilworth, S.J., Kutzarova, D., Lancien, G., Randrianarivony, N.L.: Equivalent norms with the property (β) of Rolewicz. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM **111**(1), 101–113 (2017)
12. Dutrieux, Y.: Géométrie non linéaire des espaces de Banach. Ph.D. thesis, Université Paris 6 (2002)
13. Dutta, S., Godard, A.: Banach spaces with property (M) and their Szlenk indices. Mediterr. J. Math. **5**(2), 211–220 (2008)
14. Fabian, M., Habala, P., Hájek, P., Montesinos, V., Zizler, V.: Banach Space Theory: The Basis for Linear and Nonlinear Analysis. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, New York (2011)

15. Fetter, H., Gamboa de Buen, B.: The James Forest. London Mathematical Society Lecture Note Series, vol. 236. Cambridge University Press, Cambridge (1997). With a foreword by Robert C. James and a prologue by Bernard Beauzamy
16. García-Lirola, L., Raja, M.: On strong asymptotic uniform smoothness and convexity. Rev. Mat. Complut. **31**(1), 131–152 (2018)
17. Ghoussoub, N., Maurey, B.: G_δ-embeddings in Hilbert space. J. Funct. Anal. **61**(1), 72–97 (1985)
18. Ghoussoub, N., Maurey, B., Schachermayer, W.: A counterexample to a problem on points of continuity in Banach spaces. Proc. Am. Math. Soc. **99**(2), 278–282 (1987)
19. Giles, J.R., Moors, W.B.: Differentiability properties of Banach spaces where the boundary of the closed unit ball has denting point properties. In: Miniconference on Probability and Analysis (Sydney, 1991). Proceedings of the Centre for Mathematics and its Applications, Australian National University, vol. 29, pp. 107–115. Australian National University, Canberra (1992)
20. Girardi, M.: The dual of the James tree space is asymptotically uniformly convex. Stud. Math. **147**(2), 119–130 (2001)
21. González, M., Gutiérrez, J.M.: The compact weak topology on a Banach space. Proc. R. Soc. Edinb. Sect. A **120**(3–4), 367–379 (1992)
22. Hájek, P., Lancien, G.: Various slicing indices on Banach spaces. Mediterr. J. Math. **4**(2), 179–190 (2007)
23. Haller, R., Pirk, K., Veeorg, T.: Daugavet- and delta-points in absolute sums of Banach spaces. J. Convex Anal. **28**(1), 41–54 (2021)
24. Harmand, P., Werner, D., Werner, W.: M-Ideals in Banach Spaces and Banach Algebras. Lecture Notes in Mathematics, vol. 1547. Springer, Berlin (1993)
25. Heinrich, S.: Ultraproducts in Banach space theory. J. Reine Angew. Math. **313**, 72–104 (1980)
26. Huff, R.: Banach spaces which are nearly uniformly convex. Rocky Mt. J. Math. **10**(4), 743–749 (1980)
27. Ivakhno, Y., Kadets, V.: Unconditional sums of spaces with bad projections. Visn. Khark. Univ. Ser. Mat. Prykl. Mat. Mekh. **645**(54), 30–35 (2004) (English)
28. James, R.C.: Uniformly non-square Banach spaces. Ann. Math. **80**, 542–550 (1964)
29. Johnson, W.B., Lindenstrauss, J., Preiss, D., Schechtman, G.: Almost Fréchet differentiability of Lipschitz mappings between infinite-dimensional Banach spaces. Proc. Lond. Math. Soc. (3) **84**(3), 711–746 (2002)
30. Jung, M., Rueda Zoca, A.: Daugavet points and delta-points in Lipschitz-free spaces. Stud. Math. **265**, 37–55 (2022)
31. Kalton, N.J., Werner, D.: Property (M), M-ideals and almost isometric structure of Banach spaces. J. Reine Angew. Math. **461**, 137–178 (1995)
32. Kato, M., Maligranda, L., Takahashi, Y.: On James and Jordan–von Neumann constants and the normal structure coefficient of Banach spaces. Stud. Math. **144**(3), 275–295 (2001)
33. Lancien, G.: On uniformly convex and uniformly Kadec–Klee renormings. Serdica Math. J. **21**(1), 1–18 (1995)
34. Lennard, C.: C_1 is uniformly Kadec–Klee. Proc. Am. Math. Soc. **109**(1), 71–77 (1990)
35. Moltó, A., Montesinos, V., Troyanski, S.: On quasi-denting points, denting faces and the geometry of the unit ball of $d(w, 1)$. Arch. Math. (Basel) **63**(1), 45–55 (1994)
36. Montesinos, V.: Drop property equals reflexivity. Stud. Math. **87**(1), 93–100 (1987)
37. Oja, E.: A note on M-ideals of compact operators. Acta et Comment. Universitatis Tartuensis **960**, 75–92 (1993)
38. Oja, E.: Geometry of Banach spaces having shrinking approximations of the identity. Trans. Am. Math. Soc. **352**(6), 2801–2823 (2000)
39. Partington, J.R.: Corrigenda to: On nearly uniformly convex Banach spaces. Math. Proc. Camb. Philos. Soc. **94**(1), 189 (1983)

40. Rolewicz, S.: On drop property. Stud. Math. **85**(1), 27–35 (1987)
41. Rolewicz, S.: On Δ-uniform convexity and drop property. Stud. Math. **87**(2), 181–191 (1987)
42. Rosenthal, H.P.: Some recent discoveries in the isomorphic theory of Banach spaces. Bull. Am. Math. Soc. **84**(5), 803–831 (1978)
43. Schachermayer, W.: Some more remarkable properties of the James-tree space. In: Lin B-L (ed) Banach Space Theory (Iowa City, IA, 1987). Contemporary Mathematics, vol. 85, pp. 465–496. American Mathematical Society, Providence (1989)
44. Shvydkoy, R.V.: Geometric aspects of the Daugavet property. J. Funct. Anal. **176**(2), 198–212 (2000)
45. Troyanski, S.L.: On some generalizations of denting points. Isr. J. Math. **88**(1–3), 175–188 (1994)
46. Veeorg, T.: Characterizations of Daugavet- and delta-points in Lipschitz-free spaces. Stud. Math. (**Accepted**). arXiv:2111.14393. https://www.impan.pl/en/publishing-house/journals-and-series/studia-mathematica/accepted