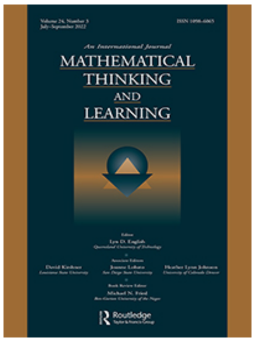

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Third grade students' multimodal mathematical reasoning when collaboratively solving combinatorial problems in small groups

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ABSTRACT

The aim of this study is to investigate Norwegian primary school students' multimodal mathematical reasoning when solving combinatorial problems. The data collection took place in four small groups of altogether thirteen 8–9 years old third-graders. Our study shows a variety of approaches used to solve the given combinatorial problems, such as count-all and grouping. These approaches were characterized by the students' use of inscriptions that displayed all combinations and inscriptions that did not display all combinations. Moreover, the students used gestures such as pointing and sliding. The students' multimodal reasoning was characterized by the ways their utterances, inscriptions, and gestures emerged and supplemented each other. The students' pointing gestures mediated the intended mathematical meaning solely when combined with inscriptions displaying all possible combinations. Sliding gestures, on the other hand, did mediate the intended mathematical meaning when their inscriptions were not displaying all possible combinations. In this latter case, the shortcomings of their inscriptions were complemented and compensated by the sliding gestures. The students' multimodal mathematical reasoning made explicit their combinatorial thinking, mediated the intended mathematical meaning, and facilitated their solving of the given combinatorial problems.

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Introduction

According to English (2005), combinatorial problems are of importance in the school curriculum. Combinatorics comprises mathematical principles of rich structure, even for the younger students, and it underlies both counting and computation (as well as probability), two main parts of early school mathematics. Moreover, combinatorial problems may facilitate young students to conjecture, think systematically, and experience the power of generalization within mathematics. Therefore, "combinatorics has an important role to play in the elementary school mathematics curriculum" (English, 2005, p. 138). We agree with Höveler's (2018) argument, that "it is essential to address combinatorial problems early with primary school pupils" (p. 82).

In our study we want to analyze primary school students' multimodal reasoning in depth when solving combinatorial problems in four small groups. We deliberately use the term 'problem' here because, as will be seen, the students' approaches to the provided tasks testify that they engage with these in accordance with how mathematical problems are defined by for instance, Schoenfeld (1985), where the difficulty for the solver of a provided problem is a function of the solver's already appropriated knowledge, experience and dispositions. One reason that students' engagement with combinatorial problems is of particular interest, is that these problems may "be solved in a variety of ways and with a variety of representational tools" (English, 2005, p. 122). We use the term

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'combinatorial problem' in line with English (2005), as indicating a problem in which the Cartesian product of two, or more, given sets is the solution, i.e. the set obtained by combining all elements of one set with each element of the other set (possibly extended to several sets).

The combinatorial problems given in our study were provided to mediate conceptualizations of multiplication, as a separate semantic model within whole number multiplication (see also, Greer, 1992) and as problems demonstrating the commutativity of multiplication. Carpenter et al. (1999) categorize such problems as symmetric in that the two (or more) factors play symmetric roles, the multiplier and the multiplicand may be swapped in combinatorial problems.

Research has documented the importance of gestures in mathematical learning processes (Arzarello et al., 2005; Edwards, 2005; Radford, 2003; Reynolds & Reeve, 2002). Studies have also shown that inscriptions may play an important mediating role in students' mathematical problem solving (Carlsen, 2009; English, 2005; Maher & Yankelewitz, 2010; Roth & McGinn, 1998). Studies have also documented that a combinatorial problem has a multiplicative structure that is quite difficult for primary students to appropriate (Borba et al., 2015; English, 2005; Höveler, 2018; Maher & Yankelewitz, 2010; Mulligan & Mitchelmore, 1997; Outhred, 1996; Zapata-Cardona, 2018b). However, Pessoa and Borba (2012) found that preschool students are able to solve combinatorial tasks such as Cartesian products. These studies document that solving of combinatorial problems might be challenging for third-graders, but still manageable for them.

The amount of research with respect to students' multimodal reasoning when solving combinatorial problems is relatively scarce. Höveler (2018) also argues that the research is sparse when it comes to scrutinizing young students' approaches to solve these types of problems. Zapata-Cardona (2018b) researched how young students approached combinatorial tasks. Her study, based on clinical interviews, revealed that the students' use of combinatorial counting identified close relationships between the students' combinatorial reasoning and multiplicative reasoning. Our study shares similarities with both Höveler's (2018) study and Zapata-Cardona's (2018b) study when it comes to research focus. However, we scrutinize the students' approaches to combinatorial problems in theoretically and methodologically different ways compared with Höveler (2018) and Zapata-Cardona (2018b).

Our study adds to the research referred above as we analyze third-graders' multimodal reasoning in small groups by way of utterances, inscriptions, and gestures when they collaboratively solve combinatorial problems. We thus seek to fill a knowledge gap, as neither the referred studies above (nor those referred below) have documented the substantial roles and interplay of utterances, inscriptions, and gestures in third-graders collective combinatorial reasoning. We thus add to results from previous research by analyzing students' use of these semiotic resources in their mathematical reasoning on combinatorial problems.

Thus, the aim of our study is to reveal the subtleties involved in four small groups of third-graders' multimodal reasoning when solving combinatorial problems. By revealing these subtleties, the purpose is to gain insights into how newcomers to combinatorial problems approach these, where difficulties emerge in the reasoning process, and how these students utilize utterances, inscriptions, and gestures to solve the problems.

In the next section, we present our overall theoretical stance, a sociocultural perspective on learning and development, before elaborating on our conceptual framework of multimodal reasoning comprising the semiotic resources utterances, inscriptions, and gestures. Finally, we give an account of the inherent mathematical operation of single-digit multiplication.

Theoretical framework

Our overall theoretical stance for studying primary students' reasoning in problem solving situations is a sociocultural perspective on learning and development (Rogoff, 1990; Säljö, 2001; Vygotsky, 1986). We thus see learning as a process of appropriating (mathematical) tools and concepts, a process nurtured and empowered in social, collaborative settings (Moschkovich, 2004; Rogoff, 1990; Wertsch, 1998). Students who appropriate concepts and tools in a mathematics class are increasing their familiarity with these, how, when, and why they are used to solve mathematical problems. For this to happen, students have to participate in a joint productive activity, establish a shared focus of attention, and ultimately develop shared meanings for these concepts and tools (Rogoff, 1990), a process called appropriation. Within this process, students need to take over "what someone else produces during joint activity for one's own use in subsequent productive activity" (Moschkovich, 2004, p. 51). We do not study the primary students' appropriation process as such. However, the students' reasoning is analyzed, as an explicit and overt constituent of the appropriation process unfolding in collaborative problem-solving situations.

With the term 'collaborative problem solving' we want to label a typical situation when three to six students in a mathematics classroom work together, communicate, share ideas and thoughts to accomplish and come up with solutions to a mathematics problem. Inspired by Schoenfeld (1985), we view a mathematics problem as a mathematics task that (1) is unfamiliar to the problem solver, i.e. the problem solver does not know immediately how to solve the task; (2) evokes an interest in the problem solver that makes her/him want to solve it; and (3) creates a relationship between the problem solver and the task, making the task a (mathematical) problem for the solver.

Multimodal mathematical reasoning

Jeannotte and Kieran (2017) argue that there currently is no consensus with respect to consistencies of mathematical reasoning. Nevertheless, they coin a conceptual model of mathematical reasoning in school mathematics. According to these researchers, mathematical reasoning is "a process of communication with others or with oneself that allows for inferring mathematical utterances from other mathematical utterances" (p. 7), a process involving both structural and processual aspects. This conceptualization has been successfully utilized by Carlsen (2018) analyzing upper secondary school students' mathematical reasoning. In our study we follow Carlsen's applications of Jeannotte and Kieran's (2017) definition of mathematical reasoning, as a process of communication featured by mediating tools such as utterances, inscriptions, and gestures.

We adopt a sound basis in the concept of mathematical reasoning launched by Jeannotte and Kieran (2017) as we further argue that in order to make sense of students' reasoning, an overall multimodal approach is needed. According to O'Halloran (2005), there are three semiotic resources, i.e. language, symbolism, and visual imagery, that function together in mathematical discourse. This co-functioning of semiotic resources is also emphasized by Lemke (2003), who claimed that mathematics "can only be learned and taught as an integral component of a larger sense-making resource system including natural language and visual representation" (p. 215). We partially adopt these authors' arguments as we analyze the ways students' oral argumentation through utterances (i.e. language), inscriptions, and gestures (symbolism and visual imagery), function together in mathematical reasoning. Therefore, "all modes of communication are attended to as part of meaning making" (Jewitt, 2006, p. 3). As will be evident, we argue that the students' reasoning is fundamentally multimodal, as the semiotic resources of utterances, inscriptions, and gestures are interchangeably used in their reasoning (see also, Bjuland et al., 2008; O'Halloran, 2005; Sáenz-Ludlow & Presmeg, 2006). However, utterances, inscriptions, and gestures do not contain any meaning per se. Rather, their meaning "arises in the context of other inscriptions and sign forms" (Roth & McGinn, 1998, p. 38). This contextual dependency of the three semiotic resources needs to be taken into account when analyzing students' collaborative problem solving, as they all contribute to the students' learning process. In a sociocultural sense, they are tools for thinking, communicating, and acting, facilitating joint activity and shared foci of attention and hence nurturing the appropriation process (Rogoff, 1990). Thus, we use the term *multimodal mathematical reasoning* to label the students' collective process of communication with others or with themselves, featured by the ways utterances, inscriptions, and gestures emerge and supplement each other, that allows for inferring mathematical utterances from other mathematical utterances (cf., Jeannotte & Kieran, 2017, p. 7)

Use of inscriptions as a part of multimodal reasoning

A semiotic resource constituting a substantial role in multimodal mathematical reasoning is the use of inscriptions (Cobb, 2002). Inscriptions are, according to Carlsen (2009), "understood as artifacts such as graphs, drawings, and mathematical symbols used for cognitive, communicative, mathematical, and problem-solving purposes in interactional settings" (p. 54). Carlsen argues that students' use of inscriptions are vital anchors in their reasoning and problem solving. According to Latour (1987), inscriptions are seen as intermediary actants in human exchanges. We use the term 'actant' to argue that inscriptions are not considered as dead objects. They play a mediating and interactive role in the students' problem-solving endeavor (Carlsen, 2009; Latour, 1987). Thus, inscriptions serve a mediating role in student problem solving between the mathematics problem given and the mathematization made by the students. An important additional point is that the inscriptions serve as externalizations of the students' thinking. Moreover, students reason by means of their inscriptions. Several studies argue that use of inscriptions is vital for students' mathematics learning (e.g., English, 2005; Maher & Yankelewitz, 2010; Roth & McGinn, 1998). For example, Maffia and Mariotti (2020) showed how inscriptions played an important role in teaching-learning sequences in a primary mathematics classroom concerning the distributive law of number multiplication.

Use of gestures as part of multimodal reasoning

An additional semiotic resource that plays a role in multimodal mathematical reasoning is the use of gestures. We define gestures in accordance with McNeill (1992), as "movements of the arms and hands [. . .] closely synchronized with the flow of speech" (p. 11). Several researchers have emphasized the role of gestures in students' mathematics learning (e.g., Arzarello et al., 2005; Edwards, 2005; Radford, 2003; Reynolds & Reeve, 2002). We follow Bjuland et al. (2008) and argue that gestures play a crucial role as mediating tools in student collaborative problem solving. Gestures are also included in Radford's (2003) notion of semiotic means of objectification: "objects, tools, linguistic devices, and signs that individuals intentionally use in social meaning-making processes to achieve a stable form of awareness, to make apparent their intentions, and to carry out their actions to attain the goal of their activities" (p. 41). A gesture, as a semiotic mean, is argued to be a substantial mediating tool in student collaborative problem solving in mathematics. The gesture fulfills an objectifying role, as it makes student reasoning apparent. Hence, gestures are used as objectifying tools, as objectification is "a process aimed at bringing something in front of someone's attention or view" (Radford, 2002, p. 14).

The inherent mathematical operation: single-digit multiplication

Various approaches to single-digit multiplication problems have been categorized by Sherin and Fuson (2005). Since the combinatorial problems dealt with by the groups in our study mathematically comprise single-digit multiplication, Sherin and Fuson's categorization is of relevance. These authors coin a taxonomy of six different categories (see Table 1). In our study of four groups' reasoning when solving combinatorial problems, we have partially adopted these identified categories, however with some minor adjustments (see *Analytical Process*).

Summing up, in this study we analyze students' use of utterances, inscriptions, and gestures when solving combinatorial problems in a small-group setting. We are thus inspired by O'Halloran (2005), who argues that these three semiotic resources, language (oral argumentation through utterances), symbolism (inscriptions), and visual imagery (inscriptions and gestures), function together in mathematical reasoning (see also, Lemke, 2003). These resources all play mediating roles in making the students' reasoning apparent and explicit. The students' utterances, inscriptions, and gesture all serve as intermediary actants between their mathematization and the given problems.

Previous research on students' sense making of combinatorial problems

A combinatorial problem may be modeled as a product in which every element in one set is to be combined with every element in another set. That is, if there are three elements in the one set (e.g., three jackets) and four elements in the other set (e.g., four jeans), the product becomes $3 \cdot 4 = 12$, i.e. twelve possible outfits combining one jacket and one pair of jeans. This product may easily be generalized to include three or more sets.

English (1991, 1993) researched students' approaches to mathematical problems involving combination of sets. She argued that students of age 4.5 to 10 years may be able to reason and solve combinatorial problems if these are meaningfully contextualized for the children. English (1991) used the context of dressing bears and physical materials such as wood bears and paper clothes. One outfit consisted of one top and one pair of jeans and the tasks varied from having two tops and three jeans (and vice versa) to three tops and three jeans. English found that the students used six different strategies increasing in sophistication in order to solve the combinatorial problems, ranging from random selection of objects via a trial-and-error procedure to a systematic pattern approach where one of the objects was held constant, i.e. odometer pattern. An odometer approach is a combinatorial, systematic pattern strategy when combining objects in two (or more) sets. One object in the first set is held constant and successively combined with every object in the second set. Then the second object in the first set is held constant and successively combined with every object in the second set, etc.

However, English (1991) found that the younger children (4.5–6) were only able to use the less efficient strategies while the older children (7–10) were able to use the more advanced strategies to solve the various combinatorial problems. English (1993) study of 7 to 12 years old children revisited the context of dressing bears, however further elaborated by also including tennis rackets to combine with various clothing. The most complex problem was to combine two tops, three jeans, and two rackets making twelve combinations. English (1993) found some of the same results as in the 1991 study, but also that the older children were able to use an odometer strategy working simultaneously with two constant objects in the most complex problems.

Outhred (1996) studied students' (Grade 1 to 4) approaches to combinatorial problems through their use of inscriptions. She gave the students a problem in the context of an ice cream store, with four different flavors and three different cone sizes. She found that the students used three different inscriptions in their approaches to find how many different ice creams one can buy: unsystematic approaches, systematic approaches without array (not all combinations are explicitly inscribed), and systematic array (all combinations are systematically inscribed). Furthermore, Outhred found that the approach called systematic array was positively correlated with a mathematically correct solution. Mulligan and Mitchelmore (1996) also studied how students (Grade 3) represented their solutions to combinatorial problems. They found that eight out of ten students were able to correctly solve the given combinatorial problems when they used inscriptions, i.e. pictorial drawings in which the children modeled the problem concretely.

The role of student made representations, i.e. what we call inscriptions, were studied longitudinally by Maher and Yankelewitz (2010), as they analyzed how second- and third-graders argued when solving an outfit problem with three shirts and two jeans. They found that the students drew pictures of the shirts and jeans to find the possible combinations. As third-graders the students also wrote down first letter abbreviations and drew line diagrams. As second-graders the students were not able to arrive at a correct answer, while as third-graders they all arrived at a correct answer. The students themselves argued that the drawn lines ensured that they had reasoned correctly. Maher and Yankelewitz argued that their findings suggested "the importance of encouraging the use of students' personal representations in building solutions" (p. 25), or what English (2005) called "representational fluency" (p. 134).

Van Bommel and Palmér (2018) investigated how six-year-olds dealt with a combinatorial problem both in an analogue format and a digital format. The children were shown three plastic bears with different color, physically and digitally. The problem was formulated as in how many different ways the three toy bears could sit in a sofa. Results of this study were also communicated partially in Palmér and van Bommel (2018). They found that the children were making duplications of solutions as well as incomplete solutions. However, the children, when working with the digital application, were more systematic in their approach. Thus, more complete solutions were created, and fewer duplications were made. Interestingly, Palmér and van Bommel discussed the children's interpretations of the word 'different,' as all but one child, when solving the problem by drawing, made "exactly three permutations with each bear sitting one time at each place" (p. 571). According to the researchers, the children plausibly interpreted 'different' as meaning different in all aspects. Keeping one of the bears as constant and changing the two other bears' seating were thus not thought of as a 'different' way of sitting in the sofa. These studies showed the children's initial appropriation of the concept of combinatorics. Furthermore, the six-year-olds were able to engage in a process of becoming steadily more systematic in approaching such combinatorial problems, afforded by technology.

Based on identification of the various approaches to single-digit multiplication the students used, as well as identification of which inscriptions and gestures they used, we formulated the following research question: What characterizes four groups of Grade 3 students' multimodal mathematical reasoning when solving two given combinatorial problems?

The study – methods and analytical approach

In this study we chose a qualitative research strategy (Bryman, 2016). We introduced mathematical problems for the students to solve. However, we did not intend to make any changes to the mathematics teaching regularly taking place, except that we wanted to scrutinize how these students approached the given combinatorial problems. Our study involved videotaped observations of four groups in order to come up with answers to our research question. The first author collected the data, adopting an overt but passive researcher role. This gave us opportunities to intensively scrutinize their multimodal mathematical reasoning when solving combinatorial problems.

Analytically we adopted a dialogical approach (Linell, 1998), i.e. we see the group's multimodal mathematical reasoning as taking the form of a dialogue jointly produced by the students in a group. Each contribution was analyzed sequentially, i.e. its meaning is fundamentally rooted according to previous and consecutive contributions. Moreover, the analysis of each dialogue takes as a point of departure that the dialogue took place in a school cultural setting in which the students sought to make sense of the mathematical tool of combinatorial problems. This dialogical approach, being a form of discourse analysis, derives from the epistemological stance of dialogism. This stance emphasizes interaction, language, contextual issues as well as interactional contributions of others as substantial for people's sense making and learning. Moreover, the dialogical approach also emphasizes the importance of externalization of thinking, i.e. students' attempts to make their reasoning explicit in front of each other. The dialogs were translated into English. The inscriptions presented are stylized versions due to the English translation.

Context and research participants

The study reported here originates from the research conducted by Wathne (2008). Wathne studied collaborative group work of thirteen Grade 3 (8–9 years old) and Grade 4 (9–10 years old) students in Norway, particularly focusing on the students' ways of solving combinatorial problems. Informed consent to participate in the study was given by each of the thirteen participants and their parents.

When the data collection in our study took place, the thirteen students were in the end of their third year of schooling. That means that they were eight and nine years old at the time of data collection. The mathematics teaching in this class was organized as follows: The teacher usually started each lesson with rehearsal of homework, then a new mathematical theme, concept(s), and/or procedure(s) were introduced. Afterward, the students were collaboratively, in small groups composed by the mathematics teacher, solving mathematical problems corresponding with the theme introduced by the teacher. The groups were working in separate rooms. The students were used to socially interact through argumentation and problem solving. The data collection thus took place in a natural setting from the students' point of view and the sessions that were videotaped did not differ in any way from the common ones, except that the problem-solving phases were videotaped.

Data was collected from the ordinary mathematics teaching in this third-grade class. Four small groups' collaborative problem solving was videotaped, one group at a time and for 45–60 minutes in each group. At their disposal, the students shared one piece of A3 paper and a pencil. The four groups were composed by the 13 students that gave their consent to participate in the study. The data analyzed emerged from data collection at four consecutive days in the end of the third-graders' academic year. The data analyzed for this particular study emerged from all four groups.

At the time of data collection, the students were familiar with one-digit multiplication. Combinatorics was at the time of data collection not explicitly mentioned in the mathematics curriculum for elementary school. However, the students had, in an earlier phase of the data collection of Wathne (2008), sparsely encountered combinatorial problems like the ones introduced for the purpose of our study.

Each group engaged with two combinatorial problems in the group sessions videotaped. Additionally, copies of students' inscriptions at the shared paper were collected, all complementing the video data. There were four students in group A and three students in group B, C, and D, and they were by the teacher randomly assigned to a group except that both genders were to be represented. The data collection took place in a public school in an urban area of Norway. The students' real names are anonymized.

The combinatorial problems given

The combinatorial problems that the groups engaged with in this study were designed by one of the researchers and the design was inspired by the conducted research of English (1991, 1993, 2005) and Outhred (1996). In our study we analyzed how the student groups solved the combinatorial problems called "Breads" and "Cards." Both problems were originally formulated in Norwegian in an ta and written form, and paper and pencils were provided. The problems read as follows, when translated to English:

Breads

You may choose between wholemeal bread (WM), rye bread (R), and white bread (W). These types of breads come in round loaves and square loaves. How many different loaves can you make?

Cards

You may choose between cards that are either green (g) or yellow (y). The cards have either Christmas (C), Birthday (B) or Easter (E) greetings, and the cards have either silver (S) or gold (G) letterings. How many different greeting cards can you make?

These are combinatorial problems that in the literature are called two-dimensional (Breads) and three-dimensional (Cards) problems respectively (e.g., English, 1993).

Analytical process

Overall, we took an inductive approach in our analyses (Bryman, 2016). Our unit of analysis was a group of students' multimodal mathematical reasoning. The Breads and Cards problems were our analytical point of departure, and we identified excerpts in which the groups discussed these problems. The second analytical step was to align the groups' ways of reasoning with the approaches to single-digit multiplication (see also, Sherin & Fuson, 2005; Table 1) used by the students. We thus scrutinized the ways the students found the various combinations involved in the two problems. In the analyses we identified three different approaches that the students used, *Count-all, Grouping* (Count-by), and *Learned products*. These approaches were partially adopted from Sherin and Fuson (2005), however with some minor adjustments.

Thirdly, we identified whether the groups used inscriptions when solving the two given problems. Furthermore, we analyzed what kind of inscriptions the students used according to the categories of Outhred (1996), i.e. systematic inscriptions without array (not all combinations are explicitly inscribed), and systematic inscriptions with array (all combinations are systematically inscribed). Finally, the fourth analytical step was to analyze the groups' use of gestures such as pointing and sliding (see also, McNeill, 1992). In all cases, the groups' discussions took various directions before they came up with the reasoning analyzed.

To exemplify our analytical steps, we draw on one excerpt that is not analyzed in the Results section. We chose not to include this excerpt in the analysis due to its similarity with what we call Excerpt 3 in the Results section. In this example, where group B discussed the Cards problem, i.e. our first analytical step, we observed the use of a grouping approach (see excerpt below, move 7), i.e. our second analytical step. Furthermore, we observed that the students used inscriptions (see, Figure 1, e.g., move 1), i.e. our third step of analysis. Finally, we also identified that the students made use of gestures (e.g., move 3), i.e. our fourth analytical step:

(1) Ben: I may choose three (*Points* at the card C S y in Figure 1)
(2) Bodil: It is those (*Points* at the inscription in Figure 1)
(3) Berit: I can explain. These cards are not identical (*Sliding* one group of four different cards in Figure 1)
(4) Bodil: Am I going to count one, two?
(5) Berit: No, that is one card, right (*Points* at one card in Figure 1)
(6) Bodil: Yes (Bodil is explaining the different combinations of cards made up of colors, greetings and letterings)
(7) Berit: Four plus four plus four. Four three times. That's twelve. I got to twelve (*grouping approach*)

We present and analyze examples of how the four groups approached the two mentioned combinatorial problems. These two problems both testify to what English (2005) calls "problems that reflect the fundamental counting principle" (p. 125).

Results

We present one approach and associated inscriptions and gestures at a time. The reason for structuring our analyses according to the approaches used, is that our focus is on the groups' multimodal mathematical reasoning and not on what group that revealed the particular way of reasoning. Table 2 shows the variety of approaches to single-digit multiplication (see also, Sherin & Fuson, 2005) used by the students. Due to our research scope, we have sparsely analyzed those incidents where students used the approach called *Learned products* and *Other* (see, Table 2). The former approach is a way of solving the combinatorial problem that can be used when the multiplicative structure of the problem is perceived. The latter approach is a way of solving the combinatorial problem that may be used when the multiplicative structure of the problem is not perceived. Nevertheless, these two approaches are included for the sake of coherence, justice on behalf of the students, and to present a full, exhaustive account of the groups' various approaches to the two provided problems.

The count-all approach combined with a grouping approach

Two excerpts are shown exemplifying the groups' count-all and grouping approach. From Table 2 we observe that both group A and group D used these approaches to solve the given problems.

Excerpt 1

With respect to the Breads problem, group A reasoned as follows, strongly supported by systematic inscriptions with array:

(75) Anne: (Draws the inscriptions in Figure 2) We may choose between six different
(76) Alf: What do you mean by six different?
(77) Anne: Six different breads, since you can choose between round and square, and then it is, then they are not equal in a way. One, two, three, four, five, six (points at each of the drawn breads while counting, see, Figure 2)
(78) Astri: Excellent, Anne
(79) Alf: What?
(80) Anne: We may choose between six different breads
(81) Alf: Okay
(82) Arnie: Yes, it is three round ones (points at the drawn round shapes) and three squares (points at the drawn squared shapes)
(83) Anne: Yes
(84) Arnie: Then it is six
(85) Alf: I got to six
(86) Astri: Yes

Anne (75) drew and wrote all the ways the breads, wholemeal, rye, and white, can be combined with round and square loaves, and she started off the dialogue by claiming that they can choose between six different breads. Alf (76) did not seem to follow Anne's reasoning, possibly because she did not mention the word "breads" in her claim. However, Anne (77) then counted all the different ways they can combine the types of bread with the types of loaves, by pointing successively on the inscribed round and squared shapes (see, Figure 2). Thus, the count-all approach is evident in this case, made explicit through the pointing gestures. According to Outhred (1996), the students thus used systematic inscriptions with array. The inscriptions already included a systematic array, making sliding gestures to count all combinations unnecessary.

Astri (78) seemed to make sense of Anne's utterance, while Alf (79) did not seem to follow the argument. Anne (80) thus repeated her arguments by making explicit the conclusion based on her reasoning. However, Anne's argument is not clear about how to arrive at this conclusion. Alf (81) probably did not follow the argument, but simply accepted Anne's conclusion. Arnie (82) confirmed Anne's conclusion, however by dividing the altogether six breads into three round breads and three squared breads. Arnie thus justified the argument of why they made six breads altogether. Arnie's solution may be interpreted as a grouping approach (3 + 3 breads), making the group's reasoning combining two approaches (see also, Sherin & Fuson, 2005), or even an odometer approach (see also, English, 1991). Arnie's perceiving of the problem's structure deviates from the inscription per se. We interpret this to indicate that Arnie perceives the multiplicative structure of the problem. Anne (83) agreed to this grouping approach, as an alternative to her previously used count-all approach, and Arnie (84) repeated the total number of breads. Alf's (85) utterance is interpreted as making explicit that he has counted the number of breads himself and ended with the same number. Astri (86) then rounded off the group's reasoning by confirming the number of different breads as six.

From the analysis of this dialogue, we argue that utterances, inscriptions, and gestures emerge and function together. The three semiotic resources, utilized in move 77 and move 82, supplement each other in mediating the intended mathematical meaning. Furthermore, we argue that this dialogue reveals that coming up with a solution to the given problem through multimodal mathematical reasoning is indeed a collective process.

Excerpt 2

With respect to the Cards problem, group D reasoned in the following way, strongly supported by systematic inscriptions without array:

(43) Doris: (Draws the inscriptions in Figure 3) One, two (sliding combinations in Figure 3)
(44) Dina: Three (sliding combinations in Figure 3)
(45) Doris: Four, five, six, seven, eight, nine, ten, eleven (sliding combinations in Figure 3)
(46) Dan: Twelve. You have counted those, haven't you? (points at some place in Figure 3)
(47) Doris: Yes
(48) Dan: It becomes eleven (The researcher asks the students to check their answer to the problem. Doris then starts counting once more)
(49) Doris: One card, two cards, three cards, four cards, five cards, six cards, seven cards, eight cards, nine cards, ten cards, eleven cards, twelve cards (sliding every combination in Figure 3. She focused at one color and one greeting at a time while she systematically varied the letterings)
(50) Dan: Yes, twelve
(51) Dina: Yes

Doris drew all the objects in the Cards problem (see, Figure 3), green and yellow card, Christmas, Birthday and Easter greetings, silver and gold letterings. Then Doris (43) initiated the counting, followed by Dina (44), then continued by Doris (45). They used sliding gestures to count the different combinations of the objects. However, from the video it is not possible to detect the pattern in their sliding gestures. Doris (45) ended up with the number 11. After some discussion whether it is 11 or 12 combinations (46–48), Dan and Doris concluded with 11 different cards. Upon the researcher's request, Doris started to count all the different combinations of objects over again. This time Doris (49) took advantage of the inscriptions and sliding gestures when she used the count-all approach, this time also in a rhythmic way (the underlined words were emphasized by Doris). In her sliding, Doris focused at one color and one greeting at a time while she systematically varied the letterings. The green card was systematically combined with each of the greetings. Then each of the greetings was combined with each of the letterings. This process was repeated with the yellow card. Thus, the sliding gestures took the pattern *g-C-S, g-B-S, g-E-S; g-C-G, g-B-G, g-E-G; y-C-S, y-B-S, y-E-S; y-C-G, y-B-G, y-E-G*. This exemplifies an odometer pattern approach (English, 1991). Dan (50) agreed to Doris' reasoning by repeating the total number of cards. Dina (51) also agreed to this number.

Furthermore, the analysis of the group's reasoning documents how the combination of the sliding gestures and the use of systematic inscriptions without array eventually made Doris successful in counting all the twelve different cards possible. Doris in fact used an approach that combined a count-all approach with a grouping approach. All combinations were counted and groups of three combinations were emphasized. The contributions are collectively seen to document the group's multimodal mathematical reasoning in this case. The conclusive utterances in moves 50–51 seem to indicate that the group has reached consensus about the correct number of cards possible with the given numbers of colors, greetings, and letterings.

In Excerpt 1 the pointing gestures were intimately related to the inscriptions made as this facilitated counting all combinations. In Excerpt 2 the sliding gestures were intimately related to the inscriptions made as this facilitated counting all combinations. The sliding gestures were afforded by having the inscriptions systematically present as array. By sliding each combination of color, greeting, and lettering, the odometer strategy becomes evident. The analysis reveals in what ways the odometer strategy unfolds in this case. By analyzing the use of inscriptions and gestures in addition to utterances, we achieve detailed insights into the students' multimodal way of reasoning mathematically. However, the pointing gestures used in Excerpt 1 only established shared foci of attention amongst the group members. The sliding gestures used in Excerpt 2 provided justifications of the reasoning. Nevertheless, both analyses show evidence of how the semiotic resources of inscriptions and gestures emerged and functioned together with utterances in the groups' reasoning (see also, O'Halloran, 2005).

The grouping approach

Two excerpts are shown exemplifying the groups' grouping approach. From Table 2 we observe that both group B and group C used this approach to solve the given combinatorial problems.

Excerpt 3

With respect to the Breads problem, group B reasoned as follows, strongly supported by systematic inscriptions without array:

(21) Berit: (Draws the inscriptions in Figure 4) You can buy two white (points at Figure 4), that white (sliding this combination in Figure 4) and that white (sliding this combination in Figure 4). Then you can buy two rye
(22) Ben: Four
(23) Berit: And wholemeal you can buy two of as well (sliding this combination in Figure 4). That is six all together, right?
(24) Bodil: Yes, okay
(25) Ben: Yes

In her reasoning, Berit (21, 23) used repeated addition, 2 + 2 = 4, 4 + 2 = 6, together with the inscriptions and the sliding gestures. Ben (22) supported Berit in her reasoning. Berit was sliding her finger between the rectangle labeled *W* and the round form and the squared form respectively (see, Figure 4). Bodil (24) and Ben (25) confirmed their agreement with Berit's claim. This variant of the grouping approach, repeated addition, differs from the count-all approach in that not every number between 1 and 6 is represented. Berit's counting was grouping two and two combinations by way of her sliding gestures and use of the inscriptions. Moreover, her sliding gestures provided justifications for her reasoning by making her argument explicit. Once again, the semiotic resources of utterances, inscriptions, and gestures emerged and supplemented each other in the students' reasoning (see also, O'Halloran, 2005).

The gestures used complemented the inscriptions used in Excerpt 2 and Excerpt 3 in such a way that the shortcomings of the inscriptions, i.e. not explicitly displaying all combinations (systematic without array), were compensated by deliberate use of sliding gestures. The inscriptions together with the sliding gestures demonstrated a systematic array approach (Outhred, 1996). In the dialogue in Excerpt 3 we observe that the contributions supplement each other, comprising a collective multimodal mathematical reasoning within the group.

Excerpt 4

Group C dealt with the Cards problem using systematic inscriptions without array:

(Carl, Chris and Cato draw the inscriptions in Figure 5. From their discussion before move 109, it is not possible to make sense of their reasoning.)
(109) Carl: No, I think it is twelve
(110) Cato: I also think it is twelve
(111) Cris: I did it just before. Those three (points at greetings C, B and E on his sheet, see, Figure 5). Then it is three (points at Gy) and then I have silver (points at Sg). Then it is three there, which is six. Then I do that once more, so it becomes twelve
(112) Cato: Now I didn't understand anything

(113) Cris: I might possibly explain it to you. So those become three (points at Gy, see, Figure 5) and here there also are three (points at Sy). Here there also are three (points at Sg), nine, and three there (points at Gg), twelve

(114) Cato: I am fed up working with this problem (This comment ends the group's reasoning on the Cards problem)

The group wrote down all the four combinations of colors and cards (*Sg, Gg, Gy, Sy*), together with the third object, greetings (*C, B, E*), separately at the bottom (see, Figure 5). Both Carl (109) and Cato (110) suggested that the total number of cards were 12. Nevertheless, Cris (111) reasoned by way of the inscriptions in Figure 5 supported by pointing gestures. He was grouping three and three combinations as the greetings *C, B*, and *E* firstly are grouped with *Gy*, secondly with *Sg*. One half of the four combinations of colors and cards was then used, so he could just double the six combinations to get to twelve combinations altogether. This quite sophisticated way of reasoning about the Cards problem seemed not to communicate well with Cato. It seemed clear that Cato (112) was confused by Cris' reasoning. Cris (113) then sought to make explicit his way of reasoning once more. This time he counted by three while pointing at the inscriptions successively. Three cards for each of the letters (greetings) *C, B*, and *E*, for each of the combinations *Gy, Sy, Sg*, and *Gg* respectively. Deviating from the three previous excerpts, there were no explicit utterances communicating agreement to Cris' reasoning, since Cato's (114) utterance ended the group's reasoning.

Cris' reasoning may also be argued to exemplify an odometer pattern approach (English, 1991). Cris (111) held one object constant, the three greetings, and got three different cards by pointing at *Gy* $(3 \cdot 1)$ and three new ones by pointing at *Sg* $(3 \cdot 1)$ and got $3 + 3 = 6$ cards. Then Cris doubled the six cards and got 12 cards by way of his combinatorial reasoning. He repeated what he just did, when considering *Sy* and *Gg*. This doubling probably contributed to the confusion expressed by Cato (112). However, Cris (113) utilized an odometer approach when attempting to make his reasoning explicit. He firstly pointed at *Gy*, holding these two objects constant, and got three cards by combining this object with the three greetings $(1 \cdot 3)$. This systematic approach was then repeated for the three other groups of two objects ($(1 \cdot 3)$ three times), grouping the number of cards simultaneously $(3 + 3 + 3 + 3 = 12)$.

The analyses of these two excerpts, Excerpt 3 and Excerpt 4, show that the grouping approach was accompanied by both sliding gestures and pointing gestures. However, from a multimodal mathematical reasoning point of view, it is worth emphasizing that the pointing gestures did not seem to serve as justifications of Cris' utterance in Excerpt 4. The pointing gestures did not manage to mediate the intended mathematical meaning between the semiotic means used (see, also Radford, 2003), preventing the complementation of utterances, inscriptions, and gestures into a systematic array approach. The lack of a mediating function of the pointing gestures used in Excerpt 4 contrasts with the justification function of the sliding gestures used in Excerpt 3.

The learned products approach

This approach is based on multiplicative reasoning as two factors are combined with their product, and the solution emerges rapidly. This way of solving the combinatorial problem can only be used when the multiplicative structure of the problem is perceived. The approach is implicitly used and differs from count-all and grouping in that only the result is uttered. With respect to the Breads problem, Cris in group C exemplified this approach by quite immediately upon his facing of the problem said: "That is six." In group D, again with respect to the Breads problem, one of the students just said: "six." The mathematical reasoning in these two cases is implicit, and we may thus only imagine that the students mentally multiply 2 times 3 or vice versa.

The other approach

With respect to the Cards problem, group A used an approach that differed from the three others in that the students counted the objects in each set rather than the combinations:

Excerpt 5

(135) Are: We can choose between a green and a yellow card. That is two. Then we can choose between
(136) Astri: Birthday, Easter and Christmas greeting
(137) Anne: That is three, and three plus two is five. Then we can choose between gold and silver. And if we add two it becomes seven
(138) Astri: That is seven ways

The group's focus is on the number of possible colors, greetings, and letterings on the cards. The group's reasoning is based on adding the number of objects in each set of the problem, 2 cards + 3 greetings + 2 letterings = 7 ways to make a card rather than multiplying the number of objects in each set to get the number of different cards possible. The students in group A did neither recognize nor perceive the multiplicative and combinatorial structure of the problem. This phenomenon is observed by Mulligan and Mitchelmore (1997) as well, as relatively common amongst third-graders' reasoning on combinatorial problems. Hence, the logic of their reasoning did not lead to a proper solution of the given problem. This may have been caused by the increase in mathematical complexity from the Breads problem, i.e. a two-dimensional problem, where group A came up with a correct solution, to the Cards problem, i.e. a three-dimensional problem, where this incorrect solution emerged. Furthermore, in this case group A did not have an inscription as a vital anchor in their reasoning and problem solving (see also, Carlsen, 2009; Latour, 1987)

Discussion

We set out in this study to come up with plausible answers to the research question: What characterizes four groups of Grade 3 students' multimodal mathematical reasoning when solving two given combinatorial problems? The overall, summative answer to this research question is as follows: The students' collective reasoning was characterized by the interplay of utterances, inscriptions, and gestures. These semiotic means of objectification (Radford, 2003) emerged and supplemented each other, and thus they collectively contributed to efforts in creating shared foci of attention and shared meanings amongst the group members. The students' made their assertions reasonable and accountable in front of fellow students in the small group. Hence, these semiotic resources all contributed to the students' multimodal mathematical reasoning (Lemke, 2003; O'Halloran, 2005).

The analyses of the groups' dialogs show that the groups used a variety of approaches when solving combinatorial problems (Sherin & Fuson, 2005). The students in group A and group B reasoned by way of inscriptions with respect to the Bread problem, and group B, group C and group D all reasoned by way of inscriptions with respect to the Cards problem. The inscriptions were thus fundamental in the groups' reasoning (Mulligan & Mitchelmore, 1997; Outhred, 1996).

In a similar manner as was shown in the study of Carlsen (2009), we argue that the inscriptions served as intermediaries (Latour, 1987) in the groups' mathematical reasoning. These artifacts mediated mathematical meaning between the combinatorial problem and the mathematization. These inscriptions, accompanied by gestures, evidently showed their substantial role as semiotic means of objectification (Radford, 2003) in the groups' reasoning (Jeannotte & Kieran, 2017).

Furthermore, the students' reasoning was characterized by the use of gestures, in particular the gestures called pointing and sliding. By pointing at various parts of their inscriptions and sliding their index finger from one part to another, together with utterances (McNeill, 1992), the students mediated their mathematical meaning. Thus, gestures, accompanied by inscriptions, played a fundamental role in the students' collaborative problem solving (see, also Bjuland et al., 2008; Lemke, 2003). Nevertheless, the analyses document that the pointing gestures predominantly established shared foci of attention amongst the group members. The sliding gestures, however, were used by students to justify their reasoning, i.e. these gestures mediated the students' mathematical thinking by taking into account the various combinations of objects in the given problems and systematically justified how objects from one group were combined with objects in a second group. The sliding gestures established both shared foci of attention *and* shared meanings (cf., Rogoff, 1990). Thus, detailed insights into the ways the students used the odometer pattern (English, 1991, 1993) in their multimodal mathematical reasoning were revealed.

As mentioned, Van Bommel and Palmér (2018) showed that six-year-olds became unsystematic in their approaches to solving a combinatorial problem. However, these researchers identify that these young children were able to engage with relatively sophisticated mathematical problems. Even though their combinatorial problems differ from the ones given in our study, we observe that students being three years older than the ones in van Bommel and Palmér's study, still face difficulties in solving combinatorial problems.

When scrutinizing the interplay of utterances, inscriptions, and gestures in more detail, our study particularly reveals the following: When students used inscriptions characterized as systematic inscriptions with array (Outhred, 1996), accompanied with pointing gestures as the analysis of excerpt 1 reveals, the students' multimodal reasoning was made explicit and they seemed to make sense of each other's mathematical arguments. The pointing gestures confirmed the way the students counted the various combinations.

When students' made inscriptions characterized as systematic inscriptions without array (Outhred, 1996), other modes of communication were needed to make the mathematical reasoning more explicit. From the analysis of Excerpt 4, it is revealed that the pointing gestures did not manage to complement the inscriptions in ways making the students' mathematical reasoning explicit. However, from the analyses of Excerpt 2 and Excerpt 3, it is revealed that the sliding gestures did complement the inscriptions in ways making the students' multimodal reasoning explicit (O'Halloran, 2005). The students thus experienced difficulties in their reasoning when what they said, the inscriptions and the gestures used did not explicitly complement each other. Nevertheless, when these three semiotic resources (O'Halloran, 2005) were utilized together, complemented each other, and explicitly mediated the students' reasoning, the collaborating students did realize the mathematical arguments behind solutions to the combinatorial problems.

Our results show ways that utterances, inscriptions, and gestures such as pointing and sliding emerge and function together in combinatorial reasoning. The students' multimodal reasoning made explicit their combinatorial thinking, mediated the intended mathematical meaning, and facilitated their solving of the given combinatorial problems. The insights revealed from our analyses, both are in accordance with previously documented results (Borba et al., 2015; Höveler, 2018; Lemke, 2003; O'Halloran, 2005) and go beyond these. The results of our study increase our insights into third-graders' combinatorial thinking as the results document the complexities involved when students are to solve combinatorial problems.

Lemke (2003) and O'Halloran (2005) argued for the importance of analyzing how semiotic resources like inscriptions and gestures play fundamental roles in the learning of mathematics. English (1991, 1993) found that students used an odometer strategy when solving two-dimensional and three-dimensional combinatorial problems. However, in our study we add to what is already know from the literature by analyzing students' use of semiotic resources in their mathematical reasoning on combinatorial problems.

The results of our study partially confirm the results of English (1991, 1993) when it comes to the adoption of an odometer strategy. English (1991, 1993) based her study on individual student work with manipulative materials. However, our study also adds to the insights provided by English as we in our study reveal the ways students collaboratively use gestures and inscriptions in their mathematical reasoning with respect to the provided combinatorial problems. By analyzing the use of gestures and inscriptions, co-functioning with their utterances, we get detailed insights into the particularities of their mathematical reasoning, insights that go beyond the results of English (1991, 1993) and beyond the more general statements of Lemke (2003) and O'Halloran (2005).

More research is needed to achieve further insights into the use of these modalities in small-group problem solving (see also, English, 2005). Moreover, we argue, in accordance with Höveler (2018), that there is a need for paying more attention to combinatorial problems in the mathematics teaching in the first years of schooling, as this is a rich mathematical structure. Our results suggest that it is fortunate to be systematic when combining sets, to make purposeful inscriptions that display the sets to be combined, but also to explicitly writing down the various combinations are fruitful ways to solve such problems. Finally, using gestures to make your thinking explicit when reasoning on the combinatorial problems is communicatively and collaboratively purposeful.

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No potential conflict of interest was reported by the author(s).

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Appendix 1

Table 1. Six categories in students' approaches to multiplication problems (Sherin & Fuson, 2005).

Name of category	Description
Count-all	Every number between one and the total is represented. Very often manipulatives or fingers are used as counting tools.
Additive calculation	Not every number between one and the total is represented. Addition of the numbers in each group is used, e.g., 4 + 4 = 8, 8 + 4 = 12, 12 + 4 = 16.
Count-by	Counting in groups, i.e. counting several at once, n, 2 n, 3 n, 4 n etc.
Pattern-based	The student recognizes a known pattern in the numbers involved.
Learned products	The solution comes rapidly, no oral verbalization except the result.
Hybrids	Combinations of the five other categories.

Appendix 2

Table 2. Overview of the groups' approaches to the given problems.

Approach	Group A		Group B		Group C		Group D	
	Breads	Cards	Breads	Cards	Breads	Cards	Breads	Cards
Count-all	X							X
Grouping	X		X	X		X		X
Learned products					X		X	
Other		X						

Appendix 3

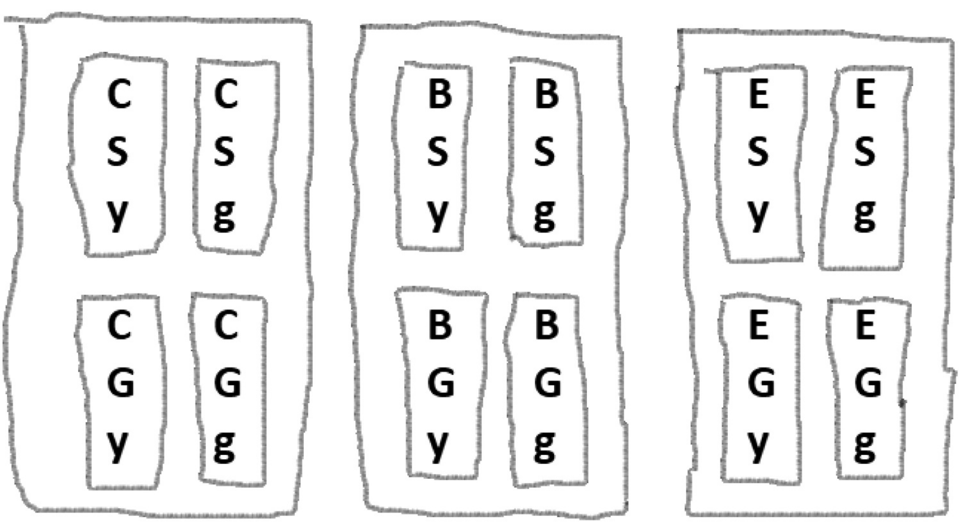

Figure 1. Group B's inscription with respect to the Cards problem.

Appendix 4

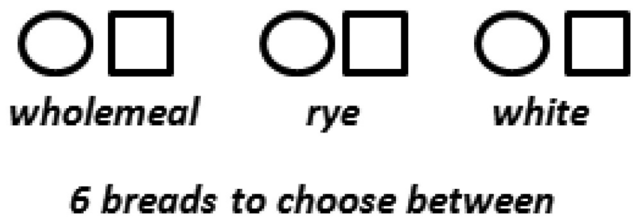

Figure 2. Group A's inscriptions with respect to the Breads problem.

Appendix 5

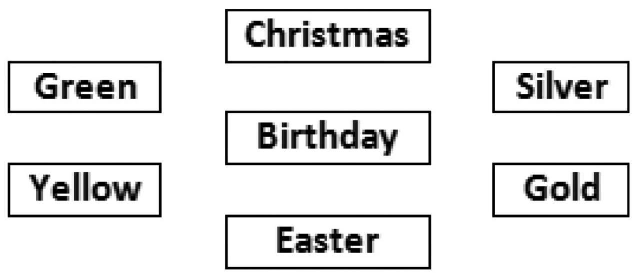

Figure 3. Group D's inscriptions with respect to the Cards problem.

Appendix 6

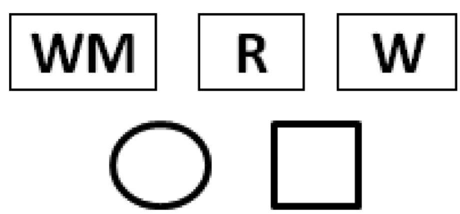

Figure 4. Group B's inscriptions with respect to the Breads problem.

Appendix 7

Figure 5. Group C's inscriptions with respect to the Cards problem.