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Convergent dynamics of optimal nonlinear damping control

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Abstract: Following the Demidovich's concept and definition of convergent systems, we analyze the optimal nonlinear damping control which was recently proposed for the second-order systems. Targeting the problem of servo-regulation, and more specifically output tracking of the $\mathcal{C}^1$ reference trajectories, it is shown that all solutions of the proposed control system are globally uniformly asymptotically stable. The existence of the unique limit solution in the coordinates origin of the control error and its time derivative are shown in the sense Demidovich's convergent dynamics. Explanatory numerical examples are provided to accompany the analysis.

Keywords: Convergent dynamics, optimal nonlinear damping, tracking control, convergence analysis, stability analysis

1. INTRODUCTION

Unlike the linear feedback systems, the nonlinear controllers and plants are mostly served with some less universal methods and tools for analysis and design. Depending on complexity of the underlying nonlinear dynamics, the local or global approach, and not least the control (or application) specification e.g. whether a set-point or tracking problem (Isidori [1995]) are of interest, the stability and convergence analysis can require more specific approaches than those which are based on the standard Lyapunov stability techniques, cf. e.g. Khalil [2002], Sastry [2013]. For analyzing the behavior of nonlinear control systems subject to the external inputs, such as reference or disturbance values, one needs to prove the existence and global asymptotic stability of a solution, along which the output tracking can be guaranteed. The so-called incremental stability (Angeli [2002]) and contraction analysis (Lohmiller and Slotine [1998]), or more generally contraction theory cf. e.g. Sontag [2010], provide a means of showing that all system trajectories converge uniformly to a unique solution for which the output tracking error is zero.

Being motivated by the Demidovich [1967] concept and definition of the convergent systems, reviewed in Pavlov et al. [2004], the following note analyzes and develops the output regulation properties of the optimal nonlinear damping control proposed recently in Ruderman [2021].

2. PRELIMINARIES

2.1 Convergent dynamics

In this section, we briefly recall the main definition and properties of a convergent system dynamics, according to Demidovich [1967], while we will closely follow the developments and notations given in Pavlov et al. [2004].

For a large class of n-dimensional nonlinear systems

$$\dot{x} = f(x,t), \tag{1}$$

where the state vector $x \in \mathbb{R}^n$, with $2 \leq n < \infty$, is continuous in time t, and $f(\cdot)$ is the vector field which is differentiable in x, the following notion of *convergent systems* can be given according to Demidovich [1967].

Definition 1. The system (1) is said to be convergent if for all initial conditions $t_0 \in \mathbb{R}$, $\bar{x}_0 \in \mathbb{R}^n$ there exists a solution $\bar{x}(t) = x(t, t_0, \bar{x}_0)$ which satisfies:

(i) $\bar{x}(t)$ is well-defined and bounded for all $t \in (-\infty, \infty)$;
(ii) $\bar{x}(t)$ is globally asymptotically stable.

Such solution $\bar{x}(t)$ is called a *limit solution*, to which all other solutions of the system (1) converge as $t \to \infty$. In other words, all solutions of a convergent system 'forget' their initial conditions after some transient time, which depends on exogenous values like the reference or disturbance, and thus converge asymptotically to $\bar{x}(t)$.

Remark 1. If a globally asymptotically stable limit solution exists, it may be non-unique. Yet if $\bar{x}(t)$ is the single solution defined and bounded for all $t \in (-\infty, \infty)$, then the system (1) is said to be *uniformly convergent.*

The uniform convergence requires the system (1) to have an unique limit solution $\bar{x}(t)$, like this is a case for asymptotically stable linear systems. Otherwise, a non-uniformly convergent system might have also another globally asymptotically stable solutions $\tilde{x}(t)$, bounded for all $t \in (-\infty, \infty)$. As the system is convergent, for any pair of such solutions it is valid $\|\bar{x}(t) - \tilde{x}(t)\| \to 0$ as $t \to \infty$.

Remark 2. The system (1) is, moreover, *exponentially convergent* if it is uniformly convergent and the limit solution $\bar{x}(t)$ is globally exponentially stable.

For further details on the uniform, asymptotic, and exponential stability properties of the system solutions we refer to the seminal literature, like e.g. Khalil [2002], Sastry [2013]. The existence and uniqueness of a limit solution of the system (1) has an essential application to the output regulation and tracking problems, see e.g. Isidori [1995],

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Huang [2004]. Here, for the given reference signal of a closed-loop system, one can seek for demonstrating the control system (1) is convergent, i.e. has an asymptotically stable limit solution along which the regulated output control error remains zero. The property of a system to be convergent follows from the sufficient condition, given by Demidovich [1967], which is formulated in the following theorem, cf. with Pavlov et al. [2004]:

Theorem 1. Consider the system (1). Suppose, for some positive definite matrix $P = P^T > 0$ the matrix

$$J(x,t) := \frac{1}{2}\Big(P\,\frac{\partial f}{\partial x}(x,t) + \Big[\frac{\partial f}{\partial x}(x,t)\Big]^T P\Big) \qquad (2)$$

is negative definite uniformly in $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ and $|f(0,t)| \leq \text{const} < +\infty$ for all $t \in \mathbb{R}$. Then the system (1) is convergent.

The proof of the Theorem 1 can be found in Pavlov et al. [2004]. Note that a particular case $f(0,t) \equiv 0$ of the Theorem 1 implies the well celebrated stability theorem by Krasovskii [1963]. The uniform negative definiteness of (2) implies a vanishing difference between any two solutions $x_i(t)$ and $x_{ii}(t)$ of the system (1). Note that for an exponentially convergent system (cf. Remark 2) it means

$$\big|x_i(t) - x_{ii}(t)\big| < \alpha \exp\big(-\beta(t-t_0)\big)\big|x_i(t_0) - x_{ii}(t_0)\big| \quad (3)$$

for all $t > t_0$, and the convergence constants $\alpha, \beta > 0$ have the same values for all pairs of solutions $\big[x_i(t),\, x_{ii}(t)\big]$.

2.2 Optimal nonlinear damping control

In the following, we will summarize the optimal nonlinear damping control, insofar as it is necessary for the main results provided in Section 3, while for the recently introduced control approach self and for its properties we refer to Ruderman [2021].

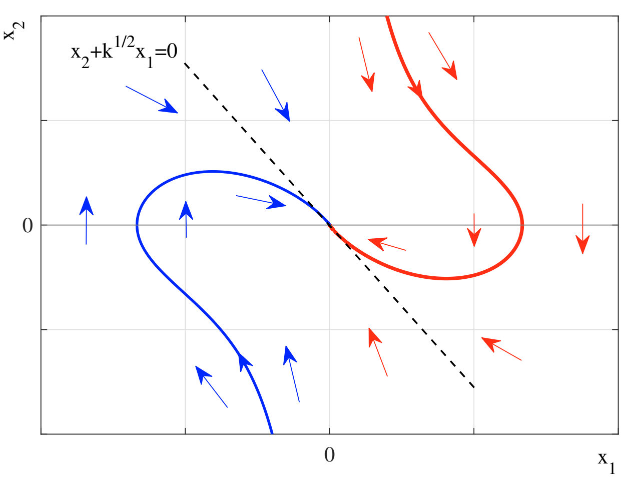

Fig. 1. Phase portrait of the control system (4), (5).

The second-order closed-loop control system with an optimal nonlinear damping is written as in Ruderman [2021]

$$\dot{x}_1 = x_2, \qquad (4)$$

$$\dot{x}_2 = -k x_1 - x_2^2 |x_1|^{-1} \mathrm{sign}(x_2), \qquad (5)$$

where $k > 0$ is the single arbitrary control design parameter. The system (4), (5), where x_1 is the output of interest, is globally asymptotically stable and converges to the unique equilibrium in the origin. This occurs (i) along an attractor of trajectories

$$x_2 + \sqrt{k}x_1 = 0 \qquad (6)$$

in vicinity to the origin, and (ii) without crossing the x_2-axis, see Figure 1. Note that the (ii)-nd property prevents singularity (otherwise owing to $x_1 = 0$) in the solutions of (4), (5), and that for all initial conditions $x_1(t_0) \neq 0 \wedge x_2(t_0) \in \mathbb{R}$, and for all trajectories outside the origin i.e. $\|x_1, x_2\|(t) \neq 0$. It is worth noting that the control system (4), (5) assumes an unperturbed system dynamics, so that the robustness against external perturbations is the subject of our future works. Other, relevant for the recent work, remarks follow below.

Remark 3. The output convergence of the control system (4), (5) is quadratic on the logarithmic scale $\log|x_1|$. Therefor, the control constitutes a significantly faster alternative to a standard proportional-derivative controller which, with for same proportional gain factor k, converges only linearly on the logarithmic scale $\log|x_1|$, cf. with [Ruderman 2021, Fig. 5].

It is also worth noting that the nonlinear damping law is entirely independent of the proportional feedback gain k. The latter solely scales the state trajectories in the (t, x_2)-coordinates, see examples depicted in Figure 2.

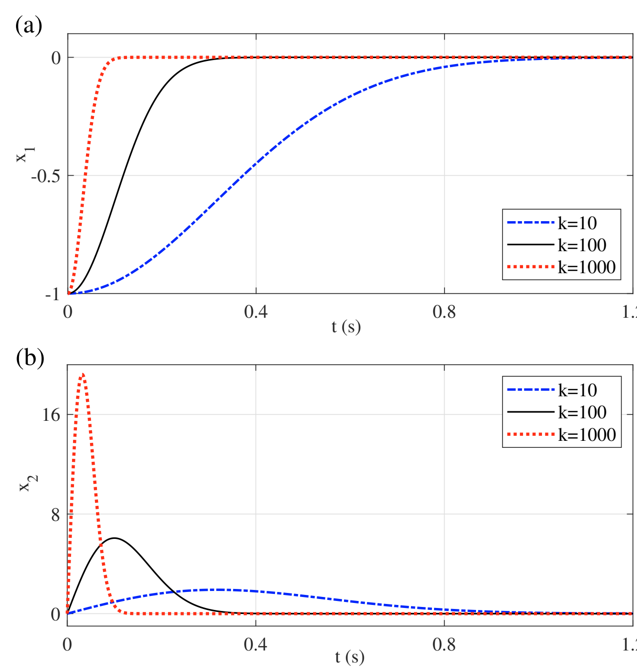

Fig. 2. State trajectories $x_1(t)$ in (a) and $x_2(t)$ in (b) for the varying values of the control gain $k = [10,\,100,\,1000]$.

Remark 4. The closed-loop control system (4), (5) allows, in addition, for a bounded control action $|\dot{x}_2| < S$, with $S = \text{const} > 0$ to be the control saturation which is yet affecting neither the stability nor convergence performance of the state trajectories, see Ruderman [2021].

Although a saturated control of (4), (5) enables bypassing singularity in the solution of a set-point problem, the output tracking problem, on the other hand, requires us to allow for $x_1 = 0 \wedge x_2 \neq 0$ at all times $t_0 \leq t \to \infty$.

For ensuring a non-singularity of the solution in the entire $(x_1, x_2) \in \mathbb{R}^2$ state-space, we will next make a necessary regularization of the nonlinear damping term in (5). With keeping this in mind, we are now in the position to formulate the main results of the recent work.

3. MAIN RESULTS

For output tracking of the reference trajectory $r(t) \in \mathcal{C}^1$, we introduce the error state $e_1 = x_1 - r$. Its time derivative is $e_2 = x_2 - \dot{r}$, respectively. Note that for an output tracking of $\mathcal{C}^1$-trajectories, one can assume $\ddot{r}(t) = 0$ for $t > \tau$, while $t \leq \tau$ characterizes certain transient phase where $\dot{r} \neq$ const. In the sense of a motion control, for instance, the time $t \leq \tau$ will correspond to the transient phases of a system acceleration or deceleration when moving. If a reference trajectory $r(t)$ contains multiple, but finite in time, transient phases with $\ddot{r}(t) \neq 0$, they will appear as temporary perturbations upon which the convergent dynamics of the control error, i.e. $\|e_1, e_2\| \to 0$, must be guaranteed for $t > \tau$.

With the above introduced states of the control error and the steady-state reference in mind, i.e. $\ddot{r} = 0$, the closed-loop control system (4), (5) can be rewritten as

$$\dot{e}_1 = e_2, \tag{7}$$

$$\dot{e}_2 = -k e_1 - \frac{|e_2|\, e_2}{|e_1| + \mu}. \tag{8}$$

Note that the introduced here regularization term $0 < \mu \ll k$ does not act as an additional design parameter, yet it prevents singularity in solutions of the system (4), (5), cf. Section 2.2. Evaluating the Jacobian of $f(x, t)$ with $x = [e_1,\, e_2]^T$, cf. (7), (8) and (1), one obtains

$$\frac{\partial f}{\partial x} = \tag{9}$$

$$= \begin{bmatrix} 0 & 1 \\ -k + |e_2|\, e_2 \operatorname{sign}(e_1)/\bigl(|e_1| + \mu\bigr)^2 & -2|e_2|/\bigl(|e_1| + \mu\bigr) \end{bmatrix}.$$

Then, suggesting the positive definite matrix

$$P = \frac{1}{2} \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}, \tag{10}$$

one can show that the matrix $J(x, t)$, which is the solution of (2), is negative definite and, correspondingly, the Theorem 1 holds. For proving it, we substitute (9) and (10) into (2) and evaluate the matrix definiteness as

$$x^T J(x, t)\, x = -\frac{3}{4} \frac{|e_2|\, e_2^2 \bigl(|e_1| + 2\mu\bigr)}{\bigl(e_1 + \mu \operatorname{sign}(e_1)\bigr)^2} \leq 0 \quad \forall\, x \neq 0. \tag{11}$$

Note that the obtained inequality (11) proves only the negative *semi-definiteness* of $J(x, t)$, since it is evident that $x^T J(x, t)\, x = 0$ for $e_2 = 0 \wedge e_1 \neq 0$. Yet it appears possible to show that $[e_1,\, e_2] = 0$ is the unique limit solution by evaluating the $\dot{e}_2$ dynamics at $e_2 = 0$. Substituting $e_2 = 0$ into (8) results in $\dot{e}_2 = -k e_1$. It implies that $[e_1 \neq 0,\, e_2 = 0](t)$ cannot be a limit (correspondingly steady-state) solution, since any trajectory will be repulsed away from $e_2 = 0$ as long as $e_1 \neq 0$. Hence, the closed-loop control system (7), (8) reveals to be the uniformly convergent one. Respectively, the origin of the control error and its time derivative, i.e. $[e_1, e_2](t) = 0 \equiv \bar{x}$, is the unique limit solution for all times $t_0 < \tau < t$ and that independently of the initial conditions t_0 and $[e_1, e_2](t_0)$.

Remark 5. When assuming a quadratic Lyapunov function candidate

$$V(x) = x^T P\, x = \frac{1}{2} k e_1^2 + \frac{1}{2} e_2^2, \tag{12}$$

which represents the total energy level (i.e. potential energy plus kinetic energy) of the system (7), (8), its time derivative results in

$$\frac{d}{dt} V(x) = -\frac{|e_2|\, e_2^2}{|e_1| + \mu}. \tag{13}$$

Thus, the rate at which the control system (7), (8) reduces its energy is cubic in the error rate, i.e. $\sim |e_2|^3$, and hyperbolic in the error size, i.e. $\sim |e_1|^{-1}$, cf. Figure 3.

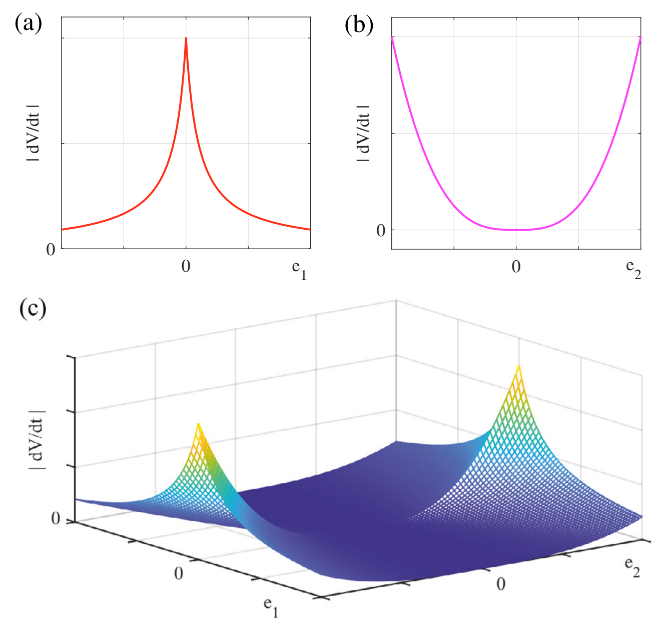

Fig. 3. Energy reduction rate $|\dot{V}|$ of the system (7), (8): depending on e_1 in (a), depending on e_2 in (b), and as overall error-states function according to (13) in (c).

It can be recognized, cf. Figure 3 (a), that the regularization factor μ prevents an infinite energy rate and, thus, ensures a finite control action as $|e_1| \to 0$. At the same time, a hyperbolic energy rate allows to accelerate the convergence as $|e_1| \to 0$. The cubic dependency of the energy rate from the error rate, on the contrary, enables the control to react faster to the error dynamics, cf. Figure 3 (b). It reveals as relevant in case of, for example, non-steady trajectory phases (i.e. $\ddot{r}(t) \neq 0$) or sudden external perturbations which can provoke the higher $|e_2|$-values.

Numerical examples

Following numerical examples are provided for the implemented system (7), (8), while assuming $k = 100$ and $\mu = 0.0001$ parameters and $r(t) \in \mathcal{C}^1$ reference trajectories. Most simple forward-Euler solver is used.

First, the output regulated trajectories are shown for the different initial values $x_0 \equiv [x_1, x_2](t_0)$:

$$x_0 = \bigl\{(0.5, 50),\, (0.1, 20),\, (1, 0),\, (1.5, -30),\, (0.3, -20)\bigr\}.$$

The assigned reference trajectory is a linear slope $r(t) = t$. The output response $x_1(t)$ under control is depicted in Figure 4 (a). The corresponding phase portrait of the error states, i.e. (e_1, e_2), is depicted Figure 4 (b).

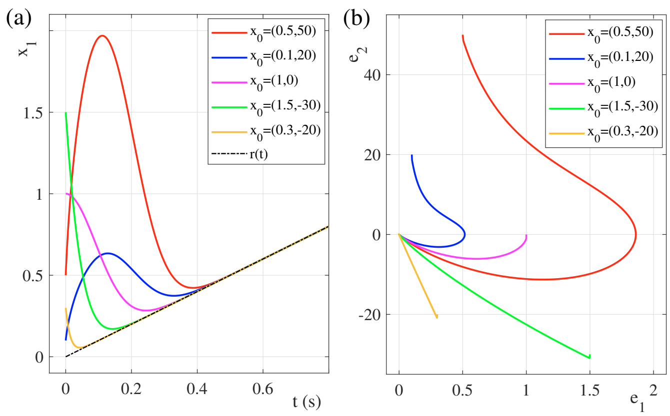

Fig. 4. Trajectories of the system (7), (8), with $k = 100$, $\mu = 0.0001$, for different initial values $x_0 \equiv [x_1, x_2](t_0)$: the output $x_1(t)$ versus reference $r(t)$ in (a), phase portrait of the error states in (b).

Next, we demonstrate the control performance of the output tracking, when $r(t)$ is only piecewise $\mathcal{C}^1$ and contains the finite phases where $\dot{r}(t) \neq$ const. Furthermore, in order to emphasize a practical control applicability, both $x_1(t)$ and $x_2(t)$ signals, used for the feedback control in (8), are subject to a non-correlated bandlimited white-noise. The output response $x_1(t)$ under control is depicted in Figure 5 (a) over the reference trajectory. The $x_2(t)$ state,

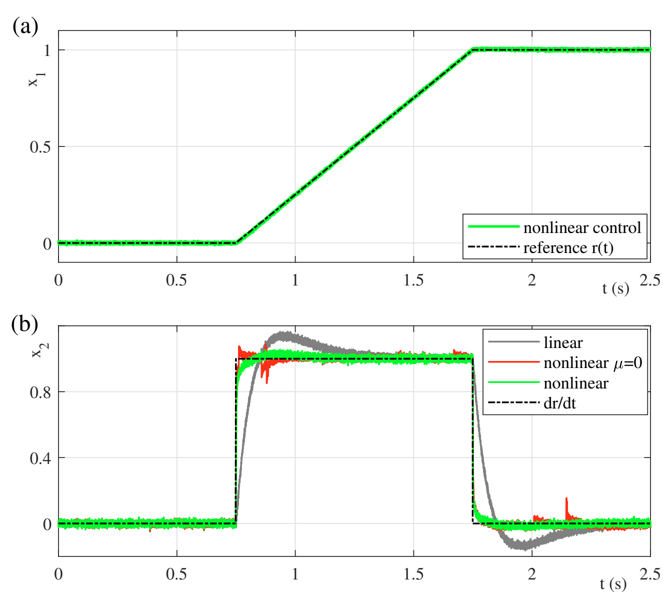

Fig. 5. Trajectories of the system (7), (8), with $k = 100$, $\mu = 0.0001$: the output $x_1(t)$ versus reference $r(t)$ in (a), the $x_2(t)$ state in (b) – compared with a case without regularization (i.e. $\mu = 0$) and with a critically damped proportional-derivative linear controller.

which corresponds to the relative velocity if, for example, a motion control is intended, is depicted in Figure 5 (b). For the sake of comparison, the case without regularization, i.e. $\mu = 0$, is also shown. Moreover, the $x_2(t)$-response of a critically damped proportional-derivative linear control is also demonstrated, for the sake of a fair comparison. Here, the assigned proportional-derivative linear controller with the same $k = 100$, features the $\dot{e}_2 = -100e_1 - 20e_2$ error dynamics instead of (8), respectively. Recall that the -20 gain (i.e. D-gain of a PD controller) corresponds to the critically damped second-order linear system.

4. CONCLUSIONS

The Demidovich's concept of the convergent system dynamics was used for analyzing convergence properties of the optimal nonlinear damping control, which was recently introduced in Ruderman [2021]. The existence of an unique limit solution in the coordinates origin of the output tracking error and its time derivative was demonstrated. A regularization term was introduced, in augmentation to the original control law of Ruderman [2021], which prevents singularity in the state solutions when an output zero crossing occurs outside of the origin. The obtained optimal nonlinear damping control is globally uniformly asymptotically stable and suitable for the output tracking of $\mathcal{C}^1$ reference trajectories. The provided analysis and results can be relevant for using the optimal nonlinear damping control as an alternative to the standard proportional-derivative (PD) controller. The optimal nonlinear damping control performs as significantly faster converging and is, moreover, also robust against the measurement noise in the both feedback signals.

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