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Model approximation for two-dimensional Markovian jump systems with state-delays and imperfect mode information

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Abstract This paper is concerned with the problem of $\mathscr{H}_\infty$ model approximation for a class of two-dimensional (2-D) discrete-time Markovian jump linear systems with state-delays and imperfect mode information. The 2-D system is described by the well-known Fornasini-Marchesini local state-space model, and the imperfect mode information in the Markov chain simultaneously involves the exactly known, partially unknown and uncertain transition probabilities. By using the characteristics of the transition probability matrices, together with the convexification of uncertain domains, a new $\mathscr{H}_\infty$ performance analysis criterion for the underlying system is firstly derived, and then two approaches, namely, the convex linearisation approach and iterative approach, to the $\mathscr{H}_\infty$ model approximation synthesis are developed. The solutions to the problem are formulated in terms of strict linear matrix inequalities (LMIs) or a sequential minimization problem subject to LMI constraints. Finally, simulation studies are provided to illustrate the effectiveness of the proposed design methods.

Keywords Two-dimensional systems · Markovian jump systems · Model approximation · Imperfect mode information · State-delays

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1 Introduction

In recent years, there have been rapidly growing interests in the analysis and synthesis of two-dimensional (2-D) systems Roesser (1975). The studies of 2-D systems are motivated by their extensive applications in many modern engineering fields, such as process control (thermal processes, gas absorption, water stream heating), multi-dimensional digital filtering, and image processing (enhancement, deblurring, seismographic data processing) Kaczorek (1985); Lu and Antoniou (1992); Lin (1999); Lin et al. (2001) and so on. Correspondingly, various issues on 2-D systems, including stability analysis, stabilization, and filtering design, have been investigated, and a large volume of open literature has been available Xu et al. (2002, 2003, 2005); Xie et al. (2002); Lam et al. (2004); Paszke et al. (2004); Hoang et al. (2005); Peng and Guan (2009).

On the other hand, Markovian jump linear systems (MJLSs) have great practical applications due to the forceful modeling ability of the Markov process in diverse communities, e.g. manufacturing systems, power systems and networked control systems, where random failure, repairs and sudden environment changes may occur in Markov chains Boukas (2005); Wang et al. (2009, 2010). It is known that some MJLSs have inherent 2-D structure, e.g. information propagation occurs from pass to pass and along a given pass in a gas absorption Wu et al. (2008). Therefore, 2-D MJLSs emerge as a more reasonable description to account for the parameter jumping phenomenon in the two independent directions of information propagation Roesser (1975). As a key factor, transition probabilities (TPs) in the Markovian jumping process determine the system behavior to a large extent, and many studies on the analysis and synthesis of MJLSs have been carried out in the context of completely known information on TPs Wu et al. (2008); Xu et al. (2007); Liu et al. (2008). However, in practice, imperfect mode information are often encountered especially when efforts to measure or estimate the mode information are costly or time-consuming. In such case, it is of both theoretical merit and practical interest, from control perspectives, to further study more general jump systems with imperfect mode information Wang et al. (2010); Karan et al. (2006); Souza (2006); Zhang and Boukas (2009); Zhao et al. (2011). To mention a few, the authors in Karan et al. (2006) addressed the robust stability analysis problem for a class of MJLSs with norm-bounded uncertain TPs; The author in Souza (2006) investigated the robust stability analysis and stabilization problems for a class of MJLSs with polytopic uncertain TPs; The authors in Zhang and Boukas (2009) considered the $\mathscr{H}_\infty$ filter synthesis problem for a class of MJLSs with partially unknown TPs. Nevertheless, the aforementioned results are only referred to one-dimensional (1-D) systems.

In addition, many complex physical systems can be usually modeled as high or even infinite order mathematical models, which poses serious difficulties to the analysis and design of the concerning systems. A logical step to take is to reduce the high-order models to some simple lower-order models according to some given criteria Ghafoor and Sreeram (2008). Model approximation/reduction thus plays an important role in the process of control system design. During the past few decades, various efficient model approximation schemes have been proposed, involving the Hankel norm approximation method Zhou (1995); Gao et al. (2004), the $\mathscr{H}_2$ approach Yan and Lam (1999), the $\mathscr{H}_\infty$ approach Zhang et al. (2003); Xu et al. (2005); Wu et al. (2006); Gao et al. (2006); Wu and Zheng (2009) and the $\mathscr{L}_2$–$\mathscr{L}_\infty$ approach Lam et al. (2005). More recently, the linear matrix inequality (LMI) technique has been applied to deal with the model approximation problem for different classes of systems Zhang et al. (2003); Wu and Zheng (2009); Lam et al. (2005). However, from the authors' best knowledge, few results have been reported on the model approximation for 2-D MJLSs with state-delays and imperfect mode information, which simultaneously comprises the exactly known, partially unknown and uncertain TPs. This motivates us for the present study.

According to the issues mentioned above, in this paper, we will tackle the problem of $\mathscr{H}_\infty$ model approximation for a class of 2-D discrete-time MJLSs with state-delays and imperfect mode information. The mathematical model of the 2-D system is described by the Fornasini-Marchesini local state-space (FMLSS) model, and the imperfect mode information simultaneously considers the exactly known, partially unknown and polytopic-type uncertain TPs, which is a more practical scenario. By fully exploiting the properties of the transition probability matrices (TPMs), together with the convexification of uncertain domains, a new $\mathscr{H}_\infty$ performance analysis criterion for the model error system will be firstly derived. To solve the model approximation problem, two distinctly different approaches will be then developed. The first approach converts the model approximation into a convex optimization problem via a linearisation procedure, and the second one is based on the cone complementarity linearisation (CCL) method El Ghaoui et al. (1997); Qiu et al. (2012), which casts the model approximation into a sequential minimization problem subject to LMI constraints. Finally, simulation studies will be performed to show the effectiveness of the proposed synthesis methods.

Notations The notations used throughout the paper are standard. $\mathbf{R}^n$ and $\mathbf{R}^{m\times n}$ denote, respectively, the n-dimensional Euclidean space, and the set of all $m \times n$ real matrices; $\mathbb{N}^+$ represents the sets of positive integers; the notation $P > 0$ means that P is real symmetric and positive definite; $\mathbf{I}$ and $\mathbf{0}$ represent the identity matrix and a zero matrix, respectively; $(\mathcal{S}, \mathcal{F}, \mathcal{P})$ denotes a complete probability space, in which $\mathcal{S}$ is the sample space, $\mathcal{F}$ is the σ algebra of subsets of the sample space, and $\mathcal{P}$ is the probability measure on $\mathcal{F}$; $\mathbb{E}[\cdot]$ stands for the mathematical expectation; $\|\cdot\|$ refers to the Euclidean norm of a vector or its induced norm of a matrix; $l_2\{[0,\infty),[0,\infty)\}$ denotes the space of square summable sequences on $\{[0,\infty),[0,\infty)\}$. Matrices, if not explicitly stated, are assumed to have appropriate dimensions for algebra operations.

2 Problem formulation and preliminaries

Consider the following 2-D discrete-time MJLSs, defined on a complete probability space $(\mathcal{S}, \mathcal{F}, \mathcal{P})$ and described by the FMLSS model with state-delays,

$$
\begin{aligned}
(\Sigma):\ x(i+1,j+1) &= A_1(r(i,j+1))x(i,j+1) + A_2(r(i+1,j))x(i+1,j)\\
&\quad + A_{d1}(r(i,j+1))x(i-d_1,j+1) + A_{d2}(r(i+1,j))x(i+1,j-d_2)\\
&\quad + B_1(r(i,j+1))u(i,j+1) + B_2(r(i+1,j))u(i+1,j)\\
z(i,j) &= C(r(i,j))x(i,j) + B_3(r(i,j))u(i,j), \qquad (1)
\end{aligned}
$$

where $x(i,j) \in \mathbf{R}^{n_x}$ is the state vector; $z(i,j) \in \mathbf{R}^{n_z}$ is the output vector; $u(i,j) \in \mathbf{R}^{n_u}$ is the input vector which belongs to $l_2\{[0,\infty),[0,\infty)\}$; and d_1 and d_2 are two constant positive integers representing delays along the vertical and horizontal directions, respectively. $A_1(r(i,j+1))$, $A_2(r(i+1,j))$, $A_{d1}(r(i,j+1))$, $A_{d2}(r(i+1,j))$, $B_1(r(i,j+1))$, $B_2(r(i+1,j))$, $C(r(i,j))$, and $B_3(r(i,j))$ are real-valued system matrices. These matrices are functions of $r(i,j)$, which is described by a discrete-time, discrete-state homogeneous Markov chain with a finite-state space $\mathcal{I} := \{1,\ldots,N\}$ and a stationary transition probability matrix (TPM) $\Pi = [\pi_{mn}]_{N\times N}$, where

$$\pi_{mn} = \Pr(r(i+1, j+1) = n \,|r(i, j+1) = m) = \Pr(r(i+1, j+1) \\ = n \,|r(i+1, j) = m), \ \forall m, n \in \mathcal{I},$$

with $\pi_{mn} \geq 0$ and $\sum_{n=1}^{N} \pi_{mn} = 1$. For $r(i+1, j) = m \in \mathcal{I}$ or $r(i, j+1) = m \in \mathcal{I}$, the system matrices of the m-th mode are denoted by $(A_{1m}, A_{2m}, A_{d1m}, A_{d2m}, B_{1m}, B_{2m}, C_m, B_{3m})$, which are real known and with appropriate dimensions. Unless otherwise stated, similar simplification is also applied to other matrices in the following.

In this paper, the TPs of the jumping process are assumed to be *uncertain and partially accessed*, i.e., the TPM $\Pi = [\pi_{mn}]_{N\times N}$ is assumed to belong to a given polytope P_Π with vertices $\Pi_s, \ s = 1, 2, \ldots, M$, $\mathrm{P}_\Pi := \left\{\Pi \,\middle|\, \Pi = \sum_{s=1}^{M} \alpha_s \Pi_s; \alpha_s \geq 0, \sum_{s=1}^{M} \alpha_s = 1\right\}$, where $\Pi_s = [\pi_{mn}]_{N\times N}$, $m, n \in \mathcal{I}$, are given TPMs containing unknown elements still. For instance, for system (Σ) with four operation modes, the TPM may be as,

$$\begin{bmatrix} \pi_{11} & \tilde{\pi}_{12} & \hat{\pi}_{13} & \pi_{14} \\ \hat{\pi}_{21} & \pi_{22} & \tilde{\pi}_{23} & \pi_{24} \\ \pi_{31} & \hat{\pi}_{32} & \pi_{33} & \hat{\pi}_{34} \\ \pi_{41} & \tilde{\pi}_{42} & \hat{\pi}_{43} & \hat{\pi}_{44} \end{bmatrix}$$

where the elements labeled with "$\hat{}$" and "$\tilde{}$" represent the unknown information and polytopic uncertainties on TPs, respectively, and the others are known TPs. For notational clarity, $\forall m \in \mathcal{I}$, we denote $\mathcal{I} = \mathcal{I}_{\mathcal{K}}^{(m)} \cup \mathcal{I}_{\mathcal{UC}}^{(m)} \cup \mathcal{I}_{\mathcal{UK}}^{(m)}$ as follows,

$$\begin{aligned} \mathcal{I}_{\mathcal{K}}^{(m)} &:= \{n : \pi_{mn} \text{ is known}\}, \\ \mathcal{I}_{\mathcal{UC}}^{(m)} &:= \{n : \tilde{\pi}_{mn} \text{ is uncertain}\}, \\ \mathcal{I}_{\mathcal{UK}}^{(m)} &:= \{n : \hat{\pi}_{mn} \text{ is unknown}\}. \end{aligned}$$

Moreover, if $\mathcal{I}_{\mathcal{K}}^{(m)} \neq \emptyset$ and $\mathcal{I}_{\mathcal{UC}}^{(m)} \neq \emptyset$, it is further described as

$$\begin{aligned} \mathcal{I}_{\mathcal{K}}^{(m)} &:= \{\mathcal{K}_{1_{(m)}}, \ldots, \mathcal{K}_{t_{(m)}}\}, \ \forall 1 \leq t_{(m)} \leq N-2 \\ \mathcal{I}_{\mathcal{UC}}^{(m)} &:= \{\mathcal{U}_{1_{(m)}}, \ldots, \mathcal{U}_{v_{(m)}}\}, \ \forall 1 \leq v_{(m)} \leq N \end{aligned}$$

where $\mathcal{K}_{t_{(m)}} \in \mathbb{N}^+$ represents the $t_{(m)}$-th known element with the index $\mathcal{K}_{t_{(m)}}$ in the m-th row of the TPM, and $\mathcal{U}_{v_{(m)}} \in \mathbb{N}^+$ represents the $v_{(m)}$-th uncertain element with the index $\mathcal{U}_{v_{(m)}}$ in the m-th row of the TPM. Obviously, $1 \leq t_{(m)} + v_{(m)} \leq N$. Also, we denote

$$\pi_{\mathcal{UK}}^{(ms)} := \sum\nolimits_{n\in\mathcal{I}_{\mathcal{UK}}^{(m)}} \hat{\pi}_{mn} = 1 - \sum\nolimits_{n\in\mathcal{I}_{\mathcal{K}}^{(m)}} \pi_{mn} - \sum\nolimits_{n\in\mathcal{I}_{\mathcal{UC}}^{(m)}} \tilde{\pi}_{mn}^{(s)}, \tag{2}$$

where $\tilde{\pi}_{mn}^{(s)}$ represents an uncertain TP in the s-th polytope, $\forall s = 1, \ldots, M$.

The boundary conditions of system (Σ) in (1) are defined by,

$$\begin{aligned} &\{x(i, j) = \phi(i, j), \quad \forall j \geq 0, \ -d_1 \leq i \leq 0\}; \\ &\{x(i, j) = \varphi(i, j), \quad \forall i \geq 0, \ -d_2 \leq j \leq 0\}; \\ &\phi(0, 0) = \varphi(0, 0). \end{aligned} \tag{3}$$

Throughout this paper, the following assumptions are made.

Assumption 1 System (Σ) in (1) is stochastically stable.

Assumption 2 The boundary conditions satisfy

$$\lim_{T_1\to\infty}\mathbb{E}\left\{\sum_{j=0}^{T_1}\sum_{i=-d_1}^{0}(\phi^{\mathrm{T}}(i,j)\phi(i,j))\right\}+\lim_{T_2\to\infty}\mathbb{E}\left\{\sum_{i=0}^{T_2}\sum_{j=-d_2}^{0}(\varphi^{\mathrm{T}}(i,j)\varphi(i,j))\right\}<\infty. \tag{4}$$

To approximate the original 2-D MJLS (1), in this paper, we are interested in designing the following mode-dependent reduced-order model,

$$\begin{aligned}\hat{x}(i+1,j+1) &= A_{r1}(r(i,j+1))\hat{x}(i,j+1)+A_{r2}(r(i+1,j))\hat{x}(i+1,j)\\ &\quad+A_{rd1}(r(i,j+1))\hat{x}(i-d_1,j+1)+A_{rd2}(r(i+1,j))\hat{x}(i+1,j-d_2)\\ &\quad+B_{r1}(r(i,j+1))u(i,j+1)+B_{r2}(r(i+1,j))u(i+1,j)\\ \hat{z}(i,j) &= C_r(r(i,j))\hat{x}(i,j)+B_{r3}(r(i,j))u(i,j)\\ \hat{x}(i,j) &= 0,\ \text{for } i\le 0 \text{ or } j\le 0,\end{aligned} \tag{5}$$

where $\hat{x}(i,j)\in\mathbf{R}^{n_r}$ $(n_r<n_x)$, $\hat{z}(i,j)\in\mathbf{R}^{n_z}$, and $\{A_{r1}(r(i,j+1)), A_{r2}(r(i+1,j)), A_{rd1}(r(i,j+1)), A_{rd2}(r(i+1,j))\}\in\mathbf{R}^{n_r\times n_r}$, $\{B_{r1}(r(i,j+1)), B_{r2}(r(i+1,j))\}\in\mathbf{R}^{n_r\times n_u}$, $C_r(r(i,j))\in\mathbf{R}^{n_z\times n_r}$ and $B_{r3}(r(i,j))\in\mathbf{R}^{n_z\times n_u}$ are the gains of the reduced-order models to be determined.

Define $\bar{x}(i,j) := \left[x^{\mathrm{T}}(i,j)\ \hat{x}^{\mathrm{T}}(i,j)\right]^{\mathrm{T}}$, and $\bar{z}(i,j) := z(i,j)-\hat{z}(i,j)$. Then, by augmenting (1) and (5) the model error dynamics can be represented as follows,

$$\begin{aligned}(\bar{\Sigma}):\ \bar{x}(i+1,j+1) &= \bar{A}_1(r(i,j+1))\bar{x}(i,j+1)+\bar{A}_2(r(i+1,j))\bar{x}(i+1,j)\\ &\quad+\bar{A}_{d1}(r(i,j+1))\bar{x}(i-d_1,j+1)+\bar{A}_{d2}(r(i+1,j))\bar{x}(i+1,j-d_2)\\ &\quad+\bar{B}_1(r(i,j+1))u(i,j+1)+\bar{B}_2(r(i+1,j))u(i+1,j)\\ \bar{z}(i,j) &= \bar{C}(r(i,j))\bar{x}(i,j)+\bar{B}_3(r(i,j))u(i,j),\end{aligned} \tag{6}$$

where

$$\begin{aligned}\bar{A}_1(r(i,j+1)) &:= \begin{bmatrix}A_1(r(i,j+1)) & \mathbf{0}\\ \mathbf{0} & A_{r1}(r(i,j+1))\end{bmatrix},\\ \bar{A}_2(r(i+1,j)) &:= \begin{bmatrix}A_2(r(i+1,j)) & \mathbf{0}\\ \mathbf{0} & A_{r2}(r(i+1,j))\end{bmatrix},\\ \bar{A}_{d1}(r(i,j+1)) &:= \begin{bmatrix}A_{d1}(r(i,j+1)) & \mathbf{0}\\ \mathbf{0} & A_{rd1}(r(i,j+1))\end{bmatrix},\\ \bar{A}_{d2}(r(i+1,j)) &:= \begin{bmatrix}A_{d2}(r(i+1,j)) & \mathbf{0}\\ \mathbf{0} & A_{rd2}(r(i+1,j))\end{bmatrix},\\ \bar{B}_1(r(i,j+1)) &:= \begin{bmatrix}B_1(r(i,j+1))\\ B_{r1}(r(i,j+1))\end{bmatrix},\ \bar{B}_2(r(i+1,j)) := \begin{bmatrix}B_2(r(i+1,j))\\ B_{r2}(r(i+1,j))\end{bmatrix},\\ \bar{C}(r(i,j)) &:= \begin{bmatrix}C(r(i,j)) & -C_r(r(i,j))\end{bmatrix},\ \bar{B}_3(r(i,j)) := B_3(r(i,j))-B_{r3}(r(i,j)).\end{aligned} \tag{7}$$

Before proceeding further, we introduce the following definitions.

Definition 1 System (6) is said to be stochastically stable if for $u(i,j)=0$ and the boundary conditions satisfying (4), the following condition holds

$$\mathbb{E}\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(\|\bar{x}(i,j+1)\|^2+\|\bar{x}(i+1,j)\|^2)\right\}<\infty.$$

Definition 2 Given a scalar $\gamma > 0$, system (6) is said to be stochastically stable with an $\mathscr{H}_\infty$ disturbance attenuation performance index γ if it is stochastically stable with $u(i, j) = 0$, and under zero boundary conditions $\phi(i, j) = \varphi(i, j) = 0$ in (3), satisfies

$$\|\tilde{z}\|_{\mathbb{E}_2} < \gamma \, \|\tilde{u}\|_2 \,,$$

for all non-zero $u \in l_2\{[0, \infty), [0, \infty)\}$, where

$$\|\tilde{z}\|_{\mathbb{E}_2} := \sqrt{\mathbb{E}\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(\|\bar{z}(i, j+1)\|^2 + \|\bar{z}(i+1, j)\|^2\right)\right\}},$$

$$\|\tilde{u}\|_2 := \sqrt{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(\|u(i, j+1)\|^2 + \|u(i+1, j)\|^2\right)}. \tag{8}$$

Thus, the purpose of this paper is to design a mode-dependent reduced-order model in the form of (5) to approximate the system (1), such that the model error system $(\bar{\Sigma})$ in (6) with imperfect mode information is stochastically stable with a prescribed $\mathscr{H}_\infty$ performance index γ.

3 Main results

In this section, based on a Markovian Lyapunov-Krasovskii functional (MLKF), a new bounded real lemma (BRL) for the 2-D MJLS (6) with imperfect mode information will be firstly proposed. Then, two distinctly different approaches will be developed to solve the $\mathscr{H}_\infty$ model approximation problem formulated in the above section.

3.1 $\mathscr{H}_\infty$ performance analysis

In this subsection, by fully exploring the properties of the TPMs, together with the convexification of uncertain domains, an $\mathscr{H}_\infty$ performance analysis criterion for the model error system (6) with imperfect mode information is derived, which will play a key role in solving the $\mathscr{H}_\infty$ model approximation synthesis problem.

Proposition 1 *The 2-D MJLS* (6) *with state-delays and imperfect mode information is stochastically stable with a guaranteed $\mathscr{H}_\infty$ performance γ, if there exist positive-definite symmetric matrices $\{P_{1m}, P_{2m}, Q_1, Q_2\} \in \mathbf{R}^{(n_x+n_r)\times(n_x+n_r)}$, such that the following matrix inequalities hold,*

$$\mathscr{A}_m^{\mathrm{T}} \mathscr{P}_n^{(s)} \mathscr{A}_m + \mathscr{C}_m^{\mathrm{T}} \mathscr{C}_m + \Theta_m < 0, \; m \in \mathcal{I}, \; n \in \mathcal{I}_{\mathcal{UK}}^{(m)}, \; s = 1, \ldots, M, \tag{9}$$

where

$$\Theta_m := \mathrm{diag}\{-P_{1m} + Q_1, -P_{2m} + Q_2, -Q_1, -Q_2, -\gamma^2\mathbf{I}, -\gamma^2\mathbf{I}\},$$

$$\mathscr{A}_m := \begin{bmatrix} \bar{A}_{1m} & \bar{A}_{2m} & \bar{A}_{d1m} & \bar{A}_{d2m} & \bar{B}_{1m} & \bar{B}_{2m} \end{bmatrix},$$

$$\mathscr{C}_m := \begin{bmatrix} \bar{C}_m & \mathbf{0} & \mathbf{0}_{n_z\times 2(n_x+n_r)} & \bar{B}_{3m} & \mathbf{0} \\ \mathbf{0} & \bar{C}_m & \mathbf{0}_{n_z\times 2(n_x+n_r)} & \mathbf{0} & \bar{B}_{3m} \end{bmatrix},$$

$$\mathscr{P}_n^{(s)} := \sum_{n\in\mathcal{I}_{\mathcal{K}}^{(m)}} \pi_{mn}(P_{1n}+P_{2n}) + \sum_{n\in\mathcal{I}_{\mathcal{UC}}^{(m)}} \tilde{\pi}_{mn}^{(s)}(P_{1n}+P_{2n}) + \pi_{\mathcal{UK}}^{(ms)} \underbrace{(P_{1n}+P_{2n})}_{n\in\mathcal{I}_{\mathcal{UK}}^{(m)}},$$

$$\pi_{\mathcal{UK}}^{(ms)} := 1 - \sum_{n\in\mathcal{I}_{\mathcal{K}}^{(m)}} \pi_{mn} - \sum_{n\in\mathcal{I}_{\mathcal{UC}}^{(m)}} \tilde{\pi}_{mn}^{(s)}. \tag{10}$$

Proof Consider the following MLKF for the 2-D MJLS (6),

$$V(i,j) := \sum_{k=1}^{2} V_k(\bar{x}(i,j+1), r(i,j+1)) + \sum_{k=3}^{4} V_k(\bar{x}(i+1,j), r(i+1,j)), \tag{11}$$

where

$$\begin{aligned}
V_1(\bar{x}(i,j+1), r(i,j+1)) &:= \bar{x}^{\mathrm{T}}(i,j+1)P_1(r(i,j+1))\bar{x}(i,j+1),\\
V_2(\bar{x}(i,j+1), r(i,j+1)) &:= \sum_{k=i-d_1}^{i-1} \bar{x}^{\mathrm{T}}(k,j+1)Q_1\bar{x}(k,j+1),\\
V_3(\bar{x}(i+1,j), r(i+1,j)) &:= \bar{x}^{\mathrm{T}}(i+1,j)P_2(r(i+1,j))\bar{x}(i+1,j),\\
V_4(\bar{x}(i+1,j), r(i+1,j)) &:= \sum_{k=j-d_2}^{j-1} \bar{x}^{\mathrm{T}}(i+1,k)Q_2\bar{x}(i+1,k).
\end{aligned} \tag{12}$$

Then, based on the MLKF defined in (11), it is known that the following condition

$$\Omega := \Delta V(i,j) + \|\tilde{z}\|_{\mathbb{E}_2}^2 - \gamma^2 \|\tilde{u}\|_2^2 < 0, \tag{13}$$

where

$$\begin{aligned}
\Delta V(i,j) := \mathbb{E}&\left\{\sum_{k=1}^{2} V_k(\bar{x}(i+1,j+1), r(i+1,j+1)) | \bar{x}(i,j+1), r(i,j+1)=m\right\}\\
&+ \mathbb{E}\left\{\sum_{k=3}^{4} V_k(\bar{x}(i+1,j+1), r(i+1,j+1)) | \bar{x}(i+1,j), r(i+1,j)=m\right\}\\
&- \sum_{k=1}^{2} V_k(\bar{x}(i,j+1), r(i,j+1)) - \sum_{k=3}^{4} V_k(\bar{x}(i+1,j), r(i+1,j)),
\end{aligned}$$

and $\|\tilde{z}\|_{\mathbb{E}_2}$ and $\|\tilde{u}\|_2$ are defined in (8), guarantees that the 2-D model error system (6) is stochastically stable with an $\mathscr{H}_\infty$ performance γ under zero boundary conditions for any nonzero $u(i,j) \in l_2\{[0,\infty),[0,\infty)\}$.

Taking the time difference of $V(i,j)$ along the trajectories of 2-D system in (6), yields

$$\begin{aligned}
\Delta V_1 &:= \mathbb{E}[V_1(\bar{x}(i+1,j+1), r(i+1,j+1)) | \bar{x}(i,j+1), r(i,j+1)=m]\\
&\quad - V_1(\bar{x}(i,j+1), r(i,j+1))\\
&= \bar{x}^{\mathrm{T}}(i+1,j+1)\left(\sum_{n\in\mathcal{I}} \pi_{mn}P_{1n}\right)\bar{x}(i+1,j+1) - \bar{x}^{\mathrm{T}}(i,j+1)P_{1m}\bar{x}(i,j+1)
\end{aligned}$$

$$
\begin{aligned}
&= \bar{x}^{\mathrm{T}}(i+1,j+1)\left(\sum_{n\in\mathcal{I}_{\mathcal{K}}^{(m)}}\pi_{mn}P_{1n}+\sum_{n\in\mathcal{I}_{\mathcal{UC}}^{(m)}}\left(\sum_{s=1}^{M}\alpha_s\tilde{\pi}_{mn}^{(s)}\right)P_{1n}+\sum_{n\in\mathcal{I}_{\mathcal{UK}}^{(m)}}\hat{\pi}_{mn}P_{1n}\right)\\
&\quad\times\bar{x}(i+1,j+1)-\bar{x}^{\mathrm{T}}(i,j+1)P_{1m}\bar{x}(i,j+1), \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
\Delta V_2 &:= \mathbb{E}[\,V_2(\bar{x}(i+1,j+1),r(i+1,j+1))|\,\bar{x}(i,j+1),r(i,j+1)=m]\\
&\quad - V_2(\bar{x}(i,j+1),r(i,j+1))\\
&= \bar{x}^{\mathrm{T}}(i,j+1)Q_1\bar{x}(i,j+1)-\bar{x}^{\mathrm{T}}(i-d_1,j+1)Q_1\bar{x}(i-d_1,j+1), \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
\Delta V_3 &:= \mathbb{E}[\,V_3(\bar{x}(i+1,j+1),r(i+1,j+1))|\,\bar{x}(i+1,j),r(i+1,j)=m]\\
&\quad -V_3(\bar{x}(i+1,j),r(i+1,j))\\
&= \bar{x}^{\mathrm{T}}(i+1,j+1)\left(\sum_{n\in\mathcal{I}}\pi_{mn}P_{2n}\right)\bar{x}(i+1,j+1)-\bar{x}^{\mathrm{T}}(i+1,j)P_{2m}\bar{x}(i+1,j)\\
&= \bar{x}^{\mathrm{T}}(i+1,j+1)\left(\sum_{n\in\mathcal{I}_{\mathcal{K}}^{(m)}}\pi_{mn}P_{2n}+\sum_{n\in\mathcal{I}_{\mathcal{UC}}^{(m)}}\left(\sum_{s=1}^{M}\alpha_s\tilde{\pi}_{mn}^{(s)}\right)P_{2n}+\sum_{n\in\mathcal{I}_{\mathcal{UK}}^{(m)}}\hat{\pi}_{mn}P_{2n}\right)\\
&\quad\times\bar{x}(i+1,j+1)-\bar{x}^{\mathrm{T}}(i+1,j)P_{2m}\bar{x}(i+1,j), \qquad (16)
\end{aligned}
$$

$$
\begin{aligned}
\Delta V_4 &:= \mathbb{E}[\,V_4(\bar{x}(i+1,j+1),r(i+1,j+1))|\,\bar{x}(i+1,j),r(i+1,j)=m]\\
&\quad -V_4(\bar{x}(i+1,j),r(i+1,j))\\
&= \bar{x}^{\mathrm{T}}(i+1,j)Q_2\bar{x}(i+1,j)-\bar{x}^{\mathrm{T}}(i+1,j-d_2)Q_2\bar{x}(i+1,j-d_2). \qquad (17)
\end{aligned}
$$

Therefore, based on the MLKF defined in (11), together with consideration of (6) and (14)–(17), we have,

$$
\Omega = \varsigma^{\mathrm{T}}(i,j)\left[\mathscr{A}_m^{\mathrm{T}}\left(\bar{\mathscr{P}}_{1n}+\bar{\mathscr{P}}_{2n}\right)\mathscr{A}_m+\mathscr{C}_m^{\mathrm{T}}\mathscr{C}_m+\Theta_m\right]\varsigma(i,j),\ m,n\in\mathcal{I}, \qquad (18)
$$

where

$$
\begin{aligned}
\varsigma(i,j) &:= [\,\bar{x}^{\mathrm{T}}(i,j+1)\ \bar{x}^{\mathrm{T}}(i+1,j)\ \bar{x}^{\mathrm{T}}(i-d_1,j+1)\\
&\qquad\quad \bar{x}^{\mathrm{T}}(i+1,j-d_2)\ u^{\mathrm{T}}(i,j+1)\ u^{\mathrm{T}}(i+1,j)\,]^{\mathrm{T}},\\
\Theta_m &:= \mathrm{diag}\{-P_{1m}+Q_1,-P_{2m}+Q_2,-Q_1,-Q_2,-\gamma^2\mathbf{I},-\gamma^2\mathbf{I}\},\\
\mathscr{A}_m &:= \begin{bmatrix}\bar{A}_{1m} & \bar{A}_{2m} & \bar{A}_{d1m} & \bar{A}_{d2m} & \bar{B}_{1m} & \bar{B}_{2m}\end{bmatrix},\\
\mathscr{C}_m &:= \begin{bmatrix}\bar{C}_m & \mathbf{0} & \mathbf{0}_{n_z\times 2(n_x+n_r)} & \bar{B}_{3m} & \mathbf{0}\\ \mathbf{0} & \bar{C}_m & \mathbf{0}_{n_z\times 2(n_x+n_r)} & \mathbf{0} & \bar{B}_{3m}\end{bmatrix},\\
\bar{\mathscr{P}}_{ln} &:= \sum_{n\in\mathcal{I}_{\mathcal{K}}^{(m)}}\pi_{mn}P_{ln}+\sum_{n\in\mathcal{I}_{\mathcal{UC}}^{(m)}}\left(\sum_{s=1}^{M}\alpha_s\tilde{\pi}_{mn}^{(s)}\right)P_{ln}+\sum_{n\in\mathcal{I}_{\mathcal{UK}}^{(m)}}\hat{\pi}_{mn}P_{ln},\ l=1,2. \qquad (19)
\end{aligned}
$$

Considering the fact that $0 \leq \alpha_s \leq 1$, $\sum_{s=1}^{M} \alpha_s = 1$, and $0 \leq \frac{\hat{\pi}_{mn}}{\pi_{\mathcal{UK}}^{(ms)}} \leq 1$, $\sum_{n \in \mathcal{I}_{\mathcal{UK}}^{(m)}} \frac{\hat{\pi}_{mn}}{\pi_{\mathcal{UK}}^{(ms)}} = 1$, (18) can be rewritten as,

$$\Omega = \sum_{s=1}^{M} \alpha_s \sum_{n \in \mathcal{I}_{\mathcal{UK}}^{(m)}} \frac{\hat{\pi}_{mn}}{\pi_{\mathcal{UK}}^{(ms)}} \left[\varsigma^{\mathrm{T}}(i,j) \left[\mathscr{A}_m^{\mathrm{T}} \mathscr{P}_n^{(s)} \mathscr{A}_m + \mathscr{C}_m^{\mathrm{T}} \mathscr{C}_m + \Theta_m \right] \varsigma(i,j) \right],$$
$$m \in \mathcal{I},\ n \in \mathcal{I}_{\mathcal{UK}}^{(m)},\ s = 1, \ldots, M, \tag{20}$$

where

$$\mathscr{P}_n^{(s)} := \sum_{n \in \mathcal{I}_{\mathcal{K}}^{(m)}} \pi_{mn}(P_{1n} + P_{2n}) + \sum_{n \in \mathcal{I}_{\mathcal{UC}}^{(m)}} \tilde{\pi}_{mn}^{(s)}(P_{1n} + P_{2n}) + \pi_{\mathcal{UK}}^{(ms)} \underbrace{(P_{1n} + P_{2n})}_{n \in \mathcal{I}_{\mathcal{UK}}^{(m)}},$$
$$\pi_{\mathcal{UK}}^{(ms)} := 1 - \sum_{n \in \mathcal{I}_{\mathcal{K}}^{(m)}} \pi_{mn} - \sum_{n \in \mathcal{I}_{\mathcal{UC}}^{(m)}} \tilde{\pi}_{mn}^{(s)}. \tag{21}$$

According to (20), it is easy to see that (13) holds if and only if $\forall s = 1, \ldots, M$,

$$\varsigma^{\mathrm{T}}(i,j) \left[\mathscr{A}_m^{\mathrm{T}} \mathscr{P}_n^{(s)} \mathscr{A}_m + \mathscr{C}_m^{\mathrm{T}} \mathscr{C}_m + \Theta_m \right] \varsigma(i,j) < 0,\ m \in \mathcal{I},\ n \in \mathcal{I}_{\mathcal{UK}}^{(m)}, \tag{22}$$

which is implied by the condition (9). This completes the proof. □

Remark 1 By fully utilizing the properties of TPMs, together with the convexification of uncertain domains, a new $\mathscr{H}_\infty$ performance analysis criterion has been derived for the 2-D MJLS (6) with state-delays and imperfect mode information in Proposition 1. It is noted that the conditions given in (9) are nonconvex due to the presence of product terms between the Lyapunov matrices and system matrices, which brings some troubles in the solutions of model approximation synthesis problem. By applying some decoupling techniques, in the following, two distinctly different approaches to solve the $\mathscr{H}_\infty$ model approximation problem will be developed.

3.2 Model approximation via convex linearisation approach

In this subsection, via a linearisation procedure, a unified framework for the solvability of the $\mathscr{H}_\infty$ model approximation problem will be proposed. It will be shown that the parametrised representations of the approximation model gains can be constructed in terms of the feasible solutions to a set of strict LMIs.

Theorem 1 *Consider the 2-D MJLS (1) with state-delays and imperfect mode information, and the approximation model in the form of (5). The model error system in (6) is stochastically stable with an $\mathscr{H}_\infty$ performance γ, if there exist positive-definite symmetric matrices $\{P_{1m}, P_{2m}, Q_1, Q_2\} \in \mathbf{R}^{(n_x+n_r)\times(n_x+n_r)}$, and matrices $G_{m(1)} \in \mathbf{R}^{n_x \times n_x}$, $G_{m(3)} \in \mathbf{R}^{n_r \times n_x}$, $\{G_{m(2)}, \bar{A}_{r1m}, \bar{A}_{r2m}, \bar{A}_{rd1m}, \bar{A}_{rd2m}\} \in \mathbf{R}^{n_r \times n_r}$, $\{\bar{B}_{r1m}, \bar{B}_{r2m}\} \in \mathbf{R}^{n_r \times n_u}$, $C_{rm} \in \mathbf{R}^{n_z \times n_r}$, and $B_{r3m} \in \mathbf{R}^{n_z \times n_u}$, $m \in \mathcal{I}$, such that the following LMIs hold,*

$$\begin{bmatrix} \mathscr{P}_n^{(s)} - \mathtt{Sym}\{G_m\} & \mathbf{0} & \bar{\mathscr{A}}_m \\ * & -\mathbf{I} & \mathscr{C}_m \\ * & * & \Theta_m \end{bmatrix} < 0,\ m \in \mathcal{I},\ n \in \mathcal{I}_{\mathcal{UK}}^{(m)},\ s = 1, \ldots, M, \tag{23}$$

where

$$\Theta_m := \mathrm{diag}\{-P_{1m} + Q_1, -P_{2m} + Q_2, -Q_1, -Q_2, -\gamma^2\mathbf{I}, -\gamma^2\mathbf{I}\},$$

$$\bar{\mathscr{A}}_m := \begin{bmatrix} \bar{\mathcal{A}}_{1m} & \bar{\mathcal{A}}_{2m} & \bar{\mathcal{A}}_{d1m} & \bar{\mathcal{A}}_{d2m} & \bar{\mathcal{B}}_{1m} & \bar{\mathcal{B}}_{2m} \end{bmatrix},$$

$$\mathscr{C}_m := \begin{bmatrix} \bar{C}_m & \mathbf{0} & \mathbf{0}_{n_z \times 2(n_x+n_r)} & \bar{B}_{3m} & \mathbf{0} \\ \mathbf{0} & \bar{C}_m & \mathbf{0}_{n_z \times 2(n_x+n_r)} & \mathbf{0} & \bar{B}_{3m} \end{bmatrix},$$

$$G_m := \begin{bmatrix} G_{m(1)} & HG_{m(2)} \\ G_{m(3)} & G_{m(2)} \end{bmatrix}, \ H := \begin{bmatrix} \mathbf{I}_{n_r} \\ \mathbf{0}_{(n_x-n_r)\times n_r} \end{bmatrix},$$

$$\bar{\mathcal{A}}_{1m} := \begin{bmatrix} G_{m(1)}A_{1m} & H\bar{A}_{r1m} \\ G_{m(3)}A_{1m} & \bar{A}_{r1m} \end{bmatrix}, \ \bar{\mathcal{A}}_{2m} := \begin{bmatrix} G_{m(1)}A_{2m} & H\bar{A}_{r2m} \\ G_{m(3)}A_{2m} & \bar{A}_{r2m} \end{bmatrix},$$

$$\bar{\mathcal{A}}_{d1m} := \begin{bmatrix} G_{m(1)}A_{d1m} & H\bar{A}_{rd1m} \\ G_{m(3)}A_{d1m} & \bar{A}_{rd1m} \end{bmatrix}, \ \bar{\mathcal{A}}_{d2m} := \begin{bmatrix} G_{m(1)}A_{d2m} & H\bar{A}_{rd2m} \\ G_{m(3)}A_{d2m} & \bar{A}_{rd2m} \end{bmatrix},$$

$$\bar{\mathcal{B}}_{1m} := \begin{bmatrix} G_{m(1)}B_{1m} + H\bar{B}_{r1m} \\ G_{m(3)}B_{1m} + \bar{B}_{r1m} \end{bmatrix}, \ \bar{\mathcal{B}}_{2m} := \begin{bmatrix} G_{m(1)}B_{2m} + H\bar{B}_{r2m} \\ G_{m(3)}B_{2m} + \bar{B}_{r2m} \end{bmatrix},$$

$$\mathscr{P}_n^{(s)} := \sum_{n\in\mathcal{I}_{\mathcal{K}}^{(m)}} \pi_{mn}(P_{1n} + P_{2n}) + \sum_{n\in\mathcal{I}_{\mathcal{UC}}^{(m)}} \tilde{\pi}_{mn}^{(s)}(P_{1n} + P_{2n}) + \pi_{\mathcal{UK}}^{(ms)} \underbrace{(P_{1n} + P_{2n})}_{n\in\mathcal{I}_{\mathcal{UK}}^{(m)}}, \tag{24}$$

with $\bar{C}_m$, $\bar{B}_{3m}$, *and* $\pi_{\mathcal{UK}}^{(ms)}$ *defined in* (7) *and* (10), *respectively.*

Moreover, if the above conditions have a set of feasible solutions ($P_{1m}, P_{2m}, Q_1, Q_2, G_{m(1)}, G_{m(2)}, G_{m(3)}, \bar{A}_{r1m}, \bar{A}_{r2m}, \bar{A}_{rd1m}, \bar{A}_{rd2m}, \bar{B}_{r1m}, \bar{B}_{r2m}, C_{rm}, B_{r3m}$), *then an admissible* n_r*-order approximation model in the form of* (5) *can be constructed as,*

$$\begin{aligned} A_{r1m} &= G_{m(2)}^{-1}\bar{A}_{r1m}, \ A_{r2m} = G_{m(2)}^{-1}\bar{A}_{r2m}, \ A_{rd1m} = G_{m(2)}^{-1}\bar{A}_{rd1m}, \\ A_{rd2m} &= G_{m(2)}^{-1}\bar{A}_{rd2m}, \ B_{r1m} = G_{m(2)}^{-1}\bar{B}_{r1m}, \ B_{r2m} = G_{m(2)}^{-1}\bar{B}_{r2m}, \end{aligned} \tag{25}$$

and C_{rm} *and* B_{r3m} *can be obtained directly from* (23).

Proof By Schur complement, (9) is equivalent to

$$\begin{bmatrix} -\left(\mathscr{P}_n^{(s)}\right)^{-1} & \mathbf{0} & \mathscr{A}_m \\ * & -\mathbf{I} & \mathscr{C}_m \\ * & * & \Theta_m \end{bmatrix} < 0, \ m \in \mathcal{I}, \ n \in \mathcal{I}_{\mathcal{UK}}^{(m)}, \ s = 1, \ldots, M, \tag{26}$$

where $\mathscr{A}_m$, $\mathscr{C}_m$, $\mathscr{P}_n^{(s)}$ and Θ_m are defined in (10).

Performing a congruent transformation to (26) by $\mathrm{diag}\{G_m, \mathbf{I}_{2n_z}, \mathbf{I}_{(4n_x+4n_r+2n_u)}\}$, it follows from

$$\left(\mathscr{P}_n^{(s)} - G_m\right)^{\mathrm{T}} \left(\mathscr{P}_n^{(s)}\right)^{-1} \left(\mathscr{P}_n^{(s)} - G_m\right) \geq 0, \tag{27}$$

that

$$-G_m^{\mathrm{T}} \left(\mathscr{P}_n^{(s)}\right)^{-1} G_m \leq \mathscr{P}_n^{(s)} - G_m - G_m^{\mathrm{T}}. \tag{28}$$

Based on (28), we have that the following inequality implies (26),

$$\begin{bmatrix} \mathscr{P}_n^{(s)} - \mathrm{Sym}\{G_m\} & \mathbf{0} & G_m \mathscr{A}_m \\ * & -\mathbf{I} & \mathscr{C}_m \\ * & * & \Theta_m \end{bmatrix} < 0,\ m \in \mathcal{I},\ n \in \mathcal{I}_{\mathcal{UK}}^{(m)},\ s = 1, \ldots, M. \tag{29}$$

For simplicity in the model approximation synthesis procedure, we first specify the slack variables as,

$$G_m = \begin{bmatrix} G_{m(1)} & HG_{(2)} \\ G_{m(3)} & G_{(4)} \end{bmatrix},\ m \in \mathcal{I}, \tag{30}$$

where $H := \begin{bmatrix} \mathbf{I}_{n_r} & \mathbf{0}_{n_r \times (n_x - n_r)} \end{bmatrix}^{\mathrm{T}}$, $G_{m(1)} \in \mathbf{R}^{n_x \times n_x}$, $G_{m(3)} \in \mathbf{R}^{n_r \times n_x}$, and $\{G_{(2)}, G_{(4)}\} \in \mathbf{R}^{n_r \times n_r}$. Then, for matrix inequality linearisation purpose, similar to Zhang and Boukas (2009); Gao et al. (2006), performing a congruent transformation to

$$\begin{bmatrix} G_{m(1)} + G_{m(1)}^{\mathrm{T}} & HG_{(2)} + G_{m(3)}^{\mathrm{T}} \\ * & G_{(4)} + G_{(4)}^{\mathrm{T}} \end{bmatrix} \tag{31}$$

by diag $\left\{\mathbf{I}_{n_x}, G_{(2)} G_{(4)}^{-1}\right\}$ yields,

$$\begin{bmatrix} G_{m(1)} + G_{m(1)}^{\mathrm{T}} & HG_{(2)} G_{(4)}^{-\mathrm{T}} G_{(2)}^{\mathrm{T}} + G_{m(3)}^{\mathrm{T}} G_{(4)}^{-\mathrm{T}} G_{(2)}^{\mathrm{T}} \\ * & G_{(2)} G_{(4)}^{-\mathrm{T}} G_{(2)}^{\mathrm{T}} + G_{(2)} G_{(4)}^{-1} G_{(2)}^{\mathrm{T}} \end{bmatrix}$$

$$:= \begin{bmatrix} G_{m(1)} + G_{m(1)}^{\mathrm{T}} & H\underline{\underline{\bar{G}_{(2)}}} + \bar{G}_{m(3)}^{\mathrm{T}} \\ * & \underline{\underline{\bar{G}_{(2)} + \bar{G}_{(2)}^{\mathrm{T}}}} \end{bmatrix}. \tag{32}$$

Thus, instead of (30), one can directly specify the matrix G_m of the following form *without loss of generality*,

$$G_m = \begin{bmatrix} G_{m(1)} & HG_{m(2)} \\ G_{m(3)} & G_{m(2)} \end{bmatrix},\ m \in \mathcal{I}. \tag{33}$$

It is noted that in this way the matrix variable $G_{m(2)}$ can be absorbed by the approximation model gain variables A_{r1m}, A_{r2m}, A_{rd1m}, A_{rd2m}, B_{r1m} and B_{r2m} by introducing

$$\begin{aligned} &\bar{A}_{r1m} := G_{m(2)} A_{r1m},\ \bar{A}_{r2m} := G_{m(2)} A_{r2m},\ \bar{A}_{rd1m} := G_{m(2)} A_{rd1m}, \\ &\bar{A}_{rd2m} := G_{m(2)} A_{rd2m},\ \bar{B}_{r1m} := G_{m(2)} B_{r1m},\ \bar{B}_{r2m} := G_{m(2)} B_{r2m}. \end{aligned} \tag{34}$$

This feature enables one to make no congruent transformation to the original matrix inequality and all the slack variables can be set as Markovian switching.

Then, by substituting the matrix G_m defined in (33) into (29), together with consideration of (34), one readily obtains (23).

On the other hand, the condition in (23) implies that $-G_{m(2)} - G_{m(2)}^{\mathrm{T}} < 0$, which means that $G_{m(2)}$ is nonsingular. Then, the approximation model gains can be constructed by (25). The proof is thus completed. □

Remark 2 Theorem 1 provides a sufficient condition on the feasibility of $\mathscr{H}_\infty$ model approximation synthesis problem for the 2-D MJLSs with state-delays and imperfect mode information. A desired approximation model can be determined by solving the following convex optimization problem.

Problem MALA (model approximation via linearisation approach)
Minimize γ subject to (23) for $P_{1m},\ P_{2m},\ Q_1,\ Q_2,\ G_{m(1)},\ G_{m(2)},\ G_{m(3)},\ \bar{A}_{r1m},\ \bar{A}_{r2m},\ \bar{A}_{rd1m},\ \bar{A}_{rd2m},\ \bar{B}_{r1m},\ \bar{B}_{r2m},\ C_{rm},\ B_{r3m},\ m \in \mathcal{I}$.

It is noted that in order to obtain the strict LMIs-based conditions in Theorem 1, a relaxation matrix H is imposed in the slack variable $G_m, m \in \mathcal{I}$. This structural constraint unavoidably brings some degree of design conservatism. To reduce the design conservatism, in the following subsection, we will resort to an iterative approach to solve the model approximation problem.

3.3 Model approximation via iterative approach

As stated in the previous subsection, the design conservatism of Theorem 1 is mainly induced by a structural constraint on the slack matrices. Another approach to the model approximation synthesis depends heavily upon the celebrated elimination/projection lemma, which casts the model approximation problem into some LMI conditions plus equality constraints Wu et al. (2006); Wu and Zheng (2009). However, it is worth pointing out that due to the imperfect mode information in 2-D MJLSs, the indices $m \in \mathcal{I}, n \in \mathcal{I}_{\mathcal{UK}}^{(m)}$, and $s = 1, 2, \ldots, M$ are simultaneously involved in (9) in Proposition 1. Thus, the so-called projection approach Wu et al. (2006); Wu and Zheng (2009) can not be utilized to derive the approximation model gains in (9). In other words, for the 2-D MJLS in (1) with imperfect mode information, the approximation model (5) can not be obtained by the projection approach as proposed in Wu et al. (2006); Wu and Zheng (2009). Alternatively, in this subsection we may resort to a direct approach to separate the Lyapunov matrices from the system matrices. The result is summarized in the following theorem.

Theorem 2 *Consider the 2-D MJLS (1) with state-delays and imperfect mode information, and the approximation model in the form of (5). The model error system in (6) is stochastically stable with an $\mathscr{H}_\infty$ performance γ, if there exist positive-definite symmetric matrices $\{P_{1m}, P_{2m}, Q_1, Q_2, X_m\} \in \mathbf{R}^{(n_x+n_r)\times(n_x+n_r)}$, and matrices $\{A_{r1m}, A_{r2m}, A_{rd1m}, A_{rd2m}\} \in \mathbf{R}^{n_r\times n_r}$, $\{B_{r1m}, B_{r2m}\} \in \mathbf{R}^{n_r\times n_u}$, $C_{rm} \in \mathbf{R}^{n_z\times n_r}$, and $B_{r3m} \in \mathbf{R}^{n_z\times n_u}$, $m \in \mathcal{I}$, such that*

$$\begin{bmatrix} \mathscr{X}_n^{(m)} & \mathbf{0} & \mathcal{F}_m^{(s)}\mathscr{A}_m \\ * & -\mathbf{I} & \mathscr{C}_m \\ * & * & \Theta_m \end{bmatrix} < 0,\ n \in \mathcal{I}_{\mathcal{UK}}^{(m)},\ s = 1, \ldots, M, \tag{35}$$

$$(P_{1m} + P_{2m})X_m = \mathbf{I},\ m \in \mathcal{I}, \tag{36}$$

where

$$\mathscr{X}_n^{(m)} := \operatorname{diag}\{X_{\mathcal{K}_{1(m)}}, \ldots, X_{\mathcal{K}_{t(m)}}, X_{\mathcal{U}_{1(m)}}, \ldots, X_{\mathcal{U}_{v(m)}}, X_n\},$$

$$\mathcal{F}_m^{(s)} := \left[\sqrt{\pi_{m\mathcal{K}_{1(m)}}}\mathbf{I}, \ldots, \sqrt{\pi_{m\mathcal{K}_{t(m)}}}\mathbf{I}, \sqrt{\tilde{\pi}_{m\mathcal{U}_{1(m)}}^{(s)}}\mathbf{I}, \ldots, \sqrt{\tilde{\pi}_{m\mathcal{U}_{v(m)}}^{(s)}}\mathbf{I}, \sqrt{\pi_{\mathcal{UK}}^{(ms)}}\mathbf{I}\right]^{\mathrm{T}},$$

$$\mathscr{A}_m := \big[\,\mathcal{A}_{1m} + E_1\mathscr{G}_{rm}F_1 \quad \mathcal{A}_{2m} + E_2\mathscr{G}_{rm}F_1 \quad \mathcal{A}_{d1m} + E_1\mathscr{G}_{rm}F_2 \quad \mathcal{A}_{d2m} + E_2\mathscr{G}_{rm}F_2$$
$$\mathcal{B}_{1m} + E_1\mathscr{G}_{rm}F_3 \quad \mathcal{B}_{2m} + E_2\mathscr{G}_{rm}F_3\,\big],$$

$$\mathscr{C}_m := \begin{bmatrix} \mathcal{C}_m + E_3\mathscr{G}_{rm}F_1 & \mathbf{0} & \mathbf{0}_{n_z\times 2(n_x+n_r)} & \mathcal{B}_{3m} + E_3\mathscr{G}_{rm}F_3 & \mathbf{0} \\ \mathbf{0} & \mathcal{C}_m + E_3\mathscr{G}_{rm}F_1 & \mathbf{0}_{n_z\times 2(n_x+n_r)} & \mathbf{0} & \mathcal{B}_{3m} + E_3\mathscr{G}_{rm}F_3 \end{bmatrix},$$

$$
\begin{aligned}
\mathscr{G}_{rm} &:= \begin{bmatrix} A_{r1m} & A_{rd1m} & B_{r1m} \\ A_{r2m} & A_{rd2m} & B_{r2m} \\ C_{rm} & \mathbf{0} & B_{r3m} \end{bmatrix}, \ \mathcal{A}_{1m} := \begin{bmatrix} A_{1m} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \ \mathcal{A}_{2m} := \begin{bmatrix} A_{2m} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \\
\mathcal{A}_{d1m} &:= \begin{bmatrix} A_{d1m} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \ \mathcal{A}_{d2m} := \begin{bmatrix} A_{d2m} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \ \mathcal{B}_{1m} := \begin{bmatrix} B_{1m} \\ \mathbf{0} \end{bmatrix}, \ \mathcal{B}_{2m} := \begin{bmatrix} B_{2m} \\ \mathbf{0} \end{bmatrix}, \\
\mathcal{C}_m &:= \begin{bmatrix} C_m & \mathbf{0} \end{bmatrix}, \ \mathcal{B}_{3m} := B_{3m}, \\
E_1 &:= \begin{bmatrix} \mathbf{0}_{n_x \times n_r} & \mathbf{0} \\ \mathbf{I} & \mathbf{0}_{n_r \times (n_r + n_z)} \end{bmatrix}, \ E_2 := \begin{bmatrix} \mathbf{0}_{n_x \times n_r} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n_r} & \mathbf{0}_{n_r \times n_z} \end{bmatrix}, \\
F_1 &:= \begin{bmatrix} \mathbf{0}_{n_r \times n_x} & \mathbf{I} \\ \mathbf{0} & \mathbf{0}_{(n_r + n_u) \times n_r} \end{bmatrix}, \ F_2 := \begin{bmatrix} \mathbf{0}_{n_x \times n_r} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n_r} & \mathbf{0}_{n_r \times n_u} \end{bmatrix}^{\mathrm{T}}, \\
E_3 &:= \begin{bmatrix} \mathbf{0}_{n_z \times 2n_r} & -\mathbf{I}_{n_z} \end{bmatrix}, \ F_3 := \begin{bmatrix} \mathbf{0}_{n_u \times 2n_r} & \mathbf{I}_{n_u} \end{bmatrix}^{\mathrm{T}}, \\
\Theta_m &:= \operatorname{diag}\{-P_{1m} + Q_1, -P_{2m} + Q_2, -Q_1, -Q_2, -\gamma^2 \mathbf{I}, -\gamma^2 \mathbf{I}\}, \\
\pi_{\mathcal{UK}}^{(ms)} &:= 1 - \sum_{n \in \mathcal{I}_{\mathcal{K}}^{(m)}} \pi_{mn} - \sum_{n \in \mathcal{I}_{\mathcal{UC}}^{(m)}} \tilde{\pi}_{mn}^{(s)}.
\end{aligned} \tag{37}
$$

Proof From Proposition 1, it is known that for $m \in \mathcal{I}$, there exists an approximation model in the form of (5) such that the model error system (6) is stochastically stable with a guaranteed $\mathscr{H}_\infty$ performance γ if there exist matrices $(P_{1m}, P_{2m}, Q_1, Q_2)$ satisfying (9). First, we rewrite the matrices defined in (7) in the following compact form,

$$
\begin{aligned}
\bar{A}_{1m} &:= \mathcal{A}_{1m} + E_1 \mathscr{G}_{rm} F_1, \ \bar{A}_{2m} := \mathcal{A}_{2m} + E_2 \mathscr{G}_{rm} F_1, \ \bar{A}_{d1m} := \mathcal{A}_{d1m} + E_1 \mathscr{G}_{rm} F_2, \\
\bar{A}_{d2m} &:= \mathcal{A}_{d2m} + E_2 \mathscr{G}_{rm} F_2, \ \bar{B}_{1m} := \mathcal{B}_{1m} + E_1 \mathscr{G}_{rm} F_3, \ \bar{B}_{2m} := \mathcal{B}_{2m} + E_2 \mathscr{G}_{rm} F_3, \\
\bar{C}_m &:= \mathcal{C}_m + E_3 \mathscr{G}_{rm} F_1, \ \bar{B}_{3m} := \mathcal{B}_{3m} + E_3 \mathscr{G}_{rm} F_3,
\end{aligned} \tag{38}
$$

where $\mathscr{G}_{rm}, \mathcal{A}_{1m}, \mathcal{A}_{2m}, \mathcal{A}_{d1m}, \mathcal{A}_{d2m}, \mathcal{B}_{1m}, \mathcal{B}_{2m}, \mathcal{C}_m, \mathcal{B}_{3m}, E_1, E_2, E_3, F_1, F_2$ and F_3 are defined in (37).

Then, by Schur complement to (9), and with the notations $\mathcal{I}_{\mathcal{K}}^{(m)} := \{\mathcal{K}_{1_{(m)}}, \ldots, \mathcal{K}_{t_{(m)}}\}$, $\mathcal{I}_{\mathcal{UC}}^{(m)} := \{\mathcal{U}_{1_{(m)}}, \ldots, \mathcal{U}_{v_{(m)}}\}$, we have that (9) is equivalent to

$$
\begin{bmatrix} \bar{\mathscr{P}}_n^{(m)} & \mathbf{0} & \digamma_m^{(s)} \mathscr{A}_m \\ * & -\mathbf{I} & \mathscr{C}_m \\ * & * & \Theta_m \end{bmatrix} < 0, \ m \in \mathcal{I}, \ n \in \mathcal{I}_{\mathcal{UK}}^{(m)}, \ s = 1, \ldots, M, \tag{39}
$$

where

$$
\begin{aligned}
\bar{\mathscr{P}}_n^{(m)} &:= \operatorname{diag}\Big\{ \left(P_{1\mathcal{K}_{1_{(m)}}} + P_{2\mathcal{K}_{1_{(m)}}}\right)^{-1}, \ldots, \left(P_{1\mathcal{K}_{t_{(m)}}} + P_{2\mathcal{K}_{t_{(m)}}}\right)^{-1}, \\
&\qquad \left(P_{1\mathcal{U}_{1_{(m)}}} + P_{2\mathcal{U}_{1_{(m)}}}\right)^{-1}, \ldots, \left(P_{1\mathcal{U}_{v_{(m)}}} + P_{2\mathcal{U}_{v_{(m)}}}\right)^{-1}, (P_{1n} + P_{2n})^{-1} \Big\}, \\
\digamma_m^{(s)} &:= \left[\sqrt{\pi_{m\mathcal{K}_{1_{(m)}}}} \mathbf{I}, \ldots, \sqrt{\pi_{m\mathcal{K}_{t_{(m)}}}} \mathbf{I}, \sqrt{\tilde{\pi}_{m\mathcal{U}_{1_{(m)}}}^{(s)}} \mathbf{I}, \ldots, \sqrt{\tilde{\pi}_{m\mathcal{U}_{v_{(m)}}}^{(s)}} \mathbf{I}, \sqrt{\pi_{\mathcal{UK}}^{(ms)}} \mathbf{I} \right]^{\mathrm{T}}.
\end{aligned} \tag{40}
$$

Setting $X_m := (P_{1m} + P_{2m})^{-1}$ and considering (40), it is easy to see that (39) is equivalent to (35) and (36). This completes the proof. □

Remark 3 Theorem 2 provides another sufficient condition for testing the solvability of n_r-order $\mathscr{H}_\infty$ model approximation synthesis for 2-D MJLS (1) with state-delays and imperfect mode information. It is noted that the condition in Theorem 2 is not strictly an LMI due to the matrix equality in (36). However, with the CCL algorithm in El Ghaoui et al. (1997); Qiu et al. (2012), we can resolve this nonconvex feasibility problem by formulating it into a sequential optimization problem subject to LMI constraints. The basic idea of CCL algorithm is that if the LMI $\begin{bmatrix} P & \mathbf{I} \\ \mathbf{I} & X \end{bmatrix} \geq 0$ is feasible with the $n \times n$ matrix variables $P > 0$ and $X > 0$, then $Trace(PX) \geq n$, and $Trace(PX) = n$ if and only if $PX = \mathbf{I}$.

Based on the above discussions and using a CCL technique, the nonconvex feasibility problem given in (35) and (36) is converted into the following minimization problem that involves LMI conditions.

Problem MAIA (model approximation via iterative approach).

Minimize Trace $\left(\sum_{m=1}^{N} (P_{1m} + P_{2m})X_m\right)$ subject to (35) and

$$\begin{bmatrix} P_{1m} + P_{2m} & \mathbf{I} \\ * & X_m \end{bmatrix} \geq 0, \ \forall m \in \mathcal{I}. \tag{41}$$

Then, the suboptimal performance of γ can be obtained by the following algorithm. The convergence of this algorithm is guaranteed in terms of similar results in El Ghaoui et al. (1997); Qiu et al. (2012).

Algorithm MAIA: Suboptimal performance of γ

Step 1. Choose a sufficiently large initial $\gamma > 0$, such that there exists a feasible solution to (35) and (41). Set $\gamma_0 = \gamma$.

Step 2. Find a feasible set $(P_{1m}^{(0)}, P_{2m}^{(0)}, X_m^{(0)}, Q_1^{(0)}, Q_2^{(0)}, A_{r1m}^{(0)}, A_{r2m}^{(0)}, A_{rd1m}^{(0)}, A_{rd2m}^{(0)}, B_{r1m}^{(0)}, B_{r2m}^{(0)}, C_{rm}^{(0)}, B_{r3m}^{(0)}, \forall m \in \mathcal{I})$ that satisfies the conditions in (35) and (41). Set $q = 0$.

Step 3. Solving the following LMI problem over the variables P_{1m}, P_{2m}, X_m, Q_1, Q_2, A_{r1m}, A_{r2m}, A_{rd1m}, A_{rd2m}, B_{r1m}, B_{r2m}, C_{rm}, B_{r3m},

$$\text{Minimize Trace}\left(\sum_{m=1}^{N}\left((P_{1m}^{(q)} + P_{2m}^{(q)})X_m + (P_{1m} + P_{2m})X_m^{(q)}\right)\right) \text{subject to (35) and (41).} \tag{42}$$

Set $P_{1m}^{(q+1)} = P_{1m}$, $P_{2m}^{(q+1)} = P_{2m}$ and $X_m^{(q+1)} = X_m$.

Step 4. Substituting the gains $A_{r1m}, A_{r2m}, A_{rd1m}, A_{rd2m}, B_{r1m}, B_{r2m}, C_{rm}, B_{r3m}$ obtained in Step 3 into (9) and if the LMIs in (9) are feasible with respect to the variables P_{1m}, P_{2m}, Q_1 and Q_2, then set $\gamma_0 = \gamma$ and return to Step 2 after decreasing γ to some extent. If (9) are infeasible within the maximum number of iterations allowed, then exit. Otherwise, set $q = q + 1$, and go to Step 3.

Remark 4 It is worth mentioning that the conditions given in Theorem 1 are strictly convex, and thus can be readily solved with commercially available software. The design conservatism of Theorem 1 mainly originates from the bounding inequality in (28) with the slack variable of a structural constraint in (33). Thus, Theorem 1 is only a sufficient condition of the performance analysis criterion in (9). In the iterative approach, however, the conditions given in (35) and (41) are equivalent to the corresponding performance analysis results in (9). This is the main advantage of Theorem 2 over Theorem 1. However, the numerical computation cost involved in Algorithm MAIA is also much larger than that involved in Theorem 1

(MALA), especially when the number of iterations increases. It will also be shown in the simulation section that Theorem 2 is generally less conservative than Theorem 1 but with more computational burden.

4 Simulation studies

In this section, we use a simulation example to demonstrate the effectiveness of the proposed model approximation method to 2-D MJLSs.

Consider a 2-D MJLS with state-delays in the form of (1) with parameters as follows,

$$\left[\begin{array}{c|c|c} A_{11} & A_{d11} & B_{11} \\ \hline A_{21} & A_{d21} & B_{21} \\ \hline C_1 & & B_{31} \end{array}\right] = \left[\begin{array}{cccc|cccc|c} 0 & -0.5 & 0 & 0 & 0 & -0.1 & 0 & 0 & 0.4 \\ 0 & 0 & -0.5 & 0 & 0 & 0 & -0.1 & 0 & 0.8 \\ 0 & 0 & 0 & -0.25 & 0 & 0 & 0 & 0 & 0.1 \\ -0.15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -0.5 & 0 & -0.5 & 0 & -0.1 & 0 & -0.1 & 0.2 \\ 0 & 0 & -0.5 & 0 & 0 & 0 & -0.1 & 0 & 0 \\ -0.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & -0.25 & 0 & 0 & 0 & 0 & -0.3 \\ \hline 0.1 & 0.3 & 0.8 & -0.5 & & & & & 0.5 \end{array}\right],$$

$$\left[\begin{array}{c|c|c} A_{12} & A_{d12} & B_{12} \\ \hline A_{22} & A_{d22} & B_{22} \\ \hline C_2 & & B_{32} \end{array}\right] = \left[\begin{array}{cccc|cccc|c} 0 & -0.2 & 0 & 0 & 0 & -0.05 & 0 & 0 & 0.9 \\ 0 & 0 & -0.2 & 0 & 0 & 0 & -0.05 & 0 & -0.5 \\ 0 & 0 & 0 & -0.16 & 0 & 0 & 0 & 0.05 & 0.8 \\ 0.16 & 0 & 0 & 0.24 & 0 & 0 & 0.05 & 0 & 1 \\ \hline 0 & -0.5 & 0 & -0.5 & 0 & -0.05 & 0 & -0.12 & 0.6 \\ 0 & -0.24 & 0 & -0.24 & 0 & 0 & -0.05 & 0 & 0 \\ 0.2 & 0 & 0 & 0.24 & 0 & 0 & 0.05 & 0 & 0.6 \\ 0 & 0 & 0 & -0.2 & 0.05 & 0 & 0 & 0 & -0.6 \\ \hline 0.5 & 0 & -0.3 & 0.2 & & & & & 0.2 \end{array}\right],$$

$$\left[\begin{array}{c|c|c} A_{13} & A_{d13} & B_{13} \\ \hline A_{23} & A_{d23} & B_{23} \\ \hline C_3 & & B_{33} \end{array}\right] = \left[\begin{array}{cccc|cccc|c} -0.4 & -0.2 & 0 & 0.2 & 0 & -0.12 & 0 & 0 & 0.5 \\ 0.1 & 0 & -0.4 & 0 & 0 & 0 & -0.12 & 0 & -0.5 \\ 0 & 0.1 & 0 & 0.2 & 0 & 0 & 0 & 0.08 & 0.4 \\ 0.2 & 0 & 0 & -0.2 & 0 & 0 & 0.08 & 0 & 1 \\ \hline 0.2 & -0.12 & 0 & -0.12 & 0 & -0.12 & 0 & -0.12 & 0.6 \\ 0 & -0.2 & -0.12 & 0 & 0 & 0 & -0.12 & 0 & 0 \\ 0.2 & 0 & -0.16 & 0 & 0 & 0 & 0.06 & 0 & 0.4 \\ 0 & 0.16 & 0 & 0.3 & 0 & 0.08 & 0 & 0 & -0.5 \\ \hline 0.4 & 0 & -0.6 & 0.6 & & & & & 0.4 \end{array}\right],$$

$$\left[\begin{array}{c|c|c} A_{14} & A_{d14} & B_{14} \\ \hline A_{24} & A_{d24} & B_{24} \\ \hline C_4 & & B_{34} \end{array}\right] = \left[\begin{array}{cccc|cccc|c} 0 & -0.4 & 0 & 0 & 0 & -0.1 & 0 & 0 & 0.3 \\ 0 & 0 & -0.4 & 0 & 0 & 0 & -0.1 & 0 & 0.5 \\ 0 & 0 & 0 & -0.2 & 0 & 0 & 0 & 0 & 0 \\ -0.12 & 0 & 0 & 0 & 0 & 0 & 0 & -0.06 & -0.2 \\ \hline 0 & -0.4 & 0 & -0.4 & 0 & -0.1 & 0 & -0.1 & 0.4 \\ 0 & 0 & -0.4 & 0 & 0 & 0 & -0.1 & 0 & 0 \\ -0.16 & 0 & 0 & 0 & 0 & 0 & 0 & -0.06 & 0.2 \\ 0 & 0 & 0 & -0.24 & 0 & 0 & 0 & 0 & 0.3 \\ \hline 0.3 & 0.2 & 0.6 & -0.5 & & & & & 0.5 \end{array}\right].$$

Four different cases for the TPM are given in Table 1, where the TPs labeled with "ˆ" and "˜" represent the unknown and uncertain elements, respectively. Specifically, Case 1, Case 2, Case 3, and Case 4 stand for the completely known TPs, imperfect mode information (including known, partially unknown and uncertain TPs), partially unknown TPs, and completely unknown TPs, respectively.

Table 1 Four different TPMs

Case 1: Completely known TPM	Case 2: Imperfect TPM1
$\begin{bmatrix} 0.3 & 0.2 & 0.1 & 0.4 \\ 0.3 & 0.2 & 0.3 & 0.2 \\ 0.1 & 0.5 & 0.3 & 0.1 \\ 0.2 & 0.2 & 0.1 & 0.5 \end{bmatrix}$	$\begin{bmatrix} 0.3 & 0.2 & 0.1 & 0.4 \\ \hat{\pi}_{21} & \hat{\pi}_{22} & 0.3 & 0.2 \\ \hat{\pi}_{31} & \hat{\pi}_{32} & \hat{\pi}_{33} & \hat{\pi}_{34} \\ 0.2 & \hat{\pi}_{42} & \hat{\pi}_{43} & \hat{\pi}_{44} \end{bmatrix}$
Case 3: Imperfect TPM2	Case 4: Completely unknown TPM
$\begin{bmatrix} 0.3 & 0.2 & 0.1 & 0.4 \\ \hat{\pi}_{21} & \hat{\pi}_{22} & 0.3 & 0.2 \\ \hat{\pi}_{31} & \hat{\pi}_{32} & \hat{\pi}_{33} & \hat{\pi}_{34} \\ 0.2 & \hat{\pi}_{42} & \hat{\pi}_{43} & \hat{\pi}_{44} \end{bmatrix}$	$\begin{bmatrix} \hat{\pi}_{11} & \hat{\pi}_{12} & \hat{\pi}_{13} & \hat{\pi}_{14} \\ \hat{\pi}_{21} & \hat{\pi}_{22} & \hat{\pi}_{23} & \hat{\pi}_{24} \\ \hat{\pi}_{31} & \hat{\pi}_{32} & \hat{\pi}_{33} & \hat{\pi}_{34} \\ \hat{\pi}_{41} & \hat{\pi}_{42} & \hat{\pi}_{43} & \hat{\pi}_{44} \end{bmatrix}$

Table 2 Comparison of minimum $\mathscr{H}_\infty$ performance for different TPMs

TPMs	Three-order		Two-order		One-order	
	Theorem 1 (MALA)	Theorem 2 (MAIA)	Theorem 1 (MALA)	Theorem 2 (MAIA)	Theorem 1 (MALA)	Theorem 2 (MAIA)
Case 1	1.3014	0.9	1.5670	1	1.6634	1.3
Case 2	1.8750	1.4	2.2072	1.5	2.2890	1.8
Case 3	2.3404	1.7	2.6960	1.9	2.8472	2.2
Case 4	3.2954	2.8	3.9645	3.3	4.3270	3.7

For Case 2, it is assumed that the uncertain TPs comprise three vertices $\Pi_s,\ s = 1, 2, 3$, where the third rows $\Pi_{s(3)},\ s = 1, 2, 3$, are given by

$$\Pi_{1(3)} = \left[\, \hat{\pi}_{31}\ 0.2\ \hat{\pi}_{33}\ 0.4 \,\right],$$

$$\Pi_{2(3)} = \left[\, \hat{\pi}_{31}\ 0.5\ \hat{\pi}_{33}\ 0.3 \,\right],$$

$$\Pi_{3(3)} = \left[\, \hat{\pi}_{31}\ 0.3\ \hat{\pi}_{33}\ 0.1 \,\right],$$

and the other rows in the three vertices are given with the same elements, that is

$$\Pi_{s(1)} = \left[\, 0.3\ 0.2\ 0.1\ 0.4 \,\right],$$

$$\Pi_{s(2)} = \left[\, \hat{\pi}_{21}\ \hat{\pi}_{22}\ 0.3\ 0.2 \,\right],$$

$$\Pi_{s(4)} = \left[\, 0.2\ \hat{\pi}_{42}\ \hat{\pi}_{43}\ \hat{\pi}_{44} \,\right],\ s = 1, 2, 3.$$

The objective is to design a reduced-order model of the form (5) to approximate the above system such that the 2-D model error system (6) is stochastically stable with an $\mathscr{H}_\infty$ performance γ. By solving the problems MALA and MAIA with the maximum number of iterations allowed as 40, a detailed comparison between the minimum $\mathscr{H}_\infty$ performance indices $\gamma_{\min}$ obtained based on Theorems 1 and 2 is summarized in Table 2. By inspection of Table 2, it is easy to see that the results based on Theorem 2 (MAIA) are much less conservative than those based on Theorem 1 (MALA). It is also shown from Tables 1 and 2 that the more information on TPs is available, the better $\mathscr{H}_\infty$ performance can be obtained, which is effective to reduce the design conservatism. Therefore, the introduction of the uncertain TPs is necessary and significant.

Specifically, for $n_r = 2$, we obtain $\gamma_{\min} = 2.2072$ by Theorem 1 with imperfect TPM1 shown in Table 1, and the two-order model parameters are given by,

$$\left[\begin{array}{c|c|c} A_{r11} & A_{rd11} & B_{r11} \\ \hline A_{r21} & A_{rd21} & B_{r21} \\ \hline C_{r1} & & B_{r31} \end{array}\right] = \left[\begin{array}{cc|cc|c} 0.0144 & -0.3696 & 0.0008 & -0.0300 & -0.5806 \\ 0.0531 & 0.3009 & -0.0020 & 0.0373 & -3.1155 \\ \hline -0.3281 & -0.3916 & -0.0098 & 0.0153 & -1.1580 \\ -0.0470 & -0.0137 & -0.0042 & 0.0029 & 1.5450 \\ \hline -0.3670 & -0.1127 & & & 0.1905 \end{array}\right],$$

$$\left[\begin{array}{c|c|c} A_{r12} & A_{rd12} & B_{r12} \\ \hline A_{r22} & A_{rd22} & B_{r22} \\ \hline C_{r2} & & B_{r32} \end{array}\right] = \left[\begin{array}{cc|cc|c} -0.0124 & -0.0804 & 0.0002 & -0.0122 & -1.1958 \\ -0.0142 & 0.0307 & -0.0007 & 0.0080 & 1.2120 \\ \hline -0.0700 & -0.1604 & -0.0197 & 0.0304 & -2.1266 \\ 0.0124 & -0.0818 & 0.0179 & -0.0266 & 0.5815 \\ \hline -0.3554 & 0.0772 & & & 0.5801 \end{array}\right],$$

$$\left[\begin{array}{c|c|c} A_{r13} & A_{rd13} & B_{r13} \\ \hline A_{r23} & A_{rd23} & B_{r23} \\ \hline C_{r3} & & B_{r33} \end{array}\right] = \left[\begin{array}{cc|cc|c} -0.2166 & -0.1689 & 0.0007 & -0.0308 & -0.4424 \\ -0.1656 & -0.0145 & -0.0024 & 0.0431 & 0.7975 \\ \hline 0.2238 & -0.0535 & -0.0083 & 0.0121 & -2.3599 \\ -0.0314 & -0.0320 & 0.0095 & -0.0156 & -0.1542 \\ \hline -0.0897 & 0.1612 & & & 0.1385 \end{array}\right],$$

$$\left[\begin{array}{c|c|c} A_{r14} & A_{rd14} & B_{r14} \\ \hline A_{r24} & A_{rd24} & B_{r24} \\ \hline C_{r4} & & B_{r34} \end{array}\right] = \left[\begin{array}{cc|cc|c} 0.0510 & -0.4391 & 0.0007 & -0.0305 & -1.1082 \\ -0.0289 & 0.3892 & -0.0016 & 0.0381 & -1.5092 \\ \hline -0.1714 & -0.1138 & -0.0133 & 0.0200 & -1.3469 \\ -0.1113 & -0.0227 & -0.0073 & 0.0105 & 0.5674 \\ \hline -0.3626 & -0.2193 & & & 0.4410 \end{array}\right].$$

For $n_r = 2$, we obtain $\gamma = 1.5$ by solving the Algorithm MAIA after 40 iterations with imperfect TPM1 shown in Table 1, and the two-order model parameters are given by,

$$\left[\begin{array}{c|c|c} A_{r11} & A_{rd11} & B_{r11} \\ \hline A_{r21} & A_{rd21} & B_{r21} \\ \hline C_{r1} & & B_{r31} \end{array}\right] = \left[\begin{array}{cc|cc|c} -0.0836 & -0.2639 & -0.0201 & -0.0439 & -1.0852 \\ 0.0023 & 0.2601 & -0.0150 & 0.0251 & -0.8008 \\ \hline -0.1013 & -0.0694 & -0.0187 & 0.0426 & -0.6650 \\ -0.0940 & -0.1282 & -0.0075 & 0.0306 & -0.4783 \\ \hline -0.4129 & -0.1966 & & & 0.2871 \end{array}\right],$$

$$\left[\begin{array}{c|c|c} A_{r12} & A_{rd12} & B_{r12} \\ \hline A_{r22} & A_{rd22} & B_{r22} \\ \hline C_{r2} & & B_{r32} \end{array}\right] = \left[\begin{array}{cc|cc|c} -0.0516 & -0.0104 & -0.0097 & -0.0211 & -1.7817 \\ -0.1369 & 0.1520 & -0.0304 & -0.0301 & 2.0060 \\ \hline -0.0412 & -0.0509 & -0.0178 & 0.0718 & -1.2370 \\ 0.0507 & -0.1461 & -0.0015 & -0.0436 & -2.2016 \\ \hline -0.2852 & 0.1028 & & & 0.3190 \end{array}\right],$$

$$\left[\begin{array}{c|c|c} A_{r13} & A_{rd13} & B_{r13} \\ \hline A_{r23} & A_{rd23} & B_{r23} \\ \hline C_{r3} & & B_{r33} \end{array}\right] = \left[\begin{array}{cc|cc|c} -0.2750 & -0.2253 & -0.0353 & -0.0654 & -0.6949 \\ -0.1390 & -0.1134 & -0.0289 & -0.0172 & 2.0973 \\ \hline 0.1214 & 0.0335 & -0.0088 & 0.0252 & -1.5701 \\ -0.0458 & 0.1194 & 0.0161 & -0.0538 & -1.2259 \\ \hline -0.0309 & 0.3206 & & & 0.4700 \end{array}\right],$$

$$\left[\begin{array}{c|c|c} A_{r14} & A_{rd14} & B_{r14} \\ \hline A_{r24} & A_{rd24} & B_{r24} \\ \hline C_{r4} & & B_{r34} \end{array}\right] = \left[\begin{array}{cc|cc|c} -0.0923 & -0.1168 & -0.0232 & -0.0379 & -0.9278 \\ 0.0658 & 0.3308 & -0.0050 & 0.0011 & -0.3380 \\ \hline -0.1338 & 0.0774 & -0.0248 & 0.0524 & -1.0029 \\ -0.0862 & -0.1913 & -0.0050 & 0.0250 & 1.2849 \\ \hline -0.3808 & -0.2745 & & & 0.2460 \end{array}\right].$$

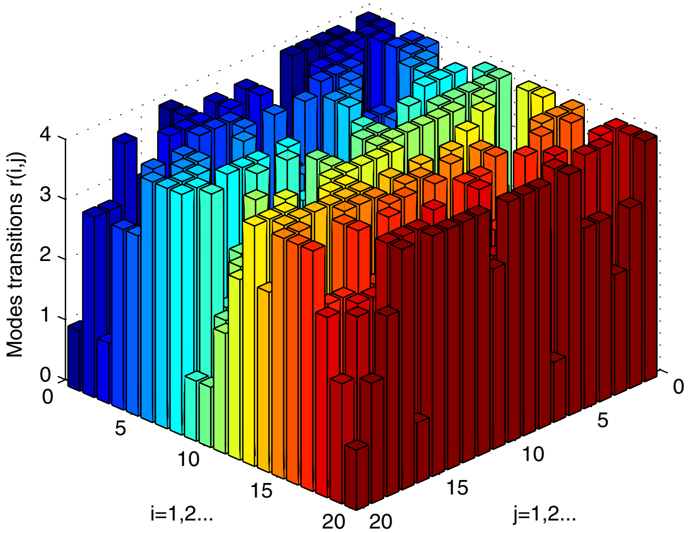


Fig. 1 One possible system mode evolution

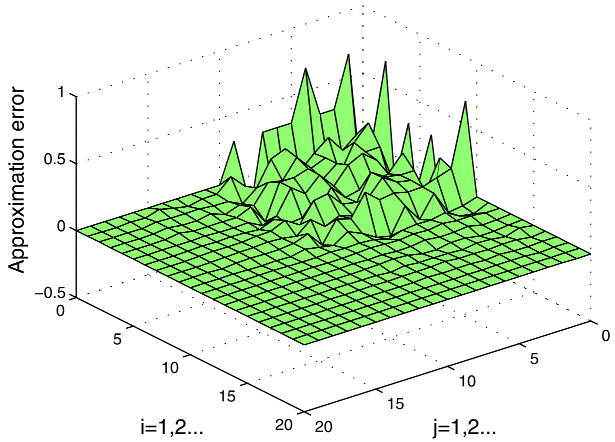


Fig. 2 Output error between the original system and the three-order approximation model with imperfect TPM1 based on Theorem 1 (MALA)

The feasible solutions for the other cases are omitted for brevity.

In order to further illustrate the effectiveness of the proposed approach, we present the simulation results with the above obtained reduced-order models under Case 2 in Table 1. Let the boundary conditions be

$$x(t,i) = x(i,t) = \begin{cases} \begin{bmatrix} -1 \ 1.4 \ 0.5 \ 0.4 \end{bmatrix}^{\mathrm{T}}, & 0 \le i \le 10, \\ \begin{bmatrix} 0 \ 0 \ 0 \ 0 \end{bmatrix}^{\mathrm{T}}, & i > 10, \end{cases}$$

where $-5 \le t \le 0$, and choose the delays $d_1 = 5$ (vertical direction), $d_2 = 5$ (horizontal direction), and the disturbance input $w(i, j)$ as

$$w(i, j) = \begin{cases} 0.2, \ 0 \le i, j \le 10, \\ 0, \quad \text{otherwise.} \end{cases}$$

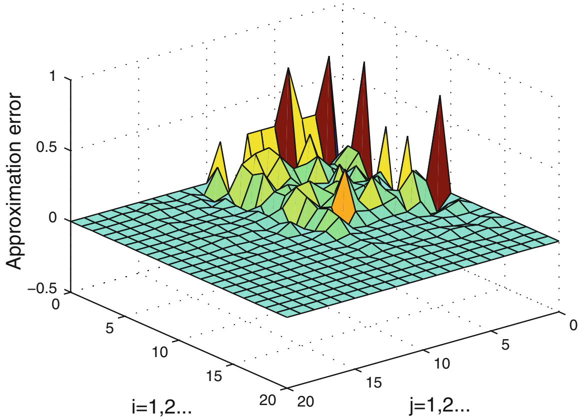


Fig. 3 Output error between the original system and the two-order approximation model with imperfect TPM1 based on Theorem 1 (MALA)

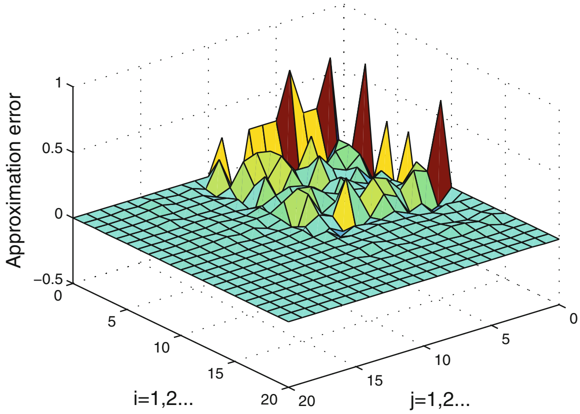


Fig. 4 Output error between the original system and the one-order approximation model with imperfect TPM1 based on Theorem 1 (MALA)

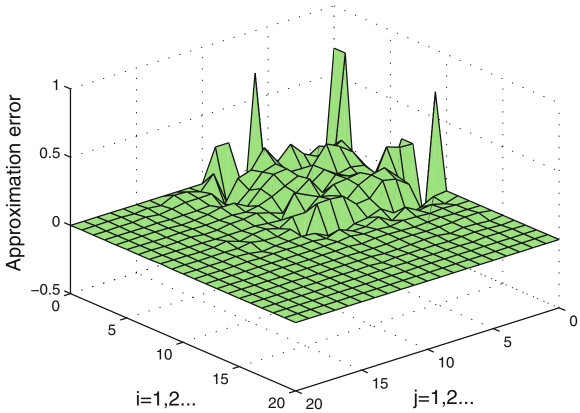


Fig. 5 Output error between the original system and the three-order approximation model with imperfect TPM1 based on Theorem 2 (MAIA)

With one possible realization of the Markovian jumping mode shown in Fig. 1, the output errors between the original system and the approximation models based on Theorem 1 (MALA) and Theorem 2 (MAIA) are displayed in Figs. 2, 3, 4 and 5, 6, 7, respectively. It can be clearly observed from the simulation curves that, despite the existence of the imperfect TPs, the obtained reduced-order models can approximate the original system very well.

Fig. 6 Output error between the original system and the two-order approximation model with imperfect TPM1 based on Theorem 2 (MAIA)

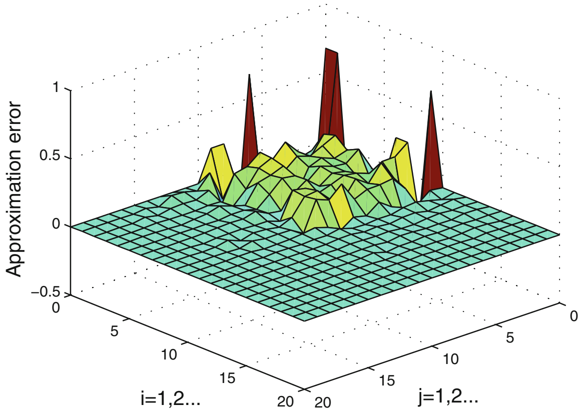
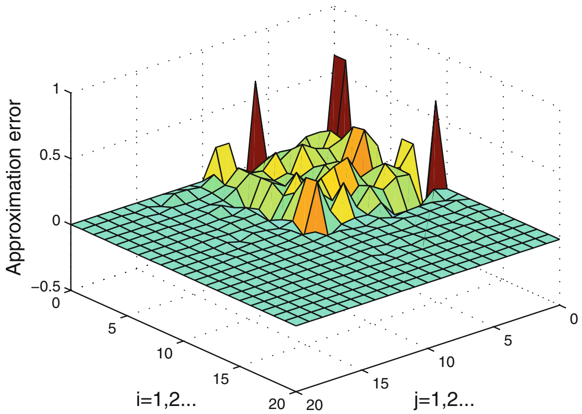


Fig. 7 Output error between the original system and the one-order approximation model with imperfect TPM1 based on Theorem 2 (MAIA)



5 Conclusions

This paper has addressed the problem of $\mathscr{H}_\infty$ model approximation for a class of 2-D MJLSs with state-delays and imperfect mode information. By fully exploring the properties of the TPMs, together with the convexification of uncertain domains, a new $\mathscr{H}_\infty$ performance analysis criterion for the 2-D model error system has been firstly developed. Then, two distinctly different approaches, namely, the convex linearisation approach and iterative approach, have been developed to solve the model approximation problem. It has been shown that the desired approximation models can be obtained by solving a set of strict LMIs or a sequential minimization problem subject to LMI constraints. Simulation studies have been given to illustrate the effectiveness of the proposed methods.

Finally, it is worth mentioning that applications of the proposed results to some real-world complex 2-D systems such as the image data processing and transmission Roesser (1975), thermal processes Xie et al. (2002), gas absorption Xu et al. (2005), and water stream heating Hoang et al. (2005) etc., are part of our future works.

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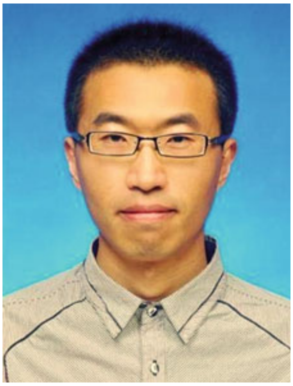

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