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Positive L_1 Observer Design for Positive Switched Systems

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Abstract This paper investigates the problem of L_1 observer design for positive switched systems. Firstly, a new kind of positive L_1 observer is proposed for positive switched linear delay-free systems with observable and unobservable subsystems. Based on the average dwell time approach, a sufficient condition is proposed to ensure the existence of the positive L_1 observer. Under the condition obtained, the estimated error converges to zero exponentially, and the L_1-gain from the disturbance input to the estimated error is less than a prescribed level. Then the proposed design result is extended to positive switched systems with mixed time-varying delays, where the mixed time-varying delays are presented in the form of discrete delay and distributed delay. Finally, two numerical examples are given to demonstrate the feasibility of the obtained results.

Keywords Positive observer · Switched systems · Exponential stability · Time-varying delays · Average dwell time

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1 Introduction

Switched systems, a classical type of hybrid dynamical systems, consist of a family of subsystems and a switching signal. The switching signal coordinates the operation of the various subsystems to define which one to be activated during a certain interval. Switched systems have numerous applications in the control of mechanical systems, the automotive industry, aircraft, air traffic control, switching power converters, chemical processes, and other fields [5,19,20,38–40].

Positive systems mean that their states and outputs are nonnegative whenever the initial conditions and inputs are nonnegative [12,21]. And a positive switched system is a switched system in which each subsystem is itself a positive system. Recently, positive switched systems have been highlighted and investigated by many researchers due to the broad applications in communication systems [30,31], formation flying [11], the viral mutation dynamics under drug treatment [21] and systems theories [1,13,14,27,28]. However, it should be pointed out that to obtain some results on positive switched systems, one has to combine both features of general switched systems and positive systems [3,4,6,15,16,22,26].

On the other hand, it is necessary to design state observers for systems due to the fact that the states of systems are not all measurable in practice. Moreover, a straightforward application of available observer designs for non-positive dynamical systems to positive dynamical systems may not be applicable. This could produce a meaningless state estimation, if there was no non-negative constraint on the state estimation. Thus, imposing a positive restriction on the designed observer for positive dynamical systems is often necessary. Recently, some problems of positive observer design for positive linear systems have been investigated in [8,10,17,27,29,32,35]. It is worth pointing out that the aforementioned literatures do not consider the effect of disturbances. However, disturbance is frequently encountered in practical engineering and social systems. When designing an observer for a given system, the estimated error will be inevitably affected by the disturbance. For positive systems, the L_1-gain index [18,34] can characterize the disturbance rejection property and by means of which we can limit the effect of disturbance in a prescribed level. However, from the authors' best knowledge, the problem of positive L_1 observer design for positive switched systems has not been fully investigated, and the main aim of this paper is to shorten such a gap.

In this paper, we focus our attention on designing a positive L_1 observer for a set of positive switched linear delay-free systems, where the observable and unobservable subsystems coexist. Moreover, it should be noted that delays are universal in practice, and the existence of them may give rise to the deterioration of system performance and instability [7,23,24,33,34,36,37]. Thus, we are also interested in investigating the positive L_1 observer design problem for positive switched systems with mixed time-varying delays.

The main contributions of this paper are threefold: (1) The definition of L_1 observer is given for the first time; (2) By applying the average dwell time approach, a positive L_1 observer design scheme is presented for positive switched linear delay-free systems with both observable and unobservable subsystems; (3) The proposed observer design method for linear delay-free systems is further extended to the case of mixed time-varying delay systems.

The rest of this paper is organized as follows. The problem formulation and some necessary definitions and lemmas are reviewed in Sect. 2. In Sect. 3, based on the average dwell time approach, a sufficient condition for the existence of a positive L_1 observer is established. An extension of the result obtained in Sect. 3 to positive switched systems with mixed time-varying delays is given in Sect. 4. Two numerical examples are provided to demonstrate the effectiveness of the proposed results in Sect. 5. In Sect. 6, concluding remarks are given.

Notations: $A \succeq 0\,(\preceq, \succ, \prec)$ means that all entries of matrix A are nonnegative (non-positive, positive, negative); $A \succ B(A \succeq B)$ means $A - B \succ 0(A - B \succeq 0)$; A^T is the transpose of matrix A; $R(R_+)$ is the set of all real (positive real) numbers; R^n is n-dimensional real vector space; R_+^n is the set of all n-dimensional positive real vectors; $R^{m\times n}$ is the set of all $(m\times n)$-dimensional real matrices. For the vector $x \in R^n$, 1-norm is denoted by $\|x\| = \sum_{l=1}^{n} |x_l|$ where x_l is the l-th element of x; Given $v : R \to R^n$, the L_1 norm is defined by $\|v\|_{L_1} = \int_{t_0}^{\infty} \|v(t)\|\mathrm{d}t$; $1_n \in R^n$ denotes a column vector with n rows containing only 1 entries; $L_1[t_0, \infty)$ is the space of absolute integrable vector-valued functions on $[t_0, \infty)$, i.e., we say $z : [t_0, \infty) \to R^k$ is in $L_1[t_0, \infty)$ if $\int_{t_0}^{\infty} \|z(t)\|dt < \infty$.

2 Problem Formulation

Consider the following switched system

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + D_{\sigma(t)}w(t) + F_{\sigma(t)}u(t), \\ y(t) = C_{\sigma(t)}x(t), \end{cases} \tag{1}$$

where $x(t) \in R^n$ and $y(t) \in R^z$ denote the state and the measured output, respectively; $w(t) \in R^{n_w}$ is the disturbance input, which belongs to $L_1[t_0, \infty)$ and $u(t) \in R^{n_u}$ is the control input; $\sigma(t) : [t_0, \infty) \to \underline{m} = \{1, 2, \ldots, m\}$ is the switching signal with m being the number of subsystems; A_p, C_p, D_p, and F_p, $\forall p \in \underline{m}$, are constant matrices with appropriate dimensions.

Assumption 1 The pairs (A_p, C_p), $p \in Q \subseteq \underline{m} = \{1, 2, ..., m\}$, are observable.

Remark 1 Assumption 1 also indicates that the pairs (A_p, C_p), $p \in \bar{Q} = \underline{m} - Q$, are unobservable, that is to say,

$$(A1) \quad \text{rank} \begin{bmatrix} C_p \\ C_pA_p \\ \vdots \\ C_pA_p^{n-1} \end{bmatrix} = n, \; p \in Q \subseteq \underline{m} = \{1, 2, \ldots, m\};$$

$$(A2)\quad \text{rank}\begin{bmatrix} C_p \\ C_pA_p \\ \vdots \\ C_pA_p^{n-1} \end{bmatrix} < n,\ p \in \bar{Q} = \underline{m} - Q.$$

When $Q = \underline{m}$, Assumption 1 will degenerate into the case that all pairs (A_p, C_p) are observable, which has been investigated in [35].

Definition 1 System (1) is said to be positive if, for any initial conditions $x(t_0) \succeq 0$, any inputs $w(t) \succeq 0$ and $u(t) \succeq 0$, and any switching signals $\sigma(t)$, the corresponding trajectories $x(t) \succeq 0$ and $y(t) \succ 0$ hold for all $t \geq t_0$.

Definition 2 [25] A is called a Metzler matrix if the off-diagonal entries of matrix A are nonnegative.

Lemma 1 [2] *System (1) is positive if and only if A_p are Metzler matrices, and $C_p \succeq 0$, $D_p \succeq 0$ and $F_p \succeq 0$, $\forall p \in \underline{m}$.*

Now, we consider the following observer

$$\begin{cases} \dot{\hat{x}}(t) = A_{\sigma(t)}\hat{x}(t) + F_{\sigma(t)}u(t) + O_{\sigma(t)}(y(t) - \hat{y}(t)), \\ \hat{y}(t) = C_{\sigma(t)}\hat{x}(t), \end{cases} \tag{2}$$

or, equivalently,

$$\begin{cases} \dot{\hat{x}}(t) = (A_{\sigma(t)} - O_{\sigma(t)}C_{\sigma(t)})\hat{x}(t) + F_{\sigma(t)}u(t) + O_{\sigma(t)}C_{\sigma(t)}\hat{x}(t), \\ \hat{y}(t) = C_{\sigma(t)}\hat{x}(t), \end{cases} \tag{3}$$

where $\hat{x}(t) \in R^n$ is the estimated state vector of $x(t)$, $\hat{y}(t) \in R^z$ is the observer output, and $O_p \in R^{n\times z}$ are the observer gain matrices to be determined later.

Remark 2 For a non-positive system, the states of the designed observer are only required to converge to those of the system. However, for positive switched system (1), the positivity of the estimated state $\hat{x}(t)$ of system (2) or (3) should also be guaranteed according to [10,27,29,32,35]. Finally, according to Lemma 1, it is naturally required that $\bar{A}_p = A_p - O_pC_p$ are Metzler matrices, $F_p \succ 0$ and $O_pC_p \succ 0$, $\forall p \in m$.

Define $\tilde{x}(t) = x(t) - \hat{x}(t)$ the estimated error of the system, then we can obtain the following error switched system:

$$\dot{\tilde{x}}(t) = (A_{\sigma(t)} - O_{\sigma(t)}C_{\sigma(t)})\tilde{x}(t) + D_{\sigma(t)}w(t) \tag{4}$$

From Lemma 1, the error dynamic system (4) is a positive switched system if $\bar{A}_p = A_p - O_pC_p$ are Metzler matrices, $D_p \succeq 0$, $\forall p \in m$.

Remark 3 As stated in [35], the positivity requirement on the estimated error $\tilde{x}(t)$ is introduced to be consistent with the state observer case and to facilitate the synthesis of the desired positive observer. It should be pointed out that although this requirement may cause a certain conservatism, the positivity of $\tilde{x}(t)$ will not affect that of the estimated state $\hat{x}(t)$. If the initial condition does not hold, $\hat{x}(t)$ will still remain positive for all $t \geq t_0$.

In order to obtain the main results, we need to recall some definitions.

Definition 3 [33] Switched system (4) with $w(t) = 0$ is said to be exponentially stable under the switching signal $\sigma(t)$, if for any initial conditions, there exist constants $\kappa > 0$ and $\varepsilon > 0$ such that the solution $\tilde{x}(t)$ satisfies

$$||\tilde{x}(t)|| \leq \kappa ||\tilde{x}(t_0)|| e^{-\varepsilon(t-t_0)}, \quad \forall t \geq t_0$$

Definition 4 [9] For any switching signal $\sigma(t)$ and any $T_2 > T_1 \geq 0$, let $N_\sigma(T_1, T_2)$ denote the number of switches of $\sigma(t)$ over the interval $[T_1, T_2)$. For given $T_a > 0$ and $N_0 \geq 0$, if the inequality

$$N_\sigma(T_1, T_2) \leq N_0 + \frac{T_2 - T_1}{T_a}$$

holds, then the positive constant T_a is called the average dwell time, and N_0 is called the chattering bound.

As commonly used in the literature, we choose $N_0 = 0$ in this paper.

Definition 5 For $\lambda > 0$ and $\gamma > 0$, system (2) is said to be a positive L_1 observer of positive switched (1) if the following conditions are satisfied:

(i) system (4) is positive;
(ii) system (4) is exponentially stable when $w(t) = 0$;
(iii) under the zero initial condition, system (4) satisfies

$$\int_{t_0}^{\infty} e^{-\lambda(t-t_0)} \|\tilde{x}(t)\| \, dt \leq \gamma \int_{t_0}^{\infty} \|w(t)\| \, dt, \quad w(t) \neq 0. \tag{5}$$

Remark 4 Due to the fact that disturbance is frequently encountered in practical engineering, when designing an observer for a given system, the estimated error will be inevitably affected by the disturbance. For positive systems, the L_1-gain index can characterize the disturbance rejection property (see [34]), and it is important and necessary to consider the L_1 observer design to attenuate the effect of disturbance in a prescribed level.

Remark 5 In Definition 5, the parameter γ characterizes error switched system's suppression to exogenous disturbance, i.e., it reflects the effect of the exogenous disturbance $w(t)$ to the observer error $\tilde{x}(t)$.

The purpose of this paper is to design the observer in the form of (2) or (3), such that the states of system (2) possess positivity and exponentially converge to those of positive switched system (1). Meanwhile, the L_1-gain from the disturbance $w(t)$ to the observer error $\tilde{x}(t)$ is attenuated in a prescribed level.

3 Observer Design

An observer design scheme for positive switched systems has been proposed in [35]. It is worth pointing out that the design method given in [35] is only applicable for the case that all pairs (A_p, C_p) are observable, and it cannot be directly applied to positive switched system (1) containing some subsystems whose pairs (A_p, C_p) are unobservable. Thus in this section, we present an observer design method for positive switched system (1) satisfying Assumption 1.

Let $T^+(t_0, t)$ denote the total running time of all subsystems with observable pairs (A_p, C_p), $p \in Q$ during $[t_0, t)$. $T^-(t_0, t)$ denotes the total running time of the remainder during $[t_0, t)$.

The following theorem gives the design result.

Theorem 1 *Consider positive switched system* (1) *satisfying Assumption* 1*, for given positive constants* α, β *and* γ*, if there exist vectors,* $\nu_p \in R^n_+$ *and* $g_p \in R^n$*, and any matrices* O_p *of appropriate dimensions, such that,*

$$\bar{A}_p = A_p - O_p C_p \text{ are Metzler matrices}, \; O_p C_p \succeq 0, \quad \forall p \in \underline{m} \tag{6}$$

$$A_p^T \nu_p + \alpha \nu_p - g_p + 1_n \preceq 0, \; D_p^T \nu_p - \gamma 1_{n_w} \preceq 0, \quad \forall p \in Q \tag{7}$$

$$A_p^T \nu_p - \beta \nu_p - g_p + 1_n \preceq 0, \; D_p^T \nu_p - \gamma 1_{n_w} \preceq 0, \quad \forall p \in \bar{Q} \tag{8}$$

$$g_p \preceq C_p^T O_p^T \nu_p, \quad \forall p \in \underline{m} \tag{9}$$

where $\nu_p = [\nu_{p1}, \nu_{p2}, \ldots, \nu_{pn}]^T$*, then system* (2) *is a positive* L_1 *observer of positive switched system* (1) *for any switching signal* $\sigma(t)$ *with the following average dwell time:*

$$\inf_{t > t_0} \frac{T^-(t_0, t)}{T^+(t_0, t)} \geq \frac{\beta + \lambda}{\alpha - \lambda}, \quad T_a > T_a^* = \frac{\ln \mu}{\lambda}, \tag{10}$$

where $0 < \lambda < \alpha$ *and* $\mu \geq 1$ *satisfies*

$$\nu_p \preceq \mu \nu_q, \quad \forall p, q \in \underline{m} \tag{11}$$

Proof It follows from Lemma 1 and (6) that systems (2) and (4) are positive. We construct the following piecewise co-positive type Lyapunov function for positive system (4)

$$V(t) = V_{\sigma(t)}(t) = \tilde{x}^T(t) \nu_{\sigma(t)} \tag{12}$$

Let $t_1 < \cdots < t_l$ denote the switching instants of $\sigma(t)$ over the interval $[t_0, t)$. When $w(t) = 0$, combining (7–9) leads to

$$V_{\sigma(t)}(t) \leq \begin{cases} e^{-\alpha(t - t_k)} V_{\sigma(t_k)}(t_k) \text{ if } \sigma(t) \in Q, \; t \in [t_k, t_{k+1}) \\ e^{\beta(t - t_k)} V_{\sigma(t_k)}(t_k) \text{ if } \sigma(t) \in \overline{Q}, \; t \in [t_k, t_{k+1}) \end{cases} \tag{13}$$

From (11) and (12), at switching instants t_k, $k = 1, 2, \ldots, l$, it holds that

$$V_{\sigma(t_k)}(t_k) \leq \mu V_{\sigma(t_k^-)}(t_k^-). \tag{14}$$

By (10), (13), (14) and Definition 4, for $t \in [t_l, t_{l+1})$, it is not hard to get

$$\begin{aligned} V_{\sigma(t)}(t) &\le e^{-\alpha T^-(t_l,t)+\beta T^+(t_l,t)} V_{\sigma(t_l)}(t_l) \\ &\le \mu e^{-\alpha T^-(t_l,t)+\beta T^+(t_l,t)} V_{\sigma(t_l^-)}(t_l^-) \\ &\le \mu e^{-\alpha T^-(t_{l-1},t)+\beta T^+(t_{l-1},t)} V_{\sigma(t_{l-1})}(t_{l-1}) \\ &\le \cdots \\ &\le (\mu)^{N_\sigma(t_0,t)} e^{-\alpha T^-(t_0,t)+\beta T^+(t_0,t)} V_{\sigma(t_0)}(t_0) \\ &\le e^{-\alpha T^-(t_0,t)+\beta T^+(t_0,t)} e^{(t-t_0)\ln(\mu)/T_a} V_{\sigma(t_0)}(t_0) \\ &\le e^{-(\lambda-\ln(\mu)/T_a)(t-t_0)} V_{\sigma(t_0)}(t_0) \end{aligned} \tag{15}$$

Denoting $\varepsilon_1 = \min_{(j,p)\in \underline{n}\times \underline{m}} \{\nu_{pj}\}$ and $a = \max_{(j,p)\in \underline{n}\times m} \{\nu_{pj}\}$, $n = \{1, 2, \ldots, n\}$, yields

$$V_{\sigma(t)}(t) \ge \varepsilon_1 \|\tilde{x}(t)\| \tag{16}$$

and

$$V_{\sigma(t_0)}(t_0) \le a \|\tilde{x}(t_0)\| . \tag{17}$$

From (15–17), we get

$$\|\tilde{x}(t)\| \le \frac{a}{\varepsilon_1} e^{-(\lambda-\frac{\ln(\mu)}{T_a})(t-t_0)} \|\tilde{x}(t_0)\| . \tag{18}$$

Thus, by denoting $\kappa = a/\varepsilon_1$ and $\varepsilon = \lambda - \frac{\ln(\mu)}{T_a} > 0$, it can be obtained from (18) that

$$\|\tilde{x}\ (t)\| \le \kappa e^{-\varepsilon(t-t_0)} \|\tilde{x}(t_0)\| , \quad \forall t \ge t_0.$$

Therefore, error system (4) with $w(t) = 0$ is exponentially stable for any switching signal with average dwell time (10). In the sequel, we will consider the L_1-gain performance.

When $w(t) \neq 0$ in system (4), following the proof line above, one get from (7) to (9) that

$$V_{\sigma(t)}(t) \le \begin{cases} e^{-\alpha(t-t_k)} V_{\sigma(t_k)}(t_k) - \int_{t_k}^{t} e^{-\alpha(t-s)} \Lambda(s)\mathrm{d}s & \text{if } \sigma(t) \in Q, t \in [t_k, t_{k+1}), \\ e^{\beta(t-t_k)} V_{\sigma(t_k)}(t_k) - \int_{t_k}^{t} e^{\beta(t-s)} \Lambda(s)\mathrm{d}s & \text{if } \sigma(t) \in \bar{Q}, t \in [t_k, t_{k+1}), \end{cases} \tag{19}$$

where $\Lambda(s) = \|\tilde{x}(s)\| - \gamma \|w(s)\|$.

Then, for $t \in [t_l, t_{l+1})$, we have

$$
\begin{aligned}
V_{\sigma(t)}(t) &\leq e^{-\alpha T^-(t_l,t)+\beta T^+(t_l,t)} V_{\sigma(t_l)}(t_l) - \int_{t_l}^{t} e^{-\alpha T^-(s,t)+\beta T^+(s,t)} \Lambda(s)\mathrm{d}s \\
&\leq \mu e^{-\alpha T^-(t_l,t)+\beta T^+(t_l,t)} V_{\sigma(t_l^-)}(t_l^-) - \int_{t_l}^{t} e^{-\alpha T^-(s,t)+\beta T^+(s,t)} \Lambda(s)\mathrm{d}s \\
&\leq \mu e^{-\alpha T^-(t_{l-1},t)+\beta T^+(t_{l-1},t)} V_{\sigma(t_{l-1})}(t_{l-1}) - \int_{t_l}^{t} e^{-\alpha T^-(s,t)+\beta T^+(s,t)} \Lambda(s)\mathrm{d}s \\
&\quad -\mu \int_{t_{l-1}}^{t_l} e^{-\alpha T^-(s,t_{l-1})+\beta T^+(s,t_{l-1})} \Lambda(s)\mathrm{d}s \leq \cdots \\
&\leq (\mu)^{N_\sigma(t_0,t)} e^{-\alpha T^-(t_0,t)+\beta T^+(t_0,t)} V_{\sigma(t_0)}(t_0) \\
&\quad - \int_{t_0}^{t} (\mu)^{N_\sigma(s,t)} e^{-\alpha T^-(s,t)+\beta T^+(s,t)} \Lambda(s)\mathrm{d}s \\
&\leq e^{-\alpha T^-(t_0,t)+\beta T^+(t_0,t)} e^{(t-t_0)\ln(\mu)/T_a} V_{\sigma(t_0)}(t_0) \\
&\quad - \int_{t_0}^{t} (\mu)^{N_\sigma(s,t)} e^{-\alpha T^-(s,t)+\beta T^+(s,t)} \Lambda(s)\mathrm{d}s.
\end{aligned}
\tag{20}
$$

Under the zero initial condition, we have $V_{\sigma(t_0)}(t_0) = 0$, then (20) becomes

$$
0 \leq - \int_{t_0}^{t} (\mu)^{N_\sigma(s,t)} e^{-\alpha T^-(s,t)+\beta T^+(s,t)} \Lambda(s)\mathrm{d}s.
$$

From (10), it is obvious that

$$
-\alpha T^-(t_0,t) + \beta T^+(t_0,t) \leq -\lambda(t-t_0),
$$

It follows that

$$
\int_{t_0}^{t} e^{-\lambda(t-s)} (\mu)^{N_\sigma(s,t)} \Lambda(s)\mathrm{d}s \leq 0.
$$

That is,

$$\int_{t_0}^{t} e^{-\lambda(t-s)}(\mu)^{N_\sigma(s,t)} \|\tilde{x}(s)\| \,\mathrm{d}s \leq \gamma \int_{t_0}^{t} e^{-\lambda(t-s)}(\mu)^{N_\sigma(s,t)} \|w(s)\| \,\mathrm{d}s. \tag{21}$$

Multiplying both sides of (21) by $e^{-N_\sigma(t_0,t)\ln(\mu)}$ yields

$$\int_{t_0}^{t} e^{-\lambda(t-s)} e^{-N_\sigma(t_0,s)\ln(\mu)} \|\tilde{x}(s)\| \,\mathrm{d}s \leq \gamma \int_{t_0}^{t} e^{-\lambda(t-s)} e^{-N_\sigma(t_0,s)\ln(\mu)} \|w(s)\| \,\mathrm{d}s. \tag{22}$$

By Definition 4 and (10), one can obtain

$$\int_{t_0}^{t} e^{-\lambda(t-s)} e^{-\lambda(s-t_0)} \|\tilde{x}(s)\| \,\mathrm{d}s \leq \gamma \int_{t_0}^{t} e^{-\lambda(t-s)} \|w(s)\| \,\mathrm{d}s. \tag{23}$$

Integrating both sides of (23) from $t = t_0$ to ∞ leads to

$$\int_{t_0}^{\infty} e^{-\lambda(t-t_0)} \|\tilde{x}(s)\| \,\mathrm{d}s \leq \gamma \int_{t_0}^{\infty} \|w(s)\| \,ds.$$

Therefore, according to Definition 5, we can conclude that system (2) is a positive L_1 observer of positive switched system (1).

This completes the proof.

Remark 6 It is worth pointing out that (7) and (9) ensure the exponential stability of the error subsystem p with observable pair (A_p, C_p). However, for the subsystem p with unobservable pair (A_p, C_p), we cannot design a gain matrix such that the corresponding error subsystem is exponentially stable. To ensure the stability of error switched system (4), the error subsystem p with unobservable pair (A_p, C_p) is allowed to be unstable with bounded growth. Conditions (8, 9) guarantee that the increase rate of the error subsystem p with unobservable pair (A_p, C_p) is bounded.

Remark 7 Compared with the existing results in the literatures [8,10,17,35], the system considered in this section satisfies Assumption 1, which is universal in the application. Also, the L_1 observer is proposed for the first time to guarantee the robustness with regard to exogenous disturbance $w(t)$.

Remark 8 From Theorem 1, it can be seen that a smaller α and a larger β will be favorable to the feasibility of (7, 8). For given positive constants α and β, if (7, 8) have no feasible solution, we can adjust the parameter α to be smaller or the parameter β to be larger. Following this guideline, a solution to (7, 8) can be found.

We now present the following algorithm for constructing the positive L_1 observer.

Algorithm 1

Step 1. Input the matrices $A_p,\ C_p,\ D_p$ and F_p, $\forall p \in \underline{m}$;

Step 2. Choose parameters $\alpha > 0,\ \beta > 0$ and $\gamma > 0$, and solve (7, 8) to obtain ν_p and g_p;

Step 3. By (6) and (9), one can get the gain matrices O_p;

Step 4. Construct the positive L_1 observer (2), where O_p are the observer gain matrices obtained in Step 3.

4 An Extension

In the section, we will generalize the design method developed in previous section to positive switched systems with mixed time-varying delays described by:

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + A_{d\sigma(t)}x(t-d(t)) + B_{\sigma(t)}\int_{t-h(t)}^{t} x(s)\mathrm{d}s \\ \qquad\quad + D_{\sigma(t)}w(t) + F_{\sigma(t)}u(t), \\ x(t_0+\theta) = \phi(\theta),\ \theta \in [-H, 0], \\ y(t) = C_{\sigma(t)}x(t), \end{cases} \tag{24}$$

where $d(t)$ denotes the time-varying delay which is everywhere time-differentiable and satisfies $0 < d(t) < d_u$ and $\dot{d}(t) \le d_d < 1$ for known constants d_u and d_d; $h(t)$ denotes the time-varying distributed delay which is everywhere time-differentiable and satisfies $0 < h(t) < h_u$ and $\dot{h}(t) \le h_d < 1$ for known constants h_u and h_d; $H = \max\{d_u, h_u\}$, and $\phi(\theta)$ is a vector-valued initial continuous function defined on interval $[-H, 0]$, $H > 0$; other definitions are the same as those of system (1). Moreover, system (24) also meets Assumption 1.

Remark 9 It should be pointed out that Assumption 1 does not indicate that the subsystem $p(p \in Q)$ of system (24) is observable, and it only means that the subsystem p satisfies the rank condition (A1) in Remark 1.

Lemma 2 [18] *System* (24) *is positive if and only if A_p are Metzler matrices, and $A_{dp} \succeq 0,\ B_p \succeq 0,\ C_p \succeq 0,\ D_p \succeq 0$, and $F_p \succeq 0, \quad \forall p \in \underline{m}$.*

Similarly, we consider the following observer for system (24)

$$\begin{cases} \dot{\hat{x}}(t) = A_{\sigma(t)}\hat{x}(t) + A_{d\sigma(t)}\hat{x}(t-d(t)) + B_{\sigma(t)}\int_{t-h(t)}^{t} \hat{x}(s)ds \\ \qquad\quad + F_{\sigma(t)}u(t) + O_{\sigma(t)}(y(t) - \hat{y}(t)), \\ \hat{x}(t_0+\theta) = 0, \quad \theta \in [-H, 0], \\ \hat{y}(t) = C_{\sigma(t)}\hat{x}(t), \end{cases} \tag{25}$$

where $\hat{x}(t) \in R^n$ is the estimated state vector of $x(t)$, $\hat{y}(t) \in R^z$ is the observer output, $O_p \in R^{n\times z}$ are the observer gain matrices to be determined later; we let $\bar{A}_p = A_p - O_pC_p$, $\forall p \in m$. According to Lemma 2, it is naturally required that $\bar{A}_p$ are Metzler matrices, and $A_{dp} \succeq 0,\ B_p \succeq 0,\ F_p \succeq 0,\ O_pC_p \succeq 0$, $\forall p \in m$.

Let $\tilde{x}(t) = x(t) - \hat{x}(t)$ be the estimated error of the system, then we can obtain the following error switched system:

$$\begin{cases} \dot{\tilde{x}}(t) = (A_{\sigma(t)} - O_{\sigma(t)}C_{\sigma(t)})\tilde{x}(t) + A_{d\sigma(t)}\tilde{x}(t-d(t)) \\ \qquad + B_{\sigma(t)} \int_{t-h(t)}^{t} \tilde{x}(s)\mathrm{d}s + D_{\sigma(t)}w(t), \\ \tilde{x}(t_0+\theta) = \phi(\theta), \quad \theta \in [-H, 0]. \end{cases} \tag{26}$$

Moreover, from Lemma 2, the error dynamic system (26) is a positive switched system if $\bar{A}_p = A_p - O_pC_p$ are Metzler matrices, $A_{dp} \succeq 0$, $B_p \succeq 0$, $D_p \succeq 0$, $p \in \underline{m}$, for any initial conditions $\phi(\theta), \succeq 0\, \theta \in [-H, 0]$.

Definition 6 [33] switched system (26) with $w(t) = 0$ is said to be exponentially stable under $\sigma(t)$, if for the initial condition $\tilde{x}(t_0 + \theta) = \phi(\theta)$, $\theta \in [-H, 0]$, there exist constants $\kappa > 0$ and $\varepsilon > 0$ such that the solution $\tilde{x}(t)$ satisfies

$$\|\tilde{x}\ (t)\| \leq \kappa \left\|\tilde{x}(t_0)\right\|_c e^{-\varepsilon(t-t_0)}, \quad \forall t \geq t_0 \tag{27}$$

where $\|\tilde{x}(t_0)\|_c = \sup\limits_{-H\leq\theta\leq0} \{\|\tilde{x}(t_0+\theta)\|\}$.

To obtain the result, we first present two lemmas for the following non-switched positive system with mixed delays, which will be essential for later development:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-d(t)) + B \int_{t-h(t)}^{t} x(s)\mathrm{d}s, \\ x(t_0+\theta) = \phi(\theta), \quad \theta \in [-H, 0], \end{cases} \tag{28}$$

where A is a Metzler constant matrix, $A_d \succeq 0$ and $B \succeq 0$ are constant matrices; $d(t)$ and $h(t)$ have been defined before.

Choose the following co-positive type Lyapunov–Krasovskii functional candidate for system (28)

$$V(t, x(t)) = V_1(t, x(t)) + V_2(t, x(t)) + V_3(t, x(t)) \tag{29}$$

where

$$V_1(t, x(t)) = x^T(t)\nu, \ V_2(t, x(t)) = \int_{t-d(t)}^{t} e^{\alpha(-t+s)}x^T(s)\upsilon\mathrm{d}s,$$

$$V_3(t, x(t)) = \int_{t-h(t)}^{t}\int_{s}^{t} e^{\alpha(-t+\theta)}x^T(\theta)\vartheta\mathrm{d}\theta\mathrm{d}s,$$

and $\nu,\ \upsilon,\ \vartheta \in R_+^n$, $\alpha > 0$.

For simplicity, $V(t, x(t))$ is written as $V(t)$ in this paper.

Lemma 3 *Given a positive constant* α*, if there exist* $\nu,\ \upsilon,\ \vartheta \in R_+^n$ *such that*

$$A^T\nu + \alpha\nu + \upsilon + h_u\vartheta \preceq 0 \tag{30}$$

$$A_d^T\nu - (1-d_d)e^{-\alpha d_u}\upsilon \preceq 0 \tag{31}$$

$$B^T\nu - (1-h_d)e^{-\alpha h_u}\vartheta \preceq 0 \tag{32}$$

where $\nu = [\nu_1, \nu_2, \ldots, \nu_n]^T$, $\upsilon = [\upsilon_1, \upsilon_2, \ldots, \upsilon_n]^T$, *and* $\vartheta = [\vartheta_1, \vartheta_2, \ldots, \vartheta_n]^T$, *then along the trajectory of system* (28)*, we have*

$$V(t) \leq e^{-\alpha(t-t_0)}V(t_0).$$

Proof Along the trajectory of system (28) with co-positive type Lyapunov–Krasovskii functional (29), we have

$$\dot{V}_1(t) = \dot{x}^T(t)\nu = x^T(t)A^T\nu + x^T(t-d(t))A_d^T\nu + \left[\int\limits_{t-h(t)}^{t} x^T(s)\mathrm{d}s\right]B^T\nu$$

$$\begin{aligned}
\dot{V}_2(t) &= -\alpha\int\limits_{t-d(t)}^{t} e^{\alpha(-t+s)}x^T(s)\upsilon\mathrm{d}s + x^T(t)\upsilon - (1-\dot{d}(t))e^{-\alpha d(t)}x^T(t-d(t))\upsilon\\
&\leq -\alpha\int\limits_{t-d(t)}^{t} e^{\alpha(-t+s)}x^T(s)\upsilon\mathrm{d}s + x^T(t)\upsilon - (1-d_d)e^{-\alpha d_u}x^T(t-d(t))\upsilon\\
\dot{V}_3(t) &= -\alpha\int\limits_{t-h(t)}^{t}\int\limits_{s}^{t} e^{\alpha(-t+\theta)}x^T(\theta)\vartheta\mathrm{d}\theta\mathrm{d}s + x^T(t)\vartheta\int\limits_{t-h(t)}^{t}\mathrm{d}s\\
&\quad -(1-\dot{h}(t))\int\limits_{t-h(t)}^{t} e^{\alpha(-t+s)}x^T(s)\vartheta\mathrm{d}s\\
&\leq -\alpha\int\limits_{t-h(t)}^{t}\int\limits_{s}^{t} e^{\alpha(-t+\theta)}x^T(\theta)\vartheta d\theta\mathrm{d}s + h_u x^T(t)\vartheta\\
&\quad -(1-h_d)\int\limits_{t-h(t)}^{t} e^{-\alpha h_u}x^T(s)\vartheta\mathrm{d}s
\end{aligned}$$

Then, we have

$$\dot{V}(t) + \alpha V(t) \leq x^T(t)(A^T\nu + \alpha\nu + \upsilon + h_u\vartheta) + x^T(t - d(t))\left(A_d^T\nu - (1 - d_d)e^{-\alpha d_u}\upsilon\right) + \int_{t-h(t)}^{t} x^T(s)\left[B^T\nu - (1 - h_d)e^{-\alpha h_u}\vartheta\right]\mathrm{d}s$$

From (30) to (33), one obtains

$$\dot{V}(t) \leq -\alpha V(t). \tag{33}$$

Then, along the trajectory of system (28), we have

$$V(t) \leq e^{-\alpha(t-t_0)}V(t_0).$$

This completes the proof.

Remark 10 It should be noted that the proposed Lyapunov–Krasovskii functional (29) is different from the existing one presented in the literature [34]. In order to deal with the time-varying distributed delay, the following term is constructed:

$$V_3(t, x(t)) = \int_{t-h(t)}^{t}\int_{s}^{t} e^{\alpha(-t+\theta)}x^T(\theta)\vartheta\,\mathrm{d}\theta\,\mathrm{d}s$$

Lemma 4 *For a given positive constant β, if there exist vectors $\nu,\ \upsilon,\ \vartheta \in R_+^n$ such that*

$$A^T\nu - \beta\nu + \upsilon + h_u\vartheta \preceq 0 \tag{34}$$

$$A_d^T\nu - (1 - d_d)\upsilon \preceq 0 \tag{35}$$

$$B^T\nu - (1 - h_d)\vartheta \preceq 0 \tag{36}$$

where $\nu = [\nu_1, \nu_2, \ldots, \nu_n]^T$, $\upsilon = [\upsilon_1, \upsilon_2, \ldots, \upsilon_n]^T$, and $\vartheta = [\vartheta_1, \vartheta_2, \ldots, \vartheta_n]^T$, then along the trajectory of system (28)*, we have*

$$V(t) \leq e^{\beta(t-t_0)}V(t_0).$$

Proof Choose the following co-positive type Lyapunov–Krasovskii functional candidate for system (28)

$$V(t, x(t)) = V_1(t, x(t)) + V_2(t, x(t)) + V_3(t, x(t)) \tag{37}$$

where

$$V_1(t, x(t)) = x^T(t)\nu,\ V_2(t, x(t)) = \int\limits_{t-d(t)}^{t} e^{\beta(t-s)} x^T(s)\upsilon \mathrm{d}s,$$

$$V_3 = \int\limits_{t-h(t)}^{t} \int\limits_{s}^{t} e^{\beta(t-\theta)} x^T(\theta)\vartheta \mathrm{d}\theta \mathrm{d}s,$$

and $\nu,\ \upsilon,\ \vartheta \in R^n_+,\ \beta > 0$.

The rest proof of this lemma is similar to that of Lemma 3, and thus is omitted here. This completes the proof.

Theorem 2 *Consider positive switched system (24) satisfying Assumption 1, for given positive constants α, γ, and β, if there exist vectors ν_p, υ_p, $\vartheta_p \in R^n_+$, and $g_p \in R^n$, and any matrices O_p of appropriate dimensions, such that*

$$\bar{A}_p = A_p - O_p C_p \text{ are Metzler matrices, } O_p C_p \succeq 0, \quad \forall p \in \underline{m}, \tag{38}$$

$$A_p^T \nu_p + \alpha\nu_p - g_p + \upsilon_p + h_u\vartheta_p + 1_n \preceq 0, \quad D_p^T \nu_p - \gamma 1_{n_w} \preceq 0, \quad \forall p \in Q, \tag{39}$$

$$A_{dp}^T \nu_p - (1 - d_d)e^{-\alpha d_u}\upsilon_p \preceq 0, \quad B_p^T \nu_p - (1 - h_d)e^{-\alpha h_u}\vartheta_p \preceq 0, \quad \forall p \in Q, \tag{40}$$

$$A_p^T \nu_p - \beta\nu_p - g_p + \upsilon_p + h_u\vartheta_p + 1_n \preceq 0,\ D_p^T \nu_p - \gamma 1_{n_w} \preceq 0, \quad \forall p \in \bar{Q}, \tag{41}$$

$$A_{dp}^T \nu_p - (1 - d_d)\upsilon_p \preceq 0, \quad B_p^T \nu_p - (1 - h_d)\vartheta_p \preceq 0, \quad \forall p \in \bar{Q}, \tag{42}$$

$$g_p \preceq C_p^T O_p^T \nu_p, \quad \forall p \in m, \tag{43}$$

where $\nu_p = [\nu_{p1}, \nu_{p2}, \ldots, \nu_{pn}]^T$, $\upsilon_p = [\upsilon_{p1}, \upsilon_{p2}, \ldots, \upsilon_{pn}]^T$ and $\vartheta_p = [\vartheta_{p1}, \vartheta_{p2}, \ldots, \vartheta_{pn}]^T$, then system (25) is a positive L_1 observer of positive switched system (24) for any switching signal $\sigma(t)$ with the following average dwell time:

$$\inf_{t>t_0} \frac{T^-(t_0, t)}{T^+(t_0, t)} \geq \frac{\beta + \lambda}{\alpha - \lambda}, \quad T_a > T_a^* = \frac{\ln(\mu\varsigma)}{\lambda} \tag{44}$$

where $\varsigma = e^{(\alpha+\beta)H}$, $0 < \lambda < \alpha$, and $\mu \geq 1$ satisfy

$$\nu_p \preceq \mu\varsigma\nu_q,\ \upsilon_p \preceq \mu\upsilon_q,\ \vartheta_p \preceq \mu\vartheta_q, \quad \forall p, q \in \underline{m} \tag{45}$$

Proof It follows from Lemma 2 and (38) that systems (25) and (26) are positive. Construct the following piecewise co-positive type Lyapunov–Krasovskii functional for system (26)

$$V(t)=V_{\sigma(t)}(t)=\begin{cases}\tilde{x}^T(t)\nu_{\sigma(t)}+\int_{t-d(t)}^{t}e^{\alpha(-t+s)}\tilde{x}^T(s)\upsilon_{\sigma(t)}\mathrm{d}s\\ \quad+\int_{t-h(t)}^{t}\int_{s}^{t}e^{\alpha(-t+\theta)}\tilde{x}^T(\theta)\vartheta_{\sigma(t)}\mathrm{d}\theta\mathrm{d}s, \quad \text{if } \sigma(t)\in Q\\ \tilde{x}^T(t)\nu_{\sigma(t)}+\int_{t-d(t)}^{t}e^{\beta(t-s)}\tilde{x}^T(s)\upsilon_{\sigma(t)}\mathrm{d}s\\ \quad+\int_{t-h(t)}^{t}\int_{s}^{t}e^{\beta(t-\theta)}\tilde{x}^T(\theta)\vartheta_{\sigma(t)}\mathrm{d}\theta\mathrm{d}s, \quad \text{if } \sigma(t)\in\bar{Q}\end{cases} \tag{46}$$

Let $t_1<\cdots<t_l$ denote the switching instants of $\sigma(t)$ over the interval $[t_0,t)$. When $w(t)=0$, by Lemmas 3 and 4, one can easily have from (39) to (43) that

$$V_{\sigma(t)}(t)\leq\begin{cases}e^{-\alpha(t-t_k)}V_{\sigma(t_k)}(t_k) & \text{if } \sigma(t)\in Q,\ t\in[t_k,t_{k+1})\\ e^{\beta(t-t_k)}V_{\sigma(t_k)}(t_k) & \text{if } \sigma(t)\in\bar{Q},\ t\in[t_k,t_{k+1})\end{cases} \tag{47}$$

From (45) and (46), at switching instants $t_k,\ k=1,2,\ldots,l$, it holds that

$$V_{\sigma(t_k)}(t_k)\leq\mu\varsigma V_{\sigma(t_k^-)}(t_k^-), \tag{48}$$

where $\varsigma=e^{(\alpha+\beta)H}$.

Following the proof line of Theorem 1, we have

$$\|\tilde{x}(t)\|\leq\frac{b}{\varepsilon_1}e^{-(\lambda-\frac{\ln(\mu\varsigma)}{T_a})(t-t_0)}\sup_{-H\leq\theta\leq0}\{\|\tilde{x}(t_0+\theta)\|\}, \tag{49}$$

where

$$\varepsilon_1=\min_{(j,p)\in\underline{n}\times\underline{m}}\left\{\nu_{pj}\right\},$$

$$b=\max_{(j,p)\in\underline{n}\times\underline{m}}\left\{\nu_{pj}\right\}+e^{\beta d_u}H\max_{(j,p)\in\underline{n}\times\underline{m}}\left\{\upsilon_{pj}\right\}+e^{\beta h_u}Hh_u\max_{(j,p)\in\underline{n}\times\underline{m}}\left\{\vartheta_{pj}\right\}.$$

Thus, by denoting $\kappa=b/\varepsilon_1$ and $\varepsilon=\lambda-\frac{\ln(\mu\varsigma)}{T_a}>0$, it can be obtained from (49) that

$$\|\tilde{x}\ (t)\|\ \leq\kappa e^{-\varepsilon(t-t_0)}\,\|\tilde{x}(t_0)\|_c\,,\quad\forall t\geq t_0,$$

where $\|\tilde{x}(t_0)\|_c=\sup_{-H\leq\theta\leq0}\{\|\tilde{x}(t_0+\theta)\|\}$.

Therefore, error system (26) with $w(t)=0$ is exponentially stable for any switching signal with average dwell time (44). In the sequel, we will consider the L_1-gain performance.

When $w(t)\neq0$ in system (26), one can get from (39) to (43) that

$$V_{\sigma(t)}(t)\leq\begin{cases}e^{-\alpha(t-t_k)}V_{\sigma(t_k)}(t_k)-\int_{t_k}^{t}e^{-\alpha(t-s)}\Lambda(s)\mathrm{d}s & \text{if } \sigma(t)\in Q,\ t\in[t_k,t_{k+1}),\\ e^{\beta(t-t_k)}V_{\sigma(t_k)}(t_k)-\int_{t_k}^{t}e^{\beta(t-s)}\Lambda(s)\mathrm{d}s & \text{if } \sigma(t)\in\bar{Q},\ t\in[t_k,t_{k+1}),\end{cases} \tag{50}$$

where $\Lambda(s)=\|\tilde{x}(s)\|-\gamma\,\|w(s)\|$.

Following the proof line of Theorem 1, we have

$$\int_{t_0}^{\infty} e^{-\lambda(t-t_0)} \|\tilde{x}(s)\| \,\mathrm{d}s \le \gamma \int_{t_0}^{\infty} \|w(s)\| \,\mathrm{d}s$$

Therefore, according to Definition 5, system (25) is a positive L_1 observer of positive switched system (24).

This completes the proof. □

We now present the following algorithm to construct a positive L_1 observer for positive switched system (24).

Algorithm 2

Step 1. Input the matrices A_p, A_{dp}, B_p, C_p, D_p, and F_p, $\forall p \in m$;

Step 2. Choose the parameters $\alpha > 0$, $\beta > 0$, and $\gamma > 0$, and solve (39–42) to obtain ν_p, υ_p, ϑ_p, and g_p;

Step 3. From (38) and (43), one can get the gain matrices O_p;

Step 4. Construct the positive L_1 observer (25), where O_p are the observer gain matrices obtained in Step 3.

5 Numerical Examples

In this section, two simulation examples are provided to demonstrate the effectiveness of the proposed approaches.

Example 1 Consider positive switched system (1) with the following parameters:

$$\text{subsystem 1}: A_1 = \begin{bmatrix} -7 & 6 \\ 6 & -8 \end{bmatrix},\ C_1 = \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.2 \end{bmatrix},\ D_1 = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix},\ F_1 = \begin{bmatrix} 0.2 \\ 0.5 \end{bmatrix},$$

$$\text{subsystem 2}: A_2 = \begin{bmatrix} -3 & 1.5 \\ 1.5 & -3 \end{bmatrix},\ C_2 = \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.2 \end{bmatrix},\ D_2 = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix},\ F_2 = \begin{bmatrix} 0.2 \\ 0.6 \end{bmatrix}.$$

It is easy to verify that the pair (A_2, C_2) is unobservable. Taking $\alpha = 0.3$, $\beta = 0.5$, $\gamma = 0.5$, $\lambda = 0.15$, $u(t) = 2|\sin t|$, and $w(t) = 0.05e^{-0.05t}$, and solving (7)-(8) in Theorem 1 give rise to

$$\nu_1 = \begin{bmatrix} 0.8798 \\ 0.9612 \end{bmatrix},\ \nu_2 = \begin{bmatrix} 0.9612 \\ 0.8798 \end{bmatrix},\ g_1 = \begin{bmatrix} 1.9158 \\ -0.0803 \end{bmatrix},\ g_2 = \begin{bmatrix} -0.0020 \\ 0.4054 \end{bmatrix}.$$

Then by Step 3 in Algorithm 1, the observer gain matrices can be obtained as

$$O_1 = \begin{bmatrix} 145.5690 & -128.7996 \\ 135.5416 & -118.4527 \end{bmatrix},\ O_2 = \begin{bmatrix} 3.1476 & 2.9579 \\ 5.2511 & 1.9061 \end{bmatrix}.$$

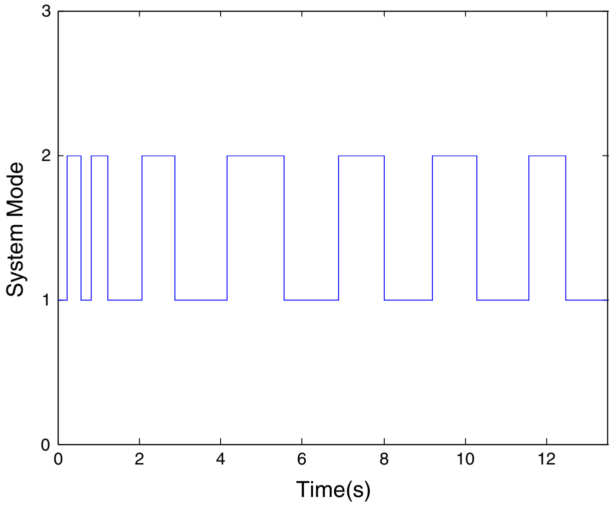

Fig. 1 Switching signal in Example 1

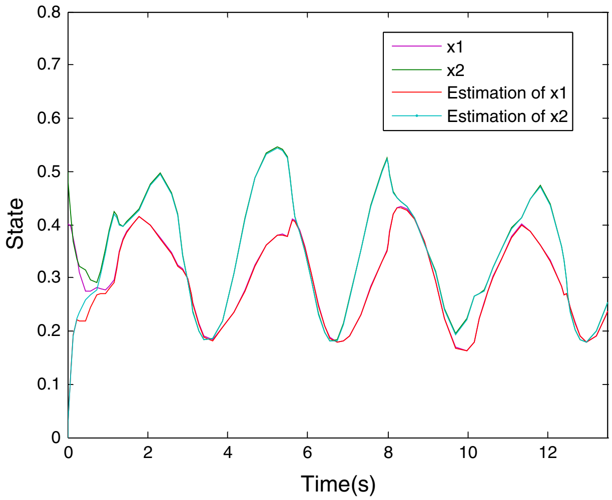

Fig. 2 State $x(t)$ of the system and its estimation $\hat{x}(t)$ in Example 1

According to (10, 11), we can get $\mu = 1.1400$, and $T_a^* = 0.8737$.

Figure 1 shows the switching signal with the average dwell time $T_a = 0.9$. The system state $x(t)$ and its estimation $\hat{x}(t)$ are shown in Fig. 2, where the initial states of the system are $x(0) = \begin{bmatrix} 0.4 & 0.5 \end{bmatrix}^T$, and the initial states of the observer are $\hat{x}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$. The estimated errors are shown in Fig. 3. From Figs. 1, 2 and 3, we can see that the states of the designed observer not only possess the positivity, but also approximate those of the original system (1).

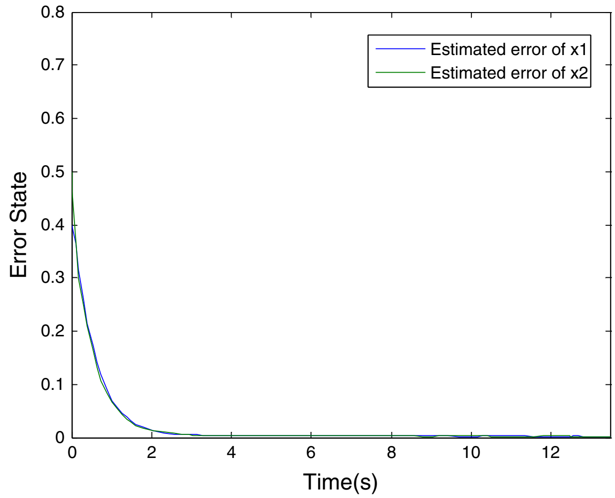

Fig. 3 Estimated errors in Example 1

Example 2 Consider positive switched system (24) with the following parameters:

$$\text{subsystem 1}: A_1 = \begin{bmatrix} -7 & 6 \\ 6 & -8 \end{bmatrix}, \ A_{d1} = \begin{bmatrix} 0.1 & 0.0 \\ 0.1 & 0.0 \end{bmatrix}, \ B_1 = \begin{bmatrix} 0.2 & 0.0 \\ 0.2 & 0.0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.2 \end{bmatrix}, \ D_1 = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \ F_1 = \begin{bmatrix} 0.2 \\ 0.5 \end{bmatrix},$$

$$\text{subsystem 2}: A_2 = \begin{bmatrix} -3 & 1.5 \\ 1.5 & -3 \end{bmatrix}, \ A_{d2} = \begin{bmatrix} 0.1 & 0.0 \\ 0.1 & 0.0 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0.2 & 0.0 \\ 0.2 & 0.0 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.2 \end{bmatrix}, \ D_2 = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \ F_2 = \begin{bmatrix} 0.2 \\ 0.6 \end{bmatrix}.$$

By Lemma 2, the trajectories of such a system will remain positive if $\phi(\theta) \succeq 0$, $\theta \in [-H, 0]$. It is easy to verify that rank $\begin{bmatrix} C_2 \\ C_2 A_2 \end{bmatrix} < 2$. Taking $\alpha = 0.3$, $\beta = 0.5$, $\gamma = 0.5$, $\lambda = 0.15$, $d(t) = 0.1\,|\sin t|$, $h(t) = 0.1\,|\sin t|$, $u(t) = 2\,|\sin t|$, and $w(t) = 0.05e^{-0.05t}$, and solving (39–42) in Theorem 2 give rise to

$$\nu_1 = \begin{bmatrix} 0.8563 \\ 0.9434 \end{bmatrix}, \ \nu_2 = \begin{bmatrix} 0.9497 \\ 0.8624 \end{bmatrix}, \ \upsilon_1 = \begin{bmatrix} 1.2638 \\ 1.1747 \end{bmatrix}, \ \upsilon_2 = \begin{bmatrix} 1.2504 \\ 1.1603 \end{bmatrix},$$

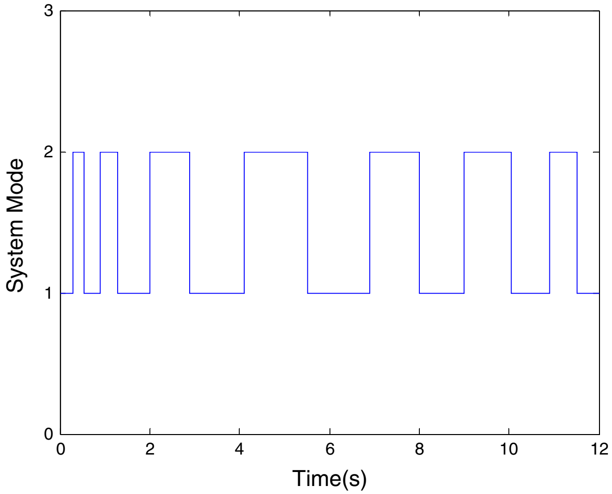

Fig. 4 Switching signal in Example 2

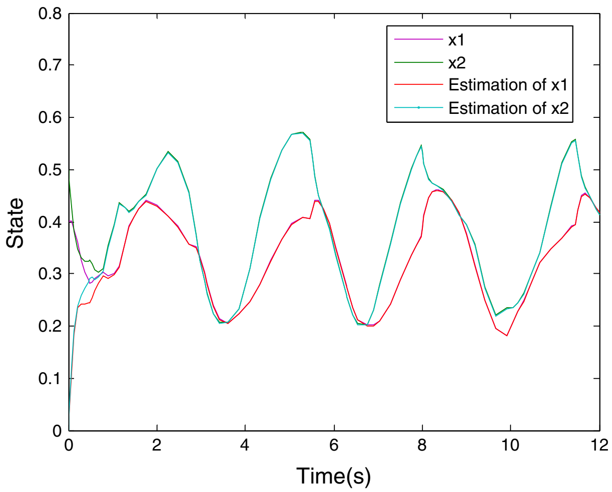

Fig. 5 State $x(t)$ of the system and its estimation $\hat{x}(t)$ in Example 2

$$\vartheta_1 = \begin{bmatrix} 1.3530 \\ 1.1747 \end{bmatrix}, \ \vartheta_2 = \begin{bmatrix} 1.3405 \\ 1.1603 \end{bmatrix}, \ g_1 = \begin{bmatrix} -3.4276 \\ -1.2712 \end{bmatrix}, \ g_2 = \begin{bmatrix} -1.4595 \\ -1.7878 \end{bmatrix}.$$

Then, by Step 3 in Algorithm 2, the observer gain matrices can be obtained as

$$O_1 = \begin{bmatrix} 148.7714 & -132.3642 \\ 183.6611 & -164.4395 \end{bmatrix}, \ O_2 = \begin{bmatrix} -116.2294 & 65.2255 \\ 51.3943 & -18.5864 \end{bmatrix}.$$

According to (44, 45), we can get $\mu = 1.0344$, $\varsigma = 1.0833$, and $T_a^* = 0.7591$.

Figure 4 shows the switching signal with the average dwell time $T_a = 0.8$. The system state $x(t)$ and its estimation $\hat{x}(t)$ are shown in Fig. 5, where the initial states of

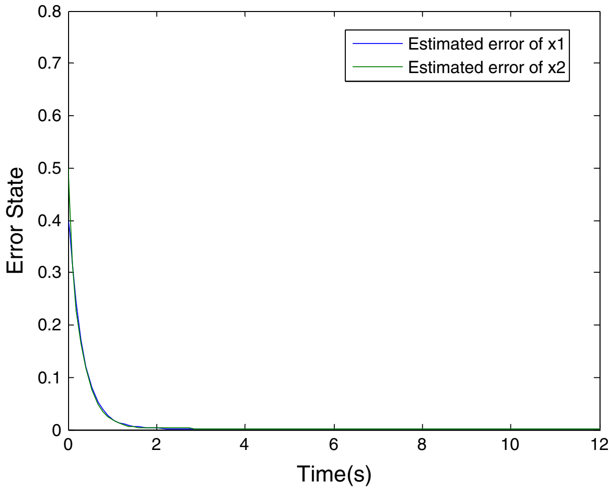

Fig. 6 Estimated errors in Example 2

the system are $x(t) = \begin{bmatrix} 0 \ 0 \end{bmatrix}^T, t \in [-H, 0), x(0) = \begin{bmatrix} 0.4 \ 0.5 \end{bmatrix}^T$, and the initial states of the observer are $\hat{x}(t) = \begin{bmatrix} 0 \ 0 \end{bmatrix}^T$, $t \in [-H, 0]$. The estimated errors are shown in Fig. 6. From Figs. 4, 5 and 6, we can see that the states of the designed observer not only possess the positivity, but also approximate those of the original system (24).

6 Conclusions

In this paper, we investigated the problem of L_1 observer design for positive switched systems. First, we have studied the positive L_1 observer design problem for positive switched delay-free systems with observable and unobservable subsystems. By constructing a piecewise co-positive type Lyapunov function and using the average dwell time approach, we proposed a positive observer design scheme. The states of the designed observer not only remain positivity but also converge to those of the original switched system, and the L_1-gain from the disturbance input to the estimate error is less than a prescribed level. Then we extended the proposed design method to positive switched systems with mixed time-varying delays. Finally, two numerical examples were presented to demonstrate the feasibility and effectiveness of the proposed method.

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