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A third integrating factor for indefinite integrals of special functions

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ABSTRACT
An integrating factor $\tilde{f}(x)$ is presented involving the terms in $y''(x)$ and $q(x)y(x)$ of the general homogenous second-order linear ordinary differential equation. The new integrating factors obey second-order differential equations, rather than being given by quadrature. The new factors provides old and new integrals for special functions which obey such differential equations. The functions considered here are cylinder functions, parabolic cylinder functions and Whittaker functions. All the integrals given have been checked using Mathematica.

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1. Introduction

In a recent paper [1], two integrating factors were considered for the general second-order homogeneous linear differential equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0. \tag{1.1}$$

Multiplication of Equation (1.1) by either of these integrating factors reduces two consecutive terms of the equation to a derivative, allowing an integral to be derived from the remaining term, and this integral can be generalized. The two leftmost terms of Equation (1.1) have an integrating factor which obeys the differential equation

$$f'(x) = p(x)f(x) \tag{1.2}$$

so that

$$f(x) = \exp\left(\int p(x)\,\mathrm{d}x\right) \tag{1.3}$$

and hence

$$f(x)q(x)y(x) = -\left[f(x)y'(x)\right]'. \tag{1.4}$$

The integrating factor $f(x)$ is identical to the Lagrangian factor introduced for Equation (1.1) in [1,3]. Multiplying both sides of Equation (1.4) by $m(f(x)y'(x))^{m-1}$ for

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$m \in \mathbb{C}$ and $m \neq 0$ gives the generalized integral

$$\int q(x) f^m(x) y'^{m-1}(x) y(x) \, \mathrm{d}x = -\frac{f^m(x) y'^m(x)}{m}. \tag{1.5}$$

For $p(x) \neq 0$, a second integrating factor [1] can be obtained for the last two terms in Equation (1.1). Expressing this equation in the form

$$y''(x) + p(x) \left[y'(x) + \frac{q(x)}{p(x)} y(x) \right] = 0 \tag{1.6}$$

then the integrating factor [1]

$$\hat{f}(x) = \exp\left(\int \frac{q(x)}{p(x)} \, \mathrm{d}x \right), \tag{1.7}$$

which obeys the differential equation

$$\hat{f}'(x) = \frac{q(x)}{p(x)} \hat{f}(x) \tag{1.8}$$

reduces Equation (1.6) to

$$\frac{\hat{f}(x) y''(x)}{p(x)} = -\left[\hat{f}(x) y(x) \right]'. \tag{1.9}$$

Multiplying Equation (1.9) by $m(\hat{f}(x)y(x))^{m-1}$ and integrating gives the generalized integral

$$\int \frac{1}{p(x)} \hat{f}^m(x) y^{m-1}(x) y''(x) \, \mathrm{d}x = -\frac{\hat{f}^m(x) y^m(x)}{m}, \tag{1.10}$$

which was considered in detail in [1].

1.1. A third integrating factor

This paper considers the case of a third integrating factor $\tilde{f}(x)$ derived for the leftmost and rightmost terms in Equation (1.1). This factor is somewhat different from the other two, and is obtained from the elementary differential identity

$$\tilde{f}(x) y''(x) - \tilde{f}''(x) y(x) = \left(\tilde{f}(x) y'(x) - \tilde{f}'(x) y(x) \right)'. \tag{1.11}$$

Multiplying Equation (1.1) by a function $\tilde{f}(x)$, initially considered arbitrary, gives

$$\tilde{f}(x) y''(x) + \tilde{f}(x) p(x) y'(x) + \tilde{f}(x) q(x) y(x) = 0, \tag{1.12}$$

and for the leftmost and rightmost terms in Equation (1.12) to reduce to a single derivative through Equation (1.11), then $\tilde{f}(x)$ must be chosen to obey the differential equation

$$\tilde{f}''(x) + q(x) \tilde{f}(x) = 0. \tag{1.13}$$

Unlike Equations (1.2) and (1.8), Equation (1.13) is second-order and hence the integrating factor $\tilde{f}(x)$ has two independent solutions. Substituting Equation (1.13) into

Equation (1.12) gives

$$\tilde{f}(x)\,p(x)\,y'(x) = \left(\tilde{f}'(x)\,y(x) - \tilde{f}(x)\,y'(x)\right)' \tag{1.14}$$

and hence gives the integral

$$\int \tilde{f}(x)\,p(x)\,y'(x)\;\mathrm{d}x = \tilde{f}'(x)\,y(x) - \tilde{f}(x)\,y'(x)\,. \tag{1.15}$$

Multiplying Equation (1.14) by $m(\tilde{f}'(x)y(x) - \tilde{f}(x)y'(x))^{m-1}$ for $m \in \mathbb{C}$ and $m \neq 0$ then gives the generalized integral

$$\int \tilde{f}(x)\,p(x)\,y'(x)\left(\tilde{f}'(x)\,y(x) - \tilde{f}(x)\,y'(x)\right)^{m-1}\;\mathrm{d}x = \frac{\left(\tilde{f}'(x)\,y(x) - \tilde{f}(x)\,y'(x)\right)^{m}}{m}. \tag{1.16}$$

Equation (1.15) is not the only manner in which an integrating factor of this type can be used. Some special functions have baseline differential equations of the form

$$y''(x) + q(x)\,y(x) = 0, \tag{1.17}$$

where $p(x)$ in Equation (1.1) is zero and $q(x)$ often can be split into two separate factors or groups of factors, so that

$$y''(x) + \left(q_1(x) + q_2(x)\right) y(x) = 0. \tag{1.18}$$

The $p(x)$ factor for any differential equation of the form (1.1) can be transformed away [2,3] to give an Equation such as (1.17). We can define an integrating factor $\tilde{f}_1(x)$ which obeys the equation

$$\tilde{f}_1''(x) + q_1(x)\tilde{f}_1(x) = 0, \tag{1.19}$$

which can be considered a fragment, or fragmentary equation, of Equation (1.18) in the sense discussed in [2,3]. Multiplying Equation (1.18) by $\tilde{f}_1(x)$ gives

$$\tilde{f}_1(x)\,q_2(x)\,y(x) = \left(\tilde{f}_1'(x)\,y(x) - \tilde{f}_1(x)\,y'(x)\right)' \tag{1.20}$$

and integrating both sides of this equation gives

$$\int \tilde{f}_1(x)\,q_2(x)\,y(x)\;\mathrm{d}x = \tilde{f}_1'(x)\,y(x) - \tilde{f}_1(x)\,y'(x)\,. \tag{1.21}$$

In a similar manner as for Equation (1.15), multiplying both sides of Equation (1.20) by $m(\tilde{f}'(x)y(x) - \tilde{f}(x)y'(x))^{m-1}$ for $m \in \mathbb{C}$ and $m \neq 0$ and integrating, gives the integral

$$\begin{aligned} &\int \tilde{f}_1(x)\,q_2(x)\,y(x)\left(\tilde{f}_1'(x)\,y(x) - \tilde{f}_1(x)\,y'(x)\right)^{m-1}\;\mathrm{d}x \\ &\quad = \frac{1}{m}\left(\tilde{f}_1'(x)\,y(x) - \tilde{f}_1(x)\,y'(x)\right)^{m}. \end{aligned} \tag{1.22}$$

Results for cylinder functions, parabolic cylinder functions and Whittaker functions are given in Sections 2–4, respectively. All integrals presented have been validated using Mathematica [4].

2. Cylinder functions

The baseline differential equation for the general cylinder function $Z_n(x) \equiv C_1 J_n(x) + C_2 Y_n(x)$ is the Bessel Equation

$$y''(x) + \frac{1}{x} y'(x) + \left(1 - \frac{n^2}{x^2}\right) y(x) = 0. \tag{2.1}$$

This equation can be readily transformed into the equation

$$y''(x) + \frac{1 \mp 2n}{x} y'(x) + y(x) = 0, \tag{2.2}$$

which has the solution $y(x) = x^{\pm n} Z_n(x)$ and for which the derivative $y'(x) = \pm x^{\pm n} Z_{n \mp 1}(x)$ is simple, or into the equation

$$y''(x) + \left(1 - \frac{n^2 - \frac{1}{4}}{x^2}\right) y(x) = 0, \tag{2.3}$$

which has $p(x) = 0$ and the general solution $y(x) = \sqrt{x} Z_n(x)$.

For Equation (2.2), the integrating factor given by Equation (1.13) can be taken to be $\tilde{f}(x) = \sin(x + \phi)$ where ϕ is an arbitrary phase factor and the principal special cases are $\tilde{f}(x) = \sin(x)$ and $\tilde{f}(x) = \cos(x)$. Substituting this solution for $\tilde{f}(x)$ into Equation (1.14), and shifting n as appropriate to give $Z_n(x)$ on the left-hand side of the equation, gives the integral

$$\int x^{\pm n} \sin(x + \phi) Z_n(x) \, dx = \frac{x^{1 \pm n}}{2n \pm 1} \left(\pm \sin(x + \phi) Z_n(x) - \cos(x + \phi) Z_{n \pm 1}(x)\right) \tag{2.4}$$

which is well known [5]. The generalization of this integral through Equation (1.16) gives

$$\int x^{m(1 \pm n) - 1} \sin(x + \phi) Z_n(x) \left(\left(\pm \sin(x + \phi) Z_n(x) - \cos(x + \phi) Z_{n \pm 1}(x)\right)\right)^{m-1} dx$$
$$= x^{m(1 \pm n)} \frac{\left(\pm \sin(x + \phi) Z_n(x) - \cos(x + \phi) Z_{n \pm 1}(x)\right)^m}{m(2n \pm 1)}. \tag{2.5}$$

Three useful fragments of Equation (2.3) which and their respective solutions for the integrating factor $\tilde{f}(x)$ are

$$y''(x) + y(x) = 0, \tag{2.6}$$

$$\tilde{f}(x) = \sin(x + \phi), \tag{2.7}$$

$$y'' - \frac{n^2 - \frac{1}{4}}{x^2} y = 0, \tag{2.8}$$

$$\tilde{f}(x) = x^{1/2 \pm n}, \tag{2.9}$$

$$y'' + \frac{1}{4x^2} y = 0, \tag{2.10}$$

$$\tilde{f}(x) = \sqrt{x} \,\text{or}\, \tilde{f}(x) = \sqrt{x} \ln(x). \tag{2.11}$$

Substituting Equation (2.7) into Equation (1.21) gives the integral

$$\int \frac{\sin(x+\phi) Z_n(x)}{x^{3/2}} \,\mathrm{d}x = \frac{4}{1-4n^2}\left(\cos(x+\phi)\sqrt{x}Z_n(x) + \sin(x+\phi)\left(\sqrt{x}Z_{n+1}(x) - \frac{\left(n+\frac{1}{2}\right)Z_n(x)}{\sqrt{x}}\right)\right), \tag{2.12}$$

which is given in [5]. The generalized form of Equation (2.12) given by Equation (1.22) is

$$\int x^{m/2-2}\sin(x+\phi) Z_n(x) \times \left(\cos(x) Z_n(x+\phi) + \sin(x+\phi)\left(Z_{n+1}(x) - \frac{\left(n+\frac{1}{2}\right)Z_n(x)}{x}\right)\right)^{m-1} \mathrm{d}x = \frac{4x^{m/2}}{m\left(1-4n^2\right)}\left(\cos(x+\phi) Z_n(x) + \sin(x+\phi)\left(Z_{n+1}(x) - \frac{\left(n+\frac{1}{2}\right)Z_n(x)}{x}\right)\right)^m. \tag{2.13}$$

Substituting Equation (2.9) into Equation (1.21) gives initially the integrals

$$\int x^{1\pm n} Z_n(x)\,\mathrm{d}x = \pm n x^{\pm n} Z_n(x) - x^{1\pm n} Z_n'(x) \tag{2.14}$$

which are reduced by the cylinder function recurrences

$$Z_n'(x) = \pm\frac{n}{x} Z_n(x) \mp Z_{n\pm 1}(x) \tag{2.15}$$

to give the elementary integrals [5]

$$\int x^{1\pm n} Z_n(x)\,\mathrm{d}x = \pm x^{1\pm n} Z_{n\pm 1}(x). \tag{2.16}$$

For this case, Equation (1.22) gives initially

$$\int x^{m(1\pm n)} Z_n(x)\, Z_{n\pm 1}^{m-1}(x)\,\mathrm{d}x = \pm\frac{x^{m(1\pm n)} Z_{n\pm 1}^m(x)}{m} \tag{2.17}$$

and shifting the index n in Equation (2.17) such that $n \to n \pm 1$ gives the integral in the form

$$\int x^{\pm mn} Z_n^{m-1}(x)\, Z_{n\mp 1}(x)\,\mathrm{d}x = \pm\frac{x^{\pm mn} Z_n^m(x)}{m} \tag{2.18}$$

which was given in [6]. Rather surprisingly, given that (2.16) is well known, Mathematica [4] cannot obtain Equations (2.18) directly.

Substituting Equations (2.11) into Equation (1.21) gives the two integrals

$$\int \frac{x^2 - n^2}{x} Z_n(x)\, \mathrm{d}x = x Z_{n+1}(x) - n Z_n(x), \tag{2.19}$$

$$\int \frac{x^2 - n^2}{x} \ln(x) Z_n(x)\, \mathrm{d}x = x \ln(x) Z_{n+1}(x) + (1 - n\ln(x)) Z_n(x) \tag{2.20}$$

and these can be generalized through Equation (1.22) to give, respectively,

$$\int x^{m-2}\left(x^2 - n^2\right) Z_n(x) \left(Z_{n+1}(x) - \frac{n}{x} Z_n(x)\right)^{m-1} \mathrm{d}x = \frac{x^m}{m}\left(Z_{n+1}(x) - \frac{n}{x} Z_n(x)\right)^m, \tag{2.21}$$

$$\int x^{m-2}\left(x^2 - n^2\right) \ln(x) Z_n(x) \left(\ln(x) Z_{n+1}(x) + \frac{1 - n\ln(x)}{x} Z_n(x)\right)^{m-1} \mathrm{d}x$$
$$= \frac{x^m}{m}\left(\ln(x) Z_{n+1}(x) + \frac{1 - n\ln(x)}{x} Z_n(x)\right)^m. \tag{2.22}$$

3. Parabolic cylinder functions

The parabolic cylinder functions $y(x) = D_n(x)$ obey the baseline differential equation

$$y''(x) + \left(n + \frac{1}{2} - \frac{x^2}{4}\right) y(x) = 0 \tag{3.1}$$

and have the recurrence relations [7]

$$D_n'(x) + \frac{x}{2} D_n(x) = n D_{n-1}(x), \tag{3.2}$$

$$D_n'(x) - \frac{x}{2} D_n(x) = -D_{n+1}(x). \tag{3.3}$$

These recurrences can be integrated to give, respectively,

$$\left(e^{x^2/4} D_n(x)\right)' = n e^{x^2/4} D_{n-1}(x), \tag{3.4}$$

$$\left(e^{-x^2/4} D_n(x)\right)' = -e^{-x^2/4} D_{n+1}(x). \tag{3.5}$$

Defining $y_1(x) = e^{x^2/4} D_n(x)$ and $y_2(x) = e^{-x^2/4} D_n(x)$ and transforming Equation (3.1) to give the differential equations obeyed by $y_1(x)$ and $y_2(x)$ results in the equations

$$y_1''(x) - x y_1'(x) + n y_1(x) = 0, \tag{3.6}$$

$$y_2''(x) + x y_2'(x) + (n+1) y_2(x) = 0. \tag{3.7}$$

Equation (3.7) is given in [7]. Equation (3.6) is similar in form to the equation obeyed by the Hermite function $H_n(x)$, which is [7]

$$y''(x) - 2x y'(x) + 2n y(x) = 0 \tag{3.8}$$

and $H_n(x/\sqrt{2})$ is a solution of Equation (3.6). The two functions are directly related by [7]

$$D_n(x) = 2^{-n/2} e^{-x^2/4} H_n\left(\frac{x}{\sqrt{2}}\right) \tag{3.9}$$

so the results obtained here for $D_n(x)$ can be transformed into results for $H_n(x)$.

The integrating factor for Equation (3.6) is

$$\tilde{f}(x) = \sin\left(\sqrt{n}x + \phi\right), \tag{3.10}$$

where ϕ is an arbitrary phase factor, with $\tilde{f}(x) = \sin(\sqrt{n}x)$ and $\tilde{f}(x) = \cos(\sqrt{n}x)$ being the main cases of interest. This factor gives through Equation (1.15), the integral

$$\begin{aligned}&\int xe^{x^2/4} \sin\left(\sqrt{n+1}x + \phi\right) D_n(x)\ \mathrm{d}x \\ &\quad = e^{x^2/4}\left(\sin\left(\sqrt{n+1}x + \phi\right) D_n(x) - \frac{\cos\left(\sqrt{n+1}x + \phi\right)}{\sqrt{n+1}} D_{n+1}(x)\right)\end{aligned} \tag{3.11}$$

which can be generalized for $m \neq 0$ through Equation (1.16) to give

$$\begin{aligned}&\int xe^{mx^2/4} \sin\left(\sqrt{n+1}x + \phi\right) D_n(x) \\ &\qquad \times \left(\sin\left(\sqrt{n+1}x + \phi\right) D_n(x) - \frac{\cos\left(\sqrt{n+1}x + \phi\right)}{\sqrt{n+1}} D_{n+1}(x)\right)^{m-1} \mathrm{d}x \\ &\quad = \frac{e^{mx^2/4}}{m}\left(\sin\left(\sqrt{n+1}x + \phi\right) D_n(x) - \frac{\cos\left(\sqrt{n}x + \phi\right)}{\sqrt{n+1}} D_{n+1}(x)\right)^m.\end{aligned} \tag{3.12}$$

The integrating factor for Equation (3.7) is

$$\tilde{f}(x) = \sin\left(\sqrt{n+1}x + \phi\right) \tag{3.13}$$

and through Equation (1.17) this gives the integral

$$\begin{aligned}&\int xe^{-\frac{x^2}{4}} \sin\left(\sqrt{n}x + \phi\right) D_n(x)\ \mathrm{d}x \\ &\quad = -e^{-\frac{x^2}{4}}\left(\sin\left(\sqrt{n}x + \phi\right) D_n(x) + \sqrt{n}\cos\left(\sqrt{n}x + \phi\right) D_{n-1}(x)\right)\end{aligned} \tag{3.14}$$

which can be generalized through Equation (1.18) for $m \neq 1$ to give

$$\begin{aligned}&\int xe^{-\frac{mx^2}{4}} \sin\left(\sqrt{n}x + \phi\right) D_n(x) \\ &\qquad \times \left(\sin\left(\sqrt{n}x + \phi\right) D_n(x) + \sqrt{n}\cos\left(\sqrt{n}x + \phi\right) D_{n-1}(x)\right)^{m-1} \mathrm{d}x \\ &\quad = -\frac{e^{-mx^2/4}}{m}\left(\sin\left(\sqrt{n}x + \phi\right) D_n(x) + \sqrt{n}\cos\left(\sqrt{n}x + \phi\right) D_{n-1}(x)\right)^m.\end{aligned} \tag{3.15}$$

Useful fragments of Equation (3.1) and their respective integrating factors $\tilde{f}(x)$ are

$$y''(x) + \left(n + \frac{1}{2}\right) y(x) = 0, \tag{3.16}$$

$$\tilde{f}(x) = \sin\left(\sqrt{n + \frac{1}{2}}x + \phi\right), \tag{3.17}$$

$$y'' - \frac{x^2}{4} y = 0, \tag{3.18}$$

$$\tilde{f}(x) = \sqrt{x}\hat{K}_{\frac{1}{4}}\left(\frac{x^2}{4}\right), \tag{3.19}$$

where $\hat{K}_{1/4}(x^2/4) \equiv C_1 K_{1/4}(x^2/4) + C_2 I_{1/4}(x^2/4)$ is a modified cylinder function.

$$y'' + \left(\frac{1}{2} - \frac{x^2}{4}\right) y = 0, \tag{3.20}$$

$$\tilde{f}(x) = e^{-\frac{x^2}{4}}; \quad \tilde{f}(x) = e^{-\frac{1}{4}x^2} \operatorname{erf}\left(\frac{ix}{\sqrt{2}}\right), \tag{3.21}$$

$$y'' + \left(-\frac{1}{2} - \frac{x^2}{4}\right) y = 0, \tag{3.22}$$

$$\tilde{f}(x) = e^{x^2/4}; \quad \tilde{f}(x) = e^{1/4x^2} \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right). \tag{3.23}$$

Substituting Equation (3.17) into Equation (1.21) gives the integral

$$\begin{aligned} &\int \sin\left(\sqrt{n + \frac{1}{2}}x + \phi\right) x^2 D_n(x)\,\mathrm{d}x \\ &= 4\left(\sin\left(\sqrt{n + \frac{1}{2}}x + \phi\right) D_n'(x) - \sqrt{n + \frac{1}{2}} \cos\left(\sqrt{n + \frac{1}{2}}x + \phi\right) D_n(x)\right), \end{aligned} \tag{3.24}$$

which is generalized by Equation (1.22) as

$$\begin{aligned} &\int \sin\left(\sqrt{n + \frac{1}{2}}x + \phi\right) x^2 D_n(x) \\ &\quad \times \left(\sin\left(\sqrt{n + \frac{1}{2}}x + \phi\right) D_n'(x) - \sqrt{n + \frac{1}{2}} \cos\left(\sqrt{n + \frac{1}{2}}x + \phi\right) D_n(x)\right)^{m-1} \mathrm{d}x \\ &= \frac{4}{m}\left(\sin\left(\sqrt{n + \frac{1}{2}}x + \phi\right) D_n'(x) - \sqrt{n + \frac{1}{2}} \cos\left(\sqrt{n + \frac{1}{2}}x + \phi\right) D_n(x)\right)^m. \end{aligned} \tag{3.25}$$

Substituting Equation (3.19) into Equation (1.21) and using the identity

$$\left(\sqrt{x}\hat{K}_{\frac{1}{4}}\left(\frac{x^2}{4}\right)\right)' = -\frac{1}{2}x^{3/2}\hat{K}_{\frac{3}{4}}\left(\frac{x^2}{4}\right) \tag{3.26}$$

gives the integral

$$\int \sqrt{x}\hat{K}_{\frac{1}{4}}\left(\frac{x^2}{4}\right) D_n(x)\,\mathrm{d}x$$
$$= -\frac{2}{2n+1}\sqrt{x}\left(\frac{x}{2}\hat{K}_{\frac{3}{4}}\left(\frac{x^2}{4}\right) D_n(x) + \hat{K}_{\frac{1}{4}}\left(\frac{x^2}{4}\right) D'_n(x)\right), \tag{3.27}$$

which is generalized by Equation (1.22) to give

$$\int x^{m/2}\hat{K}_{\frac{1}{4}}\left(\frac{x^2}{4}\right) D_n(x)$$
$$\times \left(\frac{x}{2}\hat{K}_{\frac{3}{4}}\left(\frac{x^2}{4}\right) D_n(x) + \hat{K}_{\frac{1}{4}}\left(\frac{x^2}{4}\right) D'_n(x)\right)^{m-1} \mathrm{d}x$$
$$= -\frac{2x^{m/2}}{m(2n+1)}\left(\frac{x}{2}\hat{K}_{\frac{3}{4}}\left(\frac{x^2}{4}\right) D_n(x) + \hat{K}_{\frac{1}{4}}\left(\frac{x^2}{4}\right) D'_n(x)\right)^{m}. \tag{3.28}$$

Substituting $\tilde{f}(x) = e^{-x^2/4}$ from (3.21) in Equation (1.21) and reducing using Equation (3.2) gives

$$\int e^{-x^2/4} D_n(x)\,\mathrm{d}x = -e^{-x^2/4} D_{n-1}(x), \tag{3.29}$$

which can be obtained directly from Equation (3.5). Similarly, substituting $\tilde{f}(x) = e^{x^2/4}$ from (3.23) in Equation (1.21) and reducing using Equation (3.3) gives

$$\int e^{x^2/4} D_n(x)\,\mathrm{d}x = \frac{e^{x^2/4}}{n+1} D_{n+1}(x). \tag{3.30}$$

Substituting $\tilde{f}(x) = e^{-x^2/4}\mathrm{erf}(ix/\sqrt{2})$ from (3.21) into Equation (1.21) gives

$$\int e^{-x^2/4}\mathrm{erf}\left(\frac{ix}{\sqrt{2}}\right) D_n(x)\,\mathrm{d}x = \sqrt{\frac{2}{\pi}}\frac{ie^{x^2/4}}{n} D_n(x) - D_{n-1}(x)\, e^{-\frac{1}{4}x^2}\mathrm{erf}\left(\frac{ix}{\sqrt{2}}\right). \tag{3.31}$$

Similarly, substituting $\tilde{f}(x) = e^{x^2/4}\mathrm{erf}(x/\sqrt{2})$ from (3.21) into Equation (1.21) gives

$$\int e^{x^2/4}\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right) D_n(x)\,\mathrm{d}x = \frac{1}{n+1}\left(D_{n+1}(x)\, e^{x^2/4}\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right) + \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{4}} D_n(x)\right). \tag{3.32}$$

4. Whittaker functions

The Whittaker functions $M_{\lambda,\mu}(x)$ and $W_{\lambda,\mu}(x)$ are independent solutions of the differential equation

$$y''(x) + \left(-\frac{1}{4} + \frac{\lambda}{x} + \frac{\frac{1}{4} - \mu^2}{x^2}\right) y(x) = 0 \tag{4.1}$$

and the general solution of this equation can be taken as $\hat{W}_{\lambda,\mu}(x) \equiv C_1 M_{\lambda,\mu}(x) + C_2 W_{\lambda,\mu}(x)$. Six recurrence relations for each of the functions $M_{\lambda,\mu}(x)$ and $W_{\lambda,\mu}(x)$ are given in [8,9], and these can be expressed in two groups of the form

$$\begin{aligned}&\left(e^{\pm x/2}x^{\pm\mu-1/2}\begin{Bmatrix}M_{\lambda,\mu}(x)\\W_{\lambda,\mu}(x)\end{Bmatrix}\right)'\\&\quad=C(\lambda,\mu)\,e^{\pm x/2}x^{\pm\mu-1}\begin{Bmatrix}M_{\lambda\mp1/2,\mu\mp1/2}(x)\\W_{\lambda\mp1/2,\mu\mp1/2}(x)\end{Bmatrix},\end{aligned}\tag{4.2}$$

$$\begin{aligned}&\left(e^{\pm x/2}x^{\mp\lambda}\begin{Bmatrix}M_{\lambda,\mu}(x)\\W_{\lambda,\mu}(x)\end{Bmatrix}\right)'\\&\quad=C(\lambda,\mu)\,e^{\pm x/2}x^{\mp\lambda-1}\begin{Bmatrix}M_{\lambda\pm1,\mu}(x)\\W_{\lambda\pm1,\mu}(x)\end{Bmatrix}.\end{aligned}\tag{4.3}$$

In these relations, the function $C(\lambda,\mu)$ is independent of x and varies with the particular recurrence. In Equation (4.2), the $\pm$ sign associated with μ is independent from the $\pm$ sign associated with the exponential, giving four recurrences per function, and in Equation (4.3) the $\pm$ signs are dependent, giving two recurrences per function, for a total of twelve recurrences.

Defining

$$y_1(x)\equiv e^{\pm x/2}x^{\pm\mu-1/2}\hat{W}_{\lambda,\mu}(x)=a_1(x)\,y(x)\,,\tag{4.4}$$

$$y_2(x)\equiv e^{\pm x/2}x^{\pm\lambda}\hat{W}_{\lambda,\mu}(x)=a_2(x)\,y(x)\,,\tag{4.5}$$

then Equation (4.1) can be transformed by the relation

$$y_i''(x)-\frac{2a_i'(x)}{a_i(x)}y_i'(x)+\left(2\left(\frac{a_i'(x)}{a_i(x)}\right)^2-\frac{a_i''(x)}{a_i(x)}-\frac{1}{4}+\frac{\lambda}{x}+\frac{\frac{1}{4}-\mu^2}{x^2}\right)y_i(x)=0\tag{4.6}$$

and this gives the equations satisfied by $y_1(x)$ as

$$y_1''(x)+\left(\frac{1\mp2\mu}{x}-1\right)y_1'(x)+\frac{\lambda\pm\mu-\frac{1}{2}}{x}y_1(x)=0\tag{4.7}$$

for the positive sign of the exponential in Equation (4.4) and

$$y_1''(x)+\left(\frac{1\mp2\mu}{x}+1\right)y_1'(x)+\frac{\lambda\mp\mu+\frac{1}{2}}{x}y_1(x)=0\tag{4.8}$$

for the negative sign of the exponential. The equations obeyed by $y_2(x)$ are

$$y_2''(x)\mp\left(\frac{2\lambda}{x}-1\right)y_2'(x)+\frac{\left(\lambda\pm\frac{1}{2}\right)^2-\mu^2}{x^2}y_2(x)=0,\tag{4.9}$$

where the $\pm$ sign corresponds to the $\pm$ sign in Equation (4.5). Equations (4.7)–(4.9) all have simple forms for $q(x)$ and from Equations (4.2) and (4.3) the derivatives of $y_1(x)$ and $y_2(x)$

are of simple form. The integrating factors $\tilde{f}(x)$ and their derivatives for Equations (4.7) and (4.8) are, respectively,

$$\tilde{f}(x) = \sqrt{x} Z_1 \left(2\sqrt{\lambda \pm \mu - \frac{1}{2}} \sqrt{x} \right); \quad \tilde{f}'(x) = \sqrt{\lambda \pm \mu - \frac{1}{2}} Z_0 \left(2\sqrt{\lambda \pm \mu - \frac{1}{2}} \sqrt{x} \right), \tag{4.10}$$

$$\tilde{f}(x) = \sqrt{x} Z_1 \left(2\sqrt{\lambda \mp \mu + \frac{1}{2}} \sqrt{x} \right); \quad \tilde{f}'(x) = \sqrt{\lambda \mp \mu + \frac{1}{2}} Z_0 \left(2\sqrt{\lambda \mp \mu + \frac{1}{2}} \sqrt{x} \right) \tag{4.11}$$

and the integration factors for Equation (4.9) are the solutions of the equation

$$\tilde{f}''(x) + \left(\frac{\left(\lambda \pm \frac{1}{2}\right)^2 - \mu^2}{x^2} \right) \tilde{f}(x) = 0, \tag{4.12}$$

which gives

$$\tilde{f}(x) = x^{1/2 - \sqrt{\mu^2 - \lambda(\lambda \pm 1)}}; \quad \tilde{f}'(x) = \left(\frac{1}{2} - \sqrt{\mu^2 - \lambda(\lambda \pm 1)} \right) x^{-1/2 - \sqrt{\mu^2 - \lambda(\lambda \pm 1)}}. \tag{4.13}$$

The detailed recurrence relations for $M_{\lambda,\mu}(x)$ are [8,9]

$$\left(e^{x/2} x^{\mu-1/2} M_{\lambda,\mu}(x)\right)' = 2\mu e^{x/2} x^{\mu-1} M_{\lambda-1/2,\mu-1/2}(x), \tag{4.14}$$

$$\left(e^{x/2} x^{-\mu-1/2} M_{\lambda,\mu}(x)\right)' = \frac{1/2 + \mu - \lambda}{1 + 2\mu} e^{x/2} x^{-\mu-1} M_{\lambda-1/2,\mu+1/2}(x), \tag{4.15}$$

$$\left(e^{-x/2} x^{\mu-1/2} M_{\lambda,\mu}(x)\right)' = 2\mu e^{-x/2} x^{\mu-1} M_{\lambda+1/2,\mu-1/2}(x), \tag{4.16}$$

$$\left(e^{-x/2} x^{-\mu-1/2} M_{\lambda,\mu}(x)\right)' = -\frac{\frac{1}{2} + \mu + \lambda}{1 + 2\mu} e^{-x/2} x^{-\mu-1} M_{\lambda+1/2,\mu+1/2}(x), \tag{4.17}$$

$$\left(e^{-x/2} x^{\lambda} M_{\lambda,\mu}(x)\right)' = \left(\frac{1}{2} + \mu + \lambda \right) e^{-x/2} x^{\lambda-1} M_{\lambda+1,\mu}(x), \tag{4.18}$$

$$\left(e^{x/2} x^{-\lambda} M_{\lambda,\mu}(x)\right)' = \left(\frac{1}{2} + \mu - \lambda \right) e^{x/2} x^{-\lambda-1} M_{\lambda-1,\mu}(x) \tag{4.19}$$

and the corresponding recurrences for $W_{\lambda,\mu}(x)$ are

$$\left(e^{x/2} x^{\mu-1/2} W_{\lambda,\mu}(x)\right)' = \left(\lambda + \mu - \frac{1}{2} \right) e^{x/2} x^{\mu-1} W_{\lambda-1/2,\mu-1/2}(x), \tag{4.20}$$

$$\left(e^{x/2} x^{-\mu-1/2} W_{\lambda,\mu}(x)\right)' = \left(\lambda - \mu - \frac{1}{2} \right) e^{x/2} x^{-\mu-1} W_{\lambda-1/2,\mu+1/2}(x), \tag{4.21}$$

$$\left(e^{-x/2} x^{\mu-1/2} W_{\lambda,\mu}(x)\right)' = -e^{-x/2} x^{\mu-1} W_{\lambda+1/2,\mu-1/2}(x), \tag{4.22}$$

$$\left(e^{-x/2}x^{-\mu-1/2}W_{\lambda,\mu}\left(x\right)\right)' = -e^{-x/2}x^{-\mu-1}W_{\lambda+1/2,\mu+1/2}\left(x\right), \tag{4.23}$$

$$\left(e^{-x/2}x^{\lambda}W_{\lambda,\mu}\left(x\right)\right)' = -e^{-x/2}x^{\lambda-1}W_{\lambda+1,\mu}\left(x\right), \tag{4.24}$$

$$\left(e^{x/2}x^{-\lambda}W_{\lambda,\mu}\left(x\right)\right)' = \left(\frac{1}{2}+\mu-\lambda\right)\left(\frac{1}{2}-\mu-\lambda\right)e^{x/2}x^{-\lambda-1}W_{\lambda-1,\mu}\left(x\right). \tag{4.25}$$

The integrals derived from these recurrences using Equation (1.15) are, respectively,

$$\begin{aligned}
&\int \left(x+2\mu\right)e^{x/2}x^{\mu-1}Z_1\left(2\sqrt{\lambda+\mu+\frac{1}{2}}\sqrt{x}\right)M_{\lambda,\mu}\left(x\right)\,\mathrm{d}x \\
&= e^{x/2}x^{\mu}\left(Z_1\left(2\sqrt{\lambda+\mu+\frac{1}{2}}\sqrt{x}\right)M_{\lambda,\mu}\left(x\right)\right. \\
&\quad \left. - \frac{\sqrt{\lambda+\mu+\frac{1}{2}}}{2\mu+1}Z_0\left(2\sqrt{\lambda+\mu+\frac{1}{2}}\sqrt{x}\right)M_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}\left(x\right)\right),
\end{aligned} \tag{4.26}$$

$$\begin{aligned}
&\int \left(x-2\mu\right)e^{x/2}x^{-\mu-1}Z_1\left(2\sqrt{\lambda-\mu+\frac{1}{2}}\sqrt{x}\right)M_{\lambda,\mu}\left(x\right)\,\mathrm{d}x \\
&= e^{x/2}x^{-\mu}\left(Z_1\left(2\sqrt{\lambda-\mu+\frac{1}{2}}\sqrt{x}\right)M_{\lambda,\mu}\left(x\right)\right. \\
&\quad \left. + \frac{2\mu}{\sqrt{\lambda-\mu+\frac{1}{2}}}Z_0\left(2\sqrt{\lambda-\mu+\frac{1}{2}}\sqrt{x}\right)M_{\lambda+1/2,\mu-1/2}\left(x\right)\right),
\end{aligned} \tag{4.27}$$

$$\begin{aligned}
&\int \left(x-2\mu\right)e^{-x/2}x^{\mu-1}Z_1\left(2\sqrt{\lambda-\mu-\frac{1}{2}}\sqrt{x}\right)M_{\lambda,\mu}\left(x\right)\,\mathrm{d}x \\
&= e^{-x/2}x^{\mu}\left(\frac{\sqrt{\lambda-\mu-\frac{1}{2}}}{2\mu+1}Z_0\left(2\sqrt{\lambda-\mu-\frac{1}{2}}\sqrt{x}\right)M_{\lambda-1/2,\mu+1/2}\left(x\right)\right. \\
&\quad \left. - Z_1\left(2\sqrt{\lambda-\mu-\frac{1}{2}}\sqrt{x}\right)M_{\lambda,\mu}\left(x\right)\right),
\end{aligned} \tag{4.28}$$

$$\begin{aligned}
&\int \left(x+2\mu\right)e^{-x/2}x^{-\mu-1}Z_1\left(2\sqrt{\lambda+\mu-\frac{1}{2}}\sqrt{x}\right)M_{\lambda,\mu}\left(x\right)\,\mathrm{d}x \\
&= -e^{-x/2}x^{-\mu}\left(\frac{2\mu}{\sqrt{\lambda+\mu-\frac{1}{2}}}Z_0\left(2\sqrt{\lambda+\mu-\frac{1}{2}}\sqrt{x}\right)M_{\lambda-1/2,\mu-1/2}\left(x\right)\right.
\end{aligned}$$

$$+ Z_1\left(2\sqrt{\lambda+\mu-\frac{1}{2}}\sqrt{x}\right)M_{\lambda,\mu}(x)\Bigg), \tag{4.29}$$

$$\int (x-2\lambda+2)\,x^{\lambda-5/2-\sqrt{\mu^2-\lambda(\lambda-1)}}e^{-x/2}M_{\lambda,\mu}(x)\,\mathrm{d}x$$

$$= x^{\lambda-3/2-\sqrt{\mu^2-\lambda(\lambda-1)}}e^{-x/2}\left(\frac{\frac{1}{2}-\sqrt{\mu^2-\lambda(\lambda-1)}}{\mu+\lambda-\frac{1}{2}}M_{\lambda-1,\mu}(x)-M_{\lambda,\mu}(x)\right), \tag{4.30}$$

$$\int (2\lambda+2-x)\,x^{-\lambda-5/2-\sqrt{\mu^2-\lambda(\lambda+1)}}e^{x/2}M_{\lambda,\mu}(x)\,\mathrm{d}x$$

$$= x^{-\lambda-3/2-\sqrt{\mu^2-\lambda(\lambda+1)}}e^{x/2}\left(\frac{\frac{1}{2}-\sqrt{\mu^2-\lambda(\lambda+1)}}{\mu-\lambda-\frac{1}{2}}M_{\lambda,\mu}(x)-M_{\lambda-1}(x)\right), \tag{4.31}$$

$$\int (x+2\mu)\,e^{x/2}x^{\mu-1}Z_1\left(2\sqrt{\lambda+\mu+\frac{1}{2}}\sqrt{x}\right)W_{\lambda,\mu}(x)\,\mathrm{d}x$$

$$= e^{x/2}x^{\mu}\left(Z_1\left(2\sqrt{\lambda+\mu+\frac{1}{2}}\sqrt{x}\right)W_{\lambda,\mu}(x)\right.$$

$$\left.-\frac{Z_0\left(2\sqrt{\lambda+\mu+\frac{1}{2}}\sqrt{x}\right)}{\sqrt{\lambda+\mu+\frac{1}{2}}}W_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}(x)\right), \tag{4.32}$$

$$\int (x-2\mu)\,e^{x/2}x^{-\mu-1}Z_1\left(2\sqrt{\lambda-\mu+\frac{1}{2}}\sqrt{x}\right)W_{\lambda,\mu}(x)\,\mathrm{d}x$$

$$= e^{x/2}x^{-\mu}\left(Z_1\left(2\sqrt{\lambda-\mu+\frac{1}{2}}\sqrt{x}\right)W_{\lambda,\mu}(x)\right.$$

$$\left.-\frac{Z_0\left(2\sqrt{\lambda-\mu+\frac{1}{2}}\sqrt{x}\right)}{\sqrt{\lambda-\mu+\frac{1}{2}}}W_{\lambda+1/2,\mu-1/2}(x)\right), \tag{4.33}$$

$$\int (2\mu-x)\,e^{-x/2}x^{\mu-1}Z_1\left(2\sqrt{\lambda-\mu-\frac{1}{2}}\sqrt{x}\right)W_{\lambda,\mu}(x)\,\mathrm{d}x$$

$$= e^{-x/2}x^{\mu}\left(\sqrt{\lambda-\mu-\frac{1}{2}}Z_0\left(2\sqrt{\lambda-\mu-\frac{1}{2}}\sqrt{x}\right)W_{\lambda-1/2,\mu+1/2}(x)\right.$$

$$+ Z_1\left(2\sqrt{\lambda-\mu-\frac{1}{2}}\sqrt{x}\right)W_{\lambda,\mu}(x)\Bigg), \tag{4.34}$$

$$\int(2\mu+x)\,e^{-x/2}x^{-\mu-1}Z_1\left(2\sqrt{\lambda+\mu-\frac{1}{2}}\sqrt{x}\right)W_{\lambda,\mu}(x)\,\mathrm{d}x$$
$$= -e^{-x/2}x^{-\mu}\left(\sqrt{\lambda+\mu-\frac{1}{2}}Z_0\left(2\sqrt{\lambda+\mu-\frac{1}{2}}\sqrt{x}\right)W_{\lambda-1/2,\mu-1/2}(x)\right.$$
$$\left.+ Z_1\left(2\sqrt{\lambda+\mu-\frac{1}{2}}\sqrt{x}\right)W_{\lambda,\mu}(x)\right), \tag{4.35}$$

$$\int(2\lambda-2-x)\,e^{-x/2}x^{\lambda-5/2-\sqrt{\mu^2-\lambda(\lambda-1)}}W_{\lambda,\mu}(x)\,\mathrm{d}x$$
$$= e^{-x/2}x^{\lambda-3/2-\sqrt{\mu^2-\lambda(\lambda-1)}}\left(\left(\frac{1}{2}-\sqrt{\mu^2-\lambda(\lambda-1)}\right)W_{\lambda-1,\mu}(x)+W_{\lambda,\mu}(x)\right), \tag{4.36}$$

$$\int x^{-\lambda-5/2-\sqrt{\mu^2-\lambda(\lambda+1)}}(x-2\lambda-2)\,e^{x/2}W_{\lambda,\mu}(x)\,\mathrm{d}x$$
$$= e^{x/2}x^{-\lambda-3/2-\sqrt{\mu^2-\lambda(\lambda+1)}}\left(\frac{\frac{1}{2}-\sqrt{\mu^2-\lambda(\lambda+1)}}{\mu^2-\left(\lambda+\frac{1}{2}\right)^2}W_{\lambda+1,\mu}(x)+W_{\lambda,\mu}(x)\right). \tag{4.37}$$

Integrals (4.26)–(4.37) above can be generalized using Equation (1.16).

Disclosure statement

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