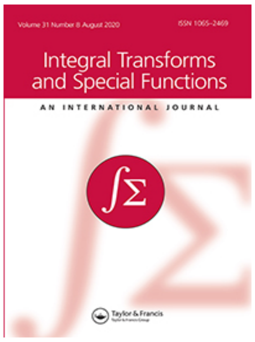

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A generalized integration formula for indefinite integrals of special functions

John T. Conway

Department of Engineering and Science, University of Agder, Grimstad, Norway

ABSTRACT

An integration formula for generating indefinite integrals which was presented in Conway JT [A Lagrangian method for deriving new indefinite integrals of special functions. Integral Transforms Spec Funct. 2015;26(10):812–824], which involves an arbitrary function $h(x)$, is generalized to include two arbitrary functions $s(x)$ and $h(x)$. In the new formula the Lagrangian factor $f(x)$ of the original formula is replaced by a new factor $\bar{f}(x)$, which is derived from $s(x)$. The use of the new formula is illustrated by deriving a number of integrals for cylinder and Whittaker functions, with several integrals combining Whittaker functions with cylinder and modified cylinder functions. Some of these integrals are among the best known results for these functions, whereas others are believed to be new. A number of new integrals of Whittaker functions are also presented which came directly from the derivatives needed to apply the integration formula. All results presented have been checked using Mathematica.

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1. Introduction

Many special functions obey the homogeneous second-order linear differential equation

$$y''(x) + p(x)\,y'(x) + q(x)\,y(x) = 0 \tag{1.1}$$

and in [1,2] an integration theorem was presented for any solution $y(x)$ of Equation (1.1). The integration formula given was

$$\int f(x)\left(h''(x) + p(x)\,h'(x) + q(x)\,h'(x)\right)y(x)\;\mathrm{d}x = f(x)\left(h'(x)\,y(x) - h(x)\,y'(x)\right) \tag{1.2}$$

where $h(x)$ is an arbitrary twice-differentiable function and $f(x)$ obeys the equation

$$f'(x) = p(x)\,f(x) \tag{1.3}$$

so that

$$f(x) = \exp\left(\int p(x)\;\mathrm{d}x\right). \tag{1.4}$$

CONTACT John T. Conway john.conway@uia.no

Suitable choices of the arbitrary function $h(x)$ in Equation (1.2) yield a large number of interesting integrals. The purpose of this paper is to generalize Equation (1.2) by introducing an additional arbitrary differentiable function $s(x)$. This is achieved by replacing Equation (1.3) with the equation

$$\bar{f}'(x) = s(x)\bar{f}(x) \tag{1.5}$$

so that

$$\bar{f}(x) = \exp\left(\int s(x)\,\mathrm{d}x\right). \tag{1.6}$$

Theorem 1.1:

$$\begin{aligned}\int \bar{f}(x)\big(h''(x) &+ \left[2s(x) - p(x)\right]h'(x)\\ &+ \left[s'(x) - p'(x) + s^2(x) - s(x)p(x) + q(x)\right]h(x)\big)y(x)\,\mathrm{d}x\\ &= \bar{f}(x)\big(h'(x)y(x) - h(x)y'(x) + h(x)\left(s(x) - p(x)\right)y(x)\big)\end{aligned} \tag{1.7}$$

where $\bar{f}(x)$ and $y(x)$ obey the respective differential equations:

$$\bar{f}'(x) = s(x)\bar{f}(x) \tag{1.8}$$

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0 \tag{1.9}$$

and $h(x), s(x), p(x)$ and $q(x)$ are arbitrary complex-valued differentiable functions of $x \in \mathbb{R}$, with $h(x)$ being at least twice-differentiable.

Proof: Arbitrary functions $h(x)$ and $y(x)$ satisfy the elementary differential identity

$$h''(x)y(x) - h(x)y''(x) = \left(h'(x)y(x) - h(x)y'(x)\right)' \tag{1.10}$$

and eliminating $y''(x)$ from Equation (1.10) using Equation (1.9) gives

$$h''(x)y(x) + h(x)\left[p(x)y'(x) + q(x)y(x)\right] = \left(h'(x)y(x) - h(x)y'(x)\right)'. \tag{1.11}$$

Multiplying Equation (1.11) by $\bar{f}(x)$ and employing Equation (1.8) gives

$$\begin{aligned}&\bar{f}(x)\left(h''(x)y(x) + h(x)\left[p(x)y'(x) + q(x)y(x)\right]\right)\\ &\quad= \left[\bar{f}(x)\left(h'(x)y(x) - h(x)y'(x)\right)\right]' - s(x)\bar{f}(x)\left(h'(x)y(x) - h(x)y'(x)\right)\end{aligned} \tag{1.12}$$

which is equivalent to

$$\begin{aligned}\bar{f}(x)\left[h''(x) + s(x)h'(x) + q(x)\right]y(x) &= \left[\bar{f}(x)\left(h'(x)y(x) - h(x)y'(x)\right)\right]'\\ &\quad+ f(x)h(x)\left(s(x) - p(x)\right)y'(x).\end{aligned} \tag{1.13}$$

The right-hand side of Equation (1.13) can be transformed to give

$$\begin{aligned}&\bar{f}(x)\left[h''(x) + s(x)h'(x) + q(x)\right]y(x)\\ &\quad= \left[\bar{f}(x)\left(h'(x)y(x) - h(x)y'(x)\right)\right]'\\ &\qquad+ \left[f(x)h(x)\left(s(x) - p(x)\right)y(x)\right]' - \left(\bar{f}(x)h(x)\left(s(x) - p(x)\right)\right)'y(x).\end{aligned} \tag{1.14}$$

The rightmost term in Equation (1.14) can be expanded, simplified with Equation (1.8), and taken over to the other side of the equation to give

$$\begin{aligned}
&\bar{f}(x)\left[h''(x)+s(x)h'(x)+q(x)\right]y(x)\\
&\quad+\bar{f}(x)\left[h(x)\left(s'(x)-p'(x)\right)+h'(x)\left(s(x)-p(x)\right)\right]y(x)\\
&\quad=\left[\bar{f}(x)\left(h'(x)y(x)-h(x)y'(x)\right)\right]'+\left[f(x)h(x)\left(s(x)-p(x)\right)y(x)\right]' \qquad (1.15)
\end{aligned}$$

and on collecting terms and integrating both sides of Equation (1.15), the theorem is proven. ■

The differential equation embedded in the integration formula (1.2) is

$$h''(x)+p(x)h'(x)+q(x)h(x)=0 \qquad (1.16)$$

and as this equation is identical to Equation (1.1), if $h(x)$ is chosen to be a solution of Equation (1.1), the integrand vanishes and no integral is generated. Instead, integrals are generated by taking $h(x)$ to be fragments of Equation (1.1) [1,2], where a fragment is derived from Equation (1.16) by deleting (or adding) one or more terms. The same procedure can be applied to Equation (1.7). The solution of the differential equation embedded in Equation (1.7) is easily obtained in terms of $y(x)$, as is proven below.

Theorem 1.2: *The solution of the differential equation*

$$h''(x)+\left(2s(x)-p(x)\right)h'(x)+\left(s'(x)-p'(x)+s^2(x)-s(x)p(x)+q(x)\right)h(x)=0 \qquad (1.17)$$

is

$$h(x)=\frac{f(x)}{\bar{f}(x)}y(x) \qquad (1.18)$$

where $y(x)$ *obeys Equation* (1.1) *and* $f(x)$ *and* $\bar{f}(x)$ *obey Equations* (1.3)–(1.6).

Proof: A function $a(x)$ relating the general solutions of Equations (1.1) and (1.17) can be defined by

$$h(x)=a(x)y(x) \qquad (1.19)$$

where the explicit form of $a(x)$ is to be determined. Substituting $y(x)=h(x)/a(x)$ into Equation (1.1) gives the equation to be obeyed by $h(x)$ as

$$\begin{aligned}
&h''(x)+\left(p(x)-2\frac{a'(x)}{a(x)}\right)h'(x)\\
&\quad+\left(\left(\frac{a'(x)}{a(x)}\right)^2-\left(\frac{a'(x)}{a(x)}\right)'-p(x)\frac{a'(x)}{a(x)}+q(x)\right)h(x)=0. \qquad (1.20)
\end{aligned}$$

For Equation (1.20) to match Equation (1.17), the coefficients of $h'(x)$ and $h(x)$ must both match. For the coefficients of $h'(x)$ to match. then

$$\frac{a'(x)}{a(x)}=p(x)-s(x) \qquad (1.21)$$

which is sufficient to determine $a(x)$ by quadrature to within an arbitrary multiplicative constant. As Equations (1.1) and (1.17) are both linear, this arbitrary constant is of no significance. From (1.21) the function $a(x)$ is given by

$$a(x) = \exp\left(\int \big(p(x) - s(x)\big)\, \mathrm{d}x\right) \tag{1.22}$$

which from Equations (1.4) and (1.6) gives

$$a(x) = \frac{f(x)}{\bar{f}(x)}. \tag{1.23}$$

Substituting Equation (1.21) into Equation (1.20) gives the coefficient of $h(x)$ in Equation (1.20) as

$$\left(\frac{a'(x)}{a(x)}\right)^2 - \left(\frac{a'(x)}{a(x)}\right)' - p(x)\frac{a'(x)}{a(x)} + q(x)$$
$$= s'(x) - p'(x) + s^2(x) - p(x)s(x) + q(x) \tag{1.24}$$

and as both coefficients match, the theorem is proven. ■

1.1. Riccati equations

For any specific special function, either $s(x)$ or $h(x)$ are first specified in Equation (1.7), and fragments of the remaining function are then chosen to give an interesting integral. Fragments of the equation obeyed by $h(x)$ are always linear, but fragments of the equation obeyed by $s(x)$ are often simple Riccati equations, chosen to tailor the function $s(x)$ to suit the specific special function. This often involves making the coefficient of the term in $h(x)$ in Equation (1.7) relatively simple. This coefficient is

$$s'(x) + s^2(x) - s(x)p(x) + q(x) - p'(x)$$

and fragments of this coefficient equated to zero give simple Riccati equations to determine $s(x)$ and hence $\bar{f}(x)$.

The new method is illustrated with sample results for known and new integrals in the following sections. Cylinder functions are considered in Section 2 and Whittaker functions in Section 3. A number of new integrals for Whittaker functions are presented which do not come directly from Equation (1.7), but arose naturally when considering the various formulas for the derivatives of Whittaker functions, which are needed to implement Equation (1.7). All results have been checked with Mathematica [3].

2. Cylinder functions

The general cylinder function $Z_n(x) \equiv C_1 J_n(x) + C_2 Y_n(x)$ obeys the Bessel equation

$$y''(x) + \frac{1}{x}y'(x) + \left(1 - \frac{n^2}{x^2}\right)y(x) = 0 \tag{2.1}$$

and for Equation (2.1), the integration formula (1.7) becomes

$$\int \bar{f}(x)\left(h''(x)+\left(2s(x)-\frac{1}{x}\right)h'(x)+\left(s'(x)+s^2(x)-\frac{s(x)}{x}+1-\frac{n^2-1}{x^2}\right)h(x)\right)$$
$$\times Z_n(x)\,\mathrm{d}x=\bar{f}(x)\left(h'(x)Z_n(x)-h(x)Z_n'(x)+h(x)\left(s(x)-\frac{1}{x}\right)Z_n(x)\right). \quad (2.2)$$

The fragment

$$s'(x)+s^2(x)+1=0 \quad (2.3)$$

has the solution

$$s(x)=-\tan(x+\phi) \quad (2.4)$$

where ϕ is an arbitrary phase factor, with the principal special cases being

$$s(x)=-\tan(x) \Rightarrow \bar{f}(x)=\cos(x) \quad (2.5)$$
$$s(x)=\cot(x) \Rightarrow \bar{f}(x)=\sin(x). \quad (2.6)$$

The integration formula corresponding to Equation (2.5) is

$$\int \cos(x)\left(h''(x)-\left(2\tan(x)+\frac{1}{x}\right)h'(x)+\left(\frac{1}{x}\tan(x)-\frac{n^2-1}{x^2}\right)h(x)\right)Z_n(x)\,\mathrm{d}x$$
$$=\cos(x)\left(h'(x)Z_n(x)-h(x)Z_n'(x)-h(x)\left(\tan(x)+\frac{1}{x}\right)Z_n(x)\right). \quad (2.7)$$

The obvious fragments to choose for Equation (2.7) are those which separate the trigonometric and non-trigonometric terms, and these are

$$-2\tan(x)h'(x)+\frac{1}{x}\tan(x)h(x)=0 \quad (2.8)$$
$$h''(x)-\frac{1}{x}h'(x)-\frac{n^2-1}{x^2}h(x)=0. \quad (2.9)$$

Equation (2.8) has the solution

$$h(x)=\sqrt{x} \quad (2.10)$$

which gives the integral

$$\int \frac{\cos(x)Z_n(x)}{x^{3/2}}\,\mathrm{d}x=\frac{4\sqrt{x}}{4n^2-1}\left(\cos(x)\left(\frac{1}{2x}Z_n(x)+Z_n'(x)\right)+\sin(x)Z_n(x)\right) \quad (2.11)$$

and the equivalent result for $s(x)=\cot(x)$ is

$$\int \frac{\sin(x)Z_n(x)}{x^{3/2}}\,\mathrm{d}x=\frac{4\sqrt{x}}{4n^2-1}\left(\sin(x)\left(\frac{1}{2x}Z_n(x)+Z_n'(x)\right)-\cos(x)Z_n(x)\right). \quad (2.12)$$

Equation (2.9) has the solutions

$$h(x)=x^{1\pm n} \quad (2.13)$$

which gives the integrals

$$\int x^{\pm n}\sin(x)\,Z_n(x)\,\mathrm{d}x = \frac{x^{\pm n}}{1\pm 2n}\left(\cos(x)\left(\mp nZ_n(x) + xZ_n'(x)\right) + x\sin(x)\,Z_n(x)\right) \tag{2.14}$$

$$\int x^{\pm n}\cos(x)\,Z_n(x)\,\mathrm{d}x = \frac{x^{\pm n}}{1\pm 2n}\left(\sin(x)\left(\pm nZ_n(x) - xZ_n'(x)\right) + x\cos(x)\,Z_n(x)\right). \tag{2.15}$$

These trigonometric integrals are well known [4] and can be derived in many different ways.

In Equation (2.2), we can take $s(x)$ to be

$$s(x) = \frac{a}{x} + b \Rightarrow \bar{f}(x) = x^a \mathrm{e}^{bx} \tag{2.16}$$

where a and b are constants which can be chosen arbitrarily. For this choice of $s(x)$, Equation (2.2) becomes

$$\begin{aligned}\int x^a\mathrm{e}^{bx}\Bigg(h''(x) + \left(\frac{2a-1}{x} + 2b\right)h'(x) \\ + \left(\frac{(a-1)^2-n^2}{x^2} + \frac{(2a-1)\,b}{x} + b^2 + 1\right)h(x)\Bigg)Z_n(x)\,\mathrm{d}x \\ = x^a\mathrm{e}^{bx}\left(h'(x)\,Z_n(x) - h(x)\,Z_n'(x) + h(x)\left(\frac{a-1}{x} + b\right)Z_n(x)\right)\end{aligned} \tag{2.17}$$

and the constants a and b can be chosen so that any two of the terms in the coefficient of $h(x)$ in this equation can be set to zero. For the choices

$$h(x) = 1;\ a = 1\pm n;\ b = 0 \tag{2.18}$$

Equation (2.17) reduces to the elementary integrals

$$\int x^{1\pm n}Z_n(x)\,\mathrm{d}x = \pm x^{1\pm n}Z_{n\pm 1}(x). \tag{2.19}$$

For the choices

$$h(x) = 1;\ a = 1\pm n;\ b = \pm i \tag{2.20}$$

where the $\pm$ signs are independent. For $b = i$, Equation (2.17) becomes

$$\int x^{\pm n}\mathrm{e}^{\mathrm{i}x}Z_n(x)\,\mathrm{d}x = \frac{x^{1\pm n}\mathrm{e}^{\mathrm{i}x}}{1\pm 2n}\left(\mp iZ_{n\pm 1}(x) + Z_n(x)\right) \tag{2.21}$$

and for $b = -i$ the equation becomes

$$\int x^{\pm n}\mathrm{e}^{-\mathrm{i}x}Z_n(x)\,\mathrm{d}x = \frac{x^{1\pm n}\mathrm{e}^{-\mathrm{i}x}}{1\pm 2n}\left(\pm iZ_{n\pm 1}(x) + Z_n(x)\right). \tag{2.22}$$

For the choices

$$h(x) = 1;\ a = \frac{1}{2}n;\ b = \pm i \tag{2.23}$$

Equation (2.17) becomes

$$\int \frac{e^{\pm ix}}{x^{3/2}} Z_n(x)\ dx = \frac{4\sqrt{x}e^{\pm ix}}{4n^2 - 1}\left(Z_n'(x) + \left(\frac{1}{2x} \mp i\right) Z_n(x)\right) \tag{2.24}$$

which is equivalent to Equations (2.11) and (2.12).

Instead of Equation (2.16), we can take $s(x)$ to be of the form

$$s(x) = \frac{a}{x} + bx \Rightarrow \bar{f}(x) = x^a \exp\left(\frac{bx^2}{2}\right) \tag{2.25}$$

and for this case with $h(x) = 1$, Equation (2.2) becomes

$$\int x^a \exp\left(\frac{bx^2}{2}\right)\left(\frac{(a-1)^2 - n^2}{x^2} + 2ab + 1 + bx^2\right) Z_n(x)\ dx$$
$$= x^a \exp\left(\frac{bx^2}{2}\right)\left(-Z_n'(x) + \left(\frac{a-1}{x} + bx\right) Z_n(x)\right) \tag{2.26}$$

and for the choices

$$a = 1 \pm n;\ b = -\frac{1}{2(1 \pm n)} \tag{2.27}$$

this gives the integral

$$\int x^{3\pm n} \exp\left(-\frac{x^2}{4(1 \pm n)}\right) Z_n(x)\ dx = 2(1 \pm n)\, x^{2\pm n} \exp\left(-\frac{x^2}{4(1 \pm n)}\right) Z_{n\pm 2}(x) \tag{2.28}$$

which was derived recently in [5,6] by quite different methods.

It is sometimes possible to choose $s(x)$ so as to include specific factors in integrals through the factor $\bar{f}(x)$. For example, for $\bar{f}(x) = \ln(x)$ we must choose

$$s(x) = \frac{1}{x \ln(x)}. \tag{2.29}$$

For this choice, Equation (2.2) becomes

$$\int \ln(x)\left(h''(x) + \left(\frac{2}{x\ln(x)} - \frac{1}{x}\right) h'(x) + \left(1 - \frac{2}{x^2 \ln(x)} - \frac{n^2 - 1}{x^2}\right) h(x)\right)$$
$$\times Z_n(x)\ dx = \ln(x)\left(h'(x) Z_n(x) - h(x) Z_n'(x) + h(x)\left(\frac{1}{x\ln(x)} - \frac{1}{x}\right) Z_n(x)\right) \tag{2.30}$$

and we can choose

$$\frac{2}{x\ln(x)} h'(x) - \frac{2}{x^2 \ln(x)} = 0 \tag{2.31}$$

which gives $h(x) = x$ and Equation (2.30) reduces to

$$\int \frac{\ln(x)}{x}\left(x^2 - n^2\right) Z_n(x)\, \mathrm{d}x = \ln(x)\left(Z_n(x) - xZ_n'(x)\right) + (1 - \ln(x))\, Z_n(x) \quad (2.32)$$

which appears to be new.

For the choices

$$s = \pm 1 \Rightarrow f(x) = \mathrm{e}^{\pm x} \quad (2.33)$$

Equation (2.2) reduces to

$$\int \mathrm{e}^{\pm x}\left(h''(x) + \left(\pm 2 - \frac{1}{x}\right) h'(x) + \left(2 \mp \frac{1}{x} - \frac{n^2 - 1}{x^2}\right) h(x)\right) Z_1(x)$$
$$= \mathrm{e}^{\pm x}\left(h'(x)\, Z_1(x) - h(x)\, Z_1'(x) + h(x)\left(\pm 1 - \frac{1}{x}\right) Z_n(x)\right). \quad (2.34)$$

For both signs of the exponential, Equation (2.34) has the fragment

$$h''(x) - \frac{1}{x} h'(x) - \frac{n^2 - 1}{x^2} = 0 \quad (2.35)$$

which has a solution $h(x) = x^{1 \pm n}$ and substituting this in Equation (2.2) gives, with some simplification, the integrals

$$\int (2x \pm 2n + 1)\, x^{\pm n} \mathrm{e}^{x} Z_n(x)\, \mathrm{d}x = x^{1 \pm n} \mathrm{e}^{x}\left(Z_n(x) \pm Z_{n \pm 1}(x)\right) \quad (2.36)$$

$$\int (2x \mp 2n - 1)\, x^{\pm n} \mathrm{e}^{-x} Z_n(x)\, \mathrm{d}x = x^{1 \pm n} \mathrm{e}^{-x}\left(\pm Z_{n \pm 1}(x) - Z_n(x)\right) \quad (2.37)$$

which appear to be new.

3. Whittaker functions

The Whittaker functions $M_{\lambda,\mu}(x)$ and $W_{\lambda,\mu}(x)$ are two independent solutions of the differential equation

$$y''(x) + \left(-\frac{1}{4} + \frac{\lambda}{x} + \frac{\frac{1}{4} - \mu^2}{x^2}\right) y(x) = 0 \quad (3.1)$$

and here the general solution of this equation will be denoted by $\hat{W}_{\lambda,\mu}(x) \equiv C_1 M_{\lambda,\mu}(x) + C_2 W_{\lambda,\mu}(x)$. For Equation (3.1) the integration theorem (1.7) becomes

$$\int \bar{f}(x)\left(h''(x) + 2s(x)\, h'(x) + \left[s'(x) + s^2(x) - \frac{1}{4} + \frac{\lambda}{x} + \frac{\frac{1}{4} - \mu^2}{x^2}\right] h(x)\right)$$
$$\times\, \hat{W}_{\lambda,\mu}(x)\, \mathrm{d}x = \bar{f}(x)\left(\left[h'(x) + h(x)\, s(x)\right] \hat{W}_{\lambda,\mu}(x) - h(x)\, \hat{W}_{\lambda,\mu}'(x)\right) \quad (3.2)$$

and in Equation (3.2) we can take $s(x)=0$ and $h(x)=\hat{W}_{\sigma,\rho}(x)\equiv C_3M_{\sigma,\rho}(x)+C_4W_{\sigma,\rho}(x)$, which gives

$$\int\left(\frac{\lambda-\sigma}{x}+\frac{\rho^2-\mu^2}{x^2}\right)\hat{W}_{\sigma,\rho}(x)\,\hat{W}_{\lambda,\mu}(x)\,\mathrm{d}x$$
$$=\hat{W}'_{\sigma,\rho}(x)\,\hat{W}_{\lambda,\mu}(x)-\hat{W}_{\sigma,\rho}(x)\,\hat{W}'_{\lambda,\mu}(x). \tag{3.3}$$

and Equation (3.3) has the two special cases

$$\int\frac{\hat{W}_{\lambda,\rho}(x)\,\hat{W}_{\lambda,\mu}(x)}{x^2}\,\mathrm{d}x=\frac{\hat{W}'_{\lambda,\rho}(x)\,\hat{W}_{\lambda,\mu}(x)-\hat{W}_{\lambda,\rho}(x)\,\hat{W}'_{\lambda,\mu}(x)}{\rho^2-\mu^2} \tag{3.4}$$

$$\int\frac{\hat{W}_{\sigma,\mu}(x)\,\hat{W}_{\lambda,\mu}(x)}{x}\,\mathrm{d}x=\frac{\hat{W}'_{\sigma,\mu}(x)\,\hat{W}_{\lambda,\mu}(x)-\hat{W}_{\sigma,\mu}(x)\,\hat{W}'_{\lambda,\mu}(x)}{\lambda-\sigma} \tag{3.5}$$

which are given in [7]. A complication in turning equations such as (3.4) and (3.5) into specific formulas is that the recurrence relations for $M_{\lambda,\mu}(x)$ and $W_{\lambda,\mu}(x)$ are different. Differential relations for $M_{\lambda,\mu}(x)$ and $W_{\lambda,\mu}(x)$ are given in [8] in the forms

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{(1/2)x}x^{\mu-1/2}M_{\lambda,\mu}(x)\right]=2\mu\mathrm{e}^{(1/2)x}x^{\mu-1}M_{\lambda-1/2,\mu-1/2}(x) \tag{3.6}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{(1/2)x}x^{-\mu-1/2}M_{\lambda,\mu}(x)\right]=\frac{\frac{1}{2}+\mu-\lambda}{1+2\mu}\mathrm{e}^{(1/2)x}x^{-\mu-1}M_{\lambda-1/2,\mu+1/2}(x) \tag{3.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{(1/2)x}x^{-\lambda}M_{\lambda,\mu}(x)\right]=\left(\frac{1}{2}+\mu-\lambda\right)\mathrm{e}^{(1/2)x}x^{-\lambda-1}M_{\lambda-1,\mu}(x) \tag{3.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{-(1/2)x}x^{\mu-1/2}M_{\lambda,\mu}(x)\right]=2\mu\mathrm{e}^{-(1/2)x}x^{\mu-1}M_{\lambda+1/2,\mu-1/2}(x) \tag{3.9}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{-(1/2)x}x^{-\mu-1/2}M_{\lambda,\mu}(x)\right]=-\frac{\frac{1}{2}+\mu+\lambda}{1+2\mu}\mathrm{e}^{-(1/2)x}x^{-\mu-1}M_{\lambda+1/2,\mu+1/2}(x) \tag{3.10}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{-(1/2)x}x^{\lambda}M_{\lambda,\mu}(x)\right]=\left(\frac{1}{2}+\mu+\lambda\right)\mathrm{e}^{-(1/2)x}x^{\lambda-1}M_{\lambda+1,\mu}(x) \tag{3.11}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{(1/2)x}x^{-\mu-1/2}W_{\lambda,\mu}(x)\right]=-\left(\frac{1}{2}+\mu-\lambda\right)\mathrm{e}^{(1/2)x}x^{-\mu-1}W_{\lambda-1/2,\mu+1/2}(x) \tag{3.12}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{(1/2)x}x^{\mu-1/2}W_{\lambda,\mu}(x)\right]=-\left(\frac{1}{2}-\mu-\lambda\right)\mathrm{e}^{(1/2)x}x^{\mu-1}W_{\lambda-1/2,\mu-1/2}(x) \tag{3.13}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{(1/2)x}x^{-\lambda}W_{\lambda,\mu}(x)\right]=\left(\frac{1}{2}+\mu-\lambda\right)\left(\frac{1}{2}-\mu-\lambda\right)\mathrm{e}^{(1/2)x}x^{-\lambda-1}W_{\lambda-1,\mu}(x) \tag{3.14}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{-(1/2)x}x^{-\mu-1/2}W_{\lambda,\mu}(x)\right]=-\mathrm{e}^{-(1/2)x}x^{-\mu-1}W_{\lambda+1/2,\mu+1/2}(x) \tag{3.15}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{-(1/2)x}x^{\mu-1/2}W_{\lambda,\mu}(x)\right]=-\mathrm{e}^{-(1/2)x}x^{\mu-1}W_{\lambda+1/2,\mu-1/2}(x) \tag{3.16}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{-(1/2)x}x^{\lambda}W_{\lambda,\mu}(x)\right] = -\mathrm{e}^{-(1/2)x}x^{\lambda-1}W_{\lambda+1,\mu}(x) \tag{3.17}$$

These relations give explicit expressions for the derivatives $M'_{\lambda,x}(x)$ and $W'_{\lambda,\mu}(x)$ as

$$M'_{\lambda,\mu}(x) = -\left(\frac{1}{2} + \frac{\mu-\frac{1}{2}}{x}\right)M_{\lambda,\mu}(x) + \frac{2\mu}{\sqrt{x}}M_{\lambda-1/2,\mu-1/2}(x) \tag{3.18}$$

$$M'_{\lambda,\mu}(x) = \left(\frac{\mu+\frac{1}{2}}{x} - \frac{1}{2}\right)M_{\lambda,\mu}(x) + \frac{\frac{1}{2}+\mu-\lambda}{(1+2\mu)\sqrt{x}}M_{\lambda-1/2,\mu+1/2}(x) \tag{3.19}$$

$$M'_{\lambda,\mu}(x) = \left(\frac{\lambda}{x} - \frac{1}{2}\right)M_{\lambda,\mu}(x) + \left(\frac{1}{2}+\mu-\lambda\right)\frac{M_{\lambda-1,\mu}(x)}{x} \tag{3.20}$$

$$M'_{\lambda,\mu}(x) = \left(\frac{1}{2} - \frac{\mu-\frac{1}{2}}{x}\right)M_{\lambda,\mu}(x) + \frac{2\mu}{\sqrt{x}}M_{\lambda+1/2,\mu-1/2}(x) \tag{3.21}$$

$$M'_{\lambda,\mu}(x) = \left(\frac{\mu+\frac{1}{2}}{x} + \frac{1}{2}\right)M_{\lambda,\mu}(x) - \frac{\frac{1}{2}+\mu+\lambda}{(1+2\mu)\sqrt{x}}M_{\lambda+1/2,\mu+1/2}(x) \tag{3.22}$$

$$M'_{\lambda,\mu}(x) = \left(\frac{1}{2} - \frac{\lambda}{x}\right)M_{\lambda,\mu}(x) + \left(\frac{1}{2}+\mu+\lambda\right)\frac{M_{\lambda+1,\mu}(x)}{x} \tag{3.23}$$

$$W'_{\lambda,\mu}(x) = \left(\frac{\mu+\frac{1}{2}}{x} - \frac{1}{2}\right)W_{\lambda,\mu}(x) - \left(\frac{1}{2}+\mu-\lambda\right)\frac{W_{\lambda-1/2,\mu+1/2}(x)}{\sqrt{x}} \tag{3.24}$$

$$W'_{\lambda,\mu}(x) = -\left(\frac{1}{2} + \frac{\mu-\frac{1}{2}}{x}\right)W_{\lambda,\mu}(x) - \left(\frac{1}{2}-\mu-\lambda\right)\frac{W_{\lambda-1/2,\mu-1/2}(x)}{\sqrt{x}} \tag{3.25}$$

$$W'_{\lambda,\mu}(x) = \left(\frac{\lambda}{x} - \frac{1}{2}\right)W_{\lambda,\mu}(x) + \left(\frac{1}{2}+\mu-\lambda\right)\left(\frac{1}{2}-\mu-\lambda\right)\frac{W_{\lambda-1,\mu}(x)}{x} \tag{3.26}$$

$$W'_{\lambda,\mu}(x) = \left(\frac{1}{2} + \frac{\mu+\frac{1}{2}}{x}\right)W_{\lambda,\mu}(x) - \frac{W_{\lambda+1/2,\mu+1/2}(x)}{\sqrt{x}} \tag{3.27}$$

$$W'_{\lambda,\mu}(x) = \left(\frac{1}{2} - \frac{\mu-\frac{1}{2}}{x}\right)W_{\lambda,\mu}(x) - \frac{W_{\lambda+1/2,\mu-1/2}(x)}{\sqrt{x}} \tag{3.28}$$

$$W'_{\lambda,\mu}(x) = \left(\frac{1}{2} - \frac{\lambda}{x}\right)W_{\lambda,\mu}(x) - \frac{W_{\lambda+1,\mu}(x)}{x}. \tag{3.29}$$

In Mathematica [3], the defaults for $M'_{\lambda,\mu}(x)$ and $W'_{\lambda,\mu}(x)$ are Equations (3.23) and (3.29), respectively. When evaluating the derivative $\hat{W}_{\lambda,\mu}(x)$ of a general Whittaker function, any one of the appropriate derivative formulas can be used for $M'_{\lambda,\mu}(x)$ and $W'_{\lambda,\mu}(x)$, which means there are 36 equally valid formulas for $\hat{W}_{\lambda,\mu}(x)$.

3.1. Generalized integrals

The relations (3.6)–(3.17) are all of the form

$$u'(x) = v(x) \tag{3.30}$$

where $u(x)$ and $v(x)$ are simple factors involving a single Whittaker function. Multiplying both sides of Equation (3.30) by $mu(x)^{m-1}$, where $m \in \mathbb{C}$, and integrating both sides gives integrals of the form

$$\int u^{m-1}(x)\, v(x)\, \mathrm{d}x = \frac{u^m(x)}{m} \text{ for } m \neq 0. \tag{3.31}$$

Integrals of this type were given for cylinder functions in [9]. Applying Equation (3.31) to Equations (3.6)–(3.17), and shifting the parameters as appropriate, gives the respective generalized integrals for $m \neq 0$

$$\int \mathrm{e}^{mx/2} x^{m\mu-1/2} M^{m-1}_{\lambda+1/2,\mu+1/2}(x)\, M_{\lambda,\mu}(x)\, \mathrm{d}x = \mathrm{e}^{mx/2} x^{m\mu} \frac{M^m_{\lambda+1/2,\mu+1/2}(x)}{(2\mu+1)\, m} \tag{3.32}$$

$$\int \mathrm{e}^{mx/2} x^{-m\mu-1/2} M^{m-1}_{\lambda+1/2,\mu-1/2}(x)\, M_{\lambda,\mu}(x)\, \mathrm{d}x = \mathrm{e}^{mx/2} x^{-m\mu} \frac{2\mu M^m_{\lambda+1/2,\mu-1/2}(x)}{\left(\mu-\lambda-\frac{1}{2}\right) m} \tag{3.33}$$

$$\int \mathrm{e}^{mx/2} x^{-m(\lambda+1)-1} M^{m-1}_{\lambda+1,\mu}(x)\, M_{\lambda,\mu}(x)\, \mathrm{d}x = \mathrm{e}^{mx/2} x^{-m(\lambda+1)} \frac{M^m_{\lambda+1,\mu}(x)}{\left(\mu-\lambda-\frac{1}{2}\right) m} \tag{3.34}$$

$$\int \mathrm{e}^{-(mx/2)} x^{m\mu-1/2} M^{m-1}_{\lambda-1/2,\mu+1/2}(x)\, M_{\lambda,\mu}(x)\, \mathrm{d}x = \mathrm{e}^{-(mx/2)} x^{m\mu} \frac{M^m_{\lambda-1/2,\mu+1/2}(x)}{(2\mu+1)\, m} \tag{3.35}$$

$$\int \mathrm{e}^{-(mx/2)} x^{-m\mu-1/2} M^{m-1}_{\lambda-1/2,\mu-1/2}(x)\, M_{\lambda,\mu}(x)\, \mathrm{d}x = -\mathrm{e}^{-(mx/2)} x^{-m\mu} \frac{2\mu M^m_{\lambda-1/2,\mu-1/2}(x)}{\left(\mu+\lambda-\frac{1}{2}\right) m} \tag{3.36}$$

$$\int \mathrm{e}^{-(mx/2)} x^{m(\lambda-1)-1} M^{m-1}_{\lambda-1,\mu}(x)\, M_{\lambda,\mu}(x)\, \mathrm{d}x = \mathrm{e}^{-(mx/2)} x^{m(\lambda-1)} \frac{M^m_{\lambda-1,\mu}(x)}{\left(\mu+\lambda-\frac{1}{2}\right) m} \tag{3.37}$$

$$\int \mathrm{e}^{mx/2} x^{-m\mu-1/2} W^{m-1}_{\lambda+1/2,\mu-1/2}(x)\, W_{\lambda,\mu}(x)\, \mathrm{d}x = \mathrm{e}^{mx/2} x^{-m\mu} \frac{W^m_{\lambda+1/2,\mu-1/2}(x)}{\left(\lambda-\mu+\frac{1}{2}\right) m} \tag{3.38}$$

$$\int \mathrm{e}^{mx/2} x^{m\mu-1/2} W^{m-1}_{\lambda+1/2,\mu+1/2}(x)\, W_{\lambda,\mu}(x)\, \mathrm{d}x = \mathrm{e}^{mx/2} x^{m\mu} \frac{W^m_{\lambda+1/2,\mu+1/2}(x)}{\left(\mu+\lambda+\frac{1}{2}\right) m} \tag{3.39}$$

$$\int \mathrm{e}^{mx/2} x^{-m(\lambda+1)-1} W^{m-1}_{\lambda+1,\mu}(x)\, W_{\lambda,\mu}(x)\, \mathrm{d}x = \frac{\mathrm{e}^{mx/2} x^{-m(\lambda+1)} W^m_{\lambda+1,\mu}(x)}{\left(\lambda-\mu+\frac{1}{2}\right)\left(\lambda+\mu+\frac{1}{2}\right) m} \tag{3.40}$$

$$\int \mathrm{e}^{-(mx/2)} x^{-m\mu-1/2} W^{m-1}_{\lambda-1/2,\mu-1/2}(x)\, W_{\lambda,\mu}(x)\, \mathrm{d}x = -\mathrm{e}^{-(mx/2)} x^{-m\mu} \frac{W^m_{\lambda-1/2,\mu-1/2}(x)}{m} \tag{3.41}$$

$$\int e^{-(mx/2)} x^{m\mu-1/2} W^{m-1}_{\lambda-1/2,\mu+1/2}(x)\, W_{\lambda,\mu}(x)\, dx = -e^{-(mx/2)} x^{m\mu} \frac{W^{m}_{\lambda-1/2,\mu+1/2}(x)}{m} \quad (3.42)$$

$$\int e^{-(mx/2)} x^{m(\lambda-1)-1} W^{m-1}_{\lambda-1,\mu}(x)\, W_{\lambda,\mu}(x)\, dx = -e^{-(mx/2)} x^{m(\lambda-1)} \frac{W^{m}_{\lambda-1,\mu}(x)}{m}. \quad (3.43)$$

Equations (3.32)–(3.43) for $m = 1$ are given in [7], though there are typographical errors in the equivalents of (3.32) and (3.33).

Dividing both sides of Equations (3.18)–(3.29) by $M_{\lambda,\mu}(x)$ or $W_{\lambda,\mu}(x)$ as appropriate, and integrating both sides of the resulting equations give the logarithmic quotient integrals

$$\int \frac{M_{\lambda,\mu}(x)}{\sqrt{x} M_{\lambda+1/2,\mu+1/2}(x)}\, dx = \frac{1}{2\mu+1} \ln\left(x^{\mu} e^{x/2} M_{\lambda+1/2,\mu+1/2}(x)\right) \quad (3.44)$$

$$\int \frac{M_{\lambda,\mu}(x)}{\sqrt{x} M_{\lambda+1/2,\mu-1/2}(x)}\, dx = \frac{2\mu}{\mu-\lambda-\frac{1}{2}} \ln\left(x^{-\mu} e^{x/2} M_{\lambda+1/2,\mu-1/2}(x)\right) \quad (3.45)$$

$$\int \frac{M_{\lambda,\mu}(x)}{x M_{\lambda+1,\mu}(x)}\, dx = \frac{\ln\left(x^{-\lambda-1} e^{x/2} M_{\lambda+1,\mu}(x)\right)}{\mu-\lambda-\frac{1}{2}} \quad (3.46)$$

$$\int \frac{M_{\lambda,\mu}(x)}{\sqrt{x} M_{\lambda-1/2,\mu+1/2}(x)}\, dx = \frac{1}{2\mu+1} \ln\left(x^{\mu} e^{-(x/2)} M_{\lambda-1/2,\mu+1/2}(x)\right) \quad (3.47)$$

$$\int \frac{M_{\lambda,\mu}(x)}{\sqrt{x} M_{\lambda-1/2,\mu-1/2}(x)}\, dx = -\frac{2\mu}{\mu+\lambda-\frac{1}{2}} \ln\left(x^{-\mu} e^{-(x/2)} M_{\lambda-1/2,\mu-1/2}(x)\right) \quad (3.48)$$

$$\int \frac{M_{\lambda,\mu}(x)}{x M_{\lambda-1,\mu}(x)}\, dx = \frac{\ln\left(x^{\lambda-1} e^{-(x/2)} M_{\lambda-1,\mu}(x)\right)}{\mu+\lambda-\frac{1}{2}}. \quad (3.49)$$

$$\int \frac{W_{\lambda,\mu}(x)}{\sqrt{x} W_{\lambda+1/2,\mu-1/2}(x)}\, dx = \frac{\ln\left(x^{-\mu} e^{x/2} W_{\lambda+1/2,\mu-1/2}(x)\right)}{\lambda-\mu+\frac{1}{2}} \quad (3.50)$$

$$\int \frac{W_{\lambda,\mu}(x)}{\sqrt{x} W_{\lambda+1/2,\mu+1/2}(x)}\, dx = \frac{\ln\left(x^{\mu} e^{x/2} W_{\lambda+1/2,\mu+1/2}(x)\right)}{\mu+\lambda+\frac{1}{2}} \quad (3.51)$$

$$\int \frac{W_{\lambda,\mu}(x)}{x W_{\lambda+1,\mu}(x)}\, dx = \frac{\ln\left(x^{-\lambda-1} e^{x/2} W_{\lambda+1,\mu}(x)\right)}{\left(\lambda-\mu+\frac{1}{2}\right)\left(\lambda+\mu+\frac{1}{2}\right)} \quad (3.52)$$

$$\int \frac{W_{\lambda,\mu}(x)}{\sqrt{x} W_{\lambda-1/2,\mu-1/2}(x)}\, dx = -\ln\left(x^{-\mu} e^{-(x/2)} W_{\lambda-1/2,\mu-1/2}(x)\right) \quad (3.53)$$

$$\int \frac{W_{\lambda,\mu}(x)}{\sqrt{x} W_{\lambda-1/2,\mu+1/2}(x)}\, dx = -\ln\left(x^{\mu} e^{-(x/2)} W_{\lambda-1/2,\mu+1/2}(x)\right) \quad (3.54)$$

$$\int \frac{W_{\lambda,\mu}(x)}{x W_{\lambda-1,\mu}(x)}\, dx = -\ln\left(x^{\lambda-1} e^{-(x/2)} W_{\lambda-1,\mu}(x)\right). \quad (3.55)$$

The logarithmic integrals (3.44)–(3.55) are the equivalent of the integrals (3.32)–(3.43) for the forbidden value $m = 0$.

3.2. Integrals from the integration formula

The choice of the arbitrary function $s(x)$ in Equation (2.16) for cylinder functions is also useful for Whittaker functions. For this choice the terms within the square bracket in Equation (3.2) become

$$\frac{\left(a-\frac{1}{2}\right)^2-\mu^2}{x^2}+\frac{2ab+\lambda}{x}+b^2-\frac{1}{4} \tag{3.56}$$

and a and b can be chosen such that any two terms in (3.56) are zero. For $h(x)=1$, Equation (3.2) reduces to

$$\begin{aligned}\int x^a \mathrm{e}^{bx}\left(\frac{\left(a-\frac{1}{2}\right)^2-\mu^2}{x^2}+\frac{2ab+\lambda}{x}+b^2-\frac{1}{4}\right)\hat{W}_{\lambda,\mu}(x)\,\mathrm{d}x \\ = x^a\mathrm{e}^{bx}\left(\left(\frac{a}{x}+b\right)\hat{W}_{\lambda,\mu}(x)-\hat{W}'_{\lambda,\mu}(x)\right).\end{aligned} \tag{3.57}$$

The choices $a=\frac{1}{2}\pm\mu$ and $b=\pm\frac{1}{2}$, where the $\pm$ signs are independent, give the four integrals

$$\int x^{\pm\mu-1/2}\mathrm{e}^{\pm x/2}\hat{W}_{\lambda,\mu}(x)\,\mathrm{d}x=\frac{x^{\pm\mu+1/2}\mathrm{e}^{\pm x/2}}{\lambda\pm\left(\frac{1}{2}\pm\mu\right)}\left(\left(\frac{\frac{1}{2}\pm\mu}{x}\pm\frac{1}{2}\right)\hat{W}_{\lambda,\mu}(x)-\hat{W}'_{\lambda,\mu}(x)\right) \tag{3.58}$$

The choices $b=\pm\frac{1}{2}$and $a=\mp\lambda$, where the $\pm$ and $\mp$ signs are not independent, give the two integrals

$$\int x^{\mp\lambda-2}\mathrm{e}^{\pm x/2}\hat{W}_{\lambda,\mu}(x)\,\mathrm{d}x=\frac{x^{\mp\lambda}\mathrm{e}^{\pm x/2}}{\left(\pm\lambda+\frac{1}{2}\right)^2-\mu^2}\left(\left(\mp\frac{\lambda}{x}\pm\frac{1}{2}\right)\hat{W}_{\lambda,\mu}(x)-\hat{W}'_{\lambda,\mu}(x)\right). \tag{3.59}$$

Equations (3.58) and (3.59) give the 12 integrals for $m=1$ in Equations (3.32)–(3.43), generalized to an arbitrary Whittaker function. The choices $a=\frac{1}{2}\pm\mu$ and $b=-\lambda/(1\pm 2\mu)$, where the $\pm$ signs are not independent, give the two integrals

$$\begin{aligned}\int x^{1/2\pm\mu}\mathrm{e}^{-\lambda x/(1\pm2\mu)}\hat{W}_{\lambda,\mu}(x)\,\mathrm{d}x \\ =\frac{x^{1/2\pm\mu}\mathrm{e}^{-\lambda x/(1\pm2\mu)}}{\frac{\lambda^2}{(1\pm2\mu)^2}-\frac{1}{4}}\left(\left(\frac{\frac{1}{2}\pm\mu}{x}-\frac{\lambda}{1\pm2\mu}\right)\hat{W}_{\lambda,\mu}(x)-\hat{W}'_{\lambda,\mu}(x)\right)\end{aligned} \tag{3.60}$$

which appear to be new.

3.3. Integrals involving Whittaker functions with cylinder and modified cylinder functions

For the choice

$$s(x)=\frac{1}{2x}\Rightarrow\bar{f}(x)=\sqrt{x} \tag{3.61}$$

Equation (3.2) becomes

$$\int \sqrt{x}\left(h''(x) + \frac{1}{x}h'(x) + \left(-\frac{1}{4} + \frac{\lambda}{x} - \frac{\mu^2}{x^2}\right)h(x)\right)\hat{W}_{\lambda,\mu}(x)\,\mathrm{d}x$$
$$= \sqrt{x}\left(\left(h'(x) + \frac{1}{2x}h(x)\right)\hat{W}_{\lambda,\mu}(x) - h(x)\,\hat{W}'_{\lambda,\mu}(x)\right) \qquad (3.62)$$

Taking $h(x)$ in Equation (3.62) to be the general cylinder function $Z_\mu(x)$, which is the solution of the equation

$$h''(x) + \frac{1}{x}h'(x) + \left(1 - \frac{\mu^2}{x^2}\right)h(x) \qquad (3.63)$$

gives the integral

$$\int \frac{4\lambda - 5x}{\sqrt{x}} Z_\mu(x)\,\hat{W}_{\lambda,\mu}(x)\,\mathrm{d}x$$
$$= 4\sqrt{x}\left(\left(Z'_\mu(x) + \frac{Z_\mu(x)}{2x}\right)\hat{W}_{\lambda,\mu}(x) - Z_\mu(x)\,\hat{W}'_{\lambda,\mu}(x)\right). \qquad (3.64)$$

For the choices

$$s(x) = \pm\frac{1}{2} \Rightarrow \bar{f}(x) = \mathrm{e}^{\pm x/2} \qquad (3.65)$$

Equation (3.2) becomes

$$\int \mathrm{e}^{\pm x/2}\left(h''(x) \pm h'(x) + \left(\frac{\lambda}{x} + \frac{\frac{1}{4} - \mu^2}{x^2}\right)h(x)\right)\hat{W}_{\lambda,\mu}(x)\,\mathrm{d}x$$
$$= \mathrm{e}^{\pm x/2}\left(\left(h'(x) \pm \frac{h(x)}{2}\right)\hat{W}_{\lambda,\mu}(x) - h(x)\,\hat{W}'_{\lambda,\mu}(x)\right) \qquad (3.66)$$

The fragments

$$\frac{\mathrm{d}^2h}{\mathrm{d}x^2} \pm \frac{\mathrm{d}h}{\mathrm{d}x} + \frac{\frac{1}{4} - \mu^2}{x^2}h = 0 \qquad (3.67)$$

have the general solutions

$$h(x) = \sqrt{x}\hat{K}_\mu\left(\frac{x}{2}\right)\mathrm{e}^{\mp x/2} \qquad (3.68)$$

where $\hat{K}_\mu(x) \equiv C_3K_\mu(x) + C_4I_\mu(x)$ is the general modified cylinder function. Substituting Equation (3.68) into Equation (3.66) gives, for both signs in Equations (3.65)–(3.68) the integral

$$\int \frac{1}{\sqrt{x}}\hat{K}_\mu\left(\frac{x}{2}\right)\hat{W}_{\lambda,\mu}(x)\,\mathrm{d}x = \frac{1}{\lambda}\left(\left(\sqrt{x}\hat{K}_\mu\left(\frac{x}{2}\right)\right)'\hat{W}_{\lambda,\mu}(x) - \sqrt{x}\hat{K}_\mu\left(\frac{x}{2}\right)\hat{W}'_{\lambda,\mu}(x)\right). \qquad (3.69)$$

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