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Research Article

Indefinite integrals for some orthogonal polynomials obtained using integrating factors

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ABSTRACT

A method has been presented recently for deriving integrals of special functions using two kinds of integrating factor for the homogeneous second-order linear differential equations which many special functions obey. The classical orthogonal polynomials are well-suited for this method, and results are given here for the Gegenbauer, Hermite and Laguerre polynomials. All the integrals presented here appear to be new and have been checked using Mathematica. Results for other orthogonal polynomials will be presented separately.

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1. Introduction

In a recent paper [1], two integrating factors $f(x)$ and $\hat{f}(x)$ were considered for the differential equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0 \tag{1.1}$$

which is obeyed by many special functions for suitable choices of the functions $p(x)$ and $q(x)$. The function $f(x)$ is identical to the Lagrangian factor introduced in [2,3] and is given by

$$f(x) = \exp\left(\int p(x)\,\mathrm{d}x\right) \tag{1.2}$$

and in [2,3] an integration involving $f(x)$ was derived

$$\int f(x)(h''(x) + p(x)h'(x) + q(x)h(x))\,\mathrm{d}x = f(x)(h'(x)y(x) - h(x)y'(x)). \tag{1.3}$$

where $h(x)$ is an arbitrary twice differentiable function. The function $f(x)$ is also the integrating factor for the first two terms of Equation (1.1), such that

$$f(x)(y''(x) + p(x)y'(x)) = (f(x)y'(x))' \tag{1.4}$$

and hence from Equation (1.1)

$$f(x)q(x)y(x) = -[f(x)y'(x)]'. \tag{1.5}$$

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Multiplying both sides of Equation (1.5) by $m(f(x)y'(x))^{m-1}$ and integrating gives [1] the general integration formula

$$\int q(x)f^m(x)y'^{m-1}(x)y(x)\,\mathrm{d}x = -\frac{f^m(x)y'^m(x)}{m} \tag{1.6}$$

which applies to all solutions $y(x)$ of Equation (1.1).

The function $\hat{f}(x)$ introduced in [1] is the integrating factor for the second two terms of Equation (1.1), such that [1]

$$\hat{f}(x)\left(y' + \frac{q(x)}{p(x)}y(x)\right) = (\hat{f}(x)y(x))' \tag{1.7}$$

and this factor can only be introduced for $p(x) \neq 0$, as $\hat{f}(x)$ is given by

$$\hat{f}(x) = \exp\left(\int \frac{q(x)}{p(x)}\,\mathrm{d}x\right). \tag{1.8}$$

Employing Equation (1.8) in Equation (1.1) gives

$$\frac{\hat{f}(x)y''(x)}{p(x)} = -[\hat{f}(x)y(x)]' \tag{1.9}$$

and multiplying both sides of Equation (1.9) by $m(\hat{f}(x)y(x))^{m-1}$ gives on integration the general (for $p(x) \neq 0$) integration formula [1]

$$\int \frac{1}{p(x)}\hat{f}^m(x)y^{m-1}(x)y''(x)\,\mathrm{d}x = -\frac{\hat{f}^m(x)y^m(x)}{m}. \tag{1.10}$$

Equations (1.6) and (1.10) were used in [1] to derive integrals of various special functions, with Equation (1.10) giving integrals of a type which seem to have been little explored previously. However, in [1] Equations (1.6) and (1.10) were not applied to the classical orthogonal polynomials, which are some of the promising cases for these formulas, and these functions are examined in Section 2 below. Section 3 examines some additional functions which were also not covered in [1]. All integrals presented have been checked with Mathematica [4].

2. Gegenbauer polynomials

The Gegenbauer polynomials $y(x) = C_n^{\lambda}(x)$ for $n \in \mathbb{N}_0$ obey the differential equation [5]

$$y''(x) - \frac{(2\lambda+1)x}{1-x^2}y'(x) + \frac{n(2\lambda+n)}{1-x^2}y(x) = 0 \tag{2.1}$$

for which

$$f(x) = (1-x^2)^{\lambda+\frac{1}{2}}; \quad \hat{f}(x) = x^{-n\frac{2\lambda+n}{2\lambda+1}}. \tag{2.2}$$

The derivatives of these polynomials are [5]

$$y'(x) = 2\lambda C_{n-1}^{\lambda+1}(x) \tag{2.3}$$

$$y''(x) = 4\lambda(\lambda+1)C_{n-2}^{\lambda+2}(x) \tag{2.4}$$

and substituting these results into Equations (1.6) and (1.10) gives the respective integrals

$$\int (1-x^2)^{m(\lambda+\frac{1}{2})-1}(C_{n-1}^{\lambda+1}(x))^{m-1}C_n^{\lambda}(x)\,\mathrm{d}x = -\frac{2\lambda(1-x^2)^{m(\lambda+\frac{1}{2})}(C_{n-1}^{\lambda+1}(x))^m}{mn(2\lambda+n)} \tag{2.5}$$

$$\int (1-x^2)x^{-mn\frac{2\lambda+n}{2\lambda+1}-1}\left(C_n^{\lambda}(x)\right)^{m-1}C_{n-2}^{\lambda+2}(x)\,\mathrm{d}x = \frac{(2\lambda+1)x^{-mn\frac{2\lambda+n}{2\lambda+1}}\left(C_n^{\lambda}(x)\right)^m}{4\lambda(\lambda+1)m}. \tag{2.6}$$

The Gegenbauer polynomials also obey the recurrences [5]

$$\left(C_n^{\lambda}(x)\right)' - \frac{(n+2\lambda)x}{1-x^2}C_n^{\lambda}(x) = -\frac{n+1}{1-x^2}C_{n+1}^{\lambda}(x) \tag{2.7}$$

$$\left(C_n^{\lambda}(x)\right)' + \frac{nx}{1-x^2}C_n^{\lambda}(x) = \frac{n+2\lambda-1}{1-x^2}C_{n-1}^{\lambda}(x) \tag{2.8}$$

and these equations can be integrated with integrating factors to give, respectively

$$\left[(1-x^2)^{-\frac{n}{2}}C_n^{\lambda}(x)\right]' = (n+2\lambda-1)\,(1-x^2)^{-\frac{n}{2}-1}C_{n-1}^{\lambda}(x) \tag{2.9}$$

$$\left[(1-x^2)^{\frac{n}{2}+\lambda}C_n^{\lambda}(x)\right]' = -(n+1)(1-x^2)^{\frac{n}{2}+\lambda-1}C_{n+1}^{\lambda}(x). \tag{2.10}$$

The differential equation obeyed by a function of the form $w(x) = a(x)y(x)$ where $y(x)$ obeys Equation (1.1) is

$$\begin{aligned} w''(x) &+ \left(p(x) - 2\frac{a'(x)}{a(x)}\right)w'(x) \\ &+ \left(2\left(\frac{a'(x)}{a(x)}\right)^2 - \frac{a''(x)}{a(x)} - p(x)\frac{a'(x)}{a(x)} + q(x)\right)w(x) = 0. \end{aligned} \tag{2.11}$$

Therefore if we define

$$y_1(x) = (1-x^2)^{-\frac{n}{2}}C_n^{\lambda}(x) \Rightarrow y_1'(x) = (n+2\lambda-1)\,(1-x^2)^{-\frac{n}{2}-1}C_{n-1}^{\lambda}(x) \tag{2.12}$$

then $y_1(x)$ obeys the differential equation

$$y_1''(x) - \frac{(2n+2\lambda+1)x}{1-x^2}y_1'(x) + \frac{n\,(n+2\lambda-1)}{(1-x^2)^2}y_1(x) = 0 \tag{2.13}$$

for which

$$f(x) = (1-x^2)^{n+\lambda+\frac{1}{2}}. \tag{2.14}$$

Similarly, defining

$$y_2(x) = (1-x^2)^{\frac{n}{2}+\lambda}C_n^{\lambda}(x) \Rightarrow y_2'(x) = -(n+1)(1-x^2)^{\frac{n}{2}+\lambda-1}C_{n+1}^{\lambda}(x) \tag{2.15}$$

then $y_2(x)$ obeys the differential equation

$$y_2''(x) + \frac{(2n+2\lambda-1)x}{1-x^2}y_2'(x) + \frac{(n+1)(n+2\lambda)}{(1-x^2)^2}y_2(x) = 0 \tag{2.16}$$

for which

$$f(x) = (1-x^2)^{-(n+\lambda-\frac{1}{2})}. \tag{2.17}$$

Substituting these results for $y_1(x)$ and $y_2(x)$ into Equation (1.6) gives the respective integrals

$$\int (1-x^2)^{m(\frac{n-1}{2}+\lambda)-1} \left(C_{n-1}^{\lambda}(x)\right)^{m-1} C_n^{\lambda}(x)\,\mathrm{d}x = -\frac{(1-x^2)^{m(\frac{n-1}{2}+\lambda)} \left(C_{n-1}^{\lambda}(x)\right)^{m}}{mn} \tag{2.18}$$

$$\int (1-x^2)^{-m(\frac{n+1}{2})-1} \left(C_{n+1}^{\lambda}(x)\right)^{m-1} C_n^{\lambda}(x)\,\mathrm{d}x = \frac{(1-x^2)^{-m(\frac{n+1}{2})} \left(C_{n+1}^{\lambda}(x)\right)^{m}}{m(n+2\lambda)}. \tag{2.19}$$

2.1. Additional relations

Six additional relations are given sequentially in [6] involving the functions $C_{2n}^{\lambda}(\sqrt{x})$ and $C_{2n+1}^{\lambda}(\sqrt{x})$, which can be used to obtain 6 pairs of general integrals from Equations (1.6) and (1.10). On grounds of space, only four of these are considered here. The first of the relations, in notation somewhat different from [6], is

$$\frac{\mathrm{d}^r}{\mathrm{d}x^r}[x^{\lambda+n+r-1}C_{2n}^{\lambda}(\sqrt{x})] = (\lambda)_r x^{\lambda+n-1}C_{2n}^{\lambda+r}(\sqrt{x}) \tag{2.20}$$

which gives the derivatives

$$y_3'(x) = [x^{\lambda+n}C_{2n}^{\lambda}(\sqrt{x})]' = \lambda x^{\lambda+n-1}C_{2n}^{\lambda+1}(\sqrt{x}) \tag{2.21}$$

$$y_4''(x) = [x^{\lambda+n+1}C_{2n}^{\lambda}(\sqrt{x})]'' = \lambda(\lambda+1)x^{\lambda+n-1}C_{2n}^{\lambda+2}(\sqrt{x}). \tag{2.22}$$

The first step in obtaining the differential equations obeyed by $y_3(x)$ and $y_4(x)$ is to first obtain the differential equation obeyed by $y(x) = C_n^{\lambda}(\sqrt{x})$. Transforming the independent variable in Equation (2.1) gives this differential equation as

$$y''(x) + \frac{1-2(\lambda+1)x}{2x(1-x)}y'(x) + \frac{n(2\lambda+n)}{4x(1-x)}y(x) = 0 \tag{2.23}$$

and for $n \to 2n$ this becomes

$$y''(x) + \frac{1-2(\lambda+1)x}{2x(1-x)}y'(x) + \frac{n(\lambda+n)}{x(1-x)}y(x) = 0 \tag{2.24}$$

and for $n \to 2n+1$

$$y''(x) + \frac{1-2(\lambda+1)x}{2x(1-x)}y'(x) + \frac{(2n+1)(2\lambda+2n+1)}{4x(1-x)}y(x) = 0. \tag{2.25}$$

From Equations (2.11) and (2.24), $y_3(x)$ obeys the differential equation

$$y_3''(x) + \frac{1-4(\lambda+n)+2(\lambda+2n-1)x}{2x(1-x)}y_3'(x) + \frac{(\lambda+n)(2\lambda+2n+1)}{2x^2(1-x)}y_3(x) = 0 \tag{2.26}$$

for which

$$f(x) = x^{\frac{1}{2}-2\lambda-2n}(1-x)^{\frac{1}{2}+\lambda} \tag{2.27}$$

and $y_4(x)$ obeys the differential equation

$$y_4''(x) + \frac{2(\lambda+2n+1)x - 3 - 4(n+\lambda)}{2x(1-x)} y_4'(x) + \frac{(2\lambda+3+2n)(\lambda+n+1) - 2(\lambda+2n+1)x}{2x^2(1-x)} y_4(x) = 0 \tag{2.28}$$

for which

$$\hat{f}(x) = x^{-\frac{(\lambda+n+1)(2\lambda+2n+3)}{4(\lambda+n)+3}} [2(2n+\lambda+1)x - 3 - 4(n+\lambda)]^{\frac{(\lambda+n)(2\lambda+2n+1)}{4(\lambda+n)+3}}. \tag{2.29}$$

Applying Equations (2.26) and (2.27) to Equation (1.6) gives the integral

$$\int x^{-m\left(n+\lambda+\frac{1}{2}\right)-1}(1-x)^{m\left(\lambda+\frac{1}{2}\right)-1}\left(C_{2n}^{\lambda+1}(\sqrt{x})\right)^{m-1} C_{2n}^{\lambda}(\sqrt{x})\,\mathrm{d}x = -\frac{2\lambda x^{-m\left(n+\lambda+\frac{1}{2}\right)}(1-x)^{m\left(\lambda+\frac{1}{2}\right)}\left(C_{2n}^{\lambda+1}(\sqrt{x})\right)^{m}}{m(\lambda+n)(2\lambda+2n+1)}. \tag{2.30}$$

Changing the dependent variable in the integral (2.30) such that $t = \sqrt{x}$ and then relabelling such that $t \to x$ in the transformed integral gives the integral

$$\int x^{-m(2n+2\lambda+1)-1}(1-x^2)^{m(\lambda+\frac{1}{2})-1}(C_{2n}^{\lambda+1}(x))^{m-1} C_{2n}^{\lambda}(x)\,\mathrm{d}x = -\frac{\lambda x^{-m(2n+2\lambda+1)}(1-x^2)^{m(\lambda+\frac{1}{2})}(C_{2n}^{\lambda+1}(x))^{m}}{m(\lambda+n)(2\lambda+2n+1)}. \tag{2.31}$$

Applying Equations (2.28) and (2.29) to Equation (1.10) gives the integral

$$\begin{aligned}\int &(1-x)x^{2m\frac{(\lambda+n+1)(\lambda+n)}{4\lambda+4n+3}-1} \\ &\times [2(2n+\lambda+1)x - 3 - 4(n+\lambda)]^{m\frac{(\lambda+n)(2\lambda+2n+1)}{4\lambda+4n+3}-1} (C_{2n}^{\lambda}(\sqrt{x}))^{m-1} C_{2n}^{\lambda+2}(\sqrt{x})\,\mathrm{d}x \\ &= -\frac{x^{2m\frac{(\lambda+n+1)(\lambda+n)}{4\lambda+4n+3}} [2(2n+\lambda+1)x - 3 - 4(n+\lambda)]^{m\frac{(\lambda+n)(2\lambda+2n+1)}{4(\lambda+n)+3}} (C_{2n}^{\lambda}(\sqrt{x}))^{m}}{2m\lambda(\lambda+1)}\end{aligned} \tag{2.32}$$

and changing the dependent variable such that $t = \sqrt{x}$ and relabelling as above gives the integral

$$\begin{aligned}\int &(1-x^2)x^{4m\frac{(\lambda+n+1)(\lambda+n)}{4\lambda+4n+3}-1} \\ &\times \left[2(2n+\lambda+1)x^2 - 3 - 4(n+\lambda)\right]^{m\frac{(\lambda+n)(2\lambda+2n+1)}{4\lambda+4n+3}-1} (C_{2n}^{\lambda}(x))^{m-1} C_{2n}^{\lambda+2}(x)\,\mathrm{d}x \\ &= -\frac{x^{4m\frac{(\lambda+n+1)(\lambda+n)}{4\lambda+4n+3}} \left[2(2n+\lambda+1)x^2 - 3 - 4(n+\lambda)\right]^{m\frac{(\lambda+n)(2\lambda+2n+1)}{4(\lambda+n)+3}} (C_{2n}^{\lambda}(x))^{m}}{4m\lambda(\lambda+1)}.\end{aligned} \tag{2.33}$$

Three other relations of this type given in [6] which will be considered here are

$$\frac{\mathrm{d}^r}{\mathrm{d}x^r}\left[x^{\lambda+n+r-\frac{1}{2}}C_{2n+1}^{\lambda}(\sqrt{x})\right]=(\lambda)_r x^{\lambda+n-\frac{1}{2}}C_{2n+1}^{\lambda+r}(\sqrt{x}) \tag{2.34}$$

$$\frac{\mathrm{d}^r}{\mathrm{d}x^r}\left[x^{r-n-1}C_{2n}^{\lambda}(\sqrt{x})\right]=(\lambda)_r x^{-n-1}C_{2n-2r}^{\lambda+r}(\sqrt{x}) \tag{2.35}$$

$$\frac{\mathrm{d}^r}{\mathrm{d}x^r}\left[x^{r-n-\frac{3}{2}}C_{2n+1}^{\lambda}(\sqrt{x})\right]=(\lambda)_r x^{-n-\frac{3}{2}}C_{2n-2r+1}^{\lambda+r}(\sqrt{x}). \tag{2.36}$$

Equation (2.34) gives two functions $y_5(x)$ and $y_6(x)$ for which the first and second derivatives, respectively, are simple. The functions, the differential equations they obey, $f(x)$, $\hat{f}(x)$ and the final resulting integrals in the form where $\sqrt{x}\to x$ are given by

$$y_5'(x)=\left[x^{\lambda+n+\frac{1}{2}}C_{2n+1}^{\lambda}(\sqrt{x})\right]'=\lambda x^{\lambda+n-\frac{1}{2}}C_{2n+1}^{\lambda+1}(\sqrt{x}) \tag{2.37}$$

$$y_5''+\frac{(2n+\lambda)x-\left(2n+2\lambda+\frac{1}{2}\right)}{x(1-x)}y_5'(x)+\frac{(n+\lambda+1)\left(n+\lambda+\frac{1}{2}\right)}{x^2(1-x)}y_5(x)=0 \tag{2.38}$$

$$f(x)=x^{-2n-2\lambda-\frac{1}{2}}(1-x)^{\lambda+\frac{1}{2}} \tag{2.39}$$

$$\begin{aligned}&\int x^{-2m(n+\lambda+1)-1}(1-x^2)^{m\left(\lambda+\frac{1}{2}\right)-1}\left(C_{2n+1}^{\lambda+1}(x)\right)^{m-1}C_{2n+1}^{\lambda}(x)\,\mathrm{d}x\\&\quad=-\frac{\lambda x^{-2m(n+\lambda+1)}(1-x^2)^{m\left(\lambda+\frac{1}{2}\right)}\left(C_{2n+1}^{\lambda+1}(x)\right)^m}{2m(n+\lambda+1)\left(n+\lambda+\frac{1}{2}\right)}\end{aligned} \tag{2.40}$$

$$y_6''(x)=\left[x^{\lambda+n+\frac{3}{2}}C_{2n+1}^{\lambda}(\sqrt{x})\right]''=\lambda(\lambda+1)x^{\lambda+n-\frac{1}{2}}C_{2n+1}^{\lambda+2}(\sqrt{x}) \tag{2.41}$$

$$\begin{aligned}&y_6''(x)+\frac{(2n+\lambda+2)x-\left(2n+2\lambda+\frac{5}{2}\right)}{x(1-x)}y_6'(x)\\&\quad+\frac{\left(n+\lambda+\frac{3}{2}\right)(n+\lambda+2)-(2n+\lambda+2)x}{x^2(1-x)}y_6(x)=0\end{aligned} \tag{2.42}$$

$$\hat{f}(x)=x^{-\frac{(n+\lambda+2)(2n+2\lambda+3)}{4n+4\lambda+5}}\left[(2n+\lambda+2)x-\left(2n+2\lambda+\frac{5}{2}\right)\right]^{\frac{(n+\lambda+1)(2n+2\lambda+1)}{4n+4\lambda+5}} \tag{2.43}$$

$$\begin{aligned}&\int(1-x^2)x^{m\frac{(2n+2\lambda+3)(2n+2\lambda+1)}{4n+4\lambda+5}-1}\left[(2n+\lambda+2)x^2-2n-2\lambda-\frac{5}{2}\right]^{m\frac{(n+\lambda+1)(2n+2\lambda+1)}{4n+4\lambda+5}-1}\\&\quad\times\left(C_{2n+1}^{\lambda}(x)\right)^{m-1}C_{2n+1}^{\lambda+2}(x)\,\mathrm{d}x\\&=-\frac{x^{m\frac{(2n+2\lambda+3)(2n+2\lambda+1)}{4n+4\lambda+5}}\left[(2n+\lambda+2)x^2-2n-2\lambda-\frac{5}{2}\right]^{m\frac{(n+\lambda+1)(2n+2\lambda+1)}{4n+4\lambda+5}}\left(C_{2n+1}^{\lambda}(x)\right)^m}{2m\lambda(\lambda+1)}.\end{aligned} \tag{2.44}$$

Equation (2.35) gives two functions $y_7(x)$ and $y_8(x)$ where

$$y_7'(x)=\left[x^{-n}C_{2n}^{\lambda}(\sqrt{x})\right]'=\lambda x^{-n-1}C_{2n-2}^{\lambda+1}(\sqrt{x}) \tag{2.45}$$

$$y_7''(x) + \frac{1+4n-(4n+2\lambda+2)\,x}{2x(1-x)}y_7'(x) + \frac{n(2n-1)}{2x^2(1-x)}y_7(x) = 0 \tag{2.46}$$

$$f(x) = x^{2n+\frac{1}{2}}(1-x)^{\lambda+\frac{1}{2}} \tag{2.47}$$

$$\int x^{m(2n-1)-1}(1-x^2)^{m\left(\lambda+\frac{1}{2}\right)-1}\left(C_{2n-2}^{\lambda+1}(x)\right)^{m-1} C_{2n}^{\lambda}(x)\,\mathrm{d}x = -\frac{\lambda x^{m(2n-1)}(1-x^2)^{m\left(\lambda+\frac{1}{2}\right)}\left(C_{2n-2}^{\lambda+1}(x)\right)^{m}}{mn(2n-1)} \tag{2.48}$$

$$y_8''(x) = \left[x^{1-n}C_{2n}^{\lambda}(\sqrt{x})\right]'' = \lambda(\lambda+1)x^{-n-1}C_{2n-4}^{\lambda+2}(\sqrt{x}) \tag{2.49}$$

$$y_8''(x) + \frac{2n-\frac{3}{2}-(2n+\lambda-1)x}{x(1-x)}y_8'(x) + \frac{(n-1)\left(n-\frac{3}{2}\right)+(2n+\lambda-1)x}{x^2(1-x)}y_8(x) = 0 \tag{2.50}$$

$$\hat{f}(x) = x^{\frac{(n-1)(2n-3)}{4n-3}}\left[2n-\frac{3}{2}-(2n+\lambda-1)x\right]^{\frac{n(1-2n)}{4n-3}} \tag{2.51}$$

$$\int (1-x^2)x^{-4m\frac{n(n-1)}{4n-3}-1}\left[2n-\frac{3}{2}-(2n+\lambda-1)x^2\right]^{m\frac{n(1-2n)}{4n-3}-1} \times (C_{2n}^{\lambda}(x))^{m-1}C_{2n-4}^{\lambda+2}(x)\,\mathrm{d}x = -\frac{x^{-4m\frac{n(n-1)}{4n-3}}\left[2n-\frac{3}{2}-(2n+\lambda-1)x^2\right]^{m\frac{n(1-2n)}{4n-3}}(C_{2n}^{\lambda}(x))^m}{2m\lambda(\lambda+1)}. \tag{2.52}$$

Equation (2.36) gives two functions $y_9(x)$ and $y_{10}(x)$ such that

$$y_9'(x) = \left[x^{-n-\frac{1}{2}}C_{2n+1}^{\lambda}(\sqrt{x})\right]' = \lambda x^{-n-\frac{3}{2}}C_{2n-1}^{\lambda+1}(\sqrt{x}) \tag{2.53}$$

$$y_9''(x) + \frac{2n+\frac{3}{2}-(2n+\lambda+2)x}{x(1-x)}y_9'(x) + \frac{n\left(n+\frac{1}{2}\right)}{x^2(1-x)}y_9(x) = 0 \tag{2.54}$$

$$f(x) = x^{2n+\frac{3}{2}}(1-x)^{\lambda+\frac{1}{2}} \tag{2.55}$$

$$\int x^{2mn-1}(1-x^2)^{m\left(\lambda+\frac{1}{2}\right)-1}\left(C_{2n-1}^{\lambda+1}(x)\right)^{m-1} C_{2n+1}^{\lambda}(x)\,\mathrm{d}x = -\frac{\lambda x^{2mn}(1-x^2)^{m\left(\lambda+\frac{1}{2}\right)}\left(C_{2n-1}^{\lambda+1}(x)\right)^{m}}{2mn\left(n+\frac{1}{2}\right)} \tag{2.56}$$

$$y_{10}''(x) = \left[x^{-n+\frac{1}{2}}C_{2n+1}^{\lambda}(\sqrt{x})\right]'' = \lambda(\lambda+1)x^{-n-\frac{3}{2}}C_{2n-3}^{\lambda+2}(\sqrt{x}) \tag{2.57}$$

$$y_{10}''(x) + \frac{4n-1-2(2n+\lambda)x}{2x(1-x)}y_{10}'(x) + \frac{n(2n-3)+1+2(2n+\lambda)x}{2x^2(1-x)}y_{10}(x) = 0 \tag{2.58}$$

$$\hat{f}(x) = x^{\frac{(n-1)(2n-1)}{4n-1}}\left[4n-1-2(2n+\lambda)x\right]^{-\frac{n(2n+1)}{4n-1}} \tag{2.59}$$

$$\int (1-x^2)x^{-m(2n-1)\frac{2n+1}{4n-1}-1}\left[4n-1-2(2n+\lambda)x^2\right]^{-m\frac{n(2n+1)}{4n-1}-1}$$
$$\times \left(C^{\lambda}_{2n+1}(x)\right)^{m-1} C^{\lambda+2}_{2n-3}(x)\,\mathrm{d}x$$
$$= -\frac{x^{-m(2n-1)\frac{2n+1}{4n-1}}\left[4n-1-2(2n+\lambda)x^2\right]^{-m\frac{n(2n+1)}{4n-1}}\left(C^{\lambda}_{2n+1}(x)\right)^{m}}{4m\lambda(\lambda+1)}. \tag{2.60}$$

3. Hermite polynomials

The Hermite polynomials $H_n(x)$ obey the differential equation [5]

$$y''(x) - 2xy'(x) + 2ny(x) = 0 \tag{3.1}$$

and the four recurrence relations for these polynomials given in [5] are equivalent to the two relations

$$H_n'(x) = 2nH_{n-1}(x) \tag{3.2}$$
$$H_n'(x) - 2xH_n(x) = -H_{n+1}(x). \tag{3.3}$$

Equation (3.3) can be integrated with an integrating factor to give

$$(e^{-x^2}H_n(x))' = -e^{-x^2}H_{n+1}(x) \tag{3.4}$$

and Equations (3.2) and (3.4) can be differentiated any number of times with the derivatives remaining simple. Defining

$$u_1(x) = H_n(x) \tag{3.5}$$

then

$$u_1'(x) = 2nH_{n-1}(x) \tag{3.6}$$
$$u_1''(x) = 4n(n-1)H_{n-2}(x) \tag{3.7}$$

and as $u_1(x)$ obeys Equation (3.1) we have

$$f(x) = e^{-x^2} \tag{3.8}$$
$$\hat{f}(x) = x^{-n}. \tag{3.9}$$

Applying Equations (3.5), (3.6) and (3.8) to Equation (1.6) gives the integral

$$\int e^{-mx^2}H^{m-1}_{n-1}(x)H_n(x)\,\mathrm{d}x = -\frac{e^{-mx^2}H^m_{n-1}(x)}{m} \tag{3.10}$$

and applying Equations (3.5), (3.7) and (3.9) to Equation (1.10) gives the integral

$$\int x^{-mn-1}H^{m-1}_n(x)H_{n-2}(x)\,\mathrm{d}x = \frac{2x^{-mn}H^m_n(x)}{4mn(n-1)}. \tag{3.11}$$

Defining

$$u_2(x) = e^{-x^2} H_n(x) \tag{3.12}$$

then

$$u_2'(x) = -e^{-x^2} H_{n+1}(x) \tag{3.13}$$

$$u_2''(x) = e^{-x^2} H_{n+2}(x) \tag{3.14}$$

where the function $u_2(x)$ obeys the equation

$$u_2''(x) + 2xu_2'(x) + 2(n+1)u_2'(x) = 0 \tag{3.15}$$

for which

$$f(x) = e^{x^2}; \quad \hat{f}(x) = x^{n+1}. \tag{3.16}$$

Applying these results to Equations (1.6) and (1.10) gives the respective integrals

$$\int H_{n+1}^{m-1}(x) H_n(x)\,\mathrm{d}x = \frac{H_{n+1}^m(x)}{2m(n+1)} \tag{3.17}$$

$$\int e^{-mx^2} x^{m(n+1)-1} H_n^{m-1}(x) H_{n+2}(x)\,\mathrm{d}x = -\frac{2e^{-mx^2} x^{m(n+1)} H_n^m(x)}{m}. \tag{3.18}$$

The relation

$$\frac{\mathrm{d}^r}{\mathrm{d}x^r}(x^{r-n-1} H_{2n}(\sqrt{x})) = \frac{(2n)!}{(2n-2r)!} x^{-n-1} H_{2n-2r}(\sqrt{x}) \quad [n \geq r] \tag{3.19}$$

is given along with some similar relations in [6] and these can be used to derive integrals similar to some of those derived for Gegenbauer polynomials in the previous section. Here only Equation (3.19) will be employed. Defining

$$u_3(x) = x^{-n} H_{2n}(\sqrt{x}) \tag{3.20}$$

then Equation (3.19) gives

$$u_3'(x) = 2n(2n-1)x^{-n-1} H_{2n-2}(\sqrt{x}) \tag{3.21}$$

and defining

$$u_4(x) = x^{-n+1} H_{2n}(\sqrt{x}) \tag{3.22}$$

then Equation (3.19) gives

$$u_4''(x) = 2n(2n-1)(2n-2)(2n-3)x^{-n-1} H_{2n-4}(\sqrt{x}). \tag{3.23}$$

When the independent variable is changed to $\sqrt{x}$ in Equation (3.1) and $n \to 2n$, this equation becomes

$$y''(x) + \left(\frac{1}{2x} - 1\right) y'(x) + \frac{n}{x} y(x) = 0 \tag{3.24}$$

with a solution $y(x) = H_{2n}(\sqrt{x})$. The function $u_3(x)$ obeys the equation

$$u_3''(x) + \left(\frac{4n+1}{2x} - 1\right) u_3'(x) + \frac{n(2n-1)}{2x^2} u_3(x) = 0 \tag{3.25}$$

for which

$$f(x) = x^{2n+\frac{1}{2}} e^{-x} \tag{3.26}$$

and the function $u_4(x)$ obeys the equation

$$u_4''(x) + \left(\frac{4n-3}{2x} - 1\right) u_4'(x) + \left(\frac{(n-1)(2n-3)}{2x^2} + \frac{1}{x}\right) u_4(x) = 0 \tag{3.27}$$

for which

$$\hat{f}(x) = x^{\frac{(n-1)(2n-3)}{4n-3}} (4n-3-2x)^{-\frac{n(2n-1)}{4n-3}}. \tag{3.28}$$

Substituting these results into Equations (1.6) and (1.10) gives the respective integrals

$$\int x^{m(2n-1)-1} e^{-mx^2} (H_{2n-2}(x))^{m-1} H_{2n}(x)\, \mathrm{d}x = -\frac{2x^{m(2n-1)} e^{-mx^2} (H_{2n-2}(x))^m}{m} \tag{3.29}$$

$$\begin{aligned} &\int x^{-4m\frac{n(n-1)}{4n-3}-1} (4n-3-2x^2)^{-m\frac{n(2n-1)}{4n-3}-1} (H_{2n}(x))^{m-1} H_{2n-4}(x)\, \mathrm{d}x \\ &= -\frac{x^{-4m\frac{n(n-1)}{4n-3}} (4n-3-2x^2)^{-m\frac{n(2n-1)}{4n-3}} (H_{2n}(x))^m}{8mn(2n-1)(2n-2)(2n-3)}. \end{aligned} \tag{3.30}$$

4. Laguerre polynomials

The Laguerre polynomials $L_n^\alpha(x)$ for $n \in N_0$ obey the differential equation

$$y''(x) + \left(\frac{\alpha+1}{x} - 1\right) y'(x) + \frac{n}{x} y(x) = 0. \tag{4.1}$$

The recurrences given in [5] are equivalent to the four relations

$$\left[L_n^\alpha(x)\right]' = -L_{n+1}^{\alpha+1}(x) \tag{4.2}$$

$$\left[L_n^\alpha(x)\right]' - L_n^\alpha(x) = -L_n^{\alpha+1}(x) \tag{4.3}$$

$$\left[L_n^\alpha(x)\right]' - \frac{n}{x} L_n^\alpha(x) = -\frac{n+\alpha}{x} L_{n-1}^\alpha(x) \tag{4.4}$$

$$\left[L_n^\alpha(x)\right]' + \frac{n+\alpha+1-x}{x} L_n^\alpha(x) = \frac{n+1}{x} L_{n+1}^\alpha(x) \tag{4.5}$$

and Equations (4.3)–(4.5) can be integrated with integrating factors to give

$$\left[e^{-x} L_n^\alpha(x)\right]' = -e^{-x} L_n^{\alpha+1}(x) \tag{4.6}$$

$$\left[x^{-n} L_n^\alpha(x)\right]' = -(n+\alpha) x^{-n-1} L_{n-1}^\alpha(x) \tag{4.7}$$

$$\left[x^{n+\alpha+1} e^{-x} L_n^\alpha(x)\right]' = (n+1) x^{n+\alpha} e^{-x} L_{n+1}^\alpha(x). \tag{4.8}$$

Equations (4.2) and (4.6) give simple derivatives on differentiation any number of times, but Equations (4.7) and (4.8) do not. The relation

$$\frac{\mathrm{d}^r}{\mathrm{d}x^r}\left(x^\alpha e^{-x} L_n^\alpha(x)\right) = \frac{(n+r)!}{n!} x^{\alpha-r} e^{-x} L_{n+r}^{\alpha-r}(x) \tag{4.9}$$

is given in [6] and will also be used here.

For

$$v_1(x) = L_n^\alpha(x) \tag{4.10}$$

then

$$v_1'(x) = -L_{n-1}^{\alpha+1}(x) \tag{4.11}$$

$$v_1''(x) = L_{n-2}^{\alpha+2}(x) \tag{4.12}$$

and $v_1(x)$ obeys Equation (4.1) for which

$$f(x) = x^{\alpha+1} e^{-x}; \quad \hat{f}(x) = (\alpha+1-x)^{-n}. \tag{4.13}$$

Applying Equations (4.1) and (4.10)–(4.13) to Equations (1.6) and (1.10) gives the respective integrals

$$\int x^{m(\alpha+1)-1} e^{-mx} \left(L_{n-1}^{\alpha+1}(x)\right)^{m-1} L_n^\alpha(x)\,\mathrm{d}x = \frac{x^{m(\alpha+1)} e^{-mx} \left(L_{n-1}^{\alpha+1}(x)\right)^m}{mn} \tag{4.14}$$

$$\int x(\alpha+1-x)^{-mn-1} \left(L_n^\alpha(x)\right)^{m-1} L_{n-2}^{\alpha+2}(x)\,\mathrm{d}x = -\frac{(\alpha+1-x)^{-mn} \left(L_n^\alpha(x)\right)^m}{m}. \tag{4.15}$$

Defining

$$v_2(x) = e^{-x} L_n^\alpha(x) \tag{4.16}$$

then

$$v_2'(x) = -e^{-x} L_n^{\alpha+1}(x) \tag{4.17}$$

$$v_2''(x) = e^{-x} L_n^{\alpha+2}(x) \tag{4.18}$$

and $v_2(x)$ obeys the differential equation

$$v_2''(x) + \left(\frac{\alpha+1}{x} + 1\right) v_2'(x) + \frac{n+\alpha+1}{x} v_2(x) = 0 \tag{4.19}$$

for which

$$f(x) = x^{\alpha+1} e^x; \hat{f}(x) = (\alpha+1+x)^{n+\alpha+1}. \tag{4.20}$$

Applying these results to Equations (1.6) and (1.10) gives the respective integrals

$$\int x^{m(\alpha+1)-1} \left(L_n^{\alpha+1}(x)\right)^{m-1} L_n^\alpha(x)\,\mathrm{d}x = \frac{x^{m(\alpha+1)} \left(L_n^{\alpha+1}(x)\right)^m}{m(n+\alpha+1)} \tag{4.21}$$

$$\int x(\alpha+1+x)^{m(n+\alpha+1)-1} e^{-mx} \left(L_n^{\alpha}(x)\right)^{m-1} L_n^{\alpha+2}(x)\,\mathrm{d}x$$
$$= -\frac{(\alpha+1+x)^{m(n+\alpha+1)} e^{-mx} \left(L_n^{\alpha}(x)\right)^{m}}{m}. \tag{4.22}$$

Defining

$$v_3(x) = x^{-n} L_n^{\alpha}(x) \tag{4.23}$$

then

$$v_3'(x) = -(n+\alpha)x^{-n-1} L_{n-1}^{\alpha}(x) \tag{4.24}$$

and $v_3(x)$ obeys the differential equation

$$v_3''(x) + \left(\frac{2n+\alpha+1}{x} - 1\right) v_3'(x) + \frac{n(n+\alpha)}{x^2} v_3(x) \tag{4.25}$$

for which

$$f(x) = x^{2n+\alpha+1} e^{-x}. \tag{4.26}$$

Applying these results to Equation (1.6) gives the integral

$$\int x^{m(n+\alpha)-1} e^{-mx} \left(L_{n-1}^{\alpha}(x)\right)^{m-1} L_n^{\alpha}(x)\,\mathrm{d}x = \frac{x^{m(n+\alpha)} e^{-mx} \left(L_{n-1}^{\alpha}(x)\right)^{m}}{mn}. \tag{4.27}$$

Defining

$$v_4(x) = x^{n+\alpha+1} e^{-x} L_n^{\alpha}(x) \tag{4.28}$$

then

$$v_4'(x) = (n+1)x^{n+\alpha} e^{-x} L_{n+1}^{\alpha}(x) \tag{4.29}$$

and $v_4(x)$ obeys the differential equation

$$v_4''(x) + \left(1 - \frac{2n+\alpha+1}{x}\right) v_4'(x) + \frac{(n+1)(n+\alpha+1)}{x^2} v_4(x) = 0 \tag{4.30}$$

for which

$$f(x) = x^{-(2n+\alpha+1)} e^{x}. \tag{4.31}$$

Applying these results to Equation (1.6) gives the integral

$$\int x^{-m(n+1)-1} \left(L_{n+1}^{\alpha}(x)\right)^{m-1} L_n^{\alpha}(x)\,\mathrm{d}x = -\frac{x^{-m(n+1)} \left(L_{n+1}^{\alpha}(x)\right)^{m}}{m(n+\alpha+1)}. \tag{4.32}$$

Defining

$$v_5(x) = x^{\alpha} e^{-x} L_n^{\alpha}(x) \tag{4.33}$$

then from Equation (4.9) we have

$$v_5'(x) = (n+1)x^{\alpha-1} e^{-x} L_{n+1}^{\alpha-1}(x) \tag{4.34}$$

$$v_5''(x) = (n+1)(n+2)x^{\alpha-2} e^{-x} L_{n+2}^{\alpha-2}(x) \tag{4.35}$$

and $v_5(x)$ obeys the differential equation

$$v_5''(x) + \left(1 + \frac{1-\alpha}{x}\right) v_5'(x) + \frac{n+1}{x} v_5(x) \tag{4.36}$$

for which

$$f(x) = x^{1-\alpha} e^x; \quad \hat{f}(x) = (1-\alpha+x)^{n+1}. \tag{4.37}$$

Applying these results to Equations (1.6) and (1.10) gives the respective integrals

$$\int \left(L_{n+1}^{\alpha-1}(x)\right)^{m-1} L_n^\alpha(x)\,\mathrm{d}x = -\frac{\left(L_{n+1}^{\alpha-1}(x)\right)^m}{m} \tag{4.38}$$

$$\int x^{m\alpha-1} e^{-mx} (1-\alpha+x)^{m(n+1)-1} \left(L_n^\alpha(x)\right)^{m-1} L_{n+2}^{\alpha-2}(x)\,\mathrm{d}x$$
$$= -\frac{x^{m\alpha} e^{-mx} (1-\alpha+x)^{m(n+1)} \left(L_n^\alpha(x)\right)^m}{m(n+1)(n+2)}. \tag{4.39}$$

As $L_n^\alpha(x)$ is the derivative of $-L_{n+1}^{\alpha-1}(x)$, Equation (4.38) can be considered to be an elementary integral.

5. Comments and conclusions

All the integrals presented here appear to be new and have been checked by differentiation using Mathematica [4]. Many more integrals of this type can be derived using this method for these three polynomials. Results for other orthogonal polynomials will be presented separately.

Disclosure statement

No potential conflict of interest was reported by the author.

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