BANACH SPACES WHERE CONVEX COMBINATIONS OF RELATIVELY WEAKLY OPEN SUBSETS OF THE UNIT BALL ARE RELATIVELY WEAKLY OPEN

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ABSTRACT. Ghoussoub, Godefroy, Maurey, and Schachermayer showed that in the positive face of the unit ball of $L_1[0,1]$, finite convex combinations of relatively weakly open subsets are relatively weakly open. We study this phenomenon in the closed unit balls of Banach spaces and call it property CWO. We introduce a geometric property, property (co), and show that if a finite-dimensional normed space X has property (co), then for any scattered locally compact Hausdorff space K, the space $C_0(K, X)$ has property CWO. Several finite-dimensional spaces are shown to have property (co). We present an example of a three-dimensional real Banach space for which $C_0(K, X)$ fails property CWO. We also obtain stability results for the properties CWO and (co), for instance, if a Banach space contains a complemented subspace isomorphic to ℓ_1, then it does not have the property CWO.

1. INTRODUCTION

In this paper we consider Banach spaces over the scalar field $\mathbb{K}$, where $\mathbb{K}$ is either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. If not mentioned explicitly the space involved could be either real or complex. By S_X, B_X, and B_X° we denote respectively the unit sphere, unit ball, and open unit ball of a Banach space X. The topological dual of X is denoted by X^*. By a slice (of the unit ball) we mean a set of the form

$$S(x^*, \varepsilon) := \{x \in B_X : \operatorname{Re} x^*(x) > 1 - \varepsilon\},$$

where $\varepsilon > 0$ and $x^* \in S_{X^*}$. A topological space K is said to be *scattered* if every non-empty subset A of K contains a point which is isolated in A.

Let $\mathcal{F} = \{f \in L_1[0,1] : f \geq 0, \|f\| = 1\}$. It was shown in [6, Remark IV.5, p. 48], that $\mathcal{F}$ has "a remarkable geometrical property": any convex combination of a finite number of relatively weakly open subsets (in

particular, slices) of $\mathcal{F}$ is still relatively weakly open. Recently, in [1], it was shown that if K is a scattered compact Hausdorff space, then the space $C(K, \mathbb{K})$ of continuous $\mathbb{K}$-valued functions on K has the property that finite convex combinations of slices of $B_{C(K,\mathbb{K})}$ are relatively weakly open in $B_{C(K,\mathbb{K})}$. Subsequently, in [7], it was shown that $C(K, \mathbb{K})$ has this property if and only if K is scattered. In fact, in [7] the result was proven for the space $C_0(K, \mathbb{K})$ of continuous $\mathbb{K}$-valued functions on K vanishing at infinity, where K is a locally compact Hausdorff space. The results in [1] are true also in this setting.

The main focus of this paper is to prove that, for some Banach spaces X, the space $C_0(K, X)$ of X-valued continuous functions on a scattered locally compact Hausdorff space K also satisfies the property that finite convex combinations of slices of $B_{C_0(K,X)}$ are relatively weakly open in $B_{C_0(K,X)}$. We will prove this by showing that even finite convex combinations of relatively weakly open subsets of the unit ball $B_{C_0(K,X)}$ are relatively weakly open in the unit ball. More specifically, we consider the following properties.

Definition 1.1. Let X be a Banach space. We say that

    (a) X has property CWO, if, for every finite convex combination C of relatively weakly open subsets of B_X, the set C is open in the relative weak topology of B_X;

    (b) X has property CWO-S, if, for every finite convex combination C of relatively weakly open subsets of B_X, every $x \in C \cap S_X$ is an interior point of C in the relative weak topology of B_X;

    (c) X has property CWO-B, if, for every finite convex combination C of relatively weakly open subsets of B_X, every $x \in C \cap B_X^\circ$ is an interior point of C in the relative weak topology of B_X.

It is clear that a Banach space X has property CWO if and only if it has both properties CWO-S and CWO-B. We will show in Theorem 5.5 that $L_1[0, 1]$ has property CWO-S, but it fails to have property CWO-B by Corollary 4.6. In fact, any ℓ_1-sum of two spaces with property CWO-S has property CWO-S, but fails to have property CWO-B [7, Theorem 2.3 and Proposition 2.1]. Let K be a scattered locally compact Hausdorff space. In Example 3.4, we give an example of a finite-dimensional Banach space X such that $C_0(K, X)$ has property CWO-B, by Theorem 2.5, but $C_0(K, X)$ fails property CWO-S, by Proposition 3.3. Thus neither of the properties CWO-S and CWO-B implies the other.

Note that, in the definition of property CWO-S, the intersection $C \cap S_X$ may be empty. Every strictly convex Banach space has property CWO-S.

Indeed, let C denote a finite convex combination of relatively weakly open subsets $W_1, \ldots, W_n$ of B_X. Then every $x \in C \cap S_X$ is an extreme point of B_X, hence $x \in \bigcap_{j=1}^{n} W_j$ which is a relatively weakly open neighbourhood of x contained in C.

We also remark that if an infinite-dimensional Banach space X has property CWO-B, then, by [10, Theorem 2.4], every finite convex combination of slices of B_X has diameter two, that is, X has the *strong diameter two property*. This is not the case for property CWO-S since, for example, ℓ_2 has this property.

Let K be a locally compact Hausdorff space. It is known that $C_0(K, X)$ can be identified with the injective tensor product $C_0(K) \hat{\otimes}_\varepsilon X$. It is also known that the (injective) tensor product $X \hat{\otimes}_\varepsilon Y$ of two Banach spaces X and Y contains one-complemented isometric copies of both X and Y. We will show in Proposition 4.3 that property CWO is inherited by one-complemented subspaces. Hence in order for $C_0(K, X)$ to have property CWO, it is necessary that both $C_0(K)$ and X have property CWO. By [7, Theorem 3.1], this implies that K must be scattered. Hence we will only consider scattered locally compact Hausdorff spaces K.

The paper is organized as follows. Section 2 is about spaces of the type $C_0(K, X)$, where K is a scattered locally compact Hausdorff space and X a finite-dimensional Banach space. We establish and discuss here a geometric condition on X, property (co), guaranteeing that $C_0(K, X)$ has property CWO; see Theorem 2.5. We prove that all strictly convex spaces have property (co). We also show that $C_0(K, X)$ has property CWO-B whenever X is finite-dimensional.

In Section 3 we show that any two-dimensional real Banach space has property (co), but there exists a three-dimensional real Banach space which fails property CWO-S. This shows that property CWO is strictly stronger than property CWO-B. We also show that if the dual of a finite-dimensional Banach space X (real or complex) is polyhedral, then X has property (co). Finally we show that both the real and complex ℓ_1^n have property (co). It should be noted in this connection that, in the complex case, $(\ell_1^n)^* = \ell_\infty^n$ is not a polyhedral space while ℓ_1^n is.

In Section 4 we prove, in Proposition 4.3, that all the CWO-properties are stable by taking one-complemented subspaces. We also show that if X contains a complemented subspace isomorphic to ℓ_1, then X does not have property CWO-B.

In Section 5 we show that c_0-sums of finite-dimensional Banach spaces with property (co) have property CWO. This result provides examples of spaces with property CWO outside the class of $C_0(K, X)$-spaces discussed in the two previous sections. We end this section by showing that the real space $L_1(\mu)$ has property CWO-S provided μ is a non-zero σ-finite (countably additive non-negative) measure.

We follow standard Banach space notation as can be found, e.g., in the book [4]. As mentioned above, we consider Banach spaces over the scalar field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We use the notation $\mathbb{T} = \{\alpha \in \mathbb{K} : |\alpha| = 1\}$ and $\mathbb{D} = \{\alpha \in \mathbb{K} : |\alpha| \leq 1\}$.

2. A GEOMETRIC CONDITION FOR BANACH SPACES X GUARANTEEING THAT $C_0(K, X)$ HAS PROPERTY CWO

Our main objective in this section is to establish a geometric condition for finite-dimensional Banach spaces X guaranteeing that the space $C_0(K, X)$, where K is a scattered locally compact Hausdorff space, has property CWO.

Definition 2.1. Let X be a Banach space. We say that a point $x \in B_X$ has *property* (co), if for every $n \in \mathbb{N}$, $n \geq 2$,

(con) whenever $x_1, \ldots, x_n \in B_X$ and $\lambda_1, \ldots, \lambda_n > 0$, $\sum_{j=1}^{n} \lambda_j = 1$, are such that $x = \sum_{j=1}^{n} \lambda_j x_j$, and $\varepsilon > 0$, there is a $\delta > 0$ such that, setting $B := B(x, \delta) \cap B_X$, there are continuous functions

$$\widehat{v}_j \colon B \to B_X, \qquad j \in \{1, \ldots, n\},$$

such that, for every $u \in B$,

$$(2.1) \qquad u = \sum_{j=1}^{n} \lambda_j \widehat{v}_j(u) \qquad \text{and} \qquad \|\widehat{v}_j(u) - x_j\| < \varepsilon, \quad j \in \{1, \ldots, n\}.$$

We say that the space X has *property* (co) if every point $x \in B_X$ has property (co).

For finite-dimensional Banach spaces, property (co) implies property CWO. We will see in Proposition 3.3 and Example 3.4 that not every finite-dimensional Banach space has property CWO.

Proposition 2.2. *Let X be a finite-dimensional Banach space with property (co). Then X has property CWO.*

Proof. Let $n \in \mathbb{N}$, let $U_1, \ldots, U_n$ be relatively weakly open subsets of B_X, and let $\lambda_1, \ldots, \lambda_n > 0$ with $\sum_{j=1}^{n} \lambda_j = 1$. If $x = \sum_{j=1}^{n} \lambda_j x_j$ where $x_j \in U_j$, then there exists an $\varepsilon > 0$ such that $B(x_j, \varepsilon) \cap B_X \subset U_j$ for every $j \in \{1, \ldots, n\}$.

By assumption, we can find a $\delta > 0$ and functions $\widehat{v}_j : B(x,\delta) \cap B_X \to B_X$ such that, for every $u \in B(x,\delta) \cap B_X$, we have $\widehat{v}_j(u) \in B(x_j,\varepsilon)$ and $\sum_{j=1}^n \lambda_j \widehat{v}_j(u) = u$. This means that $\sum_{j=1}^n \lambda_j U_j$ is relatively weakly (=norm) open in B_X. $\qquad \square$

Proposition 2.3. *Let X be a Banach space.*

    (a) *Suppose that, for a point $x \in B_X$, either x is an extreme point of B_X or $\|x\| < 1$. Then x has property (co).*

    (b) *Suppose that X is strictly convex. Then X has property (co).*

Proof. (b) follows immediately from (a); so let us prove (a). Let $n \in \mathbb{N}$, $n \geq 2$, let $x_1, \ldots, x_n \in B_X$ and $\lambda_1, \ldots, \lambda_n > 0$, $\sum_{j=1}^n \lambda_j = 1$, be such that $x = \sum_{j=1}^n \lambda_j x_j$, and let $\varepsilon > 0$.

First suppose that x is an extreme point of B_X. Then $x = x_j \in S_X$ for every j. Taking $\delta := \varepsilon$ and defining $\widehat{v}_j(u) = u$ for every $u \in B := B(x,\delta) \cap B_X$, the conditions (2.1) hold.

Now suppose that $\|x\| = 1 - \sigma$ for some $\sigma > 0$. Put $r := \frac{\varepsilon}{2}$. Choose $\delta > 0$ with $\delta < \sigma r$. Define, for every $u \in B := B(x,\delta) \cap B_X$,

$$\widehat{v}_j(u) = x_j + r(x - x_j) + (u - x), \quad j \in \{1, \ldots, n\}.$$

Since $\sum_{j=1}^n \lambda_j(x - x_j) = 0$, we get $\sum_{j=1}^n \lambda_j \widehat{v}_j(u) = u$. We also have

$$\|\widehat{v}_j(u)\| = \|rx + (1-r)x_j + (u-x)\| \leq r(1-\sigma) + (1-r) + \delta = 1 - r\sigma + \delta \leq 1;$$

hence $\widehat{v}_j : B \to B_X$. Since $\delta < \sigma r$, we have

$$\|\widehat{v}_j(u) - x_j\| = \|r(x - x_j) + (u - x)\| \leq r(1-\sigma) + r + \delta < 2r = \varepsilon,$$

and we are done. $\qquad \square$

Now comes the "core" result of this section.

Theorem 2.4. *Let K be a scattered locally compact Hausdorff space and let X be a finite-dimensional Banach space. Suppose that an element $x \in B_{C_0(K,X)}$ is such that, for every $t \in K$, the point $x(t) \in B_X$ has property (co). Then, whenever x belongs to a finite convex combination of relatively weakly open subsets of $B_{C_0(K,X)}$, the element x is an interior point of this convex combination in the relative weak topology of $B_{C_0(K,X)}$.*

Before proving Theorem 2.4, let us cash in some dividends it brings summarized in the following main theorem.

Theorem 2.5. *Let K be a scattered locally compact Hausdorff space and let X be a finite-dimensional Banach space. Then*

    (a) *$C_0(K, X)$ has property CWO-B;*

(b) *if X has property (co), then $C_0(K, X)$ has property CWO;*

Proof. (a) follows from Theorem 2.4 and Proposition 2.3, (a).

(b) follows trivially from Theorem 2.4. □

In the proof of Theorem 2.4, it is convenient to rely on the following lemma.

Lemma 2.6. *Let K be a scattered locally compact Hausdorff space and let X be a Banach space.*

(a) *Let $f \in S_{C_0(K,X)^*}$ and let $\varepsilon > 0$. Then there are $N \in \mathbb{N}$, $t_1, \ldots, t_N \in K$, and $x_1^*, \ldots, x_N^* \in X^*$ such that $\sum_{j=1}^{N} \|x_j^*\| \leq 1$ and, for the functional $g = \sum_{j=1}^{N} \delta_{t_j} \otimes x_j^* \in C_0(K, X)^*$, where*

$$g(z) = \sum_{j=1}^{N} x_j^*\big(z(t_j)\big), \quad z \in C_0(K, X),$$

one has $\|f - g\| < \varepsilon$.

(b) *Let $x \in B_{C_0(K,X)}$ and let U be a neighbourhood of x in the relative weak topology of $B_{C_0(K,X)}$. Then there are a finite subset T of K and an $\varepsilon > 0$ such that, whenever $u \in B_{C_0(K,X)}$ satisfies*

(2.2) $$\|u(t) - x(t)\| < \varepsilon \quad \text{for every } t \in T,$$

one has $u \in U$.

(c) *Suppose that X is finite-dimensional. Let $x \in B_{C_0(K,X)}$, let T be a finite subset of K, and let $\varepsilon > 0$. Then the set U of those $u \in B_{C_0(K,X)}$ that satisfy (2.2), is a neighbourhood of x in the relative weak topology of $B_{C_0(K,X)}$.*

Proof. (a). Set $Z := C_0(K, X) = C_0(K) \hat{\otimes}_\varepsilon X$. It is known that $Z^* = C_0(K)^* \hat{\otimes}_\pi X^*$ and that $B_{Z^*} = \overline{\text{conv}}\{S_{C_0(K)^*} \otimes S_{X^*}\}$ [13, Proposition 2.2]. Since K is scattered, we have $C_0(K)^* = \ell_1(K)$ and

$$B_{Z^*} = \overline{\text{conv}}\{\delta_s \otimes x^* : s \in K, x^* \in S_{X^*}\}.$$

(b). Let a finite subset $\mathcal{F} \subset S_{C_0(K,X)^*}$ and an $\varepsilon > 0$ be such that

$$\{u \in B_{C_0(K,X)} : |f(u) - f(x)| < 3\varepsilon \text{ for every } f \in \mathcal{F}\} \subset U.$$

By (a), for every $f \in \mathcal{F}$, there are $N_f \in \mathbb{N}$, $t_{f,j} \in K$, $x_{f,j}^* \in X^*$, $j \in \{1, \ldots, N_f\}$, such that $\sum_{j=1}^{N_f} \|x_{f,j}^*\| \leq 1$ and, for the functional $g_f = \sum_{j=1}^{N_f} \delta_{t_{f,j}} \otimes x_{f,j}^* \in C_0(K, X)^*$, one has $\|f - g_f\| < \varepsilon$.

Set

$$T := \{t_{f,j} : f \in \mathcal{F}, j \in \{1, \ldots, N_f\}\},$$

and suppose that $u \in B_{C_0(K,X)}$ satisfies (2.2). For every $f \in \mathcal{F}$, since

$$|g_f(u - x)| = \left| \sum_{j=1}^{N_f} x_{f,j}^*\big(u(t_{f,j}) - x(t_{f,j})\big) \right| \leq \sum_{j=1}^{N_f} \|x_{f,j}^*\| \|u(t_{f,j}) - x(t_{f,j})\|$$
$$< \varepsilon \sum_{j=1}^{N_f} \|x_{f,j}^*\| \leq \varepsilon,$$

one has

$$|f(u) - f(x)| \leq |f(u) - g_f(u)| + |g_f(u - x)| + |g_f(x) - f(x)|$$
$$\leq 2\|f - g_f\| + |g_f(u - x)|$$
$$< 2\varepsilon + \varepsilon = 3\varepsilon,$$

and it follows that $u \in U$.

(c). In finite-dimensional spaces the function that maps points to their norm is weakly continuous. Thus the mapping $u \mapsto \max_{t \in T} \|u(t) - x(t)\|$ is weakly continuous and the conclusion follows. $\qquad \square$

We are now in a position to prove Theorem 2.4.

Proof of Theorem 2.4. Let $n \in \mathbb{N}$, let $U_1, \ldots, U_n$ be relatively weakly open subsets of $B_{C_0(K,X)}$, and let $x_j \in U_j$ and $\lambda_j > 0$, $\sum_{j=1}^{n} \lambda_j = 1$, be such that $x = \sum_{j=1}^{n} \lambda_j x_j$. We are going to find a neighbourhood U of x in the relative weak topology of $B_{C_0(K,X)}$ such that $U \subset \sum_{j=1}^{n} \lambda_j U_j$.

By Lemma 2.6, (b), there are an $\varepsilon > 0$ and a finite subset T of K such that

- *whenever $u_1, \ldots, u_n \in B_{C_0(K,X)}$ are such that, for every $s \in T$,*

(2.3) $$\|u_j(s) - x_j(s)\| < \varepsilon, \quad j \in \{1, \ldots, n\},$$

one has $u_j \in U_j$, $j \in \{1, \ldots, n\}$.

For every $s \in T$, let δ_s, B_s, and $\widehat{v}_{s,j}$ be, respectively, the δ, B, and the functions $\widehat{v}_j$ from Definition 2.1 with $x = x(s)$ and $x_j = x_j(s)$. By Lemma 2.6, (c), there is a neighbourhood U of x in the relative weak topology of $B_{C_0(K,X)}$ such that, for every $u \in U$,

$$\|u(s) - x(s)\| < \delta_s \quad \text{for every } s \in T.$$

Let $u \in U$ be arbitrary. We are going to show that $u \in \sum_{j=1}^{n} \lambda_j U_j$.

For every $s \in T$, pick H_s to be a compact neighbourhood of s such that

$$u(t) \in B_s \quad \text{for every } t \in H_s.$$

We can choose the neighbourhoods H_s, $s \in T$, to be pairwise disjoint. For every $j \in \{1, \ldots, n\}$, since X is finite-dimensional, by Tietze's extension

theorem, there is a continuous function $w_j : K \to X$ such that $w_j(t) = \widehat{v}_{s,j}(u(t))$ for every $s \in T$ and every $t \in H_s$. By Urysohn's lemma, there is a $\kappa \in C_0(K, \mathbb{R})$ with values in $[0,1]$ such that $\kappa|_T = 1$ and $\operatorname{supp}\kappa \subset \bigcup_{s \in T} H_s$. Set

$$u_j := \kappa w_j + (1 - \kappa)u \in B_{C_0(K,X)}, \quad j \in \{1, \ldots, n\}.$$

Notice that $u = \sum_{j=1}^{n} \lambda_j u_j$. Indeed, if $t \notin \operatorname{supp}\kappa$, then $\kappa(t) = 0$ and thus $\sum_{j=1}^{n} \lambda_j u_j(t) = \sum_{j=1}^{n} \lambda_j u(t) = u(t)$; if $t \in \operatorname{supp}\kappa$, then $t \in H_s$ for some $s \in T$, thus

$$\sum_{j=1}^{n} \lambda_j w_j(t) = \sum_{j=1}^{n} \lambda_j \widehat{v}_{s,j}(u(t)) = u(t)$$

(because $u(t) \in B_s$), and

$$\sum_{j=1}^{n} \lambda_j u_j(t) = \sum_{j=1}^{n} \lambda_j \big(\kappa(t)w_j(t) + (1 - \kappa(t))u(t)\big)$$
$$= \kappa(t)u(t) + \big(1 - \kappa(t)\big)u(t) = u(t).$$

Also notice that, for every $j \in \{1, \ldots, n\}$ and every $s \in T$, since $u_j(s) = w_j(s) = \widehat{v}_{s,j}(u(s))$ and $u(s) \in B_s$, one has (2.3), thus $u_j \in U_j$. $\square$

3. Banach spaces with property (co)

In this section we explore Banach spaces with property (co). We give an example of a finite-dimensional Banach space, which fails property (co), and many examples of finite-dimensional Banach spaces with property (co) (see Propositions 3.5 and 3.7, and Theorem 3.8 below). We start with a characterization of property (co).

Definition 3.1. Let X be a Banach space. We say that a point $x \in B_X$ has *property* $(co2)$ if the condition (con) in Definition 2.1 is satisfied for $n = 2$.

We say that the space X has *property* $(co2)$ if every point $x \in B_X$ has property $(co2)$.

Proposition 3.2. *For a Banach space X, properties $(co2)$ and (co) are equivalent.*

Proof. It is clear that property (co) for X implies property $(co2)$. For the reverse implication, assume that X has property $(co2)$, and that $m \in \mathbb{N}$ with $m \geq 2$ is such that, whenever $x \in B_X$, the condition (con) in Definition 2.1 holds for $n = m$. It suffices to show that, whenever $x \in B_X$, the condition $(co\,n)$ holds also for $n = m + 1$. To this end, let $x \in B_X$, let $x_1, \ldots, x_{m+1} \in B_X$ and $\lambda_1, \ldots, \lambda_{m+1} > 0$, $\sum_{j=1}^{m+1} \lambda_j = 1$, be such that $x = \sum_{j=1}^{m+1} \lambda_j x_j$, and let $\varepsilon > 0$.

Setting $\lambda := \sum_{j=2}^{m+1} \lambda_j$ and $y := \sum_{j=2}^{m+1} \frac{\lambda_j}{\lambda} x_j$, observe that $\sum_{j=2}^{m+1} \frac{\lambda_j}{\lambda} = 1$, $\lambda_1 + \lambda = 1$, and $x = \lambda_1 x_1 + \lambda y$.

By our assumption, there is a $\delta_0 > 0$ such that, setting $B_0 := B(y, \delta_0) \cap B_X$, there are continuous functions $\widehat{v}_j \colon B_0 \to B_X$, $j \in \{2, \ldots, m+1\}$, such that, for every $v \in B_0$,

$$v = \sum_{j=2}^{m+1} \frac{\lambda_j}{\lambda} \widehat{v}_j(v) \quad \text{and} \quad \|\widehat{v}_j(v) - x_j\| < \varepsilon \quad \text{for every } j \in \{2, \ldots, m+1\}.$$

Since X has property $(co2)$, there is a $\delta > 0$ such that, setting $B := B(x, \delta) \cap B_X$, there are continuous functions $\widehat{u}_1 \colon B \to B_X$ and $\widehat{v} \colon B \to B_X$ such that, for every $u \in B$,

$$u = \lambda_1 \widehat{u}_1(u) + \lambda \widehat{v}(u), \quad \|\widehat{u}_1(u) - x_1\| < \varepsilon, \quad \|\widehat{v}(u) - y\| < \delta_0.$$

It remains to define, for every $j \in \{2, \ldots, m+1\}$, a function $\widehat{u}_j \colon B \to B_X$ by $\widehat{u}_j = \widehat{v}_j \circ \widehat{v}$, because in that case, for every $u \in B$, observing that $\widehat{v}(u) \in B_0$, one has

$$\|\widehat{u}_j(u) - x_j\| = \|\widehat{v}_j(\widehat{v}(u)) - x_j\| < \varepsilon \quad \text{for every } j \in \{2, \ldots, m+1\},$$

and

$$\widehat{v}(u) = \sum_{j=2}^{m+1} \frac{\lambda_j}{\lambda} \widehat{v}_j(\widehat{v}(u)) = \sum_{j=2}^{m+1} \frac{\lambda_j}{\lambda} \widehat{u}_j(u),$$

and thus

$$u = \lambda_1 \widehat{u}_1(u) + \lambda \widehat{v}(u) = \sum_{j=1}^{m+1} \lambda_j \widehat{u}_j(u),$$

which shows that X has property (co). $\qquad\square$

The following proposition indicates a class of Banach spaces X, which do not have property (co) nor does the space $C_0(K, X)$ have property CWO-S. A concrete example of a representative of this class will be given in Example 3.4.

Proposition 3.3. *Let X be a Banach space such that $\mathrm{ext}\, B_X$ is non-closed in the norm topology, and let $x \in \overline{\mathrm{ext}\, B_X} \setminus \mathrm{ext}\, B_X$. Then*

(a) *x fails property $(co2)$;*
(b) *X fails property CWO-S;*
(c) *whenever K is a locally compact Hausdorff space, the space $C_0(K, X)$ fails property CWO-S.*

Proof. (a) is obvious.

(b). By assumption there are $x_1, x_2 \in S_X$, $x_1 \neq x_2$, such that $x = \frac{1}{2} x_1 + \frac{1}{2} x_2$. Define a linear functional $g : \mathrm{span}\{x, x_1 - x_2\} \to \mathbb{K}$ by $g(x) = 0$

and $g(x_1 - x_2) = 1$ (observe that the elements x and $x_1 - x_2$ are linearly independent). Letting $x^* \in X^*$ be any norm preserving extension of $\frac{g}{\|g\|}$, one has $x^* \in S_{X^*}$, $x^*(x) = 0$, and $x^*(x_1) = 2\alpha$, $x^*(x_2) = -2\alpha$ for some $\alpha > 0$. Consider the slices

$$S_1 := \{a \in B_X : \operatorname{Re} x^*(a) > \alpha\} \quad \text{and} \quad S_2 := \{a \in B_X : \operatorname{Re}(-x^*)(a) > \alpha\}.$$

Then $x_j \in S_j$, $j \in \{1, 2\}$, and thus $x \in \frac{1}{2}S_1 + \frac{1}{2}S_2$.

Let U be an arbitrary neighbourhood of x in the relative weak topology of B_X. Then there is a $\delta > 0$ such that $B(x, \delta) \cap B_X \subset U$. We may assume that $\delta < \alpha$.

By assumption, there exists $u \in \operatorname{ext} B_X \cap B(x, \delta) \subset U$. Suppose that $u = \frac{1}{2}u_1 + \frac{1}{2}u_2$ with $u_j \in S_j$, $j \in \{1, 2\}$. Since u is an extreme point, $u_1 = u_2 = u$. But

$$|\operatorname{Re} x^*(u)| \leq |\operatorname{Re} x^*(x)| + \|u - x\| < \delta < \alpha,$$

hence $u_j \notin S_j$, $j \in \{1, 2\}$, and $u \notin \frac{1}{2}S_1 + \frac{1}{2}S_2$. Thus x is not an interior point of $\frac{1}{2}S_1 + \frac{1}{2}S_2$ in the relative weak topology of B_X.

(c) follows from (b) and Proposition 4.3 since $C_0(K, X) = C_0(K)\hat{\otimes}_\varepsilon X$ contains a one-complemented copy of X. $\qquad\square$

We now give a concrete example of the phenomenon described in Proposition 3.3.

Example 3.4. Let X be the Banach space $\mathbb{R}^3$ whose unit ball is

$$B_X = \operatorname{conv}\big((B_{\ell_2^3} - e_1) \cup B_{\ell_\infty^3} \cup (B_{\ell_2^3} + e_1)\big),$$

where $e_1 = (1, 0, 0)$. Then the point $(1, 0, 1) \in S_X$ is not an extreme point of B_X (because it lies on the line segment connecting the points $(1, -1, 1) \in S_X$ and $(1, 1, 1) \in S_X$), but it has extreme points of B_X arbitrarily close to it.

Let X be a finite-dimensional Banach space. If X is a real Banach space, then X is called *polyhedral* if $\operatorname{ext} B_X$ is a finite set. For a complex Banach space X, following [11], we say that B_X is a *complex polytope* if there exists a finite set $A \subset \operatorname{ext} B_X$ such that $\operatorname{ext} B_X = \mathbb{T} \cdot A$. We will say that X is *polyhedral* if B_X is a complex polytope.

Proposition 3.5. *Let X be a finite-dimensional Banach space such that the dual X^* is polyhedral. Then X has property $(co2)$ (and hence property (co)).*

Proof. Let $\varepsilon > 0$ and define $r := \frac{\varepsilon}{2}$. Let $\operatorname{ext} B_{X^*}/\mathbb{T} = \{\phi_1, \ldots, \phi_m\}$.

Let $x \in B_X$. Assume that $x_1, x_2 \in B_X$ and $\lambda_1, \lambda_2 > 0$, with $\lambda_1 + \lambda_2 = 1$, are such that $x = \lambda_1 x_1 + \lambda_2 x_2$. Define $J := \{n : |\phi_n(x)| < 1\}$ and

$\sigma := \min_{n \in J}(1 - |\phi_n(x)|) > 0$. Choose $\delta > 0$ such that $\delta < r\sigma$. Set $B := B(x, \delta) \cap B_X$ and define functions $\widehat{v}_1, \widehat{v}_2 : B \to B_X$ by

$$\widehat{v}_j(u) := x_j + r(x - x_j) + (u - x), \quad j \in \{1, 2\}.$$

It is trivial that $\widehat{v}_1$ and $\widehat{v}_2$ are continuous. We have

$$\lambda_1 \widehat{v}_1(u) + \lambda_2 \widehat{v}_2(u) = x + r(x - x) + (u - x) = u$$

and

$$\|\widehat{v}_j(u) - x_j\| = \|r(x - x_j) + (u - x)\| \leq r(1 - \sigma) + r + \delta < 2r = \varepsilon.$$

It remains to show that $\widehat{v}_1, \widehat{v}_2(u) \in B_X$ for all $u \in B$. Let $j \in \{1, 2\}$, $u \in B$, $n \in \{1, \ldots, m\}$, and $\alpha \in \mathbb{T}$ be arbitrary.

If $|\phi_n(x)| = 1$, then $|(\alpha \phi_n)(x)| = 1$ and

$$(\alpha \phi_n)(x) = \lambda_1 (\alpha \phi_n)(x_1) + \lambda_2 (\alpha \phi_n)(x_2).$$

This means that $(\alpha \phi_n)(x) \in \mathbb{T}$ has been written as a convex combination of elements in $\mathbb{D}$. But every point in $\mathbb{T}$ is an extreme point in $\mathbb{D}$, hence $(\alpha \phi_n)(x_j) = (\alpha \phi_n)(x)$, $j \in \{1, 2\}$. Since

$$\widehat{v}_j(u) = (1 - r)(x_j - x) + u,$$

we have

$$|(\alpha \phi_n)(\widehat{v}_j(u))| \leq (1 - r)|(\alpha \phi_n)(x_j - x)| + |\phi_n(u)| = 0 + |\phi_n(u)| \leq 1.$$

If $|\phi_n(x)| < 1$, then we use the fact that

$$\widehat{v}_j(u) = (1 - r)x_j + rx + (u - x)$$

and get

$$\begin{aligned}|(\alpha \phi_n)(\widehat{v}_j(u))| &\leq (1 - r)|\phi_n(x_j)| + r|\phi_n(x)| + |\phi_n(u - x)| \\ &\leq (1 - r) + r(1 - \sigma) + \delta \\ &\leq 1 - r\sigma + r\sigma = 1.\end{aligned}$$

In conclusion, $|\phi(\widehat{v}_j(u))| \leq 1$ for all $\phi \in \text{ext } B_{X^*}$; hence $\widehat{v}_j(u) \in B_X$. $\qquad \square$

Note that both real and complex ℓ_1^n are polyhedral, and while real ℓ_∞^n is polyhedral, complex ℓ_∞^n is not. We will however prove that complex ℓ_1^n has property (co) in Theorem 3.8 below.

Recall that, by Proposition 2.3, (a), every norm-less-than-one point of any Banach space has property (co) (and hence property $(co2)$). Next we give a necessary and sufficient condition for norm-one points in Banach spaces to have property $(co2)$, which is easier to verify that the condition from Definition 3.1 (and Definition 2.1). More precisely, we show that it is enough to define the functions from Definition 2.1 on a neighbourhood of

the norm-one point on the sphere and not a neighbourhood in the unit ball. This result will be applied to show that, for any $n \in \mathbb{N}$, the complex space ℓ_1^n has property $(co2)$.

Theorem 3.6. *Let X be a Banach space and let $x \in S_X$. The following assertions are equivalent:*

(i) *x has property $(co2)$;*
(ii) *whenever $x_1, x_2 \in S_X$, $x_1 \neq x_2$, and $\lambda_1, \lambda_2 > 0$, $\lambda_1 + \lambda_2 = 1$, are such that $x = \lambda_1 x_1 + \lambda_2 x_2$, and $\varepsilon > 0$, there is a $\delta > 0$ such that, setting $S = B(x, \delta) \cap S_X$, there are continuous functions $\mathfrak{v}_1, \mathfrak{v}_2 : S \to B_X$ such that, for every $u \in S$,*

$$(3.1) \qquad u = \lambda_1 \mathfrak{v}_1(u) + \lambda_2 \mathfrak{v}_2(u) \quad and \quad \|\mathfrak{v}_j(u) - x_j\| < \varepsilon, \quad j \in \{1, 2\}.$$

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i). Let $x_1, x_2 \in B_X$ and $\lambda_1, \lambda_2 > 0$, $\lambda_1 + \lambda_2 = 1$, be such that $x = \lambda_1 x_1 + \lambda_2 x_2$, and let $0 < \varepsilon < 1$. Then, in fact, $x_1, x_2 \in S_X$.

First consider the case when $x_1 = x_2$; then also $x = x_1$. Taking $\delta = \varepsilon$ and defining $\widehat{v}_1(u) = \widehat{v}_2(u) = u$ for every $u \in B := B(x, \delta) \cap B_X$, the conditions (2.1) hold, hence x has property $(co2)$.

Now suppose that $x_1 \neq x_2$; then, in fact, $x_1 \neq x \neq x_2$. By our assumption, there is a $\gamma \in (0, \varepsilon)$ such that, setting $S := B(x, \gamma) \cap S_X$, there are continuous functions $\mathfrak{v}_1, \mathfrak{v}_2 : S \to B_X$ satisfying (3.1) with ε replaced by $\frac{\varepsilon}{2}$ for every $u \in S$.

Set $C := \{\alpha u : \alpha \in [0, 1], u \in S\}$ and $\delta = \frac{\gamma}{4}$. Observe that $B := B(x, \delta) \cap B_X \subset C$. Indeed, suppose that $a \in B$. Since

$$\delta > \|x - a\| \geq \|x\| - \|a\| = 1 - \|a\|,$$

one has $\|a\| > 1 - \delta > \frac{1}{2}$. For $u := \frac{a}{\|a\|}$, one has $a = \|a\|u$ and $u \in S$, because

$$\|u - x\| \leq \left\|\frac{a}{\|a\|} - \frac{x}{\|a\|}\right\| + \left\|\frac{x}{\|a\|} - x\right\| = \frac{\|a - x\|}{\|a\|} + \frac{1 - \|a\|}{\|a\|} < \frac{2\delta}{\|a\|}$$
$$< 4\delta = \gamma.$$

Since every $a \in B$ has a unique representation $a = \alpha u$, where $\alpha \in (0, 1]$ and $u \in S$, the functions $\widehat{v}_1, \widehat{v}_2 : B \to B_X$ defined by

$$(3.2) \qquad \widehat{v}_j(a) = \widehat{v}_j(\alpha u) := \alpha \mathfrak{v}_j(u), \quad j \in \{1, 2\},$$

are well defined. We now show that these functions are continuous. To this end, let $\alpha_0 u_0 \in B$ ($\alpha_0 \in (0, 1]$, $u_0 \in S$) and $\beta > 0$. By the continuity of $\mathfrak{v}_1$ and $\mathfrak{v}_2$, there is a $\delta_0 > 0$ such that, whenever $u \in S$ satisfies $\|u - u_0\| < \delta_0$,

one has $\|\mathfrak{v}_j(u) - \mathfrak{v}_j(u_0)\| < \frac{\beta}{2}$, $j \in \{1, 2\}$. Suppose that $\alpha u \in B$ ($\alpha \in (0, 1]$, $u \in S$) is such that $\|\alpha u - \alpha_0 u_0\| < \min\{\frac{\delta_0}{4}, \frac{\beta}{2}\}$. Then also

$$\min\left\{\frac{\delta_0}{4}, \frac{\beta}{2}\right\} > \|\alpha u - \alpha_0 u_0\| \geq \big|\|\alpha u\| - \|\alpha_0 u_0\|\big| = |\alpha - \alpha_0|$$

and, since $\alpha = \|\alpha u\| > \frac{1}{2}$,

$$\|u - u_0\| \leq \frac{1}{\alpha}\big(\|\alpha u - \alpha_0 u_0\| + |\alpha_0 - \alpha| \, \|u_0\|\big) < 2\left(\frac{\delta_0}{4} + \frac{\delta_0}{4}\right) = \delta_0.$$

Thus

$$\|\widehat{v}_j(\alpha u) - \widehat{v}_j(\alpha_0 u_0)\| = \|\alpha \mathfrak{v}_j(u) - \alpha_0 \mathfrak{v}_j(u_0)\|$$

$$\leq |\alpha - \alpha_0| \, \|\mathfrak{v}_j(u)\| + \alpha_0 \|\mathfrak{v}_j(u) - \mathfrak{v}_j(u_0)\| < \frac{\beta}{2} + \frac{\beta}{2} = \beta.$$

It follows that the functions $\widehat{v}_1, \widehat{v}_2 : B \to B_X$ are continuous.

It remains to observe that, whenever $\alpha u \in B$ ($\alpha \in (0, 1]$, $u \in S$), one has

$$\lambda_1(\alpha \mathfrak{v}_1(u)) + \lambda_2(\alpha \mathfrak{v}_2(u)) = \alpha(\lambda_1 \mathfrak{v}_1(u) + \lambda_2 \mathfrak{v}_2(u)) = \alpha u,$$

and, since $\alpha = \|\alpha u\| > 1 - \delta$,

$$\|\alpha \mathfrak{v}_j(u) - x_j\| \leq (1 - \alpha)\|\mathfrak{v}_j(u)\| + \|\mathfrak{v}_j(u) - x_j\| < \delta + \frac{\varepsilon}{2} < \varepsilon, \quad j \in \{1, 2\}.$$

$\square$

Proposition 3.7. *Let X be a two-dimensional real Banach space. Then X has property (co).*

Proof. We are going to apply Proposition 3.2 teamed with Theorem 3.6. Let $x, x_1, x_2 \in S_X$ with $x_1 \neq x_2$ and $\lambda_1, \lambda_2 > 0$ with $\lambda_1 + \lambda_2 = 1$ be such that $x = \lambda_1 x_1 + \lambda_2 x_2$, and let $\varepsilon > 0$. We may assume that $d := \|x - x_1\| \leq \|x - x_2\|$ (or, equivalently, $\lambda_2 \leq \lambda_1$) and that $\varepsilon < d$. Set $a := \frac{x - x_1}{\|x - x_1\|}$. Observe that $d \leq 1$, and $\|x + ta\| = 1$ whenever $|t| \leq d$. We shall make use of the following claim which is easy to believe and not much harder to prove.

CLAIM. *There is a $\gamma > 0$ such that, whenever $0 < \delta \leq \gamma$, one has*

(3.3) $$S_\delta := B(x, \delta) \cap S_X = \{x + ta : t \in (-\delta, \delta)\}.$$

Letting $0 < \delta < \min\{\gamma, \frac{\lambda_1 \varepsilon}{2}, \frac{\lambda_2 \varepsilon}{2}\}$, where $\gamma > 0$ comes from Claim, we can now define functions $\widehat{v}_1, \widehat{v}_2 : S_\delta \to B_X$ by

$$\widehat{v}_1(x + ta) = x_1 + \frac{\delta}{\lambda_1}a + ta, \quad \widehat{v}_2(x + ta) = x_2 - \frac{\delta}{\lambda_2}a + ta, \qquad t \in (-\delta, \delta).$$

It remains to prove Claim. First observe that the elements x and a are linearly independent. Since all norms on X are equivalent, there is a $\gamma > 0$ such that

$$D := \left\{b_{st} := sx + ta : s, t \in \left[-\tfrac{d}{2}, \tfrac{d}{2}\right]\right\} \supset \gamma B_X$$

(observe that $\gamma \leq \frac{d}{2}$). Now suppose that $0 < \delta \leq \gamma$. Since $B(x,\delta) = x + \delta B_X \subset x + D$, every $z \in B(x,\delta)$ can be represented as $z = x + b_{st}$, where $s, t \in [-\frac{d}{2}, \frac{d}{2}]$. Since $\left| \frac{t}{1+s} \right| \leq d$, one has

$$\|x + b_{st}\| = \|(1+s)x + ta\| = (1+s)\|x + \tfrac{t}{1+s}\, a\| = 1 + s,$$

hence $\|x + b_{st}\| > 1$ if $s > 0$, and $\|x + b_{st}\| < 1$ if $s < 0$; thus

$$B(x,\delta) \cap S_X \subset \left\{ x + ta \colon t \in \left[-\tfrac{d}{2}, \tfrac{d}{2} \right] \right\}.$$

Since $\{x + ta \colon |t| \geq \delta\} \cap B(x,\delta) = \emptyset$ and $\{x + ta \colon |t| < \delta\} \subset B(x,\delta) \cap S_X$, the equality (3.3) follows. $\qquad\square$

We have already seen, in Proposition 3.5, that ℓ_1^n over the real scalars has property (co). Next we show that this is also true for complex ℓ_1^n.

Theorem 3.8. *Let $n \in \mathbb{N}$. Then the complex space ℓ_1^n has property (co).*

Proof. We are going to apply Proposition 3.2 teamed with Theorem 3.6. Let $x_0, x_1, x_2 \in S_{\ell_1^n}$ with $x_1 \neq x_2$ and $\lambda_1, \lambda_2 > 0$ with $\lambda_1 + \lambda_2 = 1$ be such that $x_0 = \lambda_1 x_1 + \lambda_2 x_2$, and let $\varepsilon > 0$. For a complex number ζ, we write $\zeta = (r, \phi)$, where r and ϕ are, respectively, the modulus and an argument of ζ. For every $j \in \{0, 1, 2\}$, let $x_j = (x_i^j)_{i=1}^n$, where $x_i^j = (r_i^j, \phi_i^j)$. We may assume that $\phi_i^0 = \phi_i^1 = \phi_i^2$ for every $i \in \{1, \ldots, n\}$ (this is because $\|x_1 + x_2\| = \|x_1\| + \|x_2\|$ yields $|x_i^1 + x_i^2| = |x_i^1| + |x_i^2|$).

For every $j \in \{0, 1, 2\}$ and every $\gamma > 0$, define

$$D_j(\gamma) := \Big\{ \big((r_i, \phi_i) \big)_{i=1}^n \in \ell_1^n \colon |r_i - r_i^j| < \gamma \text{ for every } i \in \{1, \ldots, n\}$$
$$\text{and } |\phi_i - \phi_i^j| < \gamma \text{ for every } i \in \{1, \ldots, n\} \text{ with } r_i^j \neq 0 \Big\},$$

and pick a $\gamma > 0$ such that $D_j(2\gamma) \subset B(x_j, \varepsilon)$ whenever $j \in \{1, 2\}$. We may assume that $\gamma < \pi$, and that $2\gamma < r_i^j$ whenever $j \in \{0, 1, 2\}$ and $i \in \{1, \ldots, n\}$ are such that $r_i^j > 0$.

Set

$$I_1 := \big\{ i \in \{1, \ldots, n\} \colon r_i^1 > r_i^2 \big\}, \qquad I_2 := \big\{ i \in \{1, \ldots, n\} \colon r_i^1 < r_i^2 \big\}.$$

Define $\rho_i^1 := \rho_i^2 := r_i^1 = r_i^2$ if $i \notin I_1 \cup I_2$, and

$$\rho_i^1 := r_i^1 - \frac{\gamma \lambda_2}{|I_1|} \quad \text{and} \quad \rho_i^2 := r_i^2 + \frac{\gamma \lambda_1}{|I_1|} \qquad \text{for every } i \in I_1,$$

$$\rho_i^1 := r_i^1 + \frac{\gamma \lambda_2}{|I_2|} \quad \text{and} \quad \rho_i^2 := r_i^2 - \frac{\gamma \lambda_1}{|I_2|} \qquad \text{for every } i \in I_2.$$

Observe that

(1) $\lambda_1 \rho_i^1 + \lambda_2 \rho_i^2 = \lambda_1 r_i^1 + \lambda_2 r_i^2 = r_i^0$ for every $i \in \{1, \ldots, n\}$;
(2) $\sum_{i=1}^n \rho_i^j = \sum_{i=1}^n r_i^j = 1$ for every $j \in \{1, 2\}$;

(3) $|\rho_i^j - r_i^j| < \gamma$ for every $j \in \{1, 2\}$ and every $i \in \{1, \dots, n\}$;

(4) $\rho_i^j \geq \frac{\gamma \min\{\lambda_1, \lambda_2\}}{n} =: \beta > 0$ for every $j \in \{1, 2\}$ and every $i \in \{1, \dots, n\}$ with $r_i^0 \neq 0$.

Choose a $\delta > 0$ such that $B(x_0, \delta) \subset D_0(\beta)$. For every $u \in S := B(x_0, \delta) \cap S_{\ell_1^n}$, writing $u := \left((r_i^0 + \delta_i(u), \phi_i(u)) \right)_{i=1}^n$, where $|\delta_i(u)| < \beta$ and $|\phi_i(u) - \phi_i^0| < \beta$ (here the latter inequality is dropped if $r_i^0 = 0$), define

$$\mathfrak{v}_j(u) := \left((\rho_i^j + \delta_i(u), \phi_i(u)) \right)_{i=1}^n, \quad j \in \{1, 2\}.$$

Note that $\|u\| = \sum_{i=1}^n r_i^0 + \delta_i(u) = 1 + \sum_{i=1}^n \delta_i(u)$, hence $\sum_{i=1}^n \delta_i(u) = 0$ and thus

$$\|\mathfrak{v}_j(u)\| = \sum_{i=1}^n |\rho_i^j + \delta_i(u)| = \sum_{i=1}^n \rho_i^j + \delta_i(u) = 1.$$

The functions $\mathfrak{v}_1, \mathfrak{v}_2 \colon S \to B_X$ are continuous and satisfy (3.1) for every $u \in S$. $\qquad \square$

From Theorem 3.8 (Proposition 3.5 in the real case), Proposition 2.2, and Theorem 2.5 we know that, for any scattered compact K, $C_0(K, \ell_1^n) = C_0(K) \hat{\otimes}_\varepsilon \ell_1^n$ has property CWO. A similar result does not hold for the projective tensor product.

Proposition 3.9. *Let X be a Banach space. Then $X \hat{\otimes}_\pi \ell_1^n$ fails property CWO-B.*

Proof. The proof of [7, Proposition 2.1] shows that if X and Y are Banach spaces, and $Z := X \oplus_p Y$, where $1 \leq p < \infty$, then there exists a finite convex combination of slices of B_Z which contains 0, but which fails to contain a relatively weakly open neighbourhood of 0.

The assertion follows since $X \hat{\otimes}_\pi \ell_1^n$ is isometrically isomorphic to $\ell_1^n(X)$ (see proof of [13, Example 2.6, p. 19]). $\qquad \square$

4. STABILITY RESULTS

In this section we discuss stability results of the CWO-properties in Definition 1.1. We start by showing that they all are stable by taking one-complemented subspaces, but first, let us make things easier for ourselves.

Lemma 4.1. *Let X be a Banach space.*

(a) *The following assertions are equivalent:*

    (i) *X has property CWO;*

    (ii) *whenever U_1 and U_2 are relatively weakly open subsets of B_X and $\lambda_1, \lambda_2 > 0$, $\lambda_1 + \lambda_2 = 1$, the convex combination $\lambda_1 U_1 + \lambda_2 U_2$ is open in the relative weak topology of B_X.*

(b) *The following assertions are equivalent:*
  (i) *X has property CWO-S;*
  (ii) *whenever U_1 and U_2 are relatively weakly open subsets of B_X, $\lambda_1, \lambda_2 > 0$, $\lambda_1 + \lambda_2 = 1$, and $x_1 \in U_1$, $x_2 \in U_2$ are such that $\|\lambda_1 x_1 + \lambda_2 x_2\| = 1$, the element $\lambda_1 x_2 + \lambda_2 x_2$ is an interior point of $\lambda_1 U_1 + \lambda_2 U_2$ in the relative weak topology of B_X.*

(c) *The following assertions are equivalent:*
  (i) *X has property CWO-B;*
  (ii) *whenever U_1 and U_2 are relatively weakly open subsets of B_X, $\lambda_1, \lambda_2 > 0$, $\lambda_1 + \lambda_2 = 1$, and $x_1 \in U_1$, $x_2 \in U_2$ are such that $\|\lambda_1 x_1 + \lambda_2 x_2\| < 1$, the element $\lambda_1 x_2 + \lambda_2 x_2$ is an interior point of $\lambda_1 U_1 + \lambda_2 U_2$ in the relative weak topology of B_X.*

The proof of (c), (ii) $\Rightarrow$ (i), makes use of the following lemma.

Lemma 4.2. *Let X be a Banach space, let $n \in \mathbb{N}$, let $U_1, \ldots, U_n$ be relatively weakly open subsets of B_X, and let $\lambda_1, \ldots, \lambda_n > 0$, $\sum_{j=1}^{n} \lambda_j = 1$. Then every $x \in \sum_{j=1}^{n} \lambda_j U_j$ with $\|x\| < 1$ can be written as*

$$x = \sum_{j=1}^{n} \lambda_j x_j, \quad \text{where } x_j \in U_j \text{ and } \|x_j\| < 1 \text{ for every } j \in \{1, \ldots, n\}.$$

Proof. Let $x := \sum_{j=1}^{n} \lambda_j u_j$ with $u_j \in U_j$, $j \in \{1, \ldots, n\}$, be such that $\|x\| < 1$. Choosing $r \in (0, 1)$ small enough, we have $x_j := rx + (1-r)u_j \in U_j$ and $\|x_j\| < 1$ for every $j \in \{1, \ldots, n\}$. It remains to observe that

$$\sum_{j=1}^{n} \lambda_j x_j = rx + (1 - r) \sum_{j=1}^{n} \lambda_j u_j = rx + (1 - r)x = x. \qquad \square$$

Proof of Lemma 4.1. In each of (a)–(c), the implication (i) $\Rightarrow$ (ii) is trivial. (ii) $\Rightarrow$ (i) follows easily by induction using the same idea of splitting the convex combination as in Proposition 3.2. More precisely, for (c), (ii) $\Rightarrow$ (i), one first uses Lemma 4.2. $\qquad \square$

Proposition 4.3. *Let X be a Banach space. One-complemented subspaces of X inherit each of the properties CWO, CWO-B, and CWO-S.*

Proof. Let Y be a subspace of X and $P \colon X \to X$ a projection onto Y with $\|P\| = 1$. Using Lemma 4.1, it is enough to consider

$$C_Y := \lambda_1 U_1 + \lambda_2 U_2,$$

where U_1 and U_2 are relatively weakly open subsets of B_Y and $\lambda_1, \lambda_2 > 0$ with $\lambda_1 + \lambda_2 = 1$. Since P is weak-to-weak continuous, $P^{-1}(U_1) \cap B_X$ and

$P^{-1}(U_2) \cap B_X$ are relatively weakly open in B_X. Set

$$C_X := \lambda_1 \big(P^{-1}(U_1) \cap B_X \big) + \lambda_2 \big(P^{-1}(U_2) \cap B_X \big).$$

Notice that $C_Y = C_X \cap B_Y$. This is immediate from $C_Y \subset C_X$ and $P(C_X) \subset C_Y$.

If X has property CWO, then C_X is relatively weakly open in B_X, therefore C_Y is relatively weakly open in B_Y and it follows that Y has property CWO.

For properties CWO-S and CWO-B, notice that $C_Y \cap S_Y = (C_X \cap S_X) \cap B_Y$ and $C_Y \cap B_Y^\circ = (C_X \cap B_X^\circ) \cap B_Y$. $\qquad \square$

In the case of CWO-B, we can say a lot more. Let $\lambda \geq 1$. Recall that a closed subspace Y of a Banach space X is said to be *locally λ-complemented* in X if, for every finite-dimensional subspace E of X and every $\varepsilon > 0$, there exists a linear operator $P_E : E \to Y$ with $P_E x = x$ for all $x \in E \cap Y$ and $\|P_E\| \leq \lambda + \varepsilon$. If Y is locally λ-complemented in X, then there exists an *extension operator* $\Phi : Y^* \to X^*$, that is $(\Phi y^*)(y) = y^*(y)$ for all $y \in Y$ and $y^* \in Y^*$. Note that Φ can be chosen so that $\|\Phi\| \leq \lambda$. This was shown independently by Fakhoury [5, Théorème 2.14] and Kalton [9, Theorem 3.5].

Theorem 4.4. *Let X be a Banach space with property CWO-B and let Z be an infinite-dimensional Banach space such that, for every $\varepsilon > 0$, the space X contains a locally $(1 + \varepsilon)$-complemented subspace which is $(1 + \varepsilon)$-isometric to Z. Then Z has the strong diameter two property.*

Proof. Suppose for contradiction that Z fails the strong diameter two property. Then there are $n \in \mathbb{N}$ $z_1^*, \ldots, z_n^* \in S_{Z^*}$, $\alpha > 0$, $\lambda_1, \ldots, \lambda_n > 0$ with $\sum_{i=1}^n \lambda_i = 1$, and $\rho \in (0,1)$ such that, whenever $j \in \{1, 2\}$, setting $z_{j,i}^* := (-1)^{j-1} z_i^*$ and

$$S_{j,i}^Z := S(z_{j,i}^*, \alpha) = \{z \in B_Z \colon \operatorname{Re} z_{j,i}^*(z) > 1 - \alpha\},$$

one has $\operatorname{diam}(C_j^Z) < 2\rho$, where $C_j^Z := \sum_{i=1}^n \lambda_i S_{j,i}^Z$. Observe that $\frac{1}{2}(C_1^Z + C_2^Z) \subset B(0, \rho)$, because $S_{2,i}^Z = -S_{1,i}^Z$ and thus $C_2^Z = -C_1^Z$.

Choose $\varepsilon > 0$ so that $1/(1 + \varepsilon)^2 > \max\{\rho, 1 - \alpha\}$, and let Y be a closed locally $(1 + \frac{\varepsilon}{2})$-complemented subspace of X such that there exists an isomorphism $T \in \mathcal{L}(Y, Z)$ with $\|T\| \leq 1$ and $\|T^{-1}\| < 1 + \varepsilon$. Since Y is locally $(1 + \frac{\varepsilon}{2})$-complemented, there exists an extension operator $\Phi : Y^* \to X^*$ with $\|\Phi\| \leq 1 + \frac{\varepsilon}{2}$. Consider the slices

$$S_{j,i}^X := \left\{ x \in B_X \colon \operatorname{Re}(\Phi T^* z_{j,i}^*)(x) > \frac{1}{1 + \varepsilon} \right\}$$

of B_X. To see that the sets $S_{j,i}^X$ are non-empty, observe that

$$\|\Phi T^* z_{j,i}^*\| \geq \|\Phi T^* z_{j,i}^*|_Y\| = \|T^* z_{j,i}^*\| \geq \frac{\|z_{j,i}^*\|}{\|(T^*)^{-1}\|} = \frac{1}{\|(T^{-1})^*\|} > \frac{1}{1+\varepsilon}.$$

Set $C_j^X := \sum_{i=1}^n \lambda_i S_{j,i}^X$, $j \in \{1,2\}$.

Since $0 \in \sum_{j=1}^2 \sum_{i=1}^n \frac{\lambda_i}{2} S_{j,i}^X = \frac{1}{2}(C_1^X + C_2^X)$, by courtesy of property CWO-B for X, there is a relatively weakly open subset W of B_X such that $0 \in W \subset \frac{1}{2}(C_1^X + C_2^X)$. Since $\widehat{W} := B_Y \cap W$ is non-empty (because $0 \in \widehat{W}$) and relatively weakly open in B_Y, there exists a $y \in S_Y \cap \widehat{W}$ (here we use that Y is infinite-dimensional). Now $y \in \sum_{j=1}^2 \sum_{i=1}^n \frac{\lambda_i}{2} S_{j,i}^X$, thus there are $x_{j,i} \in S_{j,i}^X$ such that $y = \sum_{j=1}^2 \sum_{i=1}^n \frac{\lambda_i}{2} x_{j,i}$. Define

$$E := \mathrm{span}\{x_{j,i} : j \in \{1,2\}, i \in \{1,\ldots,n\}\} \subset X$$

and

$$F := \mathrm{span}\{T^* z_{j,i}^* : j \in \{1,2\}, i \in \{1,\ldots,n\}\} \subset Y^*.$$

Let $I_E : E \to X$ be the natural embedding. By [12, Corollary 3.3], there exists a linear operator $P_E : E \to Y$ with $\|P_E\| \leq 1 + \varepsilon$, $P_E x = x$ for all $x \in E \cap Y$, and $y^*(P_E x) = \Phi y^*(x)$ for all $x \in E$ and $y^* \in F$. Then $y = P_E y = \sum_{j=1}^2 \sum_{i=1}^n \frac{\lambda_i}{2} P_E x_{j,i}$. For every $j \in \{1,2\}$ and every $i \in \{1,\ldots,n\}$, one has

$$y_{j,i} := \frac{1}{1+\varepsilon} P_E x_{j,i} \in \left\{ y \in B_Y : \mathrm{Re}(T^* z_{j,i}^*)(y) > \frac{1}{(1+\varepsilon)^2} \right\}.$$

Set $y_j := \sum_{i=1}^n \lambda_i y_{j,i}$, $j \in \{1,2\}$. Observing that $Ty_{j,i} \in S_{j,i}^Z$, one has $Ty_j \in C_j^Z$, thus $v := \frac{1}{2}(Ty_1 + Ty_2) \in B(0,\rho)$ and $\frac{1}{2}(y_1 + y_2) = T^{-1}v \in B(0,(1+\varepsilon)\rho)$. It follows that

$$1 = \|y\| = \left\| \sum_{j=1}^2 \sum_{i=1}^n \frac{\lambda_i}{2} P_E x_{j,i} \right\| = (1+\varepsilon) \left\| \frac{1}{2}(y_1 + y_2) \right\| \leq \rho(1+\varepsilon)^2 < 1,$$

a contradiction. □

Using the fact that if a Banach space X contains a complemented copy of ℓ_1, then it already contains, for any $\varepsilon > 0$, a $(1+\varepsilon)$-complemented subspace $(1+\varepsilon)$-isomorphic to ℓ_1 [3, Theorem 5], we immediately get

Corollary 4.5. *If X contains a complemented subspace isomorphic to ℓ_1 then X does not have property CWO-B.*

Since every non-reflexive subspace of an L-embedded space contains a complemented subspace isomorphic to ℓ_1 [8, IV.Corollary 2.3] we obtain the following.

Corollary 4.6. *Let X be an L-embedded Banach space and M a closed infinite-dimensional subspace of X. Then M does not have property CWO-B.*

In particular, $L_1[0,1]$ does not have property CWO-B.

5. The spaces $c_0(X_n)$ and $L_1(\mu)$

Let $\{X_n\}$ be a sequence of Banach spaces. Then $c_0(X_n)$ is the Banach space of all norm null sequences (x_n), where $x_n \in X_n$ for all $n \in \mathbb{N}$, with norm $\|(x_n)\| = \sup\{\|x_n\| : n \in \mathbb{N}\}$. Note that the dual Banach space of $c_0(X_n)$ is the Banach space $\ell_1(X_n^*)$ with norm $\|(x_n^*)\| = \sum_{n=1}^{\infty} \|x_n^*\|$.

We will need a lemma similar to Lemma 2.6.

Lemma 5.1. *Let $\{X_n\}$ be a sequence of finite-dimensional Banach spaces, and let $x \in B_{c_0(X_n)}$.*

    (a) *Let U be a neighbourhood of x in the relative weak topology of $B_{c_0(X_n)}$. Then there are a finite subset M of $\mathbb{N}$ and an $\varepsilon > 0$ such that, whenever $y \in B_{c_0(X_n)}$ satisfies*

(5.1) $$\|y(m) - x(m)\| < \varepsilon \quad \text{for every } n \in M,$$

    one has $y \in U$.

    (b) *Let M be a finite subset of $\mathbb{N}$, and let $\varepsilon > 0$. Then there is a neighbourhood U of x in the relative weak topology of $B_{c_0(X_n)}$ such that every $y \in U$ satisfies (5.1).*

Proof. Set $Z := c_0(X_n)$. Then $Z^* = \ell_1(X_n^*)$.

(a). Let a finite subset $\mathcal{F}$ of S_Z and an $\varepsilon > 0$ be such that

$$\{y \in B_Z : |f(y) - f(x)| < 3\varepsilon \text{ for every } f \in \mathcal{F}\} \subset U.$$

For every $f = (x_n^*)_{n=1}^{\infty} \in \mathcal{F}$, let f_N be its projection onto the first N coordinates, that is, $f_N = (z_n^*)$ where $z_n^* = x_n^*$ for $n \in \{1, \ldots, N\}$ and $z_n^* = 0$ for $n > N$. Choose N so large that $\|f - f_N\| \leq \varepsilon$ for all $f \in \mathcal{F}$, and let $M = \{1, \ldots, N\}$. Note that $\|f_N\| \leq \|f\| = 1$.

Now, if $y \in B_Z$ satisfies (5.1), then, for every $f \in \mathcal{F}$,

$$|f(y) - f(x)| \leq |f(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)| \leq 3\varepsilon,$$

and it follows that $y \in U$.

(b). For every $m \in M$, let $A_m \subset B_{X_m^*}$ be a finite $\frac{\varepsilon}{3}$-net for $B_{X_m^*}$. Set

$$U := \left\{ y \in B_{c_0(X_n)} : \left| x^* \big(y(m) - x(m) \big) \right| < \frac{\varepsilon}{3} \text{ for every } m \in M \right.$$

$$\left. \text{and every } x^* \in A_m \right\}.$$

Let $y \in U$ and $m \in M$ be arbitrary. Picking $z^* \in B_{X_m^*}$ so that

$$z^* \big(y(m) - x(m) \big) = \| y(m) - x(m) \|,$$

there is an $x^* \in A_m$ satisfying $\| z^* - x^* \| < \frac{\varepsilon}{3}$. One has

$$\begin{aligned}
\| y(m) - x(m) \| &= z^* \big(y(m) - x(m) \big) \\
&\leq \| z^* - x^* \| \| y(m) - x(m) \| + \left| x^* \big(y(m) - x(m) \big) \right| \\
&\leq \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

$\square$

Theorem 5.2. *Let $\{X_n\}$ be a sequence of finite-dimensional Banach spaces with property (co). Then $c_0(X_n)$ has property CWO.*

Proof. Set $Z := c_0(X_n)$. Let V_1 and V_2 be relatively weakly open subsets of B_Z and let $\lambda_1, \lambda_2 > 0$ with $\lambda_1 + \lambda_2 = 1$. Using Lemma 4.1, it is enough to consider the convex combination $C := \lambda_1 V_1 + \lambda_2 V_2$. Let $x = \lambda_1 x_1 + \lambda_2 x_2 \in C$ with $x_j \in V_j$. We are going to find a neighbourhood U of x in the relative weak topology of B_Z such that $U \subset C$.

By Lemma 5.1, (a), there are an $\varepsilon > 0$ and a finite subset M of $\mathbb{N}$ such that

- whenever $y_1, y_2 \in B_Z$ are such that, for every $m \in M$,

$$\| y_j(m) - x_j(m) \| < \varepsilon, \quad j \in \{1, 2\},$$

one has $y_j \in V_j$, $j \in \{1, 2\}$.

For every $m \in M$, let δ_m and $\widehat{v}_{m,j}$, be, respectively, the δ and the functions $\widehat{v}_j$ from condition (con) of Definition 2.1 with $X = X_m$, $n = 2$, $x = x(m)$, and $x_j = x_j(m)$, $j \in \{1, 2\}$. By Lemma 5.1, (b), there is a neighbourhood U of x in the relative weak topology of B_Z such that, for every $y \in U$,

$$\| y(m) - x(m) \| < \delta_m \quad \text{for every } m \in M.$$

Let $y \in U$ be arbitrary. Define $y_1, y_2 \in B_Z$ by $y_j(m) = \widehat{v}_{m,j} \big(y(m) \big)$ for every $m \in M$ and $y_j(n) = y(n)$ for every $n \in \mathbb{N} \setminus M$. Then $y = \lambda_1 y_1 + \lambda_2 y_2$ with $y_j \in V_j$, and hence $U \subset C$. $\square$

It is known that the Banach space $c_0(\ell_2^n)$ is not isomorphic to c_0 (here, by $c_0(\ell_2^n)$ we mean the space $c_0(X_n)$, where $X_n = \ell_2^n$ for every $n \in \mathbb{N}$). By the above theorem, and Proposition 2.3, (b), the space $c_0(\ell_2^n)$ has property *CWO*. In fact, we have the following result.

Corollary 5.3. *Whenever $1 \le p \le \infty$, the Banach space $c_0(\ell_p^n)$ has property CWO.*

Proof. We use Theorem 5.2 together with Theorem 3.8 if $p = 1$, with Proposition 2.3, (b), if $1 < p < \infty$, and with Proposition 3.5 if $p = \infty$. □

The above result does not hold for ℓ_∞-sums. Not even $\ell_\infty = C(\beta\mathbb{N})$ has property *CWO* since $\beta\mathbb{N}$ is not scattered (that would force ℓ_∞ to be Asplund [4, Theorem 14.25]), and $C(K)$ has property *CWO* only when K is a scattered compact Hausdorff space [7, Theorem 3.1].

In [6, Remark IV.5], it was observed that finite convex combinations of relatively weakly open subsets of the positive face of the unit ball of $L_1[0, 1]$ are still relatively weakly open. Our next theorem shows that the same (and even more) holds for the whole unit sphere.

Remark 5.4. Let μ be a non-zero σ-finite (countably additive non-negative) measure on a σ-algebra Σ of a non-empty set Ω. Then μ is atomless if and only if every finite convex combination of relatively weakly open subsets of $B_{L_1(\mu)}$ intersects the unit sphere.

Indeed, let $U_1, \ldots, U_n$ be non-empty relatively weakly open subsets of $B_{L_1(\mu)}$. By Bourgain's lemma [6, Lemma II.1], each U_j contains a finite convex combination of slices (of $B_{L_1(\mu)}$). Hence any convex combination of the U_j's contains a finite convex combination of slices. Identifying $L_1(\mu)^*$ with $L_\infty(\mu)$, one can easily show that if μ is atomless, then any convex combination of slices of $B_{L_1(\mu)}$—and thus also any convex combination of the U_j's—intersects S_X (see [1, Example 3.2] for an argument).

On the other hand, if $A \in \Sigma$ is an atom for μ, then $\frac{1}{\mu(A)}\chi_A \in B_{L_1(\mu)}$ is strongly exposed by $g := \chi_A \in B_{L_\infty(\mu)}$ (here we identify $L_1(\mu)^*$ with $L_\infty(\mu)$ again). It follows that if $\alpha > 0$ is small enough, then the slices $S_1 := S(g, \alpha)$ and $S_2 := S(-g, \alpha)$ of $B_{L_1(\mu)}$ have diameter less than 1. Now, the convex combination $C := \frac{1}{2}S_1 + \frac{1}{2}S_2$ contains 0 and has diameter less than 1; thus C does not intersect the unit sphere.

As a consequence of the results in [2] and the above remark, $L_1(\mu)$ has the Daugavet property if and only if every finite convex combination of relatively weakly open sets of its unit ball intersects the unit sphere. Next

we show that every point in such an intersection is an interior point of the corresponding convex combination in the relative weak topology of $B_{L_1(\mu)}$.

Theorem 5.5. *Let μ be a non-zero σ-finite (countably additive non-negative) measure on a sigma-algebra Σ of a non-empty set Ω. Then the real space $L_1(\mu)$ has property CWO-S.*

Proof. Let U_1 and U_2 be relatively weakly open subsets of the closed unit ball $B_{L_1(\mu)}$ of $L_1(\mu)$, and let $\lambda_1, \lambda_2 > 0$, $\lambda_1 + \lambda_2 = 1$, and $x_1 \in U_1$, $x_2 \in U_2$ be such that $\|\lambda_1 x_1 + \lambda_2 x_2\| = 1$. By Lemma 4.1 it is enough to find a neighbourhood W of $x := \lambda_1 x_1 + \lambda_2 x_2$ in the relative weak topology of $B_{L_1(\mu)}$ such that $W \subset \lambda_1 U_1 + \lambda_2 U_2$.

Throughout the proof, whenever convenient, we identify functionals in $L_1(\mu)^*$ with elements in $L_\infty(\mu)$ in the canonical way. Since $L_\infty(\mu) = \overline{\operatorname{span}}\{\chi_E \colon E \in \Sigma\}$, there are a finite collection $\mathcal{F}$ of subsets of Σ and an $\varepsilon > 0$ such that

$$V_i := \left\{ u \in B_{L_1(\mu)} : \left| \int_E (u - x_i)\, d\mu \right| < 2\varepsilon \text{ for every } E \in \mathcal{F} \right\} \subset U_i, \quad i \in \{1, 2\}.$$

We may assume that $\bigcup_{E \in \mathcal{F}} E = \Omega$, that the sets in $\mathcal{F}$ are pairwise disjoint, and that, for every $E \in \mathcal{F}$, either $x_1 \chi_E \geq 0$ a.e. and $x_2 \chi_E \geq 0$ a.e., or $x_1 \chi_E \leq 0$ a.e. and $x_2 \chi_E \leq 0$ a.e. (the latter is because if $x_1 x_2(t) < 0$ for almost every t in a set $D \in \Sigma$ with $\mu(D) > 0$, then one would have $\|x\| = \|\lambda_1 x_1 + \lambda_2 x_2\| < 1$).

Set $E_0 := \bigcup_{E \in \mathcal{F}_0} E$ where $\mathcal{F}_0 := \{E \in \mathcal{F} : \int_E x\, d\mu = 0\}$, and label the sets in $\mathcal{F} \setminus \mathcal{F}_0$ as $E_1, \ldots, E_n$ where $n \in \mathbb{N}$. For every $i \in \{1, \ldots, n\}$, set

$$\alpha_i := \int_{E_i} x_1\, d\mu, \quad \beta_i := \int_{E_i} x_2\, d\mu, \quad \text{and} \quad \gamma_i := \lambda_1 \alpha_i + \lambda_2 \beta_i = \int_{E_i} x\, d\mu.$$

Note that, since x_1 and x_2 have the same sign on each E_i, we have $\frac{\alpha_i}{\gamma_i} \geq 0$, $\frac{\beta_i}{\gamma_i} \geq 0$, and $|\gamma_i| = \int_{E_i} |x|\, d\mu$ for every $i \in \{1, \ldots, n\}$.

Set $J_0 := \{i \in \{1, \ldots, n\} : \alpha_i = 0 \text{ or } \beta_i = 0\}$, $J_1 := \{1, \ldots, n\} \setminus J_0$, and, if $J_1 \neq \emptyset$, then set $\Gamma := \sum_{i \in J_1} |\gamma_i|$. Pick $\delta > 0$ so that $Mn^2\delta < \varepsilon$ where $M := \max_{1 \leq i \leq n} \frac{\alpha_i + \beta_i}{\gamma_i}$. We may assume that if $J_1 \neq \emptyset$, then $\delta < \frac{|\gamma_i|}{2}$ and $\frac{2Mn^2\delta}{\Gamma} < \min\{\frac{\alpha_i}{\gamma_i}, \frac{\beta_i}{\gamma_i}\}$ for every $i \in J_1$.

Define

$$W := \left\{ w \in B_{L_1(\mu)} : \left| \int_{E_i} w\, d\mu - \gamma_i \right| < \delta \text{ for every } i \in \{1, \ldots, n\} \right\}.$$

Let $w \in W$ be arbitrary. For every $i \in \{0, 1, \ldots, n\}$, set $w_i = w\chi_{E_i}$ and $\eta_i := \|w_i\| - |\gamma_i|$. Then

$$|\gamma_i| + \eta_i = \|w_i\| = \int_{E_i} |w|\, d\mu \geq \left| \int_{E_i} w\, d\mu \right| > |\gamma_i| - \delta,$$

whence $\eta_i > -\delta$. On the other hand,

$$1 \geq \|w\| = \sum_{j=0}^{n} \|w_j\| \geq \sum_{j=1}^{n} |\gamma_j| + \eta_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} \eta_j > 1 + \eta_i - (n-1)\delta,$$

whence $\eta_i < (n-1)\delta$, and thus $|\eta_i| < n\delta$. Observe that if $J_1 \neq \emptyset$, then

$$\rho := \sum_{i \in J_1} \|w_i\| = \sum_{i \in J_1} \big(|\gamma_i| + \eta_i\big) > \sum_{i \in J_1} \big(|\gamma_i| - \delta\big) > \sum_{i \in J_1} \frac{|\gamma_i|}{2} = \frac{\Gamma}{2}.$$

Setting

$$c := \sum_{i=1}^{n} \frac{\beta_i - \alpha_i}{\gamma_i} \eta_i,$$

one has

$$|c| \leq M \sum_{i=1}^{n} |\eta_i| < Mn^2\delta,$$

and thus, for every $i \in J_1$,

$$\frac{|c|}{\rho} \leq \frac{2|c|}{\Gamma} < \frac{2Mn^2\delta}{\Gamma} < \min\left\{\frac{\alpha_i}{\gamma_i}, \frac{\beta_i}{\gamma_i}\right\}.$$

Define $u_0 := v_0 := w_0$ and, for every $i \in \{1, \ldots, n\}$,

$$\begin{cases} u_i := \dfrac{\alpha_i}{\gamma_i} w_i, \\[2mm] v_i := \dfrac{\beta_i}{\gamma_i} w_i \end{cases} \text{if } i \in J_0, \quad \text{and} \quad \begin{cases} u_i := \left(\dfrac{\alpha_i}{\gamma_i} + \lambda_2 \dfrac{c}{\rho}\right) w_i, \\[2mm] v_i := \left(\dfrac{\beta_i}{\gamma_i} - \lambda_1 \dfrac{c}{\rho}\right) w_i \end{cases} \text{if } i \in J_1.$$

Define $u := \sum_{i=0}^{n} u_i$ and $v := \sum_{i=0}^{n} v_i$. For every $i \in \{0, 1, \ldots, n\}$, one has $\lambda_1 u_i + \lambda_2 v_i = w_i$; thus $\lambda_1 u + \lambda_2 v = \sum_{i=0}^{n} w_i = w$. Now,

$$\begin{aligned}
\|u\| &= \|w_0\| + \sum_{i \in J_0} \frac{\alpha_i}{\gamma_i} \|w_i\| + \sum_{i \in J_1} \left(\frac{\alpha_i}{\gamma_i} + \lambda_2 \frac{c}{\rho}\right) \|w_i\| \\
&= \|w_0\| + \sum_{i=1}^{n} \frac{\alpha_i}{\gamma_i} \|w_i\| + \lambda_2 c \\
&= \|w_0\| + \sum_{i=1}^{n} \frac{\alpha_i}{\gamma_i} \big(|\gamma_i| + \eta_i\big) + \lambda_2 \sum_{i=1}^{n} \frac{\beta_i - \alpha_i}{\gamma_i} \eta_i \\
&= \|w_0\| + \sum_{i=1}^{n} |\alpha_i| + \sum_{i=1}^{n} \frac{\lambda_1 \alpha_i + \lambda_2 \beta_i}{\gamma_i} \eta_i \\
&= \|w_0\| + 1 + \sum_{i=1}^{n} \eta_i,
\end{aligned}$$

and, similarly, $\|v\| = \|w_0\| + 1 + \sum_{i=1}^{n} \eta_i$. Since

$$(5.2) \qquad \|w_0\| = \|w\| - \sum_{i=1}^{n} \|w_i\| \leq 1 - \sum_{i=1}^{n} |\gamma_i| - \sum_{i=1}^{n} \eta_i = -\sum_{i=1}^{n} \eta_i,$$

it follows that $\|u\|, \|v\| \leq 1$.

It remains to show that $u \in U_1$ and $v \in U_2$. To this end, first observe that, for every $i \in \{1, \ldots, n\}$,

$$\left| \frac{\alpha_i}{\gamma_i} \int_{E_i} w \, d\mu - \alpha_i \right| = \frac{\alpha_i}{\gamma_i} \left| \int_{E_i} w \, d\mu - \gamma_i \right| < M\delta < \varepsilon.$$

Thus $\left| \int_{E_i} u \, d\mu - \alpha_i \right| < \varepsilon$ for every $i \in J_0$. For every $i \in J_1$,

$$\left| \int_{E_i} \frac{c}{\rho} w \, d\mu \right| \leq \frac{|c|}{\rho} \int_{E_i} |w| \, d\mu = \frac{|c|}{\rho} \|w_i\| \leq |c| < Mn^2\delta < \varepsilon,$$

thus

$$\left| \int_{E_i} u \, d\mu - \alpha_i \right| \leq \left| \frac{\alpha_i}{\gamma_i} \int_{E_i} w \, d\mu - \alpha_i \right| + \lambda_2 \left| \int_{E_i} \frac{c}{\rho} w \, d\mu \right| < 2\varepsilon.$$

Since, by (5.2), $\|w_0\| \leq -\sum_{i=1}^n \eta_i < n\delta < \varepsilon$, one has, for every $E \in \mathcal{F}_0$,

$$\left| \int_E u \, d\mu - \int_E x_1 \, d\mu \right| = \left| \int_E w_0 \, d\mu \right| \leq \int_\Omega |w_0| \, d\mu = \|w_0\| < \varepsilon.$$

It follows that $u \in V_1 \subset U_1$. One can similarly show that $v \in V_2 \subset U_2$, and the proof is complete. $\qquad \square$

6. Questions

We end the paper with some natural questions.

(a) Does every finite-dimensional Banach space with property CWO have property (co)?

(b) All of our infinite-dimensional examples of spaces with property CWO contain c_0, that is, both $C_0(K, X)$ and $c_0(X_n)$ contain a copy of c_0. Must every Banach space with property CWO contain a copy of c_0?

(c) Does there exist a dual (infinite-dimensional) Banach space with property CWO?

(d) If both X and Y have property CWO, does the injective tensor product $X \widehat{\otimes}_\varepsilon Y$ also have property CWO?

We could also ask the same if both X and Y have either property CWO-S or CWO-B.

References

[1] T. A. Abrahamsen and V. Lima, *Relatively weakly open convex combinations of slices*, Proc. Amer. Math. Soc. **146** (2018), no. 10, 4421–4427. MR 3834668

[2] J. Becerra Guerrero and M. Martín, *The Daugavet property for Lindenstrauss spaces*, Methods in Banach space theory, London Math. Soc. Lecture Note Ser., vol. 337, Cambridge Univ. Press, Cambridge, 2006, pp. 91–96. MR 2326380

[3] P. N. Dowling, N. Randrianantoanina, and B. Turett, *Remarks on James's distortion theorems. II*, Bull. Austral. Math. Soc. **59** (1999), no. 3, 515–522. MR 1698052

[4] M. Fabian, P. Habala, P. Hájek, V. Montesinos, and V. Zizler, *Banach space theory*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, New York, 2011, The basis for linear and nonlinear analysis. MR 2766381 (2012h:46001)

[5] H. Fakhoury, *Sélections linéaires associées au théorème de Hahn-Banach*, J. Functional Analysis **11** (1972), 436–452. MR 0348457

[6] N. Ghoussoub, G. Godefroy, B. Maurey, and W. Schachermayer, *Some topological and geometrical structures in Banach spaces*, Mem. Amer. Math. Soc. **70** (1987), no. 378, iv+116. MR 912637

[7] R. Haller, P. Kuuseok, and M. Põldvere, *On convex combinations of slices of the unit ball in Banach spaces*, Houston J. Math. (to appear).

[8] P. Harmand, D. Werner, and W. Werner, *M-ideals in Banach spaces and Banach algebras*, Lecture Notes in Mathematics, vol. 1547, Springer-Verlag, Berlin, 1993. MR 1238713 (94k:46022)

[9] N. J. Kalton, *Locally complemented subspaces and $\mathcal{L}_p$-spaces for $0 < p < 1$*, Math. Nachr. **115** (1984), 71–97. MR 755269

[10] G. López-Pérez and A. Rueda Zoca, *Strong diameter two property and convex combination of slices reaching the unit sphere*, ArXiv e-prints (2017).

[11] N. J. Nielsen and G. H. Olsen, *Complex preduals of L_1 and subspaces of $l_\infty^n(C)$*, Math. Scand. **40** (1977), no. 2, 271–287. MR 0454597

[12] E. Oja and M. Põldvere, *Principle of local reflexivity revisited*, Proc. Amer. Math. Soc. **135** (2007), no. 4, 1081–1088 (electronic). MR 2262909

[13] R. A. Ryan, *Introduction to tensor products of Banach spaces*, Springer Monographs in Mathematics, Springer-Verlag, London, 2002. MR 2003f:46030

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