Student meaning making in elementary algebra teaching

An in-depth study of classrooms in four countries

Jorunn Reinhardtsen

UiA University of Agder

Doctoral Dissertations at the University of Agder 258

Jorunn Reinhardtsen

Student meaning making in elementary algebra teaching

An in-depth study of classrooms in four countries

Dissertation for the degree philosophiae doctor

University of Agder
Faculty of Engineering and Science
2020

Doctoral dissertations at the University of Agder 258
ISSN: 1504-9272
ISBN: 978-82-7117-958-8

Print: 07Media, Kristiansand

Acknowledgements

The making of this thesis was made possible by a group of people, here at University of Agder (UiA) and beyond, who believed in me and have contributed their time and best efforts to support me in becoming a researcher in mathematics education.

I would like to thank all the members of the VIDEOMAT project. It was a pleasure to come to know you and I found our meetings a place for rich discussions that stimulated and inspired this thesis. I would like to thank our group here at UiA, Maria Luiza Cestari, Unni Wathne and Hans Erik Borgersen for including me in the project and supporting me in every way you could. Roger Säljö and Cecilia Kilhamn, you lead the international cooperation in an egalitarian fashion allowing inexperienced researchers, such as myself, to voice our questions and our opinions, which allowed us to grow and thrive within the project. Furthermore, you were always wonderful hosts when we visited Gothenburg. Thanks to Ole Björkqvist, Ann-Sofi Röj-Lindberg and Anna-Maija Partanen. It was always enjoyable to see you and thank you for the wonderful dinner you made us, Ann-Sofi, at your welcoming home in Vasa. Special thanks to James Stigler and Karen Givvin who welcomed me to UCLA. It was such a pleasure working with you Karen, watching videos and discussing, and not to forget, enjoying the Californian sun, flora and a coffee in between work sessions.

I would like to thank all the teachers and students that participated in the VIDEOMAT project. Particularly the teachers, who so graciously cooperated with us, on top of an already overwhelming workload, and for being willing to put themselves in a position in which their classroom activity became an object of study. I have sought in the work done and hope to continue to contribute to mathematics education in a way that also serve and assist you in the very important work you do.

The writing of a thesis is an exhilarating but also, at times, taxing experience. Furthermore, it is a very personal one. The process produces not only a written thesis but is also the making of a researcher. Roger Säljö and Martin Carlsen, my supervisors, you are extraordinary, both as professionals and as warm and caring “handledare”. Surely, you took me by the hand and guided my efforts. You are part of this thesis and on that account a part of me to. Thank you for enriching and making this journey as enjoyable as it has been.

I would like to express my gratitude to Luis Radford, Anna Sfard and James Stigler for giving seminars that have been of great help in the work. Furthermore, you found time on several occasions to sit down and talk with me. Each encounter has contributed to a deeper understanding of the field and the theories under development. Thank you for your generosity. I will also express my gratitude to the discussant of my internal viva, Andreas Ryve. Thanks for giving me excellent feedback and suggestions for improvement of my thesis.

I would like to thank all my colleagues here at UiA. Fellow students in the doctoral program, thank you, for sharing your worries, challenges and your victories as you progressed. The road is shorter when you travel with someone. Thanks to the larger mathematics education community at UiA for providing a supporting and friendly environment for new colleagues.

Finally, thanks to my husband, Greg, and our four children, Sonya, Sawyer, Troy and Melissa, for putting up with a mom who at every free second sits down with the computer. You are my inspiration and I think of you always. Thanks to mom and dad, without you this could not be achieved. You have kept the house running, done homework with Melissa, taken and picked up the children as they participated in a multitude of activities every week. I am forever grateful that you created a space for me so that I could do this work.

Summary

The research presented in this thesis is based on observations and analyses of classroom data from four countries. The data were collected through a collaboration of local research teams in Finland, Norway, Sweden and the USA (California) and the shared topic of interest is the learning of elementary algebra. This thesis is concerned with the meaning making of students as they are introduced to, and engage in, tasks, symbols and ideas belonging to the highly abstract discourse of algebra. Further, as a response to the complexity of classroom learning, the thesis also seeks to advance analytical approaches for studying algebraic thinking.

In the thesis, the networking of theories is implemented as an approach (Prediger, Bikner-Ahsbahs & Arzarello, 2008) to theoretical development. Algebra learning is conceptualized as shifts in form-function relations occurring through cultural development in goal-directed collective activity (Saxe, 2012). This conceptualization is particularly useful for investigating students' learning processes in the elementary algebra classroom as major shifts in both form (the algebraic syntax versus arithmetic) and function (analytic rather than calculational) of the mathematical discourse are expected to take place.

The thesis includes two empirical studies which investigate different aspects of the development of form- function relations in elementary algebra. In the first study (Study 2), the interactions occurring in sixteen focus groups are analyzed (4 countries × 4 classrooms × 1 focus group) as the students work with the same patterning task. The study shows how students tend to use arithmetic to serve new algebraic functions in their early learning. The second study (Study 3) investigated student-student and teacher-student interactions occurring during whole-class discussions of patterning tasks over a span of four consecutive lessons. The students also in this case use arithmetic to serve new functions in the whole-class discussions. However, the students increasingly engaged with elements of the algebraic discourse, which the teacher introduces in the classroom, such as function table, variable expressions, and functional relationships.

The thesis shows that students use a meta-arithmetical discourse to create momentary and contextual algebraic objects in patterning activity, through which they were able to generalize. Specific traits of this discourse are described, and it is suggested to offer opportunities for a meaningful student learning of algebra. It

is shown how students working in a small-group, problem-solving context used arithmetic to solve an algebraic problem. The solving process is explained to occur through multimodal discursive processes, in which talk about processes is replaced with talk about objects.

The findings show how the semiotic 'ecology' emerging as students engaged with tasks in classroom activity let them touch upon different mathematical ideas. The use of a factoring discourse let students generalize in terms of number structures. In working with geometrical sequences, drawing and visualizing how the pattern grows, students encountered and discussed different rates of change. In using the cultural artifact of a function table to generalize a pattern, students touched upon a functional relationship. This points out that it is important for teachers to consider the mathematical ideas they wish to focus on in classroom patterning activity and choose tasks and semiotic means accordingly.

A genetic approach to algebra learning was developed as I engaged with the complexity of the algebra classroom and is presented in the thesis. In order to investigate a collective classroom process, the constructs *participants' positioning* and *attunement to others* were proposed. The thesis shows that different levels of inter-comprehension were achieved in teacher-student interactions. Particularly, tension in discourse is shown to shape the emerging classroom generalization practice in a distinct way.

Sammendrag

Forskningen som blir presentert i denne avhandlingen er basert på observasjoner og analyse av klasserom data fra fire land. Dataene ble samlet inn gjennom et samarbeid mellom lokale forskningsgrupper i Finland, Norge, Sverige og USA (California) og den felles interessen er læring i elementær algebra. Denne avhandlingen tar for seg elevers menings skapning ettersom de blir introdusert for, og deltar i, problemer, symboler og ideer som tilhører den høyst abstrakte algebra diskursen. Videre, som en respons til kompleksiteten ved klasserom læring, forsøker avhandlingen også å videreutvikle analytiske tilnærminger for å studere algebraisk tenkning.

En «networking» av teorier ble brukt som tilnærming til teori utvikling (Prediger, Bikner-Ahsbahs & Arzarello, 2008). Algebra læring ble koseptualisert som skift i form-funksjon relasjoner som skjer gjennom kulturell utvikling i målrettet kollektiv aktivitet (Saxe, 2012). Denne konseptualiseringen er spesielt nyttig for å undersøke student læreprosesser i elementær algebra klasserom hvor ett stort skift i både form (algebraisk symbolsk språk versus aritmetikk) og funksjon (analytisk i stedet for kalkulerende) av den matematiske diskursen er forventet å finne sted.

Denne avhandlingen inkluderer to studier som undersøker ulike aspekter av utviklingen av form-funksjon relasjoner i elementær algebra. I det første studiet (Studie 2), er interaksjonene som forekom i seksten fokus grupper (4 land x 4 klasserom x i gruppe) da elevene arbeidet med den samme mønsteroppgaven analysert. Studiet viser hvordan elever tenderer til å bruke aritmetikk i deres tidlige læring. Det andre studiet (Studie 3) undersøkte elev-elev og lærer-elev interaksjoner som forekom i hel-klasse diskusjoner. Likevel, i økende grad så tok elevene i bruk elementer av den algebraiske diskursen i økende grad som læreren introduserte i klasserommet. Disse inkluderte funksjonstabeller, bokstavuttrykk, og ett funksjonsforhold.

Forskningen viser at elever bruker en meta-aritmetisk diskurs for å danne momentane og kontekstuelle algebraiske objekter i mønster aktivitet. Disse momentane objektene muliggjorde elevenes generaliseringer. Spesifikke kvaliteter ved denne diskursen ble beskrevet og denne diskursen ses å tilby muligheter for at elever kan lære algebra på en meningsfylt måte. Det ble vist at elever som arbeidet i små grupper brukte aritmetikk til å løse et algebraisk

problem. Dette skjedde gjennom multimodale diskursive prosesser, hvor snakk om prosesser ble byttet ut med snakk om objekter.

Forskningen viser hvordan den semiotiske 'økologien' som oppsto i møtet mellom elever og oppgaver i klasseroms aktivitet gjorde at de kom i kontakt med ulike matematiske ideer. Bruken av en 'faktorerende' diskurs lot elevene generalisere i henhold til tall struktur. I arbeid med geometriske sekvenser, ved å tegne og visualisere hvordan mønsteret utvikler seg, students kom i kontakt med og diskuterte forskjellige stigningstall. Ved bruk av en funksjonstabell for å generalisere kom elevene i kontakt med funksjonsforhold. Dette peker på at det er viktig at lærere tenker nøye gjennom de matematiske ideene de ønsker å sette søkelys på i klasserom mønster aktivitet og velge oppgaver og semiotiske redskaper i samsvar med disse.

En genetisk tilnærming til algebra læring ble utviklet i møtet med kompleksiteten i algebra klasserommet. De teoretiske konseptene 'deltakeres posisjonering' og anpasning til andre' ble utviklet for å undersøke en kollektiv klasseroms prosess. Avhandlingen viser at det ble oppnådd ulike nivåer av 'inter-comprehension' i lærer-elev interaksjon. Spesielt, spenning i diskurs viste seg å forme den fremtredende klasseroms generaliserings praksisen på en bestemt måte.

Contents

1 Addressing learning in elementary algebra classrooms

This introductory chapter includes four sections. The first section (1.1) reflects on the work done in the thesis as a process regulated by theoretical considerations and empirical work in which initial research questions have been made operational and new ones have been formulated. Included here are also accounts of the background and motive for the research undertaken. Additionally, the three studies included in the thesis will be shortly summarized as elements of one coherent research undertaking. In the next section (1.2), a theoretical background for the study is presented. In section 1.3 studies focusing on collective processes of meaning making are discussed. The main aims and research questions for the thesis are formulated in the following section (1.4). Lastly, an overview of the thesis is provided (1.5).

1.1 The Journey: A resumé of the production of the thesis

This thesis is born out of an international research project named VIDEOMAT [1] (cf. section 4.3), and is profoundly shaped by this collaboration, both in terms of the data analyzed and the becoming of a researcher in mathematics education. Prior to participating as a researcher in the VIDEOMAT project I had lived in Montana (USA) for six years and completed three years of university courses, also involving teacher training and in-school observations.

The classroom data from four different countries, Finland, Sweden, Norway and the USA (California), provided a stimulating point of departure for raising interesting issues and curiosity regarding the learning of algebra and it offered the possibility for conducting varied investigations in the quest for insights. The participation in regular meetings (in-person and virtual) between research teams in the four countries, including planning for the collection of classroom data as well as subsequent analytic discussions of these, over the course of about three years, formed the foundation for the production and finalization of this thesis. It was through these discussions that the meaning making of students in introductory algebra classrooms became the focus of this thesis.

[1] The VIDEOMAT research project was made possible thanks to a grant (Project No.: 210321/F10) from NOS-OH (The Joint Committee for Nordic Research Councils for the Humanities and the Social Sciences).

The early endeavors regarding of the work with the thesis involved transcribing group discussions of students in the context of a patterning task. It quickly became clear that to grasp the students' meaning making processes, one had to attend to multimodal elements of the interactions. In terms of transcriptions, this presented a problem of reduction, i.e. determining which multi-modal elements are relevant for the mathematical meaning making processes. Therefore, the first study of this thesis (Capturing learning in classroom interaction in mathematics: Methodological considerations) addresses issues of multimodality in terms of analytical approaches, the selection of salient episodes and transcriptions when investigating learning in the mathematics classroom.

A joint book project was initiated in the VIDEOMAT project. During the project discussions it was decided that a chapter of the book was going to focus on the student perspective and include data from all the participating countries. Thus, the second paper of this thesis (The fifth lesson: Students' responses to a patterning task across the four countries), that is, a book chapter, became the first empirically based publication of the thesis. It is co-authored with Dr. Karen Givvin from UCLA, a member of the Californian research team. The writing of this chapter began with a trip to UCLA. The time together was spent watching and analyzing the video-recordings of the 16 focus groups (1 focus group × 4 classrooms × 4 countries) as the students worked with a patterning task in a problem-solving context. Finding many similarities in the groups' approaches to the patterning task within and across the data from the four countries, we decided to divide the analyses into two parts (later the analysis was expanded to include two additional parts, juxtaposing and synthesizing findings). The first part focused on the different approaches, ranging all the way from basic ones, such as counting, to more sophisticated approaches including algebraic ones. The second part was an in-depth discourse analysis of the interactions occurring in a selection of the 16 groups. Dr. Karen Givvin focused on the first part of the analysis and I focused on the second part.

The analysis of the students' discourse regarding the patterning task included two perspectives: (1) a microgenetic one, focusing on the local discourse developments that occurred during the groups' activities with the patterning task; and (2) an ontogenetic one, looking at the discussions of the groups as samples of student discourse at a specific time (i.e. when the movement from arithmetic to algebra is expected to take place in the classroom) in the long-term development of a mathematical discourse through years of schooling. The cultural-semiotic approach developed by Radford (2000, 2002) and Sfard's (2008) communicational approach were drawn upon in these analyses. In working with these approaches for understanding mathematics learning, a concern for the analytic possibilities they offer and the type of questions regarding learning that can be addressed within these emerged.

Several considerations influenced the focus of the next phase in the making of the thesis. First, one of the findings of Study 2 is that the students mainly employed a meta-arithmetical discourse in solving the patterning task and not the algebra that had been introduced during the previous four lessons. This raised issues regarding contingencies between teaching and learning in the introductory algebra classroom. Secondly, becoming familiar with Saxe's (2012) genetic approach to the cultural development of mathematical ideas, a possibility was perceived to strengthen the investigation of learning processes in algebra through a third genetic analysis, i.e. an analysis of a sociogenetic process in the introductory algebra classroom. Thirdly, Saxe's genetic perspective offered an opportunity for a well-defined coordination of different theoretical approaches, developed within mathematics education, to the analyses of learning processes in introductory algebra, thus providing the setting and a frame for a discussion regarding the theories involved. Fourth, as a thesis in mathematics education, a relevance aspect regarding the school practice of the field should be considered (Kilpatrick, 1993). An analysis focusing on whole classroom processes, including both teacher and students, contributes in this regard. Finally, and decisively, one of the Californian classrooms worked with patterning tasks throughout the four consecutive lessons video-recorded and therefore provided the opportunity to pursue the aims listed above in a coherent context of patterning activities.

Therefore, the third paper in this thesis (The emergence of a generalization practice in a 6th grade introductory algebra classroom) is a case study that focuses on whole classroom processes in introductory algebra. This study proved to be more challenging than the previous ones, since an operational description of sociogenesis in the mathematics classroom from a sociocultural perspective could not be found. Thus, such an analytical approach had to be developed. Additionally, considering whole classroom processes of solving a sequence of different patterning tasks versus student group work with one task, involves an increase in complexity, as for example one must account for the different roles of teacher and students in the processes.

This extended abstract is a methodological and theoretical reflection on the empirical analyses done previously. The methodological framework proposed by Saxe (2012) conceptualizes development as shifts in cultural forms and the cognitive functions they serve in goal-directed activity. Although Saxe applied the framework to investigate the development of mathematical ideas in a Papua New Guinea community, this is a general framework that is useful for studying learning in the mathematics classroom. I reflect on how the different mathematics education theories applied in the empirical studies facilitate the analyses of classroom learning seen as occurring through three intertwined genetic processes, i.e. microgenesis, sociogenesis and ontogenesis. Moreover, a summary of findings regarding the different genetic processes in the introductory algebra classroom, as developed in the three studies (Study I-III) are presented. Particularly, I reflect on how Saxe's (2012) genetic approach can contribute in the research of algebra teaching and learning in the classroom.

1.2 Researching the learning of elementary algebra through classroom interactions

The interest of this research is the teaching and learning of introductory algebra. Learning is investigated through classroom interaction occurring in small-group settings and in whole class discussions. The aim is to understand how students make meaning as they engage with algebraic tasks, symbols, artifacts and ideas in joint classroom activities. I see meaning making as a central notion in the building of a theory of learning that is not "deeply concerned with individual differences, notions of better and worse, more or less learning, or with comparison of these things across groups-of-individuals" (Lave 1996, p. 149).

This study does not make claims regarding specific learning outcomes but rather seeks to provide insights into the processes through which learning occurs. The first section (1.2.1) introduces the notion of meaning making in terms of subjective and collective versions, seen as rooted in classroom interaction. Meaning making is then addressed in terms of what is seen to be two interactionist agendas in educational research. These are identified in the literature in terms of being focused either on meanings as co-constructed through processes of negotiation in communicational activity or on subjective meanings in joint activity. Section 1.2.2 discuss the classroom as a community of practice with an evolving microculture. Approaches that focus on subjective meaning making in social and cultural processes are discussed in section 1.2.3.

1.2.1 Meaning making in classroom interactions

Addressing meaning making in educational research has been identified as marking a shift away from the reductionist approach to learning as propagated by the behaviorist and cognitivist tradition (von Glasersfeld, 1983, Bruner, 1990, Harré & Gillett, 1994). The choice of focusing on meaning making is to provide a rich interpretation of classroom processes. Meaning making is a central notion in a common agenda of 'teaching for understanding' (as opposed to only rote learning of procedures and facts), which unifies the constructivist and sociocultural positions on learning and development.

The analyses in this research will follow a Vygotskian view of learning and development (Vygotsky, 1978, 1986). A central idea is that learning results from social and interactional processes (Säljö, 2001). Many studies, working within different theoretical perspectives that recognize the role of the social in individual cognition, have focused on classroom interactions as the main unit of analysis. The Vygotskian view on learning and development stresses a cultural and historical perspective on what it means to be social and see social processes as primary to individual ones (Vygotsky, 1978, 1986). Within this framework, the purpose of conducting research is to investigate processes of thinking within historical, cultural and institutionalized contexts (Wertsch, 1998). In this research, the sociocultural perspective is manifested in my focus on the role of cultural artifacts and other semiotic means in the learning of algebra (cf. Kilhamn & Säljö, 2019). I use the term cultural artifact to refer to combinations of signs that are named and that are part of the mathematical discourse in accordance with Säljö (2005), i.e. equation, variable expression, function table, etc.

Vygotsky discussed meaning in terms of "word meaning" and saw that it changes in different contexts. However, he was mainly concerned with this variation in terms of the subjective connection between thought and word, which he addresses in terms of the "sense" of a word (Zinchenko, 2007). In addition to seeking to understand participants' use of words (and other semiotic means), we are also concerned with meaning as co-constructed in communicational processes. Gee (2015, p. 27) offers insights of the value of emphasizing this perspective:

> Meanings are ultimately rooted in negotiations among people in different social practices with different interests, people who share or seek to share common ground. Power plays an important role in these negotiations. The negotiation can be settled for the time being, in which case meaning become conventional and routine.

Conceptualizing the mathematics classroom as a situated community of practice (Lave, 1991), meaning making is rooted in classroom communicational processes. These processes are dependent on both teacher(s) and students. However, the teacher as a knower of mathematics, and with access to a mathematical community outside the classroom, holds a position of power in the classroom. This position is also upheld by the institution of schooling. As each new mathematical topic is introduced in the classroom, a negotiation of meanings is initiated. Thus, teaching for understanding, as part of an educational agenda, is dependent on the appropriateness and effectiveness of this negotiation. Gee (2015, p. 27) continues: "The negotiations are limited by values emanating from 'communities' or from attempts by people to establish and stabilize (perhaps only for here and now) enough common ground to agree on meaning".

In terms of the mathematics classroom, it is influenced by other communities both inside and outside of the particular school, as well as values from mathematics as a scientific discipline (representing a different community). The latter community upholds the historically produced meanings of mathematical words and signs and ways of using these in shared communication. Further, a teacher may choose to simplify a mathematical topic, by ignoring some relevant aspects and focusing on others, to achieve a common ground for the classroom community. Students, on the other hand, to avoid attention and/or conflict, or seeking the approval of the teacher, may be swift to comply with the perceived teacher agenda rather than actively voicing their perspective. Thus, educationally productive communication in the classroom is not a given.

In this study I identify and draw upon interactional studies seen as belonging to two different, but intersecting, research agendas in the literature that both approach the topic of meanings in mathematical discussions although from different perspectives. That is, one focuses on the classroom as a community of practice with its own unique identity, history, and social and socio-mathematical norms, and this research agenda is concerned with how to arrange classroom activities to optimize student learning (e.g. Lampert, 1990, Cobb, Stephan, McClain & Gravemeier, 2001). Meaning making is addressed in terms of what it means to do mathematics in that particular classroom, and studies include a concern for communication on classroom level. Thus, these studies address meaning making as co-construction in the context of a community, confined in time and space, as elaborated by Gee (2015). The second agenda identified, also views meanings as co-constructed by participants in joint activity. However, its main interest is the individual learning processes in terms of mathematics as a culturally and historically established practice. Thus, Vygotsky's subjective perspective on meaning will be elaborated on. In section 1.2.3 we focus on two approaches (Radford, 2002; Sfard, 2008) that are seen to manifest and develop important elements of this agenda.

1.2.2 A community of practice perspective on classroom practices

In this study, the teacher and the students in a classroom are conceptualized as operating within a community of practice (Lave & Wenger, 1991). This, among other things, implies that learning is not reduced to be seen as the acquisition of some pre-existing and pre-defined mathematical procedures and objects. Rather, learning is also about a way of being and acting in world. In the mathematics classroom, this implies a focus on teacher-student and student-student relations, patterns of interactions, and norms and values regarding what counts as mastery of a mathematical topic. The values and norms are in part determined by the institution of schooling and the culturally established mathematical discourse, but also by other communities that the participants are involved in. However, these values and norms are also continually negotiated by the participants in the classroom. Learning involves participation in the evolving classroom mathematical practices, and this involves both being absorbed by the practices while at the same time absorbing the culture of these practices.

This practice is organized by the society, compulsory for students and the expected outcome is that students will acquire a set of skills that are recognized as mathematical (Hatano & Wertsch, 2001). The practice in a classroom is framed by the institution of schooling and mathematics as a discipline, which both rely on particular modes of communication. These forms of communication are patterned collective activities which require reacting to certain actions with re-actions in a distinct manner (Sfard, 2008). These patterns are culturally and historically shaped. Illustrations of such patterns of the instructional discourse in the mathematics classroom have been described, for instance, by Mehan (1979) and Lemke (1990) among others, in terms of the teacher-student interaction cycle of initiation, response and evaluation (IRE) in which the teacher holds the initiative.

In the mathematics classroom these cultural modes of communication, educational and mathematical, intersect, and they are continually regenerated and altered in the classroom as each class is a unique group of people. Bishop (1985, p. 26) elaborates on the point that teaching and learning is about people:

> [E]ach classroom group is still a unique group of people–it has its own identity, its own atmosphere, its own significant events, its own pleasures and its own crises. As a result, it has its own history created by, shared between, and remembered by the people in the group.

What Bishop points to can be linked to what Cobb and Bauersfeld (1995) has termed the micro-culture of a classroom. They use traditional and reform-oriented mathematics to argue for this concept. In the traditional mathematics classroom, learning can be described as "a process of initiation into a pregiven discursive practice and occurs when students act in accord with the normative rules that constitute that practice" (op. cit., p. 6). However, contrasting traditional school mathematics with inquiry mathematics, in which students are expected to actively contribute and substantially form the classroom discourse, they contend that traditional and inquiry mathematics form two different microcultures. Looking at these microcultures from a sociocultural perspective, in which communication is constitutive of learning (Sfard, 2008), it is not only the social processes in these classrooms that are different, but also possible learning outcomes. That is, what it means to do mathematics, and the meaning given to the objects of the activity (i.e. variable, function etc.).

In this study we follow (Moschkovich, 2007, p. 25) and view mathematical practices as "simultaneously emergent in ongoing activity, and those socially, culturally, and historically produced practices that have become normative." In order to interpret collective meaning making in the classroom, we use elements of the framework of Cobb et al. (2001). However, we recast these elements in sociocultural terms and extend them by employing a Bakhtinian view of interaction. Cobb et al. focus on the development of social and socio-mathematical norms in the classroom and conceptualize collective processes in the classroom as an evolution of emergent classroom mathematical practices that are specific to particular mathematical ideas. These include a shared focus on activity, normative ways of arguing and normative ways of using cultural artifacts and other semiotic means. Simultaneously, mathematics is conceptualized as a historically and culturally established discursive practice (Sfard, 2008). That is, a discourse, distinguishable from other scientific disciplines by its objects, types of mediators used, and the rules followed by the participants. Elements of this discourse both form the background for classroom activities and shape social and individual processes in the classroom (Radford, 2002; Sfard, 2008).

1.2.3 A participationist perspective on learning

Sfard (2008) delineates a perspective on learning that view thinking and communication as two sides of the same coin. The term used for this theory, commognition, signals the close relationship that is assumed to exist between cognition and communication. It draws on discursive psychology, with roots in the late work of Wittgenstein (1952, in Harré & Gillett, 1994). Harré and Gillett (1994) explain that Wittgenstein's approach to intentionality or subjective meaning consisted of studying the use of words or other sign systems in "complex activities involving both the use of language and the use of physical tools and actions, where they were ordinarily encountered" (op. cit., p. 19). In accordance with Vygotsky's genetic law of cultural development (Vygotsky, 1978, p. 57), Harré and Gillett (1994) argue that meanings are first constructed on the social plane through inter-personal interactions and then on the intra-personal plane, which is thinking. Sfard (2008) discusses the relationship between the social and the individual in terms of two processes: (1) communalization of the individual, which refers to how the individual's communicative acts shape the communication; and (2) individualization of the collective, which refers to thinking as an individualized version of inter-personal communication.

Sfard (2008) conceptualizes mathematics as a discursive practice and classifies what she sees as family resemblances between different mathematical practices (scientific, academic and school). This includes attending to use of words and visual mediators, which are the tools for communication, as well as the outcomes of communicative activity, which are narratives and routines. These are first endowed with meaning in the social context, which the individual is part of creating, and then individualized as a personal version of the meanings co-constructed.

According to Sfard (2008), learning is seen as a growth or change in individual discourse, in terms of more closely resembling the culturally established mathematical discourse, which occurs through participation in the ‘generalized’ mathematical practice. The stories of learning told within this perspective reflects the approach to mathematics classrooms as all sharing these family resemblances. This reduction of the complexity of classrooms is particularly useful for investigating ontogenetic processes of developing mathematical ideas, which surpass one classroom community. Findings within this approach can certainly inform classroom practices. However, for the very same reason it is useful for studying ontogenesis, it is not well-fitted to analyze and understand whole classroom communication processes in terms of collective meaning making and the development of shared communication. Saxe et al. (2015) treats the classroom as a social unity and investigates the collective meaning making as a sociogenetic process in which form-function relations are reproduced and altered in communicational activity. Sociogenesis and ontogenesis intertwine in the classroom and take on directions in relation to each other’s products (Saxe, op. cit.). However, they are governed by different regulative processes. These genetic processes will be further discussed in Chapter 3.

The second approach to classroom interaction, identified as centering on meaning in terms of Vygotsky’s subjective notion of sense, is the theory of knowledge objectification (Radford, 2000; 2002; 2003). In this theory, meanings are also seen to be co-constructed in communication, but, as is the case in the commognitive perspective, the perspective mainly focuses on individual processes of learning. Learning is described as a process of becoming aware of what one did not notice before. In the classroom, learning occurs through individuals’ use of semiotic means in joint activity. That is, through “objects, artifacts, linguistic devices and signs that are intentionally used by individuals in social processes of meaning production, in order to achieve a stable form of awareness, to make apparent their intentions and to carry out their actions” (Radford, 2002, p. 14). The individual’s intentional use of semiotic means and artifacts in activity addresses subjective meaning much in the same manner as Harré and Gillett (1994) claim that Wittgenstein argued. However, Radford emphasizes that meaning is produced in the social process, in which the goal or the strived for outcome is a stable form of awareness. This is in accordance with Gee (2015), who argues that meanings are ultimately rooted in communicational processes of negotiation within communities that reach for a common ground or shared activities.

Sfard's (2008) communicational perspective and Radford's (2000; 2002) theory of knowledge objectification are seen to offer approaches to classroom interactions that are complementary in the quest to understand subjective meaning making. Sfard (2008) centers her analysis on (although she does not limit it to) linguistic communication in which recursion, as a property of language, is the explaining factor for how discourses develop. Recursion is described as a "feature of language thanks to which every legitimate linguistic construct may give rise to a new, more complex one, provided we replace some of its simpler elements with more complex linguistic constructs" (op. cit., p. 301). This has wide-ranging implications, both in terms of offering a view on the cultural mathematical discourse as a hierarchical discourse, and in terms of understanding ontogenetic development of mathematical ideas. It implies that mathematical topics in school should be organized accordingly, e.g. arithmetic should be taught prior to algebra as it includes analyses and objectification of arithmetic processes (Caspi & Sfard, 2012). Processes of reification in discursive activity are observable phenomena through which learning can be addressed.

In contrast, Radford focuses on learning as resulting from semiotic activity (Radford, 2010). His theory of knowledge objectification offers a magnifying glass on microgenetic processes of meaning making in the classroom. It approaches classroom communication through a slow-motion picture. Central to the focus on meaning making in this study, is that intentions made apparent, through for example gestures, are not translated but rather transformed, as actions are carried out through other semiotic systems (e.g. natural language or algebraic syntax). The linking between gestures, natural language, algebraic signs etc., are important elements of students' meaning making processes in algebraic activity and central to the investigation of learning in algebra (Radford, Demers, Guzmán & Cerulli, 2003).

In sum, the two theories discussed above are viewed as interactionist approaches to the classroom that focus on subjective meaning in communicational activity. The theories complement each other in the analyses of classroom interactions. Radford's theory helps one navigate through and analyze the 'sprouting of signs', while Sfard's close attention to word use and her operationalization of the notion of reification allow for an ontogenetic perspective on the communicational processes. In long term development of mathematical ideas, it is mainly words and culturally established mathematical sign systems (e.g. algebraic signs, coordinate system, graphs) that encapsulate and involve previous experiences in present activity.

In this research I draw upon perspectives identified as belonging to the two interactionist research agendas (mathematics as a cultural and historical practice and mathematics as a social practice emerging in the classroom). This enables me to investigate both subjective meaning making and meaning making as co-constructed, in terms of common ground (Gee, 2015), rooted in the same communicational activity.

1.3 Studies addressing collective classroom processes

The understanding of collective and individual classroom processes and how these are connected are of interest to any research agenda concerned with student meaning making (Hershkowitz, Hadas, Dreyfus & Schwarz, 2007). Investigations of these issues can contribute to our understanding of contingencies between teaching and learning (Saxe et al., 2015; Stephan & Rasmussen, 2002). Saxe et al. (2015) argue that the development of a micro-culture in the classroom across lessons is under-studied and under-theorized, particularly from a cultural perspective. From a classroom practice point of view, teachers plan activities and sequences of lessons implementing ideas about reasonable learning trajectories, considering students both as individuals and as forming a collective. Studies that have followed this research agenda often also build on the principles of design research (Cobb et al., 2001; Stephan & Rasmussen, 2002; Saxe et al., 2015).

Cobb et al. (2001) engaged in a teaching experiment (19 lessons) concerning linear measurement in a 1st grade classroom. The focus of the article analyzing parts of this material is on the social processes, and the findings are presented in terms of the emergence of five classroom mathematical practices (CMPs), i.e. normative ways of acting and reasoning concerning one mathematical idea. The goal was that students would engage with measurement as the accumulation of distance. This informed the delineation of the CMPs and these formed around ways of structuring space. The data was approached through initial coding and constant comparison and then conjectures about communal practices and students' reasoning were formed. These were then confirmed or rebutted by analyzing critical episodes.

Student argumentation was analyzed to determine if a CMP had been established in the classroom, i.e. mathematical ideas that previously were rebutted or questioned (therefore requiring justification) are now accepted as true. Contributions by individuals to the collective activity were interpreted in two different realms: (1) what it might say about the individual's mathematical knowing (individual processes); and (2) what it might say about normative ways of acting and reasoning in the classroom (collective processes). Individual activity is framed as interpreting and responding to others as an independent actor (reorganizing her reasoning), and, simultaneously, the individual act contributes towards the alteration or regeneration of the classroom microculture, which again informs about the individual act (as a situated act).

Cobb et al. (2001) found that new CMPs emerged as a restructuring of previous ones in a linear manner. In the first CMP, space was structured by the activity of measuring, while in the second CMP, space was structured as a property of the object. In the third CMP, the structuring of space involved the accumulation of distance. Essential to the emergence of a CMP were public discussions of different methods of measuring and differences in subjective meaning making by means of which ways of acting had to be justified. Findings concerning the interactions through which the CMPs emerged include: (1) differences in subjective meaning making came to the fore when participants encountered novel problems, and these discussions mainly remained calculational unless the teacher intervened; (2) the importance of teacher preparedness to facilitate conceptual discourse as situations for student learning; (3) the importance of teacher questioning for initiating student reflection; (4) the importance of symbolic records in student learning and the teachers' role in constructing these; and (5) the use of tools, including teacher guidance in this respect, played important roles in the emergent practices and the affordances of a particular tool depended on participation in previous CMPs. Additionally, Cobb et al. (2001) report that the analysis facilitated the evaluation of modifications to the instructions or specific teacher or researcher interventions in terms of viability, i.e. facilitating further progression in the direction stipulated in terms of the larger timeframe (19 lessons) and concern for a collective development.

Stephen and Rasmussen (2002) used the framework developed by Cobb et al. (2001) when analyzing the implementation of a design research in an introductory course in differential equations for engineers (22 class sessions). They found that six CMPs were established during this time and these were shown to emerge in a ‘network like manner’ rather than linearly as shown in Cobb et al. (op. cit.). That is, argumentation regarding several mathematical ideas evolved simultaneously contributing to the emergence of several interrelated CMPs. Stephen and Rasmussen (op. cit.) point out the importance of encouraging student justification of claims made. They found that students used the problem context to make meaning of the rationale of a claim. Thus, rather than being told the models which apply to different situations, the students reasoned and built argumentation (meaningful to them) for why a model is appropriate or not. Over time, these justifications were generalized and applied in problem solving without requiring justification.

Saxe et al. (2015) implemented a teaching experiment (19 lessons) involving integers and fractions in a 6th grade classroom. The instructional design was informed by extensive interview studies and tutorial studies with 5th grade students regarding integers and fractions. Saxe et al. approached the classroom meaning making processes focusing on the production of common ground, i.e. taken-as-shared public discourse. A common ground is regenerated and altered as participants use representational forms to solve shifting collective problems, including communicative issues.

Similar to Cobb et al. (2001), the micro-culture of a classroom is seen to include participation norms, socio-mathematical norms and normative ways of using representational forms to serve specific functions (for example use of operations in calculational processes to procure a numerical solution to a problem). In Cobb et al., social and socio-mathematical norms were considered to be relatively stable across the lessons investigated, which allowed them to focus solely on the emergence of CMPs (the equivalent in the framework of Saxe et al., 2015 are referred to as shifts in form-function relations). However, in Saxe et al. (op. cit.) changes in all three dimensions of the micro-culture were accounted for.

The analysis of the evolution of common ground is approached from a collective and an individual level of analysis. Saxe et al. (2015, p. 259) explain: “We take these levels to be mutually constitutive, with the collective level providing form and social meaning for individual activity, and individual actions creating the collective.” Thus, Saxe et al. stipulate a closer relation between subjective and collective meaning making than Cobb et al. (2001). At the collective level the evolution of participation norms and socio-mathematical norms are addressed, while at the individual level shifts in form-function relations are analyzed.

The shifts in form-function relations are seen to emerge through three genetic processes: (1) microgenesis is a moment-to-moment process in which individuals alter and reproduce form-function relations; (2) ontogenesis is the individual's development of form-function relations over time; and (3) sociogenesis is defined in terms of distribution of form-function relations across many individuals and shifts in form-function relations over time. These processes intertwine in learning and are constitutive of the evolution of a common ground. It is argued that the different timeframes of analysis can mutually support each other as follows: An analysis of microgenesis shows how individuals contribute to the generation of a common ground. An ontogenetic analysis, describing previous form-function relations, can illuminate microgenetic processes as it can point out the formation of new form-function relations, which by extension also inform the generation of a common ground. Sociogenesis is the result of multiple microgenetic processes across many individuals over time in which some form-function relations are favored over others. Thus, the analysis informs the generation of common ground in terms of distribution.

The analyses focused on participants' use of definitions as a set of representational forms when working with integers and fractions over the 19-lesson sequence. The analyses by Saxe et al. (2015) show that two socio-mathematical norms emerged on a collective level: (1) definitions should be used when explaining reasoning or justifying solutions; (2) definitions should be connected to problem context or other definitions. The analyses at the individual level show that the definitions of order, interval and unit interval, which were introduced early in the lesson sequence, were used repeatedly in the following lessons. This reoccurrence is explained in terms of students using the definitions to serve new functions. That is, faced with a novel problem a student reproduces aspects of a familiar definition (unit interval) to solve the problem (placing a number on an unmarked number line), and thus produces alterations and variants of previous uses (use the unit interval as a mental translation of the unit interval distance created by another student). This is seen to be an intrinsic property of microgenesis. The ontogenetic analysis captured long-term shifts in uses of the unit interval such as extending it to identify an improper fraction and further to construct the numerator as well as the denominator. The socio-genetic analysis shows how several students used the unit interval to translate distance in Integers Lesson 4, while some students used it to determine numerator and denominator of improper fractions in Fractions Lesson 3. In Fraction Lesson 6 students portioned the unit interval to identify equivalent fractions. In this manner, they pointed out shifts in distribution of form-function relation over time. The two socio-mathematical norms regarding use of definitions emerging, and the social position of students, are considered to contribute to the occurrence of these shifts.

The two frameworks (by Cobb et al., 2001 and Saxe et al., 2015, respectively) have similar perspectives on the microculture of a classroom. However, they take different approaches to the exploration of collective meaning making. As a product of collective activity, a CMP is a more narrowly defined construct than common ground. The CMP is centered around one mathematical idea (involving purpose, arguments and use of symbols and tools), while the common ground does not have any clearly defined boundaries and includes many aspects of classroom interactions (several mathematical ideas and social and socio-mathematical norms). The confirmation of assertions made regarding these (CMP or common ground) in terms of constituting the social situation is also based on very different premises. The conformation of a CMP is the lack of objections and needs for justification of claims made, while the conformation of a common ground is based on observed distribution among participants of identified communicational acts. Although approaching collective development using the construct of a CMP clearly has its advantages, Stephen and Rasmussen (2002) show that in more advanced mathematics classrooms CMPs overlap. That is, several mathematical ideas develop in connection to each other, which is also shown in Saxe et al. (2015). My research concerns the activity of generalizing numerical and geometrical patterns. The goal of the activity is that students should engage in algebraic thinking. Generalizing has previously been identified as one of several mathematical practices (Moschovich, 2013; NCTM 2000; Selling, 2016). Further, what constitutes an algebraic generalization in middle school patterning activity has been discussed and investigated in numerous studies (cf. Chapter 2.4). Thus, for my research the construct of a CMP is seen as an appropriate way of framing collective development.

The relationship between individual and collective processes is theorized differently in the two approaches, which in turn specifies how the collective progress is conceived of. For Cobb et al. (2001) it is a set of taken-as-shared ways of acting and reasoning that facilitate communication between independent learners and provide opportunities for restructuring their thinking. Rather than a co-created space, it consists of independent (individual) structures that appear to be similar (enough to avoid communicational challenges). In contrast, the common ground in Saxe et al. (2015) is co-created as participants tailor their representational use to serve communicational and problem-solving functions. The common ground, in which form-function relations take on social meaning in relation to social norms, conventions, artifacts and institutions, constitutes the very substance for subjective meaning making. In my research I approach the collective progress as a co-created space, in which subjective meaning making is rooted, i.e. these processes are seen to be inseparable (Sfard, 2008).

Finally, I find the genetic approach (Saxe et al., 2015) to the analysis of classroom interactions involving three different time dimensions, micro-, socio- and ontogenesis, to be comprehensive and offering a key to unravel contingencies between teaching and learning. The studies above show the complexity of undertaking an investigation of collective progress. Saxe et al. used other data sources to inquire about micro- and ontogenetic processes regarding integers and fractions to support the main study. Thus, I see the three genetic processes as offering analytic possibilities for managing the complexity of classroom learning in a way that is non-reductive.

1.4 Aims and research questions

This study aims to give empirical and theoretical contributions regarding the learning of algebra. The rationale for focusing on student meaning making in the introductory algebra classroom is to shed light on central issues of elementary algebra learning (cf. Chapter 2) from a student perspective. The study is limited in that it only considers classrooms that introduce algebra after the students have worked with arithmetic for several years, and in which the students are of similar ages (11 to 13 years old). As a further narrowing of the topic, the study investigates a generalization approach to algebra in terms of patterning activities. The focus on student meaning making contributes to the general debates within the field of school algebra by pointing out possibilities and difficulties regarding student learning of algebra in the conditions specified. Particularly, the study aims to explore the genetic relationship between arithmetic and algebra in school, and the role of cultural artifacts and other semiotic means in algebra learning (cf. Chapter 2).

The study is particularly concerned with how to conceptualize and investigate classroom learning from a sociocultural perspective, and the coordination of related theories (cf. Chapter 3), to further enlighten the development of algebraic thinking in the particular context described above. The theoretical developments and findings of this study are interpreted as shaped both by the perspective (sociocultural) taken and the specific empirical field (elementary algebra in the mathematics classroom) on which it is applied.

Keeping the empirical work focused on student meaning making, as a central theme within the sociocultural perspective, rather than applying one particular theory (e.g. Commognition, Sfard, 2008), allows for a different type of theoretical engagement. The elements of meaning making, interpreted as relevant, may not be appropriately addressed by one single theory. Thus, this study draws on several related theoretical perspectives developed within mathematics education. In order to develop a coherent approach to the analyses of classroom interactions, a more general methodology of development is incorporated, i.e. the genetic approach to cultural development as formulated by Saxe (2012). One of the aims of this study is to articulate a recontextualization (Lerman, 2010) of Saxe's approach to cognition into the field of mathematics education and to specify how it can contribute in the quest for building a comprehensive understanding of algebra learning.

From the considerations and aims described four more specific research issues have been articulated:

1. What is the nature of students' argumentations regarding numerical and geometrical patterns in elementary algebra classrooms?
2. How do students use artifacts and other semiotic means to engage with indeterminacy[2] and generalization in joint patterning activity?
3. What characterizes classroom meaning making processes in small group and whole class contexts as participants engage with algebraic ideas, tasks, artifacts and other semiotic means in joint activity?
4. How can classroom interactions be investigated to give a comprehensive understanding of learning in elementary algebra?

[2] In this research indeterminacy (Radford, 2010, 2018) is used to address what other theoretical approaches speak of when they refer to variable as a mathematical concept. The use of the analytic terminology *indeterminacy* is explicated in section 2.4.

The research issues listed here have guided the project as it progressed and have reciprocally been refined along the way. In their current form these are seen to fathom the research as a whole. The three studies all contribute to enlighten these research issues, particularly the two empirical studies contribute to all four research issues listed (Study 2 and Study 3). However, the three studies investigate research questions that are particular to each individual study.

1.5 Overview of the extended abstract

The thesis has seven chapters, an introductory chapter (Chapter 1), a literature review on algebra (Chapter 2), a genetic approach to algebra learning (Chapter 3), research methods (Chapter 4), summary of studies I-III (Chapter 5), conclusions and discussions (Chapter 6), and finally implications for instruction and further research (Chapter 7).

In Chapter 2, this research will be situated within the algebra literature in mathematics education in terms of research trends and school reforms; perspectives on school algebra and algebraic thinking; approaches to school algebra and particularly a generalization approach; and patterning activity in middle school. Thus, situating the research historically and culturally.

Chapter 3 presents the approach taken to study teaching and learning in the elementary algebra classrooms. A networking approach to the development of theory is explained and explicated. In Chapter 4 research methods are discussed. Central here is an ethnographical approach to video-analyses and the case-study design implemented. Furthermore, how classroom interactions have been analyzed is discussed. In Chapter 5 the three studies included in the thesis are presented. Chapter 6 presents and discusses the main findings of the thesis. These are presented in four parts according to how they shed light on the specific aims and research questions presented in section 1.4 and are also discussed accordingly. Finally, Chapter 7 suggests some implications for instruction and further research.

2 Algebra in school

Algebra in school is often a story of student alienation. More recently it is also one of a transformation in the way school algebra is approached that has put student meaning making and empowerment in the forefront of educational efforts (Kieran, 2007). Traditionally, arithmetic has been seen as a foundation for the learning of algebra and an appropriate topic for the early grades, while algebra requires student maturity. Research in the 1980s and 1990s shows dismal findings regarding student learning in high school algebra courses and documents what appears as a gap between arithmetical and algebraic ways of working with school mathematics (Filloy & Rojano, 1984; Herscovics & Linchevski, 1994; Kieran, 1992). These results spurred an effort to rethink school algebra that led to a movement away from emphasizing symbolic manipulations in the study of equations, to include topics such as generalized arithmetic, functions, problem solving and modeling that are seen to provide rich contexts for student meaning making regarding algebraic ideas and symbolism (Bednarz, Kieran & Lee, 1996; Kaput, 1998). Additionally, studies focusing on ways in which young students (K-grade 8) are able to participate in algebraic activity, within a re-conceptualized school algebra, show positive results which draw the traditional view of school mathematics into question (Stephens, Ellis, Blanton & Brizuela, 2017). Central to this evolving discussion is the relationship between arithmetic and algebra in school and how, in an instructional setting, one can close the gap between these two different ways of mathematizing.

Generalization activities have been emphasized in school algebra and are seen as roots of algebraic thinking (Kaput, 1998; Mason, 1996). Patterning activity involving numerical and geometrical sequences is often used to engage students in algebraic thinking, particularly in elementary and early algebra classrooms (Blanton & Kaput, 2005; Radford, 2013). In this research early algebra include efforts to promote algebraic thinking in the early grades (grades 1-5). Elementary algebra concerns a more formalized instructional approach to engage with algebra in middle school (grades 6-8), often also involving the use of algebraic syntax. In this research, I focus on the teaching and learning of elementary algebra within the context of a generalization approach. I look at this within both a problem-solving, small-group context and in teacher led whole class activity that includes elements of a functional discourse such as function table, the metaphor of function machine and algebraic expressions.

However, applying a cultural-semiotic perspective (Radford, 2002; Roth & Radford, 2010; Radford & Roth, 2011) in the analyses, I aim to contribute to central issues in school algebra that span the following content areas: (1) the relationship between arithmetic and algebra (Carraher & Schliemann, 2007); (2) the role of natural language, algebraic symbols and other semiotic means of objectification in developing algebraic thinking (Radford, Bardini & Sabena, 2007; Caspi & Sfard, 2012); and (3), more broadly, the implementation of algebraic ideas and syntax in regular middle school classrooms (Tunks & Weller, 2009).

Section 2.1 shortly discuss the field of algebra in terms of early explorations of students' difficulties with algebra and reform-efforts in school. Section 2.2 explicates the view taken in this study regarding the relationship between arithmetic and algebra in school. In the following section (2.3) perspectives on algebraic thinking are discussed. Section 2.4 discuss the role of structure in generalization activity. In section 2.5 patterning activity in elementary algebra classrooms is discussed.

2.1 Early studies and reforms in school algebra

Students' difficulties with algebraic ideas and particularly the notion of variable have been the focus of many studies in algebra learning. Küchemann (1978) found that most high school students only use variable in limited ways, such as: (1) finding its numerical value by trial and error; (2) simply replacing it for a given value; or (3) simplifying expressions by collecting like terms as if they were objects (apples and bananas). Some students were able to use a variable as a specific unknown and operate on it as if it was a number or use it as generalized number (able to take on a set of values). Only a few students were able to use variables to model general relationships or structures, activities which include an awareness of dependency.

Booth (1984; 1988), in a follow up study of Küchemann's (1978) investigations, identified reoccurring student errors regarding elementary algebra at different grade levels (8-10), and through interviews investigated why they occurred. Booth (1988) highlights the close connection between arithmetic and algebra and points out that the use of informal strategies in arithmetic, which are successful within this domain, renders algebraic problems that are solvable by recourse to a generalized arithmetical procedure or relationship, unapproachable for students. For example, counting on method, the execution of order of operations by recourse to the problem context rather than a conventional use of mathematical notation.

MacGregor and Stacey (1997), employing the categories for students' uses of letters in algebra developed by Küchemann (1978), found that Grade 7 students were able to use letters as a specific unknown and present an algebraic expression as a solution to a task. However, these results were obtained immediately after an eight weeks teaching sequence introducing algebraic notation through patterning activity, function machine and symbolic translation tasks. In contrast, the students in Küchemann's (op. cit.) study had worked within a more traditional algebra curriculum emphasizing equation solving and symbolic transformational activities, and the test was not designed in relation to a specific teaching sequence. Further, the analyses of the remaining (2000 total) students' (ages 11-15) responses to the test show that the students used letters in similar ways as documented by Küchemann (1978). Using interviews, MacGregor and Stacey (1997) show that students' unconventional uses of letters in algebra are in part due to: (1) intuitive and pragmatic reasoning; (2) students' experiences with letters in other contexts, such as abbreviations, roman numerals etc.; and (3) misguiding teaching materials. Additionally, it is shown that older students' experiences with complex reasoning in other mathematical topics interfere with their interpretation of the tasks in the given test.

Other studies suggest that the limitations of meanings that students attribute to mathematical signs such as: (1) the equal sign and operational signs signaling types of actions rather than structure; (2) letters as abbreviations, etc.; as well as (3) what constitutes a mathematical solution, i.e. the search for a numerical answer (lack of closure), are linked to their arithmetical experience (Booth, 1984; Kieran, 1981). These student interpretations become hindrances for participation in algebraic activity which necessitates structural interpretations.

Traditionally arithmetic has been taught before algebra in school and thus the bulk of empirical research has been done in this context. A notable exception is the Russian curriculum developed by Davydov and his colleagues, which first teaches algebra based on magnitudes and later introduces calculations with numbers as a case of algebra (Schmittau & Morris, 2004). In the USA, efforts are made towards implementing what is called an algebraification of the K-grade 12 curriculum (Blanton et al, 2015; Stephens et al., 2017). Foundational to this movement is the work by Kaput. Kaput (1998) saw the algebra problem in school, as documented in previous algebra studies, to be closely connected to a fragmented mathematics curriculum and the institutionalized algebra it presented, i.e. arithmetic in younger grades and algebra as separated courses in higher grades that were heavily focused on symbolic manipulations. Kaput, Carraher and Blanton (2008; Lins & Kaput, 2004) proposes that algebra should be taught from first grade as an element which integrates the arithmetical topics already in place into a coherent whole. That is, the approach to school algebra relies heavily on a perspective on algebra as generalized arithmetic, which means that generalizations are based on the realm of numbers rather than magnitudes.

However, the mainstream initial responses to the documentation of students' problems in algebra were directed at middle school in form of developing a transitional stage, pre-algebra, aiming to better prepare students for algebra courses in later grades. Broadly, these studies targeted the teaching and learning of algebra and focused on sources of meaning for algebraic objects, particularly in terms of different approaches to algebra, i.e. functional, problem solving, modeling, language, etc. (Kieran, 2007; Sutherland, Rojano, Bell & Lins, 2001). This effort broadened the view of algebra in school in a way that made it easier to envision algebra as accessible even for students in the elementary grades. This direction now comprises a series of studies that is commonly referred to as early algebra research. Its main concern is how to cultivate algebraic thinking in the elementary grades and forms its own complex field of research, which to some extent has engulfed the pre-algebra approach into a more unified vison of K-grade 8 mathematics (Kieran, 2018; Stephens et al., 2017).

The reshaping of school algebra does not only concern content and timing as calls have been made to investigate classroom cultures that support algebraic thinking (Blanton & Kaput, 2005). In this, the algebra reform movement coincides and intersects with an ongoing educational effort to revision and democratize the mathematics classroom. In broad terms this has meant a movement away from school mathematics as mainly facts and procedures to be memorized in a teacher dominated classroom, to a focus on developing student argumentation by facilitating and encouraging student contributions to the classroom discourse (Cobb, Stephan, McClain & Gravemeier, 2001; Eckert & Nilsson, 2017; Lampert, 1990). The findings in my study are seen as intimately related to both the current situation in algebra classrooms that are formed by these educational trends and the evolution of the field of school algebra.

2.2 Arithmetic and algebra in school mathematics

Forms of numerical thought are found in most human societies, including both historically and geographically distant ones (Saxe, 2012). Saxe identifies order, magnitude and arithmetic relations as fundamental numerical ideas. These ideas can be seen to play central roles in human societies, of which most of the members of the society acquire at least some rudimentary understanding as they participate in daily activities (not necessarily in schooling). In contrast, algebraic ideas have a shorter historical existence with limited geographical and sociocultural spread. These ideas evolved in more complex technological societies, among a few scholars, and are exemplary of what Vygotsky (1986) refers to as scientific concepts. These do not directly relate to experience, as spontaneous concepts do, but are more abstract and general and are mainly encountered in instructional settings or technological advanced activities.

Arithmetic ideas, as systemic organizations taught in school, are also scientific concepts, according to Vygotsky (1986). However, the fundamental numerical ideas, as identified by Saxe (2012), are likely to be encountered by children in social settings outside of and prior to schooling and are of a more spontaneous kind. Therefore, inquiries into the origin and development of arithmetical and algebraic ideas rest on different premises. The development of algebraic ideas is mainly an instructional enterprise, and the roots of these ideas are not to be sought in experiences outside of school but rather within the mathematics classroom (cf. Kilhamn & Säljö, 2019). Thus, in contrast to Saxe's (2012) fundamental arithmetic ideas such as order, magnitude and arithmetic relations, which originate in common experiences in human societies, fundamental algebraic ideas originate in experiences in school arithmetic in terms of generalized ideas such as numbers, operations and their structures.

Arithmetic and algebra offer alternative perspectives for approaching tasks in the mathematics classroom. In the introductory algebra classroom, as students are more familiar with arithmetic, I expect participants to draw on both perspectives in communication as they seek to establish a common ground for talking and acting. Here it is discussed what these perspectives are seen to entail.

Arithmetic is about the quantification of magnitudes, their transformations and relationships (Vergnaud, 1979). The school topic includes number sense (natural numbers, integers, rational numbers, etc.), operations on numbers and ways of signifying these. Students also explore the behavior of numbers and their operations through ideas such as even, odd, composite and prime numbers, and engage with several mathematical structures such as an additive structure and a multiplicative structure among others (Kieran, 2018).

Bednarz and Janvier (1996) describe characteristics of arithmetic and algebraic problems. In arithmetic problems, known data are directly connected and the challenge is to identify the nature of the relationships between them. These problems facilitate the use of arithmetic procedures as one can proceed from known data to the unknown. Thus, relationships between quantities are often not made explicit (as in a variable expression or equation) as only a sequential calculational process is presented in the solution (Vergnaud, 1982). In algebraic problems there is no direct bridging between known data, and a solution often includes manipulation of relationships identified (Bednarz & Janvier, 1996).

Thus, solutions to algebraic problems require what Radford (2018) describes as algebraic thinking, that is an analytic way of reasoning, the naming of unknowns, modeling of relationships and an ability and willingness to operate on these without being bound by real life notions. Central ideas in school algebra are unknowns, variables, equations and variable expressions, also including functions, rate of change, parameters and coefficients, providing general forms for expressing and analyzing relationships between quantities. The algebraic symbolism is central to many developments within mathematics and is superior, regarding precision and efficiency, to numbers and natural language in describing and dealing with generality and structure. The role of symbolization in the learning of algebra is one of the main discussion points in early algebra research. For some, symbolization is the hallmark of algebra, while for others algebraic thinking does not necessitate algebraic symbolism, nor does the use of this symbolism imply that students participate in an algebraic practice (Radford, 2010). Following Radford (2018), I see algebraic symbolism as a culturally evolved symbolic system that enables and facilitates certain ways of thinking. That is, symbols are not external representation of thought, but rather the availability and use of symbols in problem solving have bearing on the thinking that is elicited in the solution of a task.

Lee and Wheeler (1989) argue that an instructional goal for school mathematics is that students develop familiarity, skills and knowledge regarding both arithmetic and algebra. Including the relationship that "*one* numerical substitution may disprove an algebraic statement whereas *no finite number* of numerical substitutions can prove it" (op. cit., p. 52), and can use these ways of thinking mathematically interchangeably in problem solving. Similarly, Mason (1996) talks about two central mathematical activities as: (1) generalizing by seeing the general through the particular; and (2) specializing by seeing the particular through the general. Usiskin (1999, p. 9) argues that it is not possible to study arithmetic sufficiently "without implicitly or explicitly dealing with variables." Thus, central in elementary algebra is how the first connections between arithmetic and algebra can be forged, and how these domains can be taught in the classroom to enhance and cultivate student mathematical meaning making in a reciprocal manner.

Caspi and Sfard (2012) suggest that there are two types of tasks in which students can start to think algebraically from an arithmetical starting point. The first type of tasks are questions about an unknown involved in calculations in which the result is given. Several studies have argued that the algebraic way of solving equational problems is markedly different from students' arithmetical approaches (Bednarz & Janvier, 1996; Filloy & Rojano, 1984). The second type of tasks involves the generalization of different types of patterns. Kieran (2018) distinguishes between patterns inherent in numbers and their operations, and geometrical-numerical sequences upon which a generalization is imposed to ensure predictability. The latter is the focus of this study. Radford (2014) argues that generalization or pattern approaches to elementary algebra can be considered in terms of a continuation rather than as a rupture with arithmetic. However, he emphasizes that although there are connections between arithmetic and algebra in this type of mathematical activity, there are also differences that are important to identify to ensure the cultivation of elementary ways of thinking algebraically.

2.3 Algebraic thinking

In the introduction to this chapter three main interests of research were listed. Here I discuss different approaches to investigate these, in terms of how algebra is conceptualized and their affordances and limitations for addressing meaning making in the introductory algebra classroom. The discussion focuses on useful ways of framing an investigation of interactions in algebra that capture learning in rich environments. Furthermore, here I discuss how studies define algebraic thinking, the approach taken to research in math ed, as well as the specific findings presented. I focus the discussion on some studies which formulate main perspectives on elementary algebra that are founded on well recognized themes within algebra research, i.e. generalization, structure, the process-object duality of mathematical objects and the role of signs.

The discussion revolves around the relationship between arithmetic and algebra, which has been elaborated both in terms of rupture and continuum in ways of thinking mathematically (Hitt, Saboya & Cortés-Zavala, 2017). Educational analyses of this relationship often include a combination of two or more perspectives such as: (1) *content analysis*, a mathematician's viewpoint; (2) *didactical*, empirical analysis of students' first encounters and developments in algebra; (3) *critical historical analysis*, analysis of critical points in the phylogenies of algebra; and (4) a *learning theory*. In this section I will discuss how different studies have defined the relationship between arithmetic and algebra, including empirical findings regarding ontogenetic development of algebraic thinking. Finally, I articulate the approach to elementary algebra taken in this study.

There are differing views on how the relationship between arithmetic and algebra in school should be conceptualized. Three main perspectives can be identified: (1) the Russian curriculum, developed by Davydov and his colleagues as I mentioned above, teaches algebra based on magnitudes and later introduces calculations with numbers as a case of algebra (Schmittau & Morris, 2004); (2) a perspective that emphasizes that algebra is inherent in arithmetic and proposes that the algebraic character of arithmetic should be brought to the forefront in the classroom through the teaching of the arithmetic curriculum already in place in the elementary grades (Blanton et al, 2018; Britt & Irwin, 2011; Kaput et al., 2008); and (3) a perspective concerned with identifying characteristics of arithmetic and algebraic thinking and the different natures of these ways of mathematizing in the classroom (Caspi & Sfard, 2012; Filloy & Rojano, 1984; Herscovics & Linchevski, 1994; Radford, 2018). The Davydovian perspective will not be further discussed here as the context of my research includes students who are familiar with numbers and their operations before being introduced to algebraic ideas.

Kaput et al. (2008) defines school algebra in terms of forms of thinking that are rooted in generalization and increasingly expressed through formal algebraic syntax, as well as reasoning with symbolic forms. He also specifies content areas in which these can be cultivated, i.e. generalized arithmetic, functions, and modeling. Blanton et al. (2015; Blanton et al, 2018) adopt this perspective and formulate an explicit approach to its implementation in grades 3-

5, which is founded on didactical analyses of early algebra research, content analyses and learning progression theory. They identify four core algebraic thinking practices as generalizing, representing, justifying, and reasoning with mathematical relationships along with five content areas or big ideas in which these can be cultivated, for example a structural understanding of the equal sign. Algebraic thinking is thus defined very broadly and includes arithmetic in a rather seamless manner.

Britt and Irwin (2011), investigating school algebra in New Zeeland, propose a similar curricular approached to school algebra, in which the notion of *algebra in arithmetic* is alluded to as a steering principle. It is assumed that the focus on algebraic thinking in elementary school will enhance both the learning of arithmetic and the more formal algebra introduced in later grades.

Thus, these studies propose a perspective which delineates algebra as a continuum of arithmetic in which both forms of mathematizing can contribute to the development of a generalizing habit of mind. However, as the definition of algebraic thinking is so comprehensive and leaves no demarcation line between arithmetic and algebra, it becomes difficult to scrutinize the cognitive processes involved and the respective roles of arithmetic and algebra in the ontogenetic development of algebraic thinking.

Caspi and Sfard (2012) explored in which ways students' learning of arithmetic prepares them for the study of algebra in school. They propose a schematization of how algebraic thinking may develop, drawing on the commognitive perspective and the process-object duality (procedural-conceptual in other perspectives) in mathematics, e.g. the algebraic expression 3x+1 can be seen both as a sequence of operations and as a function (object). Mathematics is conceptualized as a specialized, cultural, discursive practice, and algebra in school, being a sub-discourse of mathematics, is defined as a meta-arithmetical discourse people employ while reflecting on numeric processes and operations. The early meta-arithmetical discourse (or algebraic thinking) can be elicited in problems involving either patterns or equations, not necessarily involving the algebraic syntax. But any question about an unknown quantity involved in calculations for which the result is given may elicit algebraic thinking. Thus, the definition of algebraic thinking is narrower than the one proposed by Kaput et al. (2008) and it is centered on specific and central algebraic ideas. Expected progress in algebraic thinking involves a movement from a processual discourse to an objectified discourse.

This is further elaborated, also involving a content analysis, through five levels of algebraic discourse. In the three, lower levels indeterminate objects are used and talked about in terms of specific unknown numbers, i.e. indeterminate numbers, while at the two upper levels indeterminate object are used to signify sets of numbers and model functional relationships, i.e. indeterminate quantities. Additionally, Caspi and Sfard (2012) draw on a critical-historical analysis that focuses on the movement from rhetoric to symbolic algebra. They propose that an informal and a formal algebraic discourse may mutually support each other's growth and ensure a meaningful learning of the algebraic symbolism.

Caspi and Sfard (2012) found that students in 5th grade and 7th grade had developed an informal meta-arithmetical discourse in their arithmetic classroom, prior to the introduction of algebra and without it being specifically targeted in teaching. Further, comparing the two student groups, they found that the seventh graders' meta-arithmetical discourse was more objectified than the one elicited from the fifth graders. Thus, Caspi and Sfard (op. cit.) show, through in-depth discourse analyses, how reflective arithmetical thinking can evolve in the arithmetic classroom and propose this as a starting point for the formal teaching and learning of algebra.

The perspective formulated by Caspi and Sfard (2012) points to the informal meta-arithmetical discourse as a continuum of arithmetic, but it also delineates ruptures between the two mathematical domains in terms of the presence and use of indeterminate objects, and the formulation and use of arithmetic processes and relationships as objects in themselves. The framework offers a skeleton structure for the analysis of ontogenetic development of algebraic thinking in the arithmetic-then-algebra context. However, the framework is sparse in terms of exploring how the connections between the informal and the formal algebraic discourse are formed, and in which ways they may support mutual growth.

Radford (2010, 2013, 2018) approaches the learning of algebra from a semiotic-cultural and sensuous cognition perspective. As Caspi and Sfard (2012), Radford defines early algebraic thinking in terms of the use of indeterminate objects mainly in pattern or equational activity. However, based on a critical historical analysis Radford & Puig (2007) emphasize that the indeterminate objects must be used in analytic activity in which embodied, verbal or symbolic formulas are deduced and not guessed (this is further explicated in relation to competing views below). Some central aspects of this approach will be briefly explained below.

In his research, Radford (2003, 2013) expands the traditional semiotic platform to not only concern formal semiotic systems but also to include the use of natural language and gestures. The signs used are not considered as mere symptoms of cognitive activity but rather as the material counterpart of thought. It is through the senses that the world can be experienced, not naturally, but as mediated by the cultural practices people participate in. Thinking and learning occur through body movements, the use of artifacts and various semiotic means, as people try to accomplish specific goals in socially mediated activity. In the algebra classroom students encounter new types of tasks and semiotic means, e.g. function table, the algebraic syntax, etc., in teacher led and goal-directed classroom activity. Radford et al. (2007) argue that such algebraic tasks and semiotic means are not neutral elements of the classroom. Rather they are bearers of historically developed patterned cognitive activity, and as such they form potential paths for learning and development.

A series of studies (Radford, 2010, 2018; Radford et al., 2007) shows that students in the algebra classroom use body movements (for example rhythm) and an array of different semiotic means, including gestures and natural language, as they engage in algebraic activity. To investigate the role and use of signs in the algebra classroom, Radford (2010) elaborates on what he terms modes of signification. There are meanings that can be intended using some signs, when other signs will not suffice. It is exactly this that makes “semiotic systems unique and unsubstitutable” (op. cit., p. 2). In other words, in processes of learning, i.e. discoveries of what has not been noticed before, a mathematical object can be perceived contextually through a gesture, drawing or in tracing a figure. In a further attempt to communicate what has been perceived, the object may be phrased in natural language. Recognized as a general mathematical object it can be implied to using mathematical terms and symbolism. Thus, the meaning given to the mathematical object in the different sign systems may not be the same. The purpose of a semiotic analysis is to disentangle the dynamics of the semiotic means and shed light on the linking between them. Thus, as the importance of natural language in the development of algebraic thinking was pointed out in the study of Caspi and Sfard (2012), the semiotic-cultural perspective offers a way of analyzing the linking between informal and formal algebraic discourse in elementary algebra research.

In terms of the algebra classroom, Radford (2010) argues that "the mathematical situation at hand and the embodied and other semiotic resources that are mobilized to tackle it in analytic ways characterize the form and generality of the algebraic thinking that is thus elicited" (p. 2). This is exemplified in a typology of algebraic thinking, drawing on data from a longitudinal study of 8^{th} and 9^{th} grade students' patterning activity, which also suggests traits of an ontogenetic development as the types specified differ in their level of generality (local or increasingly global in its applicability). The typology is further developed in Radford (2018), now labeled as types of generalizations, based on data from younger grades (mainly 4th, 5th and 6th grade, also a longitudinal study).

The first is called a factual generalization. Here, regularities or commonalities noticed are described in terms of concrete properties of the problem situation. The form thus only facilitates a local generalization. Indeterminacy do not reach the level of discourse but are present through some of their particular instances, i.e. the variable is tacitly present in the use of numbers or "facts" in a repeated calculational sequence, which can be seen as an in-action-formula (Radford, 2010). Radford (2010) shows how this type of thinking is multi-modal in nature. This type precedes the most basic level of algebraic discourse that Caspi and Sfard (2012) schematize for ontogenetic development of algebraic thinking. In their perspective, indeterminacy must reach the level of discourse. However, the second type of generalization that Radford (2018) delineates, corresponds better to their description of informal constant value algebraic discourse.

The second type is called contextual generalization. Here indeterminacy reaches the level of discourse and is not only present in action. Numbers are no longer the focus of attention as the relationship between indeterminate objects becomes the main focus. Indeterminate objects are addressed in terms of contextual elements of the task, i.e. use of generic naming such as: three times the "number"; or spatial descriptions such as: twice the "side"; or use of deictic terms. This form of generalization is what Radford (2010) calls perspectival, and it is connected to a particular way of seeing a specific situation, and therefore has to be substantially altered in order to describe a different, but mathematically similar, situation.

The third type, symbolic generalization, involves the use of algebraic symbolism and is a mode of signification that allows a much more direct observation of mathematical structures and generalities, i.e. global in its form. Radford (2018) observes a prevailing student need for including an equal sign and a variable (i.e. $n{\cdot}n=x$) in their symbolic generalizations. In that the emphasis in an algebraic expression is on structure and operations, Radford interprets this student dilemma as indicating that the movement to symbolic thinking requires a reconceptualization of numerical operations. Thus, he argues, the novelty does not lie in the introduction of the alphanumerical signs itself, as is often assumed in algebra teaching.

Caspi and Sfard (2012) offer some insights regarding the development of a structural view of operations. Some students in 7th grade were found to use terms such as *multiplied by* or *the product of* in compound verbal expressions, i.e. a result is described as an object in terms of the operation, rather than students having to perform the specific numeric calculation to find a number from which to proceed. This is interpreted as signaling that the students were well on their way of achieving a structural view of operations.

Returning to the findings of Radford (2018), he reports that although the students engaged in all the three types of generalizations exemplified in grades 4 and 5, it was only in grade 6 that the algebraic symbolism seemed to be leading some students' algebraic thinking as they did not first resort to using natural language when solving problems. However, Radford (op. cit.) argues that it is not only their symbolic functioning that had evolved but also their ways of perceiving and imagining mathematically.

The three perspectives on elementary algebra discussed all include an "eye for structure" to a certain degree. In Kaput et al. (2008) this is included as an aspect of algebraic thinking and is called reasoning with symbolic forms, i.e. seeing structure in algebraic expressions. Further, it is included in Blanton et al. (2018) in terms of developing the idea of equality. Structure sense is an integrated part of the process-object approach to the development of algebraic thinking proposed by Caspi and Sfard (2012), in which a structural view of numerical operations is seen to be a developmental milestone in early algebraic thinking. Similarly, Radford (2018) describes the need for a reconceptualization of numerical operations in order to develop symbolic thinking. However, structure sense in mathematics is not explored and developed to the same extent as generalization in these perspectives.

2.4 Structure and indeterminacy in generalization activity

Kieran (2018) takes structure as a primary focus in elementary algebra research and makes an important theoretical contribution by elaborating on notions of structure in school mathematics drawing on numerous previous studies. She contends that a structural view of arithmetic forms an important fundament for the emergence of algebraic thinking. She formulates a close link between structure and generalization (op. cit., p. 101): "there is a dual face to activity that promotes early algebraic thinking: one face looking towards generalizing, and, alternatively but complementarily, the other face looking in the opposite direction towards 'seeing through mathematical objects' and drawing out relevant structural decompositions". Kieran (op. cit.) envisions a broad definition of structure in school mathematics that goes beyond the basic properties of arithmetic and argues that the notion of structure varies with content areas. In terms of early algebra, she elaborates on notions of structure in two content areas seen as particularly relevant for the development of algebraic thinking in K-grade 8 classrooms: (1) structure in numbers and numerical operations; and (2) structure in patterning and functional activities.

In her meta-analysis, Kieran (2018) develops these notions of structure using the construct *means of structuring* previously elaborated by Freudenthal, who did not only see one encompassing mathematical structure but rather several different ones, of which Kieran (2018, p. 82) mentions "order structure, additive structure, multiplicative structure, structure according to divisors, structure according to multiples." Thus, Kieran argues that different means of structuring afford alternative but related structures, e.g. the structures according to divisors and multiples, which can be deduced from the basic structures (order, addition, and multiplication structures). Kieran describes structure in numbers and numerical operations in terms of (1) means of structuring such as "structuring according to factors, multiples, powers of 10, evens and odds, sums of 10, prime decomposition", and (2) the properties of these, i.e. the basic properties of arithmetic, the successor property, the sum of consecutive odd numbers property, the sum of even and odd numbers property, equivalence and equality properties. It is pointed out that while the structures described are inherent to numbers and numerical operations, structure is instead imposed on numerical and geometrical patterns to ensure predictability. Further, it is argued that although a functional approach to algebra, which includes patterning activity, involves additional structures such as spatial (geometrical figures) and functional (linear, quadratic, etc.), these are intimately linked to the structures of numbers and numerical operations. This is argued as follows.

Structure sense in patterning activity involves recognizing variant and invariant aspects. To mathematically generalize a figural pattern the invariant aspect, or regularity, must form a link between a spatial structure and a numerical one. If properties of the numerical structure are recognized, an expression can be deduced. Thus, Kieran (2018) contends, it is in the explication of expressions, including order of operations, that spatial and functional structures interact with the "multitude of properties and means of structuration that are related to number and numerical operations" (op. cit., p. 99) in elementary algebra activity. It is argued that the connection between structure sense in numbers and numerical operations and generalization activity is an understated and sparsely explored area of elementary algebra research. In order to develop structure-sense the classroom may engage in what Mason (1996) refers to as acts of classifying and labeling, which to him are essential parts of mathematical generalization processes.

Kieran's (2018) analysis of structure in mathematics explicates an important connection between arithmetic and algebra in early algebra. Thus, a central issue in the teaching and learning of school algebra is to explore how students can become aware of and confident in using this connection in their mathematical activity. Lee and Wheeler (1989) show that this is not easily achieved in school. They looked at grade 10 students' coordination of arithmetic and algebra in the context of generalization and justification activity. They report that only a few students move easily and productively between the two mathematical domains, while the majority show great uncertainty regarding the nature of the relationship and are hesitant to move between them. Further, they find that students tend to justify algebraic statements mainly by referencing a rule rather than known behavior of numbers. Interestingly, Lee and Wheeler (op. cit.) also point out that students need to become aware of the differences between the two domains. That is, the students must discover that although algebra aligns with arithmetic, it is not confined by the behavior of numbers. To do so, they need to become familiar with indeterminate objects.

Usiskin (1999) argues that the notion of variable, i.e. addressed in this research as indeterminacy (Radford, 2010, 2018), has many possible definitions, signifiers and symbols. He discussed these within four different approaches to school algebra, in which he argues that the role and meaning of indeterminate objects are qualitatively different. Usiskin argues that it is only within the function approach that indeterminate objects have the quality of varying. In generalized arithmetic indeterminate objects can be considered as a generalized number (cf. Küchemann, 1978), or as I see it, an indeterminate number. For example, the statement, $a + b = b + a$, is true for any value assigned to a and b, however, the generality only holds if one value is assigned at the time. In contrast, the formula (or function) $y = 3x + 1$, describe a relationship between two quantities, in which the nature of the relationship (i.e. proportional, linear, quadratic, etc.) is of main interest. Importantly, here the indeterminate quantities take on values in relation to each other, i.e. y is dependent on the value of x. Usiskin (1999) also sees the study of structures as an approach to algebra in which indeterminate objects have a different interpretation. This is relevant for the view of a numerical sequence or an algebraic expression as number structure (for example *4, 8, 12, 16*, …, perceived and described as multiples of *4*, and written as $4n$), or function structure ($3n + 1$ as a linear function, i.e. a special case of $an + b$). Here Usiskin argues that the indeterminate object is "little more than an arbitrary symbol" (op. cit., p. 11).

Kieran, Boileau and Garancon (1996) argue that a link to a functional relationship can be made in generalization activity. That is, they argue, in terms of the explicit expression students come up with as a function that determines the dependent variable from the independent one. Carraher, Martinez & Schliemann (2008) explain that a functional approach to algebra can help students to think in terms of sets of numbers, rather than particular ones. A generalizing approach and a functional approach differ mainly in terms of the mathematical ideas that are emphasized in the patterning activity. That is, a variable expression relating different quantities, can be interpreted in several ways, including formula, function and number structure (i.e., polynomials). In a functional approach, tasks often include physical quantities (i.e., age, height) and a contextual situation that describes the quality of these and their relations. The use of tables and graphs is more common as these tend to make sets of numbers and their relations more apparent to students. However, these also bring additional challenges regarding student use and interpretation. Although there is a somewhat different rationale for the two approaches to algebra, i.e. algebra as a way of deducing mathematical generalizations versus algebra in terms of independent and dependent variables in functional relations, these sometimes blend in classroom practices and educational research, as they evolve and revolve around intersecting content.

Finally, the algebraic symbolism is central to many developments within mathematics and is superior, regarding precession and efficiency, to numbers and natural language in describing and dealing with generality and structure. The role of symbolization in the learning of algebra is one of the main discussion points in early and elementary algebra research. For some, symbolization is the hallmark of algebra, while for others algebraic thinking does neither necessitate algebraic symbolism, nor does the use of this symbolism imply that students participate in an algebraic practice (Radford, 2010). Following Radford (2018), we see algebraic symbolism as a culturally evolved symbolic system that enables and facilitates certain ways of thinking. That is, symbols are not external representations of thought, but rather the availability and use of symbols in problem solving have a bearing on the thinking that is elicited in the solution of a task. In this research, I interpret the use of signs in the classroom using Radford's (2000, 2002, 2003) semiotic-cultural theory.

2.5 Patterning activity in elementary algebra

The following discussion of previous research defines what a cultural practice of algebraic generalization at the middle school level entails, i.e. analytic thinking involving deductive argumentation. The sophistication of these forms of thinking depends on the semiotic means available and elicited in task solving. It is contrasted to purely inductive arithmetic generalization, where the argumentation relies solely on examples. Previous findings and theorization regarding students' approaches to patterns, provide terminology to describe and analyze students' contributions. In my approach to generalization, as a way of introducing algebra that makes use of students' arithmetical knowledge, I follow Kieran (2018) who emphasizes the important role of structuring in pattern generalization activity both regarding number structure and function structure. Indeterminate objects are central in this activity. I draw on Usiskin (1999) to discuss indeterminacy and its varied roles and meanings in algebra, which students may encounter and engage with in the introductory algebra classroom.

For Mason (1996) generalizing in mathematics involve activities that minimize demands of attention such as "detecting sameness and difference, making distinctions, repeating and ordering, classifying and labeling" (op. cit., p. 83). For him these activities are at the roots of algebraic thinking. He emphasizes that although student exploration leading to an inductively formed conjecture is important for their meaning making, they need to mathematically verify the conjecture. That is, they need to determine if the generalization always work or for which cases it does apply. In terms of generalizing numerical and geometrical sequences, the students need to return to the source of the sequence, i.e. identifiable quantitative relationship, to justify their conjecture and not solely base it on numerical examples. This view is adopted by Radford (2008, p. 84), who explains the targeted deductive form for argumentation as consisting of "*grasping* a commonality noticed on some particulars (say p_1, p_2, p_3,…,p_k); extending or generalizing this commonality to all subsequent terms (p_{k+1}, p_{k+2}, p_{k+3}, …), and being able to use the commonality to provide a direct *expression* of any term of the sequence".

Lannin (2005) and Ellis (2007), investigating middle school students' patterning activity, argue similarly that generalization and justification processes are intimately related and are equally important in students' algebraic development. Radford (1996) sees the generalization process as essentially a proof process that moves from empirical knowledge to abstract assertions beyond what can be directly perceived. Further, he argues that not all student generalization processes in patterning activity are fruitful in terms of developing algebraic forms of thinking, and that deductive argumentations should be targeted in patterning activity in school (Radford, 2008; 2018). Ellis (2007) also emphasizes that deductive generalization is an important instructional goal but argues that other types of generalizations can serve as a bridge to this algebraic form of thinking. In her data she found that deductive forms of justification emerged only later in the instructional sessions (15 in total) and that deductive generalizations, rather than being immediate, tended to develop over time. The change in forms of justification was accompanied by students' increasingly goal-directed and creative generalization actions, as well as acts of comparison across different quantitative situations (relating to previous tasks) to establish more general assertions about linear relationships. Similarly, Lannin (2005) found that the most successful student in his study relied on an informal sense of rate of change and saw similarities between different problems based on this idea.

Lannin (2005) found that 6^{th} grade students generalized using varying strategies categorized either as non-explicit or explicit (a way of finding a specific value of the dependent variable directly). A recursive generalization, in which the next element in a sequence n is derived from the previous one $n - 1$, is a non-explicit generalization. Here I will mention some that are well documented in the literature and of which more specific strategies can be included (Lannin, 2005; Radford, 2008; Rivera & Becker, 2011): (1) *guess-and-check*, a rule is guessed working with known numerical pairs without regards to quantitative relationships in the problem context; (2) *generic object*, using a portion as a unit, or a specific figure or part of one, to determine general qualities of the quantities involved and their mathematical relationships; and (3) *quantitative relations*, using the context of the problem situation to distinguish general qualities of quantities and their relationships to determine a rule, without recourse to a generic object.

A generalization achieved through the first strategy, when verified through a few empirical instances, has been referred to as naïve induction (Radford, 2008). Lannin (2005) found that students, when asked to verify their rules, tended to use empirical examples, particularly if the generalization relied mainly on a numerical scheme for the problem situation. Students who generalized using a generic object, often overgeneralized using direct proportion (ignoring irregularities in the pattern perceived), a relationship which students tend to be familiar with and see as a convincing mathematical argument (Stacey, 1989). Lannin (2005) also found that students who developed a geometric scheme for a problem, more often remained connected to the problem context and could justify their rule by describing the general relations discerned.

The investigational work, in which students explore the quality of quantities and their relations, often through concrete numbers such as counts and measures and observe change through actions and operations, is central to student meaning making (Carraher, Martinez & Schliemann, 2008). However, it is the generalization activities described by Mason (1996) and Radford (2008) that involve the characteristics of algebraic thinking as described previously, i.e. a deductive (analytic) way of reasoning, involvement with indeterminacy (necessary to go beyond particular instances), the modeling of quantitative relationships that opens the possibility of recognizing, and working with general structures that transcend the problem situation.

In my view, and as argued by Radford (2018), the work of exploring how students form links between and come to distinguish between the two domains in their mathematical activity, and the role of signs in these processes, is central in school mathematics. This is a large area of research that goes well beyond elementary algebra. In this study I am concerned with initial connections made by teacher and students in the classroom. I find Kieran's (2018) explication of a connection between structure in numbers and numerical operations with spatial and functional structures in generalization activity a promising way to theorize about these initial connections between arithmetic and algebra. It is also particularly relevant for this study as the teacher and students, in the Californian 6th grade classroom analyzed in Study 3, had worked with multiples and factors including odd, even and prime numbers in the previous lessons leading up to the algebra lessons analyzed.

3 A genetic approach to algebra learning

The genetic approach to algebra learning, adopted in my thesis, is rooted in a sociocultural perspective on learning (Sfard, 2008) and a semiotic approach to culture (Geertz, 1973/2000). That means learning is studied as a cultural and social phenomenon in which cognitive and cultural processes are reciprocally related. These processes are mutually constitutive and evolve as participants use cultural artifacts and other semiotic means in collective and goal-directed activity (Saxe, 2012; Radford, 2000; 2002; 2003). These ideas have been discussed in theoretical developments outside and inside research on mathematics education. Here I build and expand on these ideas and present an approach to empirical inquiry. However, first the nature of the theory building is explicated.

The theory building undertaken in this research can be explained in terms of networking theories (Prediger, Bikner-Ahsbahs & Arzarello, 2008). An aim of this research is to describe the observed processes of learning in the elementary algebra classrooms in a manner that I found to be most relevant to the nature of these phenomena, using the most up-to-date resources available (in the form of employing previous scientific work). In order to do so I found it useful, and even necessary, to involve different approaches to conduct research on the learning of algebra. Prediger et al. show that most studies that network theories apply several strategies for doing so, involving varying degrees of integration. In my research the work with the theory of knowledge objectification (Radford, 2000; 2002; 2003) and commognition (Sfard, 2008) mainly involved the strategies *contrasting* and *coordinating*[3]. The strategy of contrasting theories involves comparison of different theoretical elements, but stresses differences between the theories and can reveal strong similarities and make individual strengths visible (cf. Study 2 and section 1.2.3 for a contrasting perspective on the two theories). In this manner, I want to lay a foundation for coordinating theories and provide arguments for doing so. The coordination of the two theories was done in terms of exploring a microgenetic meaning making process using the theory of knowledge objectification and using Sfard's (2008) central idea of recursion to investigate the student's discourse in terms of an ontogenetic perspective on learning. However, in employing Saxe' (2012) more general approach to cultural

[3] A coordination strategy is employed when elements from different theories are used to build a conceptual framework (Prediger et al., 2008).

development of mathematical ideas, the networking of theories also came to involve elements of the strategies *synthesizing* and *integrating locally* (cf. Prediger et al., 2008). These two strategies involve connecting stable theories in a way that allow for new theory to develop. In this latter stage, components of another theory, i.e. the emergent perspective developed by Cobb et al. (2001), were incorporated, and new theoretical constructs were developed (cf. Study 3). In the following I begin to formulate a more holistic framework for investigating algebra learning.

The framework draws on two central ideas from Saxe' (2012) approach to study the re-production and alteration of mathematical ideas in a community in Papua New Guinea. First, the cultural development of mathematical ideas can be conceptualized as shifts in form-function relation. I see this as particularly useful for framing an investigating of elementary algebra learning as it involves new forms (algebraic syntax) as well as new functions (algebraic thinking). The context of the research is an educational system that takes arithmetic as a starting point for algebra learning. Arithmetic has its own syntax and forms of thinking. Thus, the genetic relationship between arithmetic and algebra can be explored in terms of students using arithmetic and algebra for new algebraic functions as well for serving (new and old) arithmetic functions. A semiotic approach to culture is necessary to explore algebra learning in terms of shifts in form-function relations (Saxe, 2012). Radford's (2002; Roth & Radford, 2010; Radford & Roth, 2011) operationalization of a cultural semiotic approach to the classroom activity forms a foundation for the analyses of data. This is a multimodal approach to data that is concerned with the linking between students' use of different semiotic means in the objectification process, including body movements, gestures, speech and written words and symbols (cf. Study I and section 1.2.3).

Second, Saxe (2012) proposes that cultural development occurs through three interrelated and mutually constitutive genetic processes of different timeframes: microgenesis, sociogenesis and ontogenesis. Drawing on Saxe (2012; Saxe et al., 2015), these are defined regarding the mathematics classroom as follows. Microgenesis is a moment-to-moment process of creating form-function relations as teacher and students engage in goal-directed classroom activity. Sociogenesis is the process through which a classroom microculture, including specific mathematical practices (involving form-function relations), emerges and is altered and reproduced in activity over time. Ontogenesis concerns an individual's development of form-function relations over time. In any classroom act, these three genetic strands converge as teacher and students try to achieve communicational and problem-related goals in the classroom. How these are seen to converge in the elementary algebra classroom will be discussed. However, before this, the way previous theoretical developments regarding algebra learning have been incorporated will be explicated.

Radford's (2000; 2002; 2003) theory of knowledge objectification is particularly apt to investigate microgenetic processes in student group settings in the algebra classroom (cf. section 1.2.3). Radford (2002, p. 14) proposes that knowledge objectification happens through "objects, artifacts, linguistic devices and signs that are intentionally used by individuals in social processes of meaning production, in order to achieve a stable form of awareness, to make apparent their intentions and to carry out their actions". In order to study meaning making processes one must be attentive to several means of objectification, such as words, gestures, graphics and artifacts. The process of knowledge objectification is understood as the process of placing something at the center of someone's attention or view. In algebra these investigations have led to new insights. Radford (2010) argues that different sign-systems offer unique *modes of signifying*, through which meanings are transformed rather than translated. A regularity in a pattern may be noticed and expressed through the use of gestures or drawings. Regularity can be perceived and described through numbers and natural language or by applying algebraic symbolism. Further, Radford et al. (2007) argue that the different semiotic systems are not synonymous: That is, we cannot express the same meaning in gestures and natural language as we can when using algebraic expressions in terms of formula or function. However, the meanings created in the different sign-systems are not independent of each other. The details of such processes can best be captured by video-technology (cf. Study 1) and investigated through the use of slow-motion imaging. It can be time-consuming and challenging to tease out the moment-to-moment linking of different semiotic means, and, mainly, such analyses focus on a few participants' coordinated use of semiotic means. However, the knowledge acquired through such analysis can be projected onto investigations that foreground other genetic processes. Investigating classroom patterning activity, the purpose of a semiotic analysis is to investigate the dynamics and the linking of semiotic means in a genetic process, from a first awareness to more sophisticated forms of perceiving and expressing generality. Thus, the microgenetic analysis can raise awareness and help point out critical episodes both in socio- and ontogenetic analyses (both are ultimately made up of multiple microgenetic processes). However, in order to investigate genetic processes within longer timeframes and involving whole classrooms, elements of other theoretical perspectives will be better apt to capture the nature of the meaning making processes.

Sfard's (2008) discursive approach to the learning of algebra is useful for approaching the individuals' development of algebraic ideas over time (cf. section 1.2.3). Applying a process-object approach, Caspi and Sfard (2012) found that students, prior to engaging in instruction targeting algebraic thinking, developed a meta-arithmetical discourse. Drawing on the empirical analysis as well as recursion in language as fundamental in development, they propose a trajectory of algebra learning evolving through five levels of algebraic discourse (cf. section 2.2). In doing so they attempt to capture an ontogenetic trajectory detailed in particular traits of discourse, i.e. contextual, processual, granulated, objectified, etc. A microgenetic analysis, as detailed above, teases out these discursive traits in the moment-to moment classroom meaning making process. I specify that the analysis of ontogenesis in the genetic approach delineated here has not been developed to the same extent as the other genetic process. This is due to not having data to support such an analysis.

Concerning a sociogenetic analysis, elements of the two theories (Radford's and Sfard's) work together in a new way. Here the attention to participants' use of cultural artifacts and other semiotic means in collective activity (Radford, 2002) is viewed in terms of mathematical perspectives applied, conceptualized as different discourses (Sfard, 2008), i.e. calculational, rules of operations, factoring, functional, etc. Further, employing elements of the emergent perspective (Cobb et al., 2001), i.e. a classroom microculture and the notion of a classroom mathematical practice (recast in a sociocultural perspective, cf. Study 3), the discourse was also analyzed with respect to arguments and purposes perceived or explicated. Thus, I dissected the discourse in a manner that allows for an investigation of data of a larger timeframe and the coordinated use of semiotic means among a whole class. In a sociogenetic analysis, it is the collective development that is under scrutiny and what evolves is a classroom mathematical practice (CMP), consisting of normative ways of perceiving purpose, arguing and using cultural artifacts and other semiotic means. In the context of whole class patterning activity in elementary algebra, it is a classroom generalization practice (CGP) that is continually reproduced and altered. In order to describe and explain how the CGP evolves, two theoretical constructs were developed: participants' positioning and attunement to others (cf. Study 3). Participants' positioning refers to how participants use cultural artifacts and other semiotic means to re/position themselves within the joint classroom activity, i.e. taking peripheral or more central positions. Attunement to others is concerned with levels of inter-comprehension achieved in classroom conversations. For example, the same discursive object, a variable expression, may be addressed from different perspectives, rules of operations versus a functional discourse, or interlocutors may address different discursive objects, for example number structure using a factoring discourse versus the operations in an explicit expression using a functional discourse. In the first case, the conversation may continue without further notice. However, in the second case, tensions arise, and interlocutors are likely to become acutely aware of their communicational challenges. If resolved, it may lead to learning on both parts. Thus, participants' positioning and attunement to others are seen to play a role in the uptake or discontinuation of arguments, purposes and use of cultural artifacts and other semiotic means. The sociogenetic process of establishing mathematical practices will endorse some types of mathematical activity while discourage others. Thereby new microgenetic meaning making processes in the classroom are shaped, as well as individuals' ongoing ontogenetic development. On the other hand, the sociogenetic process is shaped by the individuals participating in the CGP and, thus, their previous ontogenetic development. However, individuals do not necessarily participate on equal grounds. Particularly, the relationship between teacher and students is best described as asymmetrical in that the teacher is regarded as an expert interlocutor. Subjective meaning making is more closely attended to when micro- or ontogenesis are foregrounded in the analysis. In the sociogenetic analysis collective meaning making, in a Bakthinian sense, is attended to in terms of the emergence of new forms of collective consciousness (cf. Roth & Radford, 2010; Study 3).

Finally, I offer Saxe' (2012) pictorial representation of the interplay between the genetic processes, and my interpretation and application of his theoretical constructs. Saxe explains that in "any act, the strands of microgenesis, sociogenesis, and ontogenesis converge as people use representational forms to accomplish problem-linked goals in exchanges".

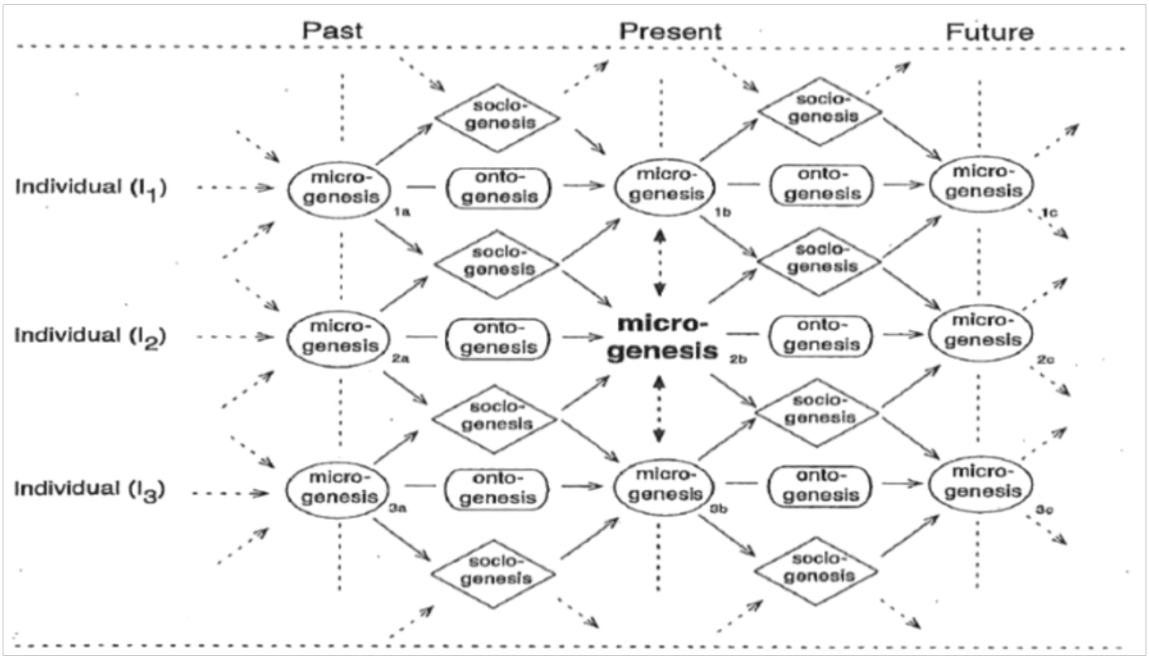


Figure 2: An illustration of the interplay of the three genetic processes (Figure copied from Saxe, 2012, p. 33)

As individuals (I_1, I_2, I_3) engage in a microgenetic process (the bold text in Figure 2) of meaning making in the classroom using different semiotic means, they bring to bear on the current situation their previous microgenetic experiences (ontogenesis). The individuals' ontogenetic development plays a role in the current microgenetic process, and the latter, in turn, play a role in the further development of the individual. However, the microgenetic processes take place in a classroom with evolving social, socio-mathematical norms and classroom mathematical practices that have bearing on the interactions. At the same time the current interaction reproduces and alters the classroom microculture.

I suggest that the genetic approach can be used in empirical investigations in various ways and not only applied to one social group, i.e. a classroom (4th grade classroom), in one particular setting (California) and within one timeframe (one school year). In a grand scope, it can be a way of connecting and supporting our understanding (meticulously produced through individual studies) of teaching and learning in algebra. However, the situated and cultural characters of these activities have to be taken account of. The research presented here is one small contribution in that respect, connecting two theories (Radford's and Sfard's) in the analyses of pattern activity in elementary algebra classrooms. Moreover, on the empirical level, the research employs and connects a cross-case analysis and a single case analysis in the treatment of classroom data from several educational systems (cf. section 4.4). It is important to focus such efforts with respect to one mathematical idea. This is in accordance with Sfard and Kieran (2001, p. 60), who argue that “interactions can only be identified by looking at the content”, and, further, that patterns of interaction evolve in direct relation to a specific mathematical topic.

4 Research methods

This research stems from an interest in elementary algebra learning and the role of algebraic syntax in interactional processes where algebra is encountered. A sociocultural perspective on learning was incorporated as it facilitates an investigation of the role of cultural artifacts as central in learning. Fundamental to this research is the assumption that culture is constituted both through cultural means and cultural practices, i.e. learning takes place at the intersection of individuals and the world (social and physical) through the means and practices that exist in culture (Cole & Gajdamaschko, 2007). Thus, in order to investigate the issue specified, a naturalistic approach to the collection of data was necessary. This chapter presents an overview of the research done including background assumptions, data collection, research design and data analysis (cf. also Kilhamn & Säljö, 2019). First, the nature of the research undertaken is discussed (2.1). Section 2.2 considers the techniques for gathering data in the VIDOEMAT project. In the next section (2.3), the case study design and the genetic approach is discussed. Finally, the operationalization of the genetic approach in the analyses of data is presented.

4.1 Background assumptions and research strategy

The inquiry of elementary algebra learning in classrooms across four countries has been guided by the interpretative paradigm. The interpretive stance taken can also be framed as belonging to the constructivist paradigm (Guba & Lincoln, 1994) as constructivism is incorporated both as an ontological and epistemological stance. That is, not only the mathematics classroom (the social world) is seen as constructed by its participants in interaction, but also the categories (language, activity, etc.) through which it can be known are constructed, and for the time being, agreed upon by those inquiring into it. These stances are paralleled regarding classroom activity, i.e. mathematics is a culturally and historically developed discourse (Sfard, 2008), and criteria for what it means to know mathematics in school vary across classrooms in time and cultural context (for example traditional versus reform oriented classrooms, cf. Cobb & Bauersfeld, 1995). Further, the results of inquiry (theory) are seen as particular and temporal answers to current aims of a specific field. Chalmers (1999, p. 168) explains:

> [A]t any stage in its development, a science will consist of some specific aims to arrive at knowledge of some specified kind, methods for arriving at those aims together with standards for judging the extent to which they have been met, and specific facts and theories that represent the current state of play as far as the realization of the aim is concerned.

In the following issues of ontology, epistemology and methodology will be discussed and form the fundament for the research undertaken.

Located within an interpretative perspective on research, the purpose of the inquiry is to make inferences into descriptive/explanatory issues (Niss, 1999). That is, the research aims to develop descriptions and explanations regarding teaching and learning in elementary algebra classrooms across four countries, asking questions such as '*how* do students engage in algebraic thinking?' and '*why* is it so?'. An important tenet of the interpretative perspective is that the natural world and people and their institutions are fundamentally different (Bryman, 2008). Bauersfeld (1992), taking an interpretivist stance, elaborates on the nature of knowledge in the human sciences:

> [O]ne of the key characteristics of the human sciences is that their knowledge does not have the predictive power of knowledge in the natural science, nor does the human sciences' knowledge accumulate in the same way, old problems reappear in new forms and require new solutions (op. cit., p. 492).

This implies that different research procedures are required in which the aim is not to explain human action in terms of causes and effects, but rather to understand it in accordance with what it means to be human and act in social realities. Bauersfeld argues that work within specific areas of mathematics education will never be completed, because societies evolve and change and people (objects of study), which behaviors we attempt to understand, are themselves trying to make sense of an ever-changing material and social world.

In mathematics education an example of a marked shift in the approaches to learning is what Lerman (2000) has identified as the *social turn*, which he defines as "the emergence into the mathematics education research community of theories that see meaning, thinking and reasoning as products of social activity" (p. 23). Another important development in this research is the use of video-technology, which has brought new opportunities but also new challenges in educational research, particularly concerning reduction of data (cf. Study 1. cf. Kilhamn & Säljö, 2019, for further discussion). The following methodological discussion takes these developments as preconditions of the research undertaken.

One of the main claims regarding affordances of a sociocultural approach to cognition is that it will foster research that is neither a-cultural nor a-historical (Hatano & Wertsch, 2001). Cole (2001) argues that historical variation needs to be considered in research, both regarding the methods for inquiry and the particular practice (e.g. mathematics teaching and learning) in which cognition is investigated. Findings can then be accurately presented in terms of these historically and culturally shaped practices. Firstly, the review in section 2.1 depicts the evolution of school algebra and the findings of this study are interpreted as intimately linked to current situations in elementary algebra classrooms. Secondly, approaches to the teaching and learning of algebra are discussed in in the sections 2.2-2.4. Taking a sociocultural approach to research, theories regarding the learning and teaching of mathematics are seen to be socially constructed discourses that are situated historically as well as in the field of mathematics education (Lerman, 2010).

The methods employed in this study are founded on a Vygotskian approach to research. Vygotsky proposed a turn away from decontextualized mentalism and behaviorism towards studying mediated action. This did not require merely an adjustment of methods used but a rethinking of methodology (Edwards, 2007). Vygotsky sought a systematic enquiry that was not reductionist, and that aimed at capturing the flow of interactions between mind and society. In more concrete terms a child is faced with a task that he cannot solve with his present capabilities. An object (or objects) is placed near the child, who was observed to often make use of it in solving the task. The object takes on the role of a sign and is drawn into the activity through the agency of the child. The child assigns signification to the object and creates temporary links in the context of his problem-solving efforts. Vygotsky explained: “In this way, we are able to study the process of accomplishing a task by the aid of specific auxiliary means” (Vygotsky, 1978, p. 74-75). Thus, the stimuli given is not seen to produce a direct response. In studying the child’s use of the object and additional means he employs, Vygotsky believed it is possible to gain insight into the inner structure and development of higher psychological processes (Engeström, 2007). Rooted in this tradition, this study seeks to investigate the emergence of algebraic thinking by accounting for and analyzing students use of algebraic artifacts and other semiotic means.

4.2 Methodological approach

Methods to study human development, and more particularly cognitive development in school like settings, have been developed in fields such as sociology, anthropology and psychology. In cross-cultural studies such as *The Six Culture Study* anthropologists gathered naturalistic data and developed new methods for analysis (LeVine, 2010). These studies contributed to the awareness of the close link between culture and cognition and a turn away from universal theories of cognitive development. Further, studies in social research pointed out the situatedness of cognition (Saxe, 1988; Lave & Wenger, 1991). Drawing on these traditions, this study seeks to investigate the genesis of algebra thinking where it is seen to emerge in contemporary society, i.e. in the elementary algebra classroom. Thus, the research sought to collect and interpret naturalistic data. This approach has traits of 'naturalistic' research such as purposeful sampling, unfolding research design and theory grounded in the data (Wellington, 2000). However, as the research aimed to investigate algebra learning across countries, the collection of data was done by national research teams and occurred in a limited time period (during two weeks in each country). Thus, in this sense the approach breaks with the traditional ethnographic approach to the collection of data, in which the researcher often spends considerable time in a tribe, organization or other social group, and where he or she interacts with members in order to achieve an 'insider's' perspective. The collection of the classroom data in each classroom in our case did not exceed two weeks and can be described as taking place through *visits* to the schools (Wellington, 2000).

Bryman (2008, p. 32) argues that "whether a qualitative study is ethnographic is to a significant extent a matter of degree." The research undertaken is ethnographic in that: (1) it sought to explore the nature of elementary algebra learning; (2) it took an initial inductive approach to classroom data; (3) a small number of cases were investigated; and (4) the analyses aimed for explicating the meanings of the participants' actions (Atkinson & Hammersley, 1994). And, more particularly, a semiotic approach to culture was incorporated (Geertz, 1973). Geertz (op. cit., p. 311) explains that in this view man is seen to be "suspended in webs of signification he himself has spun" and that culture is those webs. Thus, the work of the analysist is to sort out the structures of signification. Geertz argues that the interpretation of participants' meaning making (itself a meaning making process) starts with the collection of ethnographic data, i.e. observational accounts inscribing passing events. These are 'thick descriptions' of an interpretational nature rather than merely observational. The reliability of these accounts is closely linked to the researcher as a participant observer (Atkinson & Hammersley, 1994). In my research data from classrooms located in different cultural contexts and educational systems that I did not physically visit have been analyzed. The validity of these interpretations relies on an ethnographic approach to video data (cf. Study 1), a relatively short tradition of international video studies and the cooperation between analysts when interpreting data.

The TIMSS video study sough to document and describe national traits of teaching in three countries (Stigler & Hiebert, 2009). In order to ensure that the data would be representative of 8th grade teaching in the three countries, the researchers collected a random subsample of the full TIMMS sample. The data consisted of 231 single classroom lessons across the three countries. The data was approached by two researchers from each country, who together watched and discussed nine lessons from each country. Together they developed a common view of the nature of teaching in each country and a coding system for comparing teaching across countries. Codes were developed from discoveries, for example the Japanese teachers often asked open-ended questions, which would be presented as a hypothesis, and then criteria for the code were specified. Objectivity in interpretation is claimed and inter-coder reliability was applied. However, these codes are ultimately rooted in the interpretations emerging from the close collaboration and analytic discussions of data between researchers from the respective countries. Findings include national images of teaching, teaching as a cultural practice and a gap between countries in methods for improving teaching. Another international video-study, the Learner's Perspective Study (LPS) involving data and research teams from twelve countries, took another approach to data-collection and analysis. Clarke, Keitel and Shimizu (2006) explain that rather than comparing national teaching practices in terms of best practice, the aims were to uncover what was perceived as good practice from an insider perspective and "the meanings, mathematical and social, associated with those practices" (p. 9). Additionally, they approached teaching and learning as interdependent processes. Classrooms were selected based on national criteria for good practice and the main data included video-recordings of ten lesson sequences. An interpretative approach was taken to the data, and the findings are considered relevant in terms of diversity of perspectives rather than in terms of generalizability. My research draws on the LPS study in that it takes an interpretive approach to international video data and approaches teaching and learning as interwoven processes. However, in order to involve a comparison aspect, it draws on elements of the TIMSS video study. Particularly, comparable data were collected, and the research teams from the four different countries engaged in close collaboration when interpreting the data. This is further discussed in the treatment of data collection (cf. section 4.3). In my research the main data analyzed are video recordings, which offer other possibilities for analyses than what Geertz describes in his original account of 'thick descriptions', as the social discourse can be reconsulted repeatedly and does not relay on momentary accounts, i.e. immediate interpretations. Acknowledging that the camera neither captures what an observer would see, nor the context of social discourse, and thus cannot replace the participant observer, the argument is that the collaboration between researchers of different countries and the video data offer unique possibilities for analyses across educational systems.

This research draws heavily on the traditions of discursive psychology and social linguistics (Harré & Gillett, 1994; Gee, 2015). Rather than viewing the mind as a black box (behaviorism) or as rule-following mental processes resulting in actions from which theoretical models of the mind can be constructed (cognitivism), this perspective views the mind as discursive. That is, subjective meaning is connected to the way we use a sign, and human action is understood in terms of the discourses we participate in (Harré & Gillett, op. cit.). Thus, mind and action are not separated and by scrutinizing growth in discourse, one can investigate cognitive development (Sfard, 2008). Social meaning is interpreted in terms of a common ground, i.e. a social awareness that emerges in communication as people seek to coordinate their actions in goal-directed activity (Gee, 2015; Roth & Radford, 2010; Radford & Roth, 2011). Thus, firstly, this research describes and interprets communicational acts in the elementary algebra classroom as constitutive of algebraic thinking, not as symptoms of it. Secondly, it employs both an every-day and an expert interpretation of the classroom meaning making (Sfard, 2008; 2012). The every-day interpretation seeks to understand a communicational act in terms of familiarity with human life and relations, while the expert interpretation seeks to understand it in terms of the algebraic discourse that the participants are expected to engage in.

Finally, a qualitative research strategy was employed (Bryman, 2008). An inductive approach to linking data and theory was taken. That is, this research is not inductive in a pure sense, employing a grounded theory approach, but rather in the sense that the primary objective of the research was to describe and explain what is observed in the data. School algebra has been the focus of many previous studies and much theorization has been done. Thus, as discoveries were made working with the data these were linked to theory, which again propelled further inquiry. On another level, and when coordinating related theories, the inquiry also had an inductive tendency when working with the theories from within and when analyzing data, explanatory possibilities and limitations were discerned. Theories were coordinated in order to further analyze data. However, on both levels, the work is better described as a cycle between working inductively and deductively.

4.3 The VIDEOMAT project and data collection

The VIDEOMAT project was an international collaboration between research groups in four countries: Finland, Norway, Sweden and USA (California) (for an extensive presentation, see Kilhamn & Säljö, 2019). The research interest of the group was the introduction to algebra in school. The purpose of the project was to document, through video recordings, mathematics teaching and learning in Nordic and US classrooms in relation to a specific theme (introductory algebra).

The project took a qualitative approach to the collection of data and was concerned with teaching and learning as interwoven activities. It proved difficult to determine when the introduction to algebra occurred in the different countries by studying national curriculums and textbooks (Reinhardtsen, 2012). Thus, participating teachers were asked to alert the project members when they started teaching algebra. However, an interest in the use of letters as variables was conveyed by the project to the participating teachers as an interesting element of early algebra. This implies that we were able to collect similar empirical data in the four countries.

Purposeful selection was applied to the extent that the participating classrooms had to engage with elementary algebra during the school year. Other than that, it was a selection of convenience, i.e. volunteering classrooms near members of the national research teams. Finland, USA (California) and Sweden collected data in 6th and 7th grade classrooms. Norway collected data in 7th and 8th grade classrooms, which corresponds to the age groups in Finland and Sweden (12 and 13 years). However, the students in USA were one year younger respectively (11 and 12 years). The main data collected consisted of observation and video-recordings of four consecutive lessons in the participating classrooms (4 classrooms x 4 countries). The teachers were asked to conduct these lessons as planned. A fifth lesson was observed and video-record, in which the teachers were asked to introduce three TIMSS tasks (adapted from TIMSS & PIRLS, 2009, Figure 3) for group work in the manner he/she saw fit. These tasks were given to the teacher after the completion of the fourth lesson. The fifth lesson was included to: (1) ensure data in which students were engaged and active in problem-solving; and (2) to further facilitate comparison. Additionally, post-interviews with the teachers were made (audio recorded) and all written material from student activity, and teacher plans and notes were collected.

The empirical material collected by each country was shared among the participating countries. Additionally, the research teams created lesson graphs of all lessons (in which ordinary teaching was conducted) video-recorded in their respective countries. The first lesson in each classroom was also transcribed. Monthly meetings (virtual or in-person) between the research teams were held for the duration of the project. After the collection of data, empirical material from the lessons were presented and discussed. The students' work with the *matchstick* task (TIMSS task C, cf. Figure 3) in the fifth lesson produced rich discussions and became a focus for inquiry in the VIDEOMAT meetings. Initial coding and pre-analyses were discussed. It was during these discussions that the focus for this research, classroom work with patterning activity was formulated (cf. 1.1). These meetings and discussions were significant for the further analyses done in this research.

The work with the matchstick task, i.e. video recordings of focus groups (four from each country) and their written work, and the video recordings of four consecutive lessons in a Californian 6th grade classroom that worked with patterning tasks, form the empirical basis for this doctoral thesis.

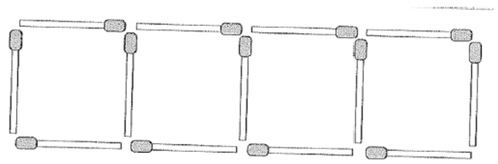
A. In Zedland, the cost of shipping a parcel is calculated by the equation:
$y = 4x + 30$, where x in the weight in grams and y is the cost in Zed dollars.
A parcel that costs 150 Zed dollars to ship can be written on the equation:
$150 = 4x + 30$
How many grams does that parcel weigh?

B. The number of jackets that Haley has is 3 more than the number Anna has.
If n is the number of jackets Haley has, how many does Anna have in terms of n?
Choose one of the alternatives and justify your answer.

a) $n - 3$
b) $n + 3$
c) $3 - n$
d) $3n$

Describe what the three other expressions would mean when Hasse has n jackets.

C.



In the figure, 13 matches were used to make 4 squares in a row.
What is the number of squares in a row that can be made in this way using 73 matches?
How do you know?

Figure 3: The TIMSS tasks (adapted from TIMSS & PIRLS, 2009)

The research teams in the four countries collected data following the guidelines described and agreed upon (cf. above). However, some variation in more specific details can be expected. In Norway we observed and collected data from five consecutive lessons in two 7th grade classrooms (school 1) and two 8th grade classrooms (school 2). I was present during all collection of data. The data collected included:

- *Observation*: The Norwegian research team had three members at the time of data collection. There was always one researcher present in the classroom, often there were two, and in a few lessons all three members were there to observe. Those who were present made field notes, however a professional camera crew were hired to take care of the video recordings.
- *Video recordings*: There were three cameras in the classroom. One focused on the teacher (controlled by a cameraman), the second focused on the classroom (still camera), and the third on a focus group (controlled by another cameraman). In the Norwegian classrooms participating in the

research, the students were seated in pairs. The teachers chose themselves one pair to be the focus group in their classroom, without any particular instructions from the researchers.

- *Audio recorded interviews with teachers*: Interviews with questions elaborated by VIDEOMAT. The teacher interview was done shortly after the video recording of his/her classroom had been completed.
- *Written material*: Folders with notepaper, provided by the researchers, were given to students at the beginning of the first lesson and collected at the end of the fifth lesson. The students were asked to place all handouts (related to the mathematical instruction) in the folder. The teachers' lesson plans and notes were collected.

In attempting to gather naturalistic empirical materials the project followed an ethnographic approach to the collection of data as described previously (4.2). Researchers took a role as an observant participant when collecting data (Wellington, 2000). That is, we observed and recorded activity but did not engage directly in the teaching-learning activities in the first four lessons. However, for the fifth lesson we took a more obtrusive role in providing three tasks for group work. Again, we did not engage directly in classroom discussions.

The goal of the project was to investigate teaching and learning in regular elementary algebra classrooms. Bringing three cameras into the classrooms, and extra persons to manage them in addition to the researchers observing the lessons, was quite an intrusion into classroom life that could possibly contribute to shaping the flow of events. The teacher may feel a pressure to 'perform' and thus design special lessons, and students may be reluctant to participate in discussions. However, the objective to videotape ordinary lessons was discussed with the teachers and they were specifically asked to conduct lessons according to plan. The students engaged in discussions and teachers reported at the end of lessons that students behaved as usually. An example of the relaxed atmosphere is the Swedish classroom reported on in Study 2, in which a paper airplane came flying in front of the camera and students were discussing everything from biceps and soccer to the mathematics task at hand.

A research project goes through different stages and ethical considerations must be attended to during the planning phase, in the process of collecting data and in the analyzing and publication of the data collected. This captures the dynamic element of research and perhaps reveals a need for flexibility in the ethical stands taken by the researchers and a willingness to reconsider those as the process of research progresses. The BERA (2011, Ethical Guidelines for Educational Research) guidelines take the concern for the person as a principal guiding the ethical considerations and the majority of the paragraphs address the concerns for the participants of the research. This will be in line with the thoughts expressed already by Kant as he sees humans as a purpose in themselves and not as a mean to achieve a desired end (Vaags, 2004). Following Kant's moral philosophy, the considerations for individual rights cannot be set aside for a greater purpose as no such exists. However, next in line as a BERA (2011) priority is knowledge. This places a burden on the researcher with a set of obligations to strive for and encourages him/her to be vigilant in the development of knowledge. Thus, doing educational research involves finding ways to predict and prevent as well as handling ethical dilemmas that may arise.

The project considered ethical issues prior and during the collection of data. Participation in the project for teachers and students followed the guidelines for voluntary informed consent (cf. NSD approval, Appendix 1). According to the objective of the VIDEOMAT project, there was no need for any form of deception in relation to the knowledge the researchers are interested in. However, when planning contact with the participants there was a discussion between the researchers of what the information given should entail. Full disclosure of plans and motives seemed to include too much information, and there was a concern that revealing the three TIMSS tasks to teachers prior to the four lessons could influence the data collection. The issue of sharing information therefore attempted to follow an ethical line between duly informing the participants and collecting data that is as "realistic" and/ or "natural" as possible (BERA, 2011).

The research may become all important to the researcher who may therefore have unrealistic expectation with respect to the involvement of participants (BERA, 2011). It was an explicit concern among researchers during collection of data to approach the teachers as professionals doing an important job under severe time constraints every day. Furthermore, the teachers were treated as experts in their classrooms as they decided what counts as an introduction to algebra. In this sense, they had a significant decision-making role in the project. These considerations for the teacher formed a starting point for communication and collaboration. It was a priority in the project to make the burden of research as light as possible for the teachers involved and gratitude for their participation was often expressed.

The data was shared only among the researchers in the project and stored in a manner that secured the anonymity of participants. Ethical considerations have been considered in choices of analytical foci. In my research, the main concern is with meaning making processes in the classrooms. Thus, value laden topics such as assessing good or bad teaching or more or less learning have been avoided. However, that does not mean that the research done is value free, only that there is less conflict between the concern for participants' wellbeing and being honest and truthful in reporting on data. Finally, anonymity of participants is secured by using pseudonyms and by not disclosing the locality of schools in the written reports.

4.4 The genetic approach and selection of cases

The research design implemented includes a cross-case study and a single case study. The case study is seen as appropriate for investigating the teaching and learning of algebra as these are culturally developed and socially situated activities (Yin, 1981). That is, processes of teaching and learning cannot be extracted from the real-life context where these occur without losing their very essence (cf. 4.1). The case study is considered as particularly apt for dealing with the nature and complexity of the phenomena in question (Stake, 2000). The genetic approach proposes to study the emergence of algebraic thinking in classroom patterning activity as three interwoven genetic processes of different time frames: micro-, socio- and ontogenesis. This was implemented as following:

- The microgenetic process was foregrounded in an analysis of the group work with the matchstick task (1 group x 4 countries x 8-15 minutes). Additionally, the students' discourse was analyzed from an ontogenetic perspective. This was done through a cross-case study (Study 2).
- The sociogenetic process was foregrounded in an analysis of a 6^{th} grade Californian classroom (4 lessons). This was a single-case study (Study 3).

The cross-case study was initiated and informed by an analysis of a larger set of data that focused on the varied problem-solving approaches of 16 groups (4 groups x 4 countries). The analysis shows that the students, in spite of coming from different countries, applied similar approaches with no more variation between groups of different countries than between groups within the same country. Further, the groups worked with several approaches (mainly three to four) going back and forth within in one group. The analysis also shows that the students mainly used arithmetic to solve the problem and not the algebra (notation, vocabulary, general ideas) they had worked with in the four previous lessons. These findings raised new questions concerning characteristics of the groups' solving processes and the nature of students' argumentation. Keeping a comparison aspect, one group from each country was chosen for a cross-case analysis. The four groups were selected so that most of the approaches previously identified would be included (with overlap, i.e. some approaches reoccur in most groups). Thus, the four cases were seen to be representative of the larger sample (Yin, 1981) but including maximum variation in that the groups were situated in different educational systems (Flyvbjerg, 2006). At this stage it was open if the in-depth discourse analysis of the group work would funnel towards pointing out differences or similarities (or both) between the groups.

The cross-case analysis done resembles a sequential direct replication design (Yin, 1981) in that: (1) one group's discussion was analyzed in full before the analysis of the next one started; (2) the analysis of each new case built upon the previous ones and sometimes led to modifications of findings, i.e. the significance of elements of student discourse was realized after the analyses of subsequent cases, occasionally also including reinterpretation of data. The first point secured the internal coherence of the analysis of one group's meaning-making process, while the second point facilitated the emergence of a more general description and explanation across the different groups' meaning making processes. The in-depth analyses across the groups revealed some differences but even more striking similarities that led to a subsequent juxtaposing of the groups' meaning making processes and a concluding synthesizes (cf. Study 2). Considering the issue of internal validity, Yin (op. cit.) argues that three or four cases are considered sufficient regarding a cross-case design, and that "once a phenomenon has been shown to occur in all cases, the concluding step is to develop a general explanation or synthesis across the cases" (p. 102). The juxtaposing of the groups revealed that one group's problem-solving process, which used algebra (equation), deviated in some important respects from the others. Thus, the synthesis done included three cases, while the fourth case became a source for contrasting. That is, the synthesis shows how students use of arithmetic to solve an algebraic problem led to an inductive solving process in which the discourse evolved from talk about concrete object to increasingly abstract mathematical ones. The fourth case provides external validity in that it shows in which ways student use of an equation to model the problem altered the meaning making process (Yin, op. cit.). It is not suggested that this research follows a systematic replication design as according to Yin, rather we point out that the analytic techniques applied in this study are reminiscent of broader more general designs.

The cross-case analysis is dependent on the robustness of the within-case design (Yin, 1981). The same aims and research questions guided the collection of data in the four countries, i.e. according to the VIDEOMAT project (cf. 4.2). Further, the analyses of the four cases revolved around following topics: (1) the nature of students' argumentation; (2) the emergence of algebraic thinking in elementary algebra classrooms; and (3) the role of semiotic means in the meaning making processes. The participants were of similar ages (12-13) and were part of regular elementary algebra classrooms in their respective countries, in which the educational systems mainly base algebra instruction on students' previous experiences in arithmetic. The unit of analysis at case level was student-student interactions. Yin (op. cit.) argues that the use of a clear design at case-level is important to avoid bias and unpredictability in the research process.

The similarities of the microgenetic meaning making processes of the three groups were so striking that the study shifted from mainly being descriptive to also seeking explanatory answers. That is, it sought to tease out why the students' discourse evolved the way it did. The theory of commognition (Sfard, 2008) was seen to offer an appropriate model for discourse development (recursion) to test out in a synthesizes of the cases. The identification and characterization of focal elements of the students' discussions completed in previous analyses formed the foundation for the final stage of synthesizing. These included thematic discourses, i.e. defined by the type of objects addressed, and discursive shifts. Thus, theoretical explanations, i.e. discursive processes such as saming, encapsulating, reification, semiotic node[4], regarding how a discourse evolves was tested out in all three cases by scrutinizing how the discursive shifts were accomplished and if these 'discursive processes' could be identified. For example, how did students go from talking about concrete objects to numbers and numerical sequences. Thus, a *surveying (the cases)* approach to the cross-case analysis was applied (Yin, 1981). However, this was done not by statistical means as portrayed by Yin, but instead by scrutinizing critical stages in discourse development (discursive shifts) occurring across cases.

---

[4] Semiotic node is a construct elaborated in Radford et al. (2007) that had been identified in the previous analyses.

The single-case study investigated a collective process of meaning making in one elementary algebra classroom that worked with pattering tasks across four lessons. The single-case study is contributing to an investigation of a larger frame than itself. That is, it was chosen and formed by the genetic approach to algebra learning and informed by previous investigations of a microgenetic process of meaning-making in patterning activity and an ontogenetic perspective on the activity. The single-case study was initiated to complement the previous cross-case analysis in specific ways by involving and attending to: (1) teacher-student interactions; (2) students' use of cultural artifacts such as variable expression, function table and the metaphor of function machine; and (3) a sociogenetic process. The Californian 6th grade classroom was chosen because it worked with patterns throughout the four lessons. It is considered as a separate case study as it involves a whole classroom community (teacher and students) and the unit of analysis is teacher-student and student-student interactions in whole class discussions. The timescale is also very different from the previous cases (8-15 minutes vs. 4 lessons). However, the same topics as in the cross-case study were investigated.

4.5 The genetic approach and data analyses

As mentioned in Chapter 3, the genetic approach suggests that algebra learning occurs through shifting form-function relations and that these need to be investigated in terms of three interwoven genetic processes of different timeframes: micro-, socio- and ontogenesis. According to Wellington (2000), this involves a triangulation of data as the results from applying one perspective can support the analysis when approaching the data from another perspective.

In order to investigate shifts in form-function relations, and particularly the role of algebraic artifacts in algebra learning, a cultural-semiotic perspective (Radford, 2002; Roth & Radford, 2010) on classroom interactions was taken. In this perspective, thought (the ideal) and body (material) are not separated; rather thinking occurs as people are moved by and move (act) upon the material and social world, that is sensuously perceiving through engaging with objects, artifacts and semiotic means in cultural activities (Radford, 2013). This aligns with Sfard's (2008) commognitive perspective, in which thinking and communicating are seen to be two sides of the same coin (cf. section 1.2.3 and Study 2). The interest of this research is student meaning making processes in algebra, resulting in both subjective meanings and collective ones. Both collective practices and individual form-function relations are reproduced and altered in interactional processes. Thus, the *unit of analysis* in this research is *student-student and teacher-student interaction occurring in small groups and whole class discussions*.

In the cross-case analysis (Study 2), a multimodal approach to analyzing student-student interaction was applied. The approach was developed in Study 1, in which the multimodal elements of the interactions that appeared relevant in a group's process of knowledge objectification (Radford, 2000; 2002; 2003) were identified. Thus, the transcribing of data was a first analytic step. In Study 2 we were concerned with analyzing the groups' meaning making processes in terms of knowledge objectification, and utterances were interpreted in terms of subjective meanings. That does not mean that individuals' utterances or other semiotic acts appearing in the group discussion were interpreted as separate entities. Rather these were interpreted as parts of a whole, i.e. joint activity, applying the concept of sequentiality (Derry et al., 2010). That is, an utterance or other semiotic acts were interpreted in terms of previous utterances and semiotic acts, and the ones that came to follow. This is in accordance with a dialogical approach to the analysis of interaction (cf. Linell, 1998). Topical sequences were identified in terms of the objects addressed in the interactions, e.g. concrete objects, patterns, arithmetic expressions. The interactions were analyzed taking account of both individual students' intents and progress as well as the groups' joint meaning making. The analysis pointed out characteristics of a microgenetic classroom process. Additionally, the interactions were analyzed from an ontogenetic perspective, applying the framework of Caspi and Sfard (2012), i.e. five levels of meta-arithmetical discourse (cf. section 2.2). On the one hand, the microgenetic analysis contributed with new insights regarding the nature of students' meta-arithmetical discourse. On the other hand, the ontogenetic perspective supplemented the microgenetic analysis by pointing to critical elements of the discourse. A central focus of the analysis was to investigate how students used arithmetic for new algebraic functions.

In the single case study (Study 3), I investigated whole class discussions in one classroom across four consecutive lessons and focused on participants' use of algebraic artifacts and other semiotic means in joint activity. The interactions were interpreted both in terms of subjective intentions and collective meanings produced. The latter being central to the study as it focused on collective development (sociogenesis). In addition to taking the same approach to discourse analysis as in Study 2, although less multimodal, this study drew particularly on the Bakhtinian idea that an utterance or semiotic act does not only belong to the individual but also to the one/s it is directed at. "The word in language is half someone else's", as Bakhtin (1981, p. 293) insisted. The argument for this is that we tailor our discourse to our conversation partners. In the next instance, the collective meaning making is seen to produce new forms of collective consciousness (Roth & Radford, 2010; Radford & Roth, 2011). Thus, the response is not a passive understanding but, rather, an active contribution to an ongoing conversation. The unit of analysis was teacher-student and student-student interactions involving either purpose, argument or the use of cultural artifacts. As the analysis focused on the collective meaning making, and in adherence to the Bakhtinian perspective, the topical sequences had to at minimum include one turn pair (initiation-response). A collective meaning making process over time (sociogenesis) is made up of multiple moment-to-moment processes. In Study 3 these microgenetic processes were analyzed with respect to the sociogenetic development. The in-dept study of students' moment-to-moment meaning making processes, as well as the conclusions concerning the students' meta-arithmetical discourse drawn in Study 2, contributed to the interpretation of the interactions in Study 3. This analysis focused on students' use of algebraic artifacts for new and old arithmetic functions as well as new algebraic functions.

In order to strive for validity and reliability at this level of interpretation, this research has followed five principles proposed by Sfard (2012). The principle of *operationality* refers to the researcher's use of theoretical constructs in the analytic process. For example, algebraic thinking was operationalized using Radford's (2002, 2010, 2018) cultural-semiotic theory and Sfard's (2008; Caspi & Sfard, 2012) discursive theory. *Completeness* refers to the analytic focus concerning mathematical content. Sfard (2012) argues that one cannot change the use of one word without having repercussions for the discourse to which it belongs. Thus, approaching patterning activity in elementary classroom I looked at changes in form-functions relations considering both arithmetic and algebra in school as entire discourses operating under different meta-rules. The *contextuality* of utterances and other semiotic acts were fundamental to the interpretations made, applying the dialogical principle of sequentiality as well as a multimodal approach. *Alternating perspectives* were employed in that critical episodes were determined from a professional viewpoint (outsider) while subjective intent and collective meaning making were analyzed employing an insider perspective (allowed by myself being a fellow human being and a participant in human discourses (cf. Geertz, 1973/2000). Finally, this research has followed the principle of *directness* in that the sayings and things done by teachers and students have been presented through transcripts rather than only my renderings of classroom events.

5 Summary of studies I-III

The three publications that are included in the thesis were previously mentioned in a narrative of the evolution of the thesis (section 1.1). The studies are summarized and presented here in chronological order:

I. Reinhardtsen, J., Carlsen, M., & Säljö, R. (2015). Capturing learning in classroom interaction in mathematics: Methodological considerations. In K. Krainer, & N. Vondrová (Eds.), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education* (pp. 1475-1481). Prague: Charles University in Prague.

II. Reinhardtsen, J., & Givvin, K. (in press). The fifth lesson: Students' responses to a patterning task across the four countries. In C. Kilhamn & R. Säljö (Eds.), *Encountering algebra. A comparative study of classrooms in Finland, Norway, Sweden and the USA* (pp. 00-00). New York: Springer.

III. Reinhardtsen, J. (submitted). The emergence of a generalization practice in a 6th grade introductory algebra classroom.

The papers are connected in that they are all concerned with the analysis of classroom interactions in terms of meaning making in patterning activity. Study 1 discusses the role of multimodality in learning processes and the act of transcribing video data as an analytical activity that involves a reduction of data. It addresses methodological issues implicated and argues that the choices made regarding reduction, the selection of salient episodes and the approach to dialog must be firmly grounded in the theoretical perspective adopted. Further, this is explicated in terms of Radford's (2002) theory of knowledge objectification and VIDEOMAT classroom data. Study 2 draws on Radford's theory, as well as the commognitive perspective by Sfard (2008), in the analyses of data from four countries. It focuses on the microgenetic process of meaning making in a small-group problem-solving context and analyzes students' approaches to a patterning task and their use of semiotic means in the solving process. Additionally, it employs an ontogenetic perspective on the students' discourse in terms of the development of algebraic thinking. Study 3 analyzes meaning making in whole class interactions across four lessons in a 6th grade Californian classroom. It investigates the role of cultural artifacts, introduced by the teacher, such as function table and variable expressions in the classroom patterning activity. It focuses on the sociogenetic process of meaning making in the classroom.

Thus, Study 1 forms a methodological foundation for the empirical papers in terms of data treatment, while Study 2 and Study 3 complement each other in two ways: (1) they investigate the role of different semiotic means, belonging to arithmetical versus algebraic discourse, in the development of algebraic thinking in small group versus whole class interactions; (2) they focus on different genetic processes in the elementary algebra classroom that are seen to intertwine in the cultural development of algebraic ideas (cf. section 3.3).

5.1 Study 1

Reinhardtsen, J., Carlsen, M., & Säljö, R. (2015). Capturing learning in classroom interaction in mathematics: Methodological considerations. In K. Krainer, & N. Vondrová (Eds.), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education* (pp. 1475-1481). Prague: Charles University in Prague.

The aim of this study was to develop a methodological approach to video-data that facilitates the analyses of student-student interactions regarding learning processes in the algebra classroom. It was motivated by the initial probing into video recordings of student small-group work with a patterning task. It became clear that in order to understand the groups' problem-solving processes one had to include some account of multimodal elements. However, as the inclusion of all information available in video data in a transcription is neither feasible nor necessary, *what* to include is a methodological question. The study scrutinizes methodological approaches taken in several interpretive studies that point out the multimodal nature of classroom learning. These make different choices regarding their approaches to dialogue, the selection of salient episodes and methods of transcribing.

The development of a methodological approach was grounded in a sociocultural perspective as the role of cultural artifacts in learning is a fundamental concern of this research. Radford's (2002) theory of knowledge objectification was chosen as a theoretical framework for the interpretation of student-student interactions in the classroom. In his perspective, learning is conceptualized as a process of becoming aware of what one could not see before. This happens in goal-directed social activity through the coordination of individuals' use of objects, artifacts, body movements, linguistic devices and signs. Accordingly, learning in the study is described as the process of becoming aware of general aspects of the displayed figural pattern by expressing observations mathematically in increasingly sophisticated forms, allowing for a mathematical solution of the task.

The methodological approach proposed is discussed regarding broader concerns with multimodality in video analysis, in terms of multimodal transcriptions and concerning the investigation of learning in classroom interactions. Multimodal elements of video-data are interpreted on two levels: (1) everyday understanding; and (2) professional interpretation. The first level of interpretation employs an everyday understanding of activity and takes the actor's perspective in locating the multimodal element. At the second level, it is interpreted from a professional perspective that is concerned with particular issues relevant to a field of inquiry. Additionally, the multimodal elements are interpreted in terms of *sequentiality*; an act is seen as motivated by prior acts and as motivating subsequent acts; and *reflexivity*; an actor frames the act to indicate its meaning.

The multimodal transcript needs to include the elements of interaction that appear to have been relevant for the participants at the time of interaction. Thus, transcribing is a hermeneutic activity as one must decide which elements constitute the situation and what is necessary to take note of to make sense of the activities as engaged in by participants. Regarding the group discussions of the patterning task, three categories of mediating resources were identified as relevant in the students' meaning making processes: (1) *inscriptions* such as drawings, tables, texts, numbers, arithmetic, algebraic (including variable/s); (2) *concretes* such as matchsticks; and (3) *gestures* such as pointing, tracing in air/figure/table, glance, rhythmic hand movement, raising hand.

In addition to the mediating resources identified, the approach specifies different features of thinking through which salient episodes can be identified. That is, elements of reasoning such as sense making, conjecturing, convincing, reflecting and generalizing; and mathematical strategies such as additive, multiplicative, equations and functional are relevant to attend to. Cultures of collaborations are also seen to play a role in the group's processes of meaning making. The paper exemplifies the approach in an analysis of a Norwegian group's work with the patterning task.

5.2 Study 2

Reinhardtsen, J., & Givvin, K. (in press). The fifth lesson: Students' responses to a patterning task across the four countries. In C. Kilhamn & R. Säljö (Eds.), *Encountering algebra. A comparative study of classrooms in Finland, Norway, Sweden and the USA* (pp. 00-00). New York, NY: Springer.

This study investigated the work of 16 focus groups, from the four countries that participated in the VIDEOMAT project, as they engaged with the matchstick task in the fifth lesson (1 focus group x 4 classrooms x 4 countries). The aim was to analyze how students in the different countries were able to participate in the algebraic discourse. The students' discourse was scrutinized through a four-part analysis. Part 1 focused on the strategies that the groups developed in order to solve the task. On the grounds of this work, one group from each country was chosen for an in-depth discourse analysis in Part 2. Part 3 juxtaposed the meaning-making processes of the four groups, and in Part 4 a synthesis of the findings was done.

The students' discourse was approached both in terms of an ontogenetic perspective on the development of algebraic thinking and in terms of a local, in time and place, discourse development, i.e. the students' microgenetic processes of meaning making. This was done through a coordination of Sfard's (2008) commognitive perspective and Radford's (2000; 2002) cultural-semiotic theory of knowledge objectification. Algebraic thinking is defined as a meta-arithmetical discourse that arises when one reflects on arithmetical processes and operations, involving the use of indeterminate objects in analytic activity. It does not necessitate the use of algebraic symbols and natural language is considered to play an important role in learning algebra meaningfully.

The first part of the analysis revealed that students developed strategies ranging from drawing and counting to more sophisticated ones involving expressions. Each group developed more than one strategy. Most groups used drawing and counting at one point in the solving process, which often became a springboard for more mathematically advanced approaches. The same strategies could be identified in groups across the four countries, with no more variation between groups of different countries than among the groups of one country. Left to themselves, the students did not use elements of the four prior algebra lessons but employed their previous arithmetical experiences to solve the task. However, some groups used algebraic expressions or equation when prompted by the teacher.

The second part include in-dept discourse analysis of one focus group from each country. The groups were chosen to include the wide range of strategies identified in Part 1. A multi-semiotic approach to the groups' moment-to-moment meaning making processes was employed. These were conceptualized as processes of knowledge objectification (Radford, 2000; 2002). Foundational to the study was that sensory-motor experiences can form the basis for abstract mathematical reasoning and that semiotic means, including gestures, are constituents of thinking. Sfard's (2008) notion of recursion in language was central to investigate how a group's discourse built on itself over the course of the problem-solving process. Recursion is the replacement of simpler utterances with more complex ones. A related notion is reification, which is the replacement of talk about processes with talk about objects.

The students' discourse was also analyzed in terms of ontogenetic development of algebraic thinking in school. These inquiries were based on five levels of algebraic discourse as depicted by Caspi & Sfard (2012), which operationalizes the process-object approach to algebra learning. Regarding school algebra, they propose three levels belonging to *constant value algebra*; processual (level 1), granular (level 2) and objectified (level 3), and two levels belonging to *variable value algebra*; processual (level 4) and objectified (level 5). Students can operate on these levels of discourse informally, using natural language, or formally, using algebraic symbolism. The findings regarding the nature of the students' discourse show that it mainly has characteristics of processual and granular levels of algebraic discourse. However, the group that employed equation in their solving process did at times operate on an objectified level.

In Part 2 of the analysis, episodes were determined according to the focal objects of students' utterances and other semiotic acts, i.e. matchsticks, numbers, expressions, etc. Talk about similar objects was seen to form a thematic discourse. A discursive shift was defined as moving from one thematic discourse to another. The juxtaposing of the four groups (Part 3 of the analysis) show that each group's solving process is unique in some respects but also shares salient features with others. Altogether five thematic discourses were identified across the four groups and named according to their focal objects: 1. Concrete objects;

2. Pattern; 3. Arithmetic expressions; 4. Equation; and 5. The irregularity. It is shown that the three groups, which did not receive any prompts from the teacher, moved from talking about concrete objects such as matchsticks and the figure, to talking about numbers and numerical/geometrical patterns, and then to talking about expressions and calculations. Importantly, features of previous thematic discourses are used in later ones and play a role in initiating discursive shifts. The fourth group shows a different solving process as it receives several prompts and use an equation from the very start.

In the synthesizes of the three groups that had similar solving trajectories (Part 4), the microgenetic processes were further scrutinized in terms of how a discourse evolves, and how new, more abstract objects are created. The three first thematic discourses (concrete objects, pattern, arithmetic expressions) were considered to form three layers of discourse according to their discursive objects: (layer 1) concrete discursive objects; (layer 2) abstract discursive objects; and (layer 3) compound abstract discursive objects. The discourse in these layers were parceled into four categories: discursive processes (saming, encapsulation, reification, semiotic node), modalities (action, gesture, inscription), word use (numbers, verbs, pronouns, nouns), and discursive objects (matchsticks, rate of change). The discursive processes explain how the abstract discursive objects are created and the layers show the nature of the discourse at each layer and how a discourse builds on itself

The study confirms that students in different school systems, 11-13 years old, develop an informal meta-arithmetical discourse, as was previously explored in Caspi and Sfard (2012), and further maps the nature of this type of discourse. The microgenetic analysis shows how a discourse builds on itself and the multimodal nature of the discursive processes laid out by Sfard (2008), i.e. reification, saming and encapsulation. The identification of semiotic nodes, through which students experience aha-moments and new objects are created, as well as the eminent role of a particular gesture (tracing how three matches make up one square), shows the multimodal and sensual character of the microgenetic process.

5.3 Study 3

Reinhardtsen, J. (submitted). The emergence of a generalization practice in a 6th grade introductory algebra classroom.

This study investigated teacher-student and student-student interactions in whole class patterning activity over the course of four consecutive lessons in a 6th grade Californian classroom. The teacher and her students worked through eight patterning problems involving both numerical and geometrical sequences. The aim was to explore contingencies between teaching and learning. This was approached in terms of a collective process of discourse development, conceptualized as the emergence of a classroom generalization practice (CGP). The purpose of the study was twofold: (1) to formulate an approach to investigate classroom interactions in terms of a sociogenetic process in the algebra classroom; and (2) to characterize this process as it unfolded in whole-class interactions. A more specific research question was formulated: How do 6th grade students use cultural artifacts and other semiotic means to engage with indeterminacy and generalization in teacher led whole class patterning activities?

Foundational to the theoretical framework elaborated is that the mathematical practice of generalizing is simultaneously viewed as a culturally and historically established practice (Sfard, 2008) and one that emerges through interactions in a classroom (Cobb et al., 2001). A CGP consists of normative ways of arguing, normative purposes of activity and normative ways of using semiotic means. I investigated participants use of cultural artifacts and other semiotic means employing Radford's (2003) cultural-semiotic perspective and Sfard's (2008) discursive perspective. In order to investigate collective versions of meaning making, I drew on Rogoff's (2008) notion of guided participation and Roth and Radford's (2010) theoretical constructs of intersubjectivity and social consciousness. A Bakhtinian view of interactions was incorporated.

In order to monitor and investigate the classroom interactions in terms of the emergence of a CGP, the constructs of *participants' positioning* and *attunement to others* were developed. It was expected that students would engage differently in the whole class discussions as they became increasingly familiar with patterning activity. Thus, it was expected that the students were to go from being peripheral participants in a generalization practice to more central participants, i.e. going from observing and carrying out secondary roles to increasingly managing such activity (Rogoff, 2008). In the data, initial and new forms of student participation, were identified and investigated regarding their uses of cultural artifacts (introduced by the teacher for this activity) and other semiotic means in these acts of re/positioning.

The second construct, attunement to others, was developed in order to account for collective versions of meaning making in the classroom. In the classroom discussions the participants, teacher and students, engage with each other's ideas. In order to do so, they adjust their perspectives with varying degrees of asymmetry, to achieve inter-comprehension. According to the classroom's social and socio-mathematical norms, it was conjectured that the teacher would adjust her discourse to the students' forms of argumentation in order to achieve intersubjectivity, while the students would attempt to make meaning of her ways of acting and respond to her evaluations of their suggestions (also in regards to each other's ideas). The discourse was investigated regarding participants' use of previous experiences and different semiotic means as they attuned to others and alterations in the resources employed was accounted for. Participants' acts of positioning and efforts to attune to each other's ideas are seen to be constitutive of the emergent CGP.

The teacher involves the students in three main phases of activity; exploring the sequence; making a function table and determining an explicit generalization in terms of operations involved; extending the generalization to all subsequent terms and creating a variable expression. The findings include an analysis of: (1) the evolution of collective versions of meaning making in terms of an emergent CGP; (2) the role of cultural artifacts and other semiotic means; and (3) a characterization of the collective classroom process. The nature of the emergent CGP is analyzed and summarized according to episodes considered to illuminate central aspects of the CGP and as being useful for the aims of the study (see Figure 4):

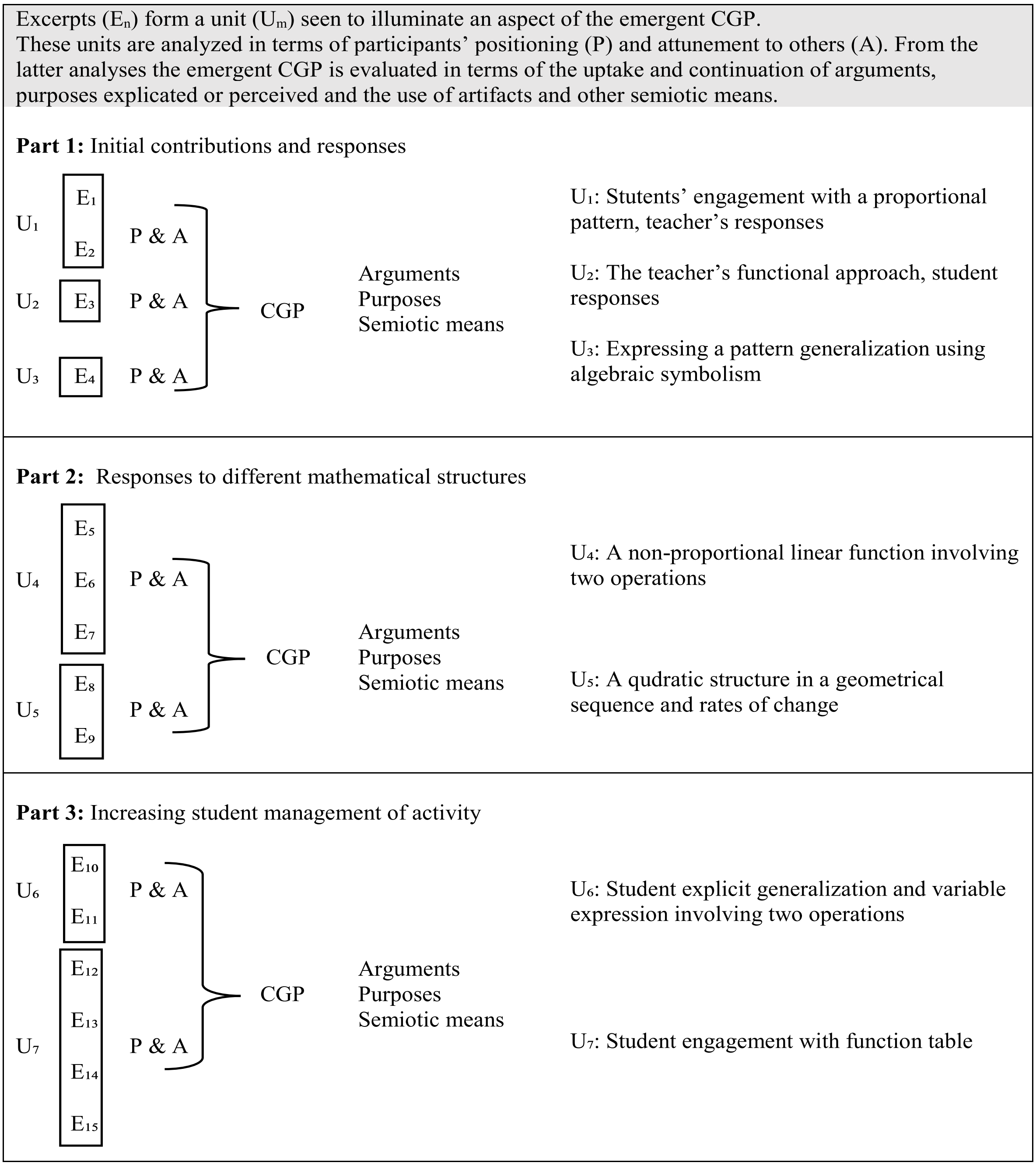


Figure 4: "Analytical approach" (Copied from Study 3, p. 17)

The CGP is seen to emerge through participants' acts of re/positioning and the attunement to others. These intertwine as participants seek to achieve social goals such as becoming full participants in the CGP and achieving communicational coherence. Attunement takes place in the immediate classroom conversation and over time. Three different levels of achieving inter-comprehension in conversations between the teacher and her students were discerned: (Level 1) a tensional discussion occurred as participants did not address the same objects (number structure versus operations); (Level 2) participants address the same objects, only from different perspectives, i.e. addressing an expression from a functional versus a calculational perspective; and (Level 3) participants address an object from the same perspective, using informal versus formal discourse. It is at the first level that participants are acutely aware of their communicative challenges. If inter-comprehension is pursued and the tension resolved, these situations could be productive in terms of new insights (for all participants). The second level occurs more frequently and seem to pass by without participants' noticing the difference in perspectives applied. The third level also occurs several times and could be productive for learning as the teacher points out mathematically important aspects using specific terminology, which could lead to an increase in student awareness.

6 Conclusions and discussion

This chapter has two sections. In the first section (6.1), findings regarding different aims and research questions are presented and discussed in separate subsections. The second section (6.2) considers limitations of the research.

6.1 Summary and discussion of findings

In order to investigate teaching and learning in elementary algebra it was determined that student meaning making is central to understand these as interrelated processes in the classroom. The nature of the data investigated, only spanning five consecutive lessons but involving 16 classrooms from four different countries, allowed for an in-depth study of interactions, and a scrutiny of discursive processes in the elementary algebra classroom across contexts. The data gathering was timed to capture the classrooms as students started to engage with algebraic symbolism. Thus, a particular interest of the study is the role of algebraic artifacts in the development of algebraic thinking. This is investigated in terms of the nature of student argumentation in generalization activity. In this study these algebraic artifacts included algebraic expressions and equations, function table and the metaphor of a function machine. However, the students' use of other semiotic means such as natural language and elements of arithmetic discourse, e.g. factoring, rules-of-operations, calculations, arithmetic expressions, was found to be central to the student meaning making and contributes to enlighten our understanding of a genetic relationship between arithmetic and algebra in school. Further, an investigation of the role of inscriptions such as a task figure and a student drawing, and the use of manipulatives and gesturing, including the linking between the very different semiotic means listed above, shed light on mathematics as ultimately a form of sensuous cognition (Radford, 2010; 2018). An interest of the research has also been to further develop and refine approaches to investigate algebra teaching and learning, including the analysis of classroom video-data.

The use of different semiotic means and the linking between them have been studied in terms of both the production of subjective and collective meanings, as these processes intertwine in the classroom (Radford, 2002; Saxe et al., 2015; Sfard, 2008). In order to scrutinize these processes in the elementary algebra classroom, it was found useful to incorporate perspectives of different timeframes. In Study 2 a coordination of Radford's (2000, 2002, 2003) theory of knowledge objectification and Sfard's (2008) commognitive perspective was found useful to scrutinize the students' discourse both from a microgenetic and an ontogenetic perspective. These theories provided necessary theoretical constructs and framings to investigate student subjective meaning making in joint activity in a small-group setting (8-15 minutes). In Study 3 the goal was to complement the previous empirical study by investigating the production of collective meanings in teacher led whole class discussions over the course of four lessons, i.e. a study of sociogenetic processes in the classroom. In accordance with a sociocultural perspective on learning, a framework was developed that proposed two theoretical constructs as useful for investigating such a process: *participants' positioning* and *attunement to others*. However, foundational in this framing is the construct of classroom norms (Cobb et al., 2001). Thus, a central feature of this thesis, in addition to empirical investigations, is the coordination of related theories in the field of mathematics education (Prediger, Bikner-Ahsbahs & Arzarello, 2008). In this research, Saxe's (2012) genetic approach to cultural development of mathematical ideas provided the grounds for a well-formulated coordination of the different theories as discussed above.

The results of this study regarding processes of learning [specific learning outcomes are not considered, cf. section 1.2] are seen as products of both the theoretical approach taken and the empirical data investigated. In order to scrutinize discourse in a microscopic fashion as explained above, it was deemed necessary and fruitful to limit the data to only include that of classroom patterning activity. The focus of this research has been algebraic thinking in processes of generalizing numeric and geometrical sequences. A particular strength of the study is that it involved 16 groups from four different countries. Thus, the studies provide a diverse and relatively large empirical material for the in-depth discourse analyses. Commonalities among the classrooms are that they are situated in institutional and cultural contexts in which: (1) numbers and their operations are considered to form a knowledge base from which to learn algebra; and (2) reform efforts regarding algebra in school, and more generally the democratization of the mathematics classroom, to varying extents, are part of the educational agenda.

The findings are presented in terms of addressing the different aims and the four research questions presented in section 1.4 and form four subsections (6.1.1-6.1.4). For the sake of clarity, the findings are presented in 11 paragraphs

(F1-F11) across these subchapters. Each subchapter is concluded with a discussion of the findings listed.

6.1.1 The nature of students' arguments and the role of cultural artifacts

With respect to the nature of student generalizing arguments, and the role of cultural artifacts and other semiotic means (cf. research questions 1 and 2, in section 1.4), the analyses show that the arguments produced are closely linked with the semiotic means elicited or made available in the tasks and through the instructional activity.

F1 In classroom explorations of numerical sequences students mainly used their knowledge of numbers and operations in their generalizing acts (cf. Study 2). However, some arguments included visual aspects such as repeating last digits in the numbers in the sequence. Students produced recursive arguments, by determining a common difference between the numbers in a sequence, and explicit arguments, by identifying the numbers in the sequence (multiples of four). Students initially did not produce explicit generalizations when working with a non-proportional structure, but as they engaged in teacher led discussions, involving the use of a function table, the students increasingly formulated explicit generalizations regarding this type of pattern. These were generated inductively, by trial-and-error using the function table, or deductively, by deducing the coefficient from the recursive relationship and then using the function table to determine the second operation. The explicit generalizations were determined by creating several arithmetic expressions (orally or written), in which the same operations were applied and checked regarding several pairs of numbers (as modeled by the teacher).

F2 In order to investigate students' meaning making in patterning activity, as a process of knowledge objectification (Radford, 2002), it was found necessary to include multimodal elements in transcriptions such as inscriptions, concretes and gestures (cf. Study 1). In classroom exploratory work with geometrical sequences, contextual and situated rates of change were central to the students' developing generalizations. As they engaged with the geometrical figures, using manipulatives, drawing, or gesturing, they got a sense of how the figure physically evolved. Although the way it changed was quantified, and later transformed to a coefficient, it retained the meaning of a factor by which change occurred. This sensuous understanding of how the task-figure evolved, often signified by gestures, was crucial to students' problem solving regarding the matchstick problem. Similarly, students' exploration of the triangle task (Task 5, Study 3) led to a lengthy discussion involving arguments for different rates of change, including constant and rising rates. In the continued discussion, an argument that visualized a new row coming out underneath the bottom row of the previous figure to make the next figure in the sequence was favored and continued. Here students also specified a constant second rate of change. Thus, the students' exploration revealed an aspect of the figural pattern that could be used to point out the mathematical structure involved (quadratic). However, it was only used to explain the recursive relationship between the numbers in the right column of the function table.

F3 Students, in most cases, were not able to make a function table on their own but participated in filling in the ones provided in the tasks or developed by the teacher (cf. Study 3). In the study, one student explained the numbers in a function table in terms of the geometrical context, including explicating the recursive relationship between consecutive numbers in the right column of the table. However, as the classroom moved to making an explicit generalization, the same student used a guess-and-check strategy in which a connection was not explicated between the explicit expression and the previous explorations of the geometrical sequence. The same disruption between empirical explorations, including subsequent recursive generalizations, and the use of a function table to determine explicit expressions, was observed repeatedly as the students increasingly managed parts of the classroom patterning activity (cf. Task 5, Study 3). Further evidence of student challenges in setting up a function table, and a somewhat mechanical use of it, is evident from the fact that students in the Californian 6th grade classroom did not make a function table in response to the matchstick task (cf. Study 2). As students did not refer to the context of the geometrical sequences when arguing for their explicit expressions, these generalizations are seen to be mainly based on numerical schemes for the problem situations. Importantly, a shift in student awareness from a focus on recursive relationships in the sequences to a focus on explicit relationships occurred when using a function table in the classroom. Thus, the function table appeared as a strong visual mediator of a cultural form of thinking, i.e. it served as a means of pointing to a functional relationship between two sets of numbers in the elementary algebra classroom. Students increasingly made expressions involving two operations using this cultural artifact. Thus, they became familiar with a non-proportional linear relationship. It is more difficult to determine the role of the metaphor of a function machine as it was only referred to by students (and teacher) in the first discussions (Tasks 1-2, Study 3). However, initially students argued using this artifact in a direct manner, i.e. in terms of a number going in and being transformed into another number that comes out. Moreover, they applied both recursive and explicit argumentation when explaining what occurs in a function machine in the specific context when only the explicit argument is appropriate.

F4 Students did not use algebraic syntax until the very end of their generalization processes (cf. Study 3). Students created variable expressions modeled on previous developed arithmetic ones and sometimes also in their image, i.e. $n \cdot n = x$. The students made meaning of these mainly in terms of arithmetic, referring to order of operations, calculations, or conventions such as $3n=3 \cdot n$. The letter used in an expression was mainly referred to as any number and talked about as a specific number that can be operated on. Thus, student use of the artifact of algebraic expression did neither appear to alter student argumentation as with the function table nor lift the generalizations above the immediate context in the way that naming did (multiples of four, squared numbers). Rather, the students engaged with the symbolism in terms of initial familiarization. However, regarding student work with the matchstick task (Study 2), one group from Sweden started the solving process by making an equation on advice from the teacher. They identified the unknown and set up the equation $3x=73$, assuming a direct-proportional relationship. In comparison to the other groups in the study, the use of the cultural artifact of equation altered how students argued regarding this problem. The students initially approached the problem in terms of mathematical structure rather than processes of drawing or calculating as did the other groups. The initial solution included only decontextualized arguments and the students did not reject a solution that included decimals. This contrasted with other groups that set up arithmetic expressions based on direct proportion (73/13, 73/4, 73/3), who rejected a decimal number as a solution. While the other groups managed to solve the problem by creating momentary and contextual objects such as numerical patterns and rates of change, the equation had permanence throughout the Swedish group's solving process as it was written down, returned to and changed (*3x=72*). Thus, it remained an anchoring point for how to think about the problem. However, as was the case in the other groups, they struggled to mathematically model the irregularity.

F5 The students dealt with indeterminate objects and generalization in several ways with or without the use of algebraic artifacts (cf. Study 2 and Study 3): (1) they formulated expressions based on a generalization of the relationship between different quantities in a problem; (2) they developed in-action-formulas in which indeterminacy was present through some of its particular instances; (3) they developed explicit generalizations using a function table touching on indeterminate quantities in terms of looking at a set of numbers and considering *all of them* at once; (4) they created an equation in which the indeterminate number (the unknown) was materialized using a letter; (5) they created algebraic expressions in which the letter was addressed as *any number* and talked about in terms of a generalized number; and (6) they used the algebraic expression to predict far elements of the sequence.

Algebraic thinking has been defined in terms of dealing with indeterminacy in analytical activity (Radford, 2010). The findings show that the students engaged with indeterminate objects explicitly and implicitly in the classroom patterning activity (F5). The object of variable is complex and has been discussed in terms of students' uses of a letter at six levels of rising sophistication (Küchemann, 1978) and in terms of having many possible roles and meanings according to the context where it is employed (Usiskin, 1999). The students made indeterminate objects material using words and algebraic syntax. They used indeterminacy in terms of replacing indeterminate objects for a given value predicting far elements of a sequence (level 2), as a specific unknown in equation (level 4) and addressed it in terms of a generalized number in algebraic expressions (level 5). Indeterminacy was also materialized in terms of using the words *all of them* (the numbers in a function table) in the context of determining a functional relationship. Here the indeterminate object takes on the meaning of variable, i.e. it takes on a set of values (Usiskin, op. cit.). These initial encounters with indeterminacy in joint classroom patterning activity are seen to offer opportunities for meaningful learning regarding the object of variable.

Deductive forms of reasoning are central to the development of algebraic thinking (Mason, 1996; Radford, 2008), while empirical explorations are important to student meaning making in algebra (Carraher et al., 2008). The findings show that students' explorations of numerical patterns mainly involved the noticing of a common difference between consecutive members while also including an identification of numbers in the sequence (F1). Their exploration of geometrical sequences elicited the use of a much wider range of semiotic means through which the notion of rate of change became prominent (F2). Radford (2008) emphasizes that the explicit expression needs to be deduced from a previously determined commonality over the members in a sequence. Commonality in terms of: (1) a common difference; (2) number identification (multiples of four, squared numbers); or (3) a rate of change, offer different possibilities for generalizing. This implies doing it: operationally (repeated addition to multiplication); structurally in terms of number structure; and function structure (Kieran, 2018). The latter two facilitate generalizing beyond the problem context, as emphasized by Mason (1996). Regarding direct proportional relationships, students often presented deductive argumentation. While working with more complex relations they often confused or disregarded a connection between recursive and explicit expressions, or a connection between explorative observations and explicit generalization made when using a function table. The latter has been described as generalizing based on a numerical scheme (cf. Lannin, 2005). Lannin found that students tended to use empirical examples to verify these generalizations. Students developed both inductive strategies, such as guess-and-check, and deductive ones involving functional argumentation, when using a function table to determine explicit expressions. However, students mainly did not connect these generalizations to previous empirical observations done while exploring a sequence.

The analyses show that the students' use of a function table in teacher led whole class patterning activity altered their initial focus on recursive relationships to a focus on explicit ones (F3). Also, the use of an equation to solve the matchstick task appeared to alter the solving process in several ways: the group approached the problem structurally, used decontextualized arguments, and mathematical objects acquired permanence (F4). The students created variable expressions modeled on previously developed arithmetic expressions.

The use of algebraic syntax as a final generalization act did not appear to increase the sophistication of the generalization as the focus remained on an operational relationship (F4). Interestingly, many students were able to create these. However, the discussions regarding these remained at a level of attempting to become familiar with the symbolism. Radford (2018) argues that the novelty for students in elementary algebra learning is not the alphanumerical signs introduced, but rather that the move to symbolic thinking requires a reconceptualization of numerical operations. He found that students' symbolic thinking in patterning activity evolved simultaneously with students' ability to mathematically perceive and imagine structure and generality. The findings of this study support these conclusions and show how students, in their explorations of patterns, touch upon relevant and useful objects for generalization such as rate of change, number structure and operational structures (cf. Kieran, 2018). These structural aspects of generalizing, if given attention in instructional activity, can support student development of algebraic thinking, including their symbolic thinking. As in previous studies (Küchemann, 1978; Caspi & Sfard, 2012), the findings indicate that the idea of equation is more readily available to students than that of variable expression. Nevertheless, several of the VIDEOMAT classrooms engaged in equational activity prior to the fifth lesson, but students did not use equation when left to themselves. Thus, as with the variable expressions, the challenge of using equation in problem solving is not only the symbolism but also the ability to recognize a problem as equational in its structure.

6.1.2 The genetic relationship between arithmetic and algebra

With regards to a genetic relationship between arithmetic and algebra in school the analyses show that students in elementary algebra participate in algebraic activity drawing largely on their experiences in arithmetic (cf. the aims of the thesis, section 1.4).

F6 The students' arithmetic experiences, which may be limited in some respects (cf. Booth, 1984, 1988), involved both challenges and opportunities for learning in algebra. The analyses showed that students' close association between multiplication and direct proportion presented a challenge when participating in the patterning activity. This limited use of multiplication made it difficult to envision a non-proportional linear relationship (cf. Study 3). Another challenge discerned was the students' focus on calculational processes rather than structure when attempting to determine a relationship between quantities. This processual approach to problems made it difficult for students to handle more complex relationships. In particular, the students struggled to mathematically model the irregularity of the pattern when working with the matchstick task. The students' focus on finding an answer or on coming up with a concrete outcome of the activity, which has been associated with arithmetic in school (cf. Booth, 1984; Kieran, 1981), contributed to a fragmentation of meaning making in the classroom generalization of sequences (Study 3).

F7 Students' acts of numerating and quantifying, and fluent movements between operations (addition/multiplication, multiplication/division), form the basic elements of students' generalizations in patterning activity (cf. Study 2 and Study 3). Their recognition of familiar numbers and application of operations show developing structural views of numbers and operations. Students formulated oral granulated expressions. The ability to do so let them model complex relationships (including two different operations relating different quantities), affording students a larger, although momentary, perspective on a problem. A student referred to rules of operations when addressing a variable expression. This is a way of making meaning of the variable expression that does not significantly separate these objects, i.e. the same operational rules are seen to apply to both arithmetic and algebraic expressions. Students' use of a factoring discourse in the patterning activity included a structural perspective on a pattern generalization (number structure). One student also used a factoring discourse to compare the two first sequences in terms of number structure, which afforded him to see the two sequences as different in terms of their elements. The students' discussions of situated and contextual rates of change when exploring geometrical sequences, offered opportunities for including a structural perspective (an all new function structure). These forms of structure sense (cf. Kieran, 2018) could lift pattern generalizations above their immediate contexts. Although the students touched upon important mathematical objects, such as indeterminacy and rate of change, in their use of an informal meta-arithmetical discourse in patterning activity, the generalizations made were mainly of a contextual and situated nature and

> the discourse was sometimes ambiguous, thus creating communicative challenges.

Caspi and Sfard (2012) define algebra in school as a meta-arithmetical discourse that people employ while reflecting on numerical processes and operations. The analyses in this research confirm that students in different school systems, 11-13 years old, develop an informal meta-arithmetical discourse. Thus, the analyses further map the nature of this type of discourse (F10). Prominent features of this discourse such as granulated expressions and contextual and momentary algebraic objects afforded means for structuring and approaching unfamiliar mathematical relationships. Thus, student engagement in arithmetic over time involves a recursion of discourse, which when elicited in patterning activity can become algebraic. Several of the student challenges in engaging in algebraic generalization (F6) were related to the change in meta-rules of discourse (cf. Sfard, 2008) as one moves from arithmetic to algebra. That is, the meta-rules change with respect to what is central to attend to in the mathematical activity, for example algorithmic and calculational processes in arithmetic versus analysis and reflection in algebra, and intended outcomes of the respective activity, numerical answers versus algebraic expressions. It was shown that although the teacher made explicit important elements of a generalization practice in the classroom patterning activity, students mainly focused on tangible outcomes such as arithmetic and variable expressions as they increasingly managed the activity.

Kieran (2018) argues that students' familiarity with arithmetic structures can facilitate the emergence of algebraic thinking. The students in my studies drew on: (1) operational structure, i.e. moving fluently between addition and multiplication including its inverse division; (2) structure according to factors, i.e. using a factoring discourse to describe differences between sequences; (3) number structure, i.e. multiples and squared numbers; and (4) multiplication structure, i.e. proportional relationship. Regarding the latter structure, my analyses showed that students mainly associated multiplication with direct proportion. This appeared as a considerable student challenge in the classroom patterning activity across classrooms. Stacey (1989) similarly identifies issues related to students' application of direct proportion in patterning activity:

> The models associated with direct proportion suggest themselves to students for strong cognitive reasons. When such an idea is found,

> students may be reluctant to question it, both because its effectiveness in supplying answers (and any answer is better than none!) and because of its simplicity (op. cit., p. 162).

Students increasingly formulated expressions modeling non-proportional linear relationship using a function table. Thus, they expanded their use of multiplication. The students used the previously explored arithmetic structures as means for structuring and generalizing geometrical sequences allowing for algebraic objects (patterns, contextual rates of change, complex oral expressions) to emerge in the discussions. Thus, the analyses shed light on how spatial, arithmetic and functional structures may interact in student patterning activity (Kieran, 2018).

6.1.3 Characteristics of classroom processes of meaning making

Regarding characteristics of classroom processes of meaning making in the elementary algebra classroom (cf. research question 3, section 1.4), the analyses address microgenetic processes (Study 2) and a sociogenetic process (Study 3). This research did not have data to support an analysis of individuals' development over time (ontogenesis). However, employing the framework of five levels of meta-arithmetical discourse developed by Caspi and Sfard (2012), an ontogenetic perspective on the nature of students' argumentation was incorporated (Study 2).

F8 The analyses of moment-to-moment meaning making in a small-group problem-solving context, across different groups of four countries solving the same task, show that although each group process had unique features, the similarities between the groups allowed for juxtaposing and synthesizing the discursive processes as they evolved (Study 2). The results showed how a discourse builds on itself, from talk about concrete objects such as matches and figures, to talk about numerical sequences and contextual rates of change, and finally to talk about complex expressions. Thus, this group work showed how students tailored numbers and operations as well as other semiotic means to serve new, algebraic functions. Also, it was shown how one group used an equation (a new form) to materialize an indeterminate object and model the matchstick problem. Additionally, the analyses of the microgenetic interactions in the 6th grade classroom showed how students used: (1) a factoring discourse to formulate explicit generalization or to compare two numerical sequences regarding number structure; (2) an operational discourse to create recursive and explicit generalizations; (3) a calculational discourse to make meaning of algebraic expressions; (4) an operational and calculational discourse to discuss different rates of change informally; (5) the function table to make recursive generalizations, inductive and deductive explicit generalizations, and touched on a functional relationship between sets of numbers; (6) gestures and visualization to determine rates of change; (7) granulated expressions to model complex relationships; (8) inscriptions to handle the irregularity of a pattern.

F9 A collective meaning making process across four lessons in one elementary algebra classroom was investigated in terms of the emergence of a classroom generalization practice (CGP, cf. Study 3). The CGP was seen to emerge through participants' acts of re/positioning and through the attunement to others. These intertwined as participants sought to achieve social goals, such as becoming full participants in the CGP and achieving communicational coherence. Attunement took place in the immediate classroom conversation and over time. Three different levels of achieving inter-comprehension in conversations between the teacher and her students were discerned. Level 1 comprised a tension in the conversation that occurred as interlocutors did not address the same objects (number structure versus operations); Level 2 comprised conversations in which interlocutors addressed the same objects, although from different perspectives. For example, they addressed an expression from a functional versus a calculational perspective; and Level 3 comprised conversations in which interlocutors addressed an object from the same perspective, using informal versus formal discourse. It was at the first level that interlocutors were acutely aware of their communicative challenges. It was conjectured that if inter-comprehension was pursued and the tension resolved, this situation could be productive in terms of offering new insights (for all participants). The second level occurred more frequently and seemed to pass by without participants' noticing the difference in perspectives applied. The third level also occurred several times, in which the teacher pointed out mathematically important aspects using specific terminology. It was conjectured that this type of situation may lead to an increase in student awareness. As students repositioned themselves and increasingly managed the different phases of classroom activity, the students voiced their subjective meaning making, which amounted to conversations that achieved different levels of inter-comprehension. The occurrence of a discourse involving tensions discouraged the use of a factoring discourse, and structural forms of argumentation fell away. A variation in forms of explanation and argumentation was accepted (Level 2 conversations) and both inductive and deductive strategies for generalizing patterns developed. Participants' use of a function table over time appeared to alter students' awareness, from focusing on a recursive relationship to focusing on an explicit one, the latter sometimes involving a functional discourse (Level 2 and Level 3 conversations). The multiplication structure (operational) is expanded to include non-proportional relationships. The making of a variable expression as an outcome of activity and the use of function table seemingly lead to an increased focus on tangible outcomes rather than on the generalizing process itself (level 2 conversations). Thus, the investigation of participants' attunement to others offered an explanation for the uptake and discontinuations of forms of argumentation.

F10 The findings regarding the nature of the students' discourse, applying an ontogenetic perspective on the development of algebraic thinking (cf. Study 2), showed that it mainly had characteristics of processual and granular levels of informal algebraic discourse. However, the group that employed an equation in their solving process did at times operate on an objectified level. Three aspects of the discourse were looked at to map out the three levels of constant algebraic discourse: (1) the mathematical models created, i.e. the in-action-formula was at a processual level, the granulated expressions was part of a granular level, while the equation was at an objectified level; (2) how relations were generalized, i.e. the listing of numbers 4 3 3 3 or use of contextual utterances such as *four in one* and *three in each* were at a processual level, an operational discourse regarding expressions were at a granular level, identification and naming in response to qualities of the structure or numbers involved was at an objectified level; and (3) how emergent objects of an algebraic nature were evoked in the problem-solving process, i.e. indeterminacy was present in the in-action-formula through some of its particular instances at a processual level, numerical patterns and a rate of change was at a granular level, while a letter as an unknown in the equation was at an objectified level.

The analyses of microgenetic processes in early algebra classrooms show how students used algebraic artifacts (new forms) and other semiotic means (familiar forms) in continuation with previous uses but also to serve new algebraic functions (F8). Saxe et al. (2015) point out the continuation and alteration of form-function relations as an intrinsic quality of the microgenetic process and show how some new form-function relations concerning integers and fractions are created in classroom activity. However, the in-depth discourse analysis done in this research fleshes out this process and points out the role of different semiotic means and the linking between them, as well as attempting to explain how the students discourse became increasingly reflective and abstract.

It was shown that the growth of the discourses of the groups could be explained by reference to Sfard's (2008) three discursive processes, i.e. by reification, saming and encapsulation (cf. Figure 3). However, these do not only consist of language use but can occur through other modalities (Study 2). For example, the same gesture takes on different meanings at different times in the solving process. It is reified as it goes from initially signifying the physical adding on of another square in the figure, to signifying a constant rate of change, and finally signifying the coefficient three. Thus, reification is not restricted to language. However, an additional discursive process was identified. That is, a semiotic node, i.e. a piece of discourse in which a range of semiotic means work together to achieve objectification (Radford et al., 2007). These discursive processes are characteristics of a microgenetic process of meaning making between peers in a problem-solving group setting that is inductive in nature, i.e. the discourse evolves from talk about concrete objects to talk about abstract ones. The analyses show the multimodal and sensuous character of such a process in elementary algebra patterning activity, which has also been shown in a series of studies Radford et al. (op. cit.), Radford (2010) and Radford (2018). The in-depth microgenetic analyses done in elementary algebra patterning activity in Study 2 complement and inform the analyses of the 6^{th} grade classroom in Study 3. The analyses further explicate how the specifics of tasks, involving text, figures, numbers, etc. and the way a task is introduced in the classroom, including the use of cultural artifacts, elicit student use of different semiotic means and were linked to the mathematical ideas the students come to engage with.

Regarding a collective classroom process of meaning making over the course of four lessons, the analyses show how the teacher and students reach different levels of inter-comprehension in conversations (microgenetic constructions) and point out possible implications for the continued emergence of the CGP. Cobb et al. (2001) point out how the students' subjective meanings came to the fore when meeting novel problems and were important for initiating conceptual discourses, often requiring intervention from the teacher. It can be argued that the situations occurring between students and the ones occurring between the teacher and the student offer different opportunities for learning. Particular to the teacher-student interaction is that students quickly accept the teacher's arguments and often does not willingly present objections or alternatives. However, if the teacher aims for intersubjectivity there is also opportunity for clarification of ambiguities as well as formalization, regulation and symbolization of discourse (cf. Study 2). Thus, as this research focuses on teacher-student interactions the findings of this study complement the findings of Cobb et al. (op. cit).

The analysis of the student discourse from an ontogenetic perspective shows its meta-arithmetical nature, corroborating the findings of Caspi and Sfard (2012). It was shown that the discourse was mainly of an informal processual or granulated nature. Opportunities and limitations for students' participation in algebraic activity were pointed to (F10). These findings were used to determine critical elements of the classroom discourse both regarding micro- and sociogenesis. Thus, recursion as a characteristic of ontogenetic development in algebra was incorporated into the classroom analyses.

6.1.4 Investigating classroom interactions in elementary algebra

To address the issue of how to investigate classroom interactions to achieve a comprehensive understanding of learning in elementary algebra (cf. research question 4, in section 1.4), a genetic approach (cf. Chapter 3), in terms of coordinating related theories, has been utilized in this research.

F11 The genetic approach addresses meaning in terms of both subjective and collective versions and involve different time perspectives on processes of meaning making, i.e. micro-, socio- and ontogenesis. Saxe' (2012) conceptualization of cultural development of mathematical ideas as shifts in form-function relations was seen as especially useful when investigating learning processes in elementary algebra as a major shift in both form and function of the mathematical discourse is expected to take place in the classroom. Radford's (2000, 2002, 2003) cultural-semiotic theory of knowledge objectification was employed to analyze the emergence of new form-function relations. This theory was found to have a well-developed framework to account for and analyze a wide range of semiotic means, and the linking between these in micro-genetic processes of meaning making. In order to employ an ontogenetic perspective on the students' discourse, Sfard's (2008) commognitive theory was incorporated. The conceptualization of different discourses as different ways of thinking, and her focus on *word use*, afforded a wider perspective and facilitated a different dissection of the students' discourse into larger units (arithmetic – algebraic, factoring – calculational, processual – objectified). Caspi and Sfard (2012) also provided an operationalization of a process-object (recursive) approach to algebra learning. In order to address collective meaning making in terms of a whole-class process across lessons, the idea of classroom norms developed by Cobb et al. (2001) was incorporated. This was useful in two respects: (1) it added a perspective on mathematical practices as emergent in classroom interactions (to the view of mathematics as a culturally and historically established practice) seen as specific to the mathematical ideas discussed; (2) it afforded an appropriate unit, in terms of grainsize, for investigating whole-class processes of meaning making. That is a classroom generalization practice involving purpose of activity, normative ways of arguing and using semiotic means. However, in order to investigate a sociogenetic classroom process from a socio-cultural perspective, a Bakhtinian view of interactions was incorporated, and the constructs of participants' positioning and attunement to others were developed. Thus, the development of the genetic approach involved a scrutinization of relevant and related theories in terms of their affordances and strengths. In terms of algebra learning, much work has been done regarding micro- and ontogenetic processes of meaning making. However, little is known concerning collective meaning making in whole-class interactions across lessons. The approach shows that addressing interactions, concerning the same mathematical content, applying the three interrelated perspectives, reveal different issues of algebra learning. The microgenetic analysis reveal the multimodal and sensual nature of algebra learning and the linking between different semiotic means in these processes. The analysis of the sociogenetic process highlights challenges in student-teacher communication as a formal algebraic/functional discourse is not yet available to students as they explore and make observations in patterning activity. Applying an ontogenetic perspective as developed by Caspi and Sfard (2012) allows for a mapping of the students' discourse in terms of levels of algebraic discourse (informal and formal versions) and points to the role of cultural artifacts in achieving an objectified level.

The approach to analyze classroom interactions in elementary algebra developed in this research is comprehensive as it addresses micro-, socio- and ontogenetic meaning making processes (Saxe, 2012). The first main contribution is that it shows how related theories (theory of knowledge objectification and commognition) can be employed to better understand learning in algebra. The second is that it pushes the quest for understanding the role of the classroom microculture in learning, an element that has largely been ignored in socio-cultural theory (Saxe et al., 2015). The analysis of collective meaning making reveals issues of algebra learning that cannot be enlightened from the other genetic perspectives (F11). Regarding micro- and ontogenetic processes in elementary algebra, the analyses confirm and connect previous developed theory and corroborate and add to previous findings, particularly regarding students' meta-arithmetical discourse. However, in order to investigate a collective process of meaning making, two constructs were developed: participants' positioning and attunement to others.

Saxe et al. (2015) conjecture that two socio-mathematical norms regarding the use of definitions and students' social position contributed to the distribution of a particular form-function relation over time. Study 3 did not account for distribution, rather it documented student re-/positioning themselves in public discourse and increasingly managing phases of classroom activity. Here the attention was on students' use of cultural artifacts and other semiotic means to accomplish these communicational acts. This resulted in analyses explicating student mathematical meaning making that can inform instructional practice (cf. Study 3). Attunement to others was constructed to deal with the spread of ideas in terms of uptake and continuation of arguments. It was argued that the levels of inter-comprehension achieved in teacher-student interaction had implications regarding the spread of ideas. Thus, I propose participants' re-/positioning and attunement to others as central aspects of a sociogenetic classroom analysis.

Another advantage point of the genetic approach as formulated in this research concerns agility. In contrast to similar approaches, i.e. Cobb et al. (2001) and Saxe et al. (2015), that center such analyses on the relationship between individual and collective development in one classroom, the genetic approach proposed here suggests that these genetic processes can be studied separately by foregrounding one and backgrounding the others, not only in the same classroom but also by studying different classrooms. This, I argue, is made possible by replacing the focus on the individual and social relation with the cultural development of one central mathematical idea. However, possibilities for such analyses outside of algebra are not explored here. The analyses presented by Saxe et al. are interesting but very complex and include so many components that the students' mathematical meaning making to some extent drown in more technical details and analysis. In contrast, the analyses in Cobb et al. (2001) bring forth the students' mathematical meaning making. However, it is still limited to simultaneous analyses of genetic processes in one classroom.

6.2 Limitations of the research

This research has been located within the interpretive paradigm in which the aims of research are to describe and explain the phenomenon observed (cf. section 4.1). Accordingly, learning processes and emerging student algebraic thinking in middle school classrooms have been described. The research also sought to advance theory to explain how these phenomena evolve in classroom interactions. The perspectives applied and the methods used in this research have bearing on the phenomena I saw and the conclusions I drew (cf. Cole, 2001). The use of video-technology has certain limitations. First, it has an angle (cannot replace a participant observer) and does not capture context (cf. section 4.2). Second, the presence of cameras and researchers in a classroom may alter classroom interactions. In the literature this is called *the Hawthorne effect* (cf. Sowder, 1998). The theoretical developments in algebra formulated by Radford (2002, 2010, 2018) and Sfard (2008) became lenses through which I looked at the data. Further, I take the view that the research presented here is a story carefully crafted in order to shed light on algebra learning in school and that there are alternative ways of telling it (Sfard, 2012). However, some stories make a better fit than others and a researcher "is always in the quest after new, more convincing versions" (op. cit., p. 7). Thus, the Bakhtinian principle of multivocality is invoked.

The classrooms studied are situated in educational systems that take arithmetic as a foundation for algebra learning and to varying extents these educational systems have implemented new approaches to school algebra as well as new more democratized forms of instruction (cf. Chapter 2). In research there has also occurred shifts regarding what is valued in algebra learning. For example, early algebra studies focused on the use of letters (Küchemann, 1978) or algebra as the study of solution methods (Filloy & Rojano, 1989), while Radford (2010) and Caspi and Sfard (2012) focused their studies on what they identify as algebraic thinking. The latter perspectives have yielded more positive views of student algebra learning. This does not necessarily mean that students have become more competent algebraists (although we certainly hope that is the case). Therefore, the conclusions drawn from my research are intrinsic to the historical and cultural context of both the classrooms studied and the perspectives on school algebra employed.

The qualitative approach to collecting data further limits the generalizability of findings. The classrooms participating in the VIDEOMAT project are a sample of convenience rather than a randomized sample. Thus, the conclusions drawn from the cross-case analysis (Study 2) point to general aspects of students' approaches to patterning activity but cannot claim representativeness. The findings drawn from the single case study (Study 3) are intrinsic to the classroom investigated (Wellington, 2000). This study points out issues concerning collective classroom development that might raise awareness and give rise to new investigations. However, the classroom process described may be quite unique, and the categories developed for investigating it are likely to undergo adjustment and refinement if applied in future studies. While the cross-case analysis draws on well-grounded theory, the single case study is a thrust into a somewhat unfamiliar territory in sociocultural research.

7 Implications

In this final chapter I consider how this research can contribute to the teaching and learning of elementary algebra. I also present some ideas and thoughts concerning how this research can stimulate new questions and future studies in school algebra. Implications for instructions are described in the first section of this chapter (7.1), while implications for further research make up the final section of this chapter (7.2).

7.1 Implications for instruction

The findings have several implications for instruction in elementary algebra. These include awareness concerning: (1) the shift in meta-rules of discourse; (2) the student need for expanding mathematical structures; (3) the use of cultural artifacts and the patterning tasks implemented in activity; (4) the limitations of inductive argumentations; and (5) the nature of micro-, socio- and ontogenetic processes in the elementary algebra classroom. The five points are discussed below.

Initiating algebraic activity in the classroom include a shift in meta-rules of discourse from that of arithmetic activity. A student awareness concerning this shift can be facilitated by a teacher focus on the generalization process in the classroom activity and by the teacher being explicit about the new meta-rules in the processes of solving tasks. Thus, teachers should avoid a student prescriptive approach to activity but rather facilitating student mathematical awareness (cf. Selling, 2016). This study highlights an emphasis on analytic and reflective activity, and as outcomes of this activity the generalizations made rather than tangible outcomes. However, engaging the students in verbally expressing the generalizations made using an everyday discourse is important. First in terms of becoming aware of indeterminacy and finding ways to address it. Second, expressing generalizations orally or in writing, can initiate discursive processes such as *regulation*, i.e. dissolving ambiguity by creating common forms of expression, *reification*, i.e. replacing talk about processes by talk about objects, and *symbolization*, i.e. the algebraic syntax is a superior form of expressing generality which comprises and standardize the discourse (Caspi & Sfard, 2012; Sfard, 2008). The teacher has a central role in identifying discursive entry points (cf. Study 2) and guiding the formalization of the students' discourse. The result of rushing to symbolization, as shown in Study 3, can lead to a student focus on determining the correct algebraic expression rooted in a somewhat superficial treatment of content, rather than encouraging student analytic activity and mathematically rich meaning making.

Mathematical structures and means for structuring appear in this study as central to the teaching and learning of elementary algebra in patterning activity. The study shows that the middle school students had begun to form number and operational structures. Indeed, these form the bases for their generalization activity. However, these may be limited in some respects and thus hinder student meaning making. Particularly, it appears necessary to find ways of addressing the close association between multiplication and proportionality that students have been found to hold (Study 2, Study 3, Stacey, 1989). Thus, teachers need to work to expand the multiplicative structure to include also non-proportional relationships. The students flexibly used the relationship between operations, e.g. addition/multiplication and multiplication/division. However, the observed student challenges in connecting recursive and explicit generalizations suggest that this flexibility is limited to calculational processes (cf. Study 3). Thus, instructional activity should aim at supporting a student reconceptualization of operations from a processual and numerical understanding to a structural and quantitative one (cf. Caspi & Sfard, 2012; Radford, 2018).

Students identified a commonality over members of a sequence in terms of operations, number identification and rate of change as a first step in a generalization process. These offer different possibilities for generalizing (cf. 6.1.1). Furthermore, it was shown that in the context of exploring geometrical sequences students engaged informally with rates of change, while working with numerical sequences students employed a factoring discourse (previous classroom topic). Student use of a function table involved a decontextualization of generalizations, a focus on explicit generalization and sometimes students touched upon a functional relationship. This implies that teachers need to consider carefully what mathematical ideas they aim to pursue in patterning activity and the means to do so in order to avoid mismatches in communication (cf. Study 3). Important to include in these considerations are: (1) the topics leading up to classroom patterning activity; (2) the choice of tasks in terms of semiotic 'ecology', i.e. semiotic content of the task itself and the semiotic means likely elicited in student activity (drawing, visualization, etc.); and (3) whether or not to introduce algebraic artifacts. In taking student meaning making as central to the generalizing activity, number structure and an all new function structure appear to be in reach for students and offer possibilities for generalizing beyond the immediate context.

The findings indicate that justification in patterning activity is not only important in itself, but also in terms of student meaning making processes. Students developed inductive approaches to determine explicit generalizations using a function table (Study 3). These generalizations appeared as separated from the students' previous explorations of geometrical sequences. Indeed, these seemed to be based only in a numerical scheme. Thus, the explorative phase and the phase of determining an expression remained separate. First, justification based on a few examples is not mathematically valid argumentation for the generalization made. Second, by insisting on deductive argumentation and justifications based on the nature of the evolving geometrical sequence, the students can engage in rich meaning making involving the generalization of a relationship between two quantities.

Finally, findings regarding the nature of the different genetic processes in the algebra classroom can inform instructional practice. The findings concerning the microgenetic process show how a sensuous understanding of a pattern can form the foundation for students' generalization. Thus, encouraging student exploration of sequences may offer access for students into the complex cultural practice of making algebraic generalizations. The research also shows how students struggle to keep track of different quantities and to determine the relationship between them, experiencing insecurity and lack of confidence in approaches developed (cf. the Norwegian group in Study 2). Therefore, teachers need to be sensitive to student meaning making processes in order to support students in reaching a conclusion. The details shown of how a discourse build on itself may inform teacher intervention in the microgenetic process. Findings regarding a sociogenetic process relevant for instruction are the possible impact and outcomes of tension in communication. If resolved it offer opportunities for learning both for the teacher (about student meaning making) and the students (about teacher and cultural meaning making). However, if it is not resolved it may have negative consequences for student meaning making. Similarly, mismatches in communication may limit student meaning making in the patterning activity. The mapping of students' informal meta-arithmetical discourse in terms of an ontogenetic trajectory can prepare teachers for making meaning of student argumentation in the classroom and to identify discursive entry-points, i.e. opportunities in discussions for supporting student algebraic thinking both in an microgenetic and an ontogenetic perspective.

7.2 Implications for research

The research points out that number structure and function structure offer opportunities rooted in students' meaning making processes to support student generalizations processes in elementary algebra. However, ways to implement these structures in instruction to support student development of algebraic thinking need to be developed. Furthermore, how students engage with these structures and in which ways these contribute to their generalizing processes need to be investigated.

The research shows that student engagement in patterning activity involves an enlargement of operational structures (e.g. non-proportional) and number structure (polynomials). Usiskin (1999) argues that it is not possible to study arithmetic sufficiently without algebra. In order to support student mathematical meaning making it seems important to investigate in which ways the growth of one discourse can support the growth of the other one.

The findings further map students' meta-arithmetical discourse as identified by Caspi and Sfard (2012). This discourse was mainly at a processual or granular level of informal algebraic discourse. Student use of an equation to solve a patterning task was interpreted to partly be at an objectified level. Furthermore, the student use of an equation altered the problem-solving process and, in some ways, distanced the solution from the problem context. Thus, the use of algebraic artifacts seems important for student development of an objectified discourse, while simultaneously there is a possibility for a decrease in student meaning making. Questions for further research is how to negate student loss of meaning when engaging with the algebraic syntax; and to what extent this can be achieved through a formalization (regulation, reification, symbolization) of the meta-arithmetical discourse.

The nature of student micro- and ontogenetic processes in algebra and how these evolve are theorized and grounded in literature. However, the theorization of a sociogenetic process in algebra lags behind. The utility and viability of the constructs *participants' positioning* and *attunement to others* need to be confirmed; and if accepted these constructs need to be adjusted and refined through new studies. Similarly, the identified levels of inter-comprehension achieved in teacher-student interactions and how these conversations shape the collective process as well as offering opportunities for subjective meaning making need further investigations. Finally, how can the genetic approach be implemented to investigate learning in other areas of school mathematics; and, thus, stimulate further explorations of sociogenetic classroom processes from a sociocultural perspective?

8 List of references

Atkinson, P., & Hammersley, M. (1994). Ethnography and participant observation. In N. Denzin & Y. Lincoln (Eds.), *Handbook of qualitative research* (pp. 248-261). Thousand Oaks, CA: Sage.

Bakhtin, M. M. (1981). *The dialogic imagination: Four essays by M. M. Bakhtin* (M. Holquist, Ed.; C. Emerson & M. Holquist, Trans.). Austin, TX: University of Texas Press.

Bauersfeld, H. (1992). A professional self-portrait [Review of the book Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics, by D. A. Grouws]. *Journal for Research in Mathematics Education, 23*(5), 483−494.

Bednarz, N., & Janvier, B. (1996). Emergence and development of algebra as a problem-solving tool: Continuities and discontinuities with arithmetic. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 115–137). Dordrecht, Holland: Kluwer Academic Publishers.

Bednarz, N., Kieran, C., & Lee, L. (1996). *Approaches to Algebra: Perspectives for Research and Teaching*. Dordrecht, Holland: Kluwer Academic Publishers.

BERA. (2011). Ethical guidelines for educational research. Retrieved 12.04.2012, from http://www.bera.ac.uk/system/files/BERA%20Ethical%20Guidelines%202011.pdf

Bishop, A. (1985). The social construction of meaning: A significant development for mathematics education. *For the Learning of Mathematics*, *5*(1), 24-28.

Blanton, M., & Kaput, J. (2005). Characterizing a classroom practice that promotes algebraic reasoning. *Journal for Research in Mathematics Education, 36*, 412–446.

Blanton, M., Brizuela, B., Stephens, A., Knuth, E., Isler, I., Gardiner, A., Stroud, R., Fonger, N., & Stylianou, D. (2018). Implementing a framework for early algebra. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12-year-olds: The global evolution of an emerging field of research and practice* (pp. 27- 49). Hamburg: Springer.

Blanton, M., Stephens, A., Knuth, E., Gardiner, A. M., Isler, I., & Kim, J.-S. (2015). The development of children's algebraic thinking: The impact of a comprehensive early algebra intervention in third grade. *Journal for Research in Mathematics Education, 46*, 39–87.

Booth, L. R. (1984). *Algebra: Children's strategies and errors*. Windsor, UK: NFER-Nelson.

Booth, L. R. (1988). Children's difficulties in beginning algebra. In A. F. Coxford & A. P. Shulte (Eds.), *The ideas of algebra, K-12 (1988 yearbook)* (pp. 20-32). Reston, VA: National Council of Teachers of Mathematics.

Britt, M. S. & Irwin (2011). Algebraic thinking with and without algebraic representation: A pathway for learning. In: J. Cai & E. Knuth (Eds), *Early algebraization: Advances in Mathematics Education*, (pp. 137−159). Berlin, Germany: Springer

Bruner, J. (1990) *Acts of meaning.* Cambridge, MA: Harvard University Press.

Bryman, A. (2008). *Social Research Methods* (3rd ed.). Oxford University Press.

Carraher, D. W., & Schliemann, A. D. (2007). Early algebra and algebraic reasoning. In F. Lester (Ed.), Handbook of research in mathematics education (pp. 669–705). Greenwich: Information Age Publishing.

Carraher, D. W., Martinez, M. V., & Schliemann, A. D. (2008). Early algebra and mathematical generalization. *ZDM - The International Journal on Mathematics Education*, *40*(1), 3–22.

Caspi, S. & Sfard, A. (2012). Spontaneous meta-arithmetic as a first step toward school algebra. *International Journal of Educational Research, 51-52,* 45-65.

Chalmers, A. F. (1999). *What is this thing called science?* Buckingham: Open University Press.

Clarke, D. J., Keitel, C., & Shimizu, Y. (Eds.). (2006). *Mathematics classrooms in twelve countries: The insider's perspective*. Rotterdam: Sense Publishers.

Cobb, P., Stephan, M., McClain, K., & Gravemeijer, K. (2001). Participating in classroom mathematical practices. *The Journal of the Learning Sciences*, *10*(1), 113–163.

Cobb, P., & Bauersfeld, H. (1995). Introduction: The coordination of psychological and sociological perspectives in mathematics education. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning* (pp. 1−16). Hillsdale, NJ: Lawrence Erlbaum Associates.

Cole, M. (2001). Remembering history in sociocultural research. *Human Development*, *44*, 166−169.

Cole, M., & Gajdamaschko, N. (2007). Vygotsky and culture. In H. Daniels, M. Cole, & J.V. Wertsch (Eds.), *The Cambridge companion to Vygotsky* (pp. 193–211). Cambridge: Cambridge University Press.

Derry, S. J., Pea, R. D., Barron, B., Engle, R. A., Ericson, F., Goldman, R., Hall, R., Koschmann, T., Lemke, J. L., Sherin, M. G., & Sherin, B. L. (2010). Conducting video research in the learning sciences: guidance on selection, analysis, technology, and ethics. *Journal of the Learning Sciences*, *19*, 3-53.

Eckert, A., & Nilsson, P. (2017). Introducing a symbolic interactionist approach on teaching mathematics: The case of revoicing as an interactional strategy in the teaching of probability. *Journal of Mathematics Teacher Education, 20*(1), 31–48.

Edwards, A. (2007). An interesting resemblance. Vygotsky, Mead, and American pragmatism. In H. Daniels, M. Cole, and J. V. Wertsch (Eds.), *The Cambridge companion to Vygotsky*, (77–100). Cambridge: Cambridge University Press.

Ellis, A. B. (2007). Connections between generalizing and justifying: Students' reasoning with linear relationships. *Journal for Research in Mathematics Education, 38*(3), 194–229.

Engeström, Y. (2007). Putting Vygotsky to work: The change laboratory as an application of double stimulation. In H. Daniels, M. Cole, & J. V. Wertsch (Eds.), *The Cambridge companion to Vygotsky* (pp. 363–382). Cambridge: Cambridge University Press.

Filloy, E., & Rojano, T. (1984). From an arithmetical thought to an algebraic thought. In J. Moser (Ed.), *Proceedings of PME-NA Vol. VI* (pp. 51– 56). Madison, WI: PME.

Flyvbjerg, B. (2006). Five misunderstandings about case-study research. *Qualitative Inquiry*, *12*(2), 219–245.

Gee, J. P. (2015). *Social linguistics and literacies: Ideology in discourses.* (5th ed). New York: Routledge.

Geertz, C. (1973/2000). *The interpretation of cultures: Selected essays*. New York: Basic Books.

Guba, E. G., & Lincoln, Y. S. (1994). Competing paradigms in qualitative research. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (pp. 105-117). Thousand Oaks, CA: Sage.

von Glasersfeld, E. (1983). Learning as constructive activity. In J. C. Bergeron, & N. Herscovics (Eds.), *Proceedings of the fifth annual meeting of the North American Group of Psychology Congress in Mathematics Education* (pp. 41−101). Montreal.

Harré, R. & Gillett, G. (1994) *The discursive mind.* Thousand Oaks, CA: Sage Publications.

Hatano, G. & Wertsch, J. V. (2001). Sociocultural approaches to cognitive development: The constitutions of culture in mind. *Human Development*, *44*, 77–83.

Herscovics, N., & Linchevski, L. (1994) A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics, 27*, 59–78.

Hershkowitz, R., Hadas, N., Dreyfus, T. & Schwarz, B. (2007). Abstracting processes, from individuals' constructing of knowledge to a group's "shared knowledge". *Mathematics Education Research Journal, 19*(2), 41–68.

Hitt, F., Saboya, M., & Cortés-Zavala, C. (2017). Rupture or continuity: The arithmetico-algebraic thinking as an alternative in a modelling process in a paper and pencil and technology environment. *Educational Studies in Mathematics, 94*(1), 97–116.

Kaput, J. J. (1998). Transforming algebra from an engine of inequity to an engine of mathematical power by "algebrafying" the K-12 curriculum. In National Council of Teachers of Mathematics & Mathematical Sciences Education Board (Eds.), *The nature and role of algebra in the K-14 curriculum: Proceedings of a national symposium*. Washington, DC: National Academy Press

Kaput, J. J., Carraher, D. W., & Blanton, M. L. (2008). *Algebra in the early grades*. New York: Routledge.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12*(3), 317-326.

Kieran, C. (1992). The learning and teaching of school algebra. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390-419). New York: MacMillan.

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 707–762). Charlotte, NC: Information Age Publishing.

Kieran, C. (2018). Seeking, using, and expressing structure in numbers and numerical operations: a fundamental path to developing early algebraic thinking. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12-year-olds: The global evolution of an emerging field of research and practice* (pp. 79–105). New York: Springer.

Kieran, C., Boileau, A., & Garançon, M. (1996). Introducing algebra by means of a technology-supported, functional approach. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra* (pp. 257–293). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Kilhamn, S. & Säljö, R. (Eds.) (in press), *Encountering algebra. A comparative study of classrooms in Finland, Norway, Sweden and the USA* (pp. 00-00). New York, NY: Springer.

Kilparick, J. (1993). Beyond face value: Assessing research in mathematics education. In G. Nissen & M. Blomhøj (Eds.), Criteria for scientific quality and relevance in the didactics of mathematics (pp. 15–34). Roskilde, Denmark: Roskilde University.

Küchemann, D. (1978). Children's understanding of numerical variables. *Mathematics in School, 7*, 23-26.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal,* 27, 29−64.

Lannin, J. K. (2005). Generalization and justification: The challenge of introducing algebraic reasoning through patterning activities. *Mathematical Thinking and Learning, 7*, 231–258.

Lave, J. (1991). Situating learning in communities of practice. In L. B. Resnick, J. M. Levine and S. D. Teasley (Eds.), *Perspectives on socially shared cognition* (pp. 63–82). Washington, DC: American Psychological Association.

Lave, J. (1996). Teaching, as learning, in practice. *Mind, Culture and Activity, 3*, 149–164.

Lave, J. & Wenger, E. (1991) *Situated learning: legitimate peripheral participation.* New York: Cambridge University Press.

Lee, L., & Wheeler, D. (1989). The arithmetic connection. *Educational Studies in Mathematics, 20*, 41–54.

Lemke, J. L. (1990). *Talking Science: Language, learning and values*. Norwood, NJ: Ablex Publishing Corporation.

Lerman, S. (2000). The social turn in mathematics education research. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 19-44). Westport, CN: Ablex.

Lerman, S. (2010). Theories of mathematics education: Is plurality a probem? In B. Sriraman & L. English (Eds.), *Theories of mathematics education* (pp. 99−109). Berlin: Springer.

LeVine, R. A. (2010). The six cultures study: Prologue to a history of a landmark project. *Journal of CrossCultural Psychology, 41*, 513-521.

Linell, P. (1998). *Approaching dialogue: Talk, interaction and contexts in dialogical perspectives*. Amsterdam: John Benjamins Publishing Company.

Lins, R., & Kaput, J. J. (2004). The early development of algebraic reasoning: The current state of the field. In K. Stacey, H. Chick, & M. Kendal (Eds.), *The future of the teaching and learning of algebra: the 12th ICMI Study* (pp. 47–70). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran & L. Lee (Eds.) *Approaches to algebra* (pp. 65-68). Dordrecht: Kluwer Academic Publishers.

Mehan, H. (1979). *Learning lessons: Social organization of the classroom*. Cambridge, MA: Harvard University Press.

Moschkovich, J. (2007). Examining mathematical discourse practices. *For the Learning of Mathematics*, *27*(1), 24-30.

Moschkovich, J. N. (2013). Principles and guidelines for equitable mathematics teaching practices and materials for English language learners. *Journal of Urban Mathematics Education, 6*(1), 45–57.

Niss, M. (1999). Aspects of the nature and state of research in mathematics education. *Educational Studies in Mathematics, 40*, 1–24.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM – The International Journal on Mathematics Education*, *40*(2), 165–178.

Radford, L. (1996). Some reflections on teaching algebra through generalization. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra* (pp. 107–111). Dordrecht: Kluwer Academic Publishers.

Radford, L. (2000). Signs and meanings in students' emergent algebraic thinking: a semiotic analysis. *Educational Studies in Mathematics*, *42*(3), 237–268.

Radford, L. (2002). The seen, the spoken and the written. A semiotic approach to the problem of objectification of mathematical knowledge. *For the Learning of Mathematics*, *22*(2), 14–23.

Radford, L. (2003). Gestures, speech and the sprouting of signs. *Mathematical Thinking and Learning*, *5*(1), 37–70.

Radford, L. (2008). Iconicity and contraction: A semiotic investigation of forms of algebraic generalizations of patterns in different contexts. *ZDM - The International Journal on Mathematics Education, 40*(1), 83–96.

Radford, L. (2010). Algebraic thinking from a cultural semiotic perspective. *Research in Mathematics Education, 12*(1), 1-19.

Radford, L. (2013). Sensuous cognition. In D. Martinovic, V. Freiman, & Z. Karadag (Eds.), *Visual mathematics and cyberlearning, mathematics education in the digital era* (Vol. 1, pp. 141–162). New York: Springer

Radford, L. (2014). Towards an embodied, cultural, and material conception of mathematics cognition. *ZDM – The International Journal of Mathematics Education, 46*(3), 349-361.

Radford, L. (2018). The emergence of symbolic algebraic thinking in primary school. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12-year-olds: The global evolution of an emerging field of research and practice* (pp. 1-23). New York: Springer.

Radford, L., Bardini, C., & Sabena, C. (2007). Perceiving the general: The multisemiotic dimension of students' algebraic activity. *Journal for Research in Mathematics Education, 38,* 507–530.

Radford, L., Demers, S., Guzmán, J. and Cerulli, M. (2003). Calculators, graphs, gestures, and the production of meaning, Proceedings of the 27 PME Conference, (Vol. 4, pp. 55-62, University of Hawaii.

Radford, L., & Puig, L. (2007). Syntax and meaning as sensuous, visual, historical forms of algebraic thinking. *Educational Studies in Mathematics, 66*, 145–164.

Radford, L. & Roth, W-M. (2011). Intercorporeality and ethical commitment: an activity perspective on learning. *Educational Studies in Mathematics*, *77*, 227-245.

Reinhardtsen, J. (2012). *The introduction of algebra: Comparative studies of textbooks in Finland, Norway, Sweden and USA*. Unpublished Master's thesis: University of Agder.

Rivera, F. D., & Becker, J. R. (2011). Formation of pattern generalization involving linear figural patterns among middle school students: Results of a three-year study. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 323 – 366). Berlin: Springer.

Rogoff, B. (2008). Observing sociocultural activity on three planes: participatory appropriation, guided participation and apprenticeship. In K. Hall, P. Murphey & J. Soler (Eds.), *Pedagogy and practice: culture and identities* (pp. 58–74). London: Sage.

Roth, W. M., & Radford, L. (2010). Re/thinking the zone of proximal development (symmetrically). *Mind, culture, and activity, 17*, 299–307.

Säljö, R. (2001). *Læring i praksis: Ett sosiokulturelt perspektiv* [Learning in practice: A sociocultural perspective], Oslo, Norway: Cappelen.

Säljö, R. (2005). Lärande och kulturella redskap: Om lärprocesser och det kollektiva minnet (Learning and cultural tools: Learning processes and collective memory). Stockholm: Norstedts Akademiska Förlag.

Saxe, G. B. (1988). Candy selling and math learning. *Educational Researcher*, *17*(6), 14–21.

Saxe, G. B. (2012). *Cultural development of mathematical ideas: Papua New Guinea studies*. New York: Cambridge University Press.

Saxe, G. B., de Kirby, K., Le, M., Sitabkhan, Y., & Kang, B. (2015). Understanding learning across lessons in classroom communities: a multi-leveled analytic approach. In A. Bikner-Ahsbahs, C. Knipping, & N. Presmeg (Eds.), *Approaches to qualitative research in mathematics education* (pp. 253–320). Dordrecht: Springer

Schmittau, J. and Morris, A. (2004). The development of algebra in the elementary mathematics curriculum of V. V. Davydov. *Mathematics Educator, 8*, 60-87.

Selling, S. K. (2016). Making mathematical practices explicit in urban middle and high school mathematics classrooms. *Journal for Research in Mathematics Education, 47*(5), 505–551.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. New York: Cambridge University Press.

Sfard, A. (2012). Introduction: Developing mathematical discourse. Some insights from communicational research. *International Journal of Educational Research, 51*, 1-9.

Sfard, A., & Kieran, C. (2001). Cognition as communication: Rethinking learning-by-talking through multi-faceted analysis of students' mathematical interactions. *Mind, Culture, & Activity, 8*, 42–76.

Sowder, J. T. (1998). Ethics in mathematics education research. In A. Sierpinska & J. Kilpatric (Eds.), *Mathematics education as research domain: A search for identity. An ICMI study* (pp. 427-442). Dordrecht, The Netherlands: Kluwer.

Stacey, K. (1989). Finding and using patterns in linear generalising problems. *Educational Studies in Mathematics, 20*(2), 147-164.

Stacey, K., & MacGregor, M. (2001). Curriculum reform and approaches to algebra. In R. Sutherland, T. Rojano, A. Bell, & R. Lins (Eds.), *Perspectives on school algebra* (pp. 141–154). Dordrecht, Netherlands: Kluwer.

Stake, R. E. (2000). Case studies. In N. K. Denzin and Y. S. Lincoln (Eds.), *Handbook of qualitative research* (pp. 435–454). Thousand Oaks, CA: Sage.

Stephens, A., Ellis, A., Blanton, M., & Brizuela, (2017). Algebraic thinking in the elementary and middle grades. In J. Cai (Ed.), *Third handbook of research in mathematics education*. Reston, VA: National Council of Teachers of Mathematics.

Stephan, M., & Rasmussen, C. (2002). Classroom mathematical practices in differential equations. *Journal of Mathematical Behavior*, 21, 459–490.

Stigler, J., & Hiebert, J. (2009). *The teaching gap*. New York: Free Press.

Sutherland, R. Rojano, T., Bell, A., & Lins, R. (2001). *Perspectives on school algebra*. Dordrecht: Kluwer Academic Publishers

Tunks, J., & Weller, K. (2009). Changing practice, changing minds, from arithmetical to algebraic thinking: an application of the concerns-based adoption model (CBAM). *Educational Studies in Mathematics, 72*, 161–183.

Usiskin, Z. (1999). Conceptions of school algebra and uses of variables. In A. F. Coxford (Ed), *The ideas of algebra, K-12* (pp. 8-19). Reston, VA: National Council of Teachers of Mathematics.

TIMSS & PIRLS (2009). TIMSS 2007 user guide for the international database: Released items, mathematics eighth grade. TIMSS & PIRLS International Study Center: Boston College. http://timss.bc.edu/timss2007/items.html

Vergnaud, G. (1979). The acquisition of arithmetical concepts. *Educational Studies in Mathematics, 10*, 263–274.

Vergnaud, G. (1982). Cognitive and developmental psychology and research in mathematics education: Some theoretical and methodological issues. *For the Learning of Mathematic*, *3*(2), 31–41.

Vygotsky, L. S. (1978). *Mind in society*: The development of higher psychological processes. Cambridge, MA: Harvard University Press.

Vygotsky, L. S. (1986). *Thought and language*: The development of higher psychological processes. Cambridge, MA: The M. I. T. Press.

Vaags, R. H. (2004). *Etikk*. Bergen: Fagbokforlaget.

Wellington, J. (2000). *Educational research: Contemporary issues and practical approaches*. London: Continuum.

Wertsch, J. V. (1998). *Mind as action*. New York: Oxford University Press.

Yin, R. (1981). The case study as a serious research strategy. *Knowledge*, *3*(1), 97–114.

Zinchenko, V. P. (2007). Thought and word: The approaches of L. S. Vygotsky and G. G. Shpet. In H. Daniels, M. Cole, & J. V. Wertch (Eds.), *The Cambridge companion to Vygotsky* (pp. 212−245). New York: Cambridge University Press.

9 Appendices

Appendix 1:

Approval from Norwegian Social Science Data Services

Norsk samfunnsvitenskapelig datatjeneste AS
NORWEGIAN SOCIAL SCIENCE DATA SERVICES

Harald Hårfagres gate 29
N-5007 Bergen
Norway
Tel: +47-55 58 21 17
Fax: +47-55 58 96 50
nsd@nsd.uib.no
www.nsd.uib.no
Org.nr. 985 321 884

Maria Luiza Cestari
Institutt for matematiske fag
Universitetet i Agder
Serviceboks 422
4604 KRISTIANSAND S

Vår dato: 14.03.2012 Vår ref:30024 / 3 / PB Deres dato: Deres ref:

TILBAKEMELDING PÅ MELDING OM BEHANDLING AV PERSONOPPLYSNINGER

Vi viser til melding om behandling av personopplysninger, mottatt 28.02.2012. Meldingen gjelder prosjektet:

30024	*Hidden Dimensions of Teaching/Learning in Mathematics: The Contribution of Video Studies to Comparative Analysis and the Development of Instruction (VIDEOMAT)*
Behandlingsansvarlig	*Universitetet i Agder, ved institusjonens øverste leder*
Daglig ansvarlig	*Maria Luiza Cestari*

Personvernombudet har vurdert prosjektet og finner at behandlingen av personopplysninger er meldepliktig i henhold til personopplysningsloven § 31. Behandlingen tilfredsstiller kravene i personopplysningsloven.

Personvernombudets vurdering forutsetter at prosjektet gjennomføres i tråd med opplysningene gitt i meldeskjemaet, korrespondanse med ombudet, eventuelle kommentarer samt personopplysningsloven og helseregisterloven med forskrifter. Behandlingen av personopplysninger kan settes i gang.

Det gjøres oppmerksom på at det skal gis ny melding dersom behandlingen endres i forhold til de opplysninger som ligger til grunn for personvernombudets vurdering. Endringsmeldinger gis via et eget skjema, http://www.nsd.uib.no/personvern/forsk_stud/skjema.html. Det skal også gis melding etter tre år dersom prosjektet fortsatt pågår. Meldinger skal skje skriftlig til ombudet.

Personvernombudet har lagt ut opplysninger om prosjektet i en offentlig database, http://www.nsd.uib.no/personvern/prosjektoversikt.jsp.

Personvernombudet vil ved prosjektets avslutning, 31.12.2013, rette en henvendelse angående status for behandlingen av personopplysninger.

Vennlig hilsen

Vigdis Namtvedt Kvalheim

Pernilla Bollman

Pernilla Bollman tlf: 55 58 24 10
Vedlegg: Prosjektvurdering

Avdelingskontorer / *District Offices:*
OSLO: NSD. Universitetet i Oslo, Postboks 1055 Blindern, 0316 Oslo. Tel: +47-22 85 52 11. nsd@uio.no
TRONDHEIM: NSD. Norges teknisk-naturvitenskapelige universitet, 7491 Trondheim. Tel: +47-73 59 19 07. kyrre.svarva@svt.ntnu.no
TROMSØ: NSD. SVF, Universitetet i Tromsø, 9037 Tromsø. Tel: +47-77 64 43 36. nsdmaa@sv.uit.no

Appendix 2:

The studies

Studies I-III

Study 1: Capturing learning in classroom interaction in mathematics: Methodological considerations.
This paper is published in the CERME 9 proceedings.

Study 2: The fifth lesson: Students' responses to a patterning task across the four countries.
This is a chapter in the book edited by C. Kilhamn & R., *Encountering algebra. A comparative study of classrooms in Finland, Norway, Sweden and the USA*. This book is currently in press.

Study 3: The emergence of a generalization practice in a 6th grade introductory algebra classroom.
This study was submitted to Journal for Research in Mathematics education 03.18.2019. It has been checked by the editor and is currently under review.

Capturing learning in classroom interaction in mathematics: Methodological considerations

Jorunn Reinhardtsen, Martin Carlsen, Roger Säljö

Capturing learning in classroom interaction in mathematics: Methodological considerations

Jorunn Reinhardtsen, Martin Carlsen and Roger Säljö

University of Agder, Kristiansand, Norway, jorunn.reinhardtsen@uia.no

This paper discusses issues of how to transcribe and analyze video-recordings when studying learning in small group work in mathematics. Since bodily features of interaction and the use of artefacts play important roles in mathematical reasoning, a multimodal approach to transcribing is necessary. Thus, the theoretical grounding for transcriptions has to be in accord with the perspective on learning adopted in the analysis. In the paper, the principles for studying what Radford (2000) refers to as knowledge objectification processes when learning mathematics will be discussed.

Keywords: Analytical approaches, knowledge objectification, multimodality.

INTRODUCTION

This paper discusses ways of doing video analyses that are relevant for understanding mathematics learning. Thus, this is a methodological paper. Our particular focus is on multimodality as a resource for learning but also as a methodological challenge for research. Analytical approaches, selection of episodes and a multimodal transcription will be discussed in light of recent developments in the field.

The background of this study is an international, comparative project called VIDEOMAT (Kilhamn & Röj-Lindberg, 2013), which studies teaching and learning of introductory algebra in four countries: Finland, Norway, Sweden and the USA. Students are between 11 and 13 years old. Data include video-recordings of lessons, written materials from student activity, teacher interviews etc. Five consecutive lessons when algebra was introduced[1] in classrooms in the four countries were documented. A group work session with a pattern task with matchsticks was selected for further investigation. This task resulted in a multitude of problem-solving strategies among students, all the way from counting to sophisticated forms of mathematical reasoning (multiplicative/generalizing).

In the literature there are many attempts to make use of multimodal analyses to understand learning processes in the context of mathematics. We will comment on some of these below. Considering that we are at an early stage of advancing knowledge through the use of multimodal approaches, we have formulated the following question for this paper: *In what ways can video recordings be transcribed and analyzed in order to study student's learning processes?*

BACKGROUND

The methodical reflections in this paper focus on classroom interaction in a problem-solving, small-group setting. A particular aim is to understand the knowledge objectification process (Radford, 2000, 2002). The object of activity in the classroom, as the students work with the matchstick task, is to develop algebraic thinking; more specifically to perceive the general nature of a pattern, and to use this insight when solving a problem. The ability to generalize is viewed as one of the most important developments in mathematical thinking.

Our analysis will follow a socio-cultural, Vygotskian view on learning and development. A central idea is that learning results from participation in social and interactional processes. Equally important is that this perspective stresses that learning and knowing are cultural phenomena.

Approaching group work in mathematics classrooms with an interest in the contributions of multimodality, the cultural-semiotic theory of learning, developed by Radford (2000), provides a promising route ahead. Radford (2002) suggests that knowledge objectification happens through semiotic activity, that is through "objects, artifacts, linguistic devices and signs that are intentionally used by individuals in social processes of meaning production, in order to achieve a stable form of awareness, to make apparent their intentions and to carry out their actions" (p. 14). The process of knowledge objectification is understood as the process of placing something at the center of someone's attention. In this study, knowledge objectification thus refers to the process of perceiving generality; the knowledge of the general nature of the pattern having a genesis and a development, and, as a further step, knowing how to express the generality mathematically and to solve the problem.

METHODOLOGICAL DEVELOPMENT IN THE STUDY OF LEARNING

The methodological deliberations by different researchers have been scrutinized. The studies analyzed all rely on naturalistic data and interpretive approaches to method, and they represent different choices in terms of data collection and analysis.

Bjuland (2002) focused on small group problem-solving in mathematics by student teachers. Data were collected by audio recordings, and the theoretical perspective utilized was dialogical, situated and socio-cognitive. The unit of analysis, referred to as an episode, was "conceived as a sequence of verbalizations focused on a special mathematical topic or idea" (p. 64), relevant to the research questions. These were then categorized according to five features of problem solving processes: sense-making, conjecturing, convincing, reflecting and generalizing.

Carlsen (2009), working in a sociocultural tradition, analyzed the appropriation of mathematical tools by students attending the final year of high school. Video recordings were used. The aim was to trace development of the student's mathematical reasoning. Relevant parts of the entire audio recorded material were transcribed in detail and subjected to in-depth analysis. The transcripts included multimodal elements in order to investigate the role of inscriptions in the appropriation process.

Radford (2000, 2002, 2012) reported on longitudinal studies involving students' group work with algebra and more specifically with patterns. This work involves methodological and theoretical developments that are interesting. Radford's research is based in a semiotic-cultural perspective on learning building on Vygotsky's view of signs as linked to and affecting our cognition. In Radford (2012), researchers took part in the process of designing the lesson material and students were organized in small groups. These sections were video recorded and student works were collected.

In his early work, Radford (2000) uses concepts from discourse analysis. He follows a three-step analysis of transcripts, a) valuing each utterance as equally important, b) contextualizing utterances, and c) including pauses and hesitations. This approach Radford (2000, p. 244) terms *situated discourse analysis*. The unit of analysis was conceived through a process of refining salient episodes through data managing by indexing and theorizing. Radford emphasizes the importance of natural language in the development of algebraic thinking and the use of algebraic symbols.

Radford, Demers, Guzmán, and Cerulli (2003) introduce the concept of *semiotic node*. This was a response to findings in many studies on the importance of gestures and artifacts in the production of graphs and algebraic expressions. Semiotic nodes are "pieces of the students' semiotic activity where action, gesture, and words work together to achieve knowledge objectification" (p. 56). The transcripts include description of gestures and the analytical tool of semiotic nodes was applied in the analysis. In Radford, Bardini, and Sabena (2007), the analysis was done in greater detail. A slow-motion, frame-by-frame, fine-grained video micro-analysis was carried out and complemented with a voice-analysis. The same kind of micro-analysis was carried out in Radford (2012), except for the voice analysis, where a multi-semiotic analysis (spoken words, written text, gestures, drawing, and symbols) was done.

Arzarello (2006) outlines a theoretical frame emphasizing the role of multimodality and embodiment in cognition. He argues for a multi-semiotic analysis of objectification processes and claims that the present semiotic frameworks cannot capture didactical processes in a satisfactory manner. Therefore, he introduced the idea of the *semiotic bundle*. In the semiotic bundle, which includes semiotic sets such as gestures, speech, written representations, as well as more formal systems, the distinctions between the sets are only made for analytical purposes while interpreted as a unitary system. The semiotic bundle is dynamic and can shift to include more or less semiotic sets as the event unfolds. The meaning of the mathematical object may not be the same in the different sets. Moreover, even if the transformation from one set to another is accomplished, the meaning the object had in the prior set may linger, and so it can take time before the concept is formalized. By looking at the data synchronically and diachronically, the genesis and evolution of the semiotic objectification process can be traced. The semiotic node introduced by Radford (2003) is similar to looking at the semiotic bundle synchronically.

Arzarello (2006) used the semiotic bundle to analyze the work of one group of five fifth-graders. Video recordings and student work were collected as part of a longitudinal research design. However, the episodes presented were chosen from a 30 minute session on problem solving. The selection process was not commented on, except by saying that four main episodes were chosen. The episodes were subjected to different analytical methods; (episode 1) synchronic analysis; (episode 2) diachronic analysis; (episode 3) synchronic + diachronic analysis; and (episode 4) diachronic analysis. The transcriptions include descriptions of gestures and pictures are presented in the analysis.

Roth and Thom (2009) looked at multimodality and learning from a phenomenological perspective. The aim of the study was to propose a new way of understanding mathematical concepts grounded in a case study. Data were collected in a second grade classroom during group work sessions in geometry. In addition, artifacts used and all work by the teacher and the students were photographed. One episode from a whole class session, lasting 69 seconds, which is called exemplary, was chosen for analysis. The episode is presented in the context of what happened before. The transcript includes details (length of pauses, pitch etc.). The episode is presented over 6 pages and several drawings depicting movements are part of the description. The authors argue that "conceptions can be understood as networks of experiences that indeterminately emerge from lived (rather than intellectual) reorganizations of embodied bodily experiences" (op. cit., p. 188).

The studies presented above are all conducted within the paradigm of interpretivism. They are ethnographic and researchers spend time observing, making field notes, and collecting students' work; the researchers are concerned with the context in which the events take place. The video and/or audio recordings are done in classrooms and are naturalistic in the sense that students are in their everyday environment engaging with mathematical activities. The studies also share a common focus on the multimodal aspects of learning, except Bjuland (2002) and Radford (2000).

METHODOLOGICAL CONSIDERATIONS

In spite of the commonalities of the studies presented, the transcripts look very different and include different features of interaction. Bezemer and Mavers (2011), investigating multimodal transcripts in research, point out that "transcripts should be judged in terms of the 'gains and losses' involved in remaking video data" (p. 204). The focus should not be on attempting to achieve representational accuracy, rather the approach should be transparent.

The studies use different analytical approaches to dialogue. Bjuland and Carlsen use the *dialogical approach* elaborated by Linell (1998), while Radford uses *situated discourse analysis*. Consequently, the process of analysis is different. Radford's first step is to look at each utterance in its own right and categorize it. As a second step, he contextualizes them. In contrast, sequentiality is central to the dialogical approach as each contribution in a dialogue gets its meaning from both prior and subsequent turns. Arzarello (2006) and Roth and Thom (2009) do not fully reveal their approach for analyzing dialogue.

An important aspect is the selection of salient episodes. Bjuland (2002) transcribed all verbalizations and then identified relevant episodes according to the analytical interest. Carlsen (2009) worked with video recordings. After several viewings, he chose 14 sessions which were roughly transcribed. Following this, relevant episodes were identified and transcribed in detail. From this sample salient episodes were chosen. Radford (2000) used *situated discourse analyses* as a first approach to the data set, which was transcribed in its entirety. The studies by Arzarello (2006) and Roth and Thom (2009) do not fully comment on the selection process.

These studies show that multimodality is an essential part of understanding how students learn mathematics. Thus, it becomes important for this branch of research to enter into a discussion on how to advance the use of multimodal methods of analysis.

CAPTURING LEARNING: SUGGESTED METHODS

The aim of this paper is to describe ways of doing video analysis that focus specifically on learning processes and which include attention to multimodality. The approach will be discussed in three sections: *video analysis*; *multimodal transcription*; *learning processes and video analysis*. Our discussion will be twofold as it a) provides arguments for the methodological choices and b) is practically oriented in that an excerpt of a multimodal transcript is included and analytical approaches are briefly exemplified.

Video analysis

The video analysis follows an interpretivist paradigm. The aim is to understand learning processes by closely following how participants engage in meaning making. Applying the notion of knowledge objectification through semiotic activity implies analyzing multimodal aspects of interaction. According to Knoblauch (2012), one has to apply two types of interpretation in order to preserve the essence of multimodal elements of interaction. The first is to interpret what is seen and heard as it appears from an everyday understanding and from the actors' point of view. The second level is the professional interpretation of the interaction.

The ethnographical aspect of this research is important in terms of the validity of interpretations. Observation of lessons, interviews with the teachers and the written materials collected improve the ability to interpret the situation. The validity of the interpretations will depend on the assumption that "people are existent and, that they have been conducting (acting) in ways that are open for reconstruction (capture) by video data" (Knoblauch, 2012, p. 73). This allows *subjective adequacy*, which means that there is a correspondence between what the researchers say and the statements by the participants. Psychological studies have shown that people often "see events similarly in terms of causal, behavioral, and thematic structures" (Derry et al., 2010, p. 7), which supports the validity of an *everyday* interpretation of interaction.

The empirical material in this study is considered to be *naturally occurring data*. We recognize that the presence of three cameras, two professionals operating them, and one to three researchers observing exert some influence on the situation. However, students today are familiar with cameras, and in consultations with the teachers after lessons they expressed that students behaved as usual.

In order to approach the complexity of the interaction in the groups, the discourse is separated into two main parts: dialogue and multimodal elements. However the two parts are interpreted as belonging to a unified system of communication and therefore seen as integral parts of meaning making. Two methodological concepts will be considered when analyzing the multimodal elements (Knoblauch, 2012, pp. 74–75): *sequentiality*, considering any action as motivated by prior actions and motivating future actions; *reflexivity*, actors do not only act but also indicate, frame or contextualize how their action is to be understood and how they have interpreted a prior action to which they are responding. These concepts correspond well with the dialogical approach which also emphasizes sequentiality.

The issue of multimodal transcription

In attempting to transcribe visual data of video recordings there is a challenge in doing adequate data reduction. The focus of our research is the interaction in the groups. Luckmann (2012, p. 32) argues that "the elements of the interaction which the analyst, based on his knowledge of social life, must assume were relevant to the participants in the original interaction, must be noted in the transcript". Knoblauch (2012, p. 75) argues that video analysis is a hermeneutic activity. "[T]he task set is not to only describe and explain non-verbal behavior". As a researcher one has to decide what knowledge is needed to make sense of a situation and to identify visible conducts constituting the situation. Therefore, multimodal transcribing is not only a preliminary stage to the analysis; the activity forms an essential part of the analysis.

The video material and the written works of students have been examined in order to understand the problem solving process through the dialogue and the semiotic actions that appear both by each individual student and as part of the joint group activities. Several multimodal elements of the interaction have been identified. These fall into three categories of use of mediating resources: *inscriptions* such as drawings, tables, texts, numbers, arithmetic, algebraic (including variable/s); *concretes* i.e. matchsticks; *gestures*

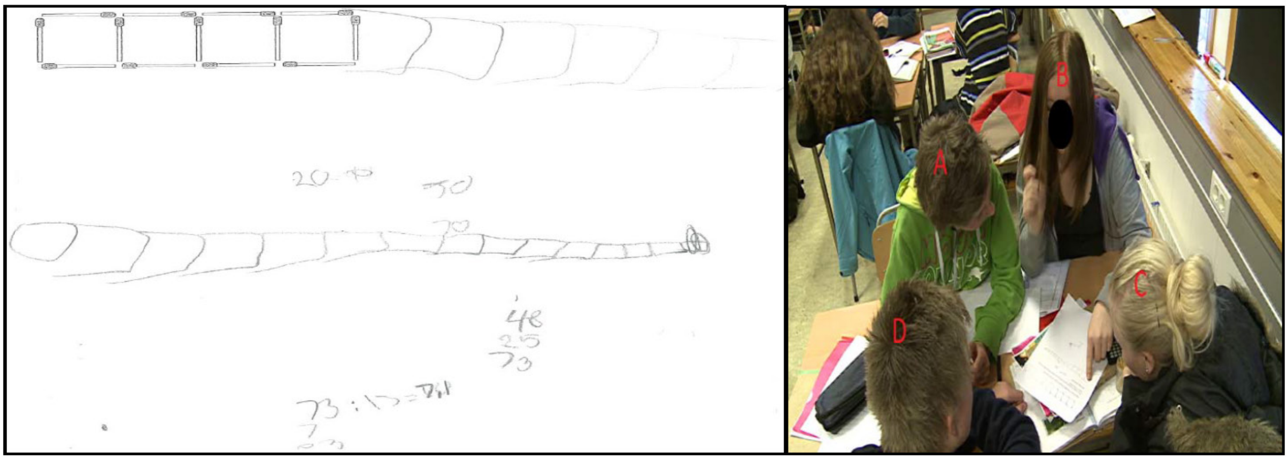


Figure 1: Representation of part of the multimodal transcription

such as pointing, tracing in air/figure/table, glance, rhythmic hand movement, raising hand.

The learning processes and video analysis

Derry and colleagues (2010) stress the importance of being systematic when selecting salient episodes. Schoenfeld (1985) parceled the dialogue according to the mode of reasoning (i.e. planning, exploration) as he expected strategic decisions to be located at the junctures between such episodes. In this study, the dialogue will also be parceled according to the mathematical strategy the students are working with. This is done in order to explore how the students' discourse on the problem evolves during the problem solving process and to reveal mechanisms which drives it. In light of these explorations, fragments of the text which show the first traceable step and its successors in the objectification process will be identified.

An excerpt from a multimodal transcription of a Norwegian group is presented below. A group of 8th grade students, Ben (A), Ann (B), Trish (C) and Sam (D), are given an algebra task (adapted from TIMSS 2007) involving matchsticks and patterns. The teacher hands out toothpicks as a material to use in order to solve the problem. Only Ann writes on the task paper. Marks indicating if the students are in the process of conjecturing (Cj) or convincing (Co) and also specifying the mathematical strategy used such as additive (A) or multiplicative (M) have been inserted into the transcript in order to show the analytical approaches to the text.

8 Trish: We can make them [*squares*] on the table. But should we just use these or? [*Trish shakes the can of toothpicks she is holding in her hand*].

9 Ann: But see, we get 7.1 [*Ann points to the division, 73 divided by 13, she has been working on*], then if you have taken ( ) then you get 7.1 squares. 1, 2, 3, 4, 5, 6, 7 [*Ann points at the squares in the task paper as she counts them and continues by pointing at imaginary squares until she reaches 7*]. So then you get less than sev…then we get, if we make 7 squares. Ok, 4.

The girls try to add a square to the figure using the toothpicks. They give it up quickly as they notice that the dimensions are different.

10 Trish: Ha..ha

11 Ann: You, this didn't work

12 Trish: We'll draw it.

13 Ann: [*She adds a square to the figure by drawing three sides in one motion, she then points at each square as she counts them*] 1, 2, 3, 4…[*adds another square in the same manner*], 5. [*starts counting the matchsticks making up the squares*] 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14… 17, 18. Ok, but see…ah…I got a good idea…look [*Now she only counts the horizontal matchsticks*] 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12…[*adds more squares using the same motion*] 13, 14…15, 16…17, 18…19, 20 [*There are now 10 squares altogether*]. So if we take [*She now counts the squares*] 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. When we have 20 rows we have [*writes 20 and then counts the vertical matchsticks silently*]…((then we have…then we have)) =

14 Sam: ((But what are we going to do with them…Ann?)).

15 Ann: = When we have 20 we have 50 pieces [*writes 50*]. Or, when we have 20, when

we have 20 such things... [*she points or taps repeatedly at the figure*].

16 Sam: It is those [*Sam holds up a toothpick*].

17 Ann: Yes, matchsticks, then we have 50 altogether [*points to the number written*], used 50 such matchsticks [*points back at the figure*] and we are going to use 73, right? =

18 Sam: Just make...

19 Ann: = So then...

20 Trish: ((really one more will be 53 and then 56))

21 Ben: ((We are going to use...))

22 Ann: No, if we have one more with 10 in it, then it becomes... =

23 Sam: ((Yes because it is four in one)).

24 Ann: = So, then we get 20 more and it becomes 70 [*writes 70*]. ((It is 1, 2, 3...so then we get 70... No, now there is too much here)) =

25 Ben: [looks at Sam and responds to his comment] ((No, it is 3, it is 4 in one and 3...1, 2, 3, 4, 5, 6, 7, 8, 9))

26 Ann: = I think I sort of lost count of it.

27 Trish: No, 70, and then you should have 1 thing more and then it becomes exactly 73.

28 Ann: Ah, but see, oh yes because 20...

29 Trish: It is really only three in each, it is only the first there is four in, and then there is only three in each the whole time [*points at the figure while she explains*].

30 Ann: But see...

31 Trish: If you do like that then...4 [*she holds her finger over the first square*]

32 Ann: 1, 2, 3. [*counts three matchsticks in the first square, then pushes away Trish's finger and starts counting in the pattern she has developed, horizontal matchsticks first and then the vertical ones*] Ok, 1, 2, 3, 4, 5, 6, 7, 8, 9 ( ) 18, 19. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20. 20. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11. 11.

While Ann is counting, Ben and Sam start paying attention to something that is going on in the classroom which is not relevant for the mathematical discussion. When the teacher approaches the group, the boys attend to solving the task again.

33 Trish: [*traces the matchsticks in the squares using the same motion as Ann used earlier when drawing new squares*] Oh, you! 73 divided by 3 and then just add 1! [*she picks up her calculator*]

34 Ann: There you said one. [While Trish is working on the calculator, Ann traces first the four matchsticks in the first square and then the 3 matchsticks in each of the following squares. She is using the same motion as earlier when drawing the squares].

35 Trish: No. [*The teacher comes over to the group, but Trish only looks at the calculator while she speaks*] 73 divided ((by 3, plus 1, 25)).

36 Ann: [*Ann looks at the teacher*] ((divided by...3. Is that right?))

In turn (9) Ann suggests a solution to the task based on a multiplicative strategy. In order to make sense of the answer she found, she turns back to the task paper and applies an additive strategy.

The marks in the text indicate important events in the problem solving process. If we focus the attention on the objectification process, we see in (20) the first verbalization of the *3+3* pattern, which is discussed and developed by Sam (23), Ben (25) and Trish (29), and finally expressed as *4+3+3*. However, in (33) we see that Trish traces the matchsticks with the same motion used by Ann that appears early in the text (13), immediately before she expresses a new conjecture for how to solve the task (33). Ann is not taking part in the discussion of the *4+3+3* pattern but seems to drive it with the gestures and the drawing she is making.

CONCLUSION

The video recordings available of 16 groups working with the same task offer an opportunity to study the role of features of thinking in the objectification process. These features, as elaborated through the empirical materials and the theoretical perspective, have been identified as: *elements of reasoning* (sense-making, conjecturing, convincing, reflecting, generalizing), *mathematical strategies* (additive, multiplicative, equations, functional), *semiotic resources* (use of language, inscriptions, concretes, gestures) and indicators of the *culture of collaboration*.

The analytical methods described are developed in order to understand how these different features of thinking are incorporated in learning processes. The ambition is to shed light on a) what role mediating tools play as students decide on mathematical strategies, b) what features of the knowledge objectification process that can be discerned, and c) what are the differences, if any, between classrooms and cultures of work in the different countries.

REFERENCES

Arzarello, F. (2006). Semiosis as a mulitimodal process. *Revista Latinoamericana de Investigacion en Matematica Educativa, 9*(1), 267–300.

Bezemer, J., & Mavers, D. (2011). Multimodal transcription as academic practice: a social semiotic perspective. *International Journal of Social Research Methodology*, *14*(3), 191–206.

Bjuland, R. (2002). *Problem solving in geometry. Reasoning processes of student teachers working in small groups. A dialogical approach.* Doctoral Dissertation. Bergen: University of Bergen.

Carlsen, M. (2009). Reasoning with paper and pencil: the role of inscriptions in student learning of geometric series. *Mathematics Education Research Journal*, *21*, 54–84.

Derry, S. J., Pea, R. D., Barron, B., Engle, R. A., Ericson, F., Goldman, R., Hall, R., Koschmann, T., Lemke, J. L., Sherin, M. G., & Sherin, B. L. (2010). Conducting video research in the learning sciences: guidance on selection, analysis, technology, and ethics. *Journal of the Learning Sciences*, *19*, 3–53.

Kilhamn, C., & Röj-Lindberg, A. S. (2013). Seeking hidden dimensions of algebra teaching through video analysis. In B. Grevholm, P. S. Hundeland, K. Juter, K. Kislenko, & P.-E. Persson (Eds.), *Nordic research in didactics of mathematic: past, present and future* (pp. 299–328). Oslo: Cappelen Damm.

Knoblauch, H. (2012). Videography, focused ethnography and video analysis. In H. Knoblauch, B. Schnettler, J. Raab, & H.-G. Soeffner (Eds.), *Video analysis: methodology and methods, qualitative audiovisual data in sociology* (pp. 69–83). Frankfurt: Peter Lang.

Linell, P. (1998). *Approaching dialogue. Talk, interaction and contexts in dialogical perspectives*. Amsterdam: John Benjamins.

Luckmann, T. (2012). Some remarks on scores in multimodal sequential analysis. In H. Knoblauch, B. Schnettler, J. Raab, & H.-G. Soeffner (Eds.), *Video analysis: methodology and methods, qualitative audiovisual data in sociology* (pp. 29–34). Frankfurt: Peter Lang.

Radford, L. (2000). Signs and meanings in students' emergent algebraic thinking: a semiotic analysis. *Educational Studies in Mathematics*, *42*, 237–268.

Radford, L. (2002). The seen, the spoken and the written. A semiotic approach to the problem of objectification of mathematical knowledge. *For the Learning of Mathematics*, *22*(2), 14–23.

Radford, L. (2012). On the development of early algebraic thinking. *PNA*, *64*, 117–133.

Radford, L., Bardini, C., & Sabena, C. (2007). Perceiving the general: the multisemiotic dimension of students' algebraic activity. *Journal for Research in Mathematics Education, 38*, 507–530.

Radford, L., Demers, S., Guzmán, J., & Cerulli, M. (2003). Calculators, graphs, gestures and the production of meaning. In N. Pateman, B. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 55–62). Honolulu: PME.

Roth, W.-M., & Thom, J. S. (2009). Bodily experience and mathematical conceptions: from classical views to a phenomenological reconceptualization. *Educational Studies in Mathematics, 70*, 175–189.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.

ENDNOTE

1. Defined in the project as when letters are introduced as variables in order to collect similar data in the four countries.