Event-Triggered Robust Adaptive Control for Discrete Time Uncertain Systems with Unmodelled Dynamics and Disturbances

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Abstract: In practice, modelling errors caused by high-order unmodelled dynamics and external disturbances are unavoidable. How to ensure the robustness of an adaptive controller with respect to such modelling errors is always a critical concern. In this paper, we consider the design of event-triggered robust adaptive control for a class of discrete time uncertain systems which involve such modelling errors and also are allowed to be non-minimum phase. Unlike some existing event-triggered control schemes, the developed controllers do not require that the measurement errors meet the corresponding input-to-state stable (ISS) condition. Global stability of the closed-loop system which means that all the signals are bounded is established in the presence of unmodelled dynamics and disturbances. Besides, in contrast to existing robust adaptive schemes, the designed adaptive controller does not involve parameters related to unmodelled dynamics and disturbances which are difficult to be chosen for ensuring such stability. An example is given to verify the effectiveness of the proposed control strategy.

1 Introduction

In recent years, the rapid development of network control technology has benefited mainly from its flexibility and ease of maintenance. However, network control systems also face a large problem, that is, communication bandwidth is limited. The control signals of the network control system are transmitted through the communication channel, but the communication channel usually needs to undertake other communication tasks and the bandwidth is limited. Therefore, reducing the frequency of signal transmission is of research value. The main idea of event-triggered control is to transmit signals as needed, rather than transmit signals in a fixed time interval. Therefore, event-triggered control has important research significance and some fruitful results have been established on event-triggered control [1–3]. Based on the small gain theory, [4] proposed a new ISS gain condition for continuous time nonlinear systems. The event-triggered strategy based on such gain conditions can avoid infinitely fast sampling phenomenon. By considering systems with quantizers, which may interact with samplers, a special event-triggered mechanism has been developed in [5]. However, it is still assumed that the control law satisfies ISS condition. So far, only limited number of results have been reported on the event-triggered control problems related to discrete-time systems. In [6] two kinds of event-triggered mechanisms, respectively involving static scheme and adaptive scheme, have been proposed. For a linear discrete time network system with uncertainties, an event-triggered based robust control strategy has been designed in [7]. Again similar to the above mentioned results on continuous time systems, a common assumption of these results is that the controller is ISS with respect to sampling error (measurement error). However, not all systems can easily be verified to meet such an assumption, especially for systems with high-order unmodelled dynamics and unknown parameters. The ISS condition can be relaxed by explicit controller design. In [8] the event-triggered rules are designed for discrete time systems with high-order unmodelled dynamics, and the stability of the system can be guaranteed even in the presence of time delay. In [9] dissipativity theory has been successfully used to analyze the stability of event-triggered control system with non-vanishing disturbances. However in these two papers, in order to design suitable controllers, a reduced-order model with known parameters needs to be available. In [10], an event-triggered control strategy has been designed to relax the ISS

assumption based on the backstepping method. However, the states of the continuous time system are required to be fully known. Note that system uncertainties can be classified to parametric uncertainties and non-parametric uncertainties. Adaptive control is an effective approach to handle the former one by using an on-line parameter estimator. However, it is rather challenging for adaptive control to handle unmodelled dynamics which is unavoidable either due to purposely ignoring some fast modes to obtain a reduced-order model for simplicity of controller design or being unable to be modelled. Thus how to address the robustness issue of adaptive control with the existence of unmodelled dynamics is both practically and theoretically important.

On the other hand, as we know, conventional parameter estimators such as estimators using a gradient estimation algorithm are obtained by minimizing certain cost functions and they are used to develop earlier conventional adaptive controllers. Such adaptive controllers, when applied to unknown plants satisfying certain ideal conditions, have some nice convergence properties [11]. However, they are non-robust against higher order unmodelled dynamics and/or external disturbances [12], which has motivated extensive research activities on robust adaptive control. To enhance their robustness, researchers have done various works. Most results focus on the modifications of estimator module, see for examples [13–20]. The major available modifications include normalization with parameter projection [14][15], sigma-modification plus normalization [16][17][18] and the use of deadzones [15] [19] [20]. These modified adaptive algorithms usually contain some critical design parameters depending on the priori knowledge of unmodelled dynamics and disturbances in order to ensure global stability [14, 15, 17, 19–22]. For example in [15] and [20], a deadzone function is introduced in parameter estimation for better properties. But the deadzone function involves an overbounding parameter that must be known, reflecting the upper bound of unmodelled dynamics gain. In addition, this parameter must be very small to achieve system stability. Obviously, it is difficult to choose the right parameter, which makes it difficult to implement the algorithm.

In this paper we design a new event-triggered output feedback robust adaptive control algorithm for a class of uncertain discrete-time systems involving ignored high-order unmodelled dynamics, which has never been considered in adaptive event-triggered control, and external disturbances. Our approach relaxes the ISS condition

and also achieves a global stability provided that the number of neglected fast poles and/or zeros is limited. The main contributions of this paper can be summarised as follows: 1)The ISS condition is removed by co-designing the new adaptive control algorithm and event-triggered mechanism. 2) In contrast to existing results on event-triggered control which do not take unmodelled dynamics and disturbances, a global stability result is achieved in their presence. 3) The designed adaptive controller does not involve parameters related to unmodelled dynamics and disturbances which are difficult to be chosen for ensuring stability. 4) Different from existing available adaptive control results, there is no minimum phase assumption on the class of systems to be controlled as simple pole assignment method is used to design the controller. But the simplicity of design and relaxation of such assumption bring much more challenges in establishing the system stability.

This paper is organised as follows. In Section 2, we give the mathematical model of the plant considered in this paper. Section 3 gives the event-triggered robust adaptive control algorithm. Section 4 presents the closed loop system stability results. Section 5 presents the simulation results. Finally, we summarize this paper in Section 6.

Notation: $\| \cdot \|$ denotes Euclidean norm

2 Problem formulations

We consider controlling a class of systems modelled as in [14] and [23], which is given as follows

$$y(k) = \bar{\theta}^T(k)\Phi(k-1) + f(k, \Phi(k-1)) + d(k) \qquad (1)$$

where $\Phi^T(k-1) = [y(k-1), ..., y(k-n-N), u(k-1), ..., u(k-n-N)]$ involving the past input and output signals which influence the current output signal $y(k)$, N is a nonnegative integer, $\bar{\theta}^T(k) = [\bar{\theta}_1(k), \bar{\theta}_2(k), \cdots, \bar{\theta}_{2(n+N)}(k)]$ is a vector consisting of unknown time-varying parameters, $f(k, \cdot)$ is a time varying nonlinear function, $d(k)$ is external disturbance of the system with an unknown upper bound.

In controller design, a simplified model is normally employed. For example, a second order linear time-invariant model is used to design a controller for a DC motor even though its actual order is higher than 2. Therefore, the following nominal reduced-order model is adopted to design the adaptive controller for the plant.

$$y(k) = \theta_*^T \phi(k-1) \qquad (2)$$

where $\phi^T(k-1) = [y(k-1), ..., y(k-n), u(k-1), ..., u(k-n)]$, $\theta_*^T = [-a_1, ..., -a_n, b_1, ..., b_n]$, a_i and b_i, $i = 1, 2, \cdots, n$ are unknown constants and n is a known integer denoting the reduced order of the system, for example 2 for the case of DC motor. Therefore, the modeling error includes two parts: the external disturbance $d(k)$ and the unmodelled dynamics $\varpi(k)$.

$$\varpi(k) = \Delta\bar{\theta}_*^T(k)\Phi(k-1) + f(k, \Phi(k-1)) \qquad (3)$$

where $\Delta\bar{\theta}_*^T(k) = [\bar{\theta}_1(k) + a_1, \cdots, \bar{\theta}_n(k) + a_n, \bar{\theta}_{n+1}(k), \cdots, \bar{\theta}_{n+N}(k), \bar{\theta}_{n+N+1}(k) - b_1, \cdots, \bar{\theta}_{2n+N}(k) - b_n, \bar{\theta}_{2n+N+1}(k), \cdots, \bar{\theta}_{2(n+N)}(k)]$ which is the difference between $\bar{\theta}(k)$ and θ_* (augmented with zeros). Through this conversion, then (1) can be rewritten as

$$y(k) = \theta_*^T \phi(k-1) + \varpi(k) + d(k) \qquad (4)$$

For the modelling error, we make some general assumptions.

Assumption 1.
$$\|\Delta\bar{\theta}_*(k)\| \leq \epsilon_1 \qquad \forall k \qquad (5)$$

$$|f(k, \Phi(k-1))| \leq \epsilon_2\|\Phi(k-1)\| \qquad \forall k \qquad (6)$$

where ϵ_1, ϵ_2 are nonnegative constants.

As N is a nonnegative integer and ϵ_1 is a small nonnegative constant, (5) implies that there are N fast poles and/or zeros (or parameter variations) have been neglected in nominal reduced-order model (2). This can be readily demonstrated.
Following Assumption 1, modeling error can be represented as

$$|\varpi(k)| \leq \epsilon\|\Phi(k-1)\| \qquad \forall k \qquad (7)$$

where $\epsilon = \epsilon_1 + \epsilon_2$.

Remark 1. *As in [23], no assumption on the knowledge of ϵ_1, ϵ_2 is made, which is different from some other approaches [14, 15, 17, 20–22]. In these papers, it is necessary to know the parameters related to the unmodelled effects for estimator modifications. The choice of such parameters is critical to establish system stability. For example, the bound ϵ should be known and large enough to satisfy (7) if a relative deadzone is used to modify parameter estimator [15, 19, 20], further it must be small enough to ensure system stability. Obviously, it is not easy to find such an ϵ.*

For external disturbance $d(k)$, we only require boundedness while the bound is not required to be known.

Assumption 2. *There exists an unknown constant D satisfying*

$$|d(k)| \leq D \qquad \forall k \qquad (8)$$

Besides, define two polynomials $A(q^{-1}), B(q^{-1})$ as follows which are derived from θ_*

$$A(q^{-1}) = 1 + a_1 q^{-1} + ... + a_n q^{-n} \qquad (9)$$

$$B(q^{-1}) = b_1 q^{-1} + ... + b_n q^{-n} \qquad (10)$$

For the nominal parameter vector θ_*, we have

Assumption 3. *θ_* lies in a known convex compact region $\mathcal{O}$ and $\mathcal{O}$ has the property that the polynomials $\hat{A}(q^{-1}), \hat{B}(q^{-1})$ induced by an arbitrary (non zero) vector $\hat{\theta}$ in $\mathcal{O}$ are uniformly coprime which means $\hat{A}(q^{-1})$ and $\hat{B}(q^{-1})$ do not have common roots in the compact region.*

Assumption 3 gives that

$$\|\theta_1 - \theta_2\| \leq k_\theta, \qquad \theta_1, \theta_2 \in \mathcal{O} \qquad (11)$$

$$\|\theta_3\| \leq k_c \qquad \theta_3 \in \mathcal{O} \qquad (12)$$

where k_θ, k_c are constants which can be unknown. k_θ represents the size of $\mathcal{O}$ and k_c represents the maximum distance from $\mathcal{O}$ to origin.

Remark 2. *Assumption 3 is to ensure the solvability of pole assignment equation given later. This assumption is normally required by pole assignment adaptive control approaches, see [15] for example and the relevant discussions there in.*

We aim to design an event-triggered adaptive controller for system (1) to follow a given reference trajectory $y^*(k)$. Because system (4) contains unknown parameter vector θ_* and modelling error involving unmodelled dynamics $\varpi(k)$ and disturbance $d(k)$, it is difficult to check and verify the ISS condition imposed in most existing event-triggered control literatures. To avoid this condition, we co-design the control law and the event-triggered mechanism to make the closed-loop system bounded input bounded output (BIBO) stable even in the presence of the unmodelled dynamics $\varpi(k)$ given in (3) and an arbitrarily bounded disturbance $d(k)$. Note that before system stability is established, the unmodelled dynamics $\varpi(k)$ cannot be assumed bounded, as its upper bound depends on system input and output according to (7). Thus the modelling error involves a bounded term $d(k)$ and also a term having unbounded effect, which makes the design and analysis challenging.

3 Event-triggered adaptive control law

In this section, a new indirect event-triggered adaptive control strategy is proposed, which includes two moduli: a parameter estimator and an event-triggered controller designed based on *Certainty Equivalence Principle* [11]. Next we introduce the two moduli separately.

3.1 Parameter estimator

Define a vector $x(k) \in R^{2(n+n_c)}$ as

$$
\begin{aligned}
x^T(k) = &[y(k-1), ..., y(k-n), ..., y(k-n-n_c) \\
&u(k-1), ..., u(k-n), ..., u(k-n-n_c)]
\end{aligned}
\tag{13}
$$

where n_c is an integer denoting the order of the controller and will be made precise later. The estimation algorithm for θ_* in the reduced-order model (2) is given by

$$
\hat{\theta}(k) = \mathcal{J}\left\{\hat{\theta}(k-1) + \frac{\phi(k-1)e(k)}{1 + x^T(k)x(k)}\right\}
\tag{14}
$$

where $\mathcal{J}$ denotes the projection operator introduced to ensure that estimated parameter vector $\hat{\theta}(k) \in \mathcal{O} \quad \forall k$, and $e(k)$ denotes prediction error which is defined as

$$
e(k) = y(k) - \phi^T(k-1)\hat{\theta}(k-1)
\tag{15}
$$

3.2 Event-triggered controller

As in [23], we use pole assignment strategy for controller design. Besides, we introduce an event-triggered mechanism to obtain the actual control signal, which can reduce the frequency of its transmission in network control.

Let $\omega(k)$ denote the signal obtained from the following pole-assignment control law, which is named as virtual control.

$$
\hat{L}(k)\omega(k) + \hat{P}(k)y(k) = \hat{P}(k)y^*(k)
\tag{16}
$$

i.e.

$$
\omega(k) = \hat{P}(k)y^*(k) - \hat{\xi}^T(k)\phi_0^T(k-1)
\tag{17}
$$

where

$$
\begin{aligned}
\hat{L}(k) &= 1 + \hat{\ell}_1(k)q^{-1} + ... + \hat{\ell}_{n_c}q^{-n_c} \\
\hat{P}(k) &= \hat{p}_1(k)q^{-1} + ... + \hat{p}_{n_c}q^{-n_c} \\
\hat{\xi}^T(k) &= [\hat{p}_1, ..., \hat{p}_{n_c}, \hat{\ell}_1, ..., \hat{\ell}_{n_c}] \\
\phi_0^T(k-1) &= [y(k-1), ..., y(k-n_c), \omega(k-1), ..., \omega(k-n_c)]
\end{aligned}
$$

Notice that we can solve algebraic equation (18) to get $\hat{L}(k), \hat{P}(k)$

$$
\hat{A}(k)\hat{L}(k) + \hat{B}(k)\hat{P}(k) = \mathcal{A}
\tag{18}
$$

where $\hat{A} = 1 + \hat{a}_1 q^{-1} + ... + \hat{a}_n q^{-n}$, $\hat{B} = \hat{b}_1 q^{-1} + ... + \hat{b}_n q^{-n}$ induced from estimator (14), $\mathcal{A}$ is a polynomial to be designed according to our needs. In order to ensure that the equation can be solved $\mathcal{A}$ must be monic with degree $n + n_c$. Besides $\mathcal{A}$ must be strictly (discrete-time) Hurwitz to achieve system stability.

From Assumption 3 and if the order n_c of the controller is not be less than n, the coefficients of $\hat{L}(k), \hat{P}(k)$ obtained from solving (18) can easily prove to be bounded [11], [15]. In our design we choose n_c as

$$
n_c = max\{n, N\}
\tag{19}
$$

where N is given in (1).

Remark 3. *Normally N is known as it is the number of purposely neglected fast poles and/or zeros of the plant during model reduction/simplification. Thus the choice of n_c is easy, unlike those parameters required by existing robust adaptive control schemes [13–20] such as ϵ in (7). Even if N is unavailable, the proposed adaptive controller in this paper can be shown to ensure local stability by following the method of analysis presented in appendixes of this paper and [23].*

The actual control signal $u(k)$ is generated from an event-triggered mechanism and is given by

$$
u(k) = \omega(k_c), \quad k_c \le k < k_{c+1}
\tag{20}
$$

$$
k_{c+1} = \inf_{k \in N}\{(k > k_c) \cap (|\omega(k) - u(k)| \ge m)\}, c \in N
\tag{21}
$$

where m is a positive constant chosen by user denoting the triggering threshold.
Let

$$
u(0) = \omega(0)
\tag{22}
$$

Then following the ideas in [10], we have the following expression for analysis purpose

$$
u(k) = \omega(k) + \lambda(k)m
\tag{23}
$$

where $\lambda(k)$ is a function and it can be shown that $|\lambda(k)| < 1$.
From (16) and (23), we have

$$
\hat{L}(k)[u(k) - \lambda(k)m] + \hat{P}(k)y(k) = \hat{P}(k)y^*(k)
\tag{24}
$$

i.e.

$$
u(k) = \hat{P}(k)y^*(k) - \hat{\xi}^T(k)\phi_1^T(k-1) + \hat{L}(k)\lambda(k)m
\tag{25}
$$

where

$$
\phi_1^T(k-1) = [y(k-1), ..., y(k-n_c), u(k-1), ..., u(k-n_c)]
$$

4 Stability results

From (19), we have

$$
N \le n_c
\tag{26}
$$

Then with the definition of $\Phi(k-1)$ and $x(k)$ in (13) we obtain

$$
\|\Phi(k-1)\| \le \|x(k)\| \qquad \forall t
\tag{27}
$$

Using (27), we can obtain the following lemma on the parameter estimator (13) - (15).

Lemma 1. *Suppose M_0 is a positive constant satisfying $D/M_0 \le \delta$. The estimator (13) - (15) applied to systems has the following properties:*
1)

$$
\tilde{e}(k) = \frac{e(k)}{(1 + x^T(k)x(k))^{1/2}}
\tag{28}
$$

$$
\in \ell_\infty
$$

2) If $\|x(\tau+1)\| > M_0, \tau = k_0, ..., k-1$ then

$$
\sum_{\tau=k_0+1}^{k}|\tilde{e}(\tau)|^2 \le \bar{k} + \alpha_1(k - k_0) + \alpha_2(k - k_0)
\tag{29}
$$

where

$$
\bar{k} = 2k_c k_\theta
\tag{30}
$$

$$
\alpha_1 = 2(k_\theta + 2\epsilon)\epsilon
\tag{31}
$$

$$
\alpha_2 = 2(k_\theta + 2\delta)\delta
\tag{32}
$$

3)

$$
\|\hat{\theta}(k) - \hat{\theta}(k-1)\| \le |\tilde{e}(k)|
\tag{33}
$$

Proof: Please see Appendix A.

Remark 4.
1) α_1 in (31) can be reduced arbitrarily small by requiring the ignored modes fast enough.
2) α_2 in (32) can be tuned to a small number by setting M_0 large enough. In fact, M_0 is introduced for stability analysis instead of a design parameter.

Now using Lemma 1, we can obtain the following stability result.

Theorem 1. *Consider the closed loop system consisting of plant* (1), *parameter estimator update laws* (13) - (15) *and the event-triggered based adaptive controller* (16) - (22). *There exists a constant $\bar{\epsilon}$ such that $\epsilon \leq \bar{\epsilon}$ ensures the closed-loop system bounded input bounded output stable (BIBO) for any bounded initial conditions.*

Proof: Please see Appendix B, as the proofs are quite involved and lengthy.

5 Simulation

In this section, we consider the following system.

$$y(k) = \bar{\theta}^T(k)\Phi(k-1) + d(k) \tag{34}$$

where $\bar{\theta}^T(k) = [2, -0.99, 0.2\sin(k), 0.5, 1]$, $\Phi(k-1) = [y(k-1), y(k-2), y(k-3), u(k-1), u(k-2)]$, $d(k) = 0.2\sin(0.1k)$. As mentioned earlier, we may not know or purposely ignore some high-order unmodelled dynamics and external disturbance to obtain reduced-order model for controller design. Thus we use the following nominal reduced-order model to design the event-triggered adaptive controller for system (34).

$$y(k) = \theta_*^T \phi(k-1) \tag{35}$$

where $\phi(k-1) = [y(k-1), y(k-2), u(k-1), u(k-2)]$, θ_*^T is a vector involving unknown parameters.

The control target is to make the output signal $y(k)$ track the reference signal $y^*(k) = 1$. Set $\mathcal{A}(q^{-1}) = [1, 0, 0, 0]^T$ which means the closed loop poles are all at the origin. The event-triggered threshold is given as $m = 0.2$ and the initial parameter estimation values are set as $\hat{\theta}(0) = [1, 0.5, 1, 0.6]^T$. Figures 1 show the actual control signal $u(k)$. And figures 2 shows the output signal $y(k)$ and the reference signal $y^*(k)$. The triggering interval time is shown in Figure 3. Clearly these results illustrate the effectiveness of the designed adaptive controller and verify the theoretical stability result established.
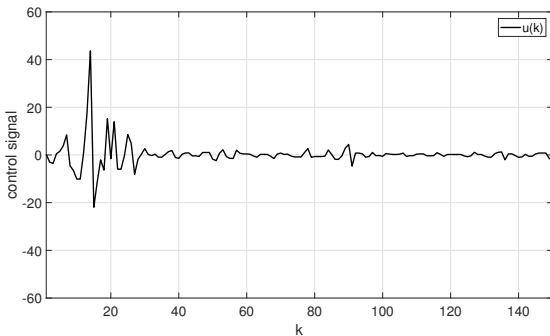
Fig. 1: control signal $u(k)$
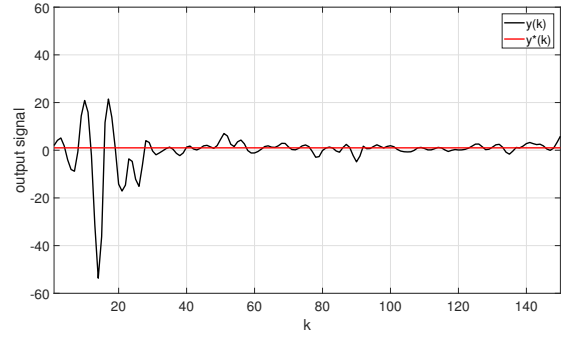
Fig. 2: output signal $y(k)$
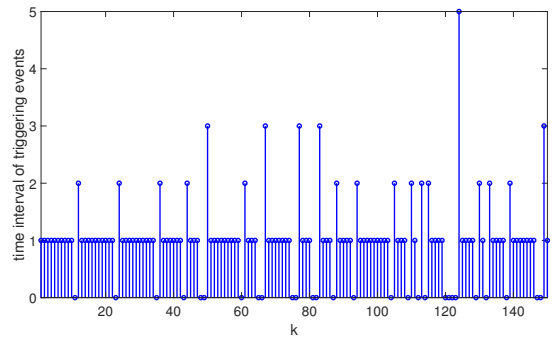
Fig. 3: triggering interval time

6 Conclusion

In this paper, we consider controlling a class of discrete time uncertain systems with unmodelled dynamics and disturbances. A new indirect event-triggered robust adaptive control algorithm has been proposed for reducing the frequency of transmission of control signal. By co-designing a suitable event-triggered mechanism and adaptive controller, the ISS condition has been removed. In addition, the controller design does not involve parameters which are difficult to be chosen based on the priori knowledge of unmodelled dynamics and disturbances. The stability analysis result shows that the event-triggered robust adaptive control algorithm can guarantee the global boundedness of all the signals in the closed loop system. Note that performance of tracking reference trajectories is not considered in this paper. This issue, which requires further research, may be addressed by employing the internal model principle as in [15] and [26]. Besides, event-triggered control for continuous-time systems with high-order unmodelled dynamics is another problem to study. Unlike discrete-time case, we cannot model the unmodelled dynamics in the way presented in this paper for continuous-time systems and inductive type of proof for stability analysis is not applicable. Thus the problem for continuous-time systems needs a different approach and deserves further investigation.

7 Acknowledgments

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Appendix A: Proof of lemma 1

1. From (4) and (15), we get

$$e(k) = -\tilde{\theta}^T(k-1)\phi(k-1) + \varpi(k) + d(k) \qquad (A36)$$

where $\tilde{\theta}(k) = \hat{\theta}(k) - \theta_*$ denoting the estimation error. Then, we can obtain the following inequality

$$
\begin{aligned}
|e(k)| &\le \|\phi(k-1)\|\|\tilde{\theta}(k-1)\| + |\varpi(k)| + D \\
&\le k_\theta\|\phi(k-1)\| + \epsilon\|\Phi(k-1)\| + D \\
&\qquad \text{using (7), (8) and (11)} \\
&\le k_\theta\|\phi(k-1)\| + \epsilon\|x(k)\| + D \\
&\qquad \text{using (27)}
\end{aligned}
\qquad (A37)
$$

Thus

$$|\tilde{e}(k)| \le \frac{k_\theta\|\phi(k-1)\| + \epsilon\|x(k)\| + D}{(1 + x^T(k)x(k))^{1/2}} \qquad (A38)$$
$$\in \ell_\infty.$$

2. Define $\hat{\theta}_{np}(k)$ as the parameter estimation before projection, which means $\hat{\theta}(k) = \mathcal{J}\{\hat{\theta}_{np}(k)\}$.
Thus

$$\hat{\theta}_{np} - \hat{\theta}(k-1) = \frac{\phi(k-1)e(k)}{1 + x^T(k)x(k)} \qquad (A39)$$

Choosing scalar function $V(k) = \tilde{\theta}^T(k)\tilde{\theta}(k)$, we have

$$
\begin{aligned}
V(k) - V(k-1) &\le \|\tilde{\theta}_{np}(k)\|^2 - \|\tilde{\theta}(k-1)\|^2 \\
&\le [\tilde{\theta}_{np}(k) - \tilde{\theta}(k-1) + 2\tilde{\theta}(k-1)] \\
&\quad [\tilde{\theta}_{np}^T(k) - \tilde{\theta}^T(k-1)] \\
&\le \frac{2\phi^T(k-1)\tilde{\theta}(k-1)e(k)}{1 + x^T(k)x(k)} \\
&\quad + \frac{|e(k)|^2}{1 + x^T(k)x(k)}
\end{aligned}
\qquad (A40)
$$

Now using (A36) gives

$$
\begin{aligned}
2\phi^T(k-1)\tilde{\theta}(k-1)e(k) &= 2(-e(k) + \varpi(k) + d(k))e(k) \\
&\le -2e^2(k) + 2|\varpi(k)||e(k)| \\
&\quad + 2|d(k)||e(k)|
\end{aligned}
\qquad (A41)
$$

From (A36), (A40) and (A41), we get

$$
\begin{aligned}
V(\tau) - V(\tau-1) &\le -\frac{e^2(\tau)}{1 + \|x(\tau)\|^2} \\
&+ \frac{2(|\phi^T(\tau-1)\tilde{\theta}(\tau-1)| + 2|\varpi(\tau)|)|\varpi(\tau)|}{1 + \|x(\tau)\|^2} \\
&+ \frac{2(2|d(\tau)| + |\phi^T(\tau-1)\tilde{\theta}(\tau-1)|)|d(\tau)|}{1 + \|x(\tau)\|^2}
\end{aligned}
\qquad (A42)
$$

Then using (7), (8), (11) and (27) we obtain

$$
\begin{aligned}
\tilde{e}^2(\tau) &\le V(\tau-1) - V(\tau) + 2(k_\theta + 2\epsilon)\epsilon \\
&+ 2(2\delta + k_\theta)\delta
\end{aligned}
\qquad (A43)
$$

Summing (A43) gives

$$
\begin{aligned}
\sum_{\tau=k_0+1}^{k} |\tilde{e}(\tau)|^2 &\le \|\tilde{\theta}(k_0)\|^2 - \|\tilde{\theta}(k)\|^2 + \alpha_1(k - k_0) \\
&+ \alpha_2(k - k_0)
\end{aligned}
\qquad (A44)
$$

Using (11) and (12), the result follows.
3.

$$
\begin{aligned}
\|\hat{\theta}(k) - \hat{\theta}(k-1)\| &\le \|\hat{\theta}_{np}(k) - \hat{\theta}(k-1)\| \\
&\le \frac{\|\phi(k-1)\||e(k)|}{1 + \|x(k)\|^2} \\
&\le |\tilde{e}(k)|
\end{aligned}
\qquad (A45)
$$

Appendix B: Proof of theorem 1

Define a posteriori prediction error ζ (see [11]) as

$$\begin{aligned} \zeta(k) =& y(k) - \phi^T(k)\hat{\theta}(k) \\ =& \hat{A}(k, q^{-1})y(k) - \hat{B}(k, q^{-1})u(k) \end{aligned} \tag{B1}$$

From (18) and after adding and substracting $\hat{L}(k)\hat{B}(k)u(k)$, we have

$$\begin{aligned} \mathcal{A}(q^{-1})y(k) =& \sum_{i=0}^{n_c} \hat{l}_i(k)[y(k-i) - \phi^T(k-i-1)\hat{\theta}(k)] \\ &+ \sum_{j=1}^{n} \hat{b}_j(k)[\hat{\xi}(k)\phi_1(k-j-1) + u(k-j)] \end{aligned} \tag{B2}$$

Similarly after adding and substracting $\hat{A}(k)\hat{P}(k)y(k)$, we have

$$\begin{aligned} \mathcal{A}(q^{-1})u(k) =& \sum_{j=0}^{n} \hat{a}_j(k)[\hat{\xi}(k)\phi_1(k-j-1) + u(k-j)] \\ &- \sum_{i=1}^{n_c} \hat{p}_i(k)[y(k-i) - \phi^T(k-i-1)\hat{\theta}(k)] \end{aligned} \tag{B3}$$

Then (24), (25) and (B1) give

$$\begin{aligned} \mathcal{A}(q^{-1})y(k) =& \sum_{i=0}^{n_c} \hat{l}_i(k)\zeta(k-i) \\ &+ \sum_{i=1}^{n_c} \hat{l}_i(k)[\hat{\theta}(k-i) - \hat{\theta}(k)]^T \phi(k-i-1) \\ &+ \sum_{j=1}^{n} \hat{b}_j(k)[\hat{\xi}(k) - \hat{\xi}(k-j)]^T \phi_1(k-j-1) \\ &+ \sum_{j=1}^{n} \hat{b}_j(k)[\hat{P}(k-j)y^*(k-j) + \hat{L}(k-j)\lambda(k-j)m] \end{aligned} \tag{B4}$$

and

$$\begin{aligned} \mathcal{A}(q^{-1})u(k) =& -\sum_{i=1}^{n_c} \hat{p}_i(k)\zeta(k-i) \\ &+ \sum_{i=1}^{n_c} \hat{p}_i(k)[\hat{\theta}(k) - \hat{\theta}(k-i)]^T \phi(k-i-1) \\ &+ \sum_{j=1}^{n} \hat{a}_j(k)[\hat{\xi}(k) - \hat{\xi}(k-j)]^T \phi_1(k-j-1) \\ &+ \sum_{j=1}^{n} \hat{a}_j(k)[\hat{P}(k-j)y^*(k-j) + \hat{L}(k-j)\lambda(k-j)m] \end{aligned} \tag{B5}$$

Let $\mathcal{A}(q^{-1}) = 1 + a_1^*(q^{-1}) + a_2^*(q^{-2}) + \cdots + a_{n+n_c}^*(q^{-n-n_c})$. Then the closed loop system can be expressed as

$$x(k+1) = Fx(k) + r(k) \tag{B6}$$

where $x(k) \in R^{2(n+n_c)}$ is given in (13) and

$$\begin{aligned} r(k) =& \mathcal{N}(k) \begin{bmatrix} \zeta(k) \\ \vdots \\ \zeta(k-n_c) \end{bmatrix} + \mathcal{M}(k) \begin{bmatrix} y^*(k) \\ \vdots \\ y^*(k-n) \end{bmatrix} \\ &+ \psi(k) \begin{bmatrix} \lambda(k) \\ \vdots \\ \lambda(k-n) \end{bmatrix} m + \Delta(k) \end{aligned} \tag{B7}$$

The form of F is

$$F = \begin{bmatrix} -a_1^* & -a_2^* & \dots & -a_{n+n_c}^* & & & & \\ 1 & & & & & & 0 & \\ & \ddots & & & & & & \\ & & 1 & 0 & & & & \\ & & & & -a_1^* & -a_2^* & \dots & -a_{n+n_c}^* \\ & & 0 & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & 1 & 0 \end{bmatrix} \tag{B8}$$

We note for later that

$$det(I - q^{-1}F) = [\mathcal{A}(q^{-1})]^2 \tag{B9}$$

and so F is exponentially stable.
$\mathcal{N}(k)$ contains controller parameters l_i, p_i; $\mathcal{M}(k), \psi(k)$ contain controller parameters l_i, p_i and model estimates a_i, b_i.
$\Delta(k)$ has the following $2(n + n_c)$ components:

$$\begin{aligned} \delta_{1i} &= \hat{l}_i(k)[\hat{\theta}(k-i) - \hat{\theta}(k)]^T \phi(k-i-1) & \text{(B10)} \\ \delta_{1j} &= \hat{b}_j(k)[\hat{\xi}(k) - \hat{\xi}(k-j)]^T \phi_1(k-j-1) & \text{(B11)} \\ \delta_{2i} &= \hat{p}_i(k)[\hat{\theta}(k) - \hat{\theta}(k-i)]^T \phi(k-i-1) & \text{(B12)} \\ \delta_{2j} &= \hat{a}_j(k)[\hat{\xi}(k) - \hat{\xi}(k-j)]^T \phi_1(k-j-1) & \text{(B13)} \end{aligned}$$

for $i = 1, ..., n_c, j = 1, ..., n$.
Now for each i and j using the Schwartz and triangle inequalities on (B10) and (B11), we get

$$\begin{aligned} |\delta_{1i}| &\leq c_{1i}\|\phi(k-i-1)\| \sum_{\tau=0}^{i-1} \|\hat{\theta}(k-\tau) - \hat{\theta}(k-\tau-1)\| \\ &\leq c_{1i}\|\phi(k-i-1)\| \sum_{\tau=0}^{i-1} \frac{|e(k-\tau)|}{(1 + \|x(k-\tau)\|^2)^{1/2}} \\ |\delta_{1j}| &\leq c_{1j}\|\phi_1(k-j-1)\| \sum_{\tau=0}^{j-1} \|\hat{\xi}(k-\tau) - \hat{\xi}(k-\tau-1)\| \\ &\leq c_{1j}\|\phi_1(k-j-1)\| \sum_{\tau=0}^{j-1} \frac{|e(k-\tau)|}{(1 + \|x(k-\tau)\|^2)^{1/2}} \end{aligned}$$

where c_{1i} and c_{1j} are constants. (In the following derivation, all $c_k, k = 1, 2, ...$ denote constants without further statement.)

By definition of $x(k)$, we have

$$\frac{\|\phi(k-i-1)\|}{(1+\|x(k-\tau)\|^2)^{1/2}} \le 1$$

$$\frac{\|\phi_1(k-i-1)\|}{(1+\|x(k-\tau)\|^2)^{1/2}} \le 1$$

for $\tau \le i-1$ and $\tau \le j-1, i = 1, ..., n_c$ and $j = 1, ..., n$. Thus

$$|\delta_{1i}| \le c_{1i} \sum_{\tau=0}^{i-1} |e(k-\tau)| \tag{B14}$$

$$|\delta_{1j}| \le c_{1j} \sum_{\tau=0}^{j-1} |e(k-\tau)| \tag{B15}$$

Bounds of the same form can be established for the components δ_{2i} and δ_{2j}. From (B1), we note

$$\zeta(k) = e(k) + [\hat{\theta}(k-1) - \hat{\theta}(k)]^T \phi(k-1) \tag{B16}$$

Thus from (33), it follows that

$$|\zeta(k)| \le 2|e(k)| \tag{B17}$$

Then from (B7), (B14), (B15) and (B17), we have

$$\|r(k)\| \le c_1 \sum_{j=0}^{n_c} |e(k-j)| + c_2 \tag{B18}$$

where we have used $n \le n_c$ and the fact that $\mathcal{N}(k), \mathcal{M}(k), \psi(k)$ and $y^*(k), m$ are bounded.
Now solving equation (B6) gives

$$x(k+1) = F^{k-k_0+1} x(k_0) + \sum_{\tau=t_0}^{k} F^{k-\tau+1} r(\tau) \tag{B19}$$

Since F is exponentially stable, $\exists \mu \in (0,1)$ such that from (B18) and (B19) we get

$$\|x(k+1)\| \le c_3 \mu^{k-k_0} \|x(k_0)\| + c_1 c_4 g(k)$$
$$+ c_2 \sum_{\tau=k_0}^{k} \mu^{k-\tau} \tag{B20}$$

where

$$g(k) = \sum_{\tau=k_0}^{k} \mu^{k-\tau} \sum_{j=0}^{n_c} |e(\tau-j)|$$

$$= \mu^{k-k_0} \sum_{j=0}^{n_c} |e(k_0 - j)| \tag{B21}$$

$$+ \sum_{\tau=k_0+1}^{k} \mu^{k-\tau} \sum_{j=0}^{n_c} |e(\tau-j)|$$

The second term in (B21) can be bounded as follows by

$$\sum_{\tau=k_0+1}^{k} \mu^{k-\tau} \sum_{j=0}^{n_c} |e(\tau-j)|$$

$$= \sum_{j=0}^{n_c} \mu^{-j} \sum_{\tau=k_0+1}^{k} \mu^{k-\tau+j} |e(\tau-j)|$$

$$= \sum_{j=0}^{n_c} \mu^{-j} \sum_{\tau=k_0-j+1}^{k-j} \mu^{k-\tau} |e(\tau)|$$

$$\le \sum_{j=0}^{n_c} \mu^{-j} \Big(\sum_{\tau=k_0+1}^{k} \mu^{k-\tau} |e(\tau)|$$

$$+ \sum_{\tau=k_0-j+1}^{k_0} \mu^{k-\tau} |e(\tau)| \Big)$$

$$\le c_5 \sum_{\tau=k_0+1}^{k} \mu^{k-\tau} |e(\tau)|$$

$$+ \mu^{k-k_0} \sum_{j=1}^{n_c} \sum_{\tau=k_0-j+1}^{k_0} \mu^{k_0-j-\tau} |e(\tau)|$$

$$\le c_5 \sum_{\tau=k_0+1}^{k} \mu^{k-\tau} |e(\tau)|$$

$$+ c_6 \mu^{k-k_0} \sum_{j=1}^{n_c} |e(k_0 - j + 1)|$$

Substituting bach through (B20) and (B21) gives

$$\|x(k+1)\| \le c_3 \mu^{k-k_0} \|x(k_0)\| + c_7 \mu^{k-k_0} \sum_{j=0}^{n_c} |e(k_0 - j)|$$

$$+ c_8 \sum_{\tau=k_0+1}^{k} \mu^{k-\tau} |e(\tau)| + c_2 \sum_{\tau=k_0}^{k} \mu^{k-\tau} \tag{B22}$$

We now use an inductive proof starting from the assumption that $\|x(\tau+1)\| \le M$ for $\tau = 0, ..., k-1$ and $k_0 \in [0, k-1]$ is specified later.
The next step is to bound the first summation term in (B22). Using (A36), (7), (8) and (11) gives

$$|e(k_0 - j)| \le k_\theta \|\phi(k_0 - j - 1)\| + \epsilon \|\Phi(k_0 - j - 1)\| + D$$

From (13), (27) and the inductive hypothesis, we get

$$\sum_{j=0}^{n_c} |e(k_0 - j)| \le c_8 \|x(k_0)\| + \bar{\epsilon} M + c_9 \tag{B23}$$

where $\bar{\epsilon}$ depends only on the parameter ϵ and integer N. Clearly we can adjust it to an arbitrarily small positive number by restricting ϵ. Then substituting (B22) into (B21) gives

$$\|x(k+1)\| \le c_{11} \mu^{k-k_0} [\|x(k_0)\| + c_1 2\bar{\epsilon} M]$$

$$+ c_8 \sum_{\tau=k_0+1}^{k} \mu^{k-\tau} (1 + \|x(\tau)\|) |\tilde{e}(\tau)| \tag{B24}$$

$$+ \sum_{\tau=k_0}^{k} \mu^{k-\tau} c_{13}$$

As in [23], time sequence Z_+ can be divided into two subsequences, i.e.

$$Z_1 = \{k | \|x(k+1)\| > M_0\}$$

$$Z_2 = \{k | \|x(k+1)\| \le M_0\}$$

Clearly, we just need to show $x(k+1)$ is bounded in Z_1. Suppose $k_0, ..., k-1 \in Z_1$ and $k_0 - 1 \in Z_2$, i.e. $\|x(k_0)\| \le M_0$. Also assume $\|x(1)\| \le M_0$. Note that this does not constrain the initial conditions, as for any given finite $x(1)$, there always exists such a M_0. Now we show that $\|x(k+1)\| \le M$.

Squaring both sides of (B24) and applying the Schwartz inequality, we get

$$\|x(k+1)\|^2 \le c_{14}\mu^{2(k-k_0)}[\|x(k_0)\|^2 + c_{15}\bar{\epsilon}^2 M^2]$$
$$+ c_{16} \sum_{\tau=k_0}^{k-1} \mu^{k-\tau}[|\tilde{e}(\tau+1)|^2 \qquad \text{(B25)}$$
$$\|x(\tau+1)\|^2 + |\tilde{e}(\tau+1)|^2 + c_{17}]$$

By multiplying two sides of equation (B25) by μ^{-k}, we can get

$$\mu^{-k}\|x(k+1)\|^2 \le s^2(k)$$
$$+ c_{16} \sum_{\tau=k_0}^{k-1} \mu^{-\tau}\|x(\tau+1)\|^2 |\tilde{e}(\tau+1)|^2 \qquad \text{(B26)}$$

where

$$s^2(k) = c_{14}\mu^{-k_0}[\|x(k_0)\|^2 + c_{15}\bar{\epsilon}^2 M^2]$$
$$+ c_{16} \sum_{\tau=k_0}^{k-1} \mu^{-\tau} c_{18} \qquad \text{(B27)}$$

By applying Gronwall Lemma in [24] to (B26), we have

$$\|x(k+1)\|^2 \le \mu^k s^2(k) + \sum_{\tau=k_0}^{k-1} \Gamma(k,\tau)\mu^\tau s^2(\tau) \qquad \text{(B28)}$$

where

$$\Gamma(k,\tau) = (\mu c_{16}|\tilde{e}(\tau+1)|^2) \prod_{\tau_1=\tau+2}^{k} \mu(1 + c_{16}|\tilde{e}(\tau_1)|^2) \quad \text{(B29)}$$

where set the product term to 1 for $\tau = k-1$. And applying the theorem of Arithmetic and Geometric Means in [25] gives

$$\prod_{i=1}^{n} a_i \le \{\frac{1}{n}\sum_{i=1}^{n} a_i\}^n \qquad \text{(B30)}$$

for a sequence of nonnegative numbers.

Applying (B30) to (B29) gives that

$$|\Gamma(k,\tau)| \le \{\frac{1}{k-\tau}[\sum_{\tau_1=\tau+1}^{k} \mu(1 + c_{16}|\tilde{e}(\tau_1)|^2)]\}^{k-\tau}$$
$$\le \{\frac{\mu c_{16}k^2}{k-\tau} + \mu(c_{16}\alpha_1 + c_{16}\alpha_2 + 1)\}^{k-\tau} \qquad \text{(B31)}$$
$$\text{using (29)}$$

Choose $\mu < \mu_c^* < 1$. Thus from equations (31) and (32) we can clearly see that $\exists \epsilon_1^*, \delta^*$ are small enough such that $\mu_c^* - \mu(1 +$

$c_{16}\alpha_2) > 0$ for $\delta \le \delta^*$ and

$$\sigma(c_{16}\alpha_1 + c_{16}\alpha_2 + 1) \le \sigma_c^* \qquad \text{(B32)}$$

for $\epsilon_1 \le \epsilon_1^*$.

From (B31)

$$|\Gamma(k,\tau)| \le (\mu_c^*)^{k-\tau}\{1 + \frac{\mu c_{16}k^2}{\mu_c^*(k-\tau)}\}^{k-\tau}$$
$$\le (\mu_c^*)^{k-\tau} exp\{\frac{\mu c_{16}k^2}{\mu_c^*}\}$$

using the inequality $(1 + \frac{1}{x})^x \le e$.

Therefore $\exists K > 1$ satisfying

$$|\Gamma(k,\tau)| \le K(\mu_c^*)^{k-\tau} \qquad \text{(B33)}$$

Next, we pay attention to $\mu^\tau s^2(\tau)$ for $\tau \ge l_0$ in (B28).

$$\mu^\tau s^2(\tau) = c_{14}\mu^{\tau-k_0}[\|x(k_0)\|^2 + c_{15}\bar{\epsilon}^2 M^2]$$
$$+ c_{16} \sum_{\tau_1=k_0}^{k-1} \mu^{\tau-\tau_1} c_{18}$$
$$\le c_{19} + c_{20}\|x(k_0)\|^2 + c_{21}\bar{\epsilon}^2 M^2$$

Substituting (B34) and (B33) into (B28), we get

$$\|x(k+1)\| \le c_{19} + c_{20}\|x(k_0)\|^2 + c_{21}\bar{\epsilon}^2 M^2$$
$$+ \sum_{\tau=k_0}^{k-1} K(\mu_c^*)^{k-\tau}[c_{19} + c_{20}\|x(k_0)\|^2 + c_{21}\bar{\epsilon}^2 M^2]$$
$$\le c_{22} + c_{23}M_0^2 + c_{24}\bar{\epsilon}^2 M^2$$
$$\text{(B34)}$$

where we have used $\|x(k_0)\| \le M_0$.

Now if $c_{24}(\bar{\epsilon}^*)^2 < 1$ and

$$M^2 > \frac{c_{22} + c_{23}M_0^2}{1 - c_{24}(\bar{\epsilon}^*)^2},$$

we have

$$\|x(k+1)\| \le M^2 \qquad \text{(B35)}$$

for all $\bar{\epsilon} \le \bar{\epsilon}^*$.

As $\bar{\epsilon} = o(\epsilon)$, so (B35) can be guaranteed if $\epsilon \le \epsilon_2^*$, where ϵ_2^* is a constant associated with $\bar{\epsilon}^*$ and chosen to ensure that $\bar{\epsilon} \le \bar{\epsilon}^*$. Thus taking $\epsilon = \min\{\epsilon_1^*, \epsilon_2^*\}$ and

$$M^2 > \max\{\frac{c_{22} + c_{23}M_0^2}{1 - c_{24}(\bar{\epsilon}^*)^2}, M_0^2\} \qquad \text{(B36)}$$

we have proved the result.

Remark 5. *For a given system, there always exists a M_0 such that $\|x(1)\| \le M_0, \|y^*(k)\|_\infty \le M_0$ and $D/M_0 \le \delta^*$ for any bounded initial conditions, set point and disturbance, where δ^* is a sufficiently small number to ensure (B32) satisfied. Note that δ^* is defined in such a way that it is independent of M_0 in establishing (B32). Since the stability condition does not depend on M_0, we do not need to know it while being aware of its existence and role as an auxiliary variable in proving our result.*