A Conclusive Analysis of the Finite-Time Behavior of the Discretized Pursuit Learning Automaton

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Abstract—This paper deals with the Finite-Time Analysis (FTA) of Learning Automata (LA), which is a topic for which very little work has been reported in the literature. This is as opposed to the asymptotic steady-state analysis for which there are, probably, scores of papers. As clarified later, unarguably, the FTA of Markov Chains, in general, and of LA, in particular, is far more complex than the asymptotic steady-state analysis. Such a FTA provides rigid bounds for the time required for the LA to attain a given convergence accuracy. We concentrate on the FTA of the Discretized Pursuit Automaton (DPA), which is probably one of the fastest and most accurate reported LA. Although such an analysis was carried out many years ago, we record that the previous work is flawed. More specifically, in all brevity, the flaw lies in the wrongly “derived” monotonic behavior of the LA after a certain number of iterations. Rather, we claim that the property that should be invoked is the submartingale property. This renders the proof to be much more involved and deep. In this paper, we rectify the flaw and re-establish the FTA based on such a submartingale phenomenon. More importantly, from the derived analysis, we are able to discover and clarify, for the first time, the underlying dilemma between the DPA’s exploitation and exploration properties. We also non-trivially confirm the existence of the optimal learning rate, which yields a better comprehension of the DPA itself.

Keywords Learning automaton, Pursuit algorithms, DPA, Finite-time analysis

I. INTRODUCTION

The field of Learning Automata (LA) has been studied as a typical model of reinforcement learning for decades. The LA operates in conjunction with an Environment, with the goal of learning the optimal action from among a set of actions offered by the Environment. LA have been applied in many dozens of fields. Conceptually, the LA works with an Environment in an iterative manner. In each iteration, the LA selects one action, which triggers either a stochastic reward or a penalty from the Environment. Then, based on the responses received from the Environment, the LA adjusts its action selection strategy in order to make a “wiser” decision in the next iteration. In such a way, the LA learns the optimal decision via repeated interactions with the Environment.

Non-Estimator LA: The development of LA has gone through three periods: the Fixed Structure Stochastic Automata (FSSA), the Variable Structure Stochastic Automata (VSSA) and the Estimator Algorithms (EAs). The FSSA are LA whose state update function and decision function are time invariant. The Tsetlin, Krylov and Krinsky automata [14] are the most notable examples of this type. As opposed to FSSA, in VSSA, either the state update function or the decision function (or both) vary with time. Interestingly, VSSA can be characterized by functions that update the probability of selecting the various actions. Typical examples of VSSA include the Linear Reward-Penalty (L

Estimator-based LA: Among the families of LA, Estimator Algorithms (EAs) are the fastest and the most accurate type of LA. They augment an action probability updating scheme with the use of estimates of the respective actions’ reward probabilities. Typically, EAs maintain running Maximum Likelihood (ML) reward probability estimates to determine whether any specific action is “better” than another. Within this family, the set of Pursuit Algorithms (PAs) were the pioneering schemes, whose design and analysis were initiated by Thathachar and Sastry [28]. The first PA was designed to operate by updating the action probabilities based on the L

2With regard to applications, the entire field of LA and stochastic learning has had a myriad of applications [8], [13], [14], [23], [29], which (apart from the many applications listed in these books) include solutions for problems in network and communications [12], [17], [22], network call admission, traffic control, quality of service routing [1], [2], [31], distributed scheduling [26], training hidden Markov models [7], neural network adaptation [11], intelligent vehicle control [30] service selection [32] and even fairly theoretical problems such as graph partitioning [20] and string taxonomy [18]. Besides these fairly generic applications, LA have been used to assist in solving numerous other optimization problems in stochastic domains, for example, in telecommunications [4], [5], [38] and in the energy sector [25], [27].
On Using Bayesian estimates: Another family of EAs, called the family of Bayesian Pursuit Algorithms (BPAs) has been designed within the Pursuit learning paradigm. It enhances the rate of convergence by substituting ML estimates with more optimistic Bayesian estimates.

Discretized EAs: To enhance the convergence, Oommen and Lanctot [21] presented the Discretized Pursuit Algorithm (DPA) by discretizing the action probability space. The DPA was shown to be superior to its continuous counterpart in terms of its rate of convergence. Besides, the DPA has the potential to be processed in a batch mode, and a fast version of the DPA was proposed in [6] to achieve a lower computational complexity. Along the same vein, the continuous BPA [33] has also been discretized to yield the Discretized BPA (DBPA) [34], with the latter being shown to be superior to its continuous counterpart in terms of its speed of convergence.

II. Proof Methodologies for LA

Like other randomized learning algorithms, in the development of LA, one of the most important aspects to validate the design of an LA is to mathematically analyze its convergence, i.e., to investigate if the LA is able to converge to the optimal action with an arbitrarily large probability. If the answer is positive, the LA is considered “e-optimal”. The mathematical techniques used for the various former-mentioned families of LA are quite distinct. The methodology for the family of FSSA involves formulating the Markov chain for the LA, computing its equilibrium probabilities, and then computing the asymptotic action selection probabilities. The proofs of convergence for VSSA involve the theory of small-step Markov processes, distance diminishing operators, and the theory of regular functions. The proofs for discretized LA involve the asymptotic analysis of the Markov chain that represents the LA in the discretized space, whence the total probability of convergence to the various actions is evaluated.

Understandably, the most difficult proofs involve the family of EAs. This is because the convergence involves two intertwined phenomena, i.e., the convergence of the reward estimates and the convergence of the action probabilities themselves. Ironically, the combination of these in the updating rule is what renders the EA fast. However, if the accuracy of the estimates are poor because of inadequate estimation (i.e., the sub-optimal actions are not sampled “enough number of times”), the convergence accuracy can be diminished, which is really a dilemma! The original proofs of convergence of PAs erroneously invoked the monotonicity property. These have since been rectified to invoke the submartingale property in [19], [33], [37].

A. Prior Flawed “Proofs” for EAs

As PAs hold the pioneering status in the development of EAs, a lot of work has been done on the analysis of the PAs [9], [10], [19], [21] and [24]. A thorough investigation on the convergence and finite-time behavior of PAs were done in [24], where the probability of selecting the optimal action was considered monotonically increasing after some time instant “t0”, and where the number of iterations that guarantees this t0 and the number of iterations that allows the LA to converge to the optimal action after t0, were bounded. Though the work in [24] was innovative and compass both the CPA and the DPA, there was a flaw in its reasoning, which rendered the analysis incorrect. To correct this flaw, new proofs for the CPA and DPA’s convergence were proposed in [36] and [35] respectively, where the authors investigated the submartingale property of the probability of selecting the optimal action, and invoked the theory of regular functions to prove that the PAs converge to the optimal action in probability. They also claimed that the new proof methodology could be extended to prove the convergence of other Pursuit-based LAs. Though the proof for the convergence of PAs has been rectified by [36] and [35], the corresponding finite-time analyses is still open.

B. Complexity of Finite-time Analysis

Most of the analysis of Markov chains and LA deal with the scenario after the transient phase of the chain has elapsed. This is because the asymptotic analysis of Markov chains is well established. In the case of ergodic Markov chains it involves determining the eigenvector associated with the eigenvalue λ = 1, and this eigenvector can be computed by an explicit eigenvalue computation or by repeatedly multiplying the underlying Markov matrix. It can also be computed by simulation and recording the ensemble average of the time after a reasonable number of observations have passed. Alternatively, in the case of absorbing Markov chains, the analysis is obtained by computing the absorption probabilities determined by the solution of the first passage probabilities (from the Kolmogorov-Chapman equations). Both of these have formed the basis for analysis even when the number of actions are large and when they are arranged hierarchically.

The Finite-time analysis of Markov chains and LA is far more complex. The reason for this is that these depend on the non-unity eigenvalues of the Markov matrix and the way by which they lead to the convergence to the asymptotic value in a geometric manner. Unlike the asymptotic case, which only involves the case when λ = 1, the Finite-time analysis involves all the eigenvalues. Due to the complexity of the situation, understandably there is very little work done on the Finite-time analysis of any family of LA, including the families of FSSA and/or VSSA.

C. Goal and Contributions of this Paper

With the above as a backdrop, we state that this is precisely the contribution of this present paper, i.e., to achieve a Finite-time analysis for a specific PA. Since we have argued on the complexity of such an analysis, we consider the process by which the specific LA can converge to the optimal action. Thereafter, by invoking the submartingale property of the underlying random process, we succeed in achieving this Finite-time analysis.
More specifically, in this paper, we aim at re-analyzing the Finite-time behavior of the DPA. That being said, we state that we are not providing a Finite-time analysis for the CPA based on submartingale property because the state space of the CPA is open and varies with time. As far as we know, this renders the analysis impossible, and the previous “proof” inaccurate. This is because one cannot prove this result by alluding to the monotonic property of the random process because the process is, quite simply, not monotonic. Our proof for the DPA is based on the new methodology used in [35] and [36], using which we succeed in obtaining bounds on the Finite-time behavior.

Thus, we can summarize the main contributions of this paper to be as follows:

- We rectify the error in the previous Finite-time analysis of the DPA, which has existed as a benchmark for over 20 years.
- We illustrate, for the first time, the dilemma between the CPA’s exploitation and exploration. Using these, and based on our theoretical study, the existence of the DPA’s optimal learning rate is justified via numerical analysis.

To pictorially clarify the main contribution of the paper, we include, in Figure 1, a schematic of how the DPA’s Finite-time analysis is achieved.

![Fig. 1: A schematic of how the proof of the DPA’s convergence is achieved.](image)

III. NOTATIONS AND A BRIEF REVIEW OF THE DPA

Before we analyze the Finite-time behavior of the DPA, we present the notations used in the DPA and the algorithm (and the proofs), as follows:

- \( r \): The number of actions.
- \( \alpha_t \): The \( t^{th} \) action that can be selected by the LA, \( \alpha_t \in \{ \alpha_1, \ldots, \alpha_r \} \).
- \( t \): Time index of the learning process and \( t \) belongs the natural numbers.
- \( \mathbf{P}(t) \): The action probability vector \( \mathbf{P}(t) = [p_1(t), p_2(t), \ldots, p_r(t)] \) at time step \( t \).
- \( p_i(t) \): The \( i^{th} \) element of the action probability vector, \( \mathbf{P}(t) \), at time step \( t \).
- \( u_i(t) \): The number of times that action \( \alpha_i \) has been rewarded by time step \( t \).
- \( v_i(t) \): The number of times that action \( \alpha_i \) has been selected by time step \( t \).

\begin{itemize}
  \item \( D = [d_1, d_2, \ldots, d_r] \): The true reward probability vector of the Environment.
  \item \( d_i \): The \( i^{th} \) element of the true reward probability vector \( D \).
  \item \( \hat{D}(t) = [\hat{d}_1(t), \hat{d}_2(t), \ldots, \hat{d}_r(t)] \): The reward probability estimates vector of the Environment at time step \( t \).
  \item \( \hat{d}_i(t) \): The \( i^{th} \) element of the reward probability estimates vector \( \hat{D}(t) \).
  \item \( m \): The index of the optimal action.
  \item \( h \): The index of the greatest element in \( \hat{D}(t) \).
  \item \( R(t) \): The response from the Environment at time step \( t \), where \( R(t) = 0 \) corresponds to a Reward, and \( R(t) = 1 \) to a Penalty.
  \item \( \Delta \): The discretized step size, where \( \Delta = \frac{1}{N} \), with \( N \) being a positive integer.
\end{itemize}

The DPA follows a “pursuit” paradigm of learning, which consists of three steps. Firstly, it maintains an action probability vector \( \mathbf{P}(t) = [p_1(t), p_2(t), \ldots, p_r(t)] \) to determine the action to be selected, where \( \sum_{i=1}^{r} p_i(t) = 1, \forall t \). More specifically, at time \( t \), an action is chosen according to the probability distribution \( \mathbf{P}(t) \). Secondly, it maintains running ML reward probability estimates to determine which action can be reckoned to be the “best” in the current iteration. Thus, it updates \( \hat{d}(t) \) based on the Environment’s response as:

\[
\begin{align*}
  u_i(t) &= u_i(t-1) + (1 - R(t)), \\
  v_i(t) &= v_i(t-1) + 1, \\
  \hat{d}_i(t) &= \frac{u_i(t)}{v_i(t)}. 
\end{align*}
\]

Thirdly, based on the response of the Environment and the knowledge of the current best action, the DPA increases the probability of selecting this action as per the Discretized \( LR_{-1} \) rules. So if \( \hat{d}_h(t) \) is the greatest element in \( \hat{D}(t) \), we update \( p(t) \) as:

- If \( R(t) = 0 \) Then
  \[ p_i(t+1) = \max\{p_i(t) - \Delta, 0\}, \forall i \neq h, \]
  \[ p_h(t+1) = 1 - \sum_{i \neq h} p_i(t+1). \]
- Else
  \[ p(t+1) = \mathbf{P}(t). \]

EndIf

We now consider the various methods by which the Finite-time analysis for the DPA has been tackled.

IV. PREVIOUS FINITE-TIME ANALYSIS ON THE DPA

To place our current result in the right perspective, it is prudent to consider the previous Finite-time analysis on the DPA submitted in [24]. In that work, \( t_0 \) is defined as the time instant after which the probability of selecting the optimal action, i.e., \( p_m(t), \forall t > t_0 \), is assumed to be monotonically increasing. The Finite-time analysis can thus be divided into two periods, namely, the one before the time instant \( t_0 \) and the one after \( t_0 \). For ease of expression, we call the learning process before time \( t_0 \) as the Learning Period (LP), and the learning process after \( t_0 \) as the Converging Period (CP). We will outline how the previous paper [24] analyzed the DPA’s behavior during both the Learning Period and the Converging Period, and then point out where the flaw of the analysis lies.

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We are grateful to the anonymous Referee who requested this figure.
We first summarize the analysis on the LP [24], where the aim is to calculate the bound of \( t_0 \). Since \( p_m(t) \) is assumed to be monotonically increasing \( \forall t > t_0 \), a certain accuracy for the reward probability estimates is required at \( t_0 \) for all actions. To achieve the required accuracy for these estimates, each action needs to be selected at least a certain number of times. We denote \( M_i(t_0), 1 \leq i \leq r \), as the number of times that each action is \emph{required} to be selected in order to achieve the accuracy of the estimate, and then define \( M = \max\{M_i(t_0), 1 \leq i \leq r\} \). Let \( n_i(t) \) be the number of times that \( \alpha_i \) has been selected by time \( t \), and \( \theta \) be a positive number close to 0. The next step is to determine the time instant \( t_0 \), after which, \( \Pr(n_i(t) \geq M) \geq 1 - \theta \) holds. Note that we have to describe the event “\( n_i(t) \geq M \)” from the perspective of the corresponding probability for the given parameter \( \theta \), due to the randomness of the behavior of the DPA. The method to achieve this is briefly explained below, while the reader is referred to [24] for additional details.

First of all, we have

\[
\Pr(n_i(t) \geq M) \geq 1 - \theta \\
\iff \Pr(n_i(t) < M) < \theta \\
\iff \sum_{j=0}^{M-1} \Pr(n_i(t) = j) < \theta \\
\iff \Pr(n_i(t) = j) < \frac{\theta}{M} \forall j,
\]

where the symbol \( \iff \) is meant to state “is equivalent to”, while the symbol \( \iff \) indicates “can be deduced by”. Obviously, if we can find the upper bound of \( \Pr(n_i(t) = j) \) as a function of \( t_0 \), we can derive the requirement of \( t_0 \) as a function of \( M \) and \( \theta \).

Clearly, \( p_i(t) \geq p_i(0) - t \Delta \) holds. Therefore, we have

\[
\Pr(\alpha_i(t) \neq \alpha_i) \leq 1 - (p_i(0) - t \Delta) = 1 - p_i(0) + t \Delta,
\]

and accordingly,

\[
\Pr(n_i(t) = j) < \binom{t}{j} (1 - p_i(0) + t \Delta)^{t-j} < t! (1 - p_i(0) + t \Delta)^{t-j} < \frac{\theta}{M}
\]

holds. Hence, considering Eq. (1) and Eq. (2), if

\[
t! (1 - (p_i(0) - t \Delta))^{t-j} < \frac{\theta}{M}
\]

holds, Eq. (1) follows.

Moreover, the authors of [24] determined that if

\[
\Delta = \frac{1}{r \times 2t}
\]

is satisfied, Eq. (3) is implied.

Given the value of \( \Delta \) as shown in Eq. (4), \( t_0 \) can be calculated as a function of \( M \) and \( \theta \). This is done by denoting \( t_0 = f(M, \theta) \) as shown in Eq. (5):

\[
f(M, \theta) = \left\lceil \frac{2M}{\ln(1/\sigma)} \ln \left[ \frac{M}{\ln(1/\sigma) \sigma} \left( \frac{M}{\theta} \right)^{1/2} \right] \right\rceil,
\]

where \( \sigma = \frac{2e-1}{2} \).

The deduction of \( f(M, \theta) \) involves quite a lot of algebraic manipulations, omitted here due to space limitations. But it can be seen as Lemma 4.1 and the corresponding proof in [24]. The final consequential result is that if \( \Delta = \frac{1}{r \times 2t} \), then till the time instant \( t_0 = f(M, \theta) \), the probability that each action is selected at least \( M \) times is greater than or equal to \( 1 - \theta \). Note that the derivation of Eq. (5) does not require any monotonic property. It just returns the required time span by considering the required number of times, \( M \), and the required probability resolution, \( \theta \), as inputs, both of which are entirely based on the nature of DPA. Therefore, Eq. (5) can still be reused in our new submartingale-based analysis, detailed in Section V.

A. “Analysis” on the Converging Period

We now revisit the “analysis” on the CP, where by this juncture, a proper learning parameter, \( \Delta \), has been determined, as shown in Eq. (4). Given \( \Delta \), and according to the updating rule of \( p_i(t) \), it is ensured\(^6\) that \( p_i(t_0) \geq \frac{p_i(0)}{2} \), which implies that before \( t_0 \), each action gets no less than half of the initial probability of being selected, and whence each action can be selected a sufficiently large number of times (for example, \( M \) times) before the time instant \( t_0 \). However, it should be noted that Eq. (4) is only a theoretical assumption for \( \Delta \), which is very conservative. In practice, the learning parameter can be much greater than the theoretical assumption.

Let \( t_1 \) be the number of iterations that the LA takes to converge after \( t_0 \). The goal is to determine \( t_1 \) such that \( p_m(t_0 + t_1) > 1 - \varepsilon \). As in [24], \( p_m(t) \), \( \forall t > t_0 \) is considered to be monotonically increasing\(^7\), and thus:

\[
p_m(t_0 + t_1) > 1 - \varepsilon \\
\iff \sum_{j \neq m} p_j(t_0 + t_1) < \varepsilon, \quad \text{Monotonicity is used.}
\]

\[
\iff \sum_{j \neq m} (p_j(t_0) - t_1 \Delta) < \varepsilon, \quad \text{Monotonicity is used.}
\]

\[
\iff p_j(t_0) - t_1 \Delta < \frac{\varepsilon}{r - 1},
\]

\[
\iff 1 - t_1 \Delta < \frac{\varepsilon}{r - 1},
\]

\[
\iff t_1 > \frac{1 - \varepsilon}{\Delta}.
\]

Hence, if after time \( t_0 \), the LA goes on running for more than \( \left\lceil \frac{1 - \varepsilon}{\Delta} \right\rceil \) iterations, then \( p_m(t) = p_m(t_0 + t_1) > 1 - \varepsilon \). Therefore, if we denote \( T_0 = t_0 + \left\lceil \frac{1 - \varepsilon}{\Delta} \right\rceil = f(M, \theta) + \left\lceil \frac{1 - \varepsilon}{\Delta} \right\rceil \), then when \( r > T_0 \), the LA converges. This leads to the finite-time instant required for the LA to converge. Again, one should note that \( T_0 \) is very conservative because of the small theoretical learning parameter \( \Delta \).

Flaw in the previous “Proof”: In the above analysis of the DPA’s finite-time behavior, the second part that focuses on the CP, has an inherent infirmity. It is based on the flawed “assertion” that attempts to prove the DPA’s \( \varepsilon \)-optimality, namely the one that claims that \( p_m(t) \) is monotonically increasing after

\(^6\)\( p_i(0) \) is most commonly set to be \( \frac{1}{2} \).

\(^7\)This is, precisely, the juncture where the argument is flawed.
t_0. This is a very strong claim that actually is unfounded. Rather, instead of the strong phenomenon of monotonicity, \( p_m(t), \ t > t_0 \), the property that does, indeed, hold true is its martingale-related property. After a certain time instant, \( p_m(t) \) is, in fact, a submartingale [35], [36], which is a weaker phenomenon than monotonicity. Thus, to render the proof accurate, we need to re-define \( t_0 \) as the time instant after which, \( p_m(t), \ t > t_0 \) is a submartingale, and analyze the converging period based on such a submartingale phenomenon.

Having clarified this, it is also prudent to crystallize the differences and similarities between [35] and this current paper. With regard to the similarity: Both [35] and this current paper abandon the futile attempt to use the monotonicity property of the random process. The difference between the two papers is that whereas [36] deals with the asymptotic property of the random process. The difference between the current paper and [35] is that whereas [36] deals with the asymptotic properties of the process, this current paper deals with the process’ finite-time behavior, which as explained above, is far more cumbersome and difficult to formalize.

V. OUR PROPOSED FINITE-TIME ANALYSIS OF THE DPA

Before we proceed, we first mention that our presently-proposed finite-time Analysis does not depend on an eigenvalue-based strategy but rather by invoking the theory of martingales. First of all, we split the convergence into two phases. The first involves the period when the estimates converge accurately enough so that the Pursuit phenomenon can take over, and the second involves the LA converging to the best action. To achieve this, we first calculate the required number of times that each action has to be selected, say \( M \). This is to guarantee that the stochastic process becomes a submartingale, which is required so as to attain the final convergence. Thereafter, even though the number \( M \) is known, we do not yet know the time instant at which all the actions will have been selected at least \( M \) times. To estimate this time, we devise a scheme to compute the quantity \( t_0 \), after which each action will have been selected at least \( M \) times with the high probability \( 1 - \theta \), where due to the randomness of the learning process, \( t_0 \) can only be bounded probabilistically with a parameter \( \theta \). Since the quantity \( t_0 \) is calculated, we can guarantee that after \( t_0 \), the process becomes a submartingale with a high probability. Thereafter, we show how we can calculate the average time required in the convergence stage. These steps lead to a very interesting proof, and amazingly fascinating results that are quite distinct from the state-of-the-art.

As the submartingale property is weaker than the property of monotonicity, we are not able to bound \( t_1 \) but rather, provide a conservative bound for its expected value. The new finite-time analysis has two parts. The first phase (Section V-A) is to calculate \( t_0 \), where \( p_m(t) \) becomes a submartingale. The second phase (Section V-B) is to derive \( E[t_1] \). Therefore, if we denote \( T_0 = t_0 + E[t_1] \), we can assert that on the average, (i.e., in the Expected sense), the LA converges when \( t > T_0 \).

A. Analysis of the Submartingale \( p_m(t), \ t > t_0 \), and \( t_0 \)

In this subsection, we briefly introduce the submartingale property of \( p_m(t), \ t > t_0 \), and then proceed to derive the time span required for the random process to become a submartingale, i.e., \( t_0 \).

We define \( q(t) \) as the probability that the reward probability estimate of the optimal action is the greatest among all the reward probability estimates at time \( t \), i.e.,

\[
q(t) = Pr(\hat{d}_m(t) > \hat{d}_i(t)), \forall i \neq m. \tag{7}
\]

According to the description of the DPA algorithm, we see that the quantity \( q(t) \) increases as \( p_m(t) \) grows. Therefore, based on the action probability updating rules of the DPA where we have worked within the Reward-Inaction paradigm:

\[
E[p_m(t+1)|P(t)] = p_m(d_m(q(p_m + c_i\Delta) + (1 - q)(p_m - \Delta)) + (1 - d_m)p_m + \sum_{j \neq m} p_j(d_j(q(p_m + c_i\Delta) + (1 - q)(p_m - \Delta)) + (1 - d_j)p_m)
\]

\[
= \sum_{j = 1}^r p_j(d_j(q(p_m + c_i\Delta) + (1 - q)(p_m - \Delta)) + (1 - d_j)p_m)
\]

\[
= \sum_{j = 1}^r (p_d d_j q c_i \Delta - \sum_{j = 1}^r p_j d_j \Delta) + \sum_{j = 1}^r p_j d_j q \Delta + \sum_{j = 1}^r p_j p_m
\]

\[
= p_m + \sum_{j = 1}^r p_j d_j (q(c_i \Delta + \Delta) - \Delta),
\]

where, \( c_i \) represents the number of actions’ probabilities, other than \( p_m \), that have not converged to zero at time \( t \). Therefore, \( c_i \) is bounded by 0 and \( r - 1 \), and it is non-increasing as \( t \) grows. Note that the condition \( c_i = 0 \) implies that \( p_m = 1 \) holds, and this further implies that the entire random process has converged to a unit vector. However, in this phase of our analysis, our intention is to study the time span when \( c_i \) is in between 1 and \( r - 1 \). The difference between \( E[p_m(t+1)] \) and \( p_m(t) \) can be expressed as:

\[
Diff_{p_m(t)} = E[p_m(t+1)|P(t)] - p_m(t)
\]

\[
= \sum_{j = 1}^r p_j(t) d_j (q(t)(c_i \Delta + \Delta) - \Delta). \tag{8}
\]

Based on Eq. (8), \( q(t)(c_i \Delta + \Delta) - \Delta > 0 \) implies that \( q(t) > \frac{\Delta}{c_i \Delta + \Delta} \). Thus, we conclude that the sequence \( p_m(t), \forall t > t_0 \), is a submartingale if there exist a time instant \( t_0 \) such that for every time instant \( t > t_0 \), \( q(t) > \max\left(\frac{\Delta}{c_i \Delta + \Delta}\right) = \frac{1}{2} \). Therefore, if we can prove that for \( \hat{\delta} \in (0, \frac{1}{2}) \), \( \forall t > t_0 \), \( q(t) > 1 - \hat{\delta} \), it implies that for the specific value of \( \hat{\delta} = \frac{1}{2} \), \( \forall t > t_0 \), \( q(t) > 1 - \hat{\delta} \), guaranteeing the submartingale property.

Consider a two-action Environment. Without loss of generality, let \( \alpha_1 \) be the optimal action and \( \alpha_2 \) the inferior one. We are to prove that \( \forall t > t_0 \),

\[
Pr(\hat{d}_1(t) - \hat{d}_2(t) > 0) > 1 - \hat{\delta}.
\]

If we define

\[
H = d_1 - d_2,
\]

then

\[
Pr(\hat{d}_1(t) - \hat{d}_2(t) > 0) \equiv 1 - Pr(\hat{H}(t) - H \leq -H),
\]

In the interest of conciseness, \( p_m(t) \) and \( q(t) \) are respectively written as \( p_m \) and \( q \) whenever there is no confusion.
Pr(\(\tilde{d}_1(t) - \tilde{d}_2(t) > 0\)) > 1 - \(\delta\) \(\Leftrightarrow\) Pr(\(\hat{H}(t) - H \leq -H\)) \(\leq\) \(\delta\).

Hence, we can equivalently prove that

\[
Pr(\hat{H}(t) - H \leq -H | n_1(t) = n) \leq e^{-\frac{2H^2}{\nu n(t + \rho - \rho^2)}}.
\]

We thus find a proper value of \(n\) such that

\[
e^{-\frac{2H^2}{\nu n(t + \rho - \rho^2)^2}} \leq \delta,
\]

which guarantees that \(Pr(\hat{H}(t) - H \leq -H) \leq \delta\).

Considering \(n\) as the variable, we solve Eq. (9) and get

\[
e^{-\frac{2H^2}{\nu n(t + \rho - \rho^2)^2}} > \delta, \quad \text{when } t < -\frac{2\ln \delta}{H^2},
\]

and when \(t \geq -\frac{2\ln \delta}{H^2},\)

\[
e^{-\frac{2H^2}{\nu n(t + \rho - \rho^2)^2}} \begin{cases} \leq \delta, & \text{when } n_1 \leq n \leq n_2, \\ > \delta, & \text{otherwise}, \end{cases}
\]

where \(n_1\) and \(n_2\) are the two real roots of Eq. (9):

\[
n_1 = \frac{t}{2} - \frac{\sqrt{H^2 + 2H \ln \delta}}{2H}, \quad \text{and}
\]

\[
n_2 = \frac{t}{2} + \frac{\sqrt{H^2 - 2H \ln \delta}}{2H}.
\]

Therefore, to make sure that Eq. (9) holds, we need to ensure that \(\forall t > t_0, \; n_1 \leq n \leq n_2\). Omitting the algebraic manipulations here\(^9\), we present the conclusion that \(t_0 \geq -\frac{2\ln \delta}{H^2}\) and \(n \geq -\frac{\ln \delta}{H^2}\).

As the above analysis is also applicable to \(\alpha_2\), which is, indeed, symmetric to \(\alpha_1\) in this two-action Environment considered, the consequence is the following: Let us suppose that we define the time instant \(t_0\) such that within the time defined by \(t_0\), \(\alpha_1\) and \(\alpha_2\) have each been selected more than \(\left\lfloor \frac{-\ln \delta}{H^2} \right\rfloor\) times. As a result of this, in such a case:

\[
e^{-\frac{2H^2}{\nu n(t + \rho - \rho^2)^2}} \leq \delta,
\]

whence we can conclude that for the given \(\delta \in (0, \frac{1}{2})\), \(\forall t > t_0,\)

\[
q(t) = Pr(\tilde{d}_1(t) - \tilde{d}_2(t) > 0) > 1 - \delta.
\]

The result can be easily extended to the \(r\)-action Environment, where if we define:

\[
H_j = d_m - d_j, j \neq m, \\
\hat{H}_j(t) = \tilde{d}_m(t) - \tilde{d}_j(t), j \neq m.
\]

Then, given any \(\delta \in (0, \frac{1}{2})\), if we denote \(\delta^* = 1 - \sqrt[r]{1 - \delta}\), we can show that there exists a time instant \(t_0\), such that within the time defined by \(t_0\), \(\alpha_m\) has been selected more than \(\left\lfloor \frac{-\ln \delta^*}{\min(H_j)} \right\rfloor\) times, and \(\alpha_j, \; \forall j \neq m\), has been selected more than \(\left\lfloor \frac{-\ln \delta^*}{\min(H_j)} \right\rfloor\) times. Consequently, for \(\forall t > t_0, q_j(t) > 1 - \delta^*\) holds and \(q(t) \geq \prod_{j, j \neq m} q_j(t) > 1 - \delta\).

As there exists a \(t_0\) such that \(\forall t > t_0, q(t) > \frac{1}{2}\), holds, \(p_m(t), \; \forall t > t_0\), is indeed a submartingale. Thus, according to the submartingale convergence theory [14] (Page 440),

\[
p_m(\infty) = 0 \text{ or } 1.
\]

If we denote \(e_j\) as the unit vector with the \(j\)-th element being 1 and 0 otherwise, then \(p_m(\infty) = 1 \iff P(\infty) = e_m\). If we define the convergence probability

\[
\Gamma_m(P) = Pr(P(\infty) = e_m | P(0)),
\]

where \(P(0)\) is the initial action probability vector, then by utilizing the theory of Regular functions, we can prove that \(\Gamma_m(P) \rightarrow 1\) [35]. Thus the DPA being \(\varepsilon\)-optimal is proven based on the submartingale property of \(p_m(t), \; \forall t > t_0\). Besides, the number of times each action is required to be selected to achieve the reward probability estimates accuracy, \(q(t) > 1 - \delta\), can be provided by

\[
M = \left\lfloor \frac{-\ln \delta^*}{\min(H_j)} \right\rfloor, \quad \text{where } \delta^* = 1 - \sqrt[r]{1 - \delta}.
\]

9The reader is referred to [35] for additional details.

Given the \(\Delta\) and \(M\) as in Eq. (4) and Eq. (11), respectively, by the same deduction as in [24], \(t_0\) can be calculated as \(t_0 = f(M, \theta)\).

B. The Average Converging Period of the Submartingale

We now present the new analysis on the behavior of the DPA in the Converging Period. Note that in the new analysis, the fundamental difference is the definition of \(t_0\), which is, after \(t_0\), the probability sequence of \(p_m(t), \; t > t_0\), does not monotonically increase, but is a submartingale.

We shall utilize Wald’s Equality to achieve this analysis. In Wald’s Equality [15], we have \(X_i, \; i = 1, 2, \ldots, \) as a sequence of Independent and Identically Distributed (IID) random variables, and I as a random variable that is a stopping time for the sequence of \(X_i\). Let \(S_i = X_1 + X_2 + \cdots + X_i\), if \(E(I)\) and \(E(X)\) are finite, then \(E(S_i) = E(X)E(I)\).

Firstly, we set a new time system \(t' = t - t_0\), which differs from the old time system \(t\), by \(t_0\). As we are dealing with the LA’s behavior after time \(t_0\), by working within the new time system, we are able to get rid of the notation that involves \(t_0\), and to affirm that for the sequence after \(t_0\), the time index starts from the time instant 0.

Secondly, we define

\[
X(t') = \begin{cases} (r-1)\Delta, \text{ with probability } q(t') \\ -\Delta, \text{ with probability } 1 - q(t') \end{cases}.
\]

For ease of analysis, if we further freeze the update of \(q(t')\) and adopt a fixed value of \(q(t')\) when \(t' = 0\), denoted by \(q_0\), then \(X(t')\) becomes a sequence of IID random variables, whose expectation can be calculated as

\[
E(X(t')) = q_0(r-1)\Delta + (1 - q_0) \times (-\Delta) = (q_0r - 1)\Delta.
\]
Thirdly, we consider $p_m(t')$ as a random point walking between the interval of $[0, 1]$, with its initial position being at $p_m(t = t_0)$. According to the action probability updating rules of the DPA,
\[
p_m(t') = p_m(t_0) + X(1) + X(2) + \ldots + X(t'),
\]
and so, if we define $S_m = p_m(t') - p_m(t_0)$, then
\[
S_m = p_m(t') - p_m(t_0) = X(1) + X(2) + \ldots + X(t').
\]
Obviously, $p_m(t')$ stops walking when it reaches the point of 0 or 1. Accordingly, $S_m$ stops at the point of $-p_m(t_0)$ or $1 - p_m(t_0)$. The probability of $p_m(t')$ stopping at point 1 (i.e., when $S_m = 1 - p_m(t_0)$) is $\Gamma_m(P)$, and the probability of $p_m(t')$ stopping at point 0 (i.e., when $S_m = -p_m(t_0)$) is $1 - \Gamma_m(P)$.

Therefore, we calculate the expectation of $S_m$ as:
\[
E(S_m) = \Gamma_m(P)(1 - p_m(t_0)) + (1 - \Gamma_m(P))(-p_m(t_0))
\]
\[
= \Gamma_m(P) - p_m(t_0).
\]
(14)

The number of times it takes for $p_m(t')$ to reach 0 or 1 (or equivalently, for $S_m$ to reach $-p_m(t_0)$ or $1 - p_m(t_0)$) is called the Stopping Time, which, we also denote as $t_1$ in the interest of keeping the notation consistent with what has been defined in the flawed analysis. According to Wald’s Equality, the expectation of the Stopping Time $t_1$ is:
\[
E(t_1) = \frac{E(S_m)}{E(X(t'))} = \frac{\Gamma_m(P) - p_m(t_0)}{(q_0r - 1)\Delta}.
\]
(15)

We thus have given an explicit expression for $E(t_1)$.

The consequence of the above is the following: We see from Eq. (15) that if $\Gamma_m(P)$, $p_m(t_0)$, $q_0$, and $\Delta$ can be determined, $E(t_1)$ can be estimated yielding a rough idea about the number of time instances it takes for the LA to converge.

In summary, the main hurdles that we encountered when arriving at these results are the following:

1) To calculate $M$, i.e., the required number of times that each action has to be selected for the given probability parameter $\theta$. This is to guarantee that the stochastic process becomes a submartingale. With $M$ and $\theta$ available, we can bound the time that the stochastic process requires to become a submartingale. The key to this involves invoking Hoeffding’s Inequality.

2) To estimate the mean time of the convergence period for the given probability parameter $\Gamma_m(P)$. When the process becomes a submartingale, it is necessary to bound, in a statistical sense, the time horizon that the process requires to converge. The key to obtain this estimate is to invoke Wald’s Equality.

$10\) $p_m(t = t_0)$ is $p_m(t_0)$ in the old time system. Without causing any confusion, we consider $p_m(t' = t_0) = p_m(t = t_0)$ a constant, and will write them simply as $p_m(t_0)$ in the rest of the paper.

$11\) $\Gamma_m(P)$ is precisely the probability of the absorbing LA converging to the unit vector $e_0$. This is exactly the notation used in the literature [14].

$12\) We are grateful to the Anonymous Referee who requested this and the following Sub-section.

C. Details of the Terms Involved

Although the various terms have been derived above, we believe that it will be prudent to explain their respective significances. Thus, in this section, we shall examine these four items one by one.

1) We first explain the meaning of the term $\Gamma_m(P)$. $\Gamma_m(P)$ is the probability of the LA converging to the correct unit vector, and so its value must be as close to unity as desired to guarantee $\varepsilon$-optimality. Generally speaking, one uses a value of $\Gamma_m(P)$ which is smaller than $1 - \theta$. The reason for this is that $1 - \theta$ is the lower bound of the probability that each action has been selected a sufficient number of times in order to make the process a submartingale. Once the process exhibits the property of a submartingale, it is possible for the LA to become $\varepsilon$-optimal with accuracy $\Gamma_m(P)$.

2) We now consider the learning parameter $\Delta$ and how it is related to Eq. (4). To reach a given converging probability, the parameter that one can adjust is $\Delta$. This is because for every $\Delta$, there exists a corresponding $\Gamma_m(P)$. The smaller the value that $\Delta$ is, the closer $\Gamma_m(P)$ is to unity. From a pragmatic perspective, a small $\Delta$ ensures that each action is selected enough number of times before the LA converges, and this, in turn, ensures that the estimation of each reward probability is accurate enough to rank the actions properly. Once the ranking of the actions becomes accurate enough, the LA will be more likely to converge to the best action, thus enhancing the converging probability, $\Gamma_m(P)$. This rationale is precisely what Eq. (4) is based on. Indeed, if $\Delta$ is defined as in Eq. (4), the given converging probability can be guaranteed. Therefore, we set $\Delta = \frac{1}{r^{\varepsilon^2}}$. Since $t$ is required to be greater than $t_0$, we can configure $t$ as $t_0 + 1$.

3) The next question is that of setting $p_m(t_0)$. Observe that as mentioned in Section IV, $p_1(0)$ is configured as $\frac{1}{r}$, as is customarily done. If the step size is $\Delta = \frac{1}{r^{2(0+1)}}$, at time instant $t_0$, in the worst case, this action probability will become:
\[
p_m(t_0) = \frac{1}{r} - \Delta t_0 = \frac{1}{r} - \frac{t_0}{2(r(t_0 + 1))} = \frac{1}{2r} + \frac{1}{2r(t_0 + 1)}.
\]
(16)

This means that in the worst case, the probability $p_m(t_0)$ is decreasing for all those $t_0$ iterations. In other words, $p_m(t_0) \geq \frac{1}{2} + \frac{1}{2r(t_0 + 1)}$ holds. We can thus set $p_m(t_0) = \frac{1}{2r} + \frac{1}{2r(t_0 + 1)}$.

4) The last quantity to be determined is $q_0 < 1$. The value of $q_0 < 1$ needs only to be greater than $\frac{1}{r}$ to keep $p_m(t)$, $\forall t > t_0$ to be a submartingale. Note also that $q_0 \geq 1 - \delta$ holds. Thus, $q_0$ can be set to any value in the interval of $(\max(1 - \delta, \frac{1}{r}), 1)$. The equality, where it is set to $1 - \delta$, is when the process is neither a submartingale nor a supermartingale, but a "pure" martingale.

In summary, for the required $\Gamma_m(P)$, if we set:
\[
\Delta = \frac{1}{r^{2(0+1)}}, \quad p_m(t_0) = \frac{1}{2r} + \frac{1}{2r(t_0 + 1)}.
\]
$q_0$ to be any value from the interval $(\max\{1 - \delta, \frac{1}{2}\}, 1)$, the quantity $E(t_1)$ can be calculated. This would thus yield a rough estimate as to how much time the LA would spend after time $t_0$ to converge.

Note that as $\Delta = \frac{1}{r \times 2(r^r + 1)}$ is a very conservative assumption and $p_m(t_0) = \frac{1}{2} + \frac{1}{2 r (\theta + 1)}$ is the lowest value that $p_m(t_0)$ can be. In addition, $q(t')$ will increase as the iterations proceed while the value utilized in the calculation, i.e., $q_0$ is when $t' = 0$, which is also very conservative. Thus the results stated for $E(t_1)$ and $t_0 + E(t_1)$ are very conservative.

VI. NUMERICAL RESULTS

In the previous sections, we derived closed-form expressions for the number of iterations needed for the DPA (with its specified resolution parameters) to converge. We now formally summarize the sequence of tasks that this entails, and then discuss the result of the bound itself by providing certain concrete values for the parameters. The three resolution parameters utilized in our calculation are $\Gamma_m(P)$, $\theta$ and $\delta$, where:

- $\Gamma_m(P)$ is the final accuracy of the LA after convergence;
- $\theta$ is the resolution of the probability that each action has been selected at least $M$ times till the time instant $t_0$;
- As per Item (4) above, $\delta$ can be used to derive the required number of times that an action is selected, i.e., $M$. It is also utilized for calculating $t_0$.

Based on the results of Sections IV and V, we now formally outline the procedure to calculate the bound when one uses concrete values for the parameters. To calculate the total bound, one can follow the following steps:

1) We first calculate the required number of times that each action has to be selected, i.e., $M$, by using Eq. (11). This is to guarantee that the stochastic process becomes a submartingale, which is required so as to attain the final convergence. Although $M$ is calculated by invoking Eq. (11), the required time span to fulfill this condition is not yet bounded. Thus, even though the number $M$ is known, we do not yet know the time instant at which all the actions will have been selected at least $M$ times.

2) To estimate this time, we adopt Eq. (5). This yields the quantity $t_0$, after which each action will have been selected at least $M$ times with the high probability $1 - \theta$. Note that due to the randomness of the learning process, the time instant $t_0$ can only be bounded probabilistically with a parameter $\theta$. In other words, when $t_0$ approaches infinity, all actions will surely be selected at least $M$ times, which is also indicated from Eq. (5). However, once the quantity $t_0$ is calculated, we know that after $t_0$, the process becomes a submartingale with a high probability. Thereafter, one can use Eq. (15) to calculate the average time required in the convergence stage.

Figure 2 illustrates how the time bound for the DPA's convergence has been calculated.

A. Numerical Examples for the Bounds

To clarify the above, we have numerically calculated the value of the bound for a few specific configurations.

1) In the first set of experiments (in Table I), we have fixed the final accuracy of the LA, $(\Gamma_m(P))$, and the resolution of the probability that the optimal action has to converge to, for it to be considered as being optimal at time $t_0$, $(\delta)$. We have then varied the resolution of the probability that each action has been selected at least $M$ times, $(\theta)$. We also have fixed the final accuracy of the LA, $(\Gamma_m(P))$, and resolution of the probability that each action has been selected at least $M$ times, $(\theta)$. We have then varied the resolution of the probability that the optimal action has to converge to for it to be considered as being optimal at time $t_0$, $(\delta)$.

2) In the second set of experiments (in Tables II and III), we have fixed the final accuracy of the LA, $(\Gamma_m(P))$, and resolution of the probability that each action has been selected at least $M$ times, $(\theta)$. We have then varied the resolution of the probability that the optimal action has to converge to for it to be considered as being optimal at time $t_0$, $(\delta)$.

3) In the final set of experiments (in Table IV), we have fixed the resolution of the probability that each action has been selected at least $M$ times, $(\theta)$, and the resolution of the probability that the optimal action has to converge to for it to be considered as being optimal at time $t_0$, $(\delta)$. We have then varied the final accuracy of the LA, $(\Gamma_m(P))$.

The results obtained have been recorded in Tables I, II, III and IV respectively. To generate these results, we have set the number of actions, $r$, to be 2, and the difference between the true reward probabilities, $H$, to be 0.2. For the results in Table I, we configured $\Gamma_m(P)$ as $1 - \theta - 0.001$, $\delta$ as 0.2, and $q_0$ as 0.96. The reason to configure $\Gamma_m(P)$ as $1 - \theta - 0.001$ was to make sure that $\Gamma_m(P) < 1 - \theta$ holds.

<table>
<thead>
<tr>
<th>(\theta)</th>
<th>(M)</th>
<th>(t_0)</th>
<th>(E(t_1))</th>
<th>(t_0 + E(t_1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>41</td>
<td>1.538</td>
<td>4.312</td>
<td>5.880</td>
</tr>
<tr>
<td>0.05</td>
<td>41</td>
<td>1.543</td>
<td>4.691</td>
<td>6.234</td>
</tr>
<tr>
<td>0.04</td>
<td>41</td>
<td>1.544</td>
<td>4.762</td>
<td>6.306</td>
</tr>
<tr>
<td>0.03</td>
<td>41</td>
<td>1.546</td>
<td>4.834</td>
<td>6.380</td>
</tr>
<tr>
<td>0.02</td>
<td>41</td>
<td>1.549</td>
<td>4.912</td>
<td>6.461</td>
</tr>
<tr>
<td>0.01</td>
<td>41</td>
<td>1.554</td>
<td>4.995</td>
<td>6.549</td>
</tr>
</tbody>
</table>

For the results in Table II, we configured $\Gamma_m(P)$ as 0.98, $\theta$ as 0.015, and $q_0$ as $1 - \delta + 0.001$. To further illustrate these convergence phenomena, we also illustrated the numerical

![Fig. 2: A schematic of how the time bound of the DPA’s convergence is calculated.](image-url)
results when we configured $\Gamma_m(P)$ as 0.99, $\theta$ as 0.005, and $q_0$ as $1 - \delta + 0.0005$. These results are given in Table III.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$M$</th>
<th>$t_0$</th>
<th>$E(t_1)$</th>
<th>$t_0 + E(t_1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>17</td>
<td>603</td>
<td>881,330</td>
<td>881,940</td>
</tr>
<tr>
<td>0.4</td>
<td>23</td>
<td>798</td>
<td>11,545</td>
<td>12,343</td>
</tr>
<tr>
<td>0.3</td>
<td>31</td>
<td>1,124</td>
<td>8,169</td>
<td>9,293</td>
</tr>
<tr>
<td>0.2</td>
<td>41</td>
<td>1,151</td>
<td>7,526</td>
<td>9,077</td>
</tr>
<tr>
<td>0.1</td>
<td>56</td>
<td>2,314</td>
<td>8,427</td>
<td>10,741</td>
</tr>
<tr>
<td>0.08</td>
<td>64</td>
<td>2,591</td>
<td>8,988</td>
<td>11,579</td>
</tr>
<tr>
<td>0.06</td>
<td>71</td>
<td>2,920</td>
<td>9,669</td>
<td>12,359</td>
</tr>
<tr>
<td>0.04</td>
<td>81</td>
<td>3,398</td>
<td>10,764</td>
<td>14,162</td>
</tr>
<tr>
<td>0.02</td>
<td>98</td>
<td>4,230</td>
<td>12,841</td>
<td>17,071</td>
</tr>
</tbody>
</table>

For the results in Table IV, we configured $\theta$ as $1 - \Gamma_m(P) - 0.001$, $\delta$ as 0.2, and $q_0$ as 0.98.

<table>
<thead>
<tr>
<th>$\Gamma_m(P)$</th>
<th>$M$</th>
<th>$t_0$</th>
<th>$E(t_1)$</th>
<th>$t_0 + E(t_1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.75</td>
<td>75</td>
<td>1,530</td>
<td>2,870</td>
<td>4,400</td>
</tr>
<tr>
<td>0.8</td>
<td>75</td>
<td>1,533</td>
<td>3,514</td>
<td>5,047</td>
</tr>
<tr>
<td>0.9</td>
<td>75</td>
<td>1,538</td>
<td>4,167/5</td>
<td>5,751</td>
</tr>
<tr>
<td>0.95</td>
<td>75</td>
<td>1,543</td>
<td>4,502</td>
<td>6,045</td>
</tr>
<tr>
<td>0.96</td>
<td>75</td>
<td>1,545</td>
<td>4,573</td>
<td>6,118</td>
</tr>
<tr>
<td>0.97</td>
<td>75</td>
<td>1,547</td>
<td>4,643</td>
<td>6,190</td>
</tr>
<tr>
<td>0.98</td>
<td>75</td>
<td>1,550</td>
<td>4,717</td>
<td>6,267</td>
</tr>
<tr>
<td>0.99</td>
<td>75</td>
<td>1,555</td>
<td>4,767/5</td>
<td>6,352</td>
</tr>
</tbody>
</table>

VII. DISCUSSIONS ON THE NUMERICAL BOUNDS

From Tables I and IV, we can observe that both $t_0$ and $E(t_1)$ monotonically increase as $\theta$ decreases and as $\Gamma_m(P)$ grows. In fact, $1 - \theta$ and $\Gamma_m(P)$ describe the same trend. As mentioned before, $\Gamma_m(P) \leq 1 - \theta$ holds because $1 - \theta$ describes the probability of the learning process being a submartingale from $t_0$ onward, which is the preliminary phase of the final convergence. Since we make one parameter as a function of the other parameter in the calculation, i.e., $\theta = 1 - \Gamma_m(P) - 0.001$, the trends of the results in the tables are the same. We also calculate the results when we decouple those two parameters, whence we observe the same trend. The reason of having this trend is, in general, because of the fact that the greater the desired convergence accuracy, the larger is the number of iterations that is needed.

As opposed to the results in Tables I and IV, the results in Tables II and III are quite different. As shown in Tables II and III, when $\delta$ decreases, $M$ and $t_0$ increase monotonically while $E(t_1)$ has a decreasing trend in the beginning and increases again when $\delta$ is greater than a certain value. This is a very interesting trend which reveals the underlying characteristic nature of the DPA. When $\delta$ is large, i.e., $\theta = 0.99$, $q_0$ is small. A smaller value of $q_0$ also indicates a smaller $t_0$ because less samples are required for the underlying estimation. However, a smaller value of $q_0$ also provides less accurate reward probability estimates and a consequent less proper ranking of the actions till $t_0$. In such a case, the LA is less certain about which action is the best, and thus spends a significant amount of time after $t_0$, in converging to the correct action for the given required accuracy. This, in turn, results in a larger $E(t_1)$. As $\delta$ becomes smaller, $q_0$ increases. A larger value of $q_0$ requires a larger number of samples for the estimation, and this results in a larger $t_0$. But after the time instant $t_0$, as the LA is more certain about the best action, it converges faster, resulting in a smaller $E(t_1)$. This explains the falling trend of the required number of iterations from 881,940 for $\delta = 0.5$ to 7,526 for $\delta = 0.2$, which reveals the process of balancing between the exploration and exploitation phases.

As $\delta$ becomes even smaller, i.e., when $\delta = 0.1$ and smaller, $E(t_1)$ increases again. This is due to the fact that the high resolution of the initial accuracy at $t_0$ results in a small learning rate, $\Delta$. The small value of $\Delta$ leads to a very slow convergence before and after $t_0$, which makes $E(t_1)$ large again, i.e., from 7,526 for $\delta = 0.2$ to 12,841 for $\delta = 0.02$.

In general, as $\delta$ decreases, $t_0$ increases and therefore $\Delta$ decreases. From the results of Table II and Table III, we realize that the total number of steps has a convex trend with respect to $\delta$, and thus $\Delta$. To illustrate this trend more clearly, we have also plotted the time bounds of the DPA based on the configurations of the environment given of Tables II and III. This plot is given in Figure 3. Obviously, the function is not a convex function, but has a convex envelop. Therefore, we can also understand, theoretically, that there exists an optimal\(^1\) $\Delta$, which can deliver a required overall convergence accuracy, i.e., $\Gamma_m(P)$, with a minimum number of steps.

As the total time required for the LA to converge consists of both $t_0$ and $t_1$, i.e., $t_0 + E(t_1)$, $\Delta$ needs to be carefully chosen to

\(^1\)It is worth mentioning that the result of this study indicates that although large $\delta$ (and thus large $\Delta$) can reduce $t_0$, there is no way to reduce the total number of steps in order to achieve the same given accuracy of the convergence $\Gamma_m(P)$. In reality, when $\Delta$ is large, the total number of steps is indeed reduced, but the convergence accuracy of the LA is also compromised, which does not conflict with our theoretic analysis.
minimize this sum. However, this is never easy, because there are many parameters for a given Environment, including the number of actions, $r$, the parameter, $H$, indicating the hardness of the Environment etc., and these will collectively influence the relationship between $A$ and $T_\infty(P)$. That is why the task of deciding the optimal learning rate for a specific Environment is still open. This is precisely what research in reinforcement learning has been trying to achieve for decades, namely to balance between the dilemmas of exploration and exploitation in an “optimal” way!

VIII. CONCLUSIONS

In this paper, we have concentrated on the the Finite-time Analysis of the Discrete Pursuit Automaton (DPA), which is probably one of the fastest and most accurate reported LA. This analysis does not invoke an eigenvalue-eigenvector strategy. Rather, we work with the time required for the LA to attain a certain level of confidence and then proceed to its absorption state. Although such an analysis was carried out many years ago, we have shown how that the previous monotonicity-based analysis was flawed. In this paper, we have shown that the property that should be invoked is the submartingale property, forcing the proof to be much more intricate. We have rectified the flaw and demonstrated the derivation of convergence probability based on such a submartingale phenomenon. From the derived analysis, we are able to discover and explicitly clarify, for the first time, the underlying dilemma between the DPA’s exploitation and exploration properties. We also confirm, in a non-trivial manner, the existence of the optimal learning rate, which yields a better exploration properties. We also confirm, in a non-trivial manner, the existence of the optimal learning rate, which yields a better exploration properties.

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