Adaptive Backstepping Control of Nonlinear Uncertain Systems with Quantized States

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Abstract—This paper investigates the stabilization problem for uncertain nonlinear systems with quantized states. All states in the system are quantized by a static bounded quantizer, including uniform quantizer, hysteresis-uniform quantizer and logarithmic-uniform quantizer as examples. An adaptive backstepping-based control algorithm which can handle discontinuity resulted from the state quantization and a new approach to stability analysis are developed by constructing a new compensation scheme for the effects of the state quantization. Besides showing the global ultimate boundedness of the system, the stabilization error performance is also established and can be improved by appropriately adjusting design parameters. Simulation results illustrate the effectiveness of our proposed scheme.

Keywords: Adaptive control, backstepping, state quantization, nonlinear systems, bounded quantizer.

I. INTRODUCTION

Due to its theoretical and practical importance in the study of digital control systems, hybrid systems, and networked control systems, there has been a great deal of interest in the development of quantized control systems. The main motivation comes from the observation that for many control systems, quantization is not only inevitable owing to the widespread use of digital processors that employ finite-precision arithmetic, but also useful. An important aspect is to use quantization schemes that yield sufficient precision, but require low communication rate. Much attention has been paid to quantized feedback control, in order to understand the required quantization density or information rate in stability analysis.

Quantized control of uncertain systems with input quantization has been studied by using robust approaches in [1], [2], [3] and adaptive approaches in [4], [5], [6], [7], [8], [9]. Feedback control of systems with state quantization has attracted growing interest lately in [10], [11], [12]. For a control system with state quantization, the states measurements are processed by quantizers, which are discontinuous maps from continuous spaces to finite sets. Such discontinuous property may lead the control design and stability analysis difficulty. Note that in [10], [11], [12], the systems considered are completely known. Uncertainties and nonlinearities always exist in many practical systems. Thus it is more reasonable to consider controller design for uncertain nonlinear systems. Although adaptive control of uncertain systems has received considerable interest and been widely investigated, there are still limited works devoted to adaptive control with state quantization. It is noted that adaptive control schemes for linear systems with state quantization have been reported only in [13], [14]. In [13], a supervisory control scheme for uncertain linear systems with quantized measurements has been proposed. While in [14], the adaptive control of linear systems with quantized measurements and bounded disturbances has been addressed.

Since backstepping technique was proposed, it has been widely used to design adaptive controllers for uncertain systems [15], [16], [17]. This technique has a number of advantages over the conventional approaches such as providing a promising way to improve the transient performance of adaptive systems by tuning design parameters. Because of such advantages, research on adaptive control of input quantization using backstepping technique has also received great attention, see for examples, [6], [7], [8]. So far, there is still no result available for backstepping-based adaptive stabilization of nonlinear uncertain systems with state quantization. One major difficulty to deal with the state quantization is that the backstepping technique requires differentiating virtual controls and in turn the states by applying chain rule. If the states are quantized, they become discontinuous and therefore it is difficult to analyze the resulting control system with the current backstepping based approaches. In this paper, we provide a solution to this problem by developing a new adaptive controller and a new approach to stability analysis. The quantizers considered in this paper are static and satisfy a bounded condition including a uniform quantizer, a hysteresis uniform quantizer and a combination of logarithmic and uniform quantizer as examples. The main contributions and the new approaches proposed to achieve them are summarized as follows.

- Note that existing backstepping design procedure requires recursively differentiating virtual controls and in turn the states. Thus these variables should be sufficiently smooth. In this paper, we make a significant modification of standard adaptive backstepping in the way that the virtual control laws only use the partial derivatives which are constants and depend on the control design parameters. It is this new technique that enables us to successfully

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overcome the difficulties caused by the discontinuity and the bounded uncertainties resulted from the state quantization.

- A new method is proposed to compensate for the effects of the state quantization. To handle fully unknown parameters, new parameter updating laws, which do not require any information on such unknown parameters including the knowledge on their bounds, are developed. Thus a new adaptive control scheme is developed to achieve desired stability and performances for a class of nonlinear systems. More specifically, 1) the stability in the sense of ultimate boundedness is achieved by choosing suitable design parameters; 2) The upper bound of ultimate stabilization error can be decreased by tuning some design parameters.

- Some new techniques summarized in Remarks 6 and 9 are developed to establish the above mentioned results.

The paper is organized as follows. Section II states the problem of this paper and presents the quantizers. Sections III presents the adaptive control design and analyzes the stability and performance. Simulation results are presented in Section IV to show the effectiveness of proposed controller. Finally, the paper is concluded in Section V.

II. PROBLEM STATEMENT

A. System Model

In this paper, we consider a class of nonlinear uncertain systems described as follows

$$
\begin{aligned}
x^{(n)}(t) &= u(t) + \psi\left(x, \dot{x}, \ldots, x^{(n-1)}\right) \\
&\quad + \phi^T\left(x, \dot{x}, \ldots, x^{(n-1)}\right)\theta \qquad (1) \\
x_{i+1}^q &= q(x^{(i)}), i = 0, 1, 2, \ldots, n-1 \qquad (2)
\end{aligned}
$$

where $(x(t), \dot{x}(t), \ldots, x^{(n-1)}(t))^T \in \Re^n$ and $u(t) \in \Re^1$ are the states and input of the system respectively, $\psi \in \Re^1$ and $\phi \in \Re^r$ are known nonlinear functions, $\theta \in \Re^r$ is a vector of unknown constant parameters, $q(\cdot)$ is a state quantizer and has an infinite level. Such a class of nonlinear systems have been addressed in many references, such as [16], [17], [18], [19]. It was noted in [18], [19], [20] that various practically important systems can be transformed to this structure.

In this paper, only quantized states $(q(x), q(\dot{x}), \ldots, q(x^{(n-1)})$ are measured. The feedback controller $u(t)$ in (1) only uses the quantized states, which is given by

$$
u = u(q(x), q(\dot{x}), \ldots, q(x^{(n-1)})). \qquad (3)
$$

For the development of control laws, the following assumptions are also made.

Assumption 1: The functions ψ and ϕ satisfy the global Lipschitz continuity condition such that

$$
\begin{aligned}
|\psi(y_1) - \psi(y_2)| &\leq L_\psi \parallel y_1 - y_2 \parallel \qquad (4) \\
\parallel \phi(y_1) - \phi(y_2) \parallel &\leq L_\phi \parallel y_1 - y_2 \parallel \qquad (5)
\end{aligned}
$$

where L_ψ and L_ϕ are constants, $y_1, y_2 \in \Re^n$ are real vectors. The norm $\parallel \cdot \parallel$ is defined as $\parallel y \parallel = (\sum_{j=1}^m y_j^2)^{1/2}$ for a vector $y = [y_1, \ldots, y_m]^T$. $|\cdot|$ denotes the absolute value of a scalar.

Assumption 2: Only quantized states $\left(q(x), q(\dot{x}), \ldots, q(x^{(n-1)})\right)$ are measurable and available for control design, instead of the states $(x, \dot{x}, \ldots, x^{(n-1)})$.

Assumption 3: For the closed-loop nonlinear uncertain system (1)-(3), it is assumed that its solution exists and is unique.

Note that similar assumption is also made in the area, for instances [5] and [10]. As illustrated in the example of simulation studies in Section IV, the designed adaptive controller (26) with parameter estimator (27) in Section III gives the existence and uniqueness of the solution.

The control objective is to design an adaptive controller u (3) for system (1) by utilizing only quantized states $(q(x), q(\dot{x}), \ldots, q(x^{(n-1)}))$ such that all the signals in the closed-loop system are globally uniformly bounded.

B. Quantizer

The quantizer $q(x)$ considered in this paper has the following property:

$$
|q(x) - x| \leq \delta \qquad (6)
$$

where $\delta > 0$ is the quantization bound. It can be shown that the quantizers illustrated below have the property (6).

1) Uniform quantizer: A uniform quantizer is modeled as

$$
q_u(x(t)) = \begin{cases} x_i & x_i - \frac{l}{2} \leq x < x_i + \frac{l}{2} \\ 0 & -x_0 \leq x < x_0 \\ -x_i & -x_i - \frac{l}{2} \leq x < -x_i + \frac{l}{2} \end{cases} \qquad (7)
$$

where $x_0 = \frac{l}{2}$ and $x_{i+1} = x_i + l$, l is the length of the quantization interval. $q_u(x)$ is in the set $U = \{0, \pm x_i\}$. The quantization error is bounded by (6), where $\delta \geq \frac{l}{2}$. The map of the uniform quantizer $q(u)$ for $x > 0$ is shown in Figure 1.

2) Hysteresis-uniform quantizer: The hysteresis uniform quantizer is modeled as

$$
\begin{aligned}
&q_{hu}(x(t)) \\
&= \begin{cases} x_i sgn(x), & x_i - \frac{l}{2} - h < |x| \leq x_i - \frac{l}{2} + h \\ & \quad \text{and } \dot{x} < 0, \text{or} \\ & x_i + \frac{l}{2} - h < |x| < x_i + \frac{l}{2} + h \\ & \quad \text{and } \dot{x} > 0, \text{or} \\ & x_i - \frac{l}{2} + h \leq |x| \leq x_i + \frac{l}{2} - h \\ 0, & -x_0 - h \leq x \leq x_0 + h \\ q(x(t^-)), & \dot{x} = 0 \end{cases}
\end{aligned} \qquad (8)
$$

where $x_0 = \frac{l}{2}$ and $x_{i+1} = x_i + l$, l is the length of the quantization interval, $h = p_h l$ is the hysteresis width constant and $0 < p_h \leq 0.5$ is hysteresis percentage, $q_{hu}(x)$ is in the set $U = \{0, \pm x_i\}$, x_0 determines the size of the dead-zone for $q(x)$. The quantization error of hysteresis uniform quantizer is bounded by (6), where $\delta \geq \frac{1}{2}l + h$. The map of the hysteresis uniform quantizer $q_{hu}(x(t))$ for $x > 0$ is shown in Figure 2.

3) Logarithmic-uniform quantizer: A quantizer combining a logarithmic quantizer and a uniform quantizer was developed in [8], which is modeled as

$$
q_s(x(t)) = \begin{cases} q_l(x_{th}) + q_u\left(x - x_{th}\right), & |x| \geq x_{th} \\ q_l(x) & |x| < x_{th} \end{cases} \qquad (9)
$$

where x_{th} is a positive constant specified by designer denoting the threshold to switch between the logarithmic and uniform quantizer, and $q_l(.)$ represents a logarithmic quantizer defined as below,

$$q_l(x(t)) = \begin{cases} x_i sgn(x) & \frac{x_i}{1+\rho} < |x| \le \frac{x_i}{1-\rho} \\ 0 & |x| \le \frac{x_{min}}{1+\rho} \end{cases} \quad (10)$$

where $x_i = \iota^{(1-i)}, x_{min}$ with $i = 1, 2, \ldots$ and parameter $\iota = \frac{1-\rho}{1+\rho}$ with $0 < \rho < 1$. The quantization error of quantizer $q_s(x)$ is bounded by (6), where $\delta \ge \frac{1}{2}l$ is the maximum quantization interval length. The map of the quantizer $q_s(x)$ is shown in Figure 3.
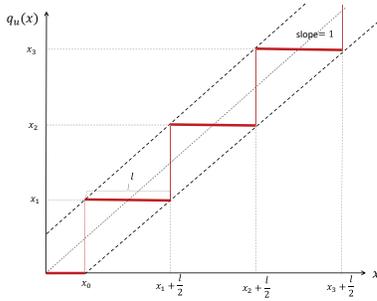

Fig. 1. The map of uniform quantizer $q_u(x)$
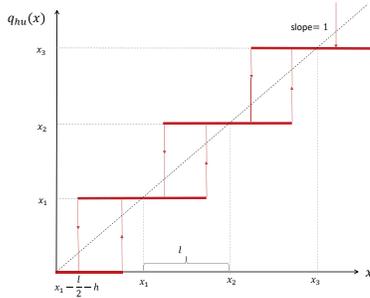

Fig. 2. The map of hysteresis-uniform quantizer $q_{hu}(x)$
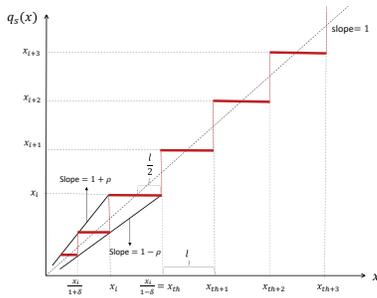

Fig. 3. The map of logarithmic-uniform quantizer $q_s(x)$

Remark 1: Note that the quantization parameters are not required to be known for our control design, such as l for uniform quantizer, l and h for hysteresis-uniform quantizer, ρ and ι for hysteresis-uniform quantizer.

The uniform quantizer $q_u(x)$ and the hysteresis-uniform $q_{hu}(x)$ have the uniformly spaced quantization levels which is optimal for uniformly distributed signal. Compared with

the uniform quantizer, the hysteresis-uniform quantizer has additional quantization levels, which are used to avoid chattering. Whenever $q_{hu}(x)$ makes a transition from one value to another, some dwell time will elapse before a new transition can occur as shown in Figure 2. The logarithmic quantizer $q_l(x)$ is non-uniform quantization, which the quantization interval is smaller near zero. But the quantization letter becomes bigger when the magnitude of the signal gets bigger, which results in unnecessary large quantization error. To overcome this problem, a logarithmic-uniform quantizer $q_s(x)$ in (9) can be used, which minimizes the average rate of communication instances, see detailed in [8]. With the logarithmic-uniform quantizer $q_s(x)$, the quantization error when $|x| > x_{th}$ remains the same as that of the logarithmic quantizer when $|x| = x_{th}$. Note that x_{th} is a user-defined parameter denoting the trade-off between system performances and communication burden, and it can be chosen according to practical applications.

III. ADAPTIVE BACKSTEPPING CONTROL

In order to design the controller using backstepping technique, system (1) is rewritten in the following form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_i &= x_{i+1}, \quad i = 1, ..., n-1 \\ \dot{x}_n &= u(t) + \psi(x_1, ..., x_n) + \theta^T \phi(x_1, ..., x_n) \end{aligned} \quad (11)$$

where $x_1 = x$, $x_i = x^{(i-1)}$, $i = 2, 3, \ldots, n$. The system states $(x_1, x_2..., x_n) \in \Re$. are quantized by a quantizer satisfying the property in (6). Only the measured quantized states $q(x_i)$, $i = 1, ..., n$ are available.

A. States are not quantized

If states x_i, $i = 1, 2, \ldots, n$ are not quantized, we begin by introducing the change of coordinates

$$\begin{aligned} z_1(x_1) &= x_1 \quad (12) \\ z_i(x_1, .., x_i) &= x_i - \alpha_{i-1}, \; i = 2, 3, \ldots, n \quad (13) \end{aligned}$$

where α_{i-1} is the virtual control function of $(x_1, .., x_{i-1})$ and will be determined at the ith step.

• *Step i ($i = 1, \ldots, n-1$)*: Following the standard backstepping design technique in [15], we choose

$$\alpha_1(x_1) = -c_1 z_1(x_1) \quad (14)$$

$$\begin{aligned} \alpha_i(x_1, .., x_i) &= -c_i z_i - z_{i-1} - \dot{\alpha}_{i-1} \\ &= -c_i z_i - z_{i-1} + \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} x_{k+1}, \\ & i = 2, ..., n-1 \end{aligned} \quad (15)$$

where $c_i, i = 2, \ldots, n-1$ are positive design parameters and $\frac{\partial \alpha_{i-1}}{\partial x_k}$ are constants which depend on $c_1, ..., c_{i-1}$. For examples,

$$\frac{\partial \alpha_1}{\partial x_1} = \frac{\partial \alpha_1}{\partial z_1} \frac{\partial z_1}{\partial x_1} = -c_1 \quad (16)$$

$$\frac{\partial \alpha_2}{\partial x_1} = -c_2 \frac{\partial z_2}{\partial x_1} - \frac{\partial z_1}{\partial x_1} = -c_2 c_1 - 1 \quad (17)$$

$$\frac{\partial \alpha_2}{\partial x_2} = -c_2 \frac{\partial z_2}{\partial x_2} + \frac{\partial \alpha_1}{\partial x_1} = -c_2 - c_1. \quad (18)$$

Considering the Lyapunov function

$$V_{n-1} = \sum_{j=1}^{n-1} \frac{1}{2} z_j^2 \tag{19}$$

then the derivative is given as

$$\dot{V}_{n-1} = -\sum_{j=1}^{n-1} c_j z_j^2 + z_{n-1} z_n \tag{20}$$

• *Step* n: If states x_i, $i = 1, 2, \ldots, n$ are not quantized, the virtual control $\alpha_n(x_1, .., x_n)$ is designed as

$$\alpha_n(x_1, .., x_n) = -c_n z_n - z_{n-1} - \dot{\alpha}_n$$
$$= -c_n z_n - z_{n-1} + \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k} x_{k+1} \tag{21}$$

where c_n is a positive design parameter and $\hat{\theta}$ is the estimate of θ. The final control $u(t)$ is chosen as

$$u(t) = \alpha_n - \psi(x_1, ..., x_n) - \hat{\theta}^T \phi(x_1, ..., x_n) \tag{22}$$
$$\dot{\hat{\theta}} = \Gamma \phi(x_1, ..., x_n) z_n, \tag{23}$$

where Γ is a positive definite matrix and $\hat{\theta}$ is the estimate of θ.

Considering the Lyapunov function

$$V = \sum_{j=1}^{n} \frac{1}{2} z_j^2 + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} \tag{24}$$

where $\tilde{\theta} = \theta - \hat{\theta}$, then the derivative is given as

$$\dot{V} = -\sum_{j=1}^{n-1} c_j z_j^2 + z_n \left(\alpha_n - \dot{\alpha}_{n-1} + z_{n-1} \right) - \tilde{\theta}^T \Gamma^{-1} \dot{\hat{\theta}}$$
$$= -\sum_{j=1}^{n} c_j z_j^2 + \tilde{\theta}^T \Gamma^{-1} \left(\Gamma \phi z_n - \dot{\hat{\theta}} \right)$$
$$= -\sum_{j=1}^{n} c_j z_j^2 \tag{25}$$

Thus we conclude that the closed-loop system without state quantization is globally asymptotically stable and the desired convergence property $\lim_{t \to \infty} z_i(t) = 0$ follows from LaSalle-Yoshizawa theorem in [15].

B. States are quantized

When states x_i, $i = 1, 2, \ldots, n$ are quantized with quantizers $q(x_i)$, choosing

$$u(t) = \bar{\alpha}_n - \bar{\psi} - \hat{\theta}^T \bar{\phi}$$
$$= -c_n \bar{z}_n - \bar{z}_{n-1} - \psi(q(x_1), ..., q(x_n))$$
$$- \hat{\theta}^T \phi(q(x_1), ..., q(x_n)) + \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k} q(x_{k+1}) \tag{26}$$

$$\dot{\hat{\theta}} = \Gamma \bar{\phi} \bar{z}_n - \Gamma k_\theta (\hat{\theta} - \theta_0)$$
$$= \Gamma \phi(q(x_1), .., q(x_n)) \bar{z}_n - \Gamma k_\theta (\hat{\theta} - \theta_0) \tag{27}$$

$$\bar{z}_1 = q(x_1) \tag{28}$$

$$\bar{z}_i = q(x_i) - \bar{\alpha}_{i-1} \tag{29}$$
$$\bar{\alpha}_1 = -c_1 \bar{z}_i \tag{30}$$
$$\bar{\alpha}_i = -c_i \bar{z}_i - \bar{z}_{i-1} + \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} q(x_{k+1}),$$
$$i = 2, ..., n \tag{31}$$

where c_i, k_θ and θ_0 are positive parameters and Γ is a positive definite matrix.

Remark 2: Note that an additional term in the form of $-\Gamma k_\theta (\hat{\theta} - \theta_0)$ is introduced in the parameter estimator (27). It will be observed from subsequent stability analysis that by adopting such modification, the following property

$$k_\theta \tilde{\theta} (\hat{\theta} - \theta_0) \leq -\frac{1}{2} k_\theta \parallel \tilde{\theta} \parallel^2 + \frac{1}{2} k_\theta \parallel (\theta - \theta_0) \parallel^2 \tag{32}$$

can be obtained which is helpful to guarantee the closed-loop system stability. Unlike [7], no any prior information about the bound of unknown parameter is required in this paper.

Remark 3: For the system with quantized states, the state x_i is not available and only the quantized state $q(x_i)$ can be used in the designed controller. If we follow the standard backstepping controller in Section III.A, the virtual control $\bar{\alpha}_i$ should be like $-c_i \bar{z}_i - \bar{z}_{i-1} + \dot{\bar{\alpha}}_{i-1}$. Note that the quantized state $(q(x_1), q(x_2)...q(x_{i-1}))$ is used in the virtual control $\bar{\alpha}_{i-1}$ which results in that the derivative of $\bar{\alpha}_{i-1}$ is discontinuous and unable to be used in the backstepping virtual control.

Remark 4: Note that the final control u in (26) and the parameter updating law in (27) utilize only the measured quantized states $q(x_i)$, $i = 1, ..., n$. One vitally important technique adopted in this paper is to use the partial derivatives $\frac{\partial \alpha_{i-1}}{\partial x_k} (i = 2, 3, \ldots, n, \ k = 1, ..., i-1)$ in the final control u in (26) and the function $\bar{\alpha}_i$ in (31), which cancels the effects caused by the previous virtual control $\bar{\alpha}_{i-1}$ in the stability analysis. Note that, as illustrated in the calculations (15)-(17), the partial derivatives $\frac{\partial \alpha_{i-1}}{\partial x_k}$ $(i = 2, 3, \ldots, n, k = 1, ..., i-1)$ are constants and depend on the control gains $(c_1, ...c_{i-1})$ chosen in each recursive step.

Remark 5: The change of coordinates z_i in (13) and the virtual control functions α_i in (15) are only used in the Lyapunov stability analysis, since the state x_i for $i = 1, \ldots, n$ is not used in the final controller and parameter estimator designed.

In order to ensure the boundedness of all signals, we first establish some preliminary results as stated in the following lemmas.

Lemma 1: The effects of state quantization are bounded as follows:

$$|\psi(q(x_1), .., q(x_n)) - \psi(x_1, .., x_n)| \leq \Delta_\psi \tag{33}$$
$$\parallel \phi(q(x_1), .., q(x_n)) - \phi(x_1, .., x_n) \parallel \leq \Delta_\phi \tag{34}$$
$$|z_i(q(x_1), .., q(x_i)) - z_i(x_1, .., x_i)| \leq \Delta_{z_i} \tag{35}$$
$$|\alpha_i(q(x_1), .., q(x_i)) - \alpha_i(x_1, .., x_i)| \leq \Delta_{\alpha_i} \tag{36}$$

where $i = 1, ..., n$, Δ_ψ and Δ_ϕ are positive constants which depend on the quantization bound δ and Lipschitz constants L_ψ and L_ϕ respectively. Δ_{z_i} is positive which depends on the quantization bound δ and control design parameters $(c_1, ..., c_{i-1})$, Δ_{α_i} is a positive constant which depends

on the quantization bound δ and control design parameters $(c_1, ..., c_i)$.

Proof: Using the property of quantizer in (6), we have

$$|q(x_i) - x_i| \leq \delta \tag{37}$$

and the Lipschitz continuous conditions for ψ and ϕ in (4) and (5) in Assumption 2, the following bounded conditions are obtained.

$$
\begin{aligned}
&|\psi(q(x_1), .., q(x_n)) - \psi(x_1, .., x_n)| \\
\leq\ & L_\psi \parallel (q(x_1), .., q(x_n)) - (x_1, .., x_n) \parallel \\
\leq\ & L_\psi \parallel (\delta, ..., \delta) \parallel = L_\psi \sqrt{n}\delta = \Delta_\psi \\
&\parallel \phi(q(x_1), .., q(x_n)) - \phi(x_1, .., x_n) \parallel \\
\leq\ & L_\phi \parallel (q(x_1), .., q(x_n)) - (x_1, .., x_n) \parallel \\
\leq\ & L_\phi \parallel (\delta, ..., \delta) \parallel = L_\phi \sqrt{n}\delta = \Delta_\phi
\end{aligned}
\tag{38,39}
$$

From (14)-(22), and (26)-(31), it is shown that

$$
\begin{aligned}
&|\bar{z}_1 - z_1(x_1)| \\
=\ & |z_1(q(x_1)) - z_1(x_1)| \\
=\ & |q(x_1) - x_1| \leq \delta \triangleq \Delta_{z_1} \\
&|\bar{\alpha}_1 - \alpha_1(x_1)| \\
=\ & |\alpha_1(q(x_1)) - \alpha_1(x_1)| \\
=\ & |-c_1(\bar{z}_1 - z_1)| \leq c_1\delta \triangleq \Delta_{\alpha_1} \\
&|\bar{z}_2 - z_2(x_1, x_2)| \\
=\ & |z_2(q(x_1), q(x_2)) - z_2(x_1, x_2)| \\
=\ & |q(x_2) - \bar{\alpha}_1 - (x_2 - \alpha_1)| \leq \delta + \Delta_{\alpha_1} \triangleq \Delta_{z_2} \\
&|\bar{\alpha}_2 - \alpha_2(x_1, x_2)| \\
=\ & |\alpha_2(q(x_1), q(x_2)) - \alpha_2(x_1, x_2)| \\
=\ & \left| -c_2(\bar{z}_2 - z_2) - (\bar{z}_1 - z_1) + \frac{\partial \alpha_1}{\partial x_1}(q(x_1) - x_1) \right| \\
\leq\ & c_2\Delta_{z_2} + \Delta_{z_1} + \left| \frac{\partial \alpha_1}{\partial x_1} \right| \delta \triangleq \Delta_{\alpha_2}
\end{aligned}
\tag{40,41,42,43}
$$

Following the same procedure based on z_i in (13), α_i in (15), $\bar{z}_i$ in (29), $\bar{\alpha}_i$ in (31), we have

$$
\begin{aligned}
&|\bar{z}_i - z_i(x_1, .., x_i)| \\
=\ & |z_i((q(x_1), .., q(x_i))) - z_i(x_1, .., x_i)| \\
\leq\ & |(q(x_i) - x_i) - (\bar{\alpha}_{i-1} - \alpha_{i-1})| \\
\leq\ & \delta + \Delta_{\alpha_{i-1}} \triangleq \Delta_{z_i} \\
&|\bar{\alpha}_i - \alpha_i(x_1, .., x_i)| \\
=\ & |\alpha_i((q(x_1), .., q(x_i))) - \alpha_i(x_1, .., x_i)| \\
\leq\ & |-c_i(z_i - \bar{z}_i) - (\bar{z}_{i-1} - z_{i-1})| \\
& + \left| \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k}(q_{k+1}(x_{k+1}) - x_{k+1}) \right| \\
\leq\ & c_i\Delta_{z_i} + \Delta_{z_{i-1}} + \sum_{k=1}^{i-1} \left| \frac{\partial \alpha_{i-1}}{\partial x_k} \right| \delta \triangleq \Delta_{\alpha_i}
\end{aligned}
\tag{44,45}
$$

∎

Lemma 2: The states $(x_1, x_2, ..., x_n)$ satisfy the following inequality,

$$\parallel (x_1, ..., x_n) \parallel \leq L_x \parallel (z_1, ..., z_n) \parallel \tag{46}$$

where L_x is a positive constant which depends on the control design parameters $(c_1, ..., c_{n-1})$.

Proof: From the definitions z_i in (12-13) and the virtual control designs α_i in (14-21), it is shown that

$$
\begin{aligned}
|x_1| =\ & |z_1| \\
|\alpha_1| \leq\ & c_1|z_1| \triangleq L_{\alpha_1}|z_1| \\
|x_2| \leq\ & |z_2 + \alpha_1| \leq |z_2| + L_{\alpha_1}|z_1| \\
\leq\ & \sqrt{2}\max\{1, L_{\alpha_1}\} \parallel (z_1, z_2) \parallel \triangleq L_{x2} \parallel (z_1, z_2) \parallel \\
|\alpha_2| \leq\ & c_2|z_2| + \left| \frac{\partial \alpha_1}{\partial x_1} x_2 \right| \\
\leq\ & (c_2 + c_1 L_{x2}) \parallel (z_1, z_2) \parallel \triangleq L_{\alpha_2} \parallel (z_1, z_2) \parallel
\end{aligned}
\tag{47,48,49,50}
$$

where L_{α_1} depends on c_1, L_{α_2} depends on (c_1, c_2), and L_{x_2} depends on c_1. Following the similar procedure, we have

$$
\begin{aligned}
|x_i| \leq\ & |z_i + \alpha_{i-1}| \\
\leq\ & |z_i| + L_{\alpha_{i-1}} \parallel (z_1, ..., z_{i-1}) \parallel \\
\leq\ & (1 + L_{\alpha_{i-1}}) \parallel (z_1, ..., z_{i-1}) \parallel \\
\triangleq\ & L_{x_i} \parallel (z_1, z_2, ..., z_i) \parallel \\
|\alpha_i| \leq\ & c_i|z_i| + \left| \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} x_{j+1} \right| \\
\leq\ & \left(c_i + \left| \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \right| L_{xi} \right) \parallel (z_1, z_2, ..., z_i) \parallel \\
\triangleq\ & L_{\alpha_i} \parallel (z_1, z_2, ..., z_i) \parallel
\end{aligned}
\tag{51,52}
$$

where L_{α_i} depends on $(c_1, ..., c_i)$, and L_{x_i} depends on $(c_1, ..., c_{i-1})$. Then we have

$$
\begin{aligned}
\parallel (x_1, ..., x_n) \parallel =\ & \left(\sum_{j=1}^{n} x_j^2 \right)^{1/2} \\
\leq\ & \left(\sum_{j=1}^{n} L_{xj}^2 \parallel (z_1, z_2, ..., z_j) \parallel^2 \right)^{1/2} \\
\leq\ & \left(\sum_{j=1}^{n} L_{xj}^2 \right)^{1/2} \parallel (z_1, ..., z_j) \parallel \\
\triangleq\ & L_x \parallel (z_1, z_2, ..., z_n) \parallel
\end{aligned}
\tag{53}
$$

∎

Remark 6: The properties (33)-(36) in Lemma 1 and (46) in Lemma 2 are key steps in the stability analysis, which will be used to eliminate the effects from state quantization.

The main results are formally stated in the following theorem.

Theorem 1: Consider the closed-loop adaptive system consisting of plant (1) with state quantization satisfying the bounded property (6), the adaptive backstepping controller (26) with parameter estimator with updating law (27), the following results can be guaranteed.

1) All the closed-loop signals are globally uniformly bounded.

2) The upper bound of stabilization error $\parallel z(t) \parallel_{[0,T]}^2$

satisfies

$$\| z(t) \|_{[0,T]}^2 = \frac{1}{T} \int_0^T \| z(t) \|^2 \, dt$$

$$\leq \frac{2}{c} \left[\frac{V(0)}{T} + M \right] \qquad (54)$$

if $k_\theta > \frac{2}{c} B^2$, where

$$c = \min\{c_1, c_2, ... c_{n-1}, \frac{1}{4} c_n\} \qquad (55)$$

$$M = \frac{1}{2} k_\theta \| (\theta - \theta_0) \|^2 + \frac{1}{c_n} \Delta_{\alpha_n}^2$$

$$+ \frac{1}{c_n} \Delta_\psi^2 + \frac{1}{c_n} \| \theta \|^2 \Delta_\phi^2 \qquad (56)$$

$$V(0) = \sum_{j=1}^n \frac{1}{2} z_j^2(0) + \frac{1}{2} \tilde{\theta}(0)^T \Gamma^{-1} \tilde{\theta}(0) \qquad (57)$$

$$B = L_\phi (L_x + \sqrt{n}\delta) \Delta_{zn}. \qquad (58)$$

Proof: Considering the Lyapunov function

$$V = \sum_{j=1}^n \frac{1}{2} z_j^2 + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} \qquad (59)$$

then its derivative obtained by following the control design in (26)-(31) is given as

$$\dot{V} = -\sum_{j=1}^{n-1} c_j z_j^2 + z_{n-1} z_n - \tilde{\theta}^T \Gamma^{-1} \dot{\hat{\theta}}$$

$$+ z_n \Big(u(t) - \alpha_n + \alpha_n + \psi + \phi^T \theta - \dot{\alpha}_{n-1} \Big)$$

$$= -\sum_{j=1}^{n-1} c_j z_j^2 - \tilde{\theta}^T \Gamma^{-1} \dot{\hat{\theta}} + z_n \Big(\bar{\alpha}_n - \bar{\psi} - \hat{\theta}^T \bar{\phi}$$

$$- \alpha_n + \alpha_n + \psi + \theta^T \phi - \dot{\alpha}_{n-1} + z_{n-1} \Big)$$

$$= -\sum_{j=1}^{n-1} c_j z_j^2 + z_n \Big(\alpha_n - \dot{\alpha}_{n-1} + z_{n-1} \Big)$$

$$+ z_n \Big(\bar{\alpha}_n - \alpha_n \Big) + z_n \Big(\psi - \bar{\psi} \Big)$$

$$+ z_n \Big(\theta^T \phi - \hat{\theta}^T \bar{\phi} \Big) - \tilde{\theta}^T \Gamma^{-1} \dot{\hat{\theta}}$$

$$\leq -\sum_{j=1}^n c_j z_j^2 - \frac{1}{2} k_\theta \| \tilde{\theta} \|^2 + \frac{1}{2} k_\theta \| (\theta - \theta_0) \|^2$$

$$+ z_n \Big(\bar{\alpha}_n - \alpha_n \Big) + z_n \Big(\psi - \bar{\psi} \Big)$$

$$+ \Big(\theta^T \phi z_n - \hat{\theta}^T \bar{\phi} z_n - \tilde{\theta}^T \bar{\phi} \bar{z}_n \Big) \qquad (60)$$

where the property (32) is used. Using the properties (5), (6), (34), (35) and (46), the last term in (60) satisfies the following inequality

$$\theta^T \phi z_n - \hat{\theta}^T \bar{\phi} z_n - \tilde{\theta}^T \bar{\phi} \bar{z}_n$$

$$= \theta^T \phi z_n - \theta \bar{\phi} z_n + \tilde{\theta} \bar{\phi} z_n - \tilde{\theta} \bar{\phi} \bar{z}_n$$

$$\leq \| \theta \| |z_n| \Delta_\phi + \| \tilde{\theta} \| \| \bar{\phi} \| \Delta_{z_n}$$

$$\leq |z_n| \| \theta \| \Delta_\phi + \| \tilde{\theta} \| L_\phi \| (q(x), ..., q(x_n)) \| \Delta_{z_n}$$

$$\leq |z_n| \| \theta \| \Delta_\phi + B \| \tilde{\theta} \| \| z \| \qquad (61)$$

where $z(t) = [z_1, z_2, ..., z_n]^T$ and $B = L_\phi(L_x + \sqrt{n}\delta)\Delta_{zn}$. Using the properties (33) and (36) in Lemma 1 and (61), the derivative of V is obtained as

$$\dot{V} \leq -\sum_{j=1}^n c_j z_j^2 - \frac{1}{2} k_\theta \| \tilde{\theta} \|^2 + |z_n| \Delta_{\alpha_n} + |z_n| \Delta_\psi$$

$$+ |z_n| \| \theta \| \Delta_\phi + B \| \tilde{\theta} \| \| z \| + \frac{1}{2} k_\theta \| (\theta - \theta_0) \|^2$$

$$\leq -\sum_{j=1}^n c_j z_j^2 + \frac{3}{4} c_n z_n^2 + \frac{c}{2} \| z(t) \|^2$$

$$- \frac{1}{2} k_\theta \| \tilde{\theta} \|^2 + \frac{1}{2c} B^2 \| \tilde{\theta} \|^2 + M$$

$$\leq -\frac{c}{2} \| z(t) \|^2 - (\frac{1}{2} k_\theta - \frac{1}{2c} B^2) \| \tilde{\theta} \|^2 + M \qquad (62)$$

where c and M are defined in (55) and (56) and the Young's inequality was used as follows.

$$|z_n| \Delta_{\alpha_n} + |z_n| \Delta_\psi + |z_n| \| \theta \| \Delta_\phi$$

$$\leq \frac{3}{4} c_n |z_n|^2 + \frac{1}{c_n} \Delta_{\alpha_n}^2 + \frac{1}{c_n} \Delta_\psi^2 + \frac{1}{c_n} \| \theta \|^2 \Delta_\phi^2 \qquad (63)$$

$$B \| \tilde{\theta} \| \| z \|$$

$$\leq \frac{c}{2} \| z(t) \|^2 + \frac{1}{2c} B^2 \| \tilde{\theta} \|^2 \qquad (64)$$

Choosing

$$k_\theta > \frac{2}{c} B^2 = \frac{2}{c} L_\phi^2 (L_x + \sqrt{n}\delta)^2 \Delta_{zn}^2, \qquad (65)$$

(62) shows that

$$\dot{V} \leq -\frac{c}{2} \| z(t) \|^2 - \frac{1}{4} k_\theta \| \tilde{\theta} \|^2 + M$$

$$\leq -\sigma V + M \qquad (66)$$

where

$$\sigma = \min\{c, \frac{\frac{1}{2} k_\theta}{\lambda_{max} (\Gamma^{-1})}\} \qquad (67)$$

By direct integration of the above inequality, we have

$$V(t) \leq V(0) e^{-\sigma t} + \frac{M}{\sigma}(1 - e^{-\sigma t})$$

$$\leq V(0) + \frac{M}{\sigma} \qquad (68)$$

which shows that V is uniformly bounded. Thus the signals $z_i(t)$ and $\hat{\theta}$ are bounded. From (13), (15) and (26), it further implies that $x_i(t)$ and $u(t)$ are bounded. Therefore all the closed-loop signals are globally uniformly bounded. From (62), we have

$$\dot{V} \leq -\frac{c}{2} \| z(t) \|^2 + M \qquad (69)$$

Integrating both sides of (69) yields that

$$\| z(t) \|_{[0,T]}^2 = \frac{1}{T} \int_0^T \| z(t) \|^2 \, dt$$

$$\leq \frac{2}{c} \left[\frac{V(0) - V(T)}{T} + M \right]$$

$$\leq \frac{2}{c} \left[\frac{V(0)}{T} + M \right] \qquad (70)$$

From (55), (56) and (57), it follows that the upper bound of the overall stabilization errors in the mean square sense of (70) can be tuned by choosing suitable parameters k_θ, c_n and Γ. The state $z(t)$ satisfies the bound (70). Similarly, the bound of the parameter estimation error is obtained as

$$\| \tilde{\theta}(t) \|_{[0,T]}^2 = \frac{1}{T} \int_0^T \| \tilde{\theta}(t) \|^2 dt \leq \frac{4}{k_\theta} \left[\frac{V(0)}{T} + M \right] \quad (71)$$

∎

After establishing the main results, we now highlight main challenges in solving the problems due to state quantization and key techniques proposed to handle them in the following remarks.

Remark 7:
- One major difficulty to deal with the state quantization is that the backstepping technique requires differentiating virtual controls and in turn the states by applying chain rule. If the states are quantized, they become discontinuous and therefore it is difficult to analyze the resulting control system with the current backstepping based approaches.
- The above difficulty is overcome by not taking the derivative of $\bar{\alpha}_i$ in the controller design and stability analysis. Instead, the final control u in (26) and the virtual control law $\bar{\alpha}_i$ in (31) use the partial derivative term $\sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} q(x_{k+1})$, which avoids taking the derivative of $\bar{\alpha}_{i-1}$ in the control design.

Remark 8: By following the general framework of backstepping procedure and including an additional term in the form of $-\Gamma k_\theta(\hat{\theta} - \theta_0)$ in the parameter adaptive law (27), we manage to design the backstepping-based adaptive control law.

Remark 9: The main challenge in stability analysis is how to handle the effects caused by analyzing states $x_1, ..., x_n$, while only the quantized states $q(x_1), ..., q(x_n(t))$ are used in the designed controller. More specifically, a major difficulty in stability analysis is how to compensate for the effects from the terms $z_n(\bar{\alpha}_n - \alpha_n)$, $z_n(\psi - \hat{\psi})$ and $(\theta^T \phi z_n - \hat{\theta}^T \bar{\phi} z_n - \tilde{\theta}^T \bar{\phi} \bar{z}_n)$ in (60). By establishing the properties (33)-(36) in Lemma 1, (46) in Lemma 2 and (61), such terms are bounded by functions depending only on the state z_i and parameter estimation error $\tilde{\theta}$. Thus all these effects can be compensated by two negative terms $-\sum_{j=1}^n c_j z_j^2$ and $-\frac{1}{2} k_\theta \| \tilde{\theta} \|^2$ as shown in (62). Above new techniques enable us to successfully overcome the difficulties caused by the discontinuity of the quantized states and the bounded uncertainties resulted from the state quantization, so as to establish the results in Theorem 1.

Remark 10: As stated in Theorem 1, k_θ is chosen to satisfy (65), which depends on the quantization bound δ, the control parameters c_i, and L_ϕ. The lower bound of k_θ can be calculated with δ being known and therefore the designed adaptive controller is implementable. For simplicity, we let $\Gamma = \gamma I$. The upper bound of the overall stabilization errors in the mean square sense of (54) can be decreased by increasing γ and c_n.

Remark 11: The obtained bounds in Lemma 1 and Theorem 1 depend on the quantization bound δ. To reduce the conservatism of the results, we can design a quantizer by choosing a small quantization density.

IV. SIMULATION STUDY

In this section we consider a pendulum system from [19] as shown in Figure 4. The equation of the motion for the pendulum system is represented as

$$ml\ddot{\theta} + mgsin(\theta) + kl\dot{\theta} = u(t) \quad (72)$$

where θ denotes the angle of the pendulum, m, l and g are the mass $[kg]$, the length of the robe $[m]$, and the acceleration due to the gravity, k is an unknown friction coefficient, u represents an input torque provided by a DC motor. The states θ and $\dot{\theta}$ are quantized by a quantizer satisfying the bounding property (6). The objective is to design a control input for u to make the output θ track a reference signal $\theta_r(t) = sin(t)$.
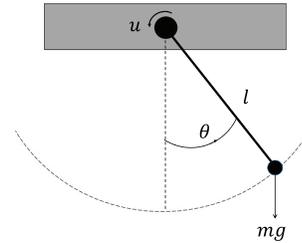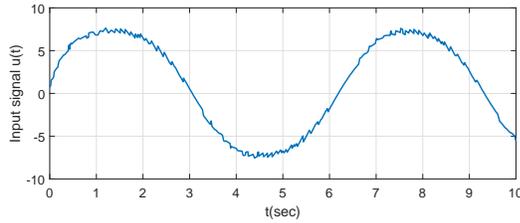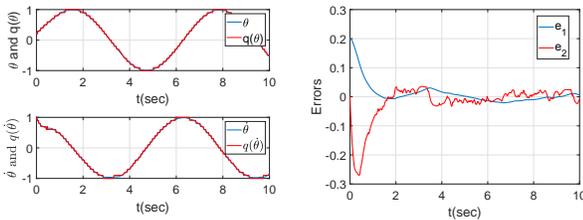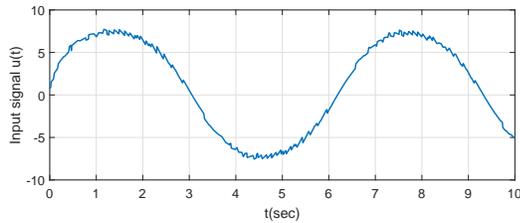

Fig. 4. Pendulum

In the simulation, we consider three quantizers: uniform quantizer in (7), hysteresis-uniform quantizer in (8) and logarithmic-uniform quantizer in (9). The quantization parameters are chosen as $l = 0.1$ for uniform quantizer, $l = 0.1$ and $p_h = 0.5$ for hysteresis-uniform quantizer, and $l = 0.1$, $\rho = 0.05$ and $x_{th} = 0.8$ for logarithmic-uniform quantizer, respectively. We choose $x_1 = \theta - \theta_r$ and $x_2 = \dot{x}_1 = \dot{\theta} - \dot{\theta}_r$. The adaptive control law (26) and the parameter estimation (27) are used where $\frac{\partial \alpha_1}{\partial x_1} = -c_1$. The initial states are chosen as $x(0) = 0.2, \dot{x}(0) = 1$ and $\hat{\theta}(0) = 0.8$. The parameters in the system (72) are selected as $m = 1$, $l = 1$, $g = 9.8$ and $k = 1$ for simulation. The design parameters are chosen as $c_3 = c_2 = 3, \gamma = 1, k_\theta = 0.1$.

The trajectories of states θ and $\dot{\theta}$ and the control input are shown in Figures 5-6 for a uniform quantizer, Figures 7-8 for a hysteresis-uniform quantizer, and Figures 9-10 for a logarithmic-uniform quantizer, respectively. Clearly, the simulation results verify our theoretical findings in Theorem 1 and show the effectiveness of our proposed control scheme.

In addition, the size of the set of outputs of the quantizer can also be calculated. For this example with the uniform quantizer with length $l = 0.1$, the state θ is bounded in $[-1\ 1]$ radian and the number of the outputs of the uniform quantizer is $round\left(\frac{\theta_{max} - \theta_{min}}{l}\right) = 20$.

V. CONCLUSION

In this paper, we develop an adaptive backstepping feedback stabilization scheme for a class of nonlinear systems with state quantization. The nonlinear functions in the system satisfy the globally Lipschitz condition. The quantizers considered in this paper are static and satisfy a bounded condition such that
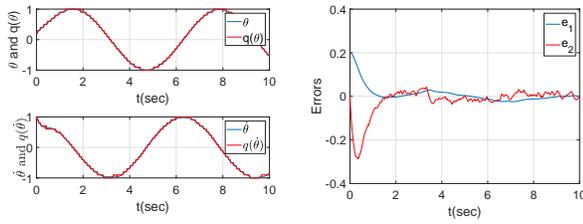
Fig. 5.   Uniform quantizer: θ and $\dot{\theta}$ and errors $\theta - \theta_r$ and $\dot{\theta} - \dot{\theta}_r$

Fig. 9.   Logarithmic-Uniform quantizer: θ and $\dot{\theta}$ and errors

Fig. 6.   Uniform quantizer: Input $u(t)$

Fig. 10.   Logarithmic-Uniform quantizer: Input $u(t)$

Fig. 7.   Uniform-Hysteresis quantizer: θ and $\dot{\theta}$ and errors

Fig. 8.   Uniform-Hysteresis quantizer: Input $u(t)$

the quantization error is bounded. It is shown that the uniform quantizer, hysteresis-uniform quantizer and logarithmic-uniform quantizer meet this bounding condition. By using backstepping approaches, a new adaptive control algorithm using only quantized states is developed by constructing a new compensation method for the effects of the state quantization. By using a new approach to stability analysis, the global ultimate boundedness of the system is obtained. The stabilization error performance is also established and can be improved by appropriately adjusting design parameters. Simulation results illustrate the effectiveness of our proposed scheme. A future work may be to relax the the global Lipschitz continuous condition for the nonlinear functions.

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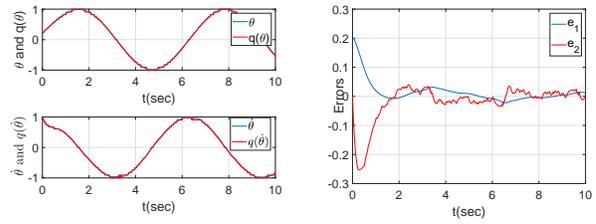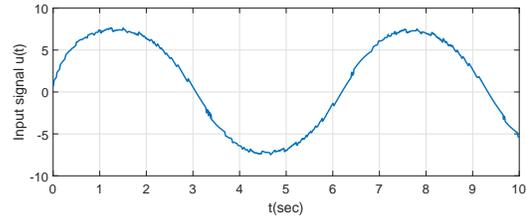
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