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Editors: Ewa Bergqvist, Magnus Österholm, Carina Granberg, and Lovisa Sumpter

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Research Reports Pr – Z

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THE USE OF NONSTANDARD PROBLEMS IN AN ODE COURSE FOR ENGINEERING STUDENTS

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We report on the design and use of 'nonstandard' problems in an ordinary differential equations (ODEs) course for engineering students. The focus of the paper is on the analysis of the development of students' mathematical discourse and conceptual understanding of Existence and Uniqueness Theorems (EUTs) from a commognitive theory perspective. Our analysis so far shows how students use familiar mathematical routines in new situations furthering their knowledge and understanding. Nonstandard problems have been a useful tool to gain insights into students' learning of mathematics.

INTRODUCTION AND BACKGROUND

The importance of ODEs in undergraduate (UG) mathematics education is widely acknowledged (Rasmussen & Wawro, 2017). However, fewer than two dozen empirical studies were published in top journals since 2004 which is "somewhat surprising given the centrality of differential equations (DEs) in the undergraduate curriculum" (ibid., p. 555). It is known that students experience difficulties with ODEs and even with the understanding of the very notions of a differential equation and its solutions (Arslan, 2010; Raychaudhuri, 2008). EUTs are among very few theoretical results included in standard ODEs courses for engineering students. Roberts (1976) emphasised that teaching EUTs makes engineering students aware of situations where solutions to initial value problems (IVPs) may not be unique or may not even exist.

Even though students experience difficulties with conceptual understanding of EUTs and their correct application (Raychaudhuri, 2007), "traditional content of DEs courses can be improved by including more activities aimed at enhancing the student's understanding of basic concepts such as DE, solution to a DE and existence and uniqueness theorem" (Arslan, 2010, p. 887). Recently, Klymchuk (2015) pointed out that students may form a habit of applying formulas or rules without checking conditions/constraints because assessment questions are often formulated so that these are automatically met. "But in real life problems not all functions and equations behave so nicely and ignoring conditions and constraints might lead to significant and costly errors" (ibid., p. 63). We use nonstandard problems to help students understand and correctly apply EUTs for DEs viewing such problems as tasks "for which students had no algorithm, well-rehearsed procedure or previously demonstrated process to follow" (Breen, O'Shea & Pfeiffer, 2013, p. 2318). Our intention is to provide challenging experiences for students studying towards mathematics-intensive degrees.

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RESEARCH SETTING AND METHODOLOGY

The research took place in an ODEs course for engineering students in their fourth year of study. The activity formed an assessed piece in the final part of the course when students acquired sufficient theoretical knowledge and good computational skills. Participation in the research was voluntary. Tutorials were attended by 50-65% of the total number of students enrolled. The lecturer of the course, one of the authors of this paper, devised a set of six problems to challenge students' conceptual understanding of the EUTs. Students were requested to work on the problems individually first (in the tutorial and at home) before discussing their solutions in small groups. In addition, each group presented their solution to one of the problems to the class. Students' written solutions (referred to as "scripts") were collected and photocopied. Group discussions were audio-recorded and transcribed. Students' scripts and group discussions of two groups, G1 and G2, (out of five recorded) form the basis of analysis for this paper. The code S12 is used to identify the student #2 in G1, etc.

We situate ourselves in an interpretative framework for data analysis using qualitative methods such as coding, interpreting and categorising (Cohen, Manion, & Morrison, 2008) to characterise students' discourse and how discourse develops. To aid and make meaningful our analysis we consider the theoretical constructs of narrative and routine of commognitive theory (Sfard, 2008). A scholarly mathematical discourse such as a definition or theorem is a narrative endorsed by the community of mathematicians. The EUTs formally introduced by the lecturer to her students represent such a mathematical narrative. Narratives are endorsed (or rejected) with the help of routines, repetitive patterns produced in creating or substantiating a narrative. Sfard distinguished three types by their aim. (1) The aim of explorations is to produce or substantiate a narrative and thus further a mathematical discourse. Explorations are further divided into construction, substantiation and recall. (2) Deeds are actions aimed at a physical change in objects. (3) Participation in rituals has the aim of creating or sustaining social bonds. Sfard also distinguishes the 'how' of a routine (the process or course of action) from the 'when' (the circumstances that evoke a person's actions and those that signal its completion), the applicability and closure conditions of a routine.

In this paper we pose the following research questions: How do students engage with nonstandard problems related to EUTs? In what ways do such problems further students' understanding of new mathematical concepts?

THE TASKS

We present the analysis of the first two problems, P1 and P2, of the assessment. All problems required the correct application of the EUTs. The theorem required for solving P1 and P2 states that if coefficients of a linear DE are continuous on a given interval, there exists a unique solution of the initial-value problem on this interval.

Students encountered 'how to verify' techniques for *particular* solutions to a DE but the requirement of P1(a) is for the *general* solution. This is an unusual problem for engineering students who are, to our knowledge and experience, almost never asked that. We make this claim by referring to standard UG engineering textbooks. It is easy to verify that the given function is a solution to the given DE (and students were able to do this) but to show that it is the general solution one has to explain the role of the arbitrary constant (method M1). To avoid this discussion, it is also possible to derive the general solution using an integrating factor or variation of constants (method M2). These are the two possible correct solutions for this problem with M1 being the method that the lecturer anticipated her students to use.

1. (a) Verify that

$$y(x) = \frac{2}{x} - \frac{C_1}{x^2}$$

is the general solution of a differential equation

$$x^2y' + 2xy = 2.$$

(b) Show that both initial conditions $y(1) = 1$ and $y(-1) = -3$ result in an identical particular solution. Does this fact violate the Existence and Uniqueness Theorem (EUT)? Explain your answer.

Figure 1: Formulation of Problem 1.

In P1(b) one may erroneously believe that two different initial conditions (ICs) give rise to the same solution $y = \frac{2}{x} - \frac{1}{x^2}$. However, since the coefficients $p(x) = \frac{2}{x}$ and $q(x) = \frac{2}{x^2}$ are not defined at $x = 0$ and are continuous on (-∞, 0) and (0, + ∞), but not on any interval including zero, there are two disjoint intervals of existence of solutions, each containing one of the initial points. The formulation of P1(b) is nonstandard for engineering students and the problem was intentionally stated in this form to create a conflict or consternation. If students go ahead and show that a given function formally satisfies the DE without paying attention to discontinuity of coefficients and the function itself at zero, they may miss the point altogether. It is for this reason the lecturer explicitly asked, "Does it violate the EUT?" and "Explain your answer".

In P2(a) students were asked to verify that a given function is the *general* solution of a second order linear DE. Again, there are two correct methods. The first is to substitute the function and its derivatives into the DE and discuss the representation for the general solution as a linear combination of two linearly independent solutions (method M3), and the second method is to integrate the DE using the substitution and reducing the order (method M4). This is an unusual problem as one cannot be sure that the given function is the general solution by merely substituting it into the DE. Usually, second order linear DEs with variable coefficients are not discussed in detail in ODE courses for engineering students but in this case the EUT can be successfully applied.

P2(b) required students to verify that the ICs cannot be satisfied. Most students did this correctly. In P2(c) it was necessary to notice that the ICs were defined at $x = 0$ which is the point of discontinuity of the coefficients of the DE. Hence, the existence and uniqueness of solutions cannot be guaranteed even though a solution may still exist and

2. (a) Verify that
$$y(x) = C_1 + C_2 x^2$$
is the general solution of a differential equation
$$xy'' - y' = 0. \qquad (1)$$
(b) Explain why there exists no particular solution of equation (1) satisfying initial conditions
$$y(0) = 0, \qquad y'(0) = 1.$$
(c) Suggest different initial conditions for this differential equation so that there will exist exactly one particular solution of a new initial value problem. Motivate your choice.

Figure 2: Formulation of Problem 2.

be unique if the conditions of the EUT are not satisfied. There are at least two ways to modify the ICs: (i) to change the initial point from $x = 0$ to any other value and use the EUT, or (ii) to modify the ICs at zero and show by direct inspection that the solution exists. In the latter case one also has to prove that the solution is unique.

ANALYSIS OF PROBLEM 1

Our analysis of P1(a) shows that students were able to demonstrate that a given function satisfies the DE as the extracts from the group discussion provided below show. However, all nine students (four in G1 and five in G2) failed to explain in their discussions and in written solutions that were submitted (with the exception of one student), why this solution was the *general* solution.

S12: Since we got the solution, I just took the derivative of that and put it into the original equation, to see that two equals two, and … that was my verification.

S13 and S14 confirmed having done the same as S12 while S11 responded,

S11: So I was the only one who actually did any work, [laughter] so I actually integrated the whole thing, and ended up with the right expression, so I think your way of doing it is a lot easier.

S13 answered "A bit more efficient at least" and all moved on to P1(b). Thus three students only showed that a given function satisfies the DE. One student integrated the DE obtaining the general solution which is a correct method (M2). However, the incomplete solution is accepted as correct (and complete) without further discussion. The discussion in G2 followed a slightly different pattern with S22 stating that it was "obvious" to differentiate the function, "put it into the DE and see if it's correct as usual." Three students (S23, S24 and S25) integrated the DE, for example,

S25: I did the same as you did, using the integrating factor, multiplying and then I just solved the equation because it's solvable ….

S24: I also solved the equation by the integrating factor, but I think it's more easy just to derivate it once and put it into the original equation and see if it's correct.

We see S24 echoing the same sentiment as S11 that method M1 is easier. However, one student intercepted to say,

S25: But there could be more solutions, they are not general solutions.

S24 answered "Yes, maybe" and all moved on to P1(b). Thus, three students reported on obtaining the general solution by integration and one of them stated that the verifying-by-substitution method was easier. Remarkably, S25 seemed to be aware that there may be a problem with it. The written (handed-in) solution of this student did contain a discussion of the constant and was the only correct solution using M1 submitted for final assessment to the lecturer. However, in the group discussion this point was not elaborated, and students agreed on the two approaches being equivalent – when they are not.

Analyses of students' scripts showed how students changed their solutions following the discussions with peers and the presentation. From a correct solution using integration (M2) to obtain the *general* solution - and thus proving what had been asked - to a verifying-by-substitution method (M1) omitting the discussion of the constant (and hence incomplete). From the analysis of the dialogue we deduce that this change occurred because students thought that M1 was easier and more efficient and not because students (with the exception of S25) realised that they were moving into a new discourse that had the aim of extending their conceptual understanding of the difference between the notions of a particular and the general solution of a DE.

P1(b) had a clear reference to the theorem to be used. The correct solution involved verification that both solutions for two different ICs were given by the same formula but these solutions were defined and continuous on two different intervals. G1 discussed P1(b) - considered "the hard part" by S11 - as follows.

S12: Undefined at zero, so we get two different curves and both solutions work. We do not have a continuous curve which happens to intersect at these two points, it's two curves that will be correct in this small area.

One student in G2 explained correctly the effect of the discontinuity.

S25. Yes, but there is a discontinuity at $x = 0$, so it [the theorem] only guarantees that for $x > 0$ and $x < 0$.

Other students in G2 provided explanations such as "So you have to have two unique solutions, one on each interval" (S24), "for two different initial conditions" (S22), and "one for the left part and one for the right part of 0" (S24). S21 appeared to struggle saying, "There could happen the chance maybe that they are the same perhaps." This indicates that the student did not understand that by the definition any solution to a DE should be at least a continuous function. P1(b) probed students' understanding of what violation implies. Four students agreed and stated explicitly that it did not violate the EUT. S21 stated, “It violates at $x = 0$” which prompted the reply,

S22. Well, but that’s not the question. The question is that if you have two identical particular solutions, does that violate the theorem. And it doesn’t.

When S21 repeated the question, S22 gave a fuller explanation that convinced S21.

Analysing students’ scripts of P1(b) we found that ten students provided complete and correct solutions. This number increased to fourteen after students had the opportunity to discuss their solutions in small groups. We tend to believe that students had time to reflect on the conditions of the EUTs and benefited from the discussion.

ANALYSIS OF PROBLEM 2

The formulation of P2(a) is similar to that of P1(a) but the nature of the problem is different due to the higher order of the DE. Most students in G1 and G2 verified by substituting the function and its derivatives in the DE that it is a solution, but failed to show that it is the general solution, not completing M3. S11 described obtaining the general solution by integration (M4) with “It’s only me that’s stupid, obviously, cos I derived the entire expression” while S13 concluded it to be “inefficient.” Students agreed that “there are different ways of showing the same thing” (S11) but considered the (incomplete) method M3 as “more efficient” (S11) or “better” (S13). The opportunity to expand their discourse was cut short at that point. In G2, S24 explained that he solved the problem by substituting the given function into the DE, and S21, S22 and S23 agreed. S25 argued that the Wronskian should be used for showing linear independence of two solutions, 1 and x^2, thus verifying that the given function is the general solution. However, S25 hesitated about the validity of this approach noticing that the Wronskian vanishes at $x = 0$ and not realising that the DE is not defined at this point. This important detail was pointed out by S24.

S25: Yes. This is what I arrive at, but I think it’s strange, that it’s a general solution when it’s still not valid. I am not too confident in this.

S24: But the original equation $xy''-y' = 0$, it isn't defined for $x = 0$.

Not all students were able to follow this argument as can be seen in students’ scripts. The exception was S25 who correctly used the Wronskian (M4) in his solution.

To explain why no particular solution can be found for the given ICs in P2(b) both groups provided similar arguments. Most students (S11, S12, S23, S24) used the ICs, obtained $C_1 = 0$ and a contradiction for the constant C_2 $(0 \cdot C_2 = 1)$. S22 reflected on the general solution considering a parabola which has an extremum at $x = 0$ where the derivative should always vanish, leading to the contradiction. S11 was expanding his discourse by looking for a reason “why it was this way”, leading him to explore:

S11: I did the same thing as well but I tried thinking why is it this way, and my sort of conclusion was that it’s in the bottom of a parabola, where the derivative always is 0.

In the final scripts three students applied ICs, obtained inconsistent system of linear equations and correctly used the EUT to explain this, while nine students only acknowledged the contradiction and provided no explanation. Two students employed a geometric argument to show that the ICs cannot be satisfied.

P2(c) provided an opportunity for students to practice the new mathematical discourse of EUTs. Students in both groups offered solutions saying they "just guessed" (S11, S12) or "made them up" (S23, S24). S11 reflected on the properties of parabolas to obtain ICs that worked. Reflecting on the multiple ways of formulating ICs students agreed that it suffices to shift the initial point to $x \neq 0$. S25 drew on the EUT to round off the discussion in G2, "I used the existence and uniqueness theorem because of discontinuity at $x = 0$, so no guarantee there, but for all other x there is a solution guaranteed", with S22 and S24 voicing agreement.

DISCUSSION

Nonstandard problems stimulated lively mathematical discussions in which students gave accounts of their solutions and collectively explored different approaches. For P1(a) students discussed two different ways they believed should verify that the given expression was the general solution. While one of the two approaches was correct, the other *could have* resulted in a correct explanation if students had recalled the definition of the general solution. Students experienced certain difficulties with the correct mathematical meaning of *particular* and *general* solutions - confirming previous research (Arslan, 2010; Raychaudhuri, 2007). We conclude that P1(a) has to be modified to call students' attention to the details of the definitions and theorems.

In P1(b), in addition to correctly produced solutions, students unexpectedly worked out collectively what does violation of the conditions of EUTs mean. They developed new understandings in this context (not foreseen by the lecturer at the stage of the task design).

P2(a) echoed some of the difficulties that students had in P1(a) with the meaning of the *general* solution. For P2(b) several students approached the solution via graphical representation which had not been discussed in this context in the course. While some students used linear algebra reasoning to arrive at the contradiction, others explained it using geometric argument. For P2(c) students correctly used the EUT to shift the ICs from $x = 0$.

In summary, we observed students' difficulties with the definitions of the particular and general solution, something the lecturer had not anticipated. We have also perceived successful use of visual mediators by students to explain why the solution to the given problem could not exist.

CONCLUSION AND FURTHER WORK

In this study we analyse students' mathematical understanding as a result of engagement with nonstandard problems, from individual scripts that show students' initial understanding to more advanced discussions in small groups and improved final solutions handed in for assessment. Students used a number of different recall routines. They employed integration of DE to obtain the general solution in P1(a) and P2(a). After group discussions most students switched to the recall routines of differentiation and substitution (appropriate for particular solutions). Thus, contrary to our expectations, students used familiar recall routines in a context where further modifications were needed. We are, therefore, going to consider how to modify formulations of P1(a) and P2(a), in particular the routine prompts used in these problems.

Our task design reflected a new, unfamiliar to students, narrative around EUTs. The lecturer used familiar words in an unusual context and described tasks differently. In our ongoing analyses we will focus on the theoretical constructs of narratives and routines including the applicability and closure conditions of routines to further our understanding of the teaching and learning process. More detailed analyses will be included in the conference presentation.

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