POLYHEDRALITY AND DECOMPOSITION

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Abstract. The aim of this note is to present two results that make the task of finding equivalent polyhedral norms on certain Banach spaces, having either a Schauder basis or an uncountable unconditional basis, easier and more transparent. The hypotheses of both results are based on decomposing the unit sphere of a Banach space into countably many pieces, such that each one satisfies certain properties. Some examples of spaces having equivalent polyhedral norms are given.

1. Introduction

The concepts of upper and lower p-estimates, where $1 < p < \infty$, play an important role when studying the geometry of Banach spaces. More precisely, using their relationship with p-convexity and concavity, it is possible to find asymptotically sharp estimates at 0 of the moduli of convexity and smoothness, and the cotype and type of the Banach lattice (see e.g. [13, Chapter 1]). We introduce an analogue of upper p-estimate in the case $p = \infty$, and in doing so we find sufficient conditions for isomorphic polyhedral renorming. In our opinion, these conditions are easier to verify in many concrete cases. Let us recall that, following V. Klee [10], a Banach space is said to be *polyhedral* when the unit balls of all of its finite-dimensional subspaces are polytopes. A Banach space X is said to be *isomorphically polyhedral* if it is isomorphic to a polyhedral space or, equivalently, if X admits an equivalent polyhedral norm.

We denote by B_X and S_X the (closed) unit ball and unit sphere of X, respectively. Let X have an unconditional basis $(e_\gamma)_{\gamma\in\Gamma}$, with corresponding biorthogonal functionals $(e^*_\gamma)_{\gamma\in\Gamma}$. Given a subset $A \subseteq \Gamma$, we define the projections

$$P_A x \;=\; \sum_{\gamma\in A} e^*_\gamma(x) e_\gamma \qquad \text{and} \qquad R_A x \;=\; x - P_A x.$$

If $(e_j)_{j=1}^\infty$ is a Schauder basis (with corresponding biorthogonals $(e^*_j)_{j=1}^\infty$), define $P_n = P_{\{1,\ldots,n\}}$ and $R_n = R_{\{1,\ldots,n\}}$. From time to time we will require the *support* of an element in X or its dual, with respect to the given basis: define

$$\operatorname{supp}(x) \;=\; \left\{\gamma \in \Gamma \;:\; e^*_\gamma(x) \neq 0\right\},$$

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for all $x \in X$ and, given $f \in X^*$, set

$$\operatorname{supp}(f) \;=\; \{\gamma \in \Gamma \;:\; f(e_\gamma) \neq 0\}\,.$$

We will also require a type of function known in approximation theory as a *modulus*, namely a non-decreasing continuous function $\omega : [0,\infty) \to [0,\infty)$ such that $\omega(0) = 0$. We present our chief definition.

Definition 1.1. We say that the Banach space X has decomposition $(*)$ (with respect to the unconditional basis $(e_\gamma)_{\gamma\in\Gamma}$ and modulus ω) if, for every $x \in X$ there exist positive numbers $c(x)$ and $d(x)$, such that the inequality

$$\|x\| \;\leqslant\; \|P_A x\| + c(x)\omega(d(x)\,\|R_A x\|_\infty), \tag{$*$}$$

holds for every subset $A \subseteq \Gamma$. Here, $\|\cdot\|_\infty$ denotes the *supremum norm* on X, i.e.

$$\|x\|_\infty \;=\; \max\left\{|e^*_\gamma(x)| \;:\; \gamma \in \Gamma\right\}.$$

Remark 1.2. It is enough that $(*)$ holds only for all $x \in S_X$. Given $x \neq 0$, we can set

$$c(x) \;=\; \|x\| \cdot c\left(\frac{x}{\|x\|}\right) \qquad \text{and} \qquad d(x) \;=\; \frac{1}{\|x\|} \cdot d\left(\frac{x}{\|x\|}\right).$$

Clearly $(*)$ holds for x if it holds for $x/\|x\|$.

Now we present our two main results.

Theorem 1.3. *Let a Banach space X have $(*)$ with respect to a symmetric basis $(e_\gamma)_{\gamma\in\Gamma}$. Then X admits an equivalent polyhedral norm.*

The proof of this theorem follows from the next result.

Theorem 1.4. *Let X be a Banach space having an unconditional basis $(e_\gamma)_{\gamma\in\Gamma}$. Let $(a_n)_{n=1}^\infty$ be a sequence of positive numbers tending to 0, such that*

$$\liminf_{n\to\infty} a_n^{-1}\big(\,\|x\| - \sup_{|A|\leqslant n} \|P_A x\|\,\big) \;<\; \infty \qquad \textit{for every } x \in X. \tag{1}$$

Then X admits an equivalent polyhedral norm.

Alternatively, if X admits a Schauder basis $(e_n)_{n=1}^\infty$, we can reach the same conclusion if we replace condition (1) by

$$\liminf_{n\to\infty} a_n^{-1}\left(\|x\| - \|P_n x\|\right) \;<\; \infty \qquad \textit{for every } x \in X. \tag{2}$$

We prove the two main results above in Section 2. Theorem 1.4 will follow from Proposition 2.3, which is a special case of the theorem restricted to monotone bases. Section 3 is devoted to examples. We will present a series of examples of Banach spaces having $(*)$, an example that exposes the difference between conditions (1) and (2) in Theorem 1.4, and an example of a non-symmetric equivalent norm on c_0 that does not satisfy condition (1) with respect to the usual basis.

We finish this section by making some observations about condition (2) above. Let us recall that $B \subseteq S_{X^*}$ is called a *boundary* of X (with respect to the norm $\|\cdot\|$) if, given $x \in X$, there exists $f \in B$ such that $f(x) = \|x\|$. In [2] and [9], it was proved that every

Banach space that has a σ-compact boundary (with respect to the norm topology) admits an equivalent polyhedral norm. We show that, in this case, condition (2) is necessary, provided that $(e_j)_{j=1}^\infty$ is shrinking.

Proposition 1.5. *Assume that X has a shrinking Schauder basis and a σ-compact boundary. Then there exists a sequence $(a_n)_{n=1}^\infty$ of positive numbers tending to 0, such that (2) holds.*

We have need of the following fact, which will be used also in Corollary 2.4.

Fact 1.6. *For every $m \in \mathbb{N}$, let $(a_{m,n})_{n=1}^\infty$ be a sequence of positive numbers such that $\lim_{n\to\infty} a_{m,n} = 0$. Then the sequence*

$$a_n \;:=\; \sum_{m=1}^{\infty} 2^{-m}\frac{a_{m,n}}{1+a_{m,n}}, \qquad n \in \mathbb{N}, \tag{3}$$

tends to 0, and

$$a_{m,n} \;\leqslant\; 2^m a_n \max_{k\in\mathbb{N}}(a_{m,k}+1),$$

for all $m, n \in \mathbb{N}$.

Proof of Proposition 1.5. Let $(K_m)_{m=1}^\infty$ be a sequence of norm compact subsets of S_{X^*}, such that $B := \bigcup_{m=1}^\infty K_m$ is a boundary. Since $(e_j)_{j=1}^\infty$ is shrinking, it is well known that $\lim_{n\to\infty} \|R_n^* f\| = 0$ for all $f \in X^*$ [12, Proposition 1.b.1]. Using the norm compactness of the K_m, $m \in \mathbb{N}$, we see that

$$a_{m,n} \;:=\; \sup_{f\in K_m} \|R_n^* f\|,$$

tends to 0 as $n \to \infty$. Let $x \in X$. As B is a boundary, there exists $m \in \mathbb{N}$ such that $f(x) = \|x\|$ for some $f \in K_m$. Given $n \in \mathbb{N}$, we have

$$\|x\| \;=\; f(x) \;=\; f(P_n x) + R_n^* f(x) \;\leqslant\; \|P_n x\| + \|R_n^* f\|\,\|x\| \;\leqslant\; \|P_n x\| + a_{m,n}\|x\|,$$

hence

$$\frac{\|x\| - \|P_n x\|}{a_{m,n}} \;\leqslant\; \|x\|,$$

for all $n \in \mathbb{N}$. Defining a_n as in (3) yields

$$\frac{\|x\| - \|P_n x\|}{a_n} \;\leqslant\; 2^m \max_{k\in\mathbb{N}}(a_{m,k}+1)\,\|x\|,$$

for all $n \in \mathbb{N}$. □

The requirement that the basis in Proposition 1.5 be shrinking is necessary for the conclusion to hold.

Example 1.7. The space c_0 with its natural norm has a countable boundary, but with respect to the summing basis of c_0, there is no sequence $(a_n)_{n=1}^\infty$ tending to 0, such that (2) holds.

Proof. Let $(e_j)_{j=1}^{\infty}$ and $(e_j^*)_{j=1}^{\infty}$ be the standard bases of c_0 and ℓ_1, respectively. The set $\{\pm e_j^* \;:\; j \in \mathbb{N}\}$ is a countable boundary of c_0 with respect to its natural norm. If $x_j := \sum_{i=1}^{j} e_i$ denotes the jth element of the summing basis of c_0, then $x_j^* = e_j^* - e_{j+1}^*$, and with respect to this basis we see that

$$P_n x \;=\; \sum_{j=1}^{n}(x(j) - x(j+1))\left(\sum_{i=1}^{j} e_i\right) \;=\; \sum_{i=1}^{n}\big(x(i) - x(n+1)\big)e_i.$$

Suppose that $x(1) = \|x\|_\infty \geqslant |x(n)| + 1$ whenever $n \geqslant 2$. Then, whenever $|x(n+1)| \leqslant \frac{1}{2}$, we have $\|P_n x\|_\infty = x(1) - x(n+1) = \|x\|_\infty - x(n+1)$. Given a sequence $(a_n)_{n=1}^{\infty}$ of positive numbers tending to 0, define $x \in c_0$ by $x(1) = \max_{n\geqslant 1}\sqrt{a_n} + 1$ and $x(n) = \sqrt{a_{n-1}}$ for $n \geqslant 2$. Fix $m \in \mathbb{N}$ such that $a_n \leqslant \frac{1}{4}$ whenever $n \geqslant m$. Then $|x(n+1)| \leqslant \frac{1}{2}$ for such n and

$$a_n^{-1}(\|x\|_\infty - \|P_n x\|_\infty) \;=\; a_n^{-1}x(n+1) \;=\; a_n^{-\frac{1}{2}} \;\to\; \infty. \qquad \square$$

2. Decompositions of the unit sphere and applications

In this section we prove Theorems 1.3 and 1.4. In a previous version of the paper, we did this by appealing to [5, Corollary 14], which is a tool for building polyhedral norms on certain Banach spaces in possession of a Markushevich basis. In accordance with the referee's suggestion, this has been replaced by Proposition 2.2, which presents two sufficient conditions for polyhedral renorming expressed in terms of decompositions of the unit sphere, and which applies to certain Banach spaces having a monotone unconditional basis or monotone Schauder basis. In addition, we provide a direct proof of this result, to benefit readers who are newcomers to this sort of work and who would otherwise have to digest quite a lot of background material. The proof is an amalgamation and reworking of those of [5, Theorem 4] and [4, Theorem 24]. For the purposes of proving the main results of this paper, we lose nothing by moving from Markushevich bases to working with the less general hypotheses of Proposition 2.2.

In order to prove Proposition 2.2, we will require the following result from [8], which we state in a modified form.

Theorem 2.1 ([8, Theorem 1]). *Let X be a Banach space and let $D \subseteq X^*$ be such that $\|x\| = \sup\{f(x) \;:\; f \in D\}$ for all $x \in X$. If there does not exist a w^*-accumulation point d of D and $x \in X$ satisfying $d(x) = \|x\| = 1$, then $\|\cdot\|$ is polyhedral.*

In their original statement D was the set of extreme points of B_{X^*}, but the result applies equally well in the more general case above (see e.g. [3, Proposition 6.11]).

Proposition 2.2.

(1) *Let X have a monotone unconditional basis $(e_\gamma)_{\gamma\in\Gamma}$ and suppose we can write*

$$S_X \;=\; \bigcup_{k=1}^{\infty} S_k,$$

where each S_k is non-empty, and find a sequence of positive integers $(n_k)_{k=1}^\infty$ in such a way that the sequence

$$b_k \ := \ \inf_{x\in S_k} \sup_{|A|=n_k} \|P_A x\|,$$

is strictly positive and converges to 1. Then X admits an equivalent polyhedral norm.

(2) *If $(e_n)_{n=1}^\infty$ is a monotone Schauder basis of X, with S_k and n_k as above, and the sequence*

$$b'_k \ := \ \inf_{x\in S_k} \|P_{n_k} x\|,$$

behaves likewise, then we reach the same conclusion.

Proof. We prove the first statement. Let $(e_\gamma)_{\gamma\in\Gamma}$ be a monotone unconditional basis of X. As the basis is monotone we can, without loss of generality, assume that $(n_k)_{k=1}^\infty$ is increasing. For convenience, set $n_0 = 0$. Set $c_k = \inf\{b_\ell \ : \ \ell \geqslant k\}$, $k \in \mathbb{N}$. Given $n \in \mathbb{N}$, define $a_n = c_k^{-1}(1+2^{-n})$, where $k \in \mathbb{N}$ is chosen in such a way that $n_{k-1} < n \leqslant n_k$. It is clear that $(a_n)_{n=1}^\infty$ is strictly decreasing and converges to 1. Given $n \in \mathbb{N}$ and a subset $A \subseteq \Gamma$ having cardinality n, define $W_A = \operatorname{span}_{\gamma\in A}(e^*_\gamma)$ and let D_A be a symmetric finite $2^{-(n+2)}$-net of S_{W_A}. By the monotonicity of the basis, given $x \in X$, there exists $f \in D_A$ satisfying

$$(1-2^{-(n+2)})\,\|P_A x\| \ \leqslant \ f(x). \tag{4}$$

Define sets

$$D_n \ = \ \bigcup \{D_A \ : \ A \subseteq \Gamma \text{ and } |A| = n\}, \qquad n \in \mathbb{N},$$

and a norm $|||\cdot|||$ by $|||x||| = \sup\{f(x) \ : \ f \in a_n D_n,\, n \in \mathbb{N}\}$, $x \in X$. First, we show that

$$\|x\| \ < \ |||x||| \ \leqslant \ a_1\|x\|,$$

whenever $x \neq 0$. It is evident that $|||x||| \leqslant a_1\|x\|$ for all x. Let $x \in S_X$. Then $x \in S_k$ for some k. We observe that

$$1 \ \leqslant \ c_k^{-1} \sup_{|A|=n_k} \|P_A x\|,$$

so there exists $A \subseteq \Gamma$, $|A| = n_k$, such that

$$(1+2^{-(n_k+1)})^{-1} \ < \ c_k^{-1}\|P_A x\|.$$

Using (4), we can choose $f \in D_A$ such that

$$(1-2^{-(n_k+2)})\,\|P_A x\| \ \leqslant \ f(x).$$

It follows that

$$\begin{aligned}|||x||| \ &\geqslant \ a_{n_k} f(x) \ \geqslant \ a_{n_k}(1-2^{-(n_k+2)})\,\|P_A x\| \\ &> \ \frac{a_{n_k} c_k (1-2^{-(n_k+2)})}{1+2^{-(n_k+1)}} \ = \ \frac{(1+2^{-n_k})(1-2^{-(n_k+2)})}{1+2^{-(n_k+1)}} \ > \ 1 \ = \ \|x\|,\end{aligned}$$

as required.

Now we claim that $d(x) < 1$ whenever $|||x||| = 1$ and d is a w^*-accumulation point of $\bigcup_{n=1}^{\infty} a_n D_n$. From Theorem 2.1 it will follow that $|||\cdot|||$ is polyhedral. Let $|||x||| = 1$ and let d be such an accumulation point. The first case is that

$$d \in \bigcap_{k=1}^{\infty} \overline{\left(\bigcup_{k=n}^{\infty} a_n D_n \right)}^{w^*}.$$

If this is so, then $\|d\| \leqslant a_k$ for all k, because $(a_n)_{n=1}^{\infty}$ is decreasing. As $a_n \to 1$, we have $\|d\| \leqslant 1$. Therefore, using the strict inequality proved above, $d(x) \leqslant \|x\| < |||x||| = 1$.

The second case is that d is a w^*-accumulation point of $a_n D_n$ for some n. Let (f_λ) be a net in D_n such that $a_n f_\lambda \neq d$ for all λ and $a_n f_\lambda$ converges to d in the w^*-topology. Given λ, let $A_\lambda \subseteq \Gamma$ such that $|A_\lambda| = n$ and $f_\lambda \in D_{A_\lambda}$. Set $A = \operatorname{supp}(d)$. We claim that $m := |A| < n$. Indeed, by w^*-convergence, whenever $\gamma \in A$, we have $\gamma \in \operatorname{supp}(f_\lambda) \subseteq A_\lambda$ for all large enough λ. Therefore $m \leqslant n$. Moreover, if $m = n$, then A_λ must take constant value A for all large enough λ. However, this cannot happen because D_A is finite and $a_n f_\lambda \neq d$ for all λ. Thus $m < n$ as claimed. We see further that $d \in a_n B_{W_A}$. Thus, by (4) again, there exists $f \in D_A$ such that

$$(1 - 2^{-(m+2)}) a_n^{-1} d(x) \;\leqslant\; (1 - 2^{-(m+2)}) \, \|P_A x\| \;\leqslant\; f(x) \;\leqslant\; a_m^{-1} |||x||| \;=\; a_m^{-1}. \tag{5}$$

Pick $k \in \mathbb{N}$ such that $n_{k-1} < m \leqslant n_k$. From (5) it follows that

$$d(x) \;\leqslant\; \frac{a_n}{a_m(1 - 2^{-(m+2)})} \;\leqslant\; \frac{1 + 2^{-n}}{(1 + 2^{-m})(1 - 2^{-(m+2)})} \;<\; \frac{1 + 2^{-n}}{1 + 2^{-(m+1)}} \;\leqslant\; 1,$$

as required (in fact it is not hard to show moreover that $|||d||| < 1$, but this fact is not required for our purposes). Our claim is proved and thus $|||\cdot|||$ is polyhedral.

The second statement, concerning Schauder bases, follows from the implication (a) $\Rightarrow$ (d) of [6, Theorem 1]. It also follows from the above by defining $c_k = \inf \{ b'_\ell \;:\; \ell \geqslant k \}$, $k \in \mathbb{N}$, letting D_n be a symmetric finite $2^{-(n+2)}$-net of $S_{P_n^* X^*}$, and making attendant simplifications thereafter. $\square$

We use Proposition 2.2 to prove the next result, which is Theorem 1.4 restricted to the case of monotone bases. We prove Theorem 1.4 straight afterwards.

Proposition 2.3.

(1) *Let X be a Banach space with a monotone unconditional basis $(e_\gamma)_{\gamma \in \Gamma}$. Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive numbers tending to* 0*, such that*

$$\liminf_{n \to \infty} a_n^{-1} \big(\|x\| - \sup_{|A|=n} \|P_A x\| \big) \;<\; \infty, \tag{6}$$

for all $x \in X$. Then X admits an equivalent polyhedral norm.

(2) *If $(e_n)_{n=1}^{\infty}$ is a monotone Schauder basis of X and $(a_n)_{n=1}^{\infty}$ is a sequence of positive numbers tending to* 0*, such that*

$$\liminf_{n \to \infty} a_n^{-1} \big(\|x\| - \|P_n x\| \big) \;<\; \infty, \tag{7}$$

then we reach the same conclusion.

Proof. We consider the first statement. Without loss of generality, we may assume that the sequence $(a_n)_{n=1}^\infty$ is non-increasing (if necessary, we can replace a_n by $a'_n := \max_{j\geqslant n} a_j$ – clearly (6) holds with respect to the a'_n). There exists an increasing sequence of positive integers $(n_k)_{k=1}^\infty$, such that the sequence $(ka_{n_k})_{k=1}^\infty$ tends to 0 and $\max_{k\geqslant 1} ka_{n_k} < 1$. From (6) it follows that, for every $x \in S_X$, there exist positive integers $m(x)$ and $\ell(x) \geqslant n_{m(x)}$ such that

$$1 \leqslant \sup_{|A|=n_{\ell(x)}} \|P_A x\| + m(x)a_{\ell(x)}. \tag{8}$$

Given $k \in \mathbb{N}$, set

$$S_k = \{x \in S_X : n_k \leqslant \ell(x) < n_{k+1}\},$$

and let $K = \{k \in \mathbb{N} : S_k \text{ is non-empty}\}$. Clearly, $S_X = \bigcup_{k\in K} S_k$.

Let $k \in K$ and $x \in S_k$. We have $n_k, n_{m(x)} \leqslant \ell(x) < n_{k+1}$. Since $(n_k)_{k=1}^\infty$ is increasing and $(a_k)_{k=1}^\infty$ is non-increasing, we get $m(x) \leqslant k$ and $a_{\ell(x)} \leqslant a_{n_k}$. Using (8) and the monotonicity of the basis we get

$$1 \leqslant \sup_{|A|=n_k} \|P_A x\| + ka_{n_k}.$$

Given $k \in K$, define

$$b_k = \inf_{x\in S_k} \sup_{|A|=n_k} \|P_A x\|.$$

We obtain $0 < 1 - ka_{n_k} \leqslant b_k \leqslant 1$. Enumerate K as an increasing sequence of positive integers $(k_j)_{j=1}^\infty$. Clearly $(S_{k_j})_{j=1}^\infty$ and $(b_{k_j})_{j=1}^\infty$ satisfy the hypotheses of Proposition 2.2 (1).

To prove the second statement, we repeat the proof above using (7), replacing instances of $\sup_{|A|=n} \|P_A x\|$ by $\|P_n x\|$ as we go, and using Proposition 2.2 (2). ☐

Proof of Theorem 1.4. First let us show how to treat the case when the basis $(e_\gamma)_{\gamma\in\Gamma}$ is unconditional. Given $x \in X$ and $\alpha \in [-1,1]^\Gamma$, we set

$$\psi(x,\alpha) = \left\| \sum_{\gamma\in\Gamma} \alpha(\gamma) e^*_\gamma(x) e_\gamma \right\|.$$

Introduce on X an equivalent norm by the formula

$$|||x||| = \sup\left\{\psi(x,\alpha) : \alpha \in [-1,1]^\Gamma\right\}.$$

Let (1) hold. We show that, for every $x \in X$, (6) holds with respect to $|||\cdot|||$.

Let $x \in X$. Since the function ψ is a continuous with respect to its second argument, and as $[-1,1]^\Gamma$ is compact, we find that $\psi(x,\cdot)$ attains its maximum at some $\beta \in [-1,1]^\Gamma$, i.e. $|||x||| = \psi(x,\beta)$. Set $y = \sum_{\gamma\in\Gamma} \beta(\gamma) e^*_\gamma(x) e_\gamma$. From the definition of $|||\cdot|||$, we know that

$$\|P_A y\| \leqslant |||P_A y||| = |||P_A x|||,$$

for every $A \subseteq \Gamma$. Hence by the monotonicity of the basis with respect to $|||\cdot|||$

$$\sup_{|A|\leqslant n} \|P_A y\| \leqslant \sup_{|A|\leqslant n} |||P_A x||| = \sup_{|A|=n} |||P_A x|||.$$

Since $|||x||| = \|y\|$, we get

$$|||x||| - \sup_{|A|=n} |||P_A x||| \;\leqslant\; \|y\| - \sup_{|A|\leqslant n} \|P_A y\|\,.$$

Bearing in mind that (1) holds for y, we have

$$\liminf_{n\to\infty} a_n^{-1}\big(|||x||| - \sup_{|A|=n} |||P_A x|||\big) \;<\; \infty,$$

thus we can apply Proposition 2.3.

In the Schauder basis case, we proceed much as above, using the equivalent norm $|||x||| = \sup_n \|P_n x\|$. We show that (2) (equivalently (7)) holds with respect to $|||\cdot|||$. Let $x \in X$. If $|||x|||$ should happen to equal $\|x\|$, we have

$$|||x||| - |||P_n x||| \;\leqslant\; \|x\| - \|P_n x\|\,, \tag{9}$$

for all $n \in \mathbb{N}$. Assume now that $|||x||| > \|x\|$. Since $\|x\| = \lim_{n\to\infty} \|P_n x\|$, we have $|||x||| = \|P_m x\|$ for some $m \in \mathbb{N}$. Since the basis $(e_j)_{j=1}^\infty$ is monotone with respect to $|||\cdot|||$ we have $|||P_n x||| = |||P_m x|||$ whenever $n \geqslant m$. Thus in this case

$$|||x||| - |||P_n x||| \;=\; 0.$$

Since (2) holds with respect to $\|\cdot\|$, this, together with (9), imply that (2) (and (7)) holds with respect to $|||\cdot|||$, and so we are in a position to apply the second part of Proposition 2.3. ☐

Now we turn our attention to the proof of Theorem 1.3. First, using Fact 1.6, the hypotheses of Theorem 1.4 (1) can be relaxed a little.

Corollary 2.4. *Let X be a Banach space having an unconditional basis $(e_\gamma)_{\gamma\in\Gamma}$. Given $m \in \mathbb{N}$, let $(a_{m,n})_{n=1}^\infty$ be a sequence of positive numbers such that $\lim_{n\to\infty} a_{m,n} = 0$ and, for every $x \in X$,*

$$\inf_{m\in\mathbb{N}}\left(\liminf_{n\to\infty} a_{m,n}^{-1}\left(\|x\| - \sup_{|A|\leqslant n}\|P_A x\|\right)\right) \;<\; \infty.$$

Then X admits a polyhedral renorming.

The following remark will be used a few times in proofs and examples to come, including that of Proposition 2.6, from which we deduce Theorem 1.3. It also allows us to simplify the expression $\sup_{|A|\leqslant n}\|P_A x\|$ in Theorem 1.4 (1), in the event that the basis of X is 1-symmetric.

Remark 2.5. Given non-zero $x \in X$, where X has an unconditional basis $(e_\gamma)_{\gamma\in\Gamma}$, we can enumerate $\mathrm{supp}(x)$ as a (finite or infinite) sequence $(\gamma_k)_{k\geqslant 1}$ of distinct points in Γ, in such a way that $|e^*_{\gamma_1}(x)| \geqslant |e^*_{\gamma_2}(x)| \geqslant |e^*_{\gamma_3}(x)| \ldots$. Set $A_n(x) = \{\gamma_1, \ldots, \gamma_n\}$ (or set $A_n(x) = \mathrm{supp}(x)$ if $|\mathrm{supp}(x)| < n$). If the basis of X is 1-symmetric then $\sup_{|A|\leqslant n}\|P_A x\|$ in condition (1) is equal to $\left\|P_{A_n(x)}x\right\|$. The choice of the γ_k, and thus the sets $A_n(x)$, may not be unique, however, said choice will not matter whenever we make use of these sets.

Let $(e_\gamma)_{\gamma\in\Gamma}$ be a normalized unconditional basis of a Banach space X. Set

$$\lambda_n \;=\; \inf\left\{\left\|\sum_{\gamma\in A} e_\gamma\right\| \;:\; A\subseteq\Gamma,\ |A|\geqslant n\right\}.$$

Assume that $(e_\gamma)_{\gamma\in\Gamma}$ is a symmetric basis. Then X is isomorphic to $c_0(\Gamma)$ if and only if the sequence $(\lambda_n)_{n=1}^\infty$ is bounded (this follows immediately from the fact that, given a normalized basis $(e_\gamma)_{\gamma\in\Gamma}$ of a Banach space having unconditional basis constant K, we have

$$K^{-1}\max_{\gamma\in A}|a_\gamma| \;\leqslant\; \left\|\sum_{\gamma\in A} a_\gamma e_\gamma\right\| \;\leqslant\; K\max_{\gamma\in A}|a_\gamma|\left\|\sum_{\gamma\in A} e_\gamma\right\|.$$

for every finite set $A\subseteq\Gamma$ and reals a_γ, $\gamma\in A$). Since $c_0(\Gamma)$ is polyhedral, Theorem 1.3 follows immediately from the next and final result of the section.

Proposition 2.6. *Let X have $(*)$ with respect to an unconditional basis $(e_\gamma)_{\gamma\in\Gamma}$ and some modulus ω. If*

$$\lim_{n\to\infty}\lambda_n \;=\; \infty, \tag{10}$$

then X admits a polyhedral renorming.

Proof. Pick $x\in X$ and define the sets $A_n(x)$ as in Remark 2.5. Given $n\in\mathbb{N}$, we have

$$\begin{aligned}
\left\|R_{A_n(x)}x\right\|_\infty &= \sup\left\{|e^*_\gamma(x)| \;:\; \gamma\in\Gamma\setminus A_n(x)\right\}\\
&\leqslant |e^*_{\gamma_n}(x)|\\
&\leqslant \lambda_n^{-1}\left\|\sum_{\gamma\in A_n(x)} e_\gamma\right\|\cdot|e^*_{\gamma_n}(x)|\\
&= \lambda_n^{-1}\left\|\sum_{\gamma\in A_n(x)} e^*_{\gamma_n}(x)e_\gamma\right\|\\
&\leqslant \lambda_n^{-1}K\left\|\sum_{k\geqslant 1} e^*_{\gamma_k}(x)e_{\gamma_k}\right\| \;=\; K\|x\|\,\lambda_n^{-1},
\end{aligned}$$

where K is the unconditional basis constant of $(e_\gamma)_{\gamma\in\Gamma}$. Since X is assumed to have $(*)$, it follows that

$$\begin{aligned}
\|x\| &\leqslant \left\|P_{A_n(x)}x\right\| + c(x)\omega(d(x)\left\|R_{A_n(x)}x\right\|_\infty)\\
&\leqslant \left\|P_{A_n(x)}x\right\| + c(x)\omega(Kd(x)\|x\|\,\lambda_n^{-1})\\
&\leqslant \sup_{|A|\leqslant n}\|P_A x\| + c(x)\omega(Kd(x)\|x\|\,\lambda_n^{-1}).
\end{aligned}\tag{11}$$

Set $a_{m,n} = m\omega(m\lambda_n^{-1})$. Given (10), we see that $\lim_{n\to\infty} a_{m,n} = 0$ for all $m\in\mathbb{N}$. From (11), it follows that

$$\|x\| \;\leqslant\; \sup_{|A|\leqslant n}\|P_A x\| + a_{m,n},$$

for all $n \in \mathbb{N}$, provided $m \geqslant \max\{c(x), Kd(x)\|x\|\}$. Now we are in a position to apply Corollary 2.4. The proof is complete. ☐

3. Examples

In our first example, we present two wide classes of Banach spaces that are quite different in character, yet share the property of having $(*)$.

Example 3.1.

(1) Let X have a normalized unconditional basis $(e_\gamma)_{\gamma\in\Gamma}$ and suppose that the set of all *summable elements* of the unit sphere
$$\left\{ f \in S_{X^*} \; : \; \sum_{\gamma\in\Gamma} |f(e_\gamma)| < \infty \right\},$$
with respect to the basis, is a boundary. Then X has $(*)$.

(2) Let M be a non-degenerate normalized Orlicz function, i.e. $M(t) > 0$ for all $t > 0$ and $M(1) = 1$. Let Γ be a set and let $h_M(\Gamma)$ be the space of all real functions x defined on Γ, such that
$$\sum_{\gamma\in\Gamma} M\left(\frac{|x(\gamma)|}{\rho}\right) \; < \; \infty,$$
for all $\rho > 0$. We equip $h_M(\Gamma)$ with the Luxemburg norm
$$\|x\| \; := \; \inf\left\{ \rho > 0 \; : \; M\left(\frac{|x(\gamma)|}{\rho}\right) \leqslant 1 \right\}.$$
The space $h_M(\Gamma)$ has $(*)$ with respect to the unit vector basis $(e_\gamma)_{\gamma\in\Gamma}$, provided
$$\lim_{t\to 0} \frac{M(Kt)}{M(t)} \; = \; \infty, \tag{12}$$
for some constant $K > 1$.

Proof.

(1) Set $\omega(t) = t$. Given $x \in X$, take $f \in B$ such that $f(x) = \|x\|$. Set $c(x) = \sum_{\gamma\in\Gamma} |f(e_\gamma)|$ and $d(x) = 1$. Given $A \subseteq \Gamma$,
$$\begin{aligned} \|x\| \; = \; f(x) \; &= \; f(P_A x) + f(R_A x) \\ &= \; f(P_A x) + \sum_{\gamma\in\Gamma\setminus A} f(e_\gamma) e^*_\gamma(x) \\ &\leqslant \; \|P_A x\| + \left(\sum_{\gamma\in\Gamma\setminus A} |f(e_\gamma)| \right) \|R_A x\|_\infty \\ &\leqslant \; \|P_A x\| + c(x) \, \|R_A x\|_\infty . \end{aligned}$$

(2) Given $t > 0$, set

$$\omega(t) \;=\; \sup\left\{\frac{M(\tau)}{M(K\tau)} \;:\; 0 < \tau \leqslant t\right\}.$$

Evidently, ω is a continuous non-decreasing function and $\lim_{t\to 0}\omega(t) = 0$. Given $x = \sum_{\gamma\in\Gamma} x(\gamma)e_\gamma \in h_M(\Gamma)$, $\|x\| = 1$, we let

$$c(x) \;=\; \sum_{\gamma\in\Gamma} M(K|x(\gamma)|),$$

and $d(x) = 1$. From the definition of $h_M(\Gamma)$, we see that $c(x)$ is finite. Let $A \subseteq \Gamma$. Since M is a convex function satisfying $M(0) = 0$, we have

$$\sum_{\gamma\in\Gamma} M(\lambda|x(\gamma)|) \;\leqslant\; \lambda\sum_{\gamma\in\Gamma} M(|x(\gamma)|),$$

whenever $0 \leqslant \lambda \leqslant 1$. In particular, as $\|P_A x\| \leqslant 1$,

$$\sum_{\gamma\in A} M(|x(\gamma)|) \;\leqslant\; \|P_A x\|\sum_{\gamma\in A} M\left(\frac{|x(\gamma)|}{\|P_A x\|}\right) \;=\; \|P_A x\|\,.$$

Therefore,

$$\begin{aligned}
\|x\| \;=\; 1 \;&=\; \sum_{\gamma\in\Gamma} M(|x(\gamma)|)\\
&=\; \sum_{\gamma\in A} M(|x(\gamma)|) + \sum_{\gamma\in\Gamma\setminus A} M(|x(\gamma)|)\\
&\leqslant\; \|P_A x\| + \left(\sup_{\gamma\in\Gamma\setminus A}\frac{M(|x(\gamma)|)}{M(K|x(\gamma)|)}\right)\sum_{\gamma\in\Gamma\setminus A} M(K|x(\gamma)|)\\
&\leqslant\; \|P_A x\| + \omega(\|R_A x\|_\infty)\sum_{\gamma\in\Gamma} M(K|x(\gamma)|)\\
&=\; \|P_A x\| + c(x)\omega(\|R_A x\|_\infty).
\end{aligned}$$

□

Remark 3.2.

(1) For the use of summable boundaries in polyhedral renorming, see [1, 7, 14].
(2) D. Leung proved that $h_M(\mathbb{N})$ admits an equivalent polyhedral norm provided M satisfies (12) [11]. For the case when Γ is an arbitrary set, see [4, 5].

Example 3.3. We consider a symmetric version of the Nakano space. Let Γ be a set and let $(p_n)_{n=1}^\infty$ be a non-decreasing sequence, with $p_1 \geqslant 1$. By $h^S_{(p_n)}(\Gamma)$ we denote the space of all real functions x defined on Γ, such that

$$\phi\left(\frac{x}{\rho}\right) \;<\; \infty,$$

for all $\rho > 0$, where

$$\phi(x) \;:=\; \sup\left\{\sum_{k=1}^{\infty} |x(\gamma_k)|^{p_k} \;:\; (\gamma_k)_{k=1}^{\infty} \text{ is a sequence of distinct points in } \Gamma\right\}.$$

Given $x \in h^S_{(p_n)}(\Gamma)$, we set

$$\|x\| \;=\; \inf\left\{\rho > 0 \;:\; \phi\left(\frac{x}{\rho}\right) \leqslant 1\right\}.$$

It is easy to see that the standard unit vectors $(e_\gamma)_{\gamma\in\Gamma}$ form an unconditional symmetric basis in $h^S_{(p_n)}(\Gamma)$. We show that $h^S_{(p_n)}(\Gamma)$ satisfies equation (1) from Theorem 1.4, provided $p_n \to \infty$.

Proof. Pick $\theta \in (0,1)$. We show that for every $x \in h^S_{(p_n)}(\Gamma)$ satisfying $\|x\| = 1$, there exists $m(x) \in \mathbb{N}$ such that

$$1 - \left\|P_{A_n(x)}x\right\| \;\leqslant\; \theta^{p_n}, \tag{13}$$

whenever $n \geqslant m(x)$, where $A_n(x)$ is any set $\{\gamma_1, \dots, \gamma_n\}$ of the form described in Remark 2.5. Setting $a_n = \theta^{p_n}$ in (13) yields (1).

As in the proof of Example 3.1 (2), as ϕ is a convex function and $\phi(0) = 0$, and $\left\|P_{A_n(x)}x\right\| \leqslant \|x\| = 1$, we have

$$\phi(P_{A_n(x)}x) \;\leqslant\; \left\|P_{A_n(x)}x\right\| \phi\left(\frac{P_{A_n(x)}x}{\left\|P_{A_n(x)}\right\|}\right) \;=\; \left\|P_{A_n(x)}x\right\|. \tag{14}$$

Given $\gamma \in \Gamma \setminus A_n(x)$, and bearing in mind that $\|\cdot\|$ is a lattice norm, we have

$$|x(\gamma)| \;\leqslant\; \left\|R_{A_n(x)}x\right\| \;\leqslant\; \|x\| \;=\; 1,$$

and therefore

$$\begin{aligned}
1 \;=\; \phi\left(\frac{R_{A_n(x)}x}{\|R_{A_n(x)}x\|}\right) \;&=\; \sum_{k=1}^{\infty}\left(\frac{|x(\gamma_{n+k})|}{\|R_{A_n(x)}x\|}\right)^{p_k} \\
&\geqslant\; \sum_{k=1}^{\infty}\left(\frac{|x(\gamma_{n+k})|}{\|R_{A_n(x)}x\|}\right)^{p_{k+n-1}} \\
&=\; \sum_{j=n+1}^{\infty}\left(\frac{|x(\gamma_j)|}{\|R_{A_n(x)}x\|}\right)^{p_{j-1}} \;\geqslant\; \sum_{j=n+1}^{\infty}\frac{|x(\gamma_j)|^{p_j}}{\left\|R_{A_n(x)}x\right\|^{p_n}},
\end{aligned}$$

which implies

$$\sum_{j=n+1}^{\infty} |x(\gamma_j)|^{p_j} \;\leqslant\; \left\|R_{A_n(x)}x\right\|^{p_n}. \tag{15}$$

There exists $m(x) \in \mathbb{N}$ such that $\left\|R_{A_n(x)}x\right\| \leqslant \theta$ whenever $n \geqslant m(x)$. Together with (14) and (15), this implies

$$\begin{aligned} 1 \;=\; \phi(x) \;&=\; \phi(P_{A_n(x)}x) + \sum_{j=n+1}^{\infty} |x(\gamma_j)|^{p_j} \\ &\leqslant\; \left\|P_{A_n(x)}x\right\| + \left\|R_{A_n(x)}x\right\|^{p_n} \;\leqslant\; \left\|P_{A_n(x)}x\right\| + \theta^{p_n}, \end{aligned}$$

whenever $n \geqslant m(x)$. □

The following examples are based on the next simple and well known fact.

Fact 3.4. *Let $(c_k)_{k=1}^n$ and $(d_k)_{k=1}^n$ be non-increasing sequences of non-negative numbers. Then*

$$\sum_{k=1}^{n} c_k d_{\pi(k)} \;\leqslant\; \sum_{k=1}^{n} c_k d_k, \tag{16}$$

whenever π is a permutation of $\{1,\ldots,n\}$.

In the next example, we expose the difference between conditions (1) and (2) of Theorem 1.4.

Example 3.5. There exists an equivalent norm $\|\cdot\|$ on c_0 that is symmetric with respect to the usual basis, such that

(1) given $x \in c_0$,

$$2^n\Big(\|x\| - \sup_{|A|\leqslant n} \|P_A x\| \Big) \;\leqslant\; 4\,\|x\|\,, \tag{17}$$

(2) but given a sequence $(a_n)_{n=1}^\infty$ of positive numbers tending to 0, there exists $y \in c_0$ such that

$$\lim_{n\to\infty} a_n^{-1}\big(\|y\| - \|P_n y\| \big) \;=\; \infty. \tag{18}$$

Proof. Consider Day's norm, defined on c_0 by

$$\|x\| \;=\; \sup\left\{ \left(\sum_{k=1}^{\infty} 2^{-k} x(j_k)^2 \right)^{\frac{1}{2}} \;:\; (j_k)_{k=1}^\infty \text{ is a sequence of distinct points in } \mathbb{N} \right\}. \tag{19}$$

(1) Pick $x \in c_0$ such that $\|x\| = 1$. We define $A_n(x)$ as in Remark 2.5. From (16), it follows that

$$\sup_{|A|\leqslant n} \|P_A x\| \;=\; \left\|P_{A_n(x)}x\right\| \;=\; \left(\sum_{k=1}^{n} 2^{-k} x(j_k)^2 \right)^{\frac{1}{2}}. \tag{20}$$

Since $|x(\gamma)| \leqslant 2\,\|x\| = 2$, we have

$$1 - \left\|P_{A_n(x)}x\right\| \;\leqslant\; 1 - \left\|P_{A_n(x)}\right\|^2 \;=\; \sum_{k=n+1}^{\infty} 2^{-k} x(j_k)^2 \;\leqslant\; 4 \sum_{k=n+1}^{\infty} 2^{-k} \;=\; 2^{2-n}.$$

Together with (20), this implies (17).

(2) Let $(a_n)_{n=1}^\infty$ be a sequence of positive numbers tending to 0. Let $(n_k)_{k=1}^\infty$ be a strictly increasing sequence of positive integers such that

$$a_n \;\leqslant\; 8^{-k}, \tag{21}$$

for all $n \geqslant n_k$. Define $x \in c_0$ by

$$x(n) \;=\; \begin{cases} 3^{\frac{1}{2}} \cdot 2^{-\frac{k}{2}} & \text{if } n = n_k, \\ 0 & \text{otherwise.} \end{cases}$$

From (19) we get $\|x\| = 1$ and

$$1 - \|P_n x\|^2 \;=\; 3 \sum_{i=k+1}^{\infty} 4^{-i} \;=\; 4^{-k},$$

whenever $n_k \leqslant n < n_{k+1}$. Hence,

$$1 - \|P_n x\| \;>\; \tfrac{1}{2}(1 - \|P_n x\|^2) \;=\; \tfrac{1}{2} 4^{-k}.$$

Using (21), we obtain $a_n^{-1}(1 - \|P_n x\|) \geqslant 2^{k-1}$ whenever $n \geqslant n_k$, which yields (18). $\square$

The final example shows that condition (1) of Theorem 1.4 can fail even on c_0, if the norm fails to be symmetric.

Example 3.6. There exists on c_0 an equivalent (non-symmetric) norm $\|\cdot\|$, with respect to which the standard basis is normalized and 1-unconditional, and having the property that given a sequence $(a_n)_{n=1}^\infty$ of positive numbers tending to 0, there exists $x \in c_0$ such that

$$\lim_{n\to\infty} a_n^{-1} \left(\|x\| - \sup_{|A| \leqslant n} \|P_A x\| \right) \;=\; \infty. \tag{22}$$

Proof. Let

$$D \;=\; \left\{ m^{-1} 2^{-k} \;:\; m, k \in \mathbb{N} \right\},$$

and let $q : \mathbb{N} \to D$ have the property that $q^{-1}(d)$ is infinite for all $d \in D$. Write $q_n = q(n)$, $n \in \mathbb{N}$. Let S be the set of all infinite subsets $L \subseteq \mathbb{N}$, such that $q_j \geqslant q_n$ whenever $j, n \in L$, $j \leqslant n$, and $\sum_{n \in L} q_n = 1$. Set

$$E \;=\; \{\pm e_n^* \;:\; n \in \mathbb{N}\} \cup \left\{ 2 \sum_{n \in L} s_n q_n e_n^* \;:\; L \in S \text{ and } s_n \in \{-1, 1\} \text{ for all } n \in \mathbb{N} \right\},$$

and define the norm

$$\|x\| \;=\; \sup \{ f(x) \;:\; f \in E \}.$$

Then $\|x\|_\infty \leqslant \|x\| \leqslant 2\|x\|_\infty$ and $\|e_n\| = 1$, as $q_n \leqslant \frac{1}{2}$ for all n, and the signs s_n in the definition of E ensure that the standard basis is 1-unconditional with respect to $\|\cdot\|$.

Given $x \in c_0$, we shall say that $|x|$ is non-increasing on its support if $|x(j)| \geqslant |x(n)|$ whenever $j, n \in \operatorname{supp}(x)$ and $j \leqslant n$. Next, we prove the following fact. Let $x \in c_0$ such that $|x|$ is non-increasing on its support, and suppose that there exists $L \in S$ such that $\mathrm{supp}(x) \subseteq L$ and

$$\|x\|_\infty \ < \ 2\sum_{j\in L} q_j|x(j)|.$$

Furthermore, let L_n be the set of the first n elements of L, and let n_0 be large enough so that

$$\|x\|_\infty \ < \ 2\sum_{j\in L_{n_0}} q_j|x(j)|.$$

Then the conclusion is that

$$\|x\| - \sup_{|A|\leqslant n} \|P_A x\| \ = \ 2\sum_{j\in L\setminus L_n} q_j|x(j)|, \tag{23}$$

whenever $n \geqslant n_0$.

To prove this fact, first we show that

$$2\sum_{j\in L} q_j|x(j)| \ = \ \|x\|\,. \tag{24}$$

One inequality is obvious. To see the other, since $\|x\|_\infty < 2\sum_{j\in L} q_j|x(j)|$, all we need to do is check that

$$\sum_{j\in M} q_j|x(j)| \ \leqslant \ \sum_{j\in L} q_j|x(j)|,$$

whenever $M \in S$, and indeed this holds, because $\mathrm{supp}(x) \subseteq L$. Next, since $|x|$ is non-increasing on its support, as is $(q_j)_{j\in L}$, given $n \geqslant n_0$ and $A \subseteq \mathbb{N}$, $|A| \leqslant n$, we have

$$\|P_A x\| \ \leqslant \ 2\sum_{j\in L_n} q_j|x(j)| \ = \ \|P_{L_n} x\|\,.$$

The equality in the line above follows because (24) holds with $P_{L_n}x$ and L_n in place of x and L, respectively. Note that

$$\|P_{L_n} x\|_\infty \ < \ 2\sum_{j\in L_n} q_j|x(j)|,$$

whenever $n \geqslant n_0$. Since $|L_n| = n$, this completes the proof of the fact.

Now let $(a_n)_{n=1}^\infty$ be a sequence of positive numbers tending to 0. Choose integers $0 = n_0 < n_1 < n_2 < \ldots$, such that

(a) $a_n \leqslant 8^{-k}$ whenever $n \geqslant n_k$, and
(b) $n_k - n_{k-1} \leqslant n_{k+1} - n_k$ for all $k \in \mathbb{N}$.

Since $q^{-1}(d)$ is infinite for all $d \in D$, it is possible to find finite sets $H_k \subseteq \mathbb{N}$ such that

(c) $\max H_k < \min H_{k+1}$,
(d) $|H_k| = n_k - n_{k-1}$ and
(e) $q_j = 2^{-k}/|H_k|$ for all $j \in H_k$.

Define $L = \bigcup_{k=1}^{\infty} H_k$. We have

$$\sum_{j \in L} q_j \;=\; \sum_{k=1}^{\infty} \sum_{j \in H_k} \frac{2^{-k}}{|H_k|} \;=\; \sum_{k=1}^{\infty} 2^{-k} \;=\; 1.$$

Together with (b) – (e) above, this ensures that $L \in S$. Now define $x \in c_0$ by

$$x(j) \;=\; \begin{cases} \frac{3}{2} \cdot 2^{-k} & \text{whenever } j \in H_k, \\ 0 & \text{otherwise.} \end{cases}$$

Then $|x| = x$ is non-increasing on its support, which equals L, and

$$2\sum_{j \in L} q_j |x(j)| \;=\; 2\sum_{k=1}^{\infty} \sum_{j \in K_k} \frac{2^{-k}}{|H_k|} \cdot \tfrac{3}{2} \cdot 2^{-k} \;=\; 3\sum_{k=1}^{\infty} 4^{-k} \;=\; 1 \;>\; \tfrac{3}{4} \;=\; |x(1)| \;=\; \|x\|_{\infty}.$$

We make the simple observation that

$$2 \sum_{j \in L_{n_2}} q_j |x(j)| \;=\; 3\left(\sum_{j \in H_1} \frac{4^{-1}}{|H_1|} + \sum_{j \in H_2} \frac{4^{-2}}{|H_2|} \right) \;=\; 3(\tfrac{1}{4} + \tfrac{1}{16}) \;>\; \|x\|_{\infty}.$$

Therefore, using equation (23), given $n \geqslant n_2$, we have

$$\|x\| - \sup_{|A| \leqslant n} \|P_A x\| \;=\; 2 \sum_{j \in L \setminus L_n} q_j x(j).$$

Given $n \geqslant n_2$, let $k \geqslant 2$ such that $n_k \leqslant n < n_{k+1}$. Then

$$\begin{aligned} \|x\| - \sup_{|A| \leqslant n} \|P_A x\| \;&=\; 2 \sum_{j \in L \setminus L_n} q_j x(j) \;\geqslant\; 2 \sum_{j \in L \setminus L_{n_{k+1}}} q_j x(j) \\ &=\; 2 \sum_{\ell=k+2}^{\infty} \sum_{j \in H_\ell} q_j x(j) \;=\; 2 \sum_{\ell=k+2}^{\infty} \tfrac{3}{2} \cdot 4^{-\ell} \;=\; 4^{-k-2}. \end{aligned}$$

Combining this with (a) above yields

$$a_n^{-1}\left(\|x\| - \sup_{|A| \leqslant n} \|P_A x\| \right) \;\geqslant\; 8^k \cdot 4^{-k-2} \;=\; 4^{k-2} \;\to\; \infty,$$

as $n \to \infty$. □

We do not know if the norm in Example 3.6 can be replaced by one that is symmetric.

Problem 3.7. *Let $X = (c_0, \|\cdot\|)$, where $\|\cdot\|$ is a symmetric equivalent norm. Does there exist a sequence $(a_n)_{n=1}^{\infty}$ of positive numbers tending to* 0*, such that Theorem 1.4* (1) *holds?*

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