RELATIVELY WEAKLY OPEN CONVEX COMBINATIONS OF SLICES

TROND A. ABRAHAMSEN AND VEGARD LIMA

Abstract. We show that c_0, and in fact $C(K)$ for any scattered compact Hausdorff space K, have the property that finite convex combinations of slices of the unit ball are relatively weakly open.

1. Introduction

Let X be a (real or complex) Banach space with unit ball B_X, unit sphere S_X, and dual X^*. Given $x^* \in S_{X^*}$ and $\varepsilon > 0$ we define a slice of B_X by

$$S(x^*, \varepsilon) := \{x \in B_X : \operatorname{Re} x^*(x) > 1 - \varepsilon\},$$

where $\operatorname{Re} x^*(x)$ denotes the real part of $x^*(x)$.

Recall the following successively stronger "big-slice concepts", defined in [3]:

Definition 1.1. A Banach space X has the

(i) *local diameter 2 property* if every slice of B_X has diameter 2.

(ii) *diameter 2 property* if every non-empty relatively weakly open subset of B_X has diameter 2.

(iii) *strong diameter 2 property* if every finite convex combination of slices of B_X has diameter 2.

By Bourgain's lemma [7, Lemma II.1] every non-empty relatively weakly open subset of B_X contains a finite convex combination of slices hence the strong diameter 2 property implies the diameter 2 property. It was shown in [4] that the two properties are not equivalent. Since a slice is relatively weakly open, the diameter 2 property implies the local diameter 2 property. Even though the converse is not true in general, as shown in [5], for some spaces it is. For example, it is known that if a Banach space X satisfies that every $x \in S_X$ is an extreme point of $B_{X^{**}}$, then every non-empty relatively weakly open subset of B_X contains a slice by Choquet's lemma (cf. e.g. Proposition 1.3 in [1]).

On a particularly sunny day at a conference at the University of Warwick in 2015, Olav Nygaard asked if the converse of Bourgain's lemma is ever true for B_X. The aim of this short note is to answer this question affirmatively by showing that c_0, and in fact $C(K)$ for any scattered compact Hausdorff space K, have the much stronger property that finite convex combinations of slices of the unit ball are relatively weakly open. See Theorems 2.3 and 2.4 below.

Date: March 19, 2019.

2010 *Mathematics Subject Classification.* 46B04, 46B20.

Key words and phrases. convex combinations of slices, relatively weakly open set, scattered compact, diameter two property.

Let us note that in general it is not true that finite convex combinations of slices of the unit ball are relatively weakly open. Indeed, for some spaces there are finite convex combinations of slices of the unit ball that do not even intersect the sphere. The Banach space ℓ_2 is one example [7, Remark IV.5]. In their proof (independent of [4]) that the strong diameter 2 property is stronger than the diameter 2 property Haller, Langemets and Põldvere [8] show that if Z is an ℓ_p-sum of two Banach spaces, $Z = X \oplus_p Y$ with $1 < p < \infty$, then for every $\lambda \in (0,1)$ there exists two slices S_1 and S_2, and a $\beta > 0$ such that $\lambda S_1 + (1-\lambda)S_2 \subset (1-\beta)B_Z$.

We should also remark that the positive part of the unit sphere of $L_1[0,1]$, $F = \{f \in L_1[0,1] : f \geq 0, \|f\| = 1\}$ is another example of a closed convex bounded subset of a Banach space that satisfies a converse to Bourgain's lemma in that finite convex combinations of slices of F are relatively weakly open [7, Remark IV.5].

The notation and conventions we use are standard and follow e.g. [6].

2. Main result

We start by recalling the following definition (see e.g. [6, Definition 14.19]).

Definition 2.1. A compact space K is said to be *scattered compact* if every closed subset $L \subset K$ has an isolated point in L.

Let K be a scattered compact Hausdorff space and consider the Banach space $C(K)$ of all (complex valued) continuous functions on K with sup-norm. Rudin [11] showed that $C(K)^* = \ell_1(K)$ in this case. Pełczyński and Semadeni [10] showed that for a compact Hausdorff space K we have $C(K)^* = \ell_1(K)$ if and only if K is scattered (= dispersed).

To prove the main result, we will need the following geometric lemma for the unit circle in the complex plane.

Lemma 2.2. *Let $\alpha, \beta \in \mathbb{R}$ such that $e^{i\alpha}$ and $e^{i\beta}$ are distinct points on the unit circle with distance $d = |e^{i\alpha} - e^{i\beta}|$. If $0 < \mu < \frac{1}{2}$, then the point $c = \mu e^{i\alpha} + (1-\mu)e^{i\beta}$ on the line segment between $e^{i\alpha}$ and $e^{i\beta}$ satisfies*

$$|c| \leq 1 - \frac{d^2\mu}{4}.$$

Proof. A straightforward calculation shows that $d^2 = 2 - 2\cos(\alpha - \beta)$ and that $|c|^2 = \mu^2 + (1-\mu)^2 + \mu(1-\mu)2\cos(\alpha-\beta)$. Hence $|c|^2 = 1 - d^2\mu(1-\mu)$. Since $\sqrt{1+x} \leq 1 + \frac{x}{2}$ for $x \geq -1$ and $\mu(1-\mu) \geq \frac{\mu}{2}$ for $\mu \in [0, \frac{1}{2}]$ we get

$$|c| = \sqrt{1 - d^2\mu(1-\mu)} \leq 1 - \frac{1}{2}d^2\mu(1-\mu) \leq 1 - \frac{d^2\mu}{4}$$

as desired. □

Theorem 2.3. *Let K be a scattered compact Hausdorff space. Then every finite convex combination of slices of the unit ball of $C(K)$ is relatively weakly open.*

Proof. Let $\{S(f_j, \varepsilon_j)\}_{j=1}^k$ be slices of $B_{C(K)}$ with $f_j \in \ell_1(K)$, $\|f_j\| = 1$, and $\varepsilon_j > 0$ for $j = 1, 2, \dots, k$. Let $\lambda_j > 0$ with $\sum_{j=1}^k \lambda_j = 1$, and consider the convex combination of these slices

$$C = \sum_{j=1}^{k} \lambda_j S(f_j, \varepsilon_j).$$

Let $x = \sum_{j=1}^{k} \lambda_j z_j \in C$ with $z_j \in S(f_j, \varepsilon_j)$. Our goal is to find a non-empty relatively weakly open neighborhood of x that is contained in C.

Let $d = \min\{\operatorname{Re} f_j(z_j) - (1-\varepsilon_j) : 1 \le j \le k\}$ and let $\eta > 0$ be such that $\eta < d/3$. Let $E \subset K$ be a finite set such that $\sum_{t \notin E} |f_j(t)| < \eta$ for $1 \le j \le k$.

Define

$$\mathcal{U} = \Big\{ y \in B_{C(K)} : |y(t) - x(t)| < \delta, t \in E \Big\}$$

where $\delta > 0$. Next we specify how δ is chosen.

Let $L = \max\{\frac{1}{\lambda_j} : j = 1, 2, \ldots, k\}$. Let

$$E_I = \{ t \in E : \text{there exists } 1 \le j_0 \le k \text{ such that } |z_{j_0}(t)| < 1 \}.$$

Define

$$\delta_I = (1+3L)^{-1} \min\{1 - |z_{j_0}(t)| : t \in E_I, |z_{j_0}(t)| < 1\}$$

if E_I is non-empty and $\delta_I = 1$ otherwise. Let

$$E_{III} = \{ t \in E \setminus E_I : \text{there exists } j \neq m \text{ such that } z_j(t) \neq z_m(t) \}$$

and define

$$D = \min_{t \in E_{III}} \min_{z_j(t) \neq z_m(t)} \{ |z_j(t) - z_m(t)|^2 \}. \tag{1}$$

Choose $0 < \rho < \min\{D/8, \eta/4L\}$. Define $\delta_{III} = D\rho(4(1+3L))^{-1}$ if E_{III} is non-empty and $\delta_{III} = 1$ otherwise. Finally we choose $\delta < \min\{\eta/6L, \delta_I, \delta_{III}\}$.

Let $y \in \mathcal{U}$. We will define $y_j \in S(f_j, \varepsilon_j)$, $j = 1, 2, \ldots, k$, and show that y can be written $y = \sum_{j=1}^{k} \lambda_j y_j \in C$.

Let $\{\mathcal{V}_t\}_{t \in E}$ be a collection of pairwise disjoint neigborhoods for the points in E chosen such that for each $t \in E$ we have $|z_j(t) - z_j(s)| < \delta$, $1 \le j \le k$, $|x(t) - x(s)| < \delta$ and $|y(t) - y(s)| < \delta$ for all $s \in \mathcal{V}_t$. If $t \in E$ is an isolated point, we let $\mathcal{V}_t = \{t\}$. Note that, in particular, we get $|x(s) - y(s)| < 3\delta$ for all $s \in V_t$.

Definition of y_j outside $\bigcup_{t \in E} \mathcal{V}_t$.

For $s \in K \setminus \cup_{t \in E} \mathcal{V}_t$ we define $y_j(s) = y(s)$ for all $1 \le j \le k$.

Definition of y_j on $\bigcup_{t \in E} \mathcal{V}_t$.

For each $t \in E$ the way we define y_j on $\mathcal{V}_t$ depends on whether $t \in E_I$, $t \in E_{III}$, or neither, so we have to consider three cases. Let $t \in E$. Choose by Urysohn's lemma a real-valued non-negative continuous function $n_t \in S_{C(K)}$ with $n_t(t) = 1$ such that $n_t(s) = 0$ off $\mathcal{V}_t$. Define $w(t) = y(t) - x(t)$ for all $t \in K$.

Case I: Assume $t \in E_I$. Then by definition of E_I there exists $1 \le j_0 \le k$ with $|z_{j_0}(t)| < 1$. Now, for $s \in \mathcal{V}_t$ let

$$y_{j_0}(s) = n_t(s)[z_{j_0}(s) + \lambda_{j_0}^{-1} w(s)] + [1 - n_t(s)] y(s)$$

and for $j \neq j_0$ we let

$$y_j(s) = n_t(s) z_j(s) + [1 - n_t(s)] y(s).$$

It is straightforward to see that $\sum_{j=1}^k \lambda_j y_j(s) = y(s)$, and that by the choice of δ

$$\begin{aligned}|z_{j_0}(s) + \lambda_{j_0}^{-1} w(s)| &\le |z_{j_0}(t)| + |z_{j_0}(s) - z_{j_0}(t)| + L|y(s) - x(s)| \\ &\le |z_{j_0}(t)| + \delta + 3L\delta < 1\end{aligned}$$

for all $s \in \mathcal{V}_t$. Thus we have $|y_j(s)| \le 1$ for every $1 \le j \le k$.

We will need that $|y_{j_0}(t) - z_{j_0}(t)| \le \lambda_{j_0}^{-1}|y(t) - x(t)| < L\delta < \eta$ and $|y_j(t) - z_j(t)| = 0$ for $j \ne j_0$.

Case II: If for all $1 \le j, m \le k$ we have $z_j(t) = z_m(t)$ with $|z_j(t)| = 1$, then $x(t) = z_j(t)$ and we can just let $y_j(s) = y(s)$ for all $1 \le j \le k$ and $s \in \mathcal{V}_t$.

We will need that $|y_j(t) - z_j(t)| = |y(t) - x(t)| < \delta < \eta$.

Case III: The remaining case is that $t \in E_{III}$, that is, $|z_j(t)| = 1$ for all $1 \le j \le k$, but not all $z_j(t)$ are equal. Order the set $\{\arg z_j(t) : 1 \le j \le k\}$ as an increasing sequence $\{\theta_1 < \theta_2 < \cdots < \theta_q\}$ and define $\theta_0 = \theta_q$. We put $A_p = \{j : \arg z_j(t) = \theta_p\}$ and $\Lambda_p = \sum_{j \in A_p} \lambda_j$.

With ρ as above we define for $1 \le p \le q$

$$c_p = \rho(e^{i\theta_{p-1}} - e^{i\theta_p}).$$

Let $s \in \mathcal{V}_t$ and define (for $j \in A_p$)

$$y_j(s) = n_t(s)\left[z_j(s) + \frac{c_p}{\Lambda_p} + \frac{w(s)}{q\Lambda_p}\right] + (1 - n_t(s))y(s).$$

We have

$$\begin{aligned}\sum_{j=1}^k \lambda_j y_j(s) &= \sum_{p=1}^q \sum_{j \in A_p} \lambda_j y_j(s) \\ &= \sum_{p=1}^q n_t(s) \sum_{j \in A_p} \lambda_j z_j(s) + \sum_{p=1}^q n_t(s) c_p + \sum_{p=1}^q n_t(s) \frac{w(s)}{q} + (1 - n_t(s))y(s) \\ &= n_t(s) \sum_{j=1}^k \lambda_j z_j(s) + n_t(s)0 + n_t(s)w(s) + (1 - n_t(s))y(s) \\ &= n_t(s)x(s) + n_t(s)(y(s) - x(s)) + y(s) - n_t(s)y(s) = y(s).\end{aligned}$$

With $\mu = \rho/\Lambda_p$

$$z_j(t) + \frac{c_p}{\Lambda_p} = e^{i\theta_p} + \mu(e^{i\theta_{p-1}} - e^{i\theta_p}) = \mu e^{i\theta_{p-1}} + (1 - \mu)e^{i\theta_p}.$$

So, by Lemma 2.2 and (1)

$$|z_j(t) + \frac{c_p}{\Lambda_p}| \le 1 - \frac{|e^{i\theta_{p-1}} - e^{i\theta_p}|^2 \rho}{4\Lambda_p} \le 1 - \frac{D\rho}{4\Lambda_p} < 1 - \frac{D\rho}{4} < 1 - (1 + 3L)\delta.$$

Hence

$$\begin{aligned}\left|z_j(s) + \frac{c_p}{\Lambda_p} + \frac{w(s)}{q\Lambda_p}\right| &\le |z_j(t) + \frac{c_p}{\Lambda_p}| + |z_j(s) - z_j(t)| + \left|\frac{w(s)}{q\Lambda_p}\right| \\ &< 1 - (1 + 3L)\delta + \delta + 3L\delta = 1.\end{aligned}$$

Thus we have $|y_j(s)| \le 1$. We will also need that

$$|y_j(t) - z_j(t)| = \left|\frac{c_p}{\Lambda_p} + \frac{w(t)}{q\Lambda_p}\right| \le \rho|e^{i\theta_{p-1}} - e^{i\theta_p}|L + 3\delta L \le 2L\rho + 3L\delta \le \eta.$$

Conclusion.

So far we have defined $y_j \in B_{C(K)}$ and shown that $y = \sum_{j=1}^k \lambda_j y_j$. Note that for each $1 \leq j \leq k$ the function y_j is continuous on K since y_j is a combination of the continuous functions z_j, y, x and n_t. Also n_t is zero off $\mathcal{V}_t$ hence $y_j = y$ on $K \setminus \cup_{t\in E}\mathcal{V}_t$.

It only remains to show that $y_j \in S(f_j, \varepsilon_j)$. We have

$$\sum_{t\notin E} |f_j(t)(y_j(t) - z_j(t))| < \eta\|y_j - z_j\| \leq 2\eta,$$

and

$$\sum_{t\in E} |f_j(t)(y_j(t) - z_j(t))| < \|f_j\|\eta < \eta.$$

Hence $|f_j(y_j - z_j)| < 3\eta$ so that

$$\operatorname{Re} f_j(y_j) \geq \operatorname{Re} f_j(z_j) - 3\eta > \operatorname{Re} f_j(z_j) - d > 1 - \varepsilon_j,$$

and we are done. □

The above theorem applies to $C[0, \alpha]$ for any infinite ordinal α, and in particular to $c = C[0, \omega]$. It should be clear that the proof also works for real scalars and that it proves the following result.

Theorem 2.4. *Every finite convex combination of slices of the unit ball of c_0 is relatively weakly open.*

3. Questions and remarks

We will end with some questions and remarks.

(i) Which Banach spaces satisfy that finite convex combinations of slices of the unit ball are relatively weakly open?
(ii) Which Banach spaces satisfy that finite convex combinations of slices of the unit ball contain a non-empty relatively weakly open neighborhood of some point in the combination?
(iii) Which Banach spaces satisfy that finite convex combinations of slices of the unit ball always have non-empty intersection with the sphere?
(iv) If finite convex combinations of slices of both B_X and B_Y are relatively weakly open, is the same true for the unit ball of $X \oplus_\infty Y$ and/or $X \oplus_1 Y$?

It is not clear that there is a connection between having relatively weakly open convex combinations of slices and the diameter two properties. But we have the following observation.

Remark 3.1. Let X be a (infinite dimensional) Banach space such that there exists a slice $S_1 = S(x^*, \varepsilon)$ of B_X with $\operatorname{diam} S_1 < 1$. Then with $S_2 = S(-x^*, \varepsilon)$ and $C = \frac{1}{2}S_1 + \frac{1}{2}S_2$ it is easy to see that $C \cap S_X = \emptyset$ hence C is a convex combination of slices which is not relatively weakly open.

Regarding Question (iii) we have the following examples of spaces where finite convex combinations of slices intersect the sphere.

Example 3.2. Finite convex combinations of slices of the unit ball of $L_1[0, 1]$ always intersect the sphere. Here slices are given by functions $g_j \in S_{L_\infty[0,1]}$. We may assume that the g_j's are simple functions and find sets $B_j \subset [0, 1]$

with $B_j \cap B_k = \emptyset$ for $j \neq k$ and $\|\chi_{B_j} g_j\|_\infty$ almost one. The functions $f_j = m(B_j)^{-1}\chi_{B_j}$ does the job (m is Lebesgue measure).

Example 3.3. Let X be a Banach space such that whenever $S_j = S(x_j^*, \varepsilon_j)$ with $x_j^* \in S_{X^*}$ and $\varepsilon_j > 0$ for $1 \leq j \leq k$, are slices of B_X, then there exists $x_j \in S_j \cap S_X$ and $y \in S_X$ such that $\|x_j \pm y\| = 1$ and $x_j + y \in S_j$.

Spaces that satisfy this condition include $\ell_\infty^c(\Gamma)$ for Γ uncountable since this space is almost square with $\varepsilon = 0$ [2, Remark 2.11]. It also includes ℓ_∞ and $C[0,1]$ since the slices there are defined by measures of bounded variation.

If X is a space with this property, then finite convex combinations of slices of B_X always intersect the sphere. Indeed, let $\lambda_j > 0$ with $\sum_{j=1}^k \lambda_j = 1$ and let $S_j = S(x_j^*, \varepsilon_j)$ be slices of B_X with $x_j^* \in S_{X^*}$ and $\varepsilon_j > 0$ for $1 \leq j \leq k$.

By assumption, there exists $x_j \in S_j \cap S_X$ and $y \in S_X$ such that $\|x_j \pm y\| = 1$ and $x_j + y \in S_j$.

Choose $y^* \in S_{X^*}$ such that $y^*(y) = 1$. Then

$$1 = \|x_j \pm y\| \geq y^*(y) \pm y^*(x_j) = 1 \pm y^*(x_j)$$

hence $y^*(x_j) = 0$. Now $\sum_{j=1}^k \lambda_j(x_j + y) \in \sum_{j=1}^k \lambda_j S_j$ and

$$\| \sum_{j=1}^k \lambda_j (x_j + y)\| \geq \sum_{j=1}^k \lambda_j y^*(y) = 1.$$

Example 3.4. If X has the Daugavet property, then finite convex combinations of weak*-slices of B_{X^*} intersect the sphere S_{X^*}. To see this let $x_j \in S_X$, $\varepsilon_j > 0$, and let $S(x_j, \varepsilon_j)$ be slices of B_{X^*} for $1 \leq j \leq k$. Consider $\sum_{j=1}^k \lambda_j S(x_j, \varepsilon_j)$ where $\lambda_j > 0$ and $\sum_{j=1}^k \lambda_j = 1$.

By using [9, Lemma 2.12] and an induction argument we can, for $1 \leq j \leq k$, find $x_j^* \in S(x_j, \varepsilon_j) \cap S_{X^*}$ such that $\| \sum_{j=1}^k \lambda_j x_j^* \| = \sum_{j=1}^k \lambda_j = 1$.

Acknowledgments. We thank Stanimir Troyanski and Olav Nygaard for fruitful conversations.

References

[1] T. A. Abrahamsen, P. Hájek, O. Nygaard, J. Talponen, and S. Troyanski, *Diameter 2 properties and convexity*, Studia Math. **232** (2016), no. 3, 227–242.

[2] T. A. Abrahamsen, J. Langemets, and V. Lima, *Almost square banach spaces*, Journal of Mathematical Analysis and Applications **434** (2016), no. 2, 1549 – 1565.

[3] T. A. Abrahamsen, V. Lima, and O. Nygaard, *Remarks on diameter* 2 *properties*, J. Convex Anal. **20** (2013), no. 1, 439–452.

[4] M. D. Acosta, J. Becerra-Guerrero, and G. López-Pérez, *Stability results of diameter two properties*, J. Convex Anal. **22** (2015), no. 1, 1–17.

[5] J. Becerra Guerrero, G. López-Pérez, and A. Rueda Zoca, *Big slices versus big relatively weakly open subsets in Banach spaces*, J. Math. Anal. Appl. **428** (2015), no. 2, 855–865. MR 3334951

[6] M. Fabian, P. Habala, P. Hájek, V. Montesinos, and V. Zizler, *Banach space theory*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, New York, 2011, The basis for linear and nonlinear analysis. MR 2766381 (2012h:46001)

[7] N. Ghoussoub, G. Godefroy, B. Maurey, and W. Schachermayer, *Some topological and geometrical structures in Banach spaces*, Mem. Amer. Math. Soc. **70** (1987), no. 378, iv+116. MR 912637

[8] R. Haller and J. Langemets, *Two remarks on diameter 2 properties*, Proc. Est. Acad. Sci. **63** (2014), no. 1, 2–7.

[9] V. M. Kadets, R. V. Shvidkoy, G. G. Sirotkin, and D. Werner, *Banach spaces with the Daugavet property*, Trans. Amer. Math. Soc. **352** (2000), no. 2, 855–873. MR 1621757 (2000c:46023)

[10] A. Pełczyński and Z. Semadeni, *Spaces of continuous functions. III. Spaces $C(\Omega)$ for Ω without perfect subsets*, Studia Math. **18** (1959), 211–222. MR 0107806

[11] W. Rudin, *Continuous functions on compact spaces without perfect subsets*, Proc. Amer. Math. Soc. **8** (1957), 39–42. MR 0085475

(T. A. Abrahamsen) Department of Mathematics, University of Agder, Postboks 422, 4604 Kristiansand, Norway.

Email address: trond.a.abrahamsen@uia.no

URL: http://home.uia.no/trondaa/index.php3

(V. Lima) NTNU, Norwegian University of Science and Technology, Aalesund, Postboks 1517, N-6025 Ålesund Norway.

Current address: Department of Engineering Sciences, University of Agder, Postboks 422, 4604 Kristiansand, Norway.

Email address: Vegard.Lima@uia.no