Stochastic Graph Filtering under Asymmetric Links in Wireless Sensor Networks

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***Abstract*—Wireless sensor networks (WSNs) are often characterized by random and asymmetric packet losses due to the wireless medium, leading to network topologies that can be modeled as random, time-varying and directed graphs. Most of existing works related to graph filtering in the context of WSNs assume that the probability of delivering an information from one node to a neighbor node is the same as in the reverse direction. This assumption is not realistic due to the typical link asymmetry in WSNs caused by interferences and background noise. In this work, we analyze the problem of applying stochastic graph filtering over random time-varying asymmetric network topologies. We show that it is possible to perform stochastic graph filtering under asymmetric links with node-variant graph filters, while optimizing a trade-off between the expected error (bias) and the variance of the error, with respect to performing graph filtering over a fixed static topology given by a certain connectivity radius of the nodes.**

***Index Terms*—Wireless sensor networks; asymmetric links; Graph signal processing; Graph filters.**

I. Introduction

Over the past few years, significant efforts have been performed to extend classical signal processing concepts to the graph setting, allowing the emergence of Graph Signal Processing field [1, 2]. In this area, of special interest is the analysis of graph signals in the graph frequency domain. In this field, graph filters (GFs) have been considered as the building tools for processing the content of graph signals. These tools have also been shown useful to analyze network data, learn data dependencies, process several tasks and solve a wide range of problems [3–7], such as distributed estimation, denoising, smoothing, classification and regression.

The design of graph filters and its distributed implementation [5, 8–11] have recently enabled their widespread use in many applications in the context of Wireless Sensor Networks (WSNs). It is important to notice that such networks are often characterized by random and asymmetric packet losses leading to model the network topologies as random, time-varying and directed graphs [12]. Very few works have focused on the problem of randomness and time-variability of the graph when applying graph filtering [9, 13]. However, none of these works consider the problem of link asymmetry when applying graph filtering since they typically assume that the probability of delivering a packet from node i to its neighboring node j is the same as in the reverse direction, from j to i. In addition to that, this probability is typically assumed to be the same for all the nodes in the network. Both assumptions are not realistic and can not be implemented under real conditions in a WSN due to interferences and background noise [12, 14].

In this work, we first analyze stochastic graph filtering with node-invariant GFs in WSNs and show that the requirement of having equal probabilities for all the links is needed in order to have an unbiased filtering, which as mentioned before, is not practical. Then, we show that it is possible to perform stochastic graph filtering under asymmetric links in WSNs with node-variant GFs while optimizing a trade-off between the expected error (bias) and the variance of the error, showing that it is possible to obtain an accurate filtering over time-varying graphs. To the best of our knowledge, this is the first work that proposes a solution to perform stochastic graph filtering in the context of WSNs under asymmetric links, while ensuring control over the resulting accuracy of filtering, in terms of bias and variance of the error.

The remainder of this paper is organized as follows. Section 2 presents the main background on graph theory and graph filters. Section 3 analyzes the problem of stochastic graph filtering when allowing asymmetric links in WSNs and section 4 proposes a solution to cope with this problem. Section 5 presents the numerical results. Section 6 concludes the paper.

Notation: Vectors (respectively matrices) are denoted by bold lowercase (uppercase) letters. The entries of a matrix $\mathbf{B}$ are denoted by b_{ij}. We denote by $\|\mathbf{v}\|_2$ the 2-norm of a vector $\mathbf{v}$. Similarly, $\|\mathbf{B}\|_2$ and $\|\mathbf{B}\|_F$ denote respectively the spectral norm and the Frobenius norm of a matrix $\mathbf{B}$. We indicate by tr(.) and diag(.), respectively, the trace operator and the diagonal matrix. The Hadamard product is indicated by $\circ$.

II. Background

Let $\mathcal{G}(\mathcal{V}, \mathcal{E})$ denote a directed graph where $\mathcal{V}$ is a set of N vertices and $\mathcal{E}$ is a set of links or edges such that if there is a link from node i to node j, then $(i, j) \in \mathcal{E}$. For any given graph $\mathcal{G}$, we define the $N \times N$ adjacency matrix $\mathbf{A}$, where $a_{ij} = 1$ if and only if $(i, j) \in \mathcal{E}$. The set of neighbors of node i is defined by $\Omega_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. The degree of node i is $d_i = \sum_{j \in \Omega_i} a_{ij}$ and $\mathbf{D}$ is the diagonal degree matrix.

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A. Graph signal, graph shift operator and graph filters

A graph signal, defined on the set of nodes of the graph, is a mapping $\mathbf{x}:\mathcal{V}\rightarrow\mathbb{R}$, and can be represented as a vector $\mathbf{x}=[x_1,...,x_N]^\top\in\mathbb{R}^N$. The i-th component x_i represents the signal value at the i-th vertex in $\mathcal{V}$. Any graph $\mathcal{G}$ can be endowed with a graph-shift operator $\mathbf{S}$, which can be represented as a matrix $\mathbf{S}\in\mathbb{R}^{N\times N}$ satisfying $s_{ij}\neq 0$ if $(i,j)\in\mathcal{E}$. There are several possible choices for the shift $\mathbf{S}$, such as the adjacency matrix $\mathbf{A}$, Laplacian matrix $\mathbf{L}=\mathbf{D}-\mathbf{A}$ and other generalizations defined on $\mathbf{L}$ [15].

A graph filter (GF) is a system $\mathbf{H}$ that takes a graph signal $\mathbf{x}$ as an input and produces another graph signal $\mathbf{y}$ as an output. A graph filter $\mathbf{H}:\mathbb{R}^N\rightarrow\mathbb{R}^N$ can be represented by an $N\times N$ matrix. GFs can be classified into two types [2, 8, 16]: *Infinite Impulse Reponse (IIR)* GFs and *Finite Impulse Response* (*FIR*) GFs. Contrary to IIR GFs, FIR GFs are designed such that their impulse responses are finite in the vertex domain. In this paper, we focus on the signal denoising application using FIR GFs, more specifically, we focus on Tikhonov denoising, by exploiting an equivalence with the ARMA filtering [9]. However, our work can be easily extended to other types of filters implemented through FIR GFs over time-varying graphs. Next, we revise the concept of FIR GFs and connect it to ARMA graph filtering for Tikhonov denoising.

1) FIR graph filters: FIR GFs can be classified as being either node-invariant or node-variant [8]:

Node-invariant FIR GF: It is a polynomial in $\mathbf{S}$ of degree L, with coefficients $\mathbf{h}=[h_0,...,h_L]^\top$. The graph signal output $\mathbf{y}$ that is generated when the node-invariant graph filter $\mathbf{H}_{inv}$ is applied, is given by:

$$\mathbf{y}=\mathbf{H}_{inv}\mathbf{x}=\sum_{l=0}^{L}h_l\,\mathbf{S}^l\mathbf{x}=\sum_{l=0}^{L}h_l\,\mathbf{x}^{(l)} \quad (1)$$

where $\mathbf{x}^{(l)}=\mathbf{S}^l\,\mathbf{x}=\mathbf{S}\,\mathbf{x}^{(l-1)}$.

Node-variant FIR GF: In this case, each node applies different weights, collected in $N\times 1$ vector $\mathbf{h}^{(l)}=[h_1^{(l)},...,h_N^{(l)}]^\top$, to the shifted signals $\mathbf{S}^l\,\mathbf{x}$. In general, node-variant GFs offer a larger number of degrees of freedom to choose the coefficients. Thus, they can be viewed as a generalization of node-invariant GFs. The graph signal output $\mathbf{y}$ that is generated when the node-variant graph filter $\mathbf{H}_{nv}$ is applied, is given by [8]:

$$\mathbf{y}=\mathbf{H}_{nv}\mathbf{x}=\sum_{l=0}^{L}\text{diag}(\mathbf{h}^{(l)})\,\mathbf{S}^l\,\mathbf{x} \quad (2)$$

2) ARMA$_1$ graph filters: The ARMA$_1$ [9] is an IIR GF with filter coefficients φ and ψ, and computed as:

$$\mathbf{y}_t=\psi\mathbf{S}\,y_{t-1}\;+\varphi\,\mathbf{x}=(\psi\mathbf{S})^t\mathbf{y}_0+\;\varphi\sum_{\tau=0}^{t-1}(\psi\mathbf{S})^\tau\,\mathbf{x} \quad (3)$$

If $\mathbf{y}_0=\mathbf{x}$, ARMA$_1$ is equivalent to the node-invariant GF of order $t=T$ by using the coefficients $[\varphi,\varphi\psi,..,\varphi\psi^{T-1},\psi^T]^\top$. It has been shown that ARMA$_1$ can solve the Tikhonov denoising problem [3], which can be formulated as [9]:

$$\mathbf{v}^*=\text{argmin}_{\mathbf{v}\in\mathbb{R}^N}\|\mathbf{x}-\mathbf{v}\|_2^2+w\,\mathbf{v}^\top\mathbf{S}\mathbf{v} \quad (4)$$

where $\mathbf{x}=\mathbf{v}+\mathbf{n}$ is a noisy graph signal, $\mathbf{v}$ the true signal assumed to be smooth with respect to the underlying graph, $\mathbf{n}$ is the noise and w the weighting factor trading smoothness and noise removal. Let $\{\lambda_n\}_{n=1}^N$ and $\{\boldsymbol{\vartheta}_n\}_{n=1}^N$ be respectively the eigenvalues and eigenvectors of $\mathbf{S}$, and ρ is the upper bound of the spectral norm of $\mathbf{S}$ *i.e.*, $\|\mathbf{S}\|_2\leq\rho$ (needed to ensure the filtering stability of ARMA$_1$ [10]). The frequency response of ARMA$_1$ is $h(\lambda_n)=\frac{\varphi}{1-\psi\lambda_n}$ subject to $|\psi|<\rho$. For $\psi{=}-w$ and $\varphi{=}1$, filtering $\mathbf{x}$ with ARMA$_1$ allows to obtain the optimal solution of the Tikhonov denoising, which is given by [9]:

$$\mathbf{v}^*=\sum_{n=1}^{N}\left(\frac{1}{1+w\lambda_n}\boldsymbol{\vartheta}_n^\top\mathbf{x}\right)\boldsymbol{\vartheta}_n \quad (5)$$

Due to the filter equivalence between FIR GFs and ARMA$_1$, FIR GFs can be used to solve the Tikhonov denoising.

III. Analysis of stochastic graph filtering with node-invariant GF

We assume that each of the N vertices included in a random graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ models a sensor node with an omnidirectional antenna, which is deployed uniformly at random over a certain area of interest. Each sensor i can communicate with its neighbors within its maximum transmission range R. We denote each graph realization as $\mathcal{G}_t(\mathcal{V},\mathcal{E}_t)$, which corresponds to different possible links connections with certain probabilities for each connection. We denote by $\mathcal{G}_0(\mathcal{V},\mathcal{E}_0)$ the particular graph realization where all the possible links within the transmission range R are active. We consider the graph filtering performed on the time-varying graphs $\mathcal{G}_t(\mathcal{V},\mathcal{E}_t)$, which are random realizations at time t of the graph $\mathcal{G}$, where the probability of establishing a link (i,j) from node i to node j at time t is p_{ij} $(0<p_{ij}\leq 1)$. Then, we assume that, for each graph realization $\mathcal{G}_t(\mathcal{V},\mathcal{E}_t)$, the set of links $\mathcal{E}_t\subseteq\mathcal{E}_0$ are activated independently across the time and generated via an i.i.d. Bernoulli process with the associated probabilities p_{ij}. We allow the links to be asymmetric in order to reflect a more realistic environment in the context of WSNs. Let $\mathbf{P}\in\mathbb{R}^{N\times N}$ denotes the connection probability matrix with entries p_{ij}. Let $\mathbf{S}$, $\mathbf{S}_t$ and $\bar{\mathbf{S}}$ denote the shift operator corresponding, respectively, to the graph $\mathcal{G}_0$, the graph $\mathcal{G}_t$ at time t and the expected graph $\bar{\mathcal{G}}$. We also assume that the spectral norm of the shift used is upper bounded, *i.e.,* $\|\mathbf{S}_t\|_2{\leq}\|\mathbf{S}\|_2{\leq}\rho$ for all t [17].

The output of a node-invariant FIR GF with coefficients ϕ_l, performed on the stochastic time-varying graph $\mathcal{G}_t$ with links established based on $\mathbf{P}$, is given by [9]:

$$\mathbf{y}_t=\sum_{l=0}^{L}\phi_l\,\boldsymbol{\Theta}(t,t-l+1)\,\mathbf{x} \quad (6)$$

where L is the order of the filter and:

$$\boldsymbol{\Theta}(t',t)=\begin{cases}\prod_{\tau=t}^{t'}\mathbf{S}_\tau & \text{if } t'\geq t\\ \mathbf{I} & \text{if } t'<t\end{cases} \quad (7)$$

The expected output of the node-invariant FIR GF performed on the expected graph $\bar{\mathcal{G}}$ for $t\geq L$ is given by [9]:

$$\bar{\mathbf{y}}_t=\mathbb{E}[\mathbf{y}_t]=\mathbb{E}[\sum_{l=0}^{L}\phi_l\left(\prod_{\tau=t}^{t-l+1}\mathbf{S}_\tau\right)\mathbf{x}]=\sum_{l=0}^{L}\phi_l\,\bar{\mathbf{S}}^l\,\mathbf{x} \quad (8)$$

Notice that the coefficients ϕ_l are intentionally used instead of h_l in order to highlight our interest in finding the coefficients ϕ_l that will provide on average the same filter output as we apply a filter with coefficients h_l over the graph $\mathcal{G}_0$.

$$\text{If} \quad \mathbf{S} = \mathbf{A} \;\; \text{then} \;\; \bar{\mathbf{S}} = \mathbb{E}[\mathbf{A}_t] = \mathbf{P} \circ \mathbf{A} = \begin{bmatrix} p_{11}a_{11} & p_{12}a_{12} & .. & p_{1N}a_{1N} \\ p_{21}a_{21} & p_{22}a_{22} & .. & .. \\ .. & .. & .. & .. \\ p_{N1}a_{N1} & p_{N2}a_{N2} & .. & p_{NN}a_{NN} \end{bmatrix}$$

The expected error (bias) is expressed as:

$$\bar{\mathbf{e}} = \mathbb{E}\big[\mathbf{y}_t - \mathbf{y}\big] = \bar{\mathbf{y}}_t - \mathbf{y} \tag{9}$$

To have $\bar{\mathbf{e}} = \mathbf{0}$ (unbiased filtering), we should have:

$$\sum_{l=0}^{L} \phi_l \, \bar{\mathbf{S}}^l \, \mathbf{x} = \sum_{l=0}^{L} h_l \, \mathbf{S}^l \, \mathbf{x} \tag{10}$$

The previous equality is also equivalent to enforcing:

$$\phi_0 = h_0 \quad \text{and} \quad \sum_{l=1}^{L} \left(\phi_l{}^{\frac{1}{l}} \, \bar{\mathbf{S}}\right)^l \mathbf{x} = \sum_{l=1}^{L} \left(h_l^{\frac{1}{l}} \, \mathbf{S}\right)^l \mathbf{x} \tag{11}$$

It can be easily seen that if $\phi_0 = h_0$ and $\phi_l{}^{\frac{1}{l}} \, \bar{\mathbf{S}} = h_l^{\frac{1}{l}} \, \mathbf{S}$ for $l \geq 1$, then the expected error $\bar{\mathbf{e}} = \mathbf{0}$. These conditions can be written as follows:

$$\phi_l{}^{\frac{1}{l}} \bar{\mathbf{S}} = \begin{bmatrix} \phi_l{}^{\frac{1}{l}} p_{11}a_{11} & \phi_l{}^{\frac{1}{l}} p_{12}a_{12} & .. & \phi_l{}^{\frac{1}{l}} p_{1N}a_{1N} \\ \phi_l{}^{\frac{1}{l}} p_{21}a_{21} & \phi_l{}^{\frac{1}{l}} p_{22}a_{22} & .. & \phi_l{}^{\frac{1}{l}} p_{2N}a_{2N} \\ .. & .. & .. & .. \\ \phi_l{}^{\frac{1}{l}} p_{N1}a_{N1} & \phi_l{}^{\frac{1}{l}} p_{N2}a_{N2} & .. & \phi_l{}^{\frac{1}{l}} p_{NN}a_{NN} \end{bmatrix} = h_l^{\frac{1}{l}} \mathbf{S} = \begin{bmatrix} h_l{}^{\frac{1}{l}} a_{11} & h_l{}^{\frac{1}{l}} a_{12} & ... & h_l{}^{\frac{1}{l}} a_{1N} \\ h_l{}^{\frac{1}{l}} a_{21} & h_l{}^{\frac{1}{l}} a_{22} & ... & h_l{}^{\frac{1}{l}} a_{2N} \\ .. & .. & .. & .. \\ h_l{}^{\frac{1}{l}} a_{N1} & h_l{}^{\frac{1}{l}} a_{N2} & ... & h_l{}^{\frac{1}{l}} a_{NN} \end{bmatrix} \tag{12}$$

This means also that for $l \geq 1$ and $\forall i, j$, we must have:

$$\phi_l{}^{\frac{1}{l}} p_{ii} a_{ii} = h_l{}^{\frac{1}{l}} a_{ii}$$

$$\phi_l{}^{\frac{1}{l}} p_{ij} a_{ij} = h_l{}^{\frac{1}{l}} a_{ij}$$

$$\phi_l{}^{\frac{1}{l}} p_{ji} a_{ji} = h_l{}^{\frac{1}{l}} a_{ji}$$

This means that in order to have $\bar{\mathbf{e}} = \mathbf{0}$, it requires to have all the links with the same probability p and the coefficients must satisfy:

$$\phi_l = p_{ij}^{-l} \;\; h_l = p_{ji}^{-l} \;\; h_l = p_{ii}^{-l} \;\; h_l = p^{-l} \, h_l \;\; \forall i, j, l \tag{13}$$

Similarly with node-variant GFs, equal probabilities enable unbiased filtering. However, it is not possible to enforce equal (or even similar) probabilities for all the links in WSNs, because interferences and background noise generate always link asymmetry. Therefore, it is not possible in practice to achieve unbiased stochastic graph filtering. Moreover, as our results show in Section V, in general, it is not possible to ensure small bias and variance with node-invariant GF. In the next section, we focus on how to implement stochastic graph filtering in WSNs under asymmetric links using node-variant GF, while controlling the bias and the variance of the error.

IV. Stochastic graph filtering in WSN under asymmetric links

In this section, we show how to perform stochastic graph filtering under asymmetric links in WSNs by using node-variant GFs, so that we can minimize a trade-off between the bias and the variance. In this paper, we assume, that each node is using broadcast communications to send data to its neighbors. Thus, every node i uses a probability q_i to establish a link towards its neighbors. The value q_i reflects in WSNs the Packet Delivery Ratio (PDR) of a given node i. The PDR can be estimated from the network environment and enforced by an appropriate cross-layer MAC protocol. The design of this protocol is out of the scope of this paper. The connection probability matrix, when nodes use broadcasting, is given by $\mathbf{Q}$ as follows:

$$\mathbf{Q} = \begin{bmatrix} q_1 & q_1 & \cdots & q_1 \\ q_2 & q_2 & \cdots & q_2 \\ \cdots & \cdots & \cdots & \cdots \\ q_N & q_N & \cdots & q_N \end{bmatrix}$$

The output of the node-variant FIR GF perfomed over the stochastic time-varying graph $\mathcal{G}_t$ with links established based on $\mathbf{Q}$ and using the $N \times 1$ vector $\boldsymbol{\phi}^{(l)} = [\phi_1^{(l)}, ..., \phi_N^{(l)}]^\top$ as filter coefficients is given by:

$$\mathbf{y}_t = \sum_{l=0}^{L} \text{diag}(\boldsymbol{\phi}^{(l)}) \; \boldsymbol{\Theta}(t, t-l+1) \; \mathbf{x} \tag{14}$$

The expected output of the FIR GF, performed over the average graph $\bar{\mathcal{G}}$ for $t \geq L$, is given by:

$$\begin{aligned} \bar{\mathbf{y}}_t = \mathbb{E}\big[\mathbf{y}_t\big] &= \mathbb{E}\Bigg[\sum_{l=0}^{L} \text{diag}(\boldsymbol{\phi}^{(l)}) \left(\prod_{\tau=t}^{t-l+1} \mathbf{S}_\tau\right) \mathbf{x}\Bigg] \\ &= \sum_{l=0}^{L} \text{diag}(\boldsymbol{\phi}^{(l)}) \; \bar{\mathbf{S}}^l \; \mathbf{x} \end{aligned} \tag{15}$$

The shift operator $\bar{\mathbf{S}}$ corresponding to the expected graph $\bar{\mathcal{G}}$ is selected such as it satisfies $\bar{\mathbf{S}}{=}\mathbb{E}[\mathbf{S}_t]{=}\mathbf{Q} \circ \mathbf{S}$ as follows:

$$\text{If } \mathbf{S} = \mathbf{A} \text{ then } \bar{\mathbf{S}} = \mathbb{E}[\mathbf{A}_t] = \mathbf{Q} \circ \mathbf{A}$$

$$\begin{aligned} \text{If } \;\mathbf{S} = \mathbf{L} \text{ then } \bar{\mathbf{S}} &{=} \mathbb{E}[\mathbf{L}_t] = \mathbb{E}[\mathbf{D}_t - \mathbf{A}_t] {=} \mathbb{E}[\mathbf{D}_t] - \mathbb{E}[\mathbf{A}_t] \\ &{=} \mathbf{Q} \circ \mathbf{D} - \mathbf{Q} \circ \mathbf{A} {=} \mathbf{Q} \circ (\mathbf{D} - \mathbf{A}) {=} \mathbf{Q} \circ \mathbf{L} \end{aligned}$$

$$\text{If } \mathbf{S}_t {=} \frac{1}{\lambda_{max}} \mathbf{L}_t - 0.5 \; (\mathbf{Q} \circ \mathbf{I}) \text{ and } \mathbf{S} {=} \frac{1}{\lambda_{max}} \; \mathbf{L} - 0.5 \; \mathbf{I} \text{ then}$$

$$\bar{\mathbf{S}} {=} \mathbb{E}[\mathbf{S}_t] {=} \mathbb{E}[\frac{1}{\lambda_{max}} \mathbf{L}_t - 0.5 \; (\mathbf{Q} \circ \mathbf{I})] {=} \frac{1}{\lambda_{max}} \mathbb{E}[\mathbf{L}_t] {-} 0.5(\mathbf{Q} \circ \mathbf{I})$$

$$= \frac{1}{\lambda_{max}} (\mathbf{Q} \circ \mathbf{L}) {-} 0.5(\mathbf{Q} \circ \mathbf{I}) {=} \mathbf{Q} \circ \left(\frac{1}{\lambda_{max}} \mathbf{L} {-} 0.5 \; \mathbf{I}\right) {=} \mathbf{Q} \circ \mathbf{S}$$

In order to have expected error $\bar{\mathbf{e}} \approx \mathbf{0}$, when applying stochastic graph filtering with node-variant GF under asymmetric links established based on $\mathbf{Q}$, we should have:

$$\bar{\mathbf{y}}_t = \sum_{l=0}^{L} \text{diag}(\boldsymbol{\phi}^{(l)}) \; (\mathbf{Q} \circ \mathbf{S})^l \; \mathbf{x} \approx \sum_{l=0}^{L} \text{diag}(\boldsymbol{h}^{(l)}) \; \mathbf{S}^l \; \mathbf{x} = \mathbf{y}$$

This means that in order to reduce the expected error and have the expected output of the stochastic time-varying filter close to the original one, one can minimize a Frobenius norm of the difference:

$$\|\mathbf{B}\|_F = \Big\| \sum_{l=0}^{L} (\text{diag}(\boldsymbol{\phi}^{(l)})\ (\mathbf{Q} \circ \mathbf{S})^l - \text{diag}(\boldsymbol{h}^{(l)})\ \mathbf{S}^l) \Big\|_F \quad (16)$$

In order to control the total Mean Square Error (MSE) of the filtering, we need to control both the bias and the variance. Next, we analyze the variance of the filtering.

A. Variance

The average of the variance across the nodes is given by:

$$\overline{var}[\mathbf{y}_t] = \text{tr}\big(\mathbb{E}[\mathbf{y}_t {\mathbf{y}_t}^H] - \mathbb{E}[\mathbf{y}_t]\mathbb{E}[\mathbf{y_t}]^H\big)/N \quad (17)$$

Proposition 1: The average variance across the nodes of the node-variant FIR GF performed on time-varying graphs, with links established based on $\mathbf{Q}$, is upper bounded by:

$$\overline{var}[\mathbf{y}_t] \leq \frac{\|\mathbf{x}\|^2}{N}\ \big(\beta_0 + \beta_1 + ... + \beta_L\ \big)^2 \quad (18)$$

where $\beta_l = \rho^l\ \|\text{diag}(\boldsymbol{\phi}^{(l)})\|_2$.
Proof: See Appendix I.

B. Pareto minimization of the expected error and variance

Our goal is to determine the optimal coefficients that minimize a pareto weighted sum of the expected error and the upper bound variance given by (18). This allows to control the overall MSE. Thus, our optimization problem can be formulated as follows:

$$\text{minimize}_{\{\boldsymbol{\phi}^{(0)},..,\boldsymbol{\phi}^{(L)}\}} \bigg(\|\mathbf{B}\|_F\bigg)^2 + \gamma\ \bigg(\sum_{l=0}^{L}\beta_l\bigg)^2 \quad (19)$$

where γ is a weighting factor between the expected error and the variance. This optimization problem can be solved efficiently since it is convex. Notice that the term $\|\mathbf{x}\|^2/N$ has been omitted because it has not an impact on the choice of the coefficients and our goal is to optimize the coefficients without the knowledge of the input graph signal.

V. Numerical results

We evaluate in MATLAB the performance of the proposed solution. A deployment of N=100 sensor nodes, randomly and uniformly distributed over a square area of side 150 m, is considered in our simulations. The transmission range of each node is fixed by default to R=70 m. We consider a Tikhonov denoising application, where the noisy smooth graph signals $\mathbf{x}$=$\mathbf{v}+\mathbf{n}$ are acquired by the WSN, with zero mean Gaussian noise $\mathbf{n}$ and 0.1 standard deviation. The shift operators used are $\mathbf{S}$=$\frac{1}{\lambda_{max}}\ \mathbf{L} - 0.5\ \mathbf{I}$ in order to provide a smaller spectral norm that can further reduce the variance. The coefficients used for filtering over time-varying graphs are obtained by our proposed solution in Section IV-B with $\gamma = 0.05$. The results are averaged over 1000 realizations.

To compare the output of graph filtering over the graph $\mathcal{G}_0$ and the time-varying graph $\mathcal{G}_t$ under asymmetric links, we use the error $\mathbf{e}$=$\mathbf{y}_t - \mathbf{y}$ and the empirical average variance of the error among all nodes and realizations $\overline{\sigma}^2_{\mathbf{e}}$=tr$(\mathbb{E}[\mathbf{e}\mathbf{e}^H])/N$, which approaches the average variance $\overline{var}[\mathbf{y}_t]$ for a sufficiently high number of realizations. In Fig. 1, the expected error $\bar{\mathbf{e}}$ and the empirical average variance $\overline{\sigma}^2_{\mathbf{e}}$ are analyzed with different average probabilities of link connectivities $\overline{q_i}$. Fig.1 (a) and (d) show that the node-variant GF significantly outperforms the node-invariant GF, when for both filters the coefficients are optimized to reduce the expected error and the variance. This can be explained by the fact that the node-variant GF has higher degrees of freedom to select the coefficients compared to the node-invariant GF, where all nodes have to use the same coefficients. Our results show also that by optimizing the coefficients of node-variant GFs, the output of graph filtering over the original graph $\mathcal{G}_0$ is very close to the one obtained for the time-varying graph $\mathcal{G}_t$. Indeed, the expected error is small and it is in the order of 10^{-2} for $\overline{q_i} \geq 0.55$. The proposed solution also provides a low empirical average variance with a value of order 10^{-3} for $\overline{q_i} \geq 0.55$. We can notice also that both the expected error and empirical average variance decrease for a higher value of $\overline{q_i}$ and a lower filter order L. When the average probability of link connectivity is low, the expected error as well as the empirical average variance are slightly affected by the transmission range R of the nodes. However, both have small values with the same previous order if $\overline{q_i} \geq 0.55$. In conclusion, for an average PDR of the nodes higher than 0.55, we observe a very low expected error as well as a very small empirical average variance by using our proposed solution. This average PDR can be considered as a realistic value in WSNs [12].

VI. Conclusion

In this work, we first show that stochastic graph filtering with node-invariant GFs requires equal probabilities for all the links to have an unbiased filtering, which it is not realistic in the context of WSNs. Then, we show how to enable the stochastic graph filtering under asymmetric links in WSNs by using node-variant GFs to provide accuracy that is, making the expected error and variance very small through optimizing the filter coefficients. As demonstrated in the numerical results, a very low expected error as well as a very small empirical average variance are obtained, when using our proposed solution.

VII. Appendix I

Given the definition of $\overline{var}[\mathbf{y}_t]$ in (17) and using the linearity of expectation and trace, we begin by computing the first term on the right side of (17) [13]:

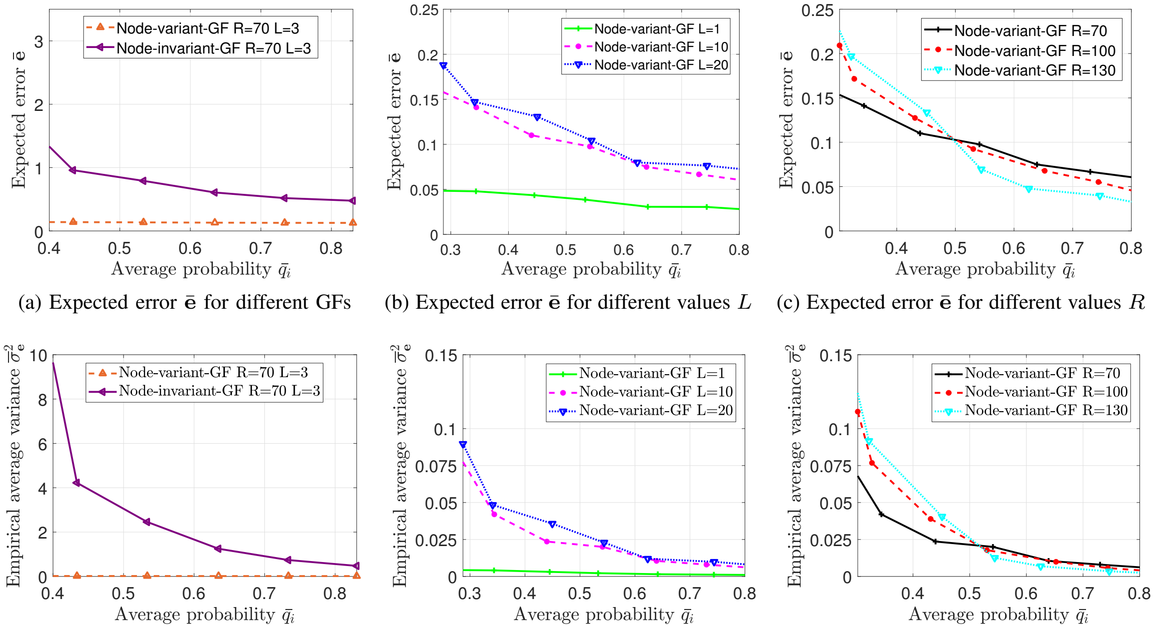
$$\text{tr}(\mathbb{E}\big[\mathbf{y}_t {\mathbf{y}_t}^H\big]) = \sum_{k=0,l=0}^{L} \Upsilon(k,l) \quad (20)$$

where:

$$\Upsilon(k,l) = \text{tr}\Bigg(\mathbb{E}\bigg[\text{diag}(\boldsymbol{\phi}^{(k)})\boldsymbol{\Theta}(t,t-k+1)\ \mathbf{x}\mathbf{x}^H \times \boldsymbol{\Theta}(t,t-l+1)^H \text{diag}(\boldsymbol{\phi}^{(l)})^H\bigg]\Bigg) \quad (21)$$

By using the commutativity property of the trace with respect to the expectation and also the cyclic property of the trace tr$(\mathbf{ABC})$ = tr$(\mathbf{CAB})$ = tr$(\mathbf{BCA})$, we have:

$$\Upsilon(k,l) = \mathbb{E}\bigg[\text{tr}\big(\ \text{diag}(\boldsymbol{\phi}^{(k)})\ \boldsymbol{\Theta}(t,t-k+1)\ \mathbf{x}\mathbf{x}^H\boldsymbol{\Theta}(t,t-l+1)^H \times diag(\boldsymbol{\phi}^{(l)})^H\big)\bigg]$$
$$= \text{tr}\big(\mathbb{E}\big[\boldsymbol{\Theta}(t,t-l+1)^H \text{diag}(\boldsymbol{\phi}^{(l)})^H \text{diag}(\boldsymbol{\phi}^{(k)})\ \boldsymbol{\Theta}(t,t-k+1)\big]\mathbf{x}\mathbf{x}^H\big) \quad (22)$$



(a) Expected error $\bar{\mathbf{e}}$ for different GFs (b) Expected error $\bar{\mathbf{e}}$ for different values L (c) Expected error $\bar{\mathbf{e}}$ for different values R

(d) $\overline{\sigma}^2_{\mathbf{e}}$ for different GFs (e) $\overline{\sigma}^2_{\mathbf{e}}$ for different values L (f) $\overline{\sigma}^2_{\mathbf{e}}$ for different values R

Fig. 1: The expected error $\bar{\mathbf{e}}$ and the empirical average variance $\overline{\sigma}^2_{\mathbf{e}}$ between graph filtering realizations perfomed on the graph $\mathcal{G}_0$ and the time-varying graph $\mathcal{G}_t$ under asymmetric links, for different GFs (node-variant and node-invariant), orders L of the filter and values of radio range R of the sensor nodes. For Figures (b) and (e) $R = 70$ and for (c) and (f) $L = 10$.

For a positive semi-definite matrix $\mathbf{B}$, $\mathbf{B} \succeq 0$ and any square matrix $\mathbf{A}$ with appropriate dimensions [18], we have:

$$\text{tr}(\mathbf{AB}) \leq 0.5\|\mathbf{A} + \mathbf{A}^H\|_2 \,\text{tr}(\mathbf{B}) \leq \|\mathbf{A}\|_2 \,\text{tr}(\mathbf{B}) \tag{23}$$

By applying the previous inequality to (22), we obtain:

$$\begin{aligned}\Upsilon(k,l) \leq & \Big\|\mathbb{E}\big[\mathbf{\Theta}(t,t-l+1)^H \text{diag}(\boldsymbol{\phi}^{(l)})^H \text{diag}(\boldsymbol{\phi}^{(k)}) \\ & \times \mathbf{\Theta}(t,t-k+1)\big]\Big\|_2 \,\text{tr}\big(\mathbf{x}\mathbf{x}^H\big)\end{aligned} \tag{24}$$

Using the Jensen's inequality of the spectral norm $\|\mathbb{E}[\mathbf{A}]\|_2 \leq \mathbb{E}[\|\mathbf{A}\|_2]$ and the sub-multiplicativity property of the spectral norm of a square matrix, $\|\mathbf{AB}\|_2 \leq \|\mathbf{A}\|_2\|\mathbf{B}\|_2$, we have:

$$\begin{aligned}\Upsilon(k,l) \leq & \mathbb{E}\big[\|\ \mathbf{\Theta}(t,t-l+1)^H \text{diag}(\boldsymbol{\phi}^{(l)})^H \text{diag}(\boldsymbol{\phi}^{(k)})\ \mathbf{\Theta}(t,t-k+1)\|_2\big] \\ & \times \text{tr}\big(\mathbf{x}\,\mathbf{x}^H\big) \\ \leq & \mathbb{E}\Big[\|\big(\prod_{\tau=t}^{t-l+1} \mathbf{S}_\tau\big)\|_2 \|\text{diag}(\boldsymbol{\phi}^{(l)})^H\|_2 \|\text{diag}(\boldsymbol{\phi}^{(k)})\|_2\ \|\big(\prod_{\tau=t}^{t-k+1} \mathbf{S}_\tau\big)\|_2\Big] \|\mathbf{x}\|^2 \\ \leq & \rho^{l+k}\ \|\text{diag}(\boldsymbol{\phi}^{(l)})\|_2\ \|\text{diag}(\boldsymbol{\phi}^{(k)})\|_2\ \ \|\mathbf{x}\|^2\end{aligned} \tag{25}$$

where ρ is the spectral radius bound *i.e,* $\|\mathbf{S}\|_2 \leq \rho$.

This implies that:

$$\begin{aligned}\text{tr}(\mathbb{E}[\mathbf{y}_t \mathbf{y}_t{}^H]) \leq & \sum_{k=0,l=0}^{L} \rho^{l+k}\ \|\text{diag}(\boldsymbol{\phi}^{(l)})\|_2\ \|\text{diag}(\boldsymbol{\phi}^{(k)})\|_2\ \ \|\mathbf{x}\|^2 \\ \leq & \ \|\mathbf{x}\|^2\big(\rho^0\|\text{diag}(\boldsymbol{\phi}^{(0)})\|_2 + \rho^1\|\text{diag}(\boldsymbol{\phi}^{(1)})\|_2 + .. \\ & + \rho^L\|\text{diag}(\boldsymbol{\phi}^{(L)})\|_2\big)^2\end{aligned} \tag{26}$$

The second term in (17) is positive *i.e.* $tr(\mathbb{E}[\mathbf{y}_t]\mathbb{E}[\mathbf{y_t}]^H) \geq 0$. By dividing both sides by N in (26) and operating, $\overline{var}[\mathbf{y}_t]$ can be bounded by expression (18).

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