Proof by induction – the role of the induction basis

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Proof by mathematical induction is a conceptually difficult, but important form of proof. The proof contains three steps and this study focuses the first one, the induction basis. The aim of the study is to explore how university students treat the induction basis in a proving task. Data were collected from 38 students' solutions to a task in a written exam and were analysed using content analysis. The results reveal that the students used different cases as the induction basis, the majority chose $n = 1$ although $n = 0$ was the preferred choice for the given task. A majority of the students used one case in their verification of the induction basis, but it was also common to use more than one case, which is superfluous for this task. Among the students who chose $n = 1$ as the initial number, a majority included more than one case in the basis step. We discuss how students' choices were influenced by the course literature and the formulation of the current task.

Introduction

Mathematical induction is an important form of mathematical proof that university students meet in the beginning of their studies. However, *proof by mathematical induction* (PMI) is conceptually difficult and there are different kinds of misconceptions that may cause problems for the students (e.g. Ernest, 1984; Ron & Dreyfus, 2004; Stylianides, Stylianides, & Philippou, 2007). In this study we focus on university students and how they treated PMI in a first course at university. Before presenting the study, we focus the structure of PMI and what previous research has taught us according to students' ways of treating such proofs.

Proof by mathematical induction

Mathematical induction is useful when you want to prove a statement that can be connected to the set of natural numbers. We exemplify this by the task used for our data collection. The task comes from a written exam:

> The number sequence a_n is defined through the recursive formula $a_n = na_{n-1} - n + 1$ for $n \geq 1$; $a_0 = 2$.
>
> a) Compute a_1, a_2, a_3, a_4 and a_5.

b) Find an explicit formula for a_n and prove by induction that it is correct. (Compare with $n!$.)

In this task, one first has to solve problems not directly connected to PMI, namely the whole a)-task and the problem to find an explicit formula in the first part of the b)-task. Here this formula is $a_n = n! + 1$. In the second part of the b)-task, PMI shall be used to prove that formula. A proof by mathematical induction can be said to contain three steps:

i) The induction basis aims to show that the statement is true for (in the example above) $n = 0$.

ii) The induction step starts with the *induction hypothesis*, which here can be expressed as "suppose k is a number for which the statement is true". Then we aim to show that this implies that $k + 1$ is a number for which the statement also is true.

iii) If step i) and step ii) hold, the induction principle claims the statement is true for every $n \geq 0$ (where n is an integer).

That the proof in itself contains three steps does not mean that every task can be solved with these three steps only. In the example above, one had to first find a closed formula that seemed to give the same result as the given recursive formula, before using PMI to prove that this closed formula actually gives the correct result for every n. There are also variations in how the three steps are applied. In the most common tasks the basis step deals with $n = 0$ or $n = 1$, but depending on what to prove you have to adapt the starting point to an adequate number or include more than one number in the induction basis.

Previous research

Ernest (1984) pronounces a number of conceptual difficulties experienced by students, and we will here focus on two of those; difficulties related to the induction basis and to the induction step respectively, and also how these two are connected in the structure of the proof.

There are different kinds of misconceptions regarding students' understanding of the induction basis. One finding is that students fail to include or do not understand the role of the induction basis. Getting the induction started, i.e. verifying the first step, is often treated as a formality without any meaning and not seen as really essential for the proof (Dubinsky, 1986; Ernest, 1984; Palla, Potari, & Spyrou, 2012), or as a preliminary activity just checking the validity of the initial case to give confidence that the statement to prove is true (Ron & Dreyfus, 2004). However, there are many examples where the induction step can be proved, but the proof fails in the induction basis, e.g. to prove that $2n + 1$ is even. There also exist uncertainties about where to start the basis step, as the misconception that the induction basis must always contain the case $n = 1$ (Stylianides et al., 2007). Connected to this is a lack of understanding regarding how many cases you need to include in the basis step and the consequences caused by the choice of starting point. Ron and Dreyfus (2004) have shown that it is not clear for all students that one has to check only for the very first case and that other checking activities are not necessary parts of the proof – except for more complicated examples where the induction basis needs to include more than one case.

A second issue is the induction step. Students often construe PMI as a method where you assume what you have to prove and then you prove it (Ernest, 1984). However, in the induction step you neither prove the statement is true for $n = k$ nor for $n = k + 1$; in fact, the truth-values of these cases are irrelevant since it is the implication "true for $n = k$ implies true for $n = k + 1$" you need to prove.

The final step of the proof is setting the results from the induction basis and the induction step together, which connects the understandings and misunderstandings due to the induction basis and induction step. Previous studies indicate that some students appear to conduct proofs without really understanding the steps involved, and that a proof has to follow a very strict scheme. In a study, some students admitted they view the basis step as nonessential, and something they did just because it was a rule stated by the teacher (Harel, 2002). Other studies showed that some students believed the induction basis had to be verified *before* the induction step for the proof to be valid (Pang & Dindyal, 2012), or that the basis step is always verifiable and thus one only needs to worry about the inductive step (Stylianides et al., 2007).

This paper is an initial report from a study aiming to explore students' understandings of PMI, and in forthcoming papers we intend to present results according to all steps of the proof. However, several researchers have identified the induction basis as one of the difficulties (e.g. Dubinsky, 1986; Ernest, 1984; Palla et al., 2012; Ron & Dreyfus, 2004; Stylianides et al., 2007), hence we here choose to focus exclusively on this initial part of the proof. Thus, this paper aims to explore how university students treat the induction basis in tasks where PMI is employed. This limitation made it possible in depth to uncover details in a crucial part of PMI and through that produce a richer description of students' different ways of handling the first step in PMI.

The context of the study

In the syllabus for compulsory school in Sweden, the word *proof* is not mentioned. However, the students shall develop their ability to apply and follow mathematical reasoning, which also is a preparation for conducting proofs. In Sweden, almost all students (98 % year 2014) continue to upper secondary school and about a fourth of the students follows the natural science or technological programme, which contain up to five courses in mathematics. In the first and third course, proofs are mentioned related to other parts of the core content, e.g. to prove and use the sine theorem. In the fourth course different methods of proof in mathematics is also an explicit part of the core content, mentioning proofs with examples from arithmetic, geometry or algebra. Although course 4 has mathematical proofs as a core content, PMI is usually not a part of the topic. However, in the fifth course one part of the core content is "Mathematical induction with concrete examples from e.g. the area of number theory" (Skolverket, 2012, p. 39). Thus, PMI is explicitly treated during course 5.

To apply for *Mathematics I*, the first mathematics course at the current university for this study, a student needs a passing grade in at least course 4 from upper secondary school. Hence, not all students have met PMI before they start Mathematics I, although they repeatedly have met proofs in general.

Mathematics I is a full time one-semester course, given at the department of mathematics at a university in Sweden. The students are aiming for a general exam in mathematics or physics, or for a teacher exam. The course has two parallel halves; algebra and calculus. PMI is included in the algebra part, which is examined mainly by a written exam at the end of the semester. PMI is introduced in *one* lecture (number 17), followed up by tutoring and task solving on PMI. In addition, one or two written hand in tasks deal with PMI. However, PMI is rarely used for proving theorems in other parts of the course. Thus, in Mathematics I, the introduction of PMI is limited to learning the method for its own sake or for future use. The current semester, the task presented in the introduction of this paper was the only task dealing with PMI in the written exam.

Regarding what number to choose as starting point in the induction basis, the course literature (Bøgvad, 2014, p. 143) uses $n = 1$ when the induction principle is established. $n = 1$ is also the most common starting point in the examples, but there are also examples with other starting points, e.g. $n = 0$ and $n = 4$. However, in 10 out of 13 exercises, the induction basis should be at $n = 1$ (including one task where both $n = 1$ and $n = 2$ are needed as basis), implying this is the usual case.

Method

In order to explore how students treat the induction basis, we chose to use data collected from students' solutions to a task of the written exam in the course Mathematics I (the task was presented above in the introduction of this paper). In total, 109 students took part in this exam, of whom eight students did not solve the current task at all, and ten students' solutions were marked with 0 points. We got permission from 38 students to use their solutions in our analyses. Of these 38 students, one gave a partly correct proof, where however the induction basis was missing; one student just presented an induction hypothesis and nothing more; while three students did not start the b)-part of the task at all. Since the focus of this paper is how students treated the induction basis, these five students will be excluded from the following analyses, which then will contain solutions from 33 students.

A content analysis (Cohen, Manion, & Morrison, 2011) of the students' solutions was undertaken. Aware of findings in previous research (e.g. Ernest, 1984; Ron & Dreyfus, 2004; Stylianides et al., 2007), we read and re-read the students' solutions, striving to identify similarities and differences in their treating of the induction basis. This content analysis generated three themes, in which each of the 33 student solutions was categorised. The first theme was whether or not the student presented a statement to be proved – recall the first part of task b) was to find a closed formula, which validity then should be proved. The second theme was what number the students chose as starting point in the induction basis (e.g. n = 0), and the third theme dealt with how many cases the students included in the basis step.

Results

In this section, we elaborate on the three themes mentioned above. We exemplify the different categories by including parts of the solutions from some of the 33 students included in the analysis. The given excerpts were chosen as representative examples of solutions in the respective category.

Did students clarify what they aimed to prove?

The first part of task b) was to find a closed formula, which was likely to give the same result as the recursive formula given in the task. Remember that the students had computed the values of a_1 to a_5 in part a), which was an obvious support when they should find the closed formula. The correct formula is, as presented above, $a_n = n! + 1$. This formula was stated by 31 of the students, e.g. one student wrote

Student A: It seems like we get the following formula for a_n, $a_n = n! + 1$.

However, one student (C) started his/her proof without giving the closed formula. That is, there was no statement to be proved, when s/he started the 'proof' by writing:

Student C: We first show the statement holds for a basis case. $n = 0 \rightarrow a_0 = 2$.

A few lines down the same student however gave the explicit formula referring to part a), and then used this formula as induction hypothesis and in the induction step. Another student (B) just began to show the (obvious) validity of the recursive formula. The first two steps presented were:

Student B: 1. $a_n = na_{n-1} - n + 1$ for $n \in [1, 5]$ as shown above.
2. $a_{n+1} = (n+1)a_n - (n + 1) + 1$ is supposed to be valid for the following n.

That is just repeating what was already given and student B also continued the 'proof' by reasoning about what came out from the recursive formula.

Starting point for the induction basis

As mentioned above, 31 students gave the correct formula ($a_n = n! + 1$), which is essential before starting the proof. However, since student B and C anyway started their proofs (see above), they have been included in the following two categorisations.

Even though it is not explicitly said in the task for which n the formula for a_n should be valid, it is implicitly given that it should be for $n \geq 0$ since the given sequence in the task starts with a_0. In the solutions analysed, 14 students included $n = 0$ in the initial step, while 17 students started at $n = 1$. We here give two examples starting at $n = 0$ and two examples starting at $n = 1$.

Student D: Check whether P is true for $n = 0$. $P(0) = a_0 = 0! + 1 = 2$. P is true for $n = 0$.

Student J: Basis step: Valid for $n \in [0, 5]$. (see above) [the student wrote "see above"]

Student F: Basis case: We check for $n = 1$: $1 + 1 = 2 = a_1$ so yes, it is true.

Student A: 1. The formula is proved for the cases 1–5. [referring to the first part of the task]

Two students started at $n = 2$. One of them did not give any motivation of his/her choice of starting point. The other student starting at $n = 2$ wrote

Student G: As basis we can use any number from task a). For example, $a_2 = 3 = 2 + 1 = 2! + 1$

Student G did neither motivate his/her choice of $n = 2$ as starting point, nor include that the formula anyway is valid for all $n \geq 0$ since s/he already had shown the equality for a_0 and a_1, which in fact is necessary for his/her proof to be complete. Despite this deficiency, the proof could be seen as valid.

The number of cases included in the basis step

As induction basis, 20 of the 33 students showed the validity of the formula for one specific case ($n = 0$, $n = 1$ or $n = 2$). Two examples were student D and F above, and two other examples are:

Student H: Basis case: We show the formula is valid for $n = 1$. $1! + 1 = 2 = a_1$

Student I: 1) Basis step: the formula is true for $n = 0$. $0! + 1 = 1 + 1 = 2$

Twelve students showed the validity for all elements from part a). Several students showed that by simply computing a_0 (or a_1) to a_5. We have above seen other forms of examples by student A and J, and yet another example is:

Student L: Basis assumption: The formula is valid for a_0–a_5 (even for a_0, since $0! = 1$, which means $0! + 1 = 2$. [referring to computations in part a) for a_1–a_5]

Finally, one student showed the validity for two cases.

Student K: $k = 0$ gives $0! + 1 = 2 = a_n$ for $n = 0$. $k = 1$ gives $1! + 1 = 2 = a_n$ for $n = 1$.

Summing up the results, focussing on the second and third theme, there are some differences in the students' choices in their solutions. Almost all students used either $n = 0$ or $n = 1$ as the first case in the induction basis. A majority of the students verified the basis for one specific case, but it was also common to use more than one case as basis. In Table 1, we combine the results from these two themes. This is a cross-table where e.g. the first column shows that of the 14 students choosing $n = 0$ as induction basis, 10 included just that case, while 4 included at least one more case.

	$n = 0$	$n = 1$	*Sum*
1 case	10	8	*18*
>1 case	4	9	*13*
Sum	*14*	*17*	*31*

Table 1: Starting point and number of cases included in the induction basis (number of students)

Here, we can notice that students who gave $n = 1$ as the first number in the induction basis also to a greater extent included more than one case in the basis step. In fact, a majority of the students starting at $n = 1$ included more than one case, while less than one third of the students starting at $n = 0$ did the same.

Discussion

The study presented in this paper is the initial part of a project about teaching and learning of PMI. Since the induction basis is the initial step of a proof by induction and this step has been identified as a difficulty (Ernest, 1984; Ron & Dreyfus, 2004; Stylianides et al., 2007), we chose in this paper to focus on the induction basis only. This narrow focus offered opportunities to a deeper exploration on students' understanding of an essential part of PMI, which is known as problematic for students.

One important finding was the variation in the students' solutions, whether $n = 0$ or $n = 1$ should be the case to verify in the induction basis. Since the recursive formula had $a_0 = 2$ as its initial value, $n = 0$ is to prefer as starting point for the proof, rather than $n = 1$. There can be various explanations for why a majority of the students anyway started with $n = 1$. Due to the course literature (Bøgvad, 2014, p. 143), the basis in the definition of PMI is conducted for $n = 1$ and most exercises start at $n = 1$ too. Hence the students are used to proofs starting at $n = 1$ and some might have the misconception that the proof always starts at checking for $n = 1$ (cf. Stylianides et al., 2007). This misconception can also depend on that students have memorised the structure of PMI and hence conduct their proof mechanically (Pang & Dindyal, 2012; Ron & Dreyfus, 2004). The task formulation may also contribute to this misconception, or at least not prevent it, since $a_0 = 2$ is already given. In addition, the task did not explicitly tell from what n to verify the formula, it just

said verify for a_n. Thus, it may not be obvious that a_0 is also computable by the closed formula and hence should be verified in the proof. The misconception that $0! = 0$ could be another possible reason to skip the case $n = 0$, since the closed formula then would not give the result $a_0 = 2$. However, we did not identify any signs of this misconception, although it cannot be ruled out.
A second finding is that over one third of the 33 students involved more than one case in their basis step, although in the current task just one case (n = 0) is needed as induction basis. This can possibly be explained by the conclusion that they are not aware of the role of the induction basis. Including more than one case, when not necessary, can be a matter of seeing the basis step as a formality (Ernest, 1984), and not understanding that …

> checking the validity of the initial case is an integral part of the proof – not a preliminary activity that is intended to shed light on the statement or to give confidence that the statement to be proved is true. (Ron & Dreyfus, 2004, p. 114)

However, the current task might encourage the adoption of including more than one case in the basis step. Before even starting the proof in the b)-part of the task, the students had to find a (closed) formula which was likely to give the correct result. Hence it is necessary to first be convinced that the formula found actually seems to coincide with the given recursive formula, i.e. "to give confidence that the statement to be proved is true" (Ron & Dreyfus, 2014, p. 114). In addition, the a)-part of the task was to, by the recursive formula, compute a_1 to a_5, which automatically gave the student five cases where the closed formula $a_n = n! + 1$ easily could be verified. Thus, that students gave more than one case as induction basis could just be a matter of that the cases were already verified. Moreover, it is not incorrect to include more than one case, though it is superfluous in the current task. It would be interesting to give almost the same task, but exclude the a)-part, give the closed formula $a_n = n! + 1$, and just ask the students to by mathematical induction prove it is correct. Possibly more students would then just verify one case in the basis step, since the initial computations of a_1 to a_5 are then not requested.

Even though the design of the task possibly had an impact on the students' tendency to include more than one case in the basis step, the results arising from combining theme two and three indicate a lack of understanding of the role of the induction basis. These results show that students who chose $n = 1$ as the (first) number in the induction basis, to a greater extent also included more than one case in the basis step. Recall that $n = 0$ was to prefer as basis. Hence, students who made one less appropriate decision were also more likely to make a second less appropriate decision. The tendency to include more than one case in the basis step indicates that the students connect the verification of the basis rather to the computations in the a)-task than to the formula to be proved. This shows a lack of understanding of the essential role of the induction basis (cf. Ernest, 1984).

Through this study, it has been possible to identify some issues about PMI. What we found most interesting was that a majority of the students chose $n = 1$

rather than $n = 0$ as induction basis and that those students also to a larger extent included more than one case in the basis step. However, when analysing written solutions to a task, it is not possible to draw deeper conclusions about how the students have reasoned when solving the task. Anyway, this study has illuminated some issues to be immersed in further research, e.g. through interviews get a clearer picture of why students include more than one case in the induction basis. Another view of the same issue is in what way the task design affects the students' solutions regarding the number of cases included in the induction basis. Hence this study has provided valuable information for the research to come.

References

Bøgvad, R. (2014). *Algebra I [8th ed.].* Stockholm University, Sweden: Department of Mathematics.

Cohen, L., Manion, L., & Morrison, K. (2011). *Research methods in education.* Milton Park, Abingdon, Oxon, United Kingdom: Routledge.

Dubinsky, E. (1986). Teaching mathematical induction I. *Journal of Mathematical Behavior, 5*, 305–317.

Ernest, P. (1984). Mathematical induction: A pedagogical discussion. *Educational Studies in Mathematics, 15*(2), 173–189.

Harel, G. (2002). The development of mathematical induction as a proof scheme: A model for DNR-based instruction. In S. R. Campbell, & R. Zaskis (Eds.), *Learning and teaching number theory: Research in cognition and instruction* (pp. 185–212). Westport, CT: Ablex Publishing.

Palla, M., Potari, D., & Spyrou, P. (2012). Secondary school students' understanding of mathematical induction: Structural characteristics and the process of proof construction. *International Journal of Science and Mathematics Education, 10*(5), 1023–1045.

Pang, A. & Dindyal, J. (2012). Students' reasoning errors in writing proof by mathematical induction. In B. Kaur & T. L. Toh (Eds.), *Reasoning, communication and connections in mathematics* (pp. 215–237). Singapore: World Scientific Publishing.

Ron, G. & Dreyfus, T. (2004). The use of models in teaching proof by mathematical induction. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 113–120). Bergen, Norway: Bergen University College.

Skolverket. (2012). *Mathematics.* [The syllabus for mathematics in upper secondary school.] Stockholm: Skolverket. Downloaded 2018-02-16 from www.skolverket.se/polopoly_fs/1.174553!/Mathematics.pdf

Stylianides, G. J., Stylianides, A. J., & Philippou, G. N. (2007). Preservice teachers' knowledge of proof by mathematical induction. *Journal of Mathematics Teacher Education, 10*(3), 145–166.