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A Behavioural Mean – Expected Shortfall Solution to the Asset Allocation Puzzle and the Time Diversification Puzzle

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Preface and Acknowledgements

This has been a long and interesting journey, albeit at times frustrating and unforgiving. However, the experience of writing this thesis—on the divergence between theory and practice—has been tremendously insightful. If I were to choose a different subject, I fear I would never be exposed to the field of behavioural economics, and in particular, behavioural finance. In fact, I have learned to appreciate the human aspects that surround the investing decision: from the way investors perceive risk, to how they tackle losses, to the other psychological factors that may or may not influence their portfolio compositions. In comparison to the normative models of portfolio choice which often impose unrealistic assumptions about the investors, I feel it is important to be aware of the human aspects that governs the portfolio decision. Therefore, I am thankful that I were able to work on this subject.

I would like to offer my special thanks to my supervisor Valeriy Zakamuline for his insights and guidance. Secondly, I would like to thank my family for the generous support and encouragement throughout the writing of this thesis. At last, my deepest admiration goes to Amanda for sticking it through.

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Abstract

The difference between the financial advice and mean-variance analysis is evident in terms of the investor's risk aversion and investment horizon. While the optimal mean-variance portfolio proportions are identical across different configurations of risk aversion and investment horizon, the financial advisers recommend a higher proportion of risky assets as the investor's risk tolerance and investment horizon increases. Consequently, mean-variance analysis is incompatible with the realities of investing.

In this thesis, we take on a behaviourist approach and attempt to explain the financial advice. Our model of choice is the mean-expected shortfall risk model (mean-LPM_1) which incorporates elements of behavioural finance such as the investor's target return and loss aversion. Previous research indicate that investors perceive risk as falling below a target return, and it is well known that investors are loss averse. That is, losses looms larger than corresponding gains to paraphrase Kahneman and Tversky (1979).

The mean-LPM_1 model is applied on two datasets that differ in terms of complexity. On both datasets, and in the presence and absence of the assumption of normally distributed returns, the model produces qualitatively similar results with the financial advise. Specifically, hypothetical mean-LPM_1 investors prefer riskier portfolios with lower loss aversions, and a longer investment horizon persuade them to hold more aggressive portfolios. Furthermore, the investors seek riskier portfolios when faced with higher target returns to have a reasonable shot at achieving those returns.

Keywords— Loss Aversion, Target Return, Expected Shortfall, Lower Partial Moments, Downside Risk, Asset Allocation Puzzle, Time Diversification Puzzle, Static Investment Horizon

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1 Introduction

1.1 Background

Canner, Mankiw, and Weil (1994) observe that the financial advisers violate the two-fund theorem by Tobin (1958). In mean-variance analysis, the two-fund theorem state that the optimal proportions are identical across efficient portfolios. However, the financial analysts recommend a higher proportion of risky assets for investors more tolerant to risk. This is the "Asset Allocation Puzzle". A notable solution to the puzzle is by De Giorgi (2011) who combines the behavioural reward risk model of De Giorgi, Hens, and Mayer (2008) with Statman's (1999) behavioural portfolio theory. De Giorgi (2011) shows that with decreasing loss aversions the portfolios become riskier, consistent with the financial advice.

In terms of a fixed investment horizon, the optimal mean variance portfolios are unaffected by the length of the horizon[1]. The financial advisers, however, advocate for a higher allocation of risky assets to younger investors who typically have longer horizons, and it is believed that the advisers subscribe to the *time diversification effect*, the notion that stocks become less risky over longer horizons (Bennyhoff, 2009). In its essence, the idea that stocks are less risky in the long run depends entirely on how risk is framed (Kritzman, 2002; Fisher and Statman, 1999). As a consequence, Fisher and Statman (1999) argue that it is more productive to examine how investors perceive risk and the other factors that influence their investment decision.

Motivated by the failures of mean-variance analysis, and the insights of Fisher and Statman (1999), this thesis opt to provide a behaviour explanation to the financial advise. As such, we concern ourselves with how advisers and practitioners view risk. Our model of choice is the mean-shortfall expectation model (mean-LPM_1), a special case of Fishburn's (1977) general mean-lower partial moments model (mean-LPM_n). The mean-LPM_1 model incorporates the concept of loss aversion (Kahneman and Tversky, 1979; Tversky and Kahneman, 1991), the notion that investors have different sensitivities to gains and losses of equal magnitude. Schmidt and Zank (2005) argue that loss aversion is synonymous with risk aversion. In this thesis, we adopt this view. The empirical observation that investors equate risk with falling below some target outcome (Kahneman & Tversky, 1979) is captured in the LPM_1 risk measure which measures the magnitude and frequency of downside deviations from a subjective

[1]The mean and variance is scaled up by the length of the investment horizon and the scaling does not affect the optimal composition.

reference point.

To test the model, we use two datasets: a classical dataset on the principal assets of stocks, bond and cash, and a more "advanced dataset" of several stock and bond funds of varying characteristics. The advanced dataset allows for a more realistic analysis. In brief, the mean-LPM_1 model applied on these datasets, both in the presence and absence of the assumption of normally distributed returns, produce portfolios consistent with the financial advise. Moreover, we identify the drivers that affect portfolio composition: lower loss aversions correspond to riskier portfolios; similarly, longer investment horizons are associated with riskier portfolios, while higher target returns reflects the need for riskier portfolios to achieve those targets.

1.2 Research Objective

The research objective is to qualitatively determine whether the mean-LPM_1 model can explain the financial advice and subsequently provide a behavioural solution to the puzzles.

1.3 Research Question

In addition to the research objective, we want to examine how the components of the mean-LPM_1 model affect the portfolio allocation. That is, how the portfolios change with longer investment horizon, loss aversion, and target return. Moreover, we want to determine if there are any portfolio differences when we assume normally distributed returns. The research questions can therefore be formulated as:

1. What is the relationship between the portfolio allocation and the loss aversion?
2. What is the relationship between the portfolio allocation and the investment horizon?
3. What is the relationship between the portfolio allocation and the target return?
4. Are there any difference in terms of portfolio composition under the assumption of normally distributed returns?

1.4 Organization of the Thesis

The literature review examines the puzzles in greater detail. Then commences a short review of the mean-variance model followed by the deficiencies of mean-variance analysis. First in terms of the expected utility theory, and then in terms of behavioural finance. The next section introduces the lower partial moments risk measures (LPM) followed by the derivation of the mean-LPM_n model. The subsection is succeeded by a presentation on the algorithm used to derive the optimal mean-LPM_n weights for $n = 1, 2$. In the end of the section we provide the rationale for studying the mean-LPM_1 model.

In the empirical section, we begin by stating the underlying assumptions that govern our research, and we provide our motivation for the specific choices we take. The analysis begins on the classical dataset under the assumption of normally distributed returns before we relax that assumption. Then we proceed to the advanced dataset, where we study a more realistic scenario involving more assets. The empirical section ends with the empirical conclusion that summarizes our findings.

In the discussion section, we reflect back on the research assumptions and explain the limitations of the study, and we discuss possible research paths for the future.

2 Literature Review

2.1 Time Diversification Puzzle

The concept that stocks are less risky over longer investment horizons is referred to as the *time diversification effect* (Bennyhoff, 2009). The theorists are unable to decide whether the concept is a fact or fallacy, so the debate is also known as the *time diversification puzzle*. But the puzzle also pertains to differences between theory and practice over longer investment horizons (Fisher & Statman, 1999). Indeed, the practitioners' portfolio recommendations is in sharp contrast to Samuelson's (1969) mathematical argument that, under certain assumptions, the portfolio decision is independent of the investment horizon. Particularly, the practical advise subscribe to the idea that younger investors with longer investment horizons should hold a larger proportion of equities than older investors with shorter horizons (Bennyhoff, 2009). Referring to Siegel's (1998) research on the historical returns on U.S. securities from 1802 to 1997, the advisers' recommendations seem appropriate if the stock performance of the past will prevail in the future. Siegel (1998) finds that equities have been historically stable over the long-term with annual compounded real returns on equities of 7 %, and approximately 7 % annual compounded real return for most major sub-periods over the sample period. Although Siegel (1998) reports that stocks are more volatile than bonds over short-term horizons, he finds that stocks never underperformed the worst performance of long-term government bonds and T-bills for periods longer or equal than 15 years. Moreover, in comparison to long-term bonds and T-bills, Siegel (1998) discovers that stocks never underperformed the inflation rate for periods longer than 15 years.

Siegel (1998) attributes the long-term stability effect of stocks to *mean reversion*—the idea that fluctuations in returns cancels out over time. Bikker and Spierdijk (2017) show that if mean reversion is present in stock returns, then successive stocks returns are negatively correlated which results in the full-period variance being less than proportional to the one-period variance (similarly to the concept of cross-asset diversification). Consequently, if mean reversion is present in stock returns, then the financial advice seems plausible if investors equate risk over the horizon in terms of the full-period variance. But the concept of mean reversion, like that of time diversification, is contested report Bianchi, Drew, and Walka (2016). After reviewing the empirical research on mean reversion, Bianchi et al. (2016) find that old research typically find evidence for mean reversion, while newer research find evidence against it.

Besides mean reversion, other return processes could entice investors to pursue aggressive stock portfolios. For instance, Kritzman (2002) assumes a random process of continuous returns. In doing so, he finds that the annualized variance, as well as the probability of incurring a loss, diminishes with longer horizons. If investors view risk as either the annualized variance or the probability of incurring a loss, then stocks become less risky over longer horizons. On the other hand, if investors equate risk with the magnitude of a disastrous loss, then risk increases with the longer horizon (Kritzman, 2002).

The financial literature on the relationship between risk and time is extensive, as scholars examine the puzzle in different set-ups and employ different measures of risk. In the option pricing theory, Bodie (1995) defines risk as the dollar cost of insuring against stock returns below the risk-free rate. He discovers, both in the presence and absence of mean reversion, that the insurance cost increases with the longer horizon. Accordingly, as the insurance cost increases with longer horizons, the implication is that risk must increase as well—a result that contradicts time diversification but not the independence of time and risk itself. The research by Merrill and Thorley (1996)—also in the option framework—arrives at the opposite conclusion of Bodie (1995). For protected equity notes and self funding market collars, they notice that the cost of insuring these securities against some minimum return decreases with time. Thus, the implication is that risk decreases too.

Panyagometh (2011) applies the downside risk measures Value at Risk (VaR) and relative VaR to the puzzle. By investigating the end value of a defined retirement portfolio, Panyagometh (2011) finds that the risk of incurring losses decreases with the longer investment horizon.

A notable contribution to debate in the expected utility paradigm is by Samuelson (1969). Under the assumptions of constant relative risk aversion, random return process, and future wealth is a function of returns only, Samuelson (1969) proves that the optimal allocation to equities is independent of the length of the investment horizon but determined only by the investor's risk aversion. Thorley (1995) arrives at the opposite conclusion of Samuelson (1969); by relaxing the assumption of constant relative risk aversion and employing a power utility function that incorporates investors with decreasing relative risk aversion, Thorley (1995) finds that the optimal allocation to equities grows with the longer horizon. A notable difference between Thorley (1995) and Samuelson (1969) relates to the statics versus dynamic approach. Thorley (1995) examines the puzzle in the single-period setting, while Samuelson (1969) examines the problem in a multi-period setting. As such, it can be argued that they are studying two conceptually different problems. Kritzman and Rich (1998) examine the time-risk relationship for different types of utility functions under different return processes and risk aversions. In summary, Kritzman and Rich (1998) discover that any conclusion on the validity of time diversification is highly sensitive to the set of assumptions imposed.

Fisher and Statman's (1999) comment that "The debate about the relationship between risk and investment horizon takes us to a dead end", captures the essence of the controversy because risk cannot be objectively defined. Alternatively, the puzzle can be viewed solely in terms of behavioural finance, an approach Fisher and Statman (1999) advocate for. In terms of behavioural finance, the assumption of constant risk aversion, for example, is empirically questionable. Kahneman and Tversky (1979) observe instead that people are risk seeking in face of losses and risk averse in face of gains. Aside from risk, Fisher and Statman (1999) also believe that factors such as self-control, cognitive errors and social responsibility can influence the investor's portfolio composition over longer horizons. Bennyhoff (2009) reasons that human capital can provide incentives for young investors to pursue stocks more aggressively than older investors with limited or depleted human capital.

2.2 Asset Allocation Puzzle

Canner et al. (1994) discovers a puzzling irregularity between the classical theory of Markowitz (1952) and the financial advisers' portfolio recommendations. The advisers violate Tobin's two-fund theorem

(Tobin, 1958) which states that every optimal mean-variance investor choose the same risky portfolio independently of their risk aversion. Thus, the proportion of bonds to stocks is necessarily identical for each investor. The financial advisers' portfolio recommendations in Canner et al. (1994), however, exhibit decreasing bond-to-stocks ratios for increasing proportions of assets in stocks. Equivalently viewed, the bond-to-stocks ratios decrease with lower aversions to risk because the financial advisers recommend a higher proportion of stocks to investors more tolerant to risk. In the financial literature, this divergence between mean-variance theory and practice is referred to as the *asset allocation puzzle* (Canner et al., 1994).

Figure 1 depicts the recommended Vanguard and Fidelity portfolios' bond-to-stocks ratios as of 2018. The advisers today, like the advisers in (Canner et al., 1994), violate the two-fund theorem. The blue curve in Figure 1 corresponds to the optimal mean-variance portfolios' bond-to-stocks ratios in the absence of a risk-free asset. As the risk aversion increases, the mean-variance investors prefer a higher proportion of bonds to stocks, or equivalently, a higher proportion of stocks to bonds with lower aversions to risk—the exact opposite result of the financial advise.

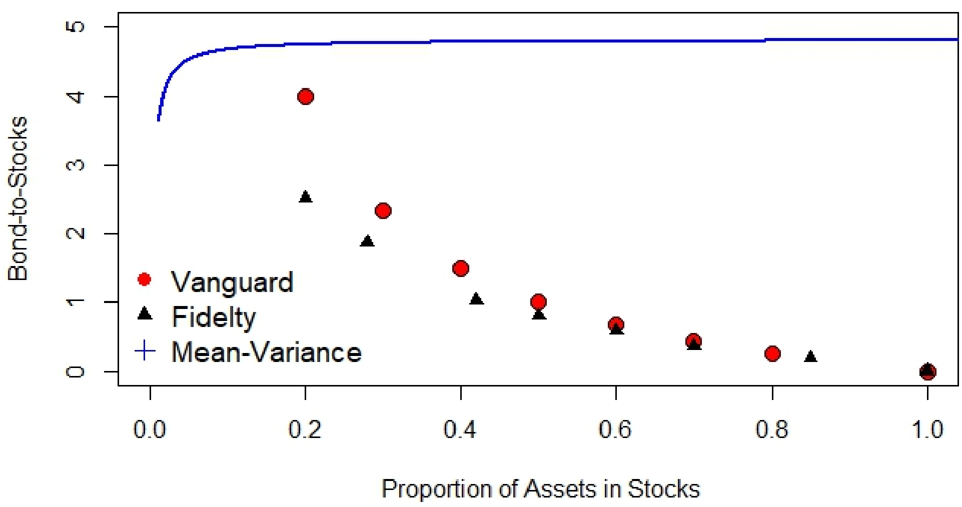

Figure 1: The Vanguard and Fidelity recommended portfolios' bond-to-stocks ratios as of 2018 and the optimal mean-variance bond-to-stocks ratios. In the absence of the risk-free asset, the mean-variance portfolios were computed by the mean-variance function derived by Pulley (1981) for risk aversion ranging from 1 to 200. The mean-variance bond-to-stocks ratios in Figure 1 are increasing with higher proportions of stocks, or equivalently, with decreasing aversions to risk—a perplexing result. The dataset used to derive the mean-variance portfolios can be found in Section 5.1

The descriptive failures of the mean-variance prompted Canner et al. (1994) to relax the mean-variance assumptions one by one, but in the end, they were not able to provide a explanation to the advisers' recommended portfolios in the mean-variance framework. Wang (2003) discovers an additional puzzle as he finds that investors with lower risk aversions are advised to hold a higher proportion of risky stocks to low-risk stocks. This violates the two fund theorem on similar grounds as before. Wang (2003) attempts to solve the main and sub-puzzle with a modified mean-variance utility function that incorporates elements of behavioural finance $U = \mu - \lambda \sum_{i=1}^{n} w_i \sigma_i^2$. The utility function incorporates the concept of loss aversion, λ, proposed in Kahneman and Tversky's (1979) prospect theory. The application of loss aversion to portfolio theory suggests that investors with lower aversions to losses will allocate more aggressively than investors with greater aversions to losses. Risk in Wang's (2003) model equates to the portfolio variance excluding covariances between assets. In behavioural finance, this approach is valid because most investors overlook the correlation between assets when constructing portfolios (Weber and Camerer, 1998; Kroll, Levy, and Rapoport, 1988). In the end, Wang's model (2003) produce qualitatively similar results as the financial advice.

Although Wang's (2003) solution incorporate elements of behavioural finance, De Giorgi's (2011) solution departs entirely from the mean-variance framework. De Giorgi (2011) applies the behavioural reward risk model of De Giorgi et al. (2008)—a model based on Kahneman and Tversky's prospect theory. The behavioural model separates unfavourable outcomes from favourable outcomes by outcomes that fall below or above some target outcome, respectively; the separation motivates the risk and reward of the portfolio. De Giorgi (2011) derives optimal portfolios for low to high target outcomes. In the context of Fisher and Statman's (1999) behavioural portfolio theory, these portfolios correspond to the investor's mental accounts, and portfolios associated with higher target outcomes represent the investor's need for upside potential, while portfolios associated with lower target returns represent the need for security or downside protection. Ignoring the correlation between the portfolios, De Giorgi (2011) assumes that the investor's global portfolio problem is to allocate wealth among these portfolios by maximizing the reward for a given risk constraint implied by the investor's loss aversion.

In the end, the behavioural model produces qualitatively similar results with the financial advise. That is, the bond-to-stocks ratio decreases with decreasing loss aversions, and in terms of Wang's (2003) sub puzzle, the ratio of large-cap to small-and-mid cap stocks decreases with decreasing loss aversions.

3 Mean-Variance Model

Drawing on the works of Merton (1972) and Markowitz (1959), this section begins with a short review on the mathematical idea behind the mean-variance model.

Markowitz (1952) seminal idea was to evaluate portfolios on a two parameter risk-reward rule with reward defined as the portfolio's expected return and risk as the portfolio's variance. According to the mean-variance rule, risk averse investors choose portfolios with the lowest variance for a given mean, or equivalently, portfolios with the highest mean for a given variance. In addition to the decision rule, Markowitz' (1952) original model assumes that investors have static probability beliefs, invest for a fixed horizon and short assets freely. Taxes, transaction costs and other indirect costs are ignored. A security's performance over a time interval $[t-1, t]$ is measured as the arithmetic return in equation (1), where P_t is the security's price on date t.

$$r_t = \frac{P_t - P_{t-1}}{P_{t-1}} \tag{1}$$

It follows from equation (1) that the arithmetic portfolio return of n securities, in equation (2), is a linearly weighted sum of the individual arithmetic returns. The portfolio's expected return and variance is given by equation (3) and (4), respectively.

$$r_{p,t} = \sum_{i=1}^{N} w_i r_{i,t} = \boldsymbol{w^T r} \tag{2}$$

$$\mu_p = \sum_{i=1}^{N} w_i u_i = \boldsymbol{w^T u} \tag{3}$$

$$\sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j \neq i}^{N} w_i^2 w_j^2 Cov[r_i, r_j] = \boldsymbol{w^T \Sigma w} \tag{4}$$

In matrix notation, $\boldsymbol{w^T} = (w_1, w_2, w_3, ..., w_n)$ is the vector of asset weights with w_i denoting the proportion of funds allocated to asset i. $\boldsymbol{r^T} = (r_1, r_2, r_3, ..., r_n)$ and $\boldsymbol{\mu^T} = (\mu_1, \mu_2, \mu_3, ..., \mu_n)$ denote the vectors of the asset returns and mean returns, respectively. $\boldsymbol{\Sigma}$ denotes the n by n co-variance matrix which is assumed to be positive definite. $\boldsymbol{w^T 1} = 1$ denotes that the portfolio weights sum up to 1. Mean-variance investors are assumed to be risk averse, so an optimal mean-variance portfolio satisfies the constrained optimisation problem (5), or equivalently, its duality.

$$\underset{\boldsymbol{w}}{\text{Minimize}} \ \ \frac{1}{2}\boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{\Sigma}\boldsymbol{w}, \ \ \text{subject to } \boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{1} = 1 \ \text{and} \ \boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{\mu} = \mu \tag{5}$$

If the asset universe includes a risk-free asset, r_f, the portfolio return and mean return is given by equations (6) and (7), while the variance remains the same as in (4). The optimisation problem in (8) reduces to problem (5) for a zero allocation to the risk-free asset, $w_0 = 0$.

$$r_p = r_f w_0 + \sum_{i=1}^{N} w_i r_i = r_f + \sum_{i=1}^{N} w_i (r_i - r_f) = r_p = \boldsymbol{w}^{\boldsymbol{T}}(\boldsymbol{r} - \boldsymbol{1}r_f) \tag{6}$$

$$\mu_p = \sum_{i=1}^{N} w_i(\mu_i - r_f) = \boldsymbol{w}^{\boldsymbol{T}}(\boldsymbol{\mu} - \boldsymbol{1}r_f) \tag{7}$$

$$\underset{\boldsymbol{w}}{\text{Minimize}} \ \ \frac{1}{2}\boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{\Sigma}\boldsymbol{w}, \ \ \text{subject to } \boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{1} = 1 \ \text{and} \ \boldsymbol{w}^{\boldsymbol{T}}(\boldsymbol{\mu} - \boldsymbol{1}r_f) = \mu \tag{8}$$

Merton (1972) shows that the constrained optimisation problems (5) and (8) lead to analytical solutions. The solutions—referred to as the efficient set of portfolios—offer the best balance of reward to risk. Expanding on the analytical solutions, the two-fund theorem by Tobin (1958) states that every linear combination of two efficient portfolios spans the efficient set of risky portfolios. Mathematically,

$$r_p = (1 - a)r_{p,1} + a r_{p,2}, \ \ \text{where} \ \ a \in \mathbb{R} \tag{9}$$

where $(1 - a)$ and a denote the proportions invested in the efficient portfolio distributions $r_{p,1}$ and $r_{p,2}$, respectively. In the presence of the risk-free asset, the two fund theorem states that a linear combination of the risk-free asset and the efficient risky portfolio r_m spans the efficient.

$$r_p = (1 - a)r_f + a r_m, \ \ \text{where} \ \ a \in \mathbb{R} \tag{10}$$

In a hypothetical world of mean-variance investors, r_m corresponds to the market portfolio of risky assets since every investor holds the same proportions of risky assets. Figure 2 depicts the risk-reward trade for the efficient set in the presence and absence of the risk-free asset.

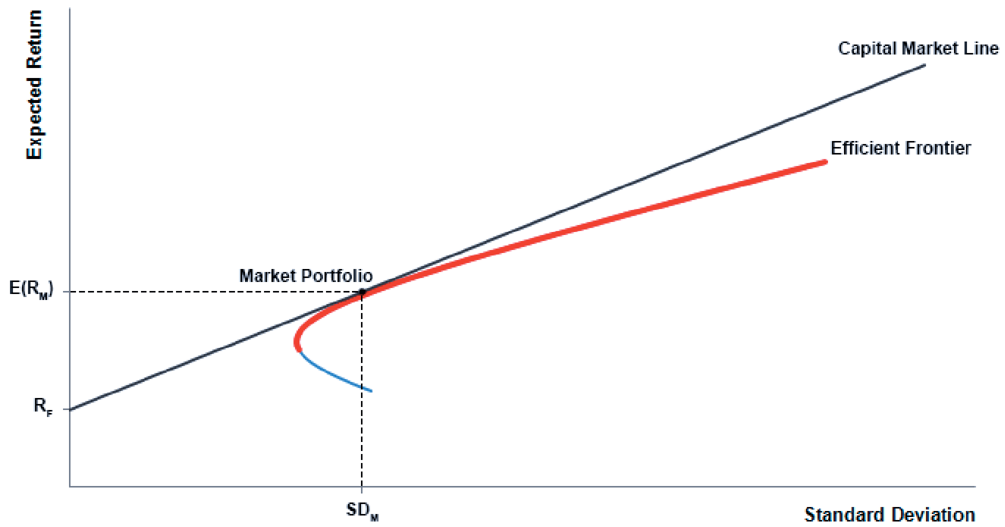

Figure 2: (σ, μ)-points on the red part of the hyperbola corresponds to the efficient set of risky portfolios in the absence of the risk-free asset. (σ, μ)-points on the Capital Market Line (CML) correspond to the efficient set of portfolios in the presence of the risk-free asset. The portfolios on the CML have the highest possible Sharpe Ratio, that is, the highest expected excess return over the risk-free asset per unit of standard deviation. The point of tangency corresponds to the standard deviation and mean return of the market portfolio. Figure 2 is taken from Manganelli (2017).

The optimisation problems (5) and (8) allow for short-sales and theoretically unbounded short positions. Canner et al. (1994) report that this assumption is likely not to hold in practise; and more interestingly, the financial advisers in their study do not even recommend short positions. Accordingly, there is a need to solve the mean-variance optimisation problem in face of short constraints, that is, for $\mathbf{0} \leq \boldsymbol{w}$. The solution to the short constrained optimization problem cannot be derived analytically. But it can be solved efficiently, since the objective function $\frac{1}{2}\boldsymbol{w}^{\boldsymbol{T}}\Sigma\boldsymbol{w}$ is quadratic, and it is convex since $\boldsymbol{\Sigma}$ is positive definite; hence, a local minimum corresponds to the global minimum. Quadratic programming methods are therefore feasible. In the programming language R, the "quadprog" package by Turlach and Weingessel (2007) implements the efficient and numerically stable dual method algorithm by Goldfarb and Idnani (1983). Instead of forming the Lagrangian for the original problem, the algorithm forms the Lagrangian for the equivalent dual problem, in terms of equivalent solutions set, and solve it instead (van de Panne & Whinston, 1964).

3.1 Deficiencies of Mean-Variance Analysis

Mean-variance preference described by the preference function $V(\mu, \sigma^2)$ with the standard assumptions (Eichner & Wagener, 2009)

$$\frac{\partial V(\mu, \sigma^2)}{\partial \mu} > 0, \quad \frac{\partial V(\mu, \sigma^2)}{\partial \sigma^2} < 0 \tag{11}$$

is not congruent with Neumann and Mogenstern's (1944) expected utility theory unless (1) the investor's utility function is quadratic, or (2) the portfolio distribution is normal. If the investor's utility function is quadratic, Danthine and Donaldson (2015) show that the congruency holds exactly for all return distributions.

$$U(r_p) = a + br_p + cr_p^2 \ \ with \ \ b > 0 \ \ and \ \ c < 0.^2 \tag{12}$$

The inequalities on the parameters guarantees that the classical assumptions of positive marginal utility and diminishing marginal utility holds. To ensure positive marginal utility $u'(r_p) > 0$, it is necessary to assume that $r_p < \frac{-b}{2c}$. Although the upper limit can be set arbitrarily high for all practical purposes, the specification implicitly determines the investor's relative and absolute risk aversion coefficients and limits the function's practicality. Moreover, the absolute risk aversion coefficient is increasing. This means that the investor prefers to reduce his proportion of risky assets as he accumulates wealth (Danthine & Donaldson, 2015). Regarding the realities of investing, it is paradoxical that investors become more risk averse with increasing wealth. In the end, it is hard to justify the assumption of quadratic utility in the expected utility paradigm.

If returns are normally distributed returns along with the classical assumptions of investor's marginal and diminishing utility, then mean-variance analysis is congruent with expected utility theory (Nelson, Ndjeunga, & Niamey, 1997). Grootveld and Hallerbach (1999) and Bawa (1975) argue, however, that the normality assumption is unrealistic since it rules out skewed distributions and returns are not unbounded below. According to Xiong and Idzorek (2011), the assumption of normality is not empirically supported because most asset classes and portfolios exhibit nonnormal return distributions; moreover, they find that extreme events occur with far greater frequency than predicted by the normal distribution. On a similar note, Levy and Duchin (2004) test the goodness of fit for 11 theoretical return

[2] $U(W) = U(W_0(1 + r_p))$ is written as $U(r_p)$ since W is completely determined by r_p.

distributions for horizons up to 4 year and report that there's at least one distribution that fits the data better than the normal distribution at every horizon.

Although Pulley (1981) shows that the congruency with expected utility theory is approximately justified in terms of small returns[3], Kahneman and Tversky (1979) finds that the axioms of expected utility theory is violated in practice. As a consequence, the behavioural validity of mean-variance preferences must be investigated on its own right. Fishburn (1977) report contention among scholars who argue that practitioners more frequently view risk as falling below some outcome or return. In comparison, the variance does not make a distinction between favourable and unfavourable deviations from the mean target. Referring to the empirical research by Kahneman and Tversky (1979), the idea of a reference or target outcome is empirically supported; people perceive losses as falling bellow a reference point. Moreover, Adams and Montesi (1995) find evidence that most corporate managers view risk this way.

According to (Fishburn, 1977), the mean portfolio return is an impractical target return because it varies from distribution to distribution. If investors do not want to be worse off than they initially were, a possibly research assumption is that investors want at least a portfolio return greater or equal to the return on some alternative perceived risk-free investment.

Mean-variance analysis also ignores the distribution's higher moments and in particular the distribution's skewness. For a non-normal distribution, consider a reflection about it's mean; it results in a distribution with the same mean and variance as the original but with different skewness (Markowitz, 1959). This result implies that mean-variance investors are indifferent to positive and negative distribution skewness. But are investors indifferent to skewness? Cumova (2004) argue on the contrary that positively skewed distributions, in comparison to negatively skewed distributions, are favoured among investors due to the potential of achieving great returns. Wen and Yang (2009) report that the skewness is an important feature of the asset price based on empirical evidence from 33 composite market indices around the world. In markets with positively skewed return distributions, Wen and Yang (2009) discovers that the risk compensation is virtually zero implying that positively skewed distributions are valuable to investors. Patton (2004) considers the impact of skewness on out-of-sample portfolio choices for investors with constant relative risk aversion and finds that the model who accounts for skewness

[3] $U(\mu, \sigma^2) = \mu - A\sigma^2$. A is the investor's constant coefficient of risk aversion. This approximation was used to compute the mean-variance portfolios in Figure 1

performs better than the model based on normally distributed returns, given no short constraints. In the end, the validity of the mean-variance model as a descriptive model of portfolio choices fails to capture essential behavioural characteristics and important information in the return distributions.

4 The Mean-LPMn Model

4.1 The Lower Partial Moments Measure (LPMn)

The failures of the variance to capture risk in the ordinary sense have prompted researchers to focus on downside risk instead—the risk associated with losses. The shortfall probability and the Value at Risk at a % probability (VaR_a) are among the simplest downside measures. VaR_a is the outcome that is only exceeded by a worse outcome a % of the times, that is, VaR_a is equal to the return distribution's a-percentile. The two risk measures are related—the shortfall risk is the a % probability of not obtaining a worse outcome than VaR_a (Cumova, 2004). These measures are simple, but simplicity comes at a cost of ignoring the distribution's higher moments. Consequently, a more comprehensive measure in these regards is Markowitz' semivariance below the mean, S_E.

$$S_E = \int_{-\infty}^{\mu} (\mu - r)^2 dF(r) \tag{13}$$

Markowitz (1959) shows that mean-S_E investors prefer positively skewed portfolios to the contrary, and under the assumption of normally distributed returns, analysis in terms of the mean and S_E is equivalent with mean-variance analysis (Markowitz, 1959).

The Lower Partial Moments (LPM) introduced by Bawa (1975) and Fishburn (1977) in equation (14)

$$LPM_n(\kappa; r) = \int_{-\infty}^{\kappa} (\kappa - r)^n dF(r) \tag{14}$$

encompasses a variety of downside risk measure—of which the variance, semivariance and the shortfall probability are but special cases—for different degrees n and target returns κ. A greater value n punishes downside deviations more harshly. As a result, n can be viewed as the parameter of risk aversion (Fishburn, 1977). For $n > 0$, Cumova (2004) shows that the LPM measures accounts for skewness and kurtosis. The LPM measures for $n = 1$ and $n = 2$ can be derived directly from the normal probability density function (Fortin & Hlouskova, 2011). Equations (15) and (16) gives the functional forms of LPM1 and LPM1 when normality is assumed.

$$LPM_1(\kappa; r) = (\kappa - \mu)T\Phi(d) + \sigma\sqrt{T}\varphi(d) \tag{15}$$

$$LPM_2(\kappa; r) = ((\kappa - \mu)^2T^2 + \sigma^2T)\Phi(d) + \sigma(\kappa - d)T\sqrt{T}\varphi(d) \tag{16}$$

$\varphi(.)$ and $\Phi(.)$ denote the probability density function and the cumulative distribution function of the standard normal variable. μ and σ^2 denote the mean and variance at the unit investment period. T is the length of the investment horizon $d = \frac{\kappa T - \mu}{\sigma}$.

4.2 The Mean-LPMn Model

In this section and throughout the rest of the thesis, we restrict our attention to hypothetical investors characterized by the utility function in equation (17).

$$u(r) = \begin{cases} r, & r \geq \kappa \\ r - \lambda(\kappa - r)^n, & r < \kappa \end{cases} \tag{17}$$

The function in equation (17) can be written compactly as in equation (18). For $n < 1$, the utility function is convex in the loss region which corresponds to risk seeking behaviour; a result that is in accordance with the empirical research of Kahneman and Tversky (1979).

$$u(r) = r - \lambda max[(\kappa - r, 0)^n] \tag{18}$$

From the utility function in (18), Fishburn (1977) shows that

$$\lambda + 1 = \frac{u(\kappa) - u(\kappa - 1)}{u(\kappa + 1) - u(\kappa)} \tag{19}$$

In absolute terms, the difference in utils for a below target return of $\kappa - 1$ is greater than the difference in utils for an above target return of $\kappa + 1$ if we assume that $\lambda > 0$ which implies $\lambda + 1 > 1$. Hence, the parameter can be interpreted as the investor's sensitivity to losses (Tversky & Kahneman, 1991). A greater loss aversion value λ corresponds to a steeper utility function in equation (18). Schmidt and

Zank (2005) argue that observed risk aversion is driven by loss aversion state and the two are essentially the same. The expected utility of equation (18) transforms into equation (20).

$$E[r - \lambda max(\kappa - r, 0)^n] = \mu - \lambda \int_{-\infty}^{\infty} max[(0, \kappa - r)]^n dF(r) =$$
$$\mu - \lambda \int_{-\infty}^{\kappa} (\kappa - r)^n dF(r) = \mu - \lambda LPM_n(\kappa; r) =$$
$$U(\mu, LPM_n) = \mu - \lambda LPM_n(\kappa; r) \tag{20}$$

The expected utility function in equation (20) can be viewed as consisting of two components: a reward and a risk component that corresponds to the portfolio's mean and the portfolio's LPM_n, respectively. By taking the partial derivatives of equation (20), the expected utility is increasing in μ and decreasing in LPM_n. Consequently, in terms of possible investment opportunities (portfolio return distributions), the optimal mean-LPM solutions are necessarily portfolios that offer the highest mean for the lowest LPM_n, or conversely, the lowest LPM_n for a given mean return. In comparison to the mean-variance model, the mean-LPM_n model incorporates key elements of behavioural finance: the concept of loss aversion, the clear separation of risk from reward, and the LPM measure's ability to account for skewness and kurtosis ($n > 0$).

4.3 Mean-LPMn Optimisation Problem and Algorithm

Foellmer and Schied (2002) prove that LPM_n is a convex risk measure for $n \geq 1$. It follows that $\mu - \lambda LPM_n(k; r)$ is concave[4]. Thus, the optimisation Problem (21) belongs to the class of convex optimisation problems. A convex function on a bounded set has a local minimum that corresponds to the global minimum; as a result, bounded convex optimisation problems can be solved efficiently (Boyd, Vandenberghe, & Grant, 1994).

$$\begin{aligned} min \;\; -(\mu - \lambda LPM_n(k; r)) \\ \text{s.t.} \\ \boldsymbol{w^T 1} - 1 = 0 \\ \boldsymbol{0} \leq \boldsymbol{w} \end{aligned} \tag{21}$$

[4]The portfolio mean is both convex and concave since it is a linear function of the vector of portfolio weights. The negative of a convex function is concave, and the sum of two concave functions is concave.

The minus sign ensures that the objective function is convex. The equality constraint reflects the fact that portfolio weights sum up to 1. Mathematically, it restricts the domain of the objective function to a bounded set. A global minimum must exist on this set, because a subset of a convex set is convex. The short constraint further restricts the domain of the objective function to nonnegative solutions.

In this thesis, we compute optimal mean-LPM_n weights in the programming language R by solving Problem (21). Since the mean-LPM_n solutions cannot be derived analytically. In R's CTRAN, we load the Rsonlp package by Ghalanos and Theussl (2012), and in particular, we use the package's solnp function to solve the constrained optimisation problem. The sonlp function is based on the algorithm by Ye (1989). In short, Ye's (1989) algorithm solves the general nonlinear Problem (22), of which Problem (21) is but a special case, by sequential quadratic programming.

$$\begin{aligned} min\ f(\boldsymbol{x}) \\ \text{s.t.} \\ g(\boldsymbol{x}) = 0 \\ I_h \leq h(\boldsymbol{x}) \leq I_h \\ I_u \leq \boldsymbol{x} \leq I_x \end{aligned} \tag{22}$$

The algorithm by Ye, 1989 involves several steps that needs to be addressed because every mean-LPM_1 portfolio in this thesis are derived from the solnp program. Consequently, the following serves a restatement of Ye's (1989) algorithm. First, the solnp add slacks to the inequality constraints and transforms Problem (22) into Problem (23).

$$\begin{aligned} min\ f(\boldsymbol{x}) \\ \text{s.t.} \\ g(\boldsymbol{x}) = 0 \\ I_u \leq \boldsymbol{x} \leq I_x \end{aligned} \tag{23}$$

At major iteration K, the solnp algorithm solves the linearly constrained Problem (24) with an augmented objective Lagrangian, a clever way to approximate a solution to Problem (23).

$$
\begin{aligned}
min\ \ f(\boldsymbol{x}) - \boldsymbol{y}^k g(\boldsymbol{x}) + (\rho/2)||g(\boldsymbol{x})||^2 \\
\text{s.t.} \\
J^k(\boldsymbol{x} - \boldsymbol{x}^k) = -g(\boldsymbol{x}^k) \\
I_u \leq \boldsymbol{x} \leq I_x
\end{aligned}
\tag{24}
$$

Here J^k is the numerical approximation to the Jacobian evaluated at $\boldsymbol{x}^k$, and $\boldsymbol{y}^k$ is the initial Lagrangian multipliers at step 0 where $\boldsymbol{y}^0 = \boldsymbol{0}$. At each major iteration K, solnp first checks whether $\boldsymbol{x}^k$ satisfy the equality constraint in Problem (24). If it is not feasible, then solnp finds a feasible vector $\boldsymbol{x}^k$. To solve Problem (24), Ye (1989) solves Problem (25) by sequential quadratic programming.

$$
\begin{aligned}
min\ \ (1/2)(\boldsymbol{x} - \boldsymbol{x}^k)^T H((\boldsymbol{x} - \boldsymbol{x}^k) + g^T(\boldsymbol{x} - \boldsymbol{x}^k) \\
\text{s.t.} \\
J^k(\boldsymbol{x} - \boldsymbol{x}^k) = -g(\boldsymbol{x}^k) \\
I_u \leq \boldsymbol{x} \leq I_x
\end{aligned}
\tag{25}
$$

Here g is the gradient and H the Hessian matrix. A solution to Problem (25) is then checked against Problem (24). If it a solution, then solnp starts major iteration $K + 1$ with $\boldsymbol{x}^{K+1}$ as the solution and with optimal Lagrange multipliers $\boldsymbol{y}^{K+1}$. Otherwise, a minor iteration starts that updates Problem (25) with a new Hessian and gradient. In the end, the process repeats until both Problem (24) and (25) are solved, or until a maximum number of iterations are reached (Ye, 1989).

If the objective is to solve for mean-$\text{LPM}_{0\leq n<1}$ in Problem (21), the solnp algorithm cannot be applied because the objective function is then non-convex. To search over rough surfaces for the global minimum, genetic algorithms can be applied. The idea behind the genetic approach is inspired by the natural evolution process (Mullen, Ardia, Gil, Windover, & Cline, 2009). The general genetic algorithm usually starts off with a random set of candidate solutions and evolve new solutions through selection and recombination operators Whitley (1994). In R, the Deoptim program by Mullen et al. (2009) implements differential evolution and can be used to solve the problem.

4.4 A Rationale for the Mean-LPM1 Model

In the empirical section we choose to study the mean-LPM_1 model. It is not immediate clear why chose this mean-LPM model instead of the mean-LPM_2 for example. In the face of losses mean-LPM_1

investors are risk-neutral, while mean-LPM_2 investors are risk-averse, as it follows from the utility function in equation (18). Referring to Kahneman and Tversky (1979), people exhibit risk-seeking behaviour in the relative loss domain. As a consequence, neither models are able to capture this aspect of investing. However, the mean-LPM_1 model fails less in this regard. On the other hand, mean-$LPM_{0<n<1}$ models satisfy risk-seeking behaviour, but these models comes at a price of uncertainty. Referring back to Section 4.3, mean-$LPM_{0<n<1}$ functions are not convex. Hence, the optimal solution to Optimization Problem 21 is most likely the best local minimum among local minima with genetic programming. Furthermore, for $n > 0$, LPM_1 is the only measure that satisfies the coherency of Artzner, Delbaen, Eber, and Heath (1999) finds Cumova (2004). Accordingly, the trade-off motivates us to study the mean-LPM_1 model.

5 Empirical Application

In the empirical section, we impose a specific set of assumptions that aims to restrict the investors' behaviours and the market they operate in. Assumptions 1, 2, and 3 directly relates to the research objective and the questions we want to answer. The other assumptions are imposed to limit the scope of the analysis, so we can focus on the essential questions. The validity of the assumptions will not be discussed here, but in Section 6 we will examine them closely.

1. The investors evaluate and choose portfolios that corresponds to the highest expected utility according to the mean-LPM_1 model.

2. The investors are loss averse, $\lambda > 0$. Loss aversion is synonymous with risk aversion (adopting Schmidt and Zank's (2005) view).

3. The investment horizon is a single static period.

4. The investors share the same beliefs about the probability return distributions. In particular, investors base their believes of the future on the historical asset returns.

5. Short selling is not allowed.

6. A risk-free asset does not exist.

7. Infinite divisibility of assets.

8. No taxes, transaction costs, or other indirect or hidden costs. Free access to information.

9. Future wealth is a function of only portfolio returns.

The goal in the empirical section is to determine, on a qualitatively level, if the mean-LPM_1 model can explain the professional financial advice with respect to the investment horizon, loss aversion, and target return, and the differences in portfolio allocations that arises if we assume normally distributed portfolio returns.

In particular, we examine the optimal mean-LPM_1 portfolios for loss aversions ranging from $\lambda = 1$ to $\lambda = 20$, by increments of 1, over the 1-year, 4-year, and 7-year long investment horizon at the constant target return of 4 %. The range in loss aversions is supposed to reflect a variety of investors. Holding the target return constant over longer horizons allows to gauge the time-effect on the optimal asset composition. In our analysis, the horizons correspond to a short, medium, and a long horizon, respectively. The horizons seem arbitrarily chosen, but in finance, as I am aware, there does not exist an agreed upon definition on what constitutes a short, medium, or long horizon. Moreover, as we examine the mean-LPM_1 model on the advanced dataset, the assets only go back to the 1980's, so to have a representative number of "long horizons" observations, we ended up with this specification. The 4 % target return corresponds to the annual mean return on the 1-month T-Bill from January 1926 to December 2011, as computed in Table 1. The assumption that investors view risk as falling below the annual mean return on the 1-month T-Bill seems plausible; the 1-month T-Bill is considered to be among the least risky investments. The assumption of a 4 % target return is eventually relaxed in favour of a more comprehensive analysis on the relationship between higher target returns and the portfolio composition over longer horizons.

The exact formulation of LPM_1 can derived directly from the normal probability density function (Fortin & Hlouskova, 2011). This facilitates the analysis of the mean-LPM_1 model under the consideration of normally distributed returns. In the first part of the analysis, we begin with the classical dataset with the assumption of normally distributed portfolio returns. Then we proceed to the nonparametric case and note any similarities or differences between the cases. In the end, we examine the optimal mean-LPM_1 portfolios given the advanced dataset under normality and in the absence of normality.

5.1 Classical Data

The classical data is obtained from Ibbotson's SBBI 2012 Classic Yearbook. The data consists of monthly prices on the 1-month Treasury bill (Cash), a market index of large-cap stocks (Stocks), and an index of long-term government bonds (Bonds). The sample period stretches from January 1927 to December 2011 (1020 monthly observations). Figure 3 displays the monthly arithmetic (nominal) returns over the sample period.

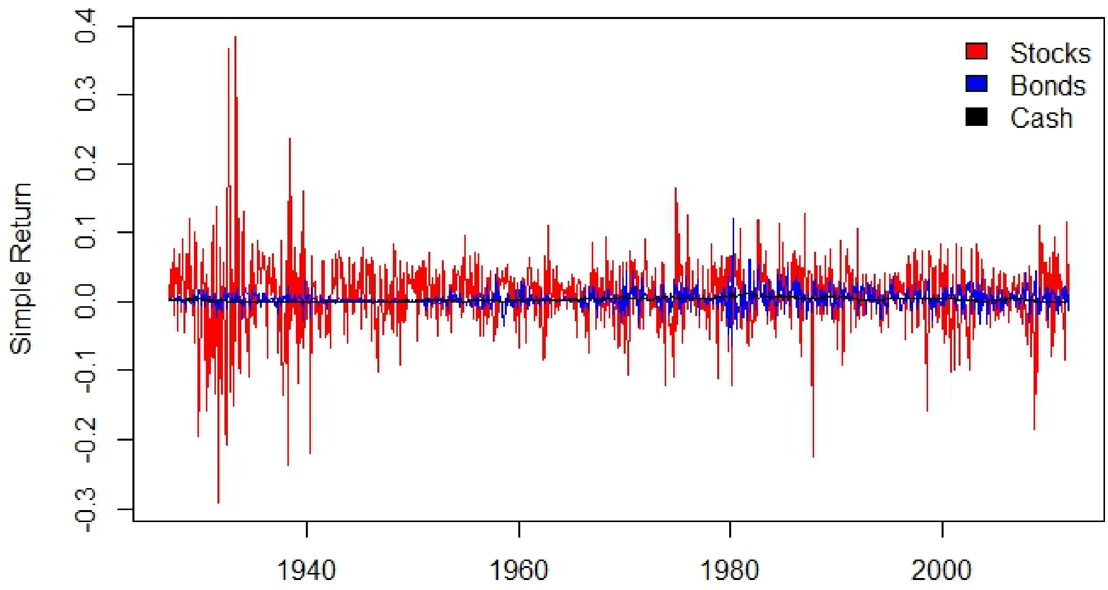

Figure 3: The monthly arithmetic returns for stocks, bonds, and cash from January 1927 to December 2011s. The unusual large fluctuations in stocks from around 1927 to approximately 1945 correspond to events such as "The Stock Market Crash of 1929" and the resulting "Great Depression", while the fluctuations around the 1940's coincide with World War 2 (Siegel, 1998). The financial crisis, as of 2008, effect on the stock returns is also notable in the figure. Evidently, monthly stock returns fluctuate more than the returns for the other assets. Over the 1-month horizon, bond and cash returns are arguably less prone to fluctuations, but compared to cash, the bond index is far from stable over the 1-month horizon.

Figure 4 depicts the cumulative log returns for the assets. Over the whole sample period, stocks offers the superior performance based on the average compounded return of 9.5 %. A dollar in stocks at the beginning of 1927 accumulates to $ 2235 in December 2011. The final values for bond and cash are insignificant in comparison.

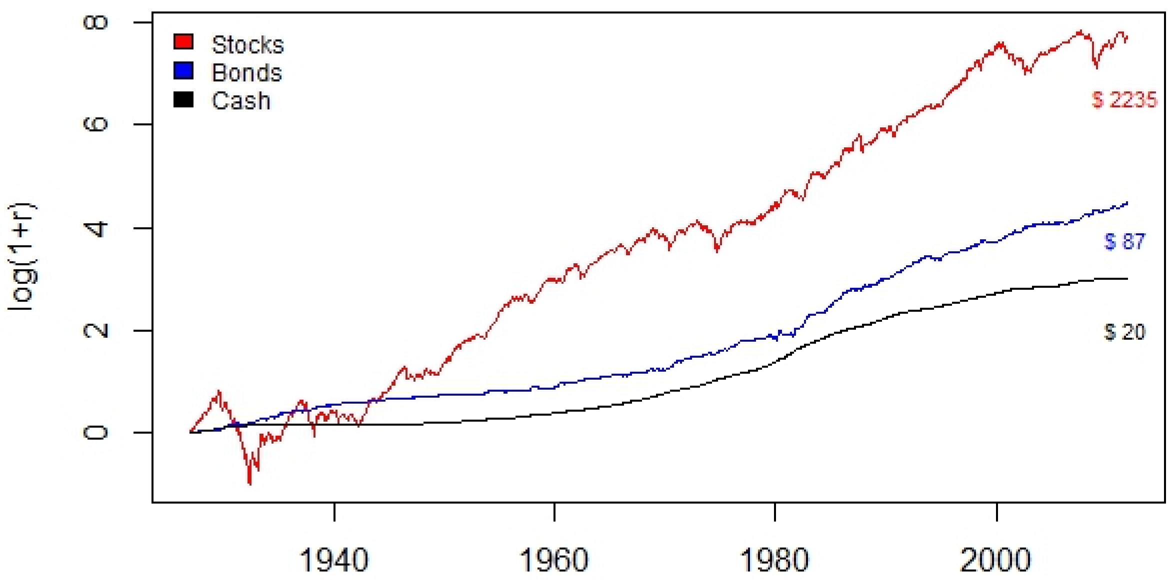

Figure 4: The cumulative log returns for stocks, bonds and cash from January 1927 to December 2011. Over the sample period, the stock index offers the superior performance; a dollar in the beginning of the sample period accumulates to $ 2235 in December 2011, corresponding to an annual compounded return of 9.5 %. The numbers for cash and bonds are only $ 87 and $ 20, respectively. The high frequency zig-zag pattern in stocks indicate short-term periods where bonds and cash outperformed stocks because bond and cash "trend" in relatively predictable patterns over the short term, given the figure (long-term as well).

Table 1 gives the annual summary statistics along with the annual correlation matrix. Bonds and cash are positively correlated; a relationship that is indicated in Figure 4. With respect to the conventional measures of risk, the annual standard deviation and the max-min spread, the large-cap stock index is—by far—the riskiest asset on a historical basis. Yet, in terms of performance measures such as the historical average return and maximum return, it is stocks that dominate the other classes.

The assets were tested for normally distributed returns in R, we tested the assets up to the 10 year horizon at 95 % confidence by applying Shapiro and Wilk (1965) test in R. Up to the 10 year horizon, we are not allowed to reject null-hypothesis that the stock index is normally distributed. Regarding bonds we can reject the null hypothesis for horizons longer than 4 years. For cash, we can reject the null-hypothesis for horizons longer than 6 years.

	Mean	Std.Dev	Skew	Kurtosis	Max	Min	Value
Stocks	0.12	0.21	-0.42	-0.05	0.58	-0.44	2234.69
Bonds	0.06	0.06	1.23	2.36	0.29	-0.05	87.49
Cash	0.04	0.03	0.97	0.86	0.15	-0.00	19.92

	Stocks	Bonds	Cash
Stocks	1.00	-0.03	-0.02
Bonds	-0.03	1.00	0.45
Cash	-0.02	0.45	1.00

Table 1: The annual correlation matrix along with the annual summary statistics for stocks, bonds, and cash from January 1927 to December 2011. By conventional risk measures such as the standard deviation and the max-min spread, the stock index is historically more volatile than bonds and cash. According to performance measures such as the the annual mean return and the observed max return, stocks are superior to the other assets.

5.1.1 Normally Distributed Returns

Figure 5 depicts the optimal mean-LPM1 portfolios associated with a short 1-year investment horizon at the 4 % target assuming normally distributed returns. In essence, the portfolios in Figure 5 are consistent with the financial advise because lower loss aversions corresponds to riskier asset compositions—if we subscribe to the notion that stocks are riskier than bonds that are riskier than cash. Although the portfolios seem to gradually become riskier with decreasing loss aversions, a significant difference can be drawn between the portfolio associated with the most loss acceptive investor and those for $\lambda > 1$. Indeed, The portfolio corresponding to $\lambda = 1$ contains entirely stocks, even at the 1 year horizon. In comparison, the portfolios corresponding to highly loss sensitive investors contains large cash holdings at this short horizon. In terms of uniqueness, lower levels of loss aversion tends to produce similar asset compositions. As such, it could be the case that as $\lambda \longrightarrow \infty$, the optimal mean-LPM$_1$ weights converges to a specific portfolio.

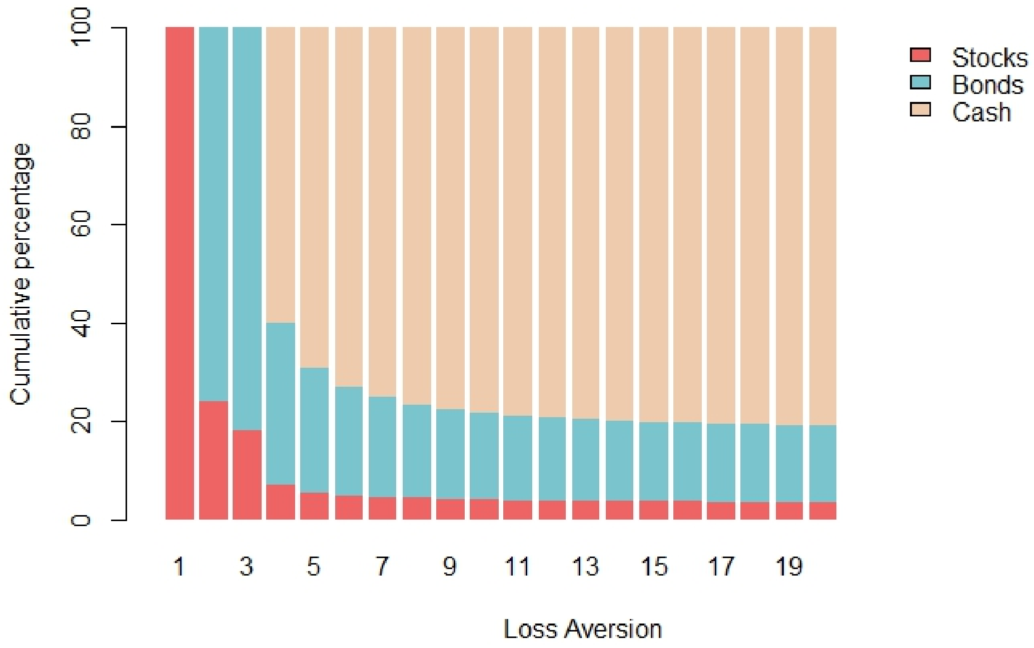

Figure 5: The optimal mean-LPM1 allocation assuming a short 1 year horizon, a target return of 4 %, and normally distributed portfolio returns. Cash intensive portfolios are associated with higher loss aversions, while stocks and bonds portfolios are associated with lower loss aversions. Moreover, the cash allocation is increasing for increasing loss aversions, but it is increasing at a diminishing rate. The stock proportion is decreasing for increasing aversions to losses, but similarly to cash, it is decreasing at a diminishing rate. Since the stock and cash proportion move in opposite directions as the loss aversion increases, the change in the bond proportion is effectively dependent on the particular level of loss aversion.

Figure 6 depicts the optimal mean-LPM1 portfolios for an investment horizon of 4 years at the target return of 4 %. The 4-year portfolios in Figure 6 is in sharp contrast to the corresponding 1-year portfolios, that is, corresponding in terms of the loss aversion, λ. The longer horizon induces every investor to hold riskier portfolios— a result consistent with the financial advise. In fact, the proportion of stocks has increased significantly at the lower levels of λ.

Cash is nonoptimal at the medium horizon except in small proportions for investors most sensitive to losses. The change in the bond allocation from the 1 to the 4 year horizons also imply that investors prefer riskier portfolios over longer horizons. For instance, at the 4 year horizon, investors with lower levels of loss aversions hold less bonds than they previously did given the 1 year horizon; similarly, investors with higher loss aversions hold more bonds at the expense of cash.

If we view the optimal 4-year portfolios in isolation, it is clear that the level of loss aversion affects the portfolio composition in the same way it did for the 1 year horizon: a lower loss aversions implies riskier portfolios and vice versa. Furthermore, as the loss aversion increases the portfolios become less distinguishable.

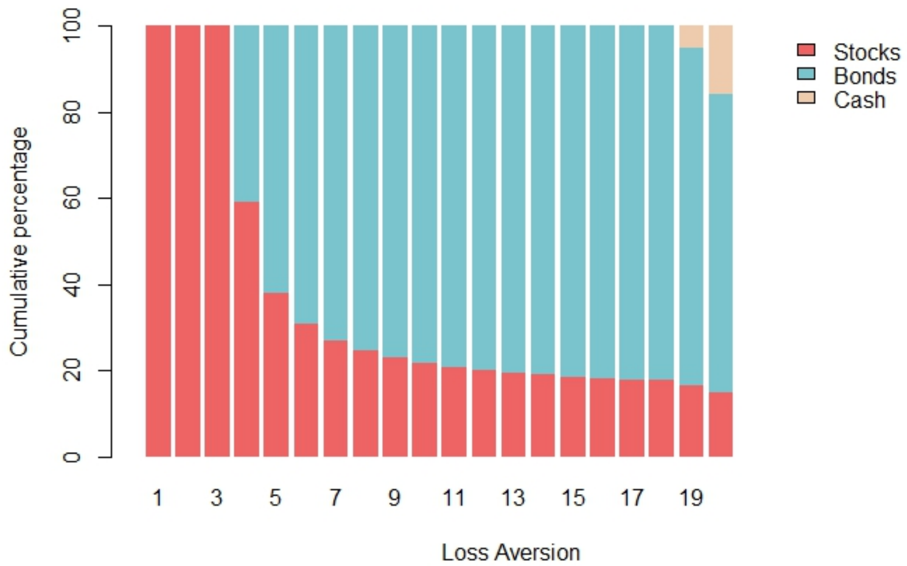

Figure 6: The optimal mean-LPM1 portfolio allocation assuming normally distributed returns, a fixed horizon of 4 years, and a target return of 4 %. Cash, except at the highest loss aversions, is not optimal at the 4 year horizon. Pure stock, or stock intensive portfolios are optimal at the lower loss aversions, but as the loss aversion increases, bonds become more attractive. Furthermore, the stock proportion decreases at a diminishing rate when the loss aversion increases, while the bond proportion increases at a diminishing rate when the loss aversion increases. As a consequence, there's little observable difference between portfolios corresponding to higher levels of loss aversions.

Figure 7 depicts the optimal portfolios over the long horizon of 7 years at the 4 % target return. In comparison to the 4-year portfolios, the 7-year portfolios include a higher proportion of stocks and a lower proportion of bonds. Referring back to the 1-year portfolios associated with the lowest loss aversions in Figure 5, the shift towards a riskier asset composition is even more evident. Clearly, investors prefer a higher allocation to stocks for longer horizons.

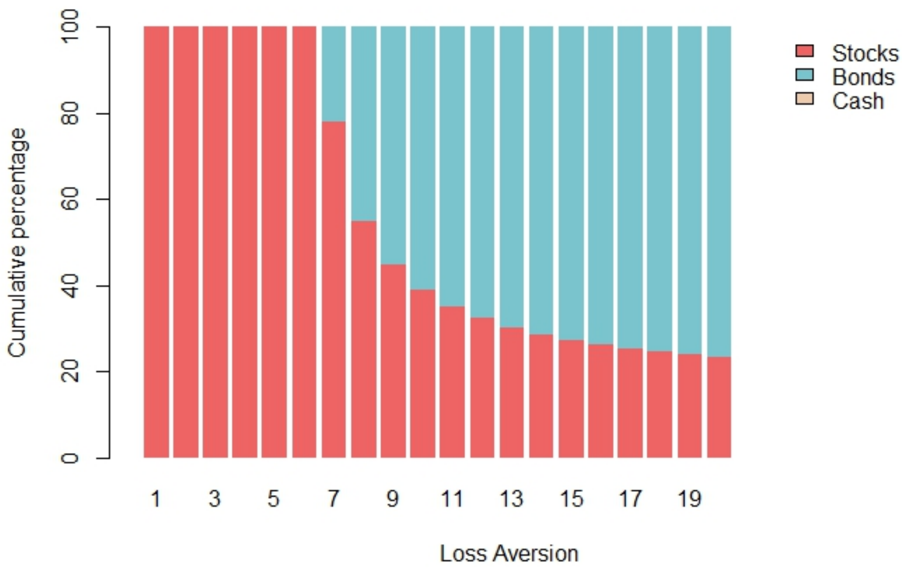

Figure 7: The optimal mean-LPM1 portfolio allocation assuming normally distributed returns, a buy-and-hold horizon of 7 years, and a target return of 4 %. Cash is not optimal at this horizon regardless of the investor's loss aversion. Pure stock and stock intensive portfolios are, however, attractive investments for investors with low and medium levels of loss aversions. As the loss aversion increases, the optimal portfolios become less stock intensive, and subsequently less risky.

Our findings suggest that the longer investment horizon entices investors to hold more stocks because purely stocks or stock intensive portfolios provide the investors with the highest expected utility. However, the convergence to stocks is more modest for portfolios corresponding to higher loss aversions. The portfolio associated with $\lambda = 6$, for instance, contains purely stocks after 7 years as the figures indicate. The portfolio corresponding with $\lambda = 20$, on the other hand, consist entirely of stock for horizons equal or longer than 15 years. Accordingly, the convergence to stocks happens faster for portfolios associated with lower levels of of loss aversion. This means that highly loss sensitive investors need sufficiently long investment periods before they commit themselves to purely stocks.

To extend the analysis, we relax the assumption of a constant target return and acknowledge that investors have different aspiration goals. Instead of directly examining how the portfolio composition changes with increasing target returns and investment horizons, we opt for an indirect approach. That is, we study how the optimal bond-to-stocks ratios change with increasing target returns and horizons. Referring back to our previous results for the 4 % target return, cash becomes increasingly insignificant at the longer horizons. Hence, for increasing target returns, it seem sensible to assume that cash becomes even less relevant[5]. Finally, Canner et al. (1994) study how the advisers portfolio's bond-to-stocks ratios changes with risk aversion. Accordingly, these insights motivates us to examine how bond-to-stocks ratios change. The "dots" in Figure 8 depict the optimal portfolios' bond-to-stocks ratios for target returns ranging from 4 % to 12 % at the the 1 year horizon.

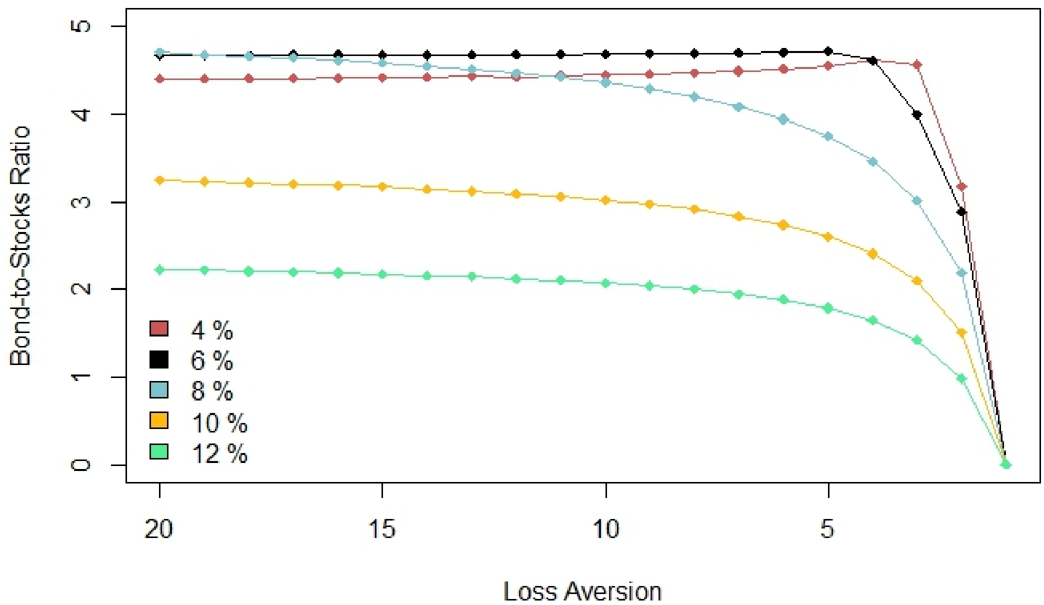

Figure 8: The optimal mean-LPM_1 bond-to-stocks ratios assuming normally distributed returns, a fixed 1 year horizon, and target returns ranging from 4 % to 12 %.

It is strictly incorrect to call the line segments connecting the dots as "bond-to-stock curves"[6]. As such, the line segments serves only as a visual aid. In Figure 8, we observe that the bond-to-stocks ratios are nearly identical for higher loss aversions at the 1 year horizon. This reflects the fact that the portfolios tend to become similar with increasing loss aversions, as was shown for the 4 % target return. Regardless of the specific target return, the bond-to-stocks ratios fall with decreasing loss aversions. Accordingly, the previous result for the 4 % case also holds for higher target returns at the 1 year horizon too.

[5]It can be shown that this is true for the dataset for hand

[6]The points on the line segments in between dots do not correspond to the any computed portfolios' optimal bond-to-stocks ratio.

Across target returns, we observe a pattern that relates to the magnitude of the target. Bond-to-stock ratios associated with higher target returns typically lie below ratios corresponding to lower target returns. This suggest that stocks become relatively more attractive investments with higher target returns in comparison to bonds. Reflecting back on the summary statistics in Table 1, bonds have limited upside potential. Hence, bonds become riskier than stocks with higher target returns due to the LPM_1 measure. Considering that stocks are also historically more rewarding, the mean-LPM_1 model favours stocks for higher target returns at all loss aversions. Figure 9 depicts the bond-to-stocks curves for an investment horizon of 4 years.

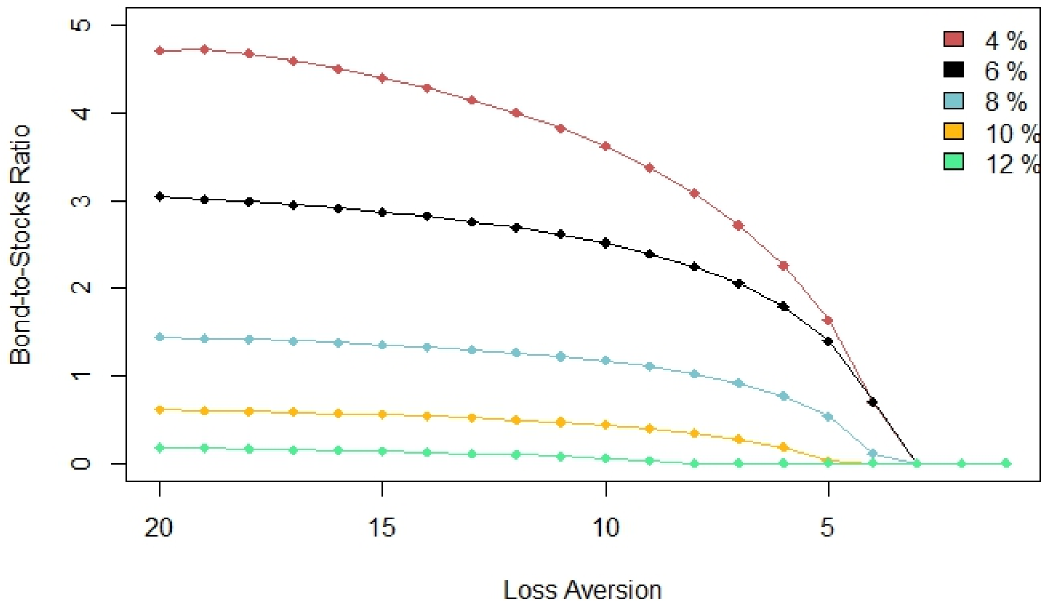

Figure 9: The optimal mean-LPM1 bond-to-stocks ratios assuming normally distributed returns, a buy-and-hold horizon of 4 years, and target returns ranging from 4 % to 12 %.

Note that Figure 9 and Figure 8 are in the same aspect configuration; consequently, it is possible to qualitatively determine how the optimal portfolio's bond-to-stock ratios change as the investment horizon lengthens. The bond-to-stock ratios in Figure 9 are below their respective 1 year ratios in Figure 8. Thus, stocks are relatively more attractive investments at the longer horizon, not only for the 4 % target return but for higher target returns as well.

Across target returns, we observe, as we did in Figure 8, that higher target returns are associated with lower bond-to-stocks ratios reflecting the fact that stocks become less-risky with increasing target returns. Moreover, the loss aversion, with respect increasing horizons and target returns, affects the stock allocation in similar ways as before; that is, loss sensitive investors prefer a higher bond to stock allocation than loss tolerant investors. However, for the highest target returns in Figure 9, we observe that the bond-to-stocks ratios are virtually 0 for all loss aversion. Indeed, the loss aversion's effect on the portfolio composition is subsequently ignored, since everyone hold stocks at these targets. Figure 10 depicts the optimal bond-to-stock ratios at the 7 year horizon.

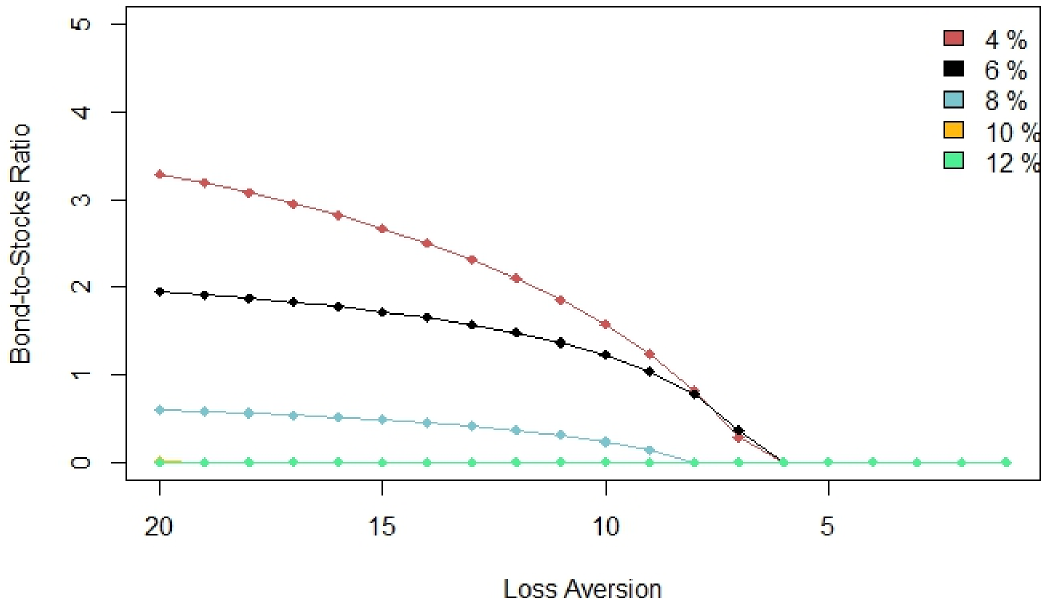

Figure 10: The optimal mean-LPM1 bond-to-stocks ratios assuming normally distributed returns, a buy-and-hold horizon of 7 years, and target returns ranging from 4 % to 12 %.

Figure 10 is in accordance with our previous results on the relationship between the asset allocation and the investor's loss aversion, time horizon, and target return: At the long investment horizon of 7 years, the bond-to-stock ratios are below their respective 4 year curves. Across target returns, a higher target return corresponds to a lower bond-to-stocks ratio. Across loss aversions, for target returns of 4, 6 and 8 %, investors with lower loss aversions prefer more stocks. At the 10 % and 12 % target returns, everyone holds stocks.

To briefly summarize our results so far, we have seen that the investment horizon, target return, and loss aversion affect the portfolio composition under the assumption of normally distributed returns. With decreasing loss aversions, the investors prefer in general less-risky portfolios, a result observed for different configurations of time and target. However, for sufficiently long target or investment periods, or a combination of the two, the loss aversion's affect on the portfolio combination is subsequently ignored because every investor do better by holding purely stocks. With increasing target returns, all investors prefer a riskier allocation, and in particular, stocks. Finally, a longer investment period induces the investors to hold more stocks.

5.1.2 Nonparametric Approach

In this section, the assumption of normally distributed returns is relaxed, and we do not assume a specific portfolio distribution. As we proceed throughout the section, an important goal, in addition to the overall goals, is to observe and comment on the similarities and differences we find with the mean-LPM_1 portfolios computed under the assumption of normally distributed returns to answer the research question in Section 1.3. Figure 11 depicts the optimal mean-LPM1 portfolios associated with a short 1-year investment horizon at the 4 % target return.

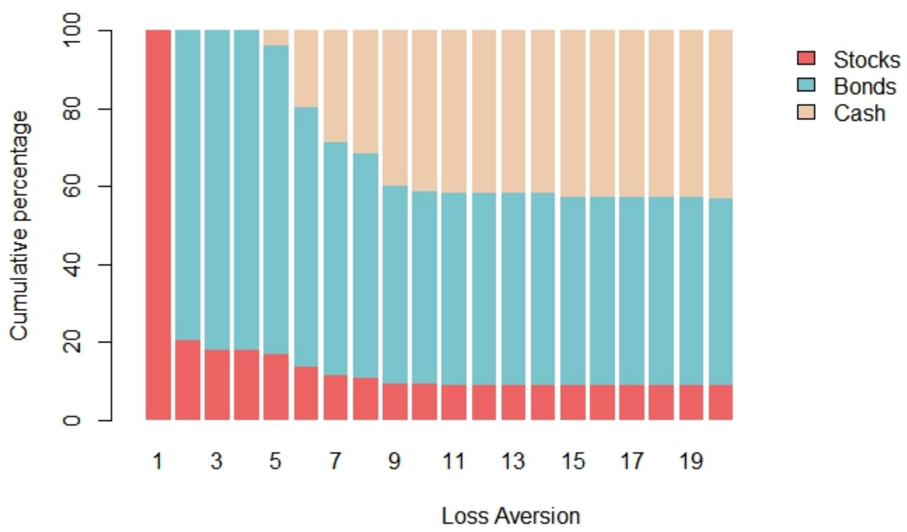

Figure 11: The optimal mean-LPM1 allocation assuming a fixed 1 year horizon, a target return of 4 %. In general, a riskier portfolio is associated with a lower loss aversion at the short 1 year horizon. For sufficiently high loss aversions cash is nonoptimal.

The portfolios in Figure 11 resemble the 1-year mean-LPM1 portfolios derived under the assumption of normally distributed returns in Figure 5. Indeed, investors with lower sensitivities to losses prefer riskier portfolios compared to more loss averse investors. Similarly, there is a difference between the portfolio corresponding to $\lambda = 1$ and the other portfolios in terms of riskiness, as the most loss acceptive investor prefers only stocks. Moreover, in terms of uniqueness, portfolios corresponding to the lowest levels of loss aversion are indistinguishable from another. A small difference relates to the cash proportion because the nonparametric portfolios include a relatively lower proportion of cash. Figure 12 depicts the optimal mean-LPM1 portfolios associated with the medium 4-year investment horizon at the 4 % target return.

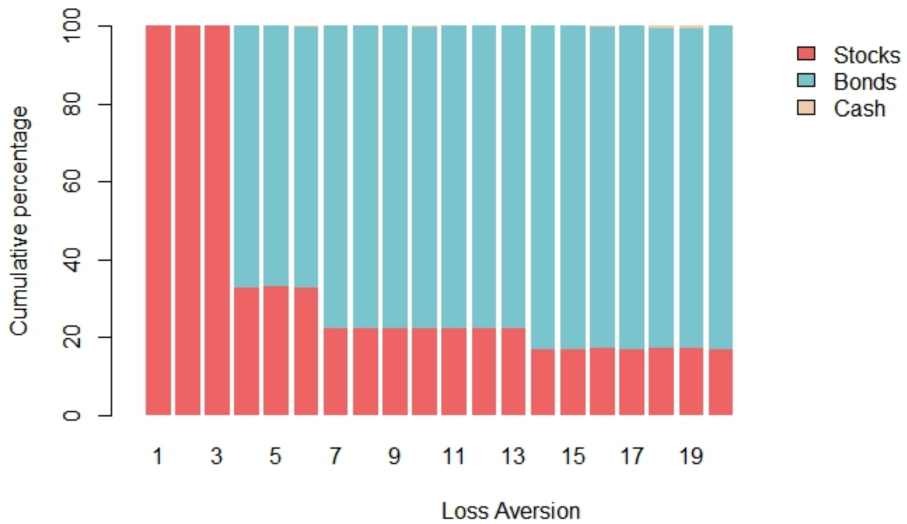

Figure 12: The optimal mean-LPM1 allocation assuming a fixed 1 year horizon, a target return of 4 %. Cash is a non-optimal asset at the 4 year horizon. In general, for decreasing loss aversions, the investors want to hold more stocks, or conversely, hold more bonds with increasing loss aversions.

Just as in the normal case, the 4-year portfolios in Figure 12 stand in sharp contrast to the corresponding 1-year portfolios in Figure 11. Clearly, a longer investment horizon prompt investors to channel their funds into riskier portfolios, and this is observed for highly loss averse investors as well. Viewed in isolation, the optimal 4-year portfolio affect the portfolio composition in the same way it did for the 1 year horizon: a lower loss aversions implies riskier portfolios and vice versa.

Figure 13 depicts the optimal portfolios over the long horizon of 7 years at the 4 % target return. Similar to the case for normally distributed returns, the 7-year portfolios include a higher proportion of stocks and a lower proportion of bonds. Referring back to the 1-year portfolios associated with the lowest loss aversions in Figure 5, the shift towards a riskier asset composition is even more evident. Again, investors prefer a higher allocation to stocks for longer horizons.

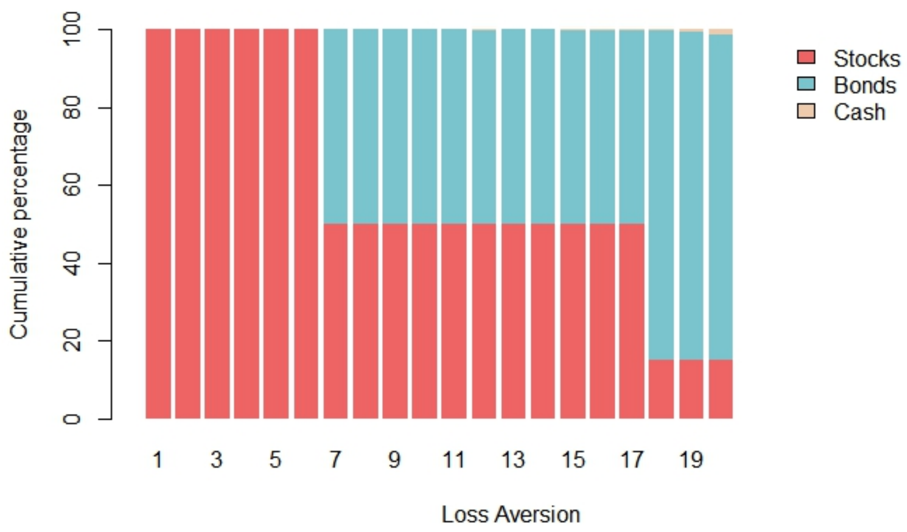

Figure 13: The optimal mean-LPM1 allocation assuming a fixed 1 year horizon, a target return of 4 %.

In comparison to the portfolios derived under the assumption of normally distributed returns, the "non parametric" portfolios are less "elegant" and looks to be almost noisy. However, this is not the case, and the differences can be explained. First, and foremost, the portfolios derived under the assumption of normally distributed returns employ the LPM measure given in equation (15). Secondly, to compute optimal portfolios for the nonparametric case, the classical dataset at the monthly frequency is transformed into a T-year frequency dataset by compounding returns, and this "new" dataset is used to compute T-year optimal mean-LPM_1 portfolios.

Figure 14 depicts the optimal bond-to-stock ratios over the short horizon of 1 year at the 4 % target return. Similarly to the normal case, the bond-to-stocks ratios are nearly identical for higher loss aversions at the 1 year horizon, reflecting the fact that the portfolios tend to become alike with increasing loss aversions. Moreover, regardless of the specific target return, the bond-to-stocks ratios falls with decreasing loss aversions. Across target returns, the familiar pattern we observed earlier continue to persist. That is, bond-to-stock ratios associated with higher target returns typically lie below ratios corresponding to lower target returns. We can interpret this observation similarly as we did under Figure 8. Figure 15 and 16 depicts the bond-to-stocks curves for the 4 and 7 year horizon, respectively. As we see from the figures, they are qualitatively similar to the ones derived under the normal case, that is to Figure 9 and Figure 10.

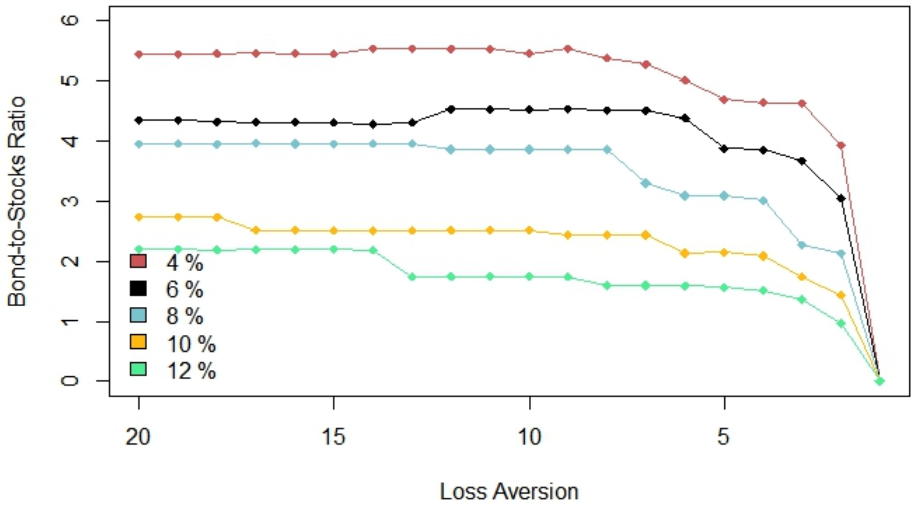

Figure 14: The optimal mean-LPM1 bond-to-stocks ratios given an investment horizon of 1 years, and target returns ranging from 4 % to 12 %.

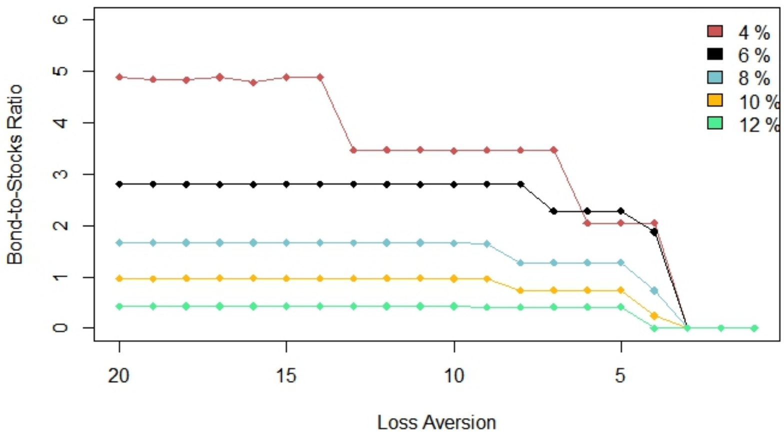

Figure 15: The optimal mean-LPM1 bond-to-stocks ratios given an investment horizon of 4 years, and target returns ranging from 4 % to 12 %.

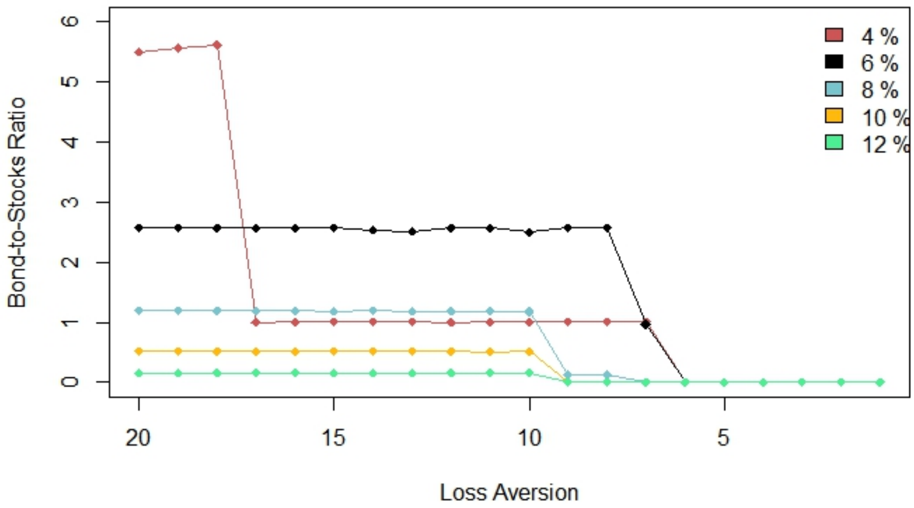

Figure 16: The optimal mean-LPM1 bond-to-stocks ratios given an investment horizon of 7 years, and target returns ranging from 4 % to 12 %.

In the end, without assuming a particular return distribution we have obtained comparable results with the mean-LPM_1 portfolios that were derived under the assumption normally distributed returns. Moreover, the nonparametric mean-LPM_1 portfolios are consistent with the financial advisers' portfolio recommendations. The advisers recommend a riskier portfolio allocation for lower risk aversions. Likewise, the hypothetical mean-LPM_1-investors prefer a riskier portfolio allocations for lower loss aversions. The advisers recommend a riskier portfolio allocation for longer time horizons. The mean-LPM_1-investors also prefer a riskier portfolio composition for longer holding periods. In addition, the mean-LPM_1-investors prefer riskier assets for higher target returns to have reasonable shot at achieving those targets.

5.2 Advanced Data

Allowing for a more realistic scenario, we test the mean-LPM1 model on a dataset containing several assets. The dataset consist of assets from Vanguard and Fidelity, and the 5-year Treasury Bill (^FVX)[7]. The assets can roughly be classified into two categories: stocks funds and bonds funds. The dataset consists of monthly prices, and the sample period stretches from January 1980 to October 2018 (468

[7]The unique Yahoo ticker refers to the fund in question, and we will refer to the assets by the ticker name.

monthly price observations).

Briefly on how Vanguard and Fidelity view the stocks funds[8]:

Vanguard Explorer Fund Investor Class (VEXPX) includes mainly small and mid-sized companies and aims for high growth. Vanguard Windsor Fund Investor Shares (VWNDX) favour large-cap stocks and aims for value. Vanguard International Growth Fund Investor Shares (VWIGX) focus on foreign stocks with high growth potential. Vanguard advise investors to have a long investment horizons and high tolerance for risk when investing in stocks. Fidelity Value Fund (FDVLX) aims for value, and Fidelity considers the fund to be of high risk.

Briefly on how Vanguard views the stocks funds:

Bonds: Vanguard Intermediate-Term Tax-Exempt Fund Investor Shares (VWITX) objective is to provide fixed income to investors. It consists of quality U.S. municipal bonds with long-term maturities of 5-6 years and carry little risk. Vanguard Long-Term Tax-Exempt Fund Investor Shares (VWLTX) is similar to VWITX, but it consists of long term bonds with maturities of 6-10 years. Vanguard High-Yield Corporate Fund Investor Shares (VWEHX) consists of high corporate "junk bonds", perceived by Vanguard to be risky but less than stocks. The 5 year Treasury-Bill (^FVX) provides fixed income to investors at the maturity date.

Table 2 gives the annual summary statistics for the assets. Table 3 depicts the annual correlations between the assets, while Figure 17 displays the cumulative log returns for the assets over the sample-period.

[8]In Appendix C, the weblinks to the assets can be found.

	Mean	Std.Dev	Skew	Kurtosis	Max	Min	Value
VWNDX	0.12	0.17	-0.81	1.74	0.45	-0.44	34.75
FDVLX	0.13	0.19	-0.61	1.94	0.57	-0.49	49.02
VMRGX	0.11	0.17	-0.65	0.50	0.39	-0.39	32.48
VEXPX	0.11	0.22	0.17	-0.19	0.58	-0.39	29.76
^FVX	0.00	0.28	0.28	-0.06	0.71	-0.64	0.27
VWEHX	0.09	0.10	0.47	0.72	0.33	-0.16	22.64
VWLTX	0.06	0.09	0.62	2.68	0.38	-0.15	9.19
VWITX	0.06	0.07	0.79	2.65	0.30	-0.08	8.82

Table 2: Annual return statistics for the funds given return data from November 1981 to November 2011. Except for the 5 year T-Bill, the stock funds are historically more volatile than the bonds funds given the standard deviation and the max-min spread. However, the stock funds offer more reward in terms of the historical mean. Over the sample period, the 5 year T-Bill underperformed the assets significantly as indicated by the summary statistics; A dollar yielded in the T-Bill yielded -73 % return over the period.

	VWNDX	FDVLX	VMRGX	VEXPX	^FVX	VWLTX	VWEHX	VWITX
VWNDX	1.00	0.89	0.78	0.74	0.29	0.72	0.29	0.20
FDVLX	0.89	1.00	0.77	0.84	0.24	0.75	0.33	0.24
VMRGX	0.78	0.77	1.00	0.82	0.27	0.57	0.22	0.16
VEXPX	0.74	0.84	0.82	1.00	0.28	0.63	0.19	0.15
^FVX	0.29	0.24	0.27	0.28	1.00	-0.09	-0.53	-0.57
VWLTX	0.72	0.75	0.57	0.63	-0.09	1.00	0.58	0.55
VWEHX	0.29	0.33	0.22	0.19	-0.53	0.58	1.00	0.98
VWITX	0.20	0.24	0.16	0.15	-0.57	0.55	0.98	1.00

Table 3: Stock funds are highly positively correlated, the bond funds are also highly correlated. Across assets we observe some weak-to medium positive correlation.

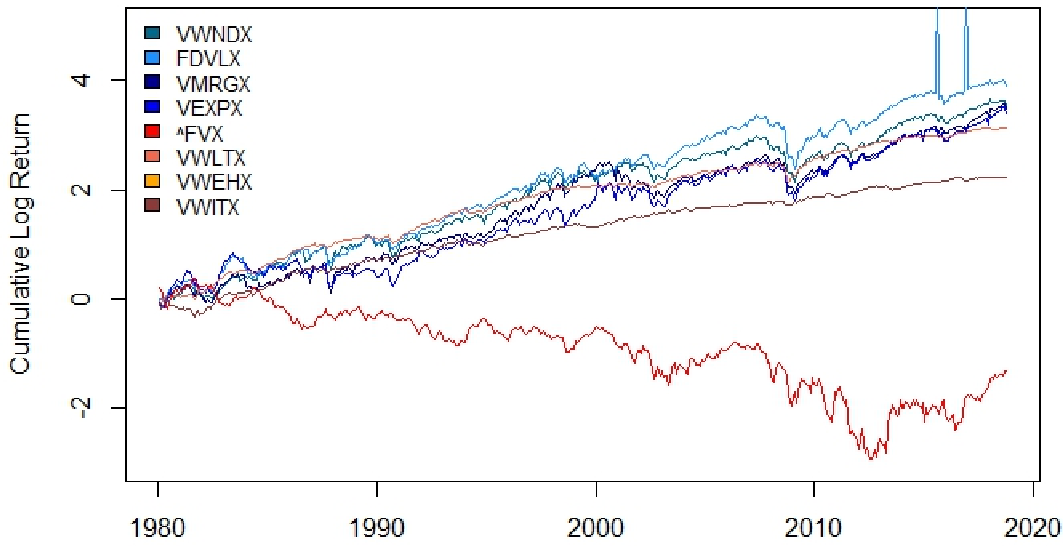

Figure 17: The cumulative log returns for the stock and bonds funds given the sample period. The stock assets move together. The irregularities in FDVLX is evident around 2010. I haven't been able to find out the reason for the spikes. Moreover, the 5-year T-bill significantly underperformed the other assets.

The 5 year T-bill underperformed the other assets by significant margins. Regarding the FDVLX stock fund, we observe two irregularities after 2010. It is a possibility that these irregularities will affect the optimal mean-LPM1 portfolios.

5.2.1 Normally Distributed Returns

The assets are coded either red or blue colours depending on the asset's security composition. Blue colours are reserved stock funds, while red colours specify bond funds. Within a colour group, the individual colours are only tags to identify the assets. Figure 18 depicts the optimal mean-LPM1 portfolios associated with a short 1-year investment horizon at the 4 % target assuming normally distributed returns.

Figure 18: The optimal mean-LPM1 allocation assuming a 1 year horizon, a target return of 4 %, and normally distributed portfolio returns. The total stock allocation is decreasing for increasing aversions to losses, but it is decreasing at a diminishing rate. The total bond allocation is increasing, albeit at a diminishing rate.

In terms of groups, the portfolios become perceivably riskier with decreasing loss aversions and vice versa. There is a notable difference between the portfolio associated with $\lambda = 1$ and the other portfolios in terms of riskiness. The portfolio with $\lambda = 1$ include stocks only, while the other portfolios include at most 20 % stocks. Accordingly, the investor most acceptive of losses prefers only stocks over the short horizon. If we allow VWITX as to play the "cash role", then cash intensive portfolios correspond to the investors most perceptive to losses. Indeed, Vanguard's VWITX appear to be the least risky asset; furthermore, it is the least risky asset by the conventional measures reported in Table 2. By viewing the assets in groups, Figure 18 is strikingly similar to the figures derived under the same 1 year configuration in Section 5.1.1 and Section 5.1.2—Figure 5 and 11, respectively.

It is important to mention that none of the portfolios include VWEHX—the corporate junk bond fund. Referring back to the Correlation Matrix 3, a correlation of 0.98 is observed between VWEHX and VWITX, and the conventional risk measures in Table 2 reports that VWEHX is riskier than VWEHX. Accordingly, the implication is that VWEHX is a redundant asset if VWITX belongs to the asset universe.

Instead of viewing the portfolios in terms of groups, we can consider them individually. Figure 18 reveals that the investors channel their funds into different types of assets, and in a way that corresponds to their loss aversion it seems. Even the investor most acceptive of losses, $\lambda = 1$, channel his funds into different stock assets; in fact, he attains a positive proportion in every stock fund available to him at the 1 year horizon. However, he seems to have a preference for Vanguard's VWNDX, followed by VEXPX, VMRGX, and Fidelity's FDVLX, in that order at the 1 year horizon. At the other extreme, the most loss perceptive investor diversify his funds primarily among the safer assets, with an even split between VWLTX and VWITX.

Figure 19 depicts the optimal mean-LPM1 portfolios associated with the medium 4-year investment. In comparison to the 1-year portfolios, the longer time horizon have shifted the portfolios towards riskier compositions. Stocks become more attractive at the expense of bonds across all portfolios, at least for portfolios corresponding to lower loss aversions. This is a result that conforms to our earlier findings.

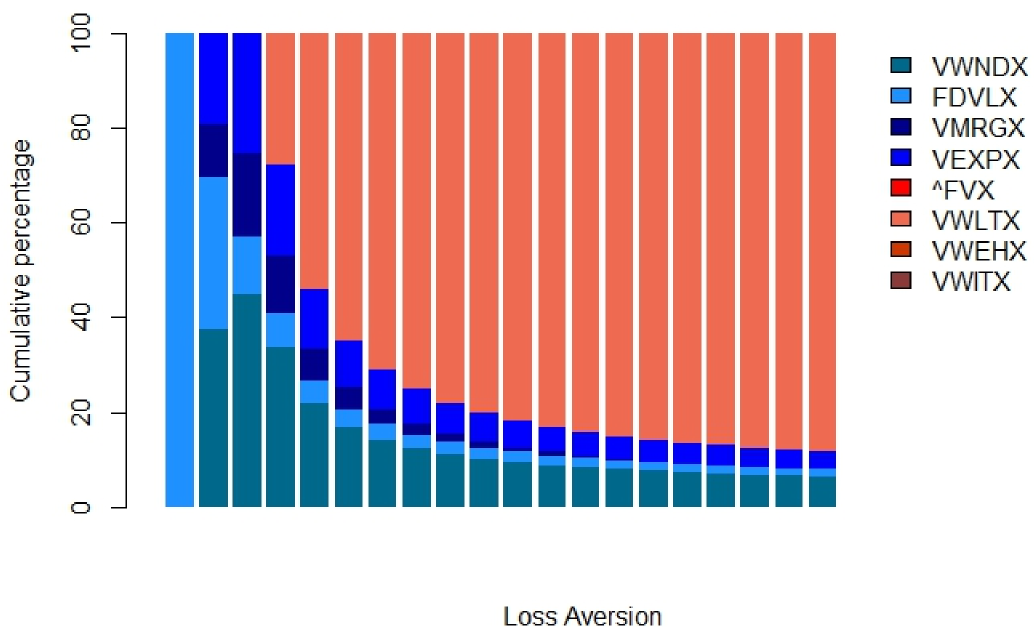

Figure 19: The optimal mean-LPM1 allocation assuming a 4 year horizon, a target return of 4 %, and normally distributed portfolio returns. Viewed in groups, the stock proportion is decreasing for increasing aversions to losses, but it is decreasing at a diminishing rate. Therefore, the bond allocation is increasing at a diminishing rate.

If we view the assets individually, then portfolios corresponding to lower levels of loss aversion also experience a dramatic shift towards riskier asset compositions. Indeed, VWITX the arguably least risky bond fund is now absent, whereas it was a significant asset over the short 1 year horizon. The longer horizon has also prompted a dramatic shift in the stock allocation, and a stock hierarchy seems to emerge. The most loss acceptive investor only prefer Fidelity's FDVLX.

Figure 20 depicts the optimal mean-LPM1 portfolios associated with a short 7-year investment horizon at the 4 % target assuming normally distributed returns.

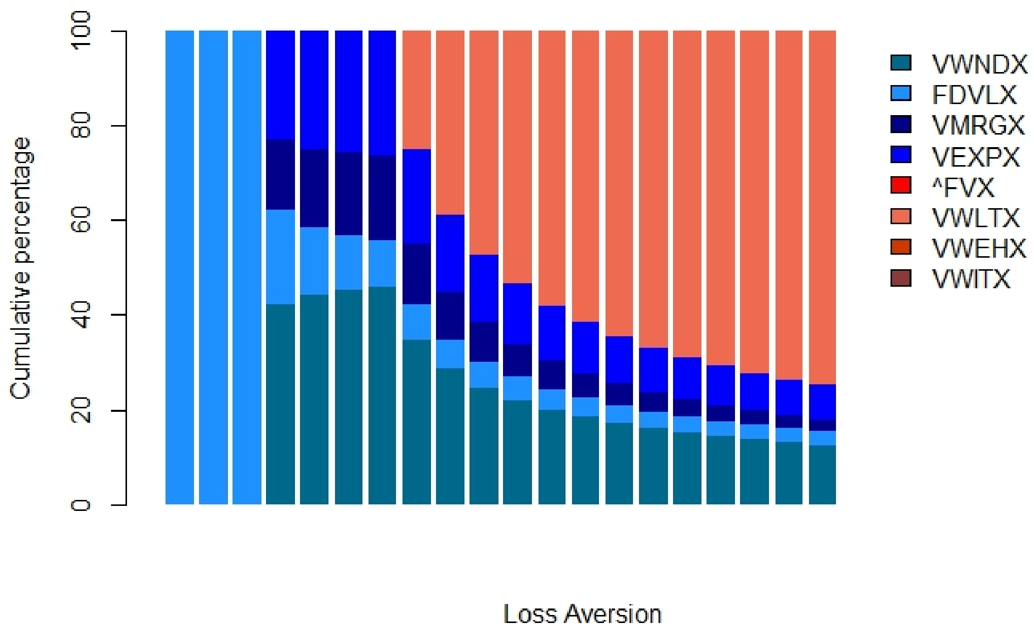

Figure 20: The optimal mean-LPM1 allocation assuming a 1 year horizon, a target return of 4 %, and normally distributed portfolio returns. Viewed in groups, the stock proportion is decreasing for increasing aversions to losses, but it is decreasing at a diminishing rate. Therefore, the bond allocation is increasing at a diminishing rate. In terms of por

If we view the assets in groups, then all the previous results apply. More interestingly, if we view the assets individually, the stock hierarchy implied in Figure 19 becomes even clearer. It is Fidelity's FDVLX, the value fund, that is provides the highest expected utility for the most loss acceptive investors. And it can be shown, given this particular dataset, that for long enough horizons, every investor hold purely FDVLX.

5.2.2 Nonparametric Approach

In this section we study the optimal asset allocation under no particular distribution. The primary goal is to determine the similarities and differences with the portfolios derived under the assumption of normality.

Figure 21 depicts the optimal mean-LPM1 portfolios associated with a short 1-year investment horizon at the 4 % target. Figure 21 is strikingly similar to Figure 18 both when we view the assets in groups and when viewed individually. Figure 22 depicts the optimal mean-LPM1 portfolios associated with a short 4-year investment. Although the portfolios in Figure 22 looks less promising than for the corresponding normality case in Figure 19, the investment horizon has clearly persuaded the investors to hold more stocks. Furthermore, FDVLX is the prefered asset as we previously found. We do not display the portfolios for the 7 year horizon because every investor hold FDVLX stocks at that horizon.

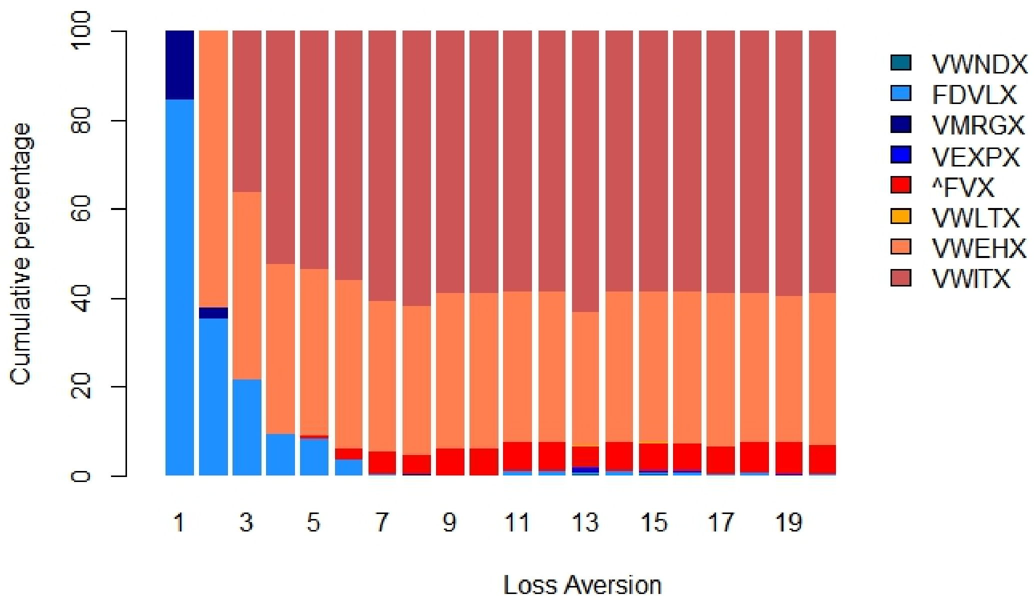

Figure 21: The optimal mean-LPM1 allocation assuming a 1 year horizon, a target return of 4 %.

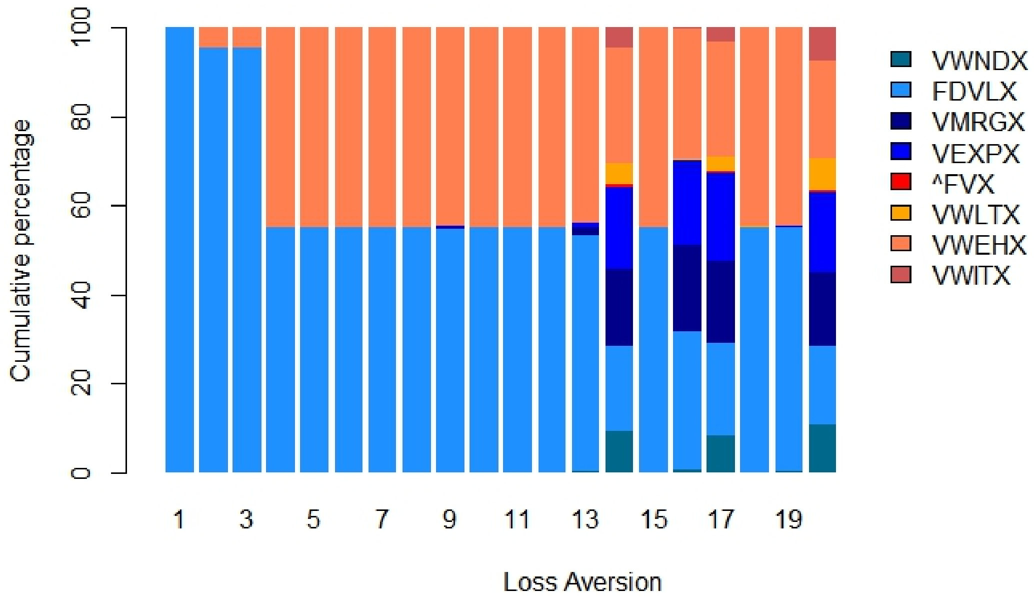

Figure 22: The optimal mean-LPM1 allocation assuming a 4 year horizon, a target return of 4 %.

5.3 Empirical Conclusion

This thesis has investigated the mean-LPM_1 model on two datasets under various configurations of the loss aversion, target return, and investment horizon. Furthermore, we have examined the model both in the presence and absence of the assumption of normally distributed portfolio returns. The main conclusion from the empirical section is that the mean-LPM_1 model produces portfolios that are qualitatively similar with the financial advise, regardless of the underlying return distribution, and specific parameter configurations.

In terms of the relationship between the portfolio allocation and the loss aversion, we observe that lower loss aversions corresponds to relatively riskier portfolios, a result observed over several configurations of the investment horizon and the target return.

In terms of the relationship between the portfolio allocation and the investment horizon, we obtain consistent results that a longer investment horizon induces a relatively riskier portfolio composition, a result that remains valid across loss aversions and target returns.

Referring back to our results, a higher target return consistently implies a riskier portfolio allocation, regardless of the investors loss aversion. Moreover, a higher target return implies a riskier portfolio over longer investment horizons as well. However, for long enough horizons, it can be shown, for both datasets, that every one hold stocks, subsequently ignoring the loss aversion and target return's effect on the portfolio composition.

With respect to the advanced dataset of many assets, we observe that investors diversify logically between assets, that is, in a way that corresponds to their loss aversion.

6 Evaluation of Model Assumptions and Discussion Pertaining Future Research

The results of the empirical section must be critically examined in light of the assumptions that govern the analysis. The most important assumption to consider is the assumption that investors base their portfolio decisions on the mean-LPM_1 model. Kahneman and Tversky (1979) obtain empirical evidence that people are risk averse in the gains domain and risk-seeking in the loss domain. However, the hypothetical mean-LPM_1 investor considered in this thesis is neither; he is risk-neutral over the domain as implied by the utility function in equation (18). Accordingly, the mean-LPM_1 model fails to implement key investor characteristics. This is arguably the model's greatest weakness. The mean-LPM_1 does not account for the investor's upside ambitions. To quote (Cumova, 2004) "...in the (μ, LPM)-portfolio model returns above the target are only input in the computation of the mean which implies (as often criticized in the literature) neutrality towards the chance of over-performing the minimal aspiration return". Indeed, the mean return does not separate favourable returns from unfavourable returns. A reward measure that focus solely on reward is the Upper Partial Moments Measure (UPM) Cumova (2004). It is conceptually similar to the LPM as it captures upward deviations from a subjective reference point. Cumova (2004) calls the (UPM, LPM)-model a logical progression over the (μ, LPM)-model. The behaviour model of De Giorgi et al., 2008 also facilitates a clean separation of risk and reward. Furthermore, it is consistent with investors who are risk-seeking in the loss domain and risk averse in the gain domain. To my knowledge, De Giorgi's (2010) behavioural model and the (UPM, LPM) model have yet to be applied over different investment horizons. Hence, to apply these models over longer investment horizons is perhaps the logical progression for future research.

The analysis does not allow for short positions in the asset. The reason why we exclude this option is that neither Vanguard, Fidelity, or the advisers in Canner et al. (1994) recommend short positions to their clients. Accordingly, if this assumption was to be relaxed, it would defeat the research purpose.

In this thesis, we have intentionally ignored Fisher and Statman's (1999) factors—factors not necessarily related to risk—that might affect the portfolio decision. Implementing social responsibility and cognitive errors would be very interesting, but it is far from obvious how one would proceed to combine these factors with a specific reward-risk model. Perhaps, the reward-risk model must be abandoned in its entirety to allow for a truer analysis.

Bennyhoff (2009) reasons that human capital can provide incentives for younger investors to pursue more aggressive portfolios, whereas this thesis assume that future wealth is a function of returns only. Indeed, this assumption is in direct violation against the realities of the world. However, it was only imposed to focus on the main theme of the thesis. Future research that include human capital is welcomed.

Throughout the thesis we have assumed a static investment horizon. That is, the investor buys the portfolio at time $t = 0$ and remains passive until he or she liquidate it at time $t = T$. Is this assumption reasonable? It depends on the type of investor and his underlying motives, an assumption we dodged so far allow for a broader analysis. However, if we consider retirement investors, then a fixed investment horizon seems reasonable. In the U.S. for example, withdrawing early amounts from the 401(k) retirement account is costly and can result in a 10 % distribution penalty tax on top of other early withdrawal costs report Ely (2017). Furthermore, it is costly to trade according to Odean, 1999 who finds that excessive trading results in lower profits. But a dynamic approach to the puzzles with a risk-reward model rooted in behavioural finance would be interesting to witness.

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8 Appendix

8.1 Appendix A: Mean-LPMn Program—Normal Returns

```
rm(list=ls(all=TRUE))
```

```
# CTRAN Packages:
library(tseries) # For get.hist.qoute function.
library(Rsolnp) # General nonlinear programming package.

norm.solver <- function(r, kappa, t, lambda, n) {
  # This function computes the optimal Mean-LPMn weights
  # for n=1, and n=2 under the assumption of normally
  # distributed returns. Here: lambda=loss aversion, t=horizon,
  # r=return data, and kappa=target return.
  nobs <- nrow(r) # The number of observations of returns.
  nAssets <- ncol(r) # The number of assets.

  MLPM <- function(pars) {
    # Objective function that computes the Mean-LPMn utility.
    # pars - is the vector of portfolio weights.
    kappa <- kappa*t # T-year target return.
    rp <- as.numeric(r%*%pars)
    mu <- mean(rp)*12 # The annual mean return.
    mup <- mu*t # The mean return over a T-year horizon.
    covmat <- cov(r)*12 # Covariance matrix.
    sigp <- sqrt(pars%*%covmat%*%pars)*sqrt(t) # The portfolio std.
    d <- (kappa-mup)/sigp
    if(n==1){ # LPM1
      lpm <- (kappa-mup)*pnorm(d) + sigp*exp(-d^2/2)/sqrt(2*pi)
    } else{ # LPM2
      lpm <- (((kappa-mu)^(2)+sigp^(2))*pnorm(d)
              + sigp*(kappa-mu)*exp(-d^2/2)/sqrt(2*pi))
    }
    eu <- mup - lambda*(mean(lpm))^(1/n) # Objective function.
```

```
  return(-eu) # With minus sign because we minimize the objective function.
}

# Equality constraints are implemented as functions.
eqfun <- function(pars) {
  return(sum(pars))
}
#Vvector for the equality constraints.
eqB <- 1 # sum of all weights

# Inequality constraints are implemented as functions.
ineqfun <- function(pars) {
  # inequality constraints: portfolio weights.
  return(pars)
}

ineqLB <- rep(0, nAssets) # Lower bound for inequality constraints.
ineqUB <- rep(1, nAssets) # upper bound for inequality constraints.

pars <- rep(1/nAssets,nAssets) # intial weights - equally weighted portfolio.

# Solve for the optimal weights.
res <- solnp(pars=pars, fun=MLPM,
             eqfun=eqfun, eqB=eqB,
             ineqfun=ineqfun, ineqLB=ineqLB, ineqUB=ineqUB)

opt.w <- round(res$pars, digits=4) # Round to 4 decimal digits.

return(opt.w)
```

```
}

norm.par <- function(r, t, kappa, n=1) {
  # This function computes the optimal weights for different
  # loss aversions and target returns under the
  # assumption of normally distributed portfolio returns.
  lambda <- 1:20 # Loss aversions.
  L <- length(lambda)
  N <- ncol(r)
  op <- matrix(0, L, N) # Return matrix.

  for(i in 1:L) { # Solve for each loss aversion.
    op[i,] <- norm.solver(r, kappa, t, lambda[i], n)
  }
  return(op)
}

# The Classical Dataset.
# Source: Ibbotson's SBBI 2012 Classic Yearbook.
# Assets : 1-month T-Bill, large-cap stock index,
# and an index of long-term government bonds.
# Unit: Arithmetic returns.
# Frequency: Monthly.
# Time Period: 1927-01-01 to 2011-12-01.
# Assumption: Normally distributed returns.


# Read the data from the directory.
data <- read.table("marketdata.txt", header = TRUE)
r <- as.matrix(data[,2:4])
```

```
dates <- data[,1]


# Example: The optimal Mean-LPM1 weights over 1 year horizon
# at a target return of 4 % assuming normally distributed returns.
# Depicted in a barplot.
x <- norm.par(r, t=1, kappa=0.04, n=1)*100 # Optimal weights.
names <- c("Stocks","Bonds","Cash")
colnames(x) <- names
rownames(x) <- 1:20
colz <- c("indianred2","cadetblue3","peachpuff2")
barplot(t(x), xlim=c(0,30), ylim =c(0,100), border=NA,
        xlab="Loss Aversion",
        ylab="Cumulative percentage", col=colz)
legend("topright", names, bty="n", fill=colz)


# Example: The optimal Mean-LPM1 bond-to-stocks ratios at the 1 year
# horizon over different target returns k.
k.4 <- norm.par(r, 1, kappa=0.04); bs.4 <- k.4[,2]/k.4[,1]
k.6 <- norm.par(r, 1, kappa=0.06); bs.6 <- k.6[,2]/k.6[,1]
k.8 <- norm.par(r, 1, kappa=0.08); bs.8 <- k.8[,2]/k.8[,1]
k.10 <- norm.par(r, 1, kappa=0.10); bs.10 <- k.10[,2]/k.10[,1]
k.12 <- norm.par(r, 1, kappa=0.12); bs.12 <- k.12[,2]/k.12[,1]

# Plot the bond-to-stocks curves.
plot(bs.4, type="o", xlim=c(20,1), ylim=c(0,5),
     ylab="Bond-to-Stocks Ratio", xlab="Loss Aversion",
     col="indianred3", pch=18)
lines(bs.6, type="o", col="black", pch=18)
```

```
lines(bs.8, type="o", col="cadetblue3", pch=18)
lines(bs.10, type="o", col="darkgoldenrod1", pch=18)
lines(bs.12, type="o", col="seagreen2", pch=18)
legend("bottomleft",c("4 %","6 %","8 %","10 %","12 %")
       ,fill=c("indianred3","black","cadetblue3",
               "darkgoldenrod1","seagreen2"), bty="n")

############################################################################

# The Advanced Dataset.
# Source: Vanguard and Fidelity funds and the 5 year T-Bill.
# Assets (Yahoo tickers): VWNDX, FDVLX, VMRGX, VEXPX, ^FVX,
# VWLTX, VWEHX, VWITX.
# Unit: $ Prices.
# Frequency: Monthly.
# Time Period: 1980-01-01 to 2018-10-01.
# Assumption: Normally distributed returns.

# Read the data from Yahoo Finance.
Symbol <- c("VWNDX","FDVLX","VMRGX","VEXPX","^FVX","VWLTX","VWEHX","VWITX")
nSymbols <- length(Symbol)
for(i in 1:nSymbols) {
  new.symbol = get.hist.quote(instrument= Symbol[i],
                              start = "1980-01-01",
                              end = "2018-11-01",
                              quote="AdjClose",
                              provider = "yahoo",
                              origin="1980-01-01",
                              compression = "m",
                              retclass="zoo")
```

```
  names(new.symbol) = Symbol[i]
  if(i==1) data = new.symbol else data = merge(data, new.symbol)
}

prices <- coredata(data)
n <- nrow(prices)

r<- apply(prices, 2, FUN=function(x){ # Convert to simple returns.
  x[2:n]/x[1:(n-1)]-1
})


# Example: The optimal portfolios at the 1 year horizon for a target return
# of 4 % assuming normally distributed returns. Depicted in a barplot.
weights <- norm.par(r, t=1, kappa=0.04, n=1)*100
rownames(weights) <- 1:20
names <- c("VWNDX","FDVLX","VMRGX","VEXPX","^FVX","VWLTX","VWEHX","VWITX")
colnames(weights) <- names # Assign names to the variables.
colz <- c("deepskyblue4","dodgerblue","darkblue","blue","red",
          "orange", "coral", "indianred3") # Assign colours.
barplot(t(weights), xlim=c(0,30), ylim =c(0,100), border=NA,
        xlab="Loss Aversion", ylab="Cumulative percentage", col=colz)
legend("topright", names, bty="n", fill=colz)


# Example: Bond-to-stocks ratio at the 1 year horizon for a target return
# of 4 % assuming normally distributed returns. Aggregating the bonds
# together and the stocks together.
weights <- norm.par(r, t=1, kappa=0.04, n=1)
stocks <- apply(weights[,1:4], 1, FUN=sum) # Total stocks weight.
```

```
bonds <- apply(weights[,5:8], 1, FUN=sum) # Total bonds weight.
bond.to.stocks <- bonds/stocks
plot(bond.to.stocks, type="o", xlim=c(20,1),
     ylab="Bond-to-Stocks Ratio", xlab="Loss Aversion",
     col="indianred3", pch=18)


# Example: The total stock allocation as a function of the loss aversion
# given a 1 year horizon and a 4 % target return under the assumption of
# normally distributed returns.
weights <- norm.par(r, t=1, kappa=0.04, n=1)*100
rownames(weights) <- 1:20
stocks <- apply(weights[,1:4], 1, FUN=sum) # Total stocks weight.
barplot(t(stocks), xlim=c(0,30), ylim =c(0,100), border=NA,
        xlab="Loss Aversion",
        ylab="Cumulative percentage", col=colz)
legend("topright", "Total Stock Allocation", bty="n", fill=colz)
```

8.2 Appendix B: Mean-LPMn Program—Nonparametric

```
rm(list=ls(all=TRUE))

# CRAN Packages:
library(tseries) # For get.hist.qoute function.
library(Rsolnp) # General nonlinear programming package.


accu.ret <- function(r, n) {
  # This function accumulates the one-period return in "r" to n-period returns.
```

```
  # r is a matrix where each column contains the returns to an asset.
  nobs = nrow(r)
  N = ncol(r)
  x = log(1 + r) # convert to log returns.
  k = as.integer((nobs/n)) # number of accumulated returns.
  y = matrix(0, nrow=k, ncol=N) # vector to hold accumulated returns.
  for(j in 1:N) {
    for(i in 1:k) {
      start = (i-1)*n + 1 # start index.
      end = i*n # end index.
      y[i,j] = sum(x[start:end,j])
    }
  }
  return(exp(y) - 1.0)
}

par.solver <- function(r, kappa, lambda, n) {
  # This function computes the optimal Mean-LPMn weights
  # for n=1, and n=2. Here: lambda=loss aversion, t=horizon,
  # r=return data, and kappa=target return.
  nobs <- nrow(r) # The number of observations of returns.
  nAssets <- ncol(r) # The number of assets.
  er <- apply(r, 2, mean) # Vector of mean returns.

  # Define custom objective function that computes the Mean-LPM1 utility.
  MLPM <- function(pars) {
    # pars - is the vector of portfolio weights.
    R <- as.numeric(r %*% pars) # Portfolio return.
    lpm <- pmax(kappa-R,0)
    eu <- mean(R) - lambda*(mean(lpm))^(1/n) # Objective function.
```

```
    return(-eu) # With minus sign because we minimize the function.
  }

  # Equality constraints are implemented as functions.
  eqfun <- function(pars) {
    return(sum(pars))
  }
  #Vvector for the equality constraints.
  eqB <- 1 # sum of all weights

  # Inequality constraints are implemented as functions.
  ineqfun <- function(pars) {
    # inequality constraints: portfolio weights.
    return(pars)
  }
  # Vectors that define lower and upper bounds for the inequality constraints.
  ineqLB <- rep(0, nAssets) # Lower bound for inequality constraints.
  ineqUB <- rep(1, nAssets) # upper bound for inequality constraints.

  pars <- rep(1/nAssets, nAssets) # intial weights - equally weighted portfolio.

  res <- solnp(pars=pars, fun=MLPM, eqfun=eqfun, eqB=eqB,
               ineqfun=ineqfun, ineqLB=ineqLB, ineqUB=ineqUB)

  opt.w <- round(res$pars, digits=4)

  return(opt.w)

}
```

```
non.par <- function(R, t, kappa, n=1) {
  # This function computes the optimal weights for different
  # loss aversions and target returns.
  R <- accu.ret(R, (12*t)) # Transform to t-year frequency.

  lambda <- 1:20  # Loss aversions.
  L <- length(lambda)
  N <- ncol(r)

  kappa <- kappa*t # T-year target return.
  op <- matrix(0, L, N) # Matrix of optimal weights, the return object.

  # Generation of optimal weights for each lambda.
  for(i in 1:L) {
    op[i,] <- par.solver(R, kappa, lambda[i], n) # solnp weights
  }
  return(op)
}

# Read the classical dataset fro the directory,
data <- read.table("marketdata.txt", header = TRUE)
r <- as.matrix(data[,2:4])

# Example: The optimal Mean-LPM1 weights over 1 year horizon
# at a target return of 4 % assuming normally distributed returns.
# Depicted in a barplot.
x <- non.par(r, t=1, kappa=0.04, n=1)*100 # Optimal weights.
names <- c("Stocks","Bonds","Cash")
colnames(x) <- names
rownames(x) <- 1:20
```

```
colz <- c("indianred2","cadetblue3","peachpuff2")
barplot(t(x), xlim=c(0,30), ylim =c(0,100), border=NA,
        xlab="Loss Aversion",
        ylab="Cumulative percentage", col=colz)
legend("topright", names, bty="n", fill=colz)



# Example: The optimal Mean-LPM1 bond-to-stocks ratios at the 1 year
# horizon over different target returns k.
k.4 <- non.par(r, 1, kappa=0.04); bs.4 <- k.4[,2]/k.4[,1]
k.6 <- non.par(r, 1, kappa=0.06); bs.6 <- k.6[,2]/k.6[,1]
k.8 <- non.par(r, 1, kappa=0.08); bs.8 <- k.8[,2]/k.8[,1]
k.10 <- non.par(r, 1, kappa=0.10); bs.10 <- k.10[,2]/k.10[,1]
k.12 <- non.par(r, 1, kappa=0.12); bs.12 <- k.12[,2]/k.12[,1]

# Plot the bond-to-stocks curves.
plot(bs.4, type="o", xlim=c(20,1), ylim=c(0,6),
     ylab="Bond-to-Stocks Ratio", xlab="Loss Aversion",
     col="indianred3", pch=18)
lines(bs.6, type="o", col="black", pch=18)
lines(bs.8, type="o", col="cadetblue3", pch=18)
lines(bs.10, type="o", col="darkgoldenrod1", pch=18)
lines(bs.12, type="o", col="seagreen2", pch=18)
legend("bottomleft",c("4 %","6 %","8 %","10 %","12 %")
       ,fill=c("indianred3","black","cadetblue3",
               "darkgoldenrod1","seagreen2"), bty="n")

############################################################################
```

```
# Read the advanced dataset from Yahoo Finance.
Symbol <- c("VWNDX","FDVLX","VMRGX","VEXPX","^FVX","VWLTX","VWEHX","VWITX")
nSymbols <- length(Symbol)
for(i in 1:nSymbols) {
  new.symbol = get.hist.quote(instrument= Symbol[i],
                              start = "1980-01-01",
                              end = "2018-11-01",
                              quote="AdjClose",
                              provider = "yahoo",
                              origin="1980-01-01",
                              compression = "m",
                              retclass="zoo")
  names(new.symbol) = Symbol[i]
  if(i==1) data = new.symbol else data = merge(data, new.symbol)
}

prices <- coredata(data)
n <- nrow(prices)

r<- apply(prices, 2, FUN=function(x){ # Convert to simple returns.
  x[2:n]/x[1:(n-1)]-1
})



# Example: The optimal portfolios at the 1 year horizon for a target return
# of 4 % assuming normally distributed returns. Depicted in a barplot.
weights <- non.par(r, t=4, kappa=0.04, n=1)*100
rownames(weights) <- 1:20
names <- c("VWNDX","FDVLX","VMRGX","VEXPX","^FVX","VWLTX","VWEHX","VWITX")
colnames(weights) <- names # Assign names to the variables.
```

```
colz <- c("deepskyblue4","dodgerblue","darkblue","blue","red",
          "orange", "coral", "indianred3") # Assign colours.
barplot(t(weights), xlim=c(0,30), ylim =c(0,100), border=NA,
        xlab="Loss Aversion", ylab="Cumulative percentage", col=colz)
legend("topright", names, bty="n", fill=colz)



# Example: The total stock allocation as a function of the loss aversion
# given a 1 year horizon and a 4 % target return.
weights <- non.par(r, t=1, kappa=0.04, n=1)*100
rownames(weights) <- 1:20
stocks <- apply(weights[,1:4], 1, FUN=sum) # Total stocks weight.
barplot(t(stocks), xlim=c(0,30), ylim =c(0,100), border=NA,
        xlab="Loss Aversion",
        ylab="Cumulative percentage", col=colz)
legend("topright", "Total Stock Allocation", bty="n", fill=colz)
```

8.3 Appendix C: Mean-Variance Program

```
rm(list=ls())

# CTRAN:
library(quadprog) # Quadratic optimization program.

eu.weights <- function(R, shorts=TRUE) {
  # Optimal weights by maximization of expected utility.
  er <- apply(R, 2, mean) # Asset mean returns.
  covmat <- cov(R) # Covariance matrix.
  n <- nrow(covmat)
  Dmat <- covmat
```

```
A <- 1:200 # Risk Aversion Parameters.
K <- length(A)
w <- matrix(0, K, n) # The output of the function.

# Compute optimal weights when shorting is allowed.
if(shorts==TRUE) {
  for(i in 1:K) {
    dvec <- er/A[i]
    Amat <- cbind(rep(-1,n))
    bvec <- -1
    # Quadratic optimization.
    result <- solve.QP(Dmat=Dmat,
                       dvec=dvec,Amat=Amat,bvec=bvec,meq=0)
    w[i,] <- round(result$solution, 6)
    if (!all(w[i,] == 0)) w[i,] <- w[i,]/sum(w[i,])
  }
  # Compute optimal weights when shorting is unallowed.
}else {
  for(i in 1:K) {
    dvec <- er/A[i]
    Amat <- cbind(rep(-1,n), diag(1,n))
    bvec <- c(-1, rep(0,n))
    # Solve the quadratic optimization problem.
    result <- solve.QP(Dmat=Dmat,
                       dvec=dvec,Amat=Amat,bvec=bvec,meq=0)
    w[i,] <- round(result$solution, 6)
    if (!all(w[i,] == 0)) w[i,] <- w[i,]/sum(w[i,])
  }
}
return(w) # Optimal portfolio weights.
```

```
}

# Load the classical dataset.
data <- read.table("marketdata.txt", header = TRUE)
r <- as.matrix(data[,2:4])

# Compute the optimal mean-var weights for risk aversions
# ranging from A=1,...,200 with short constraint.
op.weights <- eu.weights(r, shorts = TRUE)
x <- nrow(op.weights)
names <- c("Stocks","Bonds","Cash")

# Stocks and bonds recommendations from Fidelity and Vanguard.
# for different levels of risk aversions as of 24/11/2018.
# https://investor.vanguard.com/mutual-funds/list#/mutual-funds/asset-class/month-end-ret
# https://www.fidelity.com/mutual-funds/fidelity-fund-portfolios/overview
fidelity.b <- c(0.5, 0.5, 0.45, 0.40, 0.35, 0.25, 0.15, 0) # Fidelity Bonds.
fidelity.s <- c(0.2, 0.28, 0.42, 0.50, 0.60, 0.7, 0.85, 1) # Fidelity Stocks.
vanguard.b <- c(1, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3, 0.2, 0) # Vanguard Bonds.
vanguard.s <- c(0, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 1) # Vanguard Stocks.



# Compute the bond-to-stocks ratios for Fidelity, Vanguard,
# and the optimal mean-variance portfolios.
fidelity.b.s <- fidelity.b/fidelity.s # Fidelity bond-to-stocks ratios.
vanguard.b.s <- vanguard.b/vanguard.s # Vanguard bond-to-stocks ratios.
b.s <- op.weights[, 2]/op.weights[, 1] # Mean-variance bond-to-stocks ratios.

# Plot the bond-to-stocks ratios.
plot(vanguard.s, vanguard.b.s, pch=19,
```

```
     xlab="Proportion of Assets in Stocks",
     ylab="Bond-to-Stocks Ratio",
     col="red", ylim=c(0,5), cex=1.2)
points(fidelity.s, fidelity.b.s, pch=17)
lines(op.weights[,1],b.s, col="blue", lwd=2)
colours <- c("red","black","blue")
legend("bottomleft",c("Vanguard","Fidelity","Mean-Variance"),
       fill=colours, bty="n")
```

8.4 Appendix D: Descriptive Statistics Program

```
rm(list=ls())

# CTRAN Packages:

library(xtable) # Output to LaTeX.
library(tseries) #For get.hist.qoute function.



# Summary functions.

accu.ret <- function(r, n) {
  # This function accumulates the one-period
  # return in "r" to n-period returns.
  # r is a matrix where each column contains the returns to an asset.
  nobs = nrow(r)
  N = ncol(r)
  x = log(1 + r) # convert to log returns.
  k = as.integer((nobs/n)) # number of accumulated returns.
  y = matrix(0, nrow=k, ncol=N) # vector to hold accumulated returns.
  for(j in 1:N) {
```

```
    for(i in 1:k) {
      start = (i-1)*n + 1 # start index.
      end = i*n # end index.
      y[i,j] = sum(x[start:end,j])
    }
  }
  return(exp(y) - 1.0)
}

descriptive <- function(r,t=1) {
  # This function returns table of annual summary statistics
  # given monthly returns.
  r.a <- accu.ret(r,12*t)
  names <- colnames(r) # Asset names.
  Average.Return <- apply(r.a, 2, mean) # The mean asset returns.
  Std.Dev <- sqrt(diag(var(r.a))) # Standard deviation of assets.
  Skew <- apply(r.a, 2, skewness) # Skewness of assets.
  Kurtosis <- apply(r.a, 2, kurtosis) # Skewness of assets.
  Max <- apply(r.a, 2, max) # Maximum value observed for asset i.
  Min <- apply(r.a, 2, min) # Minimum value observed for asset i.
  Value <- apply(1+r, 2, prod) # End value of a dollar investment.
  cor.mat <- cor(r.a)
  descriptive <- cbind(Average.Return, Std.Dev, Skew,
                       Kurtosis, Max, Min, Value)
  descriptive <- round(descriptive, 2)
  corrmat <-  round(cor.mat,2)
  ret.list <- list(descriptive, corrmat)
  return(ret.list)
}
```

```
Accumulate <- function(r) {
  # This function acummulate log returns.
  nAssets <- ncol(r)
  r <- log(1+r) # Transform to log returns.
  r <- apply(r, 2, cumsum) # Accumulate log returns.
  return(r)
}


################### Advanced Dataset ###################

# Obtain data on assets from Yahoo Finance.
Symbol <- c("VWNDX","FDVLX","VMRGX","VEXPX","^FVX",
            "VWLTX","VWEHX","VWITX")
nSymbols <- length(Symbol)
for(i in 1:nSymbols) {
new.symbol = get.hist.quote(instrument= Symbol[i],
                              start = "1980-01-01",
                              end = "2018-11-01",
                              quote="AdjClose",
                              provider = "yahoo",
                              origin="1980-01-01",
                              compression = "m",
                              retclass="zoo")
  names(new.symbol) = Symbol[i]
  if(i==1) data = new.symbol else data = merge(data, new.symbol)
}

# Returns from February 1980 to October 2018.
dates <- seq(as.Date("1980/2/1"), to=as.Date("2018/10/1"), by = "month")
```

```
names <- names(data)
prices <- coredata(data)
n <- nrow(prices)
r<- apply(prices, 2, FUN=function(x){ # Simple arithmetic returns.
  x[2:n]/x[1:(n-1)]-1
})
colnames(r) <- names


# Compute the accumulated log returns and display them in a graph.
log.returns <- Accumulate(r)
cl <- c("deepskyblue4","dodgerblue","darkblue","blue","red",
          "coral", "tomato3", "indianred4") # Assign colours.
plot(dates, log.returns[,1], type="l",
     col=cl[1], ylim=c(-3,5), xlab='', ylab="Cumulative Log Return")
lines(dates, log.returns[,2], col=cl[2])
lines(dates, log.returns[,3], col=cl[3])
lines(dates, log.returns[,4], col=cl[4])
lines(dates, log.returns[,5], col=cl[5])
lines(dates, log.returns[,6], col=cl[6])
lines(dates, log.returns[,7], col=cl[7])
lines(dates, log.returns[,7], col=cl[8])
legend("topleft", names, fill=cl, bty="n", cex=0.8)


# Obtain summary statistics.
summary <- descriptive(r,1)
xtable(summary[[1]]) # Descriptive statistics to LaTeX.
xtable(summary[[2]]) # Correlation matrix to LaTeX.
```

```
################## Classical Dataset ##################

# Read the classical dataset.
data <- read.table("marketdata.txt", header = TRUE)
r <- as.matrix(data[,2:4])
dates <- seq(as.Date("1927/1/1"),
             to=as.Date("2011/12/1"), by = "month")
colo <- c("red","blue","black")
names <-c("Stocks","Bonds","Cash")

# Plot simple returns.
plot(dates, r[,1], type="l", col="red", xlab="",
     ylab="Simple Return")
lines(dates, r[,2], col="blue" )
lines(dates, r[,3], col="black")
legend("topright", names, fill=colo, bty="n")

# Plot cumulative log returns.
g <- Accumulate(r)
plot(dates, g[,1], type="l", col="red", xlab="",
     ylab="Cumulative Log Return")
lines(dates, g[,2], col="blue" )
lines(dates, g[,3], col="black")
legend("topleft", names, fill=colo, bty="n")

# Summary statistics in tables.
summary <- descriptive(r,1)
rownames(summary[[1]]) <- names
rownames(summary[[2]]) <- names
colnames(summary[[2]]) <- names
```

```
xtable(summary[[1]]) # Descriptive statistics.
xtable(summary[[2]]) # Annual correlation matrix.
```

8.5 Appendix E: Reflection Note

In this thesis, we take on a behaviourist approach and attempt to explain why the financial advisers recommend riskier portfolio compositions to investors with longer investment horizons and lower aversions to losses. Our motivation for the thesis subject is simple: The positive application of Markowitz' (1959) mean-variance model fails to describe the financial advice on these two dimensions. Moreover, empirical research find that people violate the expected axioms of Neumann and Morgenstern (1944) on a consistent basis.

The behaviourist approach implies that we are only interested in the investors and how they perceive risk and losses, not to some normative "gold standard". The reward-risk model which we assume the investors base their decision on, the mean-LPM_1 model to be specific, incorporates key elements of behavioural finance that are empirically supported. The key elements we are concerned about is the investor's loss aversion and the investor's target return. Our findings are consistent with the financial advice. The optimal mean-LPM_1 portfolios over longer horizons induce a riskier portfolio composition, holding the target return and loss aversion constant. The optimal mean-LPM_1 portfolios corresponding to lower loss aversions induce a riskier composition, ceteris paribus. Finally, The optimal mean-LPM_1 portfolios corresponding to higher target returns implies a riskier portfolio, all else equal. These findings are based on the model's application on two dataset that differs in terms of complexity, and in the presence and absence of normally distributed returns.

Behavioural economics, and in particular, behavioural finance is on the "up-rise" after it has been in the shadow of expected utility theory and mean-variance over all these years. The recent Nobel prize awarded to economist Richard H. Thaler is a testimony of this fact. Accordingly, this thesis relates to the broader international trend of studying human behaviours in financial settings. The financial market is internationally intertwined and it is in constant evolution, new securities emerges everyday, some more complex than the previous one. This ever changing landscape affects the market participants and subsequently their behaviours. Accordingly, it is a possibility that the mean-LPM_1 model will fail to describe the financial advise in the future. Since it is a descriptive theory, its survival depends on being able to explain the financial advise. To safeguard its survival it must be re-evaluated against the recommended portfolios, and possibly re-modified, or scrapped in its entirety.

In the short to intermediate future I do not believe this to be a realistic scenario however. To the contrary, I believe the mean-LPM_1 model could improve or innovate the existing financial products offered to the common investor. Today the financial advisers screen future clients on time horizon and risk aversion. Ignoring the time horizon, I think it could productive to screen investors on their minimum required target returns and loss aversion instead of risk aversion. Considering that investors view risk as falling below some subjective return, this seems reasonable. Moreover, I believe it is more beneficial to screen investors on losses than risk aversion, since losses are concrete and easy to understand. But, a challenge is to compute the investor's true "loss aversion", a topic not covered in the thesis. However, suppose it is unproblematic to discover the investors loss aversions, then I believe the mean-LPM_1 could serve as a practical investing tool and to help investors with portfolio decision.

The mean-LPM_1 model is the embodiment of risk mitigation because the mean-LPM_1 portfolios offer the highest reward-to-risk trade given the investor's target, loss aversion, and investment horizon. However, if the mean-LPM_1 model is applied as an investing tool in a professional setting, then it implies responsibility on the adviser's side to inform the client on the inherent dangers the tool comes with. First, the adviser's need to inform the client that it is possible he loses everything even if the portfolio is optimal. Indeed, the mean-LPM_1 portfolios are risky. In theory, the mean-LPM_1 model discover the best mean-LPM_1 portfolios over the universe of assets. However, it is impossible to include every possible financial assets in a practical setting. As a result, the derived mean-LPM_1 portfolios are in a sense not optimal. Moreover, the mean-LPM_1 model relies on historical data to compute the optimal portfolios for the future. In a sense, when we use historical data we are computing optimal portfolios for the past. This point needs to be addressed to the client, it is a possibility that the underlying asset distributions are different in the future.