Research Article **Open Access**

Trond A. Abrahamsen, Aleksander Leraand, André Martiny, and Olav Nygaard*

Two properties of Müntz spaces

https://doi.org/10.1515/dema-2017-0025
Received June 14, 2017; accepted August 21, 2017

Abstract: We show that Müntz spaces, as subspaces of $C[0, 1]$, contain asymptotically isometric copies of c_0 and that their dual spaces are octahedral.

Keywords: Müntz space; Asymptotically isometric copy of c_0; Octahedral space; Diameter 2 properties

MSC: 46E15, 46B04, 46B20, 26A99

1 Introduction

Let $\Lambda = (\lambda_k)_{k=0}^{\infty}$ be a strictly increasing sequence of non-negative real numbers and let $M(\Lambda) = \overline{\operatorname{span}}\{t^{\lambda_k}\}_{k=0}^{\infty} \subset C[0, 1]$ where $C[0, 1]$ is the space of real valued continuous functions on $[0, 1]$ endowed with the max-norm. We will call $M(\Lambda)$ a Müntz space provided $\sum_{k=1}^{\infty} 1/\lambda_k < \infty$. The name is justified by Müntz' wonderful discovery that if $\lambda_0 = 0$ then $M(\Lambda) = C[0, 1]$ if and only if $\sum_{k=1}^{\infty} 1/\lambda_k = \infty$.

It is well known that $C[0, 1]$ contains isometric copies of c_0 (see e.g. [1, p. 86] how to construct them) and that its dual space is isometric to an $L_1(\mu)$ space for some measure μ. The aim of this paper is to demonstrate that Müntz spaces inherit quite a bit of structure from $C[0, 1]$ in that they always contain asymptotically isometric copies of c_0, and that their dual spaces are always octahedral. (An $L_1(\mu)$ space is octahedral. See below for an argument.) Let us proceed by recalling the definitions of these two concepts and put them into some context.

Definition 1.1. [2, Theorem 2] A Banach space X is said to contain an *asymptotically isometric copy of* c_0 if there exist a sequence $(x_n)_{n=1}^{\infty}$ in X and constants $0 < m < M < \infty$ such that for all sequences $(t_n)_{n=1}^{\infty}$ with finitely many non zero terms

$$m \sup_n |t_n| \le \left\| \sum_n t_n x_n \right\| \le M \sup_n |t_n|,$$

and

$$\lim_{n\to\infty} \|x_n\| = M.$$

R. C. James proved a long time ago (see [3]) that X contains an almost isometric copy of c_0 as soon at is contains a copy of c_0. Note that containing an asymptotically isometric copy of c_0 is a stronger property, see e.g. [2, Example 5].

Definition 1.2. A Banach space X is said to be *octahedral* if for any finite-dimensional subspace F of X and every $\varepsilon > 0$, there exists $y \in S_X$ with

$$\|x + y\| \ge (1 - \varepsilon)(\|x\| + 1) \ \text{ for all } \ x \in F.$$

***Corresponding Author: Olav Nygaard:** Department of Mathematics, University of Agder, Postboks 422, 4604 Kristiansand, Norway, E-mail: Olav.Nygaard@uia.no
Trond A. Abrahamsen, Aleksander Leraand, André Martiny: Department of Mathematics, University of Agder, Postboks 422, 4604 Kristiansand, Norway, E-mail: trond.a.abrahamsen@uia.no, aleksander9109@gmail.com, andremartiny91@gmail.com

This concept was introduced by G. Godefroy and B. Maurey (see [4, p. 118]), and in [5] the following result can be found on page 12:

Theorem 1.3 (Deville-Godefroy). *Let X be a Banach space. Then X^* is octahedral if and only if every finite convex combination of slices of B_X has diameter 2.*

By a slice of B_X we mean a set of the form

$$S(x^*, \varepsilon) := \{x \in B_X : x^*(x) > 1 - \varepsilon, \varepsilon > 0, x^* \in S_{X^*}\}.$$

Remark 1.4. As we have mentioned, Theorem 1.3 can be found, but without proof, in [5]. Deville had proven in [4, Theorem 1 and Proposition 3] that if X is octahedral, every finite convex combination of w^*-slices of B_{X^*} has diameter 2. In the same paper he asks if the converse is true (Remark (c) on page 119). Since there is no proof included in [5], new proofs appeared, independently, in [6] and [7], in connection with a new study of spaces where all finite convex combination of slices of B_X has diameter 2.

When we show that the dual of Müntz spaces are octahedral we will use Theorem 1.3 and establish the equivalent property stated there. Note that an $L_1(\mu)$ space is octahedral. Indeed, the bidual of such a space can be written $L_1(\mu)^{**} = L_1(\mu) \oplus_1 X$ for some subspace X of $L_1(\mu)^{**}$ (see e.g. [8, IV. Example 1.1]). From here the octahedrality of $L_1(\mu)$ is a straightforward application of the Principle of Local Reflexivity.

The main reference concerning Müntz spaces is [9]. But there most of the phenomena that are studied are linked to spreading properties of Λ and not general results concerning all Müntz spaces.

We do not know of much research in the direction of our results. But we would like to mention a paper of P. Petráček ([10]), where he demonstrates that Müntz spaces are never reflexive and asks whether they can have the Radon-Nikodým property. Since the Radon-Nikodým property implies the existence of slices of arbitrarily small diameter, we now understand that Müntz spaces rather belong to the “opposite world” of Banach spaces.

See also Remark 2.9 for some more related results.

2 Results

Definition 2.1. We will say that a strictly increasing sequence of non-negative real numbers $(\lambda_k)_{k=0}^\infty$ has the *Rapid Increase Property (RIP)* if $\lambda_{k+1} \geq 2\lambda_k$ for every $k \geq 0$.

We will call a function of the form

$$p(x) = x^\alpha - x^\beta,$$

where $0 \leq \alpha < \beta$, a *spike function*.

Remark 2.2. If $\alpha > 0$ it should be clear that any spike function p satisfies $p(0) = p(1) = 0$, attains its norm on a unique point x_p, is strictly increasing on $[0, x_p]$, and strictly decreasing on $[x_p, 1]$. To visualize the arguments that come, we think it is a good idea at this stage to draw the graphs of e.g. $x^{100} - x^{200}$ and $x^{1000} - x^{20000}$.

We will need the following result below.

Lemma 2.3. *Let $(\lambda_k)_{k=0}^\infty$ be an RIP sequence and $(p_k)_{k=0}^\infty$ the sequence of corresponding spike functions $p_k(x) = x^{\lambda_k} - x^{\lambda_{k+1}}$. Then $\inf_k \|p_k\| \geq 1/4$. Moreover, the sequence $(p_k/\|p_k\|)_{k=1}^\infty$ converges to 0 weakly in $M(\Lambda)$.*

Proof. We want to find the norm of the spike function defined by

$$p_k(x) = x^{\lambda_k} - x^{\lambda_{k+1}}.$$

Observe that $r_k(x) := x^{\lambda_k} - x^{2\lambda_k} \le p_k(x)$ for all $x \in [0, 1]$. Now, by standard calculus, r_k attains its maximum at x_k where ${x_k}^{\lambda_k} = \frac{1}{2}$. Thus

$$\|p_k\| \ge r_k(x_k) = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

As $(p_k)_{k=1}^{\infty}$ converges pointwise to 0 and $\inf_k \|p_k\| \ge 1/4$, the sequence $(p_k/\|p_k\|)_{k=1}^{\infty}$ converges pointwise to 0 and thus weakly to 0 as it is bounded. □

Remark 2.4. By standard calculus one can show that the point at which p_k in Lemma 2.3 obtains its norm is $\bar{x}_k = (\lambda_k/\lambda_{k+1})^{1/(\lambda_{k+1}-\lambda_k)}$. For sufficiently large λ_k it is straightforward to show that

$$y_k := 1/(\lambda_{k+1} - \lambda_k)^{1/(\lambda_{k+1}-\lambda_k)} \le \bar{x}_k,$$

that y_k is strictly monotone, and that y_k converges to 1 ($\lambda_k \ge 3$ is sufficient).

Theorem 2.5. *The dual of any Müntz space is octahedral.*

Proof. Let $M(\Lambda)$ be a Müntz space. Let

$$C = \sum_{j=1}^{n} \mu_j S(x_j^\star, \varepsilon_j),$$

where $\sum_{j=1}^n \mu_j = 1$, $\mu_j > 0$, and $S(x_j^\star, \varepsilon_j)$, $1 \le j \le n$, is a slice of $B_{M(\Lambda)}$. We will show that the diameter of C is 2 (cf. Theorem 1.3). To this end, start with some $f \in C$ and write $f = \sum_{j=1}^n \mu_j g^j$, where $g^j \in S(x_j^\star, \varepsilon_j)$. Let $(\lambda_k)_{k=0}^{\infty}$ be an RIP subsequence of Λ (which is possible as $\sum_{k=1}^{\infty} 1/\lambda_k < \infty$) and put

$$h_k^{j+} = g^j + (1 - g^j(x_k))\frac{p_k}{\|p_k\|}$$
$$h_k^{j-} = g^j - (1 + g^j(x_k))\frac{p_k}{\|p_k\|}$$

where $(p_k)_{k=0}^{\infty}$ is the sequence of spike functions corresponding to $(\lambda_k)_{k=0}^{\infty}$ and x_k the (unique) point where p_k attains its norm. We will prove that, for any $\varepsilon > 0$, there exists a $K = K(\varepsilon)$ such that whenever $k \ge K$ we have $\frac{1}{1+2\varepsilon} h_k^{j+}$, $\frac{1}{1+2\varepsilon} h_k^{j-} \in S(x_j^\star, \varepsilon_j)$ for every $1 \le j \le n$. Then, clearly

$$\frac{1}{1+2\varepsilon} \sum_{j=1}^{n} \mu_j h_k^{j\pm} \in C,$$

and

$$\left\| \frac{1}{1+2\varepsilon} \sum_{j=1}^{n} \mu_j h_k^{j+} - \frac{1}{1+2\varepsilon} \sum_{j=1}^{n} \mu_j h_k^{j-} \right\| \ge \frac{1}{1+2\varepsilon} \left(\sum_{j=1}^{n} \mu_j [h_k^{j+}(x_k) - h_k^{j-}(x_k)] \right) = \frac{2}{1+2\varepsilon}.$$

for all $k \ge K$. Since ε is arbitrary, we can thus conclude that C has diameter 2.

To produce the $K = K(\varepsilon)$ above, note that $h_k^{j\pm}$ converges to g^j pointwise, and thus weakly since the sequences are bounded. As $U_j := \{x \in M(\Lambda) : x_j^\star(x) > 1 - 2\varepsilon_j\}$ is weakly open, each sequence $(h_k^{j\pm})_{k=0}^{\infty}$ enters U_j eventually. Since there are only a finite number of sets U_j, this entrance is uniform. So, what is left to prove is that for $\varepsilon > 0$ there exists K such that $\|h_k^{j\pm}\| \le 1 + 2\varepsilon$ whenever $k \ge K$.

Now, let $\varepsilon > 0$. Combining Remark 2.2, Remark 2.4, that $(p_k/\|p_k\|)_{k=1}^{\infty}$ converges pointwise to 0, and the continuity of g^j, we can find $K \in \mathbb{N}$ such that for all $k \ge K$ there are points $0 < a_k < x_k < b_k < 1$ such that

$$\frac{p_k(x)}{\|p_k\|} > \varepsilon \Leftrightarrow x \in (a_k, b_k),$$
$$\sup_{u,v \in (a_k,b_k)} |g^j(u) - g^j(v)| < \varepsilon, \; j = 1, \ldots, n.$$

We will see that this K does the job for the given $\varepsilon > 0$: Let $k \geq K$ and suppose $x \notin (a_k, b_k)$. Then

$$|h_k^{j+}(x)| = \left| g^j(x) + (1 - g^j(x_k)) \frac{p_k(x)}{\|p_k\|} \right| \leq |g^j(x)| + 2\varepsilon \leq 1 + 2\varepsilon.$$

If $x \in (a_k, b_k)$, observe that

$$\begin{aligned} |h_k^{j+}(x)| &\leq \left| g^j(x) + (1 - g^j(x)) \frac{p_k(x)}{\|p_k\|} \right| + |g^j(x) - g^j(x_k)| \frac{p_k(x)}{\|p_k\|} \\ &< \left| g^j(x) + (1 - g^j(x)) \frac{p_k(x)}{\|p_k\|} \right| + \varepsilon. \end{aligned}$$

Now, if $g^j(x) \geq 0$, then

$$\left| g^j(x) + (1 - g^j(x)) \frac{p_k(x)}{\|p_k\|} \right| \leq g^j(x) + (1 - g^j(x)) = 1.$$

If $g^j(x) < 0$ and $g^j(x) + (1 - g^j(x))p_k(x)/\|p_k\| \geq 0$, then

$$\left| g^j(x) + (1 - g^j(x)) \frac{p_k(x)}{\|p_k\|} \right| \leq g^j(x) + (1 - g^j(x)) = 1.$$

If $g^j(x) < 0$ and $g^j(x) + (1 - g^j(x))p_k(x)/\|p_k\| < 0$, then

$$\left| g^j(x) + (1 - g^j(x)) \frac{p_k(x)}{\|p_k\|} \right| \leq |g^j(x)| \leq 1.$$

In any case we have for $k \geq K$ and $x \in [0, 1]$ that $|h_k^{j+}(x)| \leq 1 + 2\varepsilon$. The argument that $\|h_k^{j-}\| \leq 1 + 2\varepsilon$ is similar. □

Theorem 2.6. *Müntz spaces contain asymptotically isometric copies of c_0.*

Proof. We will construct a sequence $(f_n)_{n=1}^{\infty} \subset M(\Lambda)$ and pairwise disjoint intervals $I_n = (a_n, b_n) \subset [0, 1]$ such that for all $n \in \mathbb{N}$

(i) $f_n(x) \geq 0$ for all $x \in [0, 1]$,
(ii) $\|f_n\| = 1 - 1/2^n$,
(iii) $b_n < a_{n+1}$
(iv) $f_n(x) > 1/2^{2n} \Leftrightarrow x \in I_n$,
(v) $f_n(x) < 1/2^{2m}$ whenever $m \geq n$ and $x \in I_m$.

To this end choose a subsequence of Λ with the RIP. For simplicity denote also this subsequence by $(\lambda_k)_{k=0}^{\infty}$. Let $(p_k)_{k=1}^{\infty}$ be its corresponding sequence of spike functions, and let x_k be the (unique) point in $(0, 1)$ where p_k obtains its maximum.

Now, start by letting $k_1 = 1$ and put

$$f_1 = (1 - 1/2) \frac{p_{k_1}}{\|p_{k_1}\|}.$$

Using continuity and properties of p_1, we can find an interval $I_1 = (a_1, b_1)$ such that $0 < a_1 < b_1 < 1$ and $f_1(x) > \frac{1}{2^2} \Leftrightarrow x \in I_1$. By construction f_1 satisfies the conditions (i) - (iv).

To construct f_2 we use Lemma 2.3 and Remarks 2.2 and 2.4 to find $k_2 \in \mathbb{N}$ and an interval $I_2 = (a_2, b_2)$ with $b_1 < a_2 < b_2 < 1$ such that

$$\begin{aligned} x \in I_2 &\Leftrightarrow p_{k_2}(x) > \frac{1/2^4}{1 - 1/2^2} \|p_{k_2}\|, \\ x \in I_2 &\Rightarrow p_{k_1}(x) \leq \frac{1}{2^4}. \end{aligned}$$

Let

$$f_2 = (1 - 1/2^2)\frac{p_{k_2}}{\|p_{k_2}\|}.$$

By construction f_1 now satisfies condition (v) for $m \le 2$ and f_2 satisfies conditions (i) - (iv).

To construct f_3 we use Lemma 2.3 and Remarks 2.2 and 2.4 again to find $k_3 \in \mathbb{N}$ and an interval $I_3 = (a_3, b_3)$ with $b_2 < a_3 < b_3 < 1$ such that

$$x \in I_3 \Leftrightarrow p_{k_3}(x) > \frac{1/2^6}{1 - 1/2^3}\|p_{k_3}\|,$$
$$x \in I_3 \Rightarrow p_{k_j}(x) \le \frac{1}{2^6} \quad \text{for } j = 1, 2.$$

Let

$$f_3 = (1 - 1/2^3)\frac{p_{k_3}}{\|p_{k_3}\|}.$$

By construction f_1 and f_2 now satisfy condition (v) for $m \le 3$ and f_3 satisfies conditions (i) - (iv). If we continue in the same manner we obtain a sequence $(f_n)_{n=1}^{\infty} \subset M(\Lambda)$ and a sequence of intervals $I_n = (a_n, b_n)$ which satisfies the conditions (i) - (v).

Now we will show that $(f_n)_{n=1}^{\infty}$ satisfies the requirements of Definition 1.1. To this end we need to find constants $0 < m < M < \infty$ such that given any sequence $(t_n)_{n=1}^{\infty}$ with finitely many non zero terms

$$m \sup_n |t_n| \le \|\sum_n t_n f_n\| \le M \sup_n |t_n| \tag{1}$$

and

$$\lim_{n\to\infty} \|f_n\| = M \tag{2}$$

We claim that (1) and (2) holds with $m = \frac{1}{4}$ and $M = 1$. First observe that we have $\lim_{n\to\infty} \|f_n\| = 1$ immediately from the requirements, so (2) holds for $M = 1$. In order to prove the two inequalities in (1), let $(t_n)_{n=1}^{\infty}$ be an arbitrary sequence with finitely many non zero terms. First we will prove that $1/4 \sup_n |t_n| \le \left\|\sum_n t_n f_n\right\|$. We can assume by scaling that $\sup |t_n| = 1$. Since $(t_n)_{n=1}^{\infty}$ has finitely many non zero terms, its norm is attained at, say, $n = N$, i.e. $|t_N| = 1$. Put $x_N = x_{k_N}$ where x_{k_N} is the point where p_{k_N} and thus f_N attains its norm. Then

$$\begin{aligned}\|\sum_{n\in\mathbb{N}} t_n f_n\| &\ge |t_N f_N(x_N)| - |\sum_{n\ne N} t_n f_n(x_N)| \\ &\ge 1 - \frac{1}{2^N} - \sum_{n\ne N} |f_n(x_N)| \\ &> 1 - \frac{1}{2^N} - \frac{1}{4} \ge \frac{1}{4}.\end{aligned}$$

We conclude that the left hand side of the inequality (1) holds. Now we will show the right hand side of this inequality holds, i.e. we want to prove that $|\sum_n t_n f_n(x)| \le 1$ for all $x \in [0, 1]$. Since $f_n \ge 0$ for all $n = 1, 2, \dots,$ we may assume that every t_n is positive. Now, if $x \notin \cup_n(a_n, b_n)$, we have

$$\sum_n t_n f_n(x) \le \sum_n f_n(x) \le \sum_n \frac{1}{2^{2n}} \le \frac{1}{3}.$$

If, on the other hand $x \in (a_{n'}, b_{n'})$ for some $n' \in \mathbb{N}$, then

$$\begin{aligned}\sum_n t_n f_n(x) &\le f_{n'}(x) + \sum_{n<n'} f_n(x) + \sum_{n>n'} f_n(x) \\ &\le 1 - \frac{1}{2^{n'}} + \frac{n'-1}{2^{2n'}} + \frac{1}{2^{2n'}} \\ &\le 1 + \frac{n' - 2^{n'}}{2^{2n'}} \le 1 - \frac{1}{2^{2n'}} < 1.\end{aligned}$$

These combined yields the right hand side of the inequality (1), so the proof is complete. □

A Banach space X contains an *asymptotically isometric copy of* ℓ_1 if it contains a sequence $(x_n)_{n=1}^{\infty}$ for which there exists a sequence $(\delta_n)_{n=1}^{\infty}$ in $(0, 1)$, decreasing to 0, and such that

$$\sum_{n=1}^{m}(1-\delta_n)|a_n| \le \|\sum_{n=1}^{m} a_n x_n\| \le \sum_{n=1}^{m}|a_n|$$

for each finite sequence $(a_n)_{n=1}^{m}$ in $\mathbb{R}$.

Merging ([11, Theorem 2]) and [12, Lemma 2.3] gives us that if either the Banach space X contains an asymptotically isometric copy of c_0 or if X^* is octahedral, then X^* contains an asymptotically isometric copy of ℓ_1. So, we have two ways of proving

Corollary 2.7. *$M(\Lambda)^*$ contains an asymptotically isometric copy of ℓ_1.*

Moreover, we have

Corollary 2.8. *$M(\Lambda)^{**}$ contains an isometrically isomorphic copy of $L_1[0, 1]$.*

Proof. This follows from Corollary 2.7 and [13, Theorem 2]. □

Remark 2.9 (Added in proof).

(a) One of the anonymous referees invited the authors to consider the question whether Müntz spaces also could be octahedral (as $C[0, 1]$ is). Here is a preliminary answer: Combine the so called Clarkson-Erdös-Schwartz theorem (see [9, Theorem 6.2.3]) in tandem with a result of P. Wojtaszczyk (see [14, Theorem 1]). Then we see that when Λ consists of natural numbers, $M(\Lambda)$ is isomorphic to a subspace of c_0. Since an octahedral space contains a copy of ℓ_1, we have a negative answer for a big class of Müntz spaces.

(b) We have mentioned [12, Lemma 2.3] as reference for the fact that an octahedral space contains an asymptotically isometric copy of ℓ_1. It has come to our knowledge that this result, even with the same proof, was published earlier by Yamina Yagoub-Zidi, see [15, Proposition 3.3].

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