J Fourier Anal Appl
DOI 10.1007/s00041-017-9542-x

Factorizations and Singular Value Estimates of Operators with Gelfand–Shilov and Pilipović kernels

Yuanyuan Chen[1] · **Mikael Signahl**[2] · **Joachim Toft**[1]

Received: 11 May 2016

Abstract We prove that any linear operator with kernel in a Pilipović or Gelfand–Shilov space can be factorized by two operators in the same class. We also give links on numerical approximations for such compositions. We apply these composition rules to deduce estimates of singular values and establish Schatten–von Neumann properties for such operators.

Keywords Matrices · Harmonic oscillator · Hermite functions · Kernel theorems · Schatten–von Neumann operators · Singular values

Mathematics Subject Classification Primary 47Bxx · 15A23 · 45P05; Secondary 35S05

1 Introduction

The singular values and their decays are strongly related to possibilities of obtaining suitable finite rank approximations of operators. For a linear and compact operator

Communicated by Michael Ruzhansky.

✉ Joachim Toft
joachim.toft@lnu.se

Yuanyuan Chen
yuanyuan.chen@lnu.se

Mikael Signahl
mikael.signahl@uia.no

1 Department of Mathematics, Linnæus University, Växjö, Sweden

2 Department of Mathematical Sciences, University of Agder, Kristiansand, Norway

Published online: 13 April 2017 Birkhäuser which acts between Hilbert spaces, the singular values are the eigenvalues in decreasing order of the modulus of the operator. If more generally, the linear operator T is continuous from the quasi-Banach space $\mathscr{B}_1$ to (another) quasi-Banach space $\mathscr{B}_2$, then the singular value of order $k \ge 1$ is given by

$$\boldsymbol{\sigma}_k(T) = \boldsymbol{\sigma}_k(T, \mathscr{B}_1, \mathscr{B}_2) \equiv \inf \|T - T_0\|_{\mathscr{B}_1 \to \mathscr{B}_2} \tag{1.1}$$

where the infimum is taken over all linear operators T_0 from $\mathscr{B}_1$ to $\mathscr{B}_2$ of rank at most $k-1$. (See Sect. 2 for notations). It follows that T is compact when $\boldsymbol{\sigma}_k(T)$ decreases to zero as k tends to infinity, or equivalently, T can be approximated by finite rank operators with arbitrarily small errors.

In this paper we deduce estimates of $\boldsymbol{\sigma}_k(T)$ when $\mathscr{B}_1$ and $\mathscr{B}_2$ stay between small test function spaces, denoted by $\mathcal{H}_s(\mathbf{R}^d)$ and $\mathcal{H}_{0,s}(\mathbf{R}^d)$, and their (large) duals. The spaces $\mathcal{H}_s(\mathbf{R}^d)$ and $\mathcal{H}_{0,s}(\mathbf{R}^d)$ are invariant under the Fourier transform, depend on the parameter $s \ge 0$ and are obtained by imposing certain exponential type estimates on the Hermite coefficients of the involved functions. More precisely, the set $\mathcal{H}_s(\mathbf{R}^d)$ ($\mathcal{H}_{0,s}(\mathbf{R}^d)$) consists of all

$$f = \sum_\alpha c_\alpha h_\alpha$$

such that $|c_\alpha| \lesssim e^{-c|\alpha|^{\frac{1}{2s}}}$ for some (for every) $c > 0$. It follows that $\mathcal{H}_s(\mathbf{R}^d)$ and $\mathcal{H}_{0,s}(\mathbf{R}^d)$ increase with s, and are continuously embedded and dense in $\mathscr{S}(\mathbf{R}^d)$.

In [26] the spaces $\mathcal{H}_s(\mathbf{R}^d)$ and $\mathcal{H}_{0,s}(\mathbf{R}^d)$ and their duals were characterized in different ways. For example, the images under the Bargmann transform were given, and it was proved that $f \in \mathcal{H}_s(\mathbf{R}^d)$ ($f \in \mathcal{H}_{0,s}(\mathbf{R}^d)$), if and only if f satisfies

$$|H^N f(x)| \lesssim h^N N!^{2s} \tag{1.2}$$

for some $h > 0$ (for every $h > 0$), where H is the harmonic oscillator $|x|^2 - \Delta$ on $\mathbf{R}^d$. In this context we recall that Pilipović introduced in [19] function spaces whose elements obey estimates of the form (1.2) for certain choices of s. For this reason, we call $\mathcal{H}_s(\mathbf{R}^d)$ and $\mathcal{H}_{0,s}(\mathbf{R}^d)$ the Pilipović spaces of Roumieu and Beurling type, respectively, of degree $s \ge 0$ (cf. [26]).

In [19], it is also proved that $\mathcal{H}_{s_1}(\mathbf{R}^d)$ and $\mathcal{H}_{0,s_2}(\mathbf{R}^d)$ agree with the Gelfand–Shilov spaces $\mathcal{S}_{s_1}(\mathbf{R}^d)$ and $\Sigma_{s_2}(\mathbf{R}^d)$, respectively, when $s_1 \ge \frac{1}{2}$ and $s_2 > \frac{1}{2}$, while $\mathcal{H}_{0,\frac{1}{2}}(\mathbf{R}^d)$ is different from the trivial space $\Sigma_{\frac{1}{2}}(\mathbf{R}^d) = \{0\}$. The family of Pilipović spaces therefore contains all Gelfand–Shilov spaces which are invariant under Fourier transformations.

In Sect. 5 we consider linear operators whose kernels belong to $\mathcal{H}_s(\mathbf{R}^{2d})$. Some parts of the approach here is related to the analysis in [6,7], where Schatten–von Neumann properties for certain types of integral operators on compact manifolds are deduced. We show that the singular values of such operator satisfies the estimate

$$\boldsymbol{\sigma}_k(T, \mathscr{B}_1, \mathscr{B}_2) \lesssim e^{-rk^{\frac{1}{2ds}}} \tag{1.3}$$

for some $r > 0$, when $\mathscr{B}_j$ stays between $\mathcal{H}_s(\mathbf{R}^d)$ and its dual. If the $\mathcal{H}_s$-spaces and their duals are replaced by $\mathcal{H}_{0,s}$-spaces and their duals, then we also prove that (1.3) is true for every $r > 0$. Furthermore, if $\mathcal{H}_s$-spaces and their duals are replaced by Schwartz spaces and their duals, then we prove

$$\boldsymbol{\sigma}_k(T, \mathscr{B}_1, \mathscr{B}_2) \lesssim k^{-N} \tag{1.4}$$

for every $N \geq 0$, which should be available in the literature.

These singular-value estimates are based on the fact that the operator classes here above possess convenient factorization properties, which are deduced in Sect. 4. More precisely, an operator class $\mathcal{M}$ is called a *factorization algebra*, if for every $T \in \mathcal{M}$, there exist $T_1, T_2 \in \mathcal{M}$ such that $T = T_1 \circ T_2$. (In [28] the term *decomposition algebra* is used instead of factorization algebra). Evidently, $\mathscr{L}(\mathscr{B})$, the set of continuous linear operators on the quasi-Banach space $\mathscr{B}$ is a factorization algebra, since we may choose T_1 as the identity operator and $T_2 = T$. A more challenging situation appears when $\mathcal{M}$ does not contain the identity operator, and in this situation it is easy to find operator classes which are not factorization algebras. For example, any Schatten–von Neumann class of finite order is not a factorization algebra.

If $\mathscr{B}$ above is a Hilbert space and $\mathcal{M}$ is the set of compact operators on $\mathscr{B}$, then it follows by an application of the spectral theorem that $\mathcal{M}$ is a factorization algebra. It is also well-known that the set of linear operators with kernels in the Schwartz space is a factorization algebra (see e. g. [2,17,21,28,30]). Furthemore, similar facts hold true for the set of operators with kernels in a fixed Gelfand–Shilov space (cf. [28]).

In Sect. 4 we extend the latter property such that all Pilipović spaces are included. That is, we prove that the set of operators with kernels in a fixed (but arbitrarily chosen) Pilipović space is a factorization algebra.

Since the singular values of the operators under considerations either satisfy conditions of the form (1.3) or (1.4) for every $N \geq 0$, it follows that the sequence $\{\boldsymbol{\sigma}_k(T)\}_{k=1}^{\infty}$ belongs to ℓ^p for every $p > 0$. This implies that any such operator is a Schatten–von Neumann operator of degree p for every $p > 0$.

Here we remark that the latter conclusions in the Gelfand–Shilov situation, were deduced in [28] in slightly different ways, which enables to replace the quasi-Banach spaces $\mathscr{B}_1$ and $\mathscr{B}_2$ by convenient Hilbert spaces. The main property behind the latter reduction concerns [25, Proposition 3.8], where it is proved that if $s \geq \frac{1}{2}$ and

$$\mathcal{H}_s(\mathbf{R}^d) \subseteq \mathscr{B} \subseteq \mathcal{H}_s'(\mathbf{R}^d),$$

then there are Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ such that

$$\mathcal{H}_s(\mathbf{R}^d) \subseteq \mathscr{H}_1 \subseteq \mathscr{B} \subseteq \mathscr{H}_2 \subseteq \mathcal{H}_s'(\mathbf{R}^d). \tag{1.5}$$

The Schatten–von Neumann properties are then obtained in straight-forward ways by the factorization properties in combination with the exact formulas, for Hilbert–Schmidt norms of operators acting between Hilbert spaces. We remark that extensions of (1.5) which include Pilipović spaces can be found in [5].

Our investigations also include analysis of operators with kernels in $\mathcal{H}_{\flat_\sigma}$, $\mathcal{H}_{0,\flat_\sigma}$, $\sigma > 0$, or their duals. These spaces were carefully investigated in [10,26] and the Hermite coefficients of the involved functions should be bounded by expressions of the form $h^{|\alpha|}(\alpha!)^{-\frac{1}{2\sigma}}$. In [26], these spaces are characterized in different ways. For example, it is here proved that the Bargmann transform is bijective from $\mathcal{H}_{\flat_\sigma}(\mathbf{R}^d)$ to the set of all entire functions F on $\mathbf{C}^d$ such that

$$|F(z)| \lesssim e^{c|z|^{\frac{2\sigma}{\sigma+1}}}$$

for some constant $C > 0$.

In Sect. 3 we deduce kernel theorems for operators with kernels in these spaces, or related distribution spaces. In Sect. 4 we show certain factorization properties of operators with kernels in $\mathcal{H}_{\flat_\sigma}$ or in $\mathcal{H}_{0,\flat_\sigma}$. These factorization results are slightly weaker compared to what is deduced for operators with kernels in $\mathcal{H}_s$ and $\mathcal{H}_{0,s}$ when $s \ge 0$ is real.

In Sect. 5 we apply these factorization properties to obtain singular value estimates for operators with kernels in $\mathcal{H}_{\flat_\sigma}$ or in $\mathcal{H}_{0,\flat_\sigma}$. In particular we show that if T is an operator on $L^2(\mathbf{R}^d)$ with kernel in $\mathcal{H}_{\flat_\sigma}(\mathbf{R}^{2d})$, then the singular values of T satisfy

$$\boldsymbol{\sigma}_k(T) \lesssim h^k (k!)^{-\frac{1}{2\sigma d}}$$

for some constant $h > 0$.

Finally, in Sect. 6 we apply the results from the first sections to obtain certain characterizations of operators with kernels in $\mathcal{H}_s$ and $\mathcal{H}_{0,s}$. Some arguments here involve estimates with modulation space norms, and a short introduction to modulation spaces are therefore included in Sect. 2.

2 Preliminaries

In this section we recall some basic facts. We start by discussing Pilipović spaces and their properties. Thereafter we consider suitable spaces of formal Hermite series expansions, and discuss their links with Pilipović spaces.

2.1 The Pilipović Spaces

We start by considering spaces which are obtained by suitable estimates of Gelfand–Shilov or Gevrey type when using powers of the harmonic oscillator $H \equiv |x|^2 - \Delta$, $x \in \mathbf{R}^d$.

Let $h > 0$, $s \ge 0$ and let $\mathcal{S}_{h,s}(\mathbf{R}^d)$ be the set of all $f \in C^\infty(\mathbf{R}^d)$ such that

$$\|f\|_{\mathcal{S}_{h,s}} \equiv \sup_{N \ge 0} \frac{\|H^N f\|_{L^\infty}}{h^N (N!)^{2s}} < \infty. \tag{2.1}$$

Then $\mathcal{S}_{h,s}(\mathbf{R}^d)$ is a Banach space. If h_α is the Hermite function

$$h_\alpha(x) = \pi^{-\frac{d}{4}}(-1)^{|\alpha|}(2^{|\alpha|}\alpha!)^{-\frac{1}{2}}e^{\frac{|x|^2}{2}}\big(\partial^\alpha e^{-|x|^2}\big),$$

on $\mathbf{R}^d$ of order α, then $Hh_\alpha = (2|\alpha|+d)h_\alpha$. This implies that $\mathcal{S}_{h,s}(\mathbf{R}^d)$ contains all Hermite functions when $s>0$, and if $s=0$, and $\alpha \in \mathbf{N}^d$ satisfies $2|\alpha|+d \le h$, then $h_\alpha \in \mathcal{S}_{h,s}(\mathbf{R}^d)$.

We let

$$\boldsymbol{\Sigma}_s(\mathbf{R}^d) \equiv \bigcap_{h>0} \mathcal{S}_{h,s}(\mathbf{R}^d) \quad \text{and} \quad \mathcal{S}_s(\mathbf{R}^d) \equiv \bigcup_{h>0} \mathcal{S}_{h,s}(\mathbf{R}^d)$$

and equip these spaces by projective and inductive limit topologies, respectively, of $\mathcal{S}_{h,s}(\mathbf{R}^d)$, $h>0$. (Cf. [13,18,19,26].)

In [18,19], Pilipović proved that if $s_1 \ge \frac{1}{2}$ and $s_2 > \frac{1}{2}$, then $\mathcal{S}_{s_1}(\mathbf{R}^d)$ and $\boldsymbol{\Sigma}_{s_2}(\mathbf{R}^d)$ agree with the Gelfand–Shilov spaces $\mathcal{S}_{s_1}(\mathbf{R}^d)$ and $\Sigma_{s_2}(\mathbf{R}^d)$[1], respectively, and that

$$\boldsymbol{\Sigma}_{1/2}(\mathbf{R}^d) \neq \Sigma_{1/2}(\mathbf{R}^d) = \{0\}.$$

(See e. g. [26] for notations).

By the definitions it follows that the latter relations extend into

$$\mathcal{S}_{s_1}(\mathbf{R}^d) = \mathcal{S}_{s_1}(\mathbf{R}^d), \qquad \boldsymbol{\Sigma}_{s_2}(\mathbf{R}^d) = \Sigma_{s_2}(\mathbf{R}^d), \qquad s_1 \ge \frac{1}{2},\ s_2 > \frac{1}{2}$$

and

$$\mathcal{S}_{s_1}(\mathbf{R}^d) \neq \mathcal{S}_{s_1}(\mathbf{R}^d) = \{0\}, \qquad \boldsymbol{\Sigma}_{s_2}(\mathbf{R}^d) \neq \Sigma_{s_2}(\mathbf{R}^d) = \{0\}, \qquad s_1 < \frac{1}{2},\ 0 < s_2 \le \frac{1}{2}.$$

The space $\boldsymbol{\Sigma}_s(\mathbf{R}^d)$ is called the *Pilipović space (of Beurling type) of order $s \ge 0$ on $\mathbf{R}^d$*. Similarly, $\mathcal{S}_s(\mathbf{R}^d)$ is called the *Pilipović space (of Roumieu type) of order $s \ge 0$ on $\mathbf{R}^d$*.

The dual spaces of $\mathcal{S}_{h,s}(\mathbf{R}^d)$, $\boldsymbol{\Sigma}_s(\mathbf{R}^d)$ and $\mathcal{S}_s(\mathbf{R}^d)$ are denoted by $\mathcal{S}_{h,s}'(\mathbf{R}^d)$, $\boldsymbol{\Sigma}_s'(\mathbf{R}^d)$ and $\mathcal{S}_s'(\mathbf{R}^d)$, respectively. We have

$$\boldsymbol{\Sigma}_s'(\mathbf{R}^d) = \bigcup_{h>0} \mathcal{S}_{h,s}'(\mathbf{R}^d)$$

when $s>0$ and

$$\mathcal{S}_s'(\mathbf{R}^d) = \bigcap_{h>0} \mathcal{S}_{h,s}'(\mathbf{R}^d)$$

[1] Note that the boldface characters, $\boldsymbol{\Sigma}_*$, $\mathcal{S}_*$ etc. denote Pilipović spaces, and non-boldface characters, Σ_*, $\mathcal{S}_*$ etc. denote Gelfand–Shilov spaces.

when $s \geq 0$, with inductive respectively projective limit topologies of $\mathcal{S}'_{h,s}(\mathbf{R}^d)$, $h > 0$ (cf. [26]).

2.2 Quasi-Banach Spaces, Singular Values and Schatten–von Neumann Operators

Let $\mathscr{B}$ be a vector space. A *quasi-norm* $\| \cdot \|_{\mathscr{B}}$ on $\mathscr{B}$ is a non-negative and real-valued function on $\mathscr{B}$ which is non-degenerate in the sense that

$$\|f\|_{\mathscr{B}} = 0 \qquad \Longleftrightarrow \qquad f = 0, \quad f \in \mathscr{B},$$

and fulfills

$$\|\alpha f\|_{\mathscr{B}} = |\alpha| \cdot \|f\|_{\mathscr{B}}, \qquad\qquad f \in \mathscr{B}, \ \alpha \in \mathbf{C},$$

and

$$\|f + g\|_{\mathscr{B}} \leq D(\|f\|_{\mathscr{B}} + \|g\|_{\mathscr{B}}), \qquad\qquad f, g \in \mathscr{B}, \tag{2.2}$$

for some constant $D \geq 1$ which is independent of $f, g \in \mathscr{B}$. Then $\mathscr{B}$ is a topological vector space when the topology for $\mathscr{B}$ is defined by $\| \cdot \|_{\mathscr{B}}$, and $\mathscr{B}$ is called a quasi-Banach space if $\mathscr{B}$ is complete under this topology.

Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be (quasi-)Banach spaces, and let T be a linear map between $\mathscr{B}_1$ and $\mathscr{B}_2$. Then the *singular values* of order $k \geq 1$ of T is given by (1.1), where the infimum is taken over all linear operators T_0 from $\mathscr{B}_1$ to $\mathscr{B}_2$ with rank at most $k - 1$. Therefore, $\boldsymbol{\sigma}_1(T)$ equals $\|T\|_{\mathscr{B}_1 \to \mathscr{B}_2}$, and $\boldsymbol{\sigma}_k(T)$ are non-negative and non-increasing with respect to k.

For any $p \in (0, \infty]$, $\mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)$, the set of Schatten–von Neumann operators of order p is the quasi-Banach space which consists of all linear and continuous operators T from $\mathscr{B}_1$ to $\mathscr{B}_2$ such that

$$\|T\|_{\mathscr{I}_p} = \|T\|_{\mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)} \equiv \|(\boldsymbol{\sigma}_k(T, \mathscr{B}_1, \mathscr{B}_2))_{k=1}^{\infty}\|_{l^p}$$

is finite.

2.3 Spaces of Hermite Series Expansions

Next we recall the definitions of topological vector spaces of Hermite series expansions, given in [26]. As in [26], it is convenient to use the sets $\mathbf{R}_\flat$ and $\overline{\mathbf{R}_\flat}$ when indexing our spaces.

Definition 2.1 The sets $\mathbf{R}_\flat$ and $\overline{\mathbf{R}_\flat}$ are given by

$$\mathbf{R}_\flat = \mathbf{R}_+ \bigcup_{\sigma > 0} \{\flat_\sigma\} \quad \text{and} \quad \overline{\mathbf{R}_\flat} = \mathbf{R}_\flat \bigcup \{0\}.$$

Moreover, beside the usual ordering in $\mathbf{R}$, the elements $\flat_\sigma$ in $\mathbf{R}_\flat$ and $\overline{\mathbf{R}_\flat}$ are ordered by the relations $x_1 < \flat_{\sigma_1} < \flat_{\sigma_2} < x_2$, when $\sigma_1 < \sigma_2$, $x_1 < \frac{1}{2}$ and $x_2 \ge \frac{1}{2}$ are real.

A function ϑ on a discrete set Λ is called a *weight* (on Λ) if it is real-valued and positive.

Definition 2.2 Let $p \in (0,\infty]$, $s \in \mathbf{R}_\flat$, $r \in \mathbf{R}$, ϑ be a weight on $\mathbf{N}^d$, and let

$$\vartheta_{r,s}(\alpha) \equiv \begin{cases} e^{r|\alpha|^{\frac{1}{2s}}}, & \text{when} \quad s \in \mathbf{R}_+, \\ r^{|\alpha|}(\alpha!)^{\frac{1}{2\sigma}}, & \text{when} \quad s = \flat_\sigma, \end{cases} \qquad \alpha \in \mathbf{N}^d.$$

Then,

(1) $\ell_0'(\mathbf{N}^d)$ is the set of all sequences $\{c_\alpha\}_{\alpha\in\mathbf{N}^d} \subseteq \mathbf{C}$ on $\mathbf{N}^d$;
(2) $\ell_{0,0}(\mathbf{N}^d) \equiv \{0\}$, and $\ell_0(\mathbf{N}^d)$ is the set of all sequences $\{c_\alpha\}_{\alpha\in\mathbf{N}^d} \subseteq \mathbf{C}$ such that $c_\alpha \neq 0$ for at most finite numbers of α;
(3) $\ell^p_{[\vartheta]}(\mathbf{N}^d)$ is the quasi-Banach space which consists of all sequences $\{c_\alpha\}_{\alpha\in\mathbf{N}^d} \subseteq \mathbf{C}$ such that

$$\|\{c_\alpha\}_{\alpha\in\mathbf{N}^d}\|_{\ell^p_{[\vartheta]}} \equiv \|\{c_\alpha\vartheta(\alpha)\}_{\alpha\in\mathbf{N}^d}\|_{\ell^p}$$

is finite;
(4) $\ell_{0,s}(\mathbf{N}^d) \equiv \bigcap_{r>0} \ell^p_{[\vartheta_{r,s}]}(\mathbf{N}^d)$ and $\ell_s(\mathbf{N}^d) \equiv \bigcup_{r>0} \ell^p_{[\vartheta_{r,s}]}(\mathbf{N}^d)$, with projective respectively inductive limit topologies of $\ell^p_{[\vartheta_{r,s}]}(\mathbf{N}^d)$ with respect to $r > 0$;
(5) $\ell_{0,s}'(\mathbf{N}^d) \equiv \bigcup_{r>0} \ell^p_{[1/\vartheta_{r,s}]}(\mathbf{N}^d)$ and $\ell_s'(\mathbf{N}^d) \equiv \bigcap_{r>0} \ell^p_{[1/\vartheta_{r,s}]}(\mathbf{N}^d)$, with inductive respectively projective limit topologies of $\ell^p_{[1/\vartheta_{r,s}]}(\mathbf{N}^d)$ with respect to $r > 0$.

Let $p \in (0,\infty]$, and let Ω_N be the set of all $\alpha \in \mathbf{N}^d$ such that $|\alpha| \le N$. Then the topology of $\ell_0(\mathbf{N}^d)$ is defined by the inductive limit topology of the sets

$$\left\{ \{c_\alpha\}_{\alpha\in\mathbf{N}^d} \in \ell_0'(\mathbf{N}^d)\, ;\ c_\alpha = 0 \text{ when } \alpha \notin \Omega_N \right\}$$

with respect to $N \ge 0$, and whose topology is given through the quasi-semi-norms

$$\{c_\alpha\}_{\alpha\in\mathbf{N}^d} \mapsto \|\{c_\alpha\}_{|\alpha|\le N}\|_{\ell^p(\Omega_N)}. \tag{2.3}$$

Furthermore, the topology of $\ell_0'(\mathbf{N}^d)$ is defined by the quasi-semi-norms (2.3). It follows that $\ell_0'(\mathbf{N}^d)$ is a Fréchet space, and that the topologies of $\ell_0(\mathbf{N}^d)$ and $\ell_0'(\mathbf{N}^d)$ are independent of p.

Next we consider spaces of formal Hermite series expansions

$$f = \sum_{\alpha\in\mathbf{N}^d} c_\alpha h_\alpha, \quad \{c_\alpha\}_{\alpha\in\mathbf{N}^d} \in \ell_0'(\mathbf{N}^d), \tag{2.4}$$

which correspond to

$$\ell_{0,s}(\mathbf{N}^d), \quad \ell_s(\mathbf{N}^d), \quad \ell^p_{[\vartheta]}(\mathbf{N}^d), \quad \ell'_s(\mathbf{N}^d) \quad \text{and} \quad \ell'_{0,s}(\mathbf{N}^d). \tag{2.5}$$

For that reason we consider the mapping

$$T : \{c_\alpha\}_{\alpha\in\mathbf{N}^d} \mapsto \sum_{\alpha\in\mathbf{N}^d} c_\alpha h_\alpha, \tag{2.6}$$

between sequences and formal Hermite series expansions.

Definition 2.3 Let $p \in (0, \infty]$, ϑ be a weight on $\mathbf{N}^d$, and let $s \in \overline{\mathbf{R}_\flat}$.

- The spaces

$$\mathcal{H}_{0,s}(\mathbf{R}^d), \quad \mathcal{H}_s(\mathbf{R}^d), \quad \mathcal{H}^p_{[\vartheta]}(\mathbf{R}^d), \quad \mathcal{H}'_s(\mathbf{R}^d) \quad \text{and} \quad \mathcal{H}'_{0,s}(\mathbf{R}^d) \tag{2.7}$$

  are the images of T in (2.6) under the corresponding spaces in (2.5). Furthermore, the topologies of the spaces in (2.7) are inherited from corresponding spaces in (2.5);
- The quasi-norm $\|f\|_{\mathcal{H}^p_{[\vartheta]}}$ of $f \in \mathcal{H}'_0(\mathbf{R}^d)$, is given by $\|\{c_\alpha\}_{\alpha\in\mathbf{N}^d}\|_{\ell^p_{[\vartheta]}}$, when f is given by (2.4).

By the definitions it follows that the inclusions

$$\begin{aligned} \mathcal{H}_0(\mathbf{R}^d) &\subseteq \mathcal{H}_{0,s}(\mathbf{R}^d) \subseteq \mathcal{H}_s(\mathbf{R}^d) \subseteq \mathcal{H}_{0,t}(\mathbf{R}^d) \\ &\subseteq \mathscr{S}(\mathbf{R}^d) \subseteq \mathscr{S}'(\mathbf{R}^d) \subseteq \mathcal{H}'_{0,t}(\mathbf{R}^d) \subseteq \mathcal{H}'_s(\mathbf{R}^d) \\ &\subseteq \mathcal{H}'_{0,s}(\mathbf{R}^d) \subseteq \mathcal{H}'_0(\mathbf{R}^d), \quad \text{when } s, t \in \mathbf{R}_\flat, \ s < t, \end{aligned} \tag{2.8}$$

hold true, and are in fact continuous embeddings.

Remark 2.4 By the definition it follows that T in (2.6) is a homeomorphism between any of the spaces in (2.5) and corresponding space in (2.7).

The next result shows that the spaces in Definition 2.3 essentially agrees with the Pilipović spaces. We refer to [26] for the proof.

Proposition 2.5 *Let* $0 \le s \in \mathbf{R}$. *Then* $\mathcal{H}_{0,s}(\mathbf{R}^d) = \boldsymbol{\Sigma}_s(\mathbf{R}^d)$ *and* $\mathcal{H}_s(\mathbf{R}^d) = \boldsymbol{\mathcal{S}}_s(\mathbf{R}^d)$.

Remark 2.6 Let T be given by (2.6), $p \in [1, 2]$ and let ϑ be a weight on $\mathbf{N}^d$ such that $1/\vartheta \in \ell^\infty(\mathbf{N}^d)$. Then

$$\begin{aligned} &\big(\ell_s(\mathbf{N}^d), \ell^2(\mathbf{N}^d), \ell'_s(\mathbf{N}^d)\big) \xrightarrow{T} \big(\mathcal{H}_s(\mathbf{R}^d), L^2(\mathbf{R}^d), \mathcal{H}'_s(\mathbf{R}^d)\big), \quad s \ge 0, \\ &\big(\ell_{0,s}(\mathbf{N}^d), \ell^2(\mathbf{N}^d), \ell'_{0,s}(\mathbf{N}^d)\big) \xrightarrow{T} \big(\mathcal{H}_{0,s}(\mathbf{R}^d), L^2(\mathbf{R}^d), \mathcal{H}'_{0,s}(\mathbf{R}^d)\big), \quad s > 0, \\ &\big(\ell^p_{[\vartheta]}(\mathbf{N}^d), \ell^2(\mathbf{N}^d), \ell^{p'}_{[1/\vartheta]}(\mathbf{N}^d)\big) \xrightarrow{T} \big(\mathcal{H}^p_{[\vartheta]}(\mathbf{R}^d), L^2(\mathbf{R}^d), \mathcal{H}^{p'}_{[1/\vartheta]}(\mathbf{R}^d)\big) \end{aligned}$$

are isometric bijections between Gelfand triples. (Cf. e. g. Sect. 4 in [26].)

2.4 Pseudo-Differential Operators

We let $\mathscr{F}$ be the Fourier transform on $\mathscr{S}'(\mathbf{R}^d)$, given by

$$(\mathscr{F} f)(\xi) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(x) e^{-i\langle x,\xi\rangle}\, d\xi$$

when $f \in L^1(\mathbf{R}^d)$. We also let $\mathscr{F}_2 F$ be the partial Fourier transform of $F(x,y)$ with respect to the y-variable. The Fourier transform restricts to homeomorphisms on $\mathscr{S}(\mathbf{R}^d)$, $\mathcal{H}_s(\mathbf{R}^d)$ and $\mathcal{H}_{s,0}(\mathbf{R}^d)$, and extends uniquely to homeomorphisms on corresponding duals. The same holds true for $\mathscr{F}_2$ when acting on functions and distributions on $\mathbf{R}^{2d}$ (cf. [26]).

For every $s \ge \frac{1}{2}$, real $d \times d$-matrix A and $a \in \mathcal{S}_s'(\mathbf{R}^{2d})$ (the symbol), the pseudo-differential operator $\operatorname{Op}_A(a)$ is the linear and continuous operator from $\mathcal{S}_s(\mathbf{R}^d)$ to $\mathcal{S}_s'(\mathbf{R}^d)$ with distribution kernel

$$K_{A,a}(x,y) = (2\pi)^{-d/2} (\mathscr{F}_2^{-1} a)(x - A(x-y), x-y).$$

If $a \in L^1(\mathbf{R}^{2d})$ and $f \in \mathcal{S}_s(\mathbf{R}^d)$, then $\operatorname{Op}_A(a) f$ is given by

$$\operatorname{Op}_A(a) f(x) = (2\pi)^{-d} \iint_{\mathbf{R}^{2d}} a(x - A(x-y), \xi) f(y) e^{i\langle x-y,\xi\rangle}\, dy d\xi.$$

The product for compositions of pseudo-differential operators on the symbol level is denoted by $\#_A$. This implies that if $a, b \in \mathscr{S}(\mathbf{R}^{2d})$, then $a \#_A b$ is defined by

$$\operatorname{Op}_A(a \#_A b) = \operatorname{Op}_A(a) \circ \operatorname{Op}_A(b).$$

The product $a \#_A b$ is well-defined and is uniquely extendable in different ways (see e. g. [1,4,16]).

2.5 Modulation Spaces

Next we discuss basic properties for modulation spaces, and start by recalling the conditions for the involved weight functions. A function ω on $\mathbf{R}^d$ is called a *weight* (on $\mathbf{R}^d$), if $\omega > 0$ and $\omega, 1/\omega \in L^\infty_{loc}(\mathbf{R}^d)$.

Let ω be a weight on $\mathbf{R}^d$, and set $\langle x \rangle \equiv 1 + |x|$ when $x \in \mathbf{R}^d$. Then ω is called a weight of polynomial type, if

$$\omega(x+y) \lesssim \omega(x) \langle y \rangle^N, \quad x, y \in \mathbf{R}^d, \tag{2.9}$$

for some $N \ge 0$. Here and in what follows we write $A \lesssim B$ when $A, B \ge 0$ and $A \le cB$ for a suitable constant $c > 0$. We also let $A \asymp B$ when $A \lesssim B$ and $B \lesssim A$. We let $\mathscr{P}(\mathbf{R}^d)$ be the set of all weights on $\mathbf{R}^d$ of polynomial type.

Let $\phi \in \mathscr{S}(\mathbf{R}^d)\backslash 0$ be fixed. Then the *short-time Fourier transform* $V_\phi f$ of $f \in \mathscr{S}'(\mathbf{R}^d)$ with respect to the *window function* ϕ is defined by

$$V_\phi f(x,\xi) \equiv (\mathscr{F}_2(U(f \otimes \phi)))(x,\xi) = \mathscr{F}(f\,\overline{\phi(\,\cdot\,-x)})(\xi),$$

where $(UF)(x,y) = F(y, y-x)$. If $f \in \mathscr{S}(\mathbf{R}^d)$, then it follows that

$$V_\phi f(x,\xi) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(y)\overline{\phi(y-x)}e^{-i\langle y,\xi\rangle}\,dy.$$

Let $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $p,q \in (0,\infty]$ and $\phi \in \mathscr{S}(\mathbf{R}^d)\backslash 0$ be fixed. Then the mixed Lebesgue space $L^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ consists of all measurable functions F on $\mathbf{R}^{2d}$ such that $\|F\|_{L^{p,q}_{(\omega)}} < \infty$. Here

$$\|F\|_{L^{p,q}_{(\omega)}} \equiv \|F_{p,\omega}\|_{L^q}, \quad \text{where} \quad F_{p,\omega}(\xi) \equiv \|F(\,\cdot\,,\xi)\omega(\,\cdot\,,\xi)\|_{L^p}. \tag{2.10}$$

We note that these quasi-norms might attain $+\infty$.

The *modulation space* $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ is the quasi-Banach space which consist of all $f \in \mathscr{S}'(\mathbf{R}^d)$ such that $\|f\|_{M^{p,q}_{(\omega)}} < \infty$, where

$$\|f\|_{M^{p,q}_{(\omega)}} \equiv \|V_\phi f\|_{L^{p,q}_{(\omega)}}. \tag{2.11}$$

For conveniency we set $M^{p,q} = M^{p,q}_{(\omega)}$ when $\omega \equiv 1$. We remark that the definition of $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ is independent of the choice of $\phi \in \mathscr{S}(\mathbf{R}^d)\backslash 0$ and different ϕ gives rise to equivalent quasi-norms. (See e. g. [8,9,11,14] for general properties of modulation spaces).

3 Kernel Theorems

In this section we deduce suitable kernel theorems for operators between Pilipović spaces and their duals. Since the spaces under considerations can in convenient ways be formulated in terms of Hermite series expansions, we may easily reduce ourselves to kernel results for matrix operators, in similar ways as in e. g. [20].

We begin with the following result concerning kernel properties of matrix operators. Here we identify linear operators on discrete sets by their matrices.

Proposition 3.1 *Let ϑ_k be weight functions on $\mathbf{N}^{d_k}$, $k = 1,2$, $\vartheta(\alpha,\beta) = \vartheta_1(\beta)^{-1}\vartheta_2(\alpha)$, and let T be a linear and continuous map from $\ell^1_{[\vartheta_1]}(\mathbf{N}^{d_1})$ to $\ell^\infty_{[\vartheta_2]}(\mathbf{N}^{d_2})$. Then the following is true:*

(1) *If $A \in \ell^\infty_{[\vartheta]}(\mathbf{N}^{d_2+d_1})$, then the map $f \mapsto A\cdot f$ from $\ell_0(\mathbf{N}^{d_1})$ to $\ell_0'(\mathbf{N}^{d_2})$ extends uniquely to a linear and continuous map from $\ell^1_{[\vartheta_1]}(\mathbf{N}^{d_1})$ to $\ell^\infty_{[\vartheta_2]}(\mathbf{N}^{d_2})$;*

(2) *there is a unique element $A \in \ell^\infty_{[\vartheta]}(\mathbf{N}^{d_2+d_1})$ such that $Tf = A\cdot f$ for every $f \in \ell^1_{[\vartheta_1]}(\mathbf{N}^{d_1})$. Furthermore,*

$$\|T\|_{\ell^1_{[\vartheta_1]}(\mathbf{N}^{d_1})\to\ell^\infty_{[\vartheta_2]}(\mathbf{N}^{d_2})} = \|A\|_{\ell^\infty_{[\vartheta]}}. \tag{3.1}$$

Proof The assertion (1) follows by straight-forward estimates and is left for the reader.
(2) Evidently, for some unique (matrix)

$$A = (a_{\alpha,\beta})_{(\alpha,\beta)\in\mathbf{N}^{d_2+d_1}} \in \ell_0'(\mathbf{N}^{d_2+d_1}),$$

$Tf = A\cdot f$ holds for every $f \in \ell_0(\mathbf{N}^{d_1})$. Moreover, let

$$\Omega_1 = \{\, f_1\in\ell_0(\mathbf{N}^{d_1})\,;\ \|f_1\|_{\ell^1_{[\vartheta_1]}} \le 1\,\} \quad\text{and}\quad \Omega_2 = \{\, f_2 \in \ell_0(\mathbf{N}^{d_2})\,;\ \|f_2\|_{\ell^1_{[1/\vartheta_2]}} \le 1\,\}.$$

Since $\ell_0(\mathbf{N}^{d_1})$ and $\ell_0(\mathbf{N}^{d_2})$ are dense in $\ell^1_{[\vartheta_1]}(\mathbf{N}^{d_1})$ and $\ell^1_{[1/\vartheta_2]}(\mathbf{N}^{d_2})$, respectively, we obtain

$$\begin{aligned}\|T\|_{\ell^1_{[\vartheta_1]}(\mathbf{N}^{d_1})\to\ell^\infty_{[\vartheta_2]}(\mathbf{N}^{d_2})} &= \sup_{f_1\in\Omega_1}\sup_{f_2\in\Omega_2} |(A\cdot f_1, f_2)_{\ell^2}| \\ &= \sup_{f_1\in\Omega_1}\|A\cdot f_1\|_{\ell^\infty_{[\vartheta_2]}} = \sup_{\beta\in\mathbf{N}^{d_1}}\sup_{\alpha\in\mathbf{N}^{d_2}} |a_{\alpha,\beta}\vartheta_1(\beta)^{-1}\vartheta_2(\alpha)| = \|A\|_{\ell^\infty_{[\vartheta]}},\end{aligned}$$

which gives (2). □

By the links between $\mathcal{H}^p_{[\vartheta_k]}(\mathbf{R}^{d_k})$ and $\mathcal{H}^p_{[\vartheta]}(\mathbf{R}^{d_2}\times\mathbf{R}^{d_1})$, and $\ell^p_{[\vartheta_k]}(\mathbf{N}^{d_k})$ and $\ell^p_{[\vartheta]}(\mathbf{N}^{d_2}\times\mathbf{N}^{d_1})$, respectively, the previous proposition immediately gives the following. (Cf. Remark 2.6.)

Proposition 3.2 *Let ϑ_k be weight functions on $\mathbf{N}^{d_k}$, $k = 1, 2$, $\vartheta(\alpha_2,\beta) = \vartheta_1(\beta)^{-1}\vartheta_2(\alpha_2)$, and let T be a linear and continuous map from $\mathcal{H}^1_{[\vartheta_1]}(\mathbf{R}^{d_1})$ to $\mathcal{H}^\infty_{[\vartheta_2]}(\mathbf{R}^{d_2})$. Then the following is true:*

(1) *If $K \in \mathcal{H}^\infty_{[\vartheta]}(\mathbf{R}^{d_2}\times\mathbf{R}^{d_1})$, then the map*

$$f \mapsto \big(x_2 \mapsto \langle K(x_2,\,\cdot\,), f\rangle\big) \tag{3.2}$$

from $\mathcal{H}_0(\mathbf{R}^{d_1})$ to $\mathcal{H}_0'(\mathbf{R}^{d_2})$ extends uniquely to a linear and continuous map from $\mathcal{H}^1_{[\vartheta_1]}(\mathbf{R}^{d_1})$ to $\mathcal{H}^\infty_{[\vartheta_2]}(\mathbf{R}^{d_2})$;

(2) *there is a unique element $K \in \mathcal{H}^\infty_{[\vartheta]}(\mathbf{R}^{d_2}\times\mathbf{R}^{d_1})$ such that*

$$Tf = \big(x_2 \mapsto \langle K(x_2,\,\cdot\,), f\rangle\big) \tag{3.3}$$

for every $f \in \mathcal{H}^1_{[\vartheta_1]}(\mathbf{R}^{d_1})$. Furthermore,

$$\|T\|_{\mathcal{H}^1_{[\vartheta_1]}(\mathbf{R}^{d_1})\to\mathcal{H}^\infty_{[\vartheta_2]}(\mathbf{R}^{d_2})} = \|K\|_{\mathcal{H}^\infty_{[\vartheta]}}. \tag{3.4}$$

We now have the following kernel results.

Theorem 3.3 *Let $s \in \overline{\mathbf{R}_\flat}$, and let T be the linear and continuous map from $\mathcal{H}_0(\mathbf{R}^{d_1})$ to $\mathcal{H}_0'(\mathbf{R}^{d_2})$. Then the following is true:*

(1) *if T is a linear and continuous map from $\mathcal{H}_s'(\mathbf{R}^{d_1})$ to $\mathcal{H}_s(\mathbf{R}^{d_2})$, then there is $K \in \mathcal{H}_s(\mathbf{R}^{d_2} \times \mathbf{R}^{d_1})$ such that (3.3) holds true;*
(2) *if T is a linear and continuous map from $\mathcal{H}_s(\mathbf{R}^{d_1})$ to $\mathcal{H}_s'(\mathbf{R}^{d_2})$, then there is $K \in \mathcal{H}_s'(\mathbf{R}^{d_2} \times \mathbf{R}^{d_1})$ such that (3.3) holds true.*

The same holds true if the $\mathcal{H}_s$ and $\mathcal{H}_s'$ spaces are replaced by $\mathcal{H}_{0,s}$ and $\mathcal{H}_{0,s}'$ spaces, respectively, or by $\mathscr{S}$ and $\mathscr{S}'$ spaces, respectively.

Theorem 3.4 *Let $K \in \mathcal{H}_0'(\mathbf{R}^{d_2} \times \mathbf{R}^{d_1})$, $s \in \overline{\mathbf{R}_\flat}$ and let T be the linear and continuous map from $\mathcal{H}_0(\mathbf{R}^{d_1})$ to $\mathcal{H}_0'(\mathbf{R}^{d_2})$, given by (3.2). Then the following is true:*

(1) *if $K \in \mathcal{H}_s(\mathbf{R}^{d_2} \times \mathbf{R}^{d_1})$, then T extends uniquely to a linear and continuous map from $\mathcal{H}_s'(\mathbf{R}^{d_1})$ to $\mathcal{H}_s(\mathbf{R}^{d_2})$;*
(2) *if $K \in \mathcal{H}_s'(\mathbf{R}^{d_2} \times \mathbf{R}^{d_1})$, then T extends uniquely to a linear and continuous map from $\mathcal{H}_s(\mathbf{R}^{d_1})$ to $\mathcal{H}_s'(\mathbf{R}^{d_2})$.*

The same holds true if the $\mathcal{H}_s$ and $\mathcal{H}_s'$ spaces are replaced by $\mathcal{H}_{0,s}$ and $\mathcal{H}_{0,s}'$ spaces, respectively, or by $\mathscr{S}$ and $\mathscr{S}'$ spaces, respectively.

Proof of Theorems 3.3 and 3.4 Let $p \in [1, \infty]$,

$$\vartheta_r(\alpha) = \begin{cases} e^{r|\alpha|^{\frac{1}{2s}}}, & s \in \mathbf{R}_+ \bigcup \{0\}, \\ r^{|\alpha|}(\alpha!)^{\frac{1}{2\sigma}}, & s = \flat_\sigma, \end{cases}$$

and $\sigma_r(\alpha) = \langle \alpha \rangle^r$. The results follow from Proposition 3.2, and the facts that

$$\mathcal{H}_s = \bigcup_{r>0} \mathcal{H}^p_{[\vartheta_r]}, \quad \mathcal{H}_{0,s}' = \bigcup_{r>0} \mathcal{H}^p_{[1/\vartheta_r]}, \quad \mathscr{S}' = \bigcup_{r>0} \mathcal{H}^p_{[1/\sigma_r]},$$

with suitable inductive limit topologies, and

$$\mathcal{H}_s' = \bigcap_{r>0} \mathcal{H}^p_{[1/\vartheta_r]}, \quad \mathcal{H}_{0,s} = \bigcap_{r>0} \mathcal{H}^p_{[\vartheta_r]}, \quad \mathscr{S} = \bigcap_{r>0} \mathcal{H}^p_{[\sigma_r]},$$

with suitable projective limit topologies. □

Evidently, the assertions on $\mathscr{S}$ and $\mathscr{S}'$ in Theorems 3.3 and 3.4 are well-known. For the other cases, the results are straight-forward consequences of the nuclearity of $\mathcal{H}^1_{[\vartheta]}(\mathbf{R}^{d_2} \times \mathbf{R}^{d_1})$ (cf. e. g. [12] or [29]).

For completeness we also write down some of the corresponding results in the matrix case. The proofs follow by similar arguments as for the proofs of Theorems 3.4 and 3.3, and are left for the reader. Here we recall that $\ell_{\mathscr{S}}(\mathbf{N}^{d_2} \times \mathbf{N}^{d_1})$ is the set of all matrices $A = (a_{\alpha,\beta})_{(\alpha,\beta)\in\mathbf{N}^{d_2+d_1}}$ such that

$$|a_{\alpha,\beta}| \lesssim \langle (\alpha, \beta) \rangle^{-N} \quad \text{for every} \quad N \ge 0$$

and $\ell'_{\mathscr{S}}(\mathbf{N}^{d_2} \times \mathbf{N}^{d_1})$ is the set of all such matrices such that

$$|a_{\alpha,\beta}| \lesssim \langle (\alpha, \beta) \rangle^N \quad \text{for some} \quad N \ge 0.$$

Theorem 3.5 *Let $s \in \overline{\mathbf{R}_\flat}$ be real and let T be the linear and continuous map from $\ell_0(\mathbf{N}^{d_1})$ to $\ell'_0(\mathbf{N}^{d_2})$ with matrix $A \in \ell'_0(\mathbf{N}^{d_2} \times \mathbf{N}^{d_1})$. Then the following is true:*

(1) *if $A \in \ell_s(\mathbf{N}^{d_2} \times \mathbf{N}^{d_1})$, then T extends uniquely to linear and continuous mappings from $\ell'_s(\mathbf{N}^{d_1})$ to $\ell_s(\mathbf{N}^{d_2})$;*
(2) *if $A \in \ell'_s(\mathbf{N}^{d_2} \times \mathbf{N}^{d_1})$, then T extends uniquely to linear and continuous mappings from $\ell_s(\mathbf{N}^{d_1})$ to $\ell'_s(\mathbf{N}^{d_2})$.*
(3) *if T is a linear and continuous map from $\ell'_s(\mathbf{N}^{d_1})$ to $\ell_s(\mathbf{N}^{d_2})$, then $A \in \ell_s(\mathbf{N}^{d_2} \times \mathbf{N}^{d_1})$;*
(4) *if T is a linear and continuous map from $\ell_s(\mathbf{N}^{d_1})$ to $\ell'_s(\mathbf{N}^{d_2})$, then $A \in \ell'_s(\mathbf{N}^{d_2} \times \mathbf{N}^{d_1})$.*

The same holds true if ℓ_s and their duals are replaced by $\ell_{0,s}$ and their duals, respectively, or by $\ell_{\mathscr{S}}$ and their duals, respectively.

4 Factorizations of Pilipović and Gelfand–Shilov Kernels, and Pseudo-Differential Operators

In this section we deduce convenient factorization properties for operators with kernels in Pilipović spaces.

In what follows we use the convention that if T_0 is a linear and continuous operator from $\mathcal{H}_0(\mathbf{R}^{d_1})$ to $\mathcal{H}'_0(\mathbf{R}^{d_2})$, and $g \in \mathcal{H}'_0(\mathbf{R}^{d_0})$, then $T_0 \otimes g$ is the linear and continuous operator from $\mathcal{H}_0(\mathbf{R}^{d_1})$ to $\mathcal{H}'_0(\mathbf{R}^{d_2+d_0})$, given by

$$(T_0 \otimes g) \; : \; f \mapsto (T_0 f) \otimes g.$$

In the following definition we recall that an operator T from $\mathcal{H}_0(\mathbf{R}^d)$ to $\mathcal{H}'_0(\mathbf{R}^d)$ is called *positive semi-definite*, if $(Tf, f)_{L^2} \ge 0$, for every $f \in \mathcal{H}_0(\mathbf{R}^d)$, and then we write $T \ge 0$.

Definition 4.1 Let $d_2 \ge d_1$ and let T be a linear operator from $\mathcal{H}_0(\mathbf{R}^{d_1})$ to $\mathcal{H}'_0(\mathbf{R}^{d_2})$. Then T is said to be a *Hermite diagonal operator* if $T = T_0 \otimes g$, where the Hermite functions are eigenfunctions to T_0, and either $d_2 = d_1$ and $g = 1$, or $d_2 > d_1$ and g is a Hermite function.

Moreover, if $T = T_0 \otimes g$ is a Hermite diagonal operator and T_0 is positive semi-definite, then T is said to be a *positive semi-definite Hermite diagonal operator*.

The first part of the following result can be found in [2,30] (see also [17,21] and the references therein for an elementary proof).

Theorem 4.2 *Let $s \in \mathbf{R}$, T be a linear and continuous operator from $\mathcal{H}_0(\mathbf{R}^{d_1})$ to $\mathcal{H}'_0(\mathbf{R}^{d_2})$ with the kernel K, and let $d_0 \ge \min(d_1, d_2)$. Then the following is true:*

(1) *If* $s \geq 0$ *and* $K \in \mathcal{H}_s(\mathbf{R}^{d_2+d_1})$, *then there are operators* T_1 *and* T_2 *with kernels* $K_1 \in \mathcal{H}_s(\mathbf{R}^{d_0+d_1})$ *and* $K_2 \in \mathcal{H}_s(\mathbf{R}^{d_2+d_0})$ *respectively such that* $T = T_2 \circ T_1$. *Furthermore, if* $j \in \{1, 2\}$ *is fixed and* $d_0 \geq d_j$, *then* T_j *can be chosen as a positive semi-definte Hermite diagonal operator.*
(2) *If* $s > 0$ *and* $K \in \mathcal{H}_{0,s}(\mathbf{R}^{d_2+d_1})$, *then there are operators* T_1 *and* T_2 *with kernels* $K_1 \in \mathcal{H}_{0,s}(\mathbf{R}^{d_0+d_1})$ *and* $K_2 \in \mathcal{H}_{0,s}(\mathbf{R}^{d_2+d_0})$ *respectively such that* $T = T_2 \circ T_1$. *Furthermore, if* $j \in \{1, 2\}$ *is fixed and* $d_0 \geq d_j$, *then* T_j *can be chosen as a positive semi-definte Hermite diagonal operator.*

The corresponding result for $s = \flat_\sigma$ reads:

Theorem 4.3 *Let* $\sigma > 0$, T *be a linear and continuous operator from* $\mathcal{H}_0(\mathbf{R}^{d_1})$ *to* $\mathcal{H}_0'(\mathbf{R}^{d_2})$ *with the kernel* K. *Then the following is true.*

(1) *If* $K \in \mathcal{H}_{\flat_\sigma}(\mathbf{R}^{d_2+d_1})$, *then there are operators* T_0, T_1 *and* T_2 *with kernels* $K_0 \in \mathcal{H}_{1/2}(\mathbf{R}^{d_2+d_1})$, $K_1 \in \mathcal{H}_{\flat_{2\sigma}}(\mathbf{R}^{d_1+d_1})$ *and* $K_2 \in \mathcal{H}_{\flat_{2\sigma}}(\mathbf{R}^{d_2+d_2})$, *respectively, and* $T = T_2 \circ T_0 \circ T_1$. *Furthermore,* T_1 *and* T_2 *can be chosen as positive semi-definite Hermite diagonal operators;*
(2) *If* $K \in \mathcal{H}_{0,\flat_\sigma}(\mathbf{R}^{d_2+d_1})$, *then there are operators* T_0, T_1 *and* T_2 *with kernels* $K_0 \in \mathcal{H}_{0,1/2}(\mathbf{R}^{d_2+d_1})$, $K_1 \in \mathcal{H}_{0,\flat_{2\sigma}}(\mathbf{R}^{d_1+d_1})$ *and* $K_2 \in \mathcal{H}_{0,\flat_{2\sigma}}(\mathbf{R}^{d_2+d_2})$, *respectively, and* $T = T_2 \circ T_0 \circ T_1$. *Furthermore,* T_1 *and* T_2 *can be chosen as positive semi-definite Hermite diagonal operators.*

Remark 4.4 An operator with kernel in $\mathcal{H}_s(\mathbf{R}^{2d})$ is sometimes called a regularizing operator with respect to $\mathcal{H}_s(\mathbf{R}^d)$, because it extends uniquely to a continuous map from (the large space) $\mathcal{H}_s'(\mathbf{R}^d)$ into (the small space) $\mathcal{H}_s(\mathbf{R}^d)$. Analogously, an operator with kernel in $\mathcal{H}_{0,s}(\mathbf{R}^{2d})$ ($\mathscr{S}(\mathbf{R}^{2d})$) is sometimes called a regularizing operator with respect to $\mathcal{H}_{0,s}(\mathbf{R}^d)$ ($\mathscr{S}(\mathbf{R}^d)$).

Proof of Theorem 4.2 First we assume that $d_0 = d_1$, and start to prove (1). Let $h_{d,\alpha}(x)$ be the Hermite function on $\mathbf{R}^d$ of order $\alpha \in \mathbf{N}^d$. Then K posseses the expansion

$$K(x, y) = \sum_{\alpha\in\mathbf{N}^{d_2}} \sum_{\beta\in\mathbf{N}^{d_1}} a_{\alpha,\beta} h_{d_2,\alpha}(x) h_{d_1,\beta}(y), \tag{4.1}$$

where the coefficients $a_{\alpha,\beta}$ satisfies

$$\sup_{\alpha,\beta} |a_{\alpha,\beta} e^{r(|\alpha|^{\frac{1}{2s}} + |\beta|^{\frac{1}{2s}})}| < \infty, \tag{4.2}$$

for some $r > 0$.

Now let $z \in \mathbf{R}^{d_1}$, and

$$\begin{aligned} K_{0,2}(x, z) &= \sum_{\alpha\in\mathbf{N}^{d_2}} \sum_{\beta\in\mathbf{N}^{d_1}} b_{\alpha,\beta} h_{d_2,\alpha}(x) h_{d_1,\beta}(z), \\ K_{0,1}(z, y) &= \sum_{\alpha,\beta\in\mathbf{N}^{d_1}} c_{\alpha,\beta} h_{d_1,\alpha}(z) h_{d_1,\beta}(y), \end{aligned} \tag{4.3}$$

where

$$b_{\alpha,\beta} = a_{\alpha,\beta} e^{\frac{r}{2}|\beta|^{\frac{1}{2s}}} \quad \text{and} \quad c_{\alpha,\beta} = \delta_{\alpha,\beta} e^{-\frac{r}{2}|\alpha|^{\frac{1}{2s}}}.$$

Here $\delta_{\alpha,\beta}$ is the Kronecker delta. Then it follows that

$$\int K_{0,2}(x,z)K_{0,1}(z,y)\,dz = \sum_{\alpha\in\mathbf{N}^{d_2}} \sum_{\beta\in\mathbf{N}^{d_1}} a_{\alpha,\beta} h_{d_2,\alpha}(x) h_{d_1,\beta}(y) = K(x,y).$$

Hence, if T_j is the operator with kernel $K_{0,j}$, $j = 1, 2$, then $T = T_2 \circ T_1$. Furthermore,

$$\sup_{\alpha,\beta} |b_{\alpha,\beta} e^{\frac{r}{2}(|\alpha|^{\frac{1}{2s}} + |\beta|^{\frac{1}{2s}})}| \le \sup_{\alpha,\beta} |a_{\alpha,\beta} e^{r(|\alpha|^{\frac{1}{2s}} + |\beta|^{\frac{1}{2s}})}| < \infty$$

and

$$\sup_{\alpha,\beta} |c_{\alpha,\beta} e^{\frac{r}{4}(|\alpha|^{\frac{1}{2s}} + |\beta|^{\frac{1}{2s}})}| = \sup_{\alpha} |e^{-\frac{r}{4}|\alpha|^{\frac{1}{2s}}} e^{\frac{r}{4}|\alpha|^{\frac{1}{2s}}}| < \infty.$$

This implies that $K_{0,1} \in \mathcal{H}_s(\mathbf{R}^{d_1+d_1})$ and $K_{0,2} \in \mathcal{H}_s(\mathbf{R}^{d_2+d_1})$, and (1) follows with $K_1 = K_{0,1}$ and $K_2 = K_{0,2}$, in the case $d_0 = d_1$.

In order to prove (2), we assume that $K \in \mathcal{H}_{0,s}(\mathbf{R}^{d_2+d_1})$, and we let $a_{\alpha,\beta}$ be the same as the above. Then (4.2) holds for any $r > 0$, which implies that if $n \ge 0$ is an integer, then

$$\Theta_n \equiv \sup\{\, |\beta|\,;\ |a_{\alpha,\beta}| \ge e^{-2(n+1)(|\alpha|^{\frac{1}{2s}} + |\beta|^{\frac{1}{2s}})} \text{ for some } \alpha \in \mathbf{N}^{d_2} \,\} \tag{4.4}$$

is finite.

We let

$$I_1 = \{\, \beta \in \mathbf{N}^{d_1}\,;\ |\beta| \le \Theta_1 + 1 \,\},$$

and define inductively

$$I_j = \{\, \beta \in \mathbf{N}^{d_1} \backslash I_{j-1}\,;\ |\beta| \le \Theta_j + j \,\}, \quad j \ge 2.$$

Then

$$I_j \cap I_k = \emptyset \quad \text{when} \quad j \ne k, \quad \text{and} \quad \bigcup_{j\ge 0} I_j = \mathbf{N}^{d_1},$$

and by the definitions it follows that I_j is a finite set for every j.

We also let $K_{0,1}$ and $K_{0,2}$ be given by (4.3), where, if $\beta \in I_j$,

$$b_{\alpha_2,\beta} = a_{\alpha_2,\beta} e^{j|\beta|^{\frac{1}{2s}}} \quad \text{and} \quad c_{\alpha_1,\beta} = \delta_{\alpha_1,\beta} e^{-j|\beta|^{\frac{1}{2s}}},$$

when $\alpha_1 \in \mathbf{N}^{d_1}$, $\alpha_2 \in \mathbf{N}^{d_2}$. If T_ℓ is the operator with kernel $K_{0,\ell}$ for $\ell = 1, 2$, then it follows that $T_2 \circ T_1 = T$. Furthermore, if $r > 0$, then we have

$$\sup_{\alpha,\beta} |b_{\alpha,\beta} e^{r(|\alpha|^{\frac{1}{2s}} + |\beta|^{\frac{1}{2s}})}| \le J_1 + J_2,$$

where

$$J_1 = \sup_{j \le r+1} \sup_{\alpha} \sup_{\beta \in I_j} |b_{\alpha,\beta} e^{r(|\alpha|^{\frac{1}{2s}} + |\beta|^{\frac{1}{2s}})}|, \tag{4.5}$$

and

$$J_2 = \sup_{j > r+1} \sup_{\alpha} \sup_{\beta \in I_j} |b_{\alpha,\beta} e^{r(|\alpha|^{\frac{1}{2s}} + |\beta|^{\frac{1}{2s}})}|. \tag{4.6}$$

Since only finite numbers of β is involved in the suprema in (4.5), it follows from (4.2) and the definition of $b_{\alpha,\beta}$ that J_1 is finite.

For J_2 we have

$$\begin{aligned} J_2 &= \sup_{j > r+1} \sup_{\alpha} \sup_{\beta \in I_j} |a_{\alpha,\beta} e^{r|\alpha|^{\frac{1}{2s}} + (r+j)|\beta|^{\frac{1}{2s}}}| \\ &\le \sup_{j > r+1} \sup_{\alpha} \sup_{\beta \in I_j} |e^{-2j(|\alpha|^{\frac{1}{2s}} + |\beta|^{\frac{1}{2s}})} e^{r|\alpha|^{\frac{1}{2s}} + (r+j)|\beta|^{\frac{1}{2s}}}| < \infty, \end{aligned}$$

where the first inequality follows from (4.4). Hence

$$\sup_{\alpha,\beta} |b_{\alpha,\beta} e^{r(|\alpha|^{\frac{1}{2s}} + |\beta|^{\frac{1}{2s}})}| < \infty$$

for every $r > 0$, which implies that $K_{0,2} \in \mathcal{H}_{0,s}(\mathbf{R}^{d_2+d_1})$.

If we now replace $b_{\alpha,\beta}$ with $c_{\alpha,\beta}$ in the definition of J_1 and J_2, it follows by similar arguments that both J_1 and J_2 are finite, also in this case. This gives

$$\sup_{\alpha,\beta} |c_{\alpha,\beta} e^{r(|\alpha|^{\frac{1}{2s}} + |\beta|^{\frac{1}{2s}})}| < \infty$$

for every $r > 0$. Hence $K_1 \in \mathcal{H}_{0,s}(\mathbf{R}^{d_1+d_1})$, and (2) follows in the case $d_0 = d_1$.

Next assume that $d_0 > d_1$, and let $d = d_0 - d_1 \ge 1$. Then we set

$$K_1(z_0, y) = K_{0,1}(z_1, y) h_{d,0}(z) \quad \text{and} \quad K_2(x, z_0) = K_{0,2}(x, z_1) h_{d,0}(z),$$

where $K_{0,j}$ are the same as in the first part of the proofs, $z_1 \in \mathbf{R}^{d_1}$ and $z \in \mathbf{R}^d$, giving that $z_0 = (z_1, z) \in \mathbf{R}^{d_0}$. We get

$$\int_{\mathbf{R}^{d_0}} K_2(x, z_0) K_1(z_0, y)\, dz_0 = \int_{\mathbf{R}^{d_1}} K_{0,2}(x, z_1) K_{0,1}(z_1, y)\, dz_1 = K(x, y).$$

The assertion (1) now follows in the case $d_0 > d_1$ from the implications

$$K_{0,1} \in \mathcal{H}_s(\mathbf{R}^{d_1+d_1}) \quad \Longrightarrow \quad K_1 \in \mathcal{H}_s(\mathbf{R}^{d_0+d_1})$$

and

$$K_{0,2} \in \mathcal{H}_s(\mathbf{R}^{d_2+d_1}) \quad \Longrightarrow \quad K_2 \in \mathcal{H}_s(\mathbf{R}^{d_2+d_0})$$

Since the same implications hold after the $\mathcal{H}_s$ spaces have been replaced by $\mathcal{H}_{0,s}$ spaces, the assertion (2) also follows in the case $d_0 > d_1$, and the theorem follows in the case $d_0 \geq d_1$.

It remains to prove the result in the case $d_0 \geq d_2$. By taking the adjoint, the roles of $j = 1$ and $j = 2$ are interchanged, and the result follows when $d_0 \geq d_2$ as well. The proof is complete. □

Proof of Theorem 4.3 (1) We have

$$K(x, y) = \sum_{\alpha \in \mathbf{N}^{d_2}} \sum_{\beta \in \mathbf{N}^{d_1}} a_{\alpha,\beta} h_{d_2,\alpha}(x) h_{d_1,\beta}(y), \tag{4.7}$$

where

$$\sup_{\alpha,\beta} |a_{\alpha,\beta}(\alpha!\beta!)^{\frac{1}{2\sigma}} R^{-(|\alpha|+|\beta|)}| < \infty$$

for some $R > 1$.

Let $z_j \in \mathbf{R}^{d_j}$, and

$$K_0(z_2, z_1) = \sum_{\alpha \in \mathbf{N}^{d_2}} \sum_{\beta \in \mathbf{N}^{d_1}} a_{0,\alpha,\beta} h_{d_2,\alpha}(z_2) h_{d_1,\beta}(z_1), \tag{4.8}$$

$$K_1(z_1, y) = \sum_{\alpha \in \mathbf{N}^{d_1}} \sum_{\beta \in \mathbf{N}^{d_1}} a_{1,\alpha,\beta} h_{d_1,\alpha}(z_1) h_{d_1,\beta}(y) \tag{4.9}$$

and

$$K_2(x, z_2) = \sum_{\alpha \in \mathbf{N}^{d_2}} \sum_{\beta \in \mathbf{N}^{d_2}} a_{2,\alpha,\beta} h_{d_2,\alpha}(x) h_{d_2,\beta}(z_2), \tag{4.10}$$

where

$$a_{j,\alpha,\beta} = (\alpha!)^{-\frac{1}{2\sigma}} \delta_{\alpha,\beta} R^{2|\alpha|}, \qquad \alpha, \beta \in \mathbf{N}^{d_j}, \quad j = 1, 2,$$

and

$$a_{0,\alpha,\beta} = a_{\alpha,\beta}(\alpha!\beta!)^{\frac{1}{2\sigma}} R^{-2(|\alpha|+|\beta|)}, \qquad \alpha \in \mathbf{N}^{d_2},\ \beta \in \mathbf{N}^{d_1}.$$

Then it follows that

$$\iint_{\mathbf{R}^{d_2+d_1}} K_2(x, z_2) K_0(z_2, z_1) K_1(z_1, y)\, dz_2 dz_1 = K(x, y).$$

Hence, if T_j is the operator with kernel K_j, $j = 0, 1, 2$, then $T = T_2 \circ T_0 \circ T_1$. Furthermore, the kernels lie in the claimed spaces since

$$\sup_{\alpha,\beta} |a_{j,\alpha,\beta}(\alpha!\beta!)^{\frac{1}{4\sigma}} R^{-(|\alpha|+|\beta|)}| < \infty, \quad \alpha, \beta \in \mathbf{N}^{d_j} \quad j = 1, 2,$$

and if $0 < c < \log R$, then

$$\sup_{\alpha,\beta} |a_{0,\alpha,\beta} e^{c(|\alpha|+|\beta|)}| \le \sup_{\alpha,\beta} |R^{-(|\alpha|+|\beta|)} e^{c(|\alpha|+|\beta|)}| < \infty.$$

Next we prove (2). Let $a_{\alpha,\beta}$ be as in (4.7). Then

$$\sup_{\alpha,\beta} |a_{\alpha,\beta}(\alpha!\beta!)^{\frac{1}{2\sigma}} R^{|\alpha|+|\beta|}| < \infty$$

for every $R > 1$, which implies that

$$\Theta_{1,n} \equiv \sup\{\, |\beta|\, ;\ |a_{\alpha,\beta}| \ge (n+1)^{-6(|\alpha|+|\beta|)}(\alpha!\beta!)^{-\frac{1}{2\sigma}} \text{ for some } \alpha \in \mathbf{N}^{d_2}\,\}$$

and

$$\Theta_{2,n} \equiv \sup\{\, |\alpha|\, ;\ |a_{\alpha,\beta}| \ge (n+1)^{-6(|\alpha|+|\beta|)}(\alpha!\beta!)^{-\frac{1}{2\sigma}} \text{ for some } \beta \in \mathbf{N}^{d_1}\,\}$$

are finite for every $n \ge 1$.

We let

$$I_{j,1} = \{\, \gamma \in \mathbf{N}^{d_j}\, ;\ |\gamma| \le \Theta_{j,1} + 1\,\},$$

and define inductively

$$I_{j,m} = \{\, \gamma \in \mathbf{N}^{d_j} \setminus I_{j,m-1}\, ;\ |\gamma| \le \Theta_{j,m} + m\,\}, \quad m \ge 2,\ \ j = 1, 2.$$

Then

$$I_{j,m} \cap I_{j,n} = \emptyset \quad \text{when} \quad m \ne n, \quad \text{and} \quad \bigcup_{m \ge 1} I_{j,m} = \mathbf{N}^{d_j}.$$

and by the definitions it follows that $I_{j,m}$ is a finite set for every m.

We also let K_j, $j = 0, 1, 2$ be given by (4.8)–(4.10), where

$$a_{j,\alpha,\beta} = (\alpha!)^{-\frac{1}{2\sigma}} \delta_{\alpha,\beta} m^{-|\alpha+\beta|}, \qquad \alpha \in I_{j,m}, \quad j = 1, 2,$$

and

$$a_{0,\alpha,\beta} = a_{\alpha,\beta}(\alpha!\beta!)^{\frac{1}{2\sigma}} m_2^{2|\alpha|} m_1^{2|\beta|}, \qquad \alpha \in I_{2,m_2},\ \beta \in I_{1,m_1}.$$

If T_j is the operator with kernel K_j for $j = 0, 1, 2$, then it follows that $T_2 \circ T_0 \circ T_1 = T$ and T_1, T_2 are positive semi-definite Hermite diagonal operators. The result therefore follows if we prove

$$\begin{aligned} &|a_{j,\alpha,\beta}| \lesssim r^{|\alpha+\beta|}(\alpha!\beta!)^{-\frac{1}{4\sigma}}, \ \forall r>0, \quad j=1,2, \quad \text{and} \\ &|a_{0,\alpha,\beta}| \lesssim e^{-r(|\alpha|+|\beta|)}, \qquad \forall r>0, \end{aligned} \tag{4.11}$$

and since

$$\bigcup_{m\le R+1} I_{j,m} \quad \text{and} \quad \bigcup_{m_1+m_2\le 2R} I_{2,m_2}\times I_{1,m_1}$$

are finite sets and $R > 1$ is arbitrary, it suffices to prove

$$\sup_{m>R+1}\ \sup_{\alpha\in I_{j,m}} |(\alpha!\beta!)^{\frac{1}{4\sigma}} R^{|\alpha+\beta|} a_{j,\alpha,\beta}| < \infty, \quad j=1,2, \tag{4.12}$$

and

$$\sup_{(m_1,m_2)\in M_k}\ \sup_{\alpha\in I_{2,m_2}}\ \sup_{\beta\in I_{1,m_1}} |R^{|\alpha|+|\beta|} a_{0,\alpha,\beta}| < \infty, \quad k=1,2,3, \tag{4.13}$$

where

$$\begin{aligned} M_1 &= \{\,(m_1,m_2)\in \mathbf{Z}_+^2\, ;\ m_1\ge 2R-1,\ m_2=1\,\},\\ M_2 &= \{\,(m_1,m_2)\in \mathbf{Z}_+^2\, ;\ m_2\ge 2R-1,\ m_1=1\,\} \end{aligned}$$

and

$$M_3 = \{\,(m_1,m_2)\in \mathbf{Z}_+^2\, ;\ m_1+m_2\ge 2R,\ m_1,m_2\ge 2\,\},$$

We have

$$\begin{aligned} &\sup_{m>R+1}\ \sup_{\alpha\in I_{j,m}} |(\alpha!\beta!)^{\frac{1}{4\sigma}} R^{|\alpha+\beta|} a_{j,\alpha,\beta}| \\ &\qquad\qquad = \sup_{m>R+1}\ \sup_{\alpha,\beta\in I_{j,m}} |\delta_{\alpha,\beta} R^{|\alpha+\beta|} m^{-|\alpha+\beta|}| < \infty, \quad j=1,2, \end{aligned}$$

and (4.12) follows.

Next we prove (4.13), and start with the case $k = 1$. Then $\beta \in I_{1,m_1}$ gives

$$|a_{\alpha,\beta}| \le m_1^{-6(|\alpha|+|\beta|)}(\alpha!\beta!)^{-\frac{1}{2\sigma}},$$

which, by the fact that $m_2 = 1$, implies

$$|a_{0,\alpha,\beta}| = |a_{\alpha,\beta}|(\alpha!\beta!)^{\frac{1}{2\sigma}} m_1^{2|\beta|} \le m_1^{-4(|\alpha|+|\beta|)}.$$

Hence,

$$\sup_{(m_1,m_2)\in M_1}\ \sup_{\alpha\in I_{2,1}}\ \sup_{\beta\in I_{1,m_1}} |R^{|\alpha|+|\beta|}a_{0,\alpha,\beta}| \le \sup_{m_1>R>1} R^{|\alpha|+|\beta|}m_1^{-4(|\alpha|+|\beta|)} < \infty,$$

and (4.13) follows in the case $k=1$.

In the same way, (4.13) follows in the case $k=2$.

Next we prove (4.13) in the case $k=3$. By the definitions it follows that if $\alpha\in I_{2,m_2}$, then

$$|a_{\alpha,\beta}| \le m_2^{-6(|\alpha|+|\beta|)}(\alpha!\beta!)^{-\frac{1}{2\sigma}}, \quad \forall\, \beta\in \mathbf{N}^{d_1},$$

and if $\beta\in I_{1,m_1}$, then

$$|a_{\alpha,\beta}| \le m_1^{-6(|\alpha|+|\beta|)}(\alpha!\beta!)^{-\frac{1}{2\sigma}}, \quad \forall\, \alpha\in \mathbf{N}^{d_2}.$$

Hence, if $\alpha\in I_{2,m_2}$ and $\beta\in I_{1,m_1}$, the geometric mean of the right-hand sides of the inequalities becomes

$$|a_{\alpha,\beta}| \le (m_1m_2)^{-3(|\alpha|+|\beta|)}(\alpha!\beta!)^{-\frac{1}{2\sigma}},$$

giving that

$$|a_{0,\alpha,\beta}| \le (m_1m_2)^{-(|\alpha|+|\beta|)}.$$

This gives

$$\begin{aligned}
&\sup_{(m_1,m_2)\in M_3}\ \sup_{\alpha\in I_{2,m_2}}\ \sup_{\beta\in I_{1,m_1}} |R^{|\alpha|+|\beta|}a_{0,\alpha,\beta}|\\
&\qquad \le \sup_{(m_1,m_2)\in M_3}\ \sup_{\alpha\in I_{2,m_2}}\ \sup_{\beta\in I_{1,m_1}} R^{|\alpha|+|\beta|}(m_1m_2)^{-(|\alpha|+|\beta|)} < \infty,
\end{aligned}$$

and (4.13), and thereby (4.11) follows. □

Remark 4.5 Let $\sigma>0$ and $T\ge 0$ be a Hermite diagonal operator on $L^2(\mathbf{R}^d)$ with kernel K in $\mathcal{H}_{\flat_\sigma}$. By the proof of Theorem 4.3, there are Hermite diagonal operators $T_1\ge 0$ and $T_2\ge 0$ on $L^2(\mathbf{R}^d)$ with kernels K_1 and K_2 such that

$$K_1\in \mathcal{H}_{\flat_\sigma}(\mathbf{R}^{2d}), \quad K_2\in \mathcal{H}_{1/2}(\mathbf{R}^{2d}) \quad \text{and} \quad T=T_1\circ T_2=T_2\circ T_1.$$

In fact, if K is given by (4.7) with $d_1=d_2=d$, it suffices to let K_1 and K_2 be given by (4.9) and (4.10), where

$$a_{1,\alpha,\beta}=R^{|\alpha+\beta|}(a_{\alpha,\beta})^{1/2}(\alpha!\beta!)^{-\frac{1}{4\sigma}} \quad \text{and} \quad a_{2,\alpha,\beta}=R^{-|\alpha+\beta|}(a_{\alpha,\beta})^{1/2}(\alpha!\beta!)^{\frac{1}{4\sigma}}$$

with $R\ge 1$ sufficiently large.

Remark 4.6 From the construction of K_1 and K_2 in the proofs of Theorems 4.2 and 4.3, it follows that it is not so difficult to use numerical methods for approximations of candidates to K_1 and K_2. In fact, K_1 and K_2 are formed explicitly by the elements of the matrix for T, when the Hermite functions are used as basis for $\mathscr{S}$, $\mathcal{H}_s$ and $\mathcal{H}_{0,s}$.

We finish the section by presenting some consequences in the calculus of pseudo-differential operators. The following result is an immediate consequence of Theorem 4.2 and the fact that the map $a \mapsto K_{A,a}$ is continuous and bijective on $\mathcal{S}_{s_1}(\mathbf{R}^{2d})$, and on $\Sigma_{s_2}(\mathbf{R}^{2d})$, for every $s_1 \ge \frac{1}{2}$ and $s_2 > \frac{1}{2}$.

Theorem 4.7 *Let A be a real $d \times d$-matrix, $s_1 \ge \frac{1}{2}$ and let $s_2 > \frac{1}{2}$. Then the following is true:*

(1) *if $a \in \mathcal{H}_{s_1}(\mathbf{R}^{2d})$, then there are $a_1, a_2 \in \mathcal{H}_{s_1}(\mathbf{R}^{2d})$ such that $a = a_1 \#_A a_2$;*
(2) *if $a \in \mathcal{H}_{0,s_2}(\mathbf{R}^{2d})$, then there are $a_1, a_2 \in \mathcal{H}_{0,s_2}(\mathbf{R}^{2d})$ such that $a = a_1 \#_A a_2$.*

Remark 4.8 Extensions of Theorem 4.7 to the case where s_1 and s_2 are allowed to be smaller than $\frac{1}{2}$ is not so smooth, because those Pilipović spaces which are not Gelfand–Shilov spaces, are not invariant under dilations (cf. [26, Proposition 7.4]). However, if A is a real $d \times d$ matrix and $a \in \mathscr{S}(\mathbf{R}^{2d})$ is such that the kernel $K_{A,a}$ belongs to $\mathcal{H}_s(\mathbf{R}^{2d})$, then we may apply Theorem 4.2 in this situation as well.

Therefore, let $\mathcal{G}_{A,s}(\mathbf{R}^{2d})$ ($\mathcal{G}_{0,A,s}(\mathbf{R}^{2d})$) be the set of all $a \in \mathscr{S}(\mathbf{R}^{2d})$ such that $K_{A,a} \in \mathcal{H}_s(\mathbf{R}^{2d})$ ($K_{A,a} \in \mathcal{H}_{0,s}(\mathbf{R}^{2d})$). If $a \in \mathcal{G}_{A,s}(\mathbf{R}^{2d})$ ($a \in \mathcal{G}_{0,A,s}(\mathbf{R}^{2d})$), then there are elements $a_1, a_2 \in \mathcal{G}_{A,s}(\mathbf{R}^{2d})$ ($a_1, a_2 \in \mathcal{G}_{0,A,s}(\mathbf{R}^{2d})$) such that $a = a_1 \#_A a_2$.

5 Singular Value Estimates and Schatten–von Neumann Properties for Operators with Pilipović Kernels

In this section we use Theorem 4.2 to obtain estimates of the form (1.3) for operators T with kernels in Pilipović spaces of order s, provided $\mathscr{B}_1$ and $\mathscr{B}_2$ stay between the given Pilipović space and its dual. In particular it follows that any such operator belongs to any Schatten–von Neumann class.

In the following result we show that the singular values for operators T_K with kernels K in Pilipović spaces or Schwartz spaces, obey estimates of the form

$$\sigma_k(T_K, \mathscr{B}_1, \mathscr{B}_2) \lesssim e^{-rk^{\frac{1}{2ds}}}, \tag{5.1}$$

$$\sigma_k(T_K, \mathscr{B}_1, \mathscr{B}_2) \lesssim R^k (k!)^{-\frac{1}{2\sigma d}} \tag{5.2}$$

or

$$\sigma_k(T_K, \mathscr{B}_1, \mathscr{B}_2) \lesssim k^{-N}. \tag{5.3}$$

Here $V_1 \hookrightarrow V_2$ means that the topological space V_1 is continuously embedded in the topological space V_2.

Theorem 5.1 *Let $p \in (0, \infty]$, $s \ge 0$ be real, $\sigma > 0$ and let $d = \min(d_1, d_2)$. Then the following is true:*

(1) *if $K \in \mathcal{H}_s(\mathbf{R}^{d_2+d_1})$, and $\mathscr{B}_1$ and $\mathscr{B}_2$ are quasi-Banach spaces such that $\mathscr{B}_1 \hookrightarrow \mathcal{H}'_s(\mathbf{R}^{d_1})$ and $\mathcal{H}_s(\mathbf{R}^{d_2}) \hookrightarrow \mathscr{B}_2$, then (5.1) holds for some $r > 0$. In particular, $T_K \in \mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)$;*

(2) *if* $K \in \mathcal{H}_{0,s}(\mathbf{R}^{d_2+d_1})$, *and* $\mathscr{B}_1$ *and* $\mathscr{B}_2$ *are quasi-Banach spaces such that* $\mathscr{B}_1 \hookrightarrow \mathcal{H}'_{0,s}(\mathbf{R}^{d_1})$ *and* $\mathcal{H}_{0,s}(\mathbf{R}^{d_2}) \hookrightarrow \mathscr{B}_2$, *then* (5.1) *holds for every* $r > 0$. *In particular,* $T_K \in \mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)$;
(3) *if* $K \in \mathcal{H}_{\flat_\sigma}(\mathbf{R}^{d_2+d_1})$, *and* $\mathscr{B}_1$ *and* $\mathscr{B}_2$ *are quasi-Banach spaces such that* $\mathscr{B}_1 \hookrightarrow \mathcal{H}'_{1/2}(\mathbf{R}^{d_1})$ *and* $\mathcal{H}_{1/2}(\mathbf{R}^{d_2}) \hookrightarrow \mathscr{B}_2$, *then* (5.2) *holds for some* $R > 0$. *In particular,* $T_K \in \mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)$;
(4) *if* $K \in \mathcal{H}_{0,\flat_\sigma}(\mathbf{R}^{d_2+d_1})$, *and* $\mathscr{B}_1$ *and* $\mathscr{B}_2$ *are quasi-Banach spaces such that* $\mathscr{B}_1 \hookrightarrow \mathcal{H}'_{0,1/2}(\mathbf{R}^{d_1})$ *and* $\mathcal{H}_{0,1/2}(\mathbf{R}^{d_2}) \hookrightarrow \mathscr{B}_2$, *then* (5.2) *holds for every* $R > 0$. *In particular,* $T_K \in \mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)$;
(5) *if* $K \in \mathscr{S}(\mathbf{R}^{d_2+d_1})$, *and* $\mathscr{B}_1$ *and* $\mathscr{B}_2$ *are quasi-Banach spaces such that* $\mathscr{B}_1 \hookrightarrow \mathscr{S}'(\mathbf{R}^{d_1})$ *and* $\mathscr{S}(\mathbf{R}^{d_2}) \hookrightarrow \mathscr{B}_2$, *then* (5.3) *holds for every* $N > 0$. *In particular,* $T_K \in \mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)$.

We observe that Theorem 5.1 (5) should be available in the literature.

We need some preparations for the proof. First we recall that if $\mathscr{B}_j$, $j = 0, 1, 2$, are quasi-Banach spaces and T_j are linear and continuous mappings from $\mathscr{B}_{j-1}$ to $\mathscr{B}_j$, $j = 1, 2$, then there is a constant C such that

$$\boldsymbol{\sigma}_k(T_2 \circ T_1, \mathscr{B}_0, \mathscr{B}_2) \le C\|T_1\|_{\mathscr{B}_0 \to \mathscr{B}_1} \boldsymbol{\sigma}_k(T_2, \mathscr{B}_1, \mathscr{B}_2) \tag{5.4}$$

and

$$\boldsymbol{\sigma}_k(T_2 \circ T_1, \mathscr{B}_0, \mathscr{B}_2) \le C\|T_2\|_{\mathscr{B}_1 \to \mathscr{B}_2} \boldsymbol{\sigma}_k(T_1, \mathscr{B}_0, \mathscr{B}_1). \tag{5.5}$$

In fact, if $\Omega_{j,l}(k)$ is the set of all linear operators from $\mathscr{B}_j$ to $\mathscr{B}_l$ with rank at most $k-1$, then

$$\begin{aligned}
\boldsymbol{\sigma}_k(T_2 \circ T_1, \mathscr{B}_0, \mathscr{B}_2) &= \inf_{S \in \Omega_{0,2}(k)} \|T_2 \circ T_1 - S\|_{\mathscr{B}_0 \to \mathscr{B}_2} \\
&\le \inf_{T_0 \in \Omega_{0,1}(k)} \|T_2 \circ T_1 - T_2 \circ T_0\|_{\mathscr{B}_0 \to \mathscr{B}_2} \\
&\lesssim \|T_2\|_{\mathscr{B}_1 \to \mathscr{B}_2} \left(\inf_{T_0 \in \Omega_{0,1}(k)} \|T_1 - T_0\|_{\mathscr{B}_0 \to \mathscr{B}_1} \right) \\
&= \|T_2\|_{\mathscr{B}_1 \to \mathscr{B}_2} \boldsymbol{\sigma}_k(T_1, \mathscr{B}_0, \mathscr{B}_1),
\end{aligned}$$

which gives (5.5). In the same way (5.4) is obtained. (See also [22]).

Proof of Theorem 5.1 We only prove (1), (3) and (5). The assertions (2) and (4) follow by similar arguments and are left for the reader.

(1) By Theorem 4.2 we get

$$T_K = T_{K_3} \circ T_{K_2} \circ T_{K_1}, \tag{5.6}$$

where the kernels K_1, K_2 and K_3 of the operators T_{K_1}, T_{K_2} and T_{K_3} belong to $\mathcal{H}_s(\mathbf{R}^{d_1+d_1})$, $\mathcal{H}_s(\mathbf{R}^{d_2+d_1})$ and $\mathcal{H}_s(\mathbf{R}^{d_2+d_2})$, respectively, and T_{K_2} is a positive semi-definite Hermite diagonal operator.

It follows that T_{K_1} is continuous from $\mathscr{B}_1$ to $L^2(\mathbf{R}^{d_1})$, and T_{K_3} is continuous from $L^2(\mathbf{R}^{d_2})$ to $\mathscr{B}_2$. Hence, by (5.4) and (5.5) it suffices to prove that, if $T = T_{K_2}$, then

$$\boldsymbol{\sigma}_k = \boldsymbol{\sigma}_k(T, L^2(\mathbf{R}^{d_1}), L^2(\mathbf{R}^{d_2})) \le Ce^{-ck^{\frac{1}{2ds}}}, \tag{5.7}$$

for some positive constants c and C.

By the constructions we have, setting $d_0 = |d_1 - d_2|$,

$$K_2(x,y) = K_{0,2}(x,y_1)h(y_2), \qquad y = (y_1,y_2), \qquad x, y_1 \in \mathbf{R}^d, \qquad y_2 \in \mathbf{R}^{d_0},$$

when $d_1 \ge d_2$, and

$$K_2(x,y) = h(x_1)K_{0,2}(x_2,y), \qquad x = (x_1,x_2), \qquad x_2, y \in \mathbf{R}^d, \qquad x_1 \in \mathbf{R}^{d_0}$$

when $d_2 \ge d_1$, where

$$K_{0,2}(x,y) = \sum_{\alpha\in\mathbf{N}^d} c_\alpha h_\alpha(x)h_\alpha(y), \quad x, y \in \mathbf{R}^d, \tag{5.8}$$

with

$$0 \le c_\alpha \lesssim e^{-r|\alpha|^{\frac{1}{2s}}} \tag{5.9}$$

for some constant $r > 0$. Here h is a fixed Hermite function on $\mathbf{R}^{d_0}$ when $d_0 > 0$, and $h \equiv 1$ otherwise. We observe that (5.8) describes the spectral decomposition of $T_{K_{0,2}}$, with $\{h_\alpha\}_{\alpha\in\mathbf{N}^{d_0}}$ as the orthonormal basis of eigenfunctions, and with eigenvalues $\{c_\alpha\}_{\alpha\in\mathbf{N}^{d_0}}$. Furthermore, it is evident that

$$\boldsymbol{\sigma}_k(T_{K_{0,2}}, L^2(\mathbf{R}^d), L^2(\mathbf{R}^d)) = \boldsymbol{\sigma}_k(T_{K_2}, L^2(\mathbf{R}^{d_1}), L^2(\mathbf{R}^{d_2})), \quad k \ge 1.$$

Hence it suffices to prove (5.7) in the case $d_1 = d_2 = d$.

Let $M_{N,d}$ be the number of all multi-indices $\alpha \in \mathbf{N}^d$ such that $|\alpha| \le N$. Then $M_{N,d} \asymp \langle N\rangle^d$. Since the singular values are the eigenvalues of T_{K_2} in non-increasing order, (5.9) gives

$$\boldsymbol{\sigma}_k(T_{K_2}) \lesssim e^{-rN^{\frac{1}{2s}}}$$

for some $r > 0$ when $M_{N-1,d} < k \le M_{N,d}$. For such k we also have $k \asymp N^d$, since $(N-1)^d \asymp N^d$. By combining these estimates we get

$$\boldsymbol{\sigma}_k(T_K) \lesssim e^{-rN^{\frac{1}{2s}}} \lesssim e^{-r_0k^{\frac{1}{2ds}}}$$

for some constant r_0. This gives (5.1).

(3) By Theorem 4.3 and Remark 4.5, we get

$$T_K = T_{K_{0,2}} \circ T_{K_2} \circ T_{K_0} \circ T_{K_1} \circ T_{K_{0,1}}, \tag{5.10}$$

where the corresponding kernels satisfy

$$K_{0,j} \in \mathcal{H}_{1/2}(\mathbf{R}^{d_j+d_j}), \quad K_j \in \mathcal{H}_{\flat_{2\sigma}}(\mathbf{R}^{d_j+d_j}), \quad \text{and} \quad K_0 \in \mathcal{H}_{1/2}(\mathbf{R}^{d_2+d_1}),$$

$j = 1, 2$. Furthermore, all kernels except K_0 to the operators in (5.10) are positive semi-definite Hermite diagonal operators.

It follows that

$$T_{K_{0,1}} : \mathscr{B}_1 \to L^2(\mathbf{R}^{d_1}), \quad T_{K_0} : L^2(\mathbf{R}^{d_1}) \to L^2(\mathbf{R}^{d_2})$$
$$\text{and} \quad T_{K_{0,2}} : L^2(\mathbf{R}^{d_2}) \to \mathscr{B}_2,$$

are continuous. By similar arguments as in the proof of (1), we get

$$\boldsymbol{\sigma}_k(T_{K_j}, L^2(\mathbf{R}^{d_j}), L^2(\mathbf{R}^{d_j})) \lesssim R^k (k!)^{-\frac{1}{2\sigma d_j}}, \quad j = 1, 2.$$

Hence,

$$\boldsymbol{\sigma}_k(T_K, \mathscr{B}_1, \mathscr{B}_2) \lesssim R^k (k!)^{-\frac{1}{2\sigma d_j}}, \quad j = 1, 2,$$

in view of (5.4)–(5.5). This gives (3).

(5) By [2,17,21,28,30] we get

$$T_K = T_{K_3} \circ T_{K_2} \circ T_{K_1}, \tag{5.11}$$

where the kernels K_1, K_2 and K_3 of the operators T_{K_1}, T_{K_2} and T_{K_3} belong to $\mathscr{S}(\mathbf{R}^{d_1+d_1})$, $\mathscr{S}(\mathbf{R}^{d_2+d_1})$ and $\mathscr{S}(\mathbf{R}^{d_2+d_2})$, respectively. Furthermore, we may assume that T_{K_2} is a positive semi-definite Hermite diagonal operator (cf. e. g. [28]).

It follows that T_{K_1} is continuous from $\mathscr{B}_1$ to $L^2(\mathbf{R}^{d_1})$, and T_{K_3} is continuous from $L^2(\mathbf{R}^{d_2})$ to $\mathscr{B}_2$. Hence, by (5.4) and (5.5) it suffices to prove that for every $N > 0$ there is a constant $C > 0$ such that

$$\boldsymbol{\sigma}_k = \boldsymbol{\sigma}_k(T_{K_{0,2}}, L^2(\mathbf{R}^d), L^2(\mathbf{R}^d)) \le C k^{-N}. \tag{5.12}$$

where $K_{0,2}$ is the same as in the proof of (1). By the construction, c_α in (5.8) fulfills

$$0 \le c_\alpha \lesssim \langle \alpha \rangle^{-N}$$

for every $N > 0$, and by similar arguments as in the proof of (1) we get

$$\boldsymbol{\sigma}_k \lesssim k^{-\frac{N}{d}}$$

for every N, and (5) follows.

Finally, by (5.1)–(5.3) it also follows that $\{\boldsymbol{\sigma}_k(T,\mathscr{B}_1,\mathscr{B}_2)\}$ belongs to ℓ^p for every $p>0$. This gives the second parts of (1)–(5). □

6 Discrete Characterizations of Kernels to Smoothing Operators

In this section we show that any operators with kernels in Gelfand–Shilov, Pilipović or Schwartz spaces can be characterized by convenient expansions of the form

$$K=\sum_{j=1}^{\infty}\lambda_j f_{1,j}\otimes f_{2,j},\qquad \{\lambda_j\}_{j=1}^{\infty}\subseteq \mathbf{R}_+ \tag{6.1}$$

for some

$$\{f_{k,j}\}_{j=1}^{\infty}\subseteq L^2(\mathbf{R}^{d_k}),\quad k=1,2. \tag{6.2}$$

In fact, the following result is an extension of Lemma 3.2 in [24].

Theorem 6.1 *Let $p\in[1,\infty]$ and T be a linear and continuous operator from $\mathcal{H}_0(\mathbf{R}^{d_1})$ to $\mathcal{H}_0'(\mathbf{R}^{d_2})$ with kernel K. Then the following is true:*

(1) *if $K\in\mathscr{S}(\mathbf{R}^{d_2}\times\mathbf{R}^{d_1})$, then (6.1) holds for some orthogonal sequences in (6.2) such that*

$$\sup_{j\ge1}\left(j^N\lambda_j\right)<\infty \quad \textit{and}\quad \sup_{j\ge1}\left(j^N\|x^\alpha D^\beta f_{k,j}\|_{L^p(\mathbf{R}^{d_k})}\right)<\infty \tag{6.3}$$

for $k=1,2$, $\alpha,\beta\in\mathbf{N}^d$ and every $N\ge0$.

(2) *if* $K\in C^\infty(\mathbf{R}^{d_2}\times\mathbf{R}^{d_1})$ satisfies (6.1) and (6.3) *for $k=1,2$ and every $N\ge1$, then $K\in\mathscr{S}(\mathbf{R}^{d_2}\times\mathbf{R}^{d_1})$.*

The corresponding characterizations of operators with Pilipović kernels are given in the following theorem. Here recall that the harmonic oscillator is given by $H=|x|^2-\Delta$, $x\in\mathbf{R}^d$.

Theorem 6.2 *Let $p\in[1,\infty]$, $s>0$, $d=\min(d_1,d_2)$ and T be a linear and continuous operator from $\mathcal{H}_0(\mathbf{R}^{d_1})$ to $\mathcal{H}_0'(\mathbf{R}^{d_2})$ with kernel K. Then the following is true:*

(1) *if $K\in\mathcal{H}_s(\mathbf{R}^{d_2}\times\mathbf{R}^{d_1})$ ($K\in\mathcal{H}_{0,s}(\mathbf{R}^{d_2}\times\mathbf{R}^{d_1})$), then (6.1) holds for some orthogonal sequences in (6.2) such that*

$$\sup_{j\ge1}\left(e^{r\cdot j^{\frac{1}{2ds}}}\lambda_j\right)<\infty \quad \textit{and}\quad \sup\left(\frac{e^{r\cdot j^{\frac{1}{2ds}}}\|H^N f_{k,j}\|_{L^p(\mathbf{R}^{d_k})}}{h^N(N!)^{2s}}\right)<\infty \tag{6.4}$$

for $k=1,2$ and some $h>0$ and $r>0$ (every $h>0$ and $r>0$), where the latter supremum is taken over all $j\ge0$ and $N\ge0$;

(2) if $K \in C^\infty(\mathbf{R}^{d_2} \times \mathbf{R}^{d_1})$ satisfies (6.1) and (6.4) *for* $k = 1, 2$ *and some* $r > 0$ *(every* $r > 0$*), then* $K \in \mathcal{H}_s(\mathbf{R}^{d_2} \times \mathbf{R}^{d_1})$ *(*$K \in \mathcal{H}_{0,s}(\mathbf{R}^{d_2} \times \mathbf{R}^{d_1})$*).*

We need some preparations for the proof. First we observe that $\mathcal{H}^p_{[\vartheta]}$ possesses the expected interpolation properties. (Cf. [3].)

Lemma 6.3 *Let* $\theta \in [0, 1]$, ϑ, ϑ_1 *and* ϑ_2 *be weights on* $\mathbf{N}^d$, *and let* $p, p_1, p_2 \in [1, \infty]$ *be such that*

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2} \quad \textit{and} \quad \vartheta = \vartheta_1^{1-\theta}\vartheta_2^{\theta}.$$

Then

$$(\mathcal{H}^{p_1}_{[\vartheta_1]}(\mathbf{R}^d), \mathcal{H}^{p_2}_{[\vartheta_2]}(\mathbf{R}^d))_{[\theta]} = \mathcal{H}^{p}_{[\vartheta]}(\mathbf{R}^d).$$

Proof The result follows from the fact that the map

$$\{c_\alpha\}_{\alpha \in \mathbf{N}^d} \mapsto \sum_{\alpha \in \mathbf{N}^d} c_\alpha h_\alpha$$

is bijective and isometric from $\ell^p_{[\vartheta]}(\mathbf{N}^d)$ to $\mathcal{H}^p_{[\vartheta]}(\mathbf{R}^d)$, and that

$$\left(\ell^{p_1}_{[\vartheta_1]}(\mathbf{N}^d), \ell^{p_2}_{[\vartheta_2]}(\mathbf{N}^d)\right)_{[\theta]} = \ell^{p}_{[\vartheta]}(\mathbf{N}^d).$$

□

We also need the following extension of [27, Proposition 5.5] on powers of non-negative self-adjoint operators on $L^2(\mathbf{R}^d)$.

Proposition 6.4 *Let* $s \ge 0$, $t > 0$ *and let* T *be a self-adjoint and non-negative operator on* $L^2(\mathbf{R}^d)$ *with kernel* K *in* $\mathcal{H}_s(\mathbf{R}^d \times \mathbf{R}^d)$. *Then the following is true:*

(1) *the kernel of* T^t *belongs to* $\mathcal{H}_s(\mathbf{R}^d \times \mathbf{R}^d)$;
(2) T^t *is continuous from* $\mathcal{H}_s'(\mathbf{R}^d)$ *to* $\mathcal{H}_s(\mathbf{R}^d)$.

The same holds true if the $\mathcal{H}_s$ *and* $\mathcal{H}_s'$ *spaces are replaced by* $\mathcal{H}_{0,s}$ *and* $\mathcal{H}_{0,s}'$ *spaces, respectively, or by* $\mathscr{S}$ *and* $\mathscr{S}'$ *spaces, respectively.*

Proof We only prove the result when $K \in \mathcal{H}_s(\mathbf{R}^d \times \mathbf{R}^d)$. The other cases follow by similar arguments and are left for the reader.

Let

$$\Omega = \{\, z \in \mathbf{C}\,;\ 0 < \mathrm{Re}(z) < 1 \,\}$$

and $T_0(z) = T^z$ when $z \in \overline{\Omega}$. Then the map $z \mapsto T(z)$ with values in $\mathscr{L}(L^2(\mathbf{R}^d))$ is continuous on $\overline{\Omega}$ and analytic on Ω.

Furthermore, by writing $T^z = T^x \circ T^{iy}$ when $z = x + iy$, and using that T^{iy} is bounded on $L^2(\mathbf{R}^d)$ when $y \in \mathbf{R}$, it follows from the assumptions that

$$\sup_{y\in\mathbf{R}} \|T_0(iy)\|_{L^2(\mathbf{R}^d)\to L^2(\mathbf{R}^d)} \le 1,$$
$$\sup_{y\in\mathbf{R}} \|T_0(1+iy)\|_{L^2(\mathbf{R}^d)\to \mathcal{H}^2_{[\vartheta_r]}(\mathbf{R}^d)} < \infty$$

and

$$\sup_{z\in\overline{\Omega}} \|T_0(z)\|_{L^2(\mathbf{R}^d)\to L^2(\mathbf{R}^d)} \le \sup_{0\le x\le 1} \|T^x\|_{L^2(\mathbf{R}^d)\to L^2(\mathbf{R}^d)}$$

for some $r > 0$, where $\vartheta_r(\alpha) = e^{r|\alpha|^{\frac{1}{2s}}}$.

It now follows from Lemma 6.3 and Calderon-Lion's interpolation theorem (cf. Theorem IX.20 in [20]) that T^t is continuous from $L^2(\mathbf{R}^d)$ to $\mathcal{H}^2_{[\vartheta_{rt}]}(\mathbf{R}^d)$. Duality gives that

$$T^t : L^2(\mathbf{R}^d) \to \mathcal{H}^2_{[\vartheta_{rt}]}(\mathbf{R}^d)$$

and

$$T^t : \mathcal{H}^2_{[1/\vartheta_{rt}]}(\mathbf{R}^d) \to L^2(\mathbf{R}^d)$$

are continuous. Hence, by interpolation we obtain that

$$T^t : \mathcal{H}^2_{[1/\vartheta_{rt/2}]}(\mathbf{R}^d) \to \mathcal{H}^2_{[\vartheta_{rt/2}]}(\mathbf{R}^d)$$

is continuous (cf. Remark 2.6), and the result follows from

$$\mathcal{H}_s(\mathbf{R}^d) = \bigcup_{r>0} \mathcal{H}^2_{[\vartheta_r]}(\mathbf{R}^d) \quad \text{and} \quad \mathcal{H}'_s(\mathbf{R}^d) = \bigcap_{r>0} \mathcal{H}^2_{[1/\vartheta_r]}(\mathbf{R}^d).$$

□

We also need the following characterization of Pilipović kernels. Here recall that $\mathscr{P}(\mathbf{R}^d)$ is the set of polynomially moderated weights on $\mathbf{R}^d$ (cf. Sect. 2).

Lemma 6.5 *Let $p, q \in (0,\infty]$, $\omega \in \mathscr{P}(\mathbf{R}^{2d_2} \times \mathbf{R}^{2d_1})$, $s > 0$, $K \in \mathcal{H}'_0(\mathbf{R}^{d_2} \times \mathbf{R}^{d_1})$,*

$$H_1 = |x_1|^2 - \Delta_{x_1} \quad \textit{and} \quad H_2 = |x_2|^2 - \Delta_{x_2}, \qquad x = (x_2, x_1) \in \mathbf{R}^{d_2} \times \mathbf{R}^{d_1}.$$

Also let $H = H_2 + H_1$ be the Harmonic oscillator on $\mathbf{R}^{d_2} \times \mathbf{R}^{d_1}$. Then the following conditions are equivalent:

(1) $K \in \mathcal{H}_s(\mathbf{R}^{d_2} \times \mathbf{R}^{d_1})$ *($K \in \mathcal{H}_{0,s}(\mathbf{R}^{d_2} \times \mathbf{R}^{d_1})$);*

(2) *for some* $h > 0$ *(for every* $h > 0$*) it holds*

$$\|H^N K\|_{L^2} \lesssim h^N (N!)^{2s}, \qquad N \geq 0; \tag{6.5}$$

(3) *for some* $h > 0$ *(for every* $h > 0$*) it holds*

$$\|H_1^{N_1} H_2^{N_2} K\|_{L^2} \lesssim h^{(N_1+N_2)} (N_1! N_2!)^{2s}, \qquad N_1, N_2 \geq 0; \tag{6.6}$$

(4) *for some* $h > 0$ *(for every* $h > 0$*) it holds*

$$\|H_1^{N_1} H_2^{N_2} K\|_{M^{p,q}_{(\omega)}} \lesssim h^{N_1+N_2} (N_1! N_2!)^{2s}, \qquad N_1 \geq N_{0,1},\ N_2 \geq N_{0,2}. \tag{6.7}$$

Proof The assertion (1) and (2) are equivalent in view of [26, Proposition 5.1]. Next we prove that (2) and (3) are equivalent. Assume that (6.5) holds. Since $K \in \mathcal{H}_0'(\mathbf{R}^{d_2} \times \mathbf{R}^{d_1})$, K has a formal Hermite series expansions

$$K = \sum_{\alpha_1 \in \mathbf{N}^{d_1}} \sum_{\alpha_2 \in \mathbf{N}^{d_2}} c_\alpha(K) h_{\alpha_2} \otimes h_{\alpha_1},$$

where the Hermite coefficients satisfy

$$|c_\alpha(K)| \lesssim e^{-\frac{1}{h}|\alpha|^{\frac{1}{2s}}}, \qquad \alpha = (\alpha_2, \alpha_1) \in \mathbf{N}^{d_2} \times \mathbf{N}^{d_1},$$

for some (every) $h > 0$. By Parseval's inequality we obtain

$$\begin{aligned}
&\|H_1^{N_1} H_2^{N_2} K\|_{L^2} \\
&\quad \leq \sum_{\alpha_1 \in \mathbf{N}^{d_1}} \sum_{\alpha_2 \in \mathbf{N}^{d_2}} (2|\alpha_1| + d_1)^{N_1} (2|\alpha_2| + d_2)^{N_2} e^{-\frac{1}{h}|\alpha|^{\frac{1}{2s}}} \leq I_1 \cdot I_2,
\end{aligned}$$

where

$$I_k = \sum_{\alpha_j \in \mathbf{N}^{d_k}} (2|\alpha_j| + d_k)^{N_k} e^{-\frac{1}{h_0}|\alpha_j|^{\frac{1}{2s}}}$$

with $h_0 = ch$, for some constant $c > 0$ which only depends on s.

By Lemma 5.7 in [26] and its proof we get

$$I_k \lesssim (3(4sh_0)^{2s})^{N_k} (N_k!)^{2s} \asymp h^{N_k} (N_k!)^{2s}$$

for some $h > 0$ and a combination of these estimates shows that (2) implies (3).

Assume instead that (6.6) holds. Then

$$\|H^N K\|_{L^2} = \|(H_1 + H_2)^N K\|_{L^2} \leq \sum_{k=0}^{N} \binom{N}{k} \|H_1^{N-k} H_2^k K\|_{L^2}$$

$$\lesssim h^N \sum_{k=0}^{N} \binom{N}{k} ((N-k)!k!)^{2s} \le h^N (N!)^{2s} \sum_{k=0}^{N} \binom{N}{k} = (2h)^N (N!)^{2s},$$

and it follows that (3) implies (2).

It remains to prove the equivalence between (4) and (1)–(3). First we show that

$$\|H_1^N K\|_{M^{p,q}_{(\omega)}} \lesssim h^N (N!)^{2s}, \quad N \ge N_0 \tag{6.8}$$

is independent on N_0 and ω when $p, q \ge 1$. If (6.8) is true for $N_0 = 0$, then it is also true for $N_0 > 0$. If $0 \le N \le N_0$, $N_1 = N_0 - N \ge 0$ and (6.8) holds for some $N_0 \ge 0$, then by the fact that

$$H_1^N \, : \, M^{p,q}_{(v_N \omega)}(\mathbf{R}^{d_2} \times \mathbf{R}^{d_1}) \to M^{p,q}_{(\omega)}(\mathbf{R}^{d_2} \times \mathbf{R}^{d_1}), \tag{6.9}$$

with

$$v_N(x_1, \xi_1, x_2, \xi_2) = C(1 + |x_1|^2 + |\xi_1|^2)^N, \tag{6.10}$$

is a homeomorphism (cf. e.g. [23, Theorem 3.10]), it follows that

$$\|H_1^N K\|_{M^{p,q}_{(\omega)}} \lesssim \|H_1^{N_0} K\|_{M^{p,q}_{(\omega/v_{N_1})}} \lesssim \|H_1^{N_0} K\|_{M^{p,q}_{(\omega)}} < \infty,$$

and (6.8) holds for $N_0 = 0$ as well. This shows that (6.8) is independent of $N_0 \ge 0$ when $p, q \ge 1$.

Since $\omega \in \mathscr{P}(\mathbf{R}^{2d_2} \times \mathbf{R}^{2d_1})$, there exists an integer $N_0 \ge 0$ such that

$$1/v_{N_0} \lesssim \omega \lesssim v_{N_0},$$

and then

$$\|K\|_{M^{p,q}_{(1/v_{N_0})}} \lesssim \|K\|_{M^{p,q}_{(\omega)}} \lesssim \|K\|_{M^{p,q}_{(v_{N_0})}}. \tag{6.11}$$

Hence the stated invariance follows if we prove that (6.8) holds for $\omega = v_{N_0}$, if it is true for $\omega = 1/v_{N_0}$.

Therefore, assume that (6.8) holds for $\omega = 1/v_{N_0}$. If $N \ge 2N_0$, then the bijectivity of (6.9) gives

$$
\begin{aligned}
\frac{\|H_1^N K\|_{M^{p,q}_{(v_{N_0})}}}{h^N (N!)^{2s}} &\lesssim \frac{\|H_1^{N+2N_0} K\|_{M^{p,q}_{(1/v_{N_0})}}}{h^N (N!)^{2s}} \\
&= h^{2N_0} \binom{N+2N_0}{2N_0}^{2s} ((2N_0)!)^{2s} \frac{\|H_1^{N+2N_0} K\|_{M^{p,q}_{(1/v_{N_0})}}}{h^{N+2N_0}((N+2N_0)!)^{2s}} \\
&\asymp \binom{N+2N_0}{2N_0}^{2s} \frac{\|H_1^{N+2N_0} K\|_{M^{p,q}_{(1/v_{N_0})}}}{h^{N+2N_0}((N+2N_0)!)^{2s}} \lesssim \frac{\|H_1^{N+2N_0} K\|_{M^{p,q}_{(1/v_{N_0})}}}{h_1^{N+2N_0}((N+2N_0)!)^{2s}},
\end{aligned}
$$

where $h_1 = \frac{h}{4^s}$. This shows that (6.8) is independent of ω in the case $p, q \geq 1$.

By repeating these arguments, it follows that (6.7) is independent of $N_{0,1}$, $N_{0,2}$, ω and $p, q \in [1, \infty]$. For general $p, q \in (0, \infty]$, the invariance of (6.7) with respect to $N_{0,1}$, $N_{0,2}$, ω, p and q, is now a consequence of the embeddings

$$
M^\infty_{(v_N \omega)}(\mathbf{R}^{d_2} \times \mathbf{R}^{d_1}) \subseteq M^{p,q}_{(\omega)}(\mathbf{R}^{d_2} \times \mathbf{R}^{d_1}) \subseteq M^\infty_{(\omega)}(\mathbf{R}^{d_2} \times \mathbf{R}^{d_1})
$$

when

$$
N > \frac{d_1 + d_2}{\min(p, q)}
$$

(see e. g. [11]).

The equivalence bewteen (3) and (4) now follows from these invariance properties and the fact that $L^2 = M^{2,2}$, and the result follows.

In the next proof we let ON_d be the set of all orthonormal sequences in $L^2(\mathbf{R}^d)$.

Proof of Theorems 6.1 and 6.2 We only prove Theorem 6.2 in the Roumieu case. The other cases (Theorem 6.2 in the Beurling case, and Theorem 6.1) follow by similar arguments and are left for the reader.

In the sequel we employ the same notation used in the proof of Lemma 6.5

(1) Assume that $K \in \mathcal{H}_s(\mathbf{R}^{d_2} \times \mathbf{R}^{d_1})$. By polar decomposition we have

$$
K(x_2, x_1) = \sum_{j=1}^\infty \lambda_{0,j} g_j(x_2) \overline{f_j(x_1)}, \qquad x_1 \in \mathbf{R}^{d_1},\ x_2 \in \mathbf{R}^{d_2},
$$

where $\lambda_{0,j} \geq 0$ are the singular values of T, $\{f_j\}_{j=1}^\infty \in \mathrm{ON}_{d_1}$ and $\{g_j\}_{j=1}^\infty \in \mathrm{ON}_{d_2}$. Now let K_1 and K_2 be the kernels of $T_1 \equiv (T^* \circ T)^{\frac{1}{4}}$ and $T_2 \equiv (T \circ T^*)^{\frac{1}{4}}$, respectively. Then

$$
K_1(x_2, x_1) = \sum_{j=1}^\infty \sqrt{\lambda_{0,j}}\, f_j(x_2) \overline{f_j(x_1)}, \quad x_1, x_2 \in \mathbf{R}^{d_1}
$$

and

$$K_2(x_2,x_1)=\sum_{j=1}^{\infty}\sqrt{\lambda_{0,j}}\,g_j(x_2)\overline{g_j(x_1)},\quad x_1,x_2\in\mathbf{R}^{d_2}.$$

By Theorem 5.1 we get

$$\lambda_{0,j}\lesssim e^{-rj^{\frac{1}{2ds}}}\tag{6.12}$$

for some constant $r>0$.

Since $K_1\in\mathcal{H}_s(\mathbf{R}^{d_1}\times\mathbf{R}^{d_1})$, Lemma 6.5 gives

$$\sum_{j=1}^{\infty}\sqrt{\lambda_{0,j}}\|H^Nf_j\|_{L^2}^2=\|H_1^NH_2^NK_1\|_{\mathrm{Tr}}\le\|H_1^NH_2^NK_1\|_{M^{1,1}}\lesssim h^N(N!)^{4s},$$

where $\|\cdot\|_{\mathrm{Tr}}$ is the trace-class norm. Here we have identified operators with their kernels, and used the fact that operators with kernels in $M^{1,1}(\mathbf{R}^{2d})$ are of trace-class (cf. [15,26]). Hence,

$$\lambda_{0,j}^{\frac{1}{4}}\|H^Nf_j\|_{L^2}\lesssim h_0^N(N!)^{2s},$$

where $h_0=\sqrt{h}$. Hence, if $f_{1,j}=\lambda_{0,j}^{\frac{1}{3}}f_j$ we obtain

$$\|H^Nf_{1,j}\|_{L^2}\lesssim\lambda_{0,j}^{\frac{1}{12}}h_0^N(N!)^{2s}\lesssim e^{-r\cdot j^{\frac{1}{2ds}}}h_0^N(N!)^{2s}$$

for some $r>0$. By considering K_2 instead of K_1 and letting $f_{2,j}=\lambda_{0,j}^{\frac{1}{3}}g_j$, the same computations give

$$\|H^Nf_{2,j}\|_{L^2}\lesssim e^{-r\cdot j^{\frac{1}{2ds}}}h_0^N(N!)^{2s}$$

for some $r>0$ and $h_0>0$.

The assertion now follows if we let $\lambda_j=\lambda_{0,j}^{\frac{1}{3}}$.

(2) By the assumptions and Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}\|H_1^{N_1}H_2^{N_2}K\|_{L^2}&=\left\|H_1^{N_1}H_2^{N_2}\left(\sum\lambda_jf_{1,j}\otimes f_{2,j}\right)\right\|_{L^2}\\&=\left(\iint_{\mathbf{R}^{d_2+d_1}}\left|\sum\lambda_jH_1^{N_1}f_{1,j}\otimes H_2^{N_2}f_{2,j}\right|^2dx_1dx_2\right)^{1/2}\\&\le\left(\iint_{\mathbf{R}^{d_2+d_1}}\left(\sum\lambda_j^2\right)\left(\sum|H_1^{N_1}f_{1,j}\otimes H_2^{N_2}f_{2,j}|^2\right)dx_1dx_2\right)^{1/2}\end{aligned}$$

$$\leq \sum \lambda_j \left(\sum \|H_1^{N_1} f_{1,j}\|_{L^2}^2 \|H_2^{N_2} f_{2,j}\|_{L^2}^2 \right)^{1/2}$$
$$\lesssim h^{N_1+N_2}(N_1!N_2!)^{2s} \sum e^{-r \cdot j^{1/2ds}} \lesssim h^{N_1+N_2}(N_1!N_2!)^{2s}.$$

Hence, $K \in \mathcal{H}_s(\mathbf{R}^{d_2} \times \mathbf{R}^{d_1})$ in view of Lemma 6.5. □

References

1. Bayer, D.: Bilinear time-frequency distributions and pseudodifferential operators, Thesis, Universität Wien, Wien (2010)
2. Beals, R.: Characterization of pseudodifferential operators and applications. Duke Math. J. **44**, 45–57 (1977)
3. Bergh, J., Löfström, J.: Interpolation Spaces, An Introduction. Springer, Berlin (1976)
4. Cappiello, M. Toft, J.: Pseudo-differential operators in a Gelfand–Shilov setting. Math. Nachr. **290**, 738–755 (2017)
5. Chen, Y., Signahl, M., Toft, J.: Hilbert space embeddings for Gelfand–Shilov and Pilipović spaces. In: Oberguggenberger, M., Toft, J., Vindas, J., Wahlberg P. (eds.) *Generalized Functions and Fourier Analysis*, Operator Theory: Advances and Applications, Birkhäuser Verlag, Basel, **260** (to appear)
6. Delgado, J., Ruzhansky, M.: Schatten classes on compact manifolds: kernel conditions. J. Funct. Anal. **267**, 772–798 (2014)
7. Delgado, J., Ruzhansky, M.: Kernel and symbol criteria for Schatten classes and r-nuclearity on compact manifolds. C. R. Math. Acad. Sci. Paris **352**, 779–784 (2014)
8. Feichtinger, H.G.: Modulation spaces on locally compact abelian groups. Technical report, University of Vienna, Vienna, 1983; also In: Krishna, M., Radha, R., Thangavelu S. (eds.) Wavelets and their applications, Allied Publishers Private Limited, NewDehli, pp. 99–140 (2003)
9. Feichtinger, H.G.: Modulation spaces: looking back and ahead. Sampl. Theory Signal Image Process. **5**, 109–140 (2006)
10. Fernandez, C., Galbils, A., Toft, J.: The Bargmann transform and powers of harmonic oscillator on Gelfand–Shilov subspaces. RACSAM **111**, 1–13 (2017)
11. Galperin, Y.V., Samarah, S.: Time-frequency analysis on modulation spaces $M_m^{p,q}$, $0<p,q \leq \infty$. Appl. Comput. Harmon. Anal. **16**, 1–18 (2004)
12. Gelfand, I.M., Shilov, G.E.: Generalized Functions, I–III. Academic Press, New York (1968)
13. Gramchev, T., Pilipović, S., Rodino, L.: Classes of degenerate elliptic operators in Gelfand-Shilov spaces. In: Rodino, L., Wong, M.W. (eds.) New Developments in Pseudo-Differential Operators. Operator Theory: Advances and Applications, pp. 15–31. Birkhäuser Verlag, Basel (2009)
14. Gröchenig, K.H.: Foundations of Time-Frequency Analysis. Birkhäuser, Boston (2001)
15. Gröchenig, K.H., Heil, C.: Modulation spaces and pseudo-differential operators. Integral Eq. Oper. Theory **34**, 439–457 (1999)
16. Hörmander, L.: The Analysis of Linear Partial Differential Operators. Springer, Berlin (1983)
17. Klis, C.: *A simple proof of the identity* $\mathcal{S} \circ \mathcal{S} = \mathcal{S}$ In: Proceedings of the Conference on Convergence and Generalized Functions, Katowice 1983, Preprint of the Institute of Math., Polish Academy of Sci., pp. 71–74
18. Pilipovic, S.: Generalization of Zemanian spaces of generalized functions which have orthonormal series expansions. SIAM J. Math. Anal. **17**, 477–484 (1986)
19. Pilipović, S.: Tempered ultradistributions. Boll. U.M.I. **7**, 235–251 (1988)
20. Reed, M., Simon, B.: Methods of Modern Mathematical Physics. Academic Press, London (1979)
21. Sadlok, Z.: On uniform convergence of Hermite series. Ann. Polon. Math. **43**, 207–210 (1983)

22. Simon, B.: Trace Ideals and their Applications. I, London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge (1979)
23. Signahl, M., Toft, J.: Mapping properties for the Bargmann transform on modulation spaces. J. Pseudo-Differ. Oper. Appl. **3**, 1–30 (2012)
24. Toft, J.: Regularizations, decompositions and lower bound problems in the Weyl calculus. Commun. Partial Differ. Eq. **25**, 1201–1234 (2000)
25. Toft, J.: Multiplication properties in Gelfand–Shilov pseudo-differential calculus. In: Molahajlo, S., Pilipović, S., Toft, J., Wong, M.W., Zhu, H. (eds.) Pseudo-Differential Operators, Generalized Functions and Asymptotics. Operator Theory: Advances and Applications, Birkhäuser Verlag, Basel, pp. 117–172 (2013)
26. Toft, J.: Images of function and distribution spaces under the Bargmann transform. J. Pseudo-Differ. Oper. Appl. **8**, 83–139 (2017)
27. Toft, J.: Continuity and compactness for pseudo-differential operators with symbols in quasi-Banach spaces or Hörmander classes. Anal. Appl. (2016). doi:10.1142/S0219530516500159
28. Toft, J., Khrennikov, A., Nilsson, B., Nordebo, S.: Decompositions of Gelfand–Shilov kernels into kernels of similar class. J. Math. Anal. Appl. **396**, 315–322 (2012)
29. Treves, F.: Topological Vector Spaces, Distributions and Kernels. Academic Press, New York (1967)
30. Voigt, J.: Factorization in some Fréchet algebras of differentiable functions. Studia Math. **77**, 333–348 (1984)