Preface

First of all, it has been both exciting and challenging to conduct research that deals with an important topic, namely concept images. It was Mr Pentti Haukkanen, a lecturer in Mathematics, who first suggested me this topic. Despite the fact that I have been a co-author in some mathematical articles, this education in the joint Nordic Master Program Nordima has been an unforgettable experience. A new 'world' has opened to me. This is something I can warmly recommend to those interested in teaching mathematics.

First and foremost, I would like to thank Professor John Monaghan at the University of Agder. He has guided me in a dedicated and very professional way. Knowing how busy he is with his work, I appreciate his valuable aid and flexibility especially much.

In Finland, there are three people to whom I would like to express my profound gratitude. Mr Jorma Joutsenlahti, a lecturer in Didactics of Mathematics, at the current Faculty of Education, University of Tampere, granted me admission to the programme. With him I have had many rewarding discussions. Furthermore, I have had the honour of knowing Mr. Pentti Haukkanen (current Faculty of Natural Sciences) since the very beginning of my studies at the University of Tampere. He is an excellent source of guidance in teaching and research. Last but not least, Mr Timo Tossavainen, a lecturer in Mathematics School of Applied Educational Science and Teacher Education, University of Eastern Finland, is among the top-class researchers in the field of my study. The list of reasons for thanking him is endless. I would particularly like to mention that without the thorough data he kindly offered to my use, this thesis could not probably have been done. I take this opportunity to wish him every success in his future endeavours at Luleå University of Technology .

Once again, my sincere thanks to all four of you, John, Jorma, Pentti and Timo.

Thanks to the lecturers in Mathematics, Mr Tero Luodeslampi and Mr Jukka Männistö University of Tampere Teacher Training School, for hints and a chance to conduct the survey.

It has been a joy and pleasure to be a part of the student societies both in Tampere and Kristiansand. ESN is a valuable addition to student life I would like to extend my gratitude to. With Mr Sokratis Theodoridis I have shared many rewarding and joyful times and discussions which have been nothing but positive. All the lecturers in different courses and classes that I have participated in have taught me something useful and given new insights. The administrative personnel at both universities deserve my thanks for their many efforts and favours.

The Erasmus grant and support from the Social Insurance Institution of Finland have made it financially possible for me to study full time. Such an opportunity should never be taken for granted and thus I would like to express my gratitude.

Finally, I sincerely thank my family and friends for all the support and help they have offered me during my studies. They all have volunteered selflessly in every possible way for which I greatly appreciate them.

Tampere/Kristiansand, May 2017

Juha Sillanpää

Summary

This study is titled "On concept images of monotonicity of Finnish secondary students – with comparison and reference to tertiary students". The aim is to examine: (1) What kind of aspects of monotonicity are present and, possibly, dominant in the upper secondary school students' (advanced level) concept images? (2) What are the most typical misconceptions and deficiencies in mathematical reasoning that the participants have concerning monotonicity? (3) To what extent do the results and answers to (1) and (2) compare with the results of university students? This thesis makes use of the study of Tossavainen, Haukkanen and Pesonen (2013), who investigated, *inter alia*, corresponding concept images among mathematics students from two Finnish universities. The theoretical framework is mainly based on the classic paper of Tall and Vinner (1981).

The test was carried out in Tampere in April 2016 by using the same questionnaire as in Tossavainen et al. (2013). For the sake of comparability, also the similar guidelines in scoring and categorization of concept images were principally followed in processing the responses of 26 students. The quantitative analysis of the data collected consists of descriptive statistics and Student's t-test and it was done by using SPSS.

It was found out that overall, there were so many faulty responses that erroneous concept images were most common, followed by experimental approaches, while not a single response was considered as algebraic. Among tertiary students there were more of them, but evoked mainly and superficially by imposition. There were also differences between secondary and tertiary students in the scores achieved, the formers' scores being approximately two thirds of the latters' in monotonicity and total items. Nevertheless, in this context the differences are not surprising and most of them are not significant.

Sammendrag

Denne studien heter “On concept images of monotonicity of Finnish secondary students – with comparison and reference to tertiary students”. Målet er å eksaminere: (1) Hvilke slags aspekter av monotonicitet forekommer og hvilke av dem er muligens dominerende på videregående skole elevenes (avansert nivå) konseptbilder? (2) Hva er de mest typiske misforståelser og mangler i matematisk resonnement som deltakerne har angående monotonicitet? (3) I hvilken grad sammenligner resultatene og svarene på (1) og (2) med resultatene fra universitetsstudenter? Denne oppgaven benytter studiet av Tossavainen, Haukkanen og Pesonen (2013), som blant annet undersøkte konseptbilder blant matematikkstuderende fra to finske universiteter. Det teoretiske rammeverket er hovedsakelig basert på den klassiske artikkelen av Tall og Vinner (1981).

Testen ble gjennomført i Tammerfors i april 2016 ved å bruke det samme spørreskjemaet som i Tossavainen et al. (2013). For sammenlignelighets skyld ble også de tilsvarende retningslinjene for poengberegning og kategorisering av konseptbilder fulgt i behandlingen av svar fra 26 studenter. Den kvantitative analysen av dataene samlet består av beskrivende statistikk og Students t-test, og det ble gjort ved å bruke SPSS.

Det ble funnet ut at totalt sett var det så mange feilaktige svar at feilkonceptbilder var mest vanlige, etterfulgt av eksperimentelle tilnærminger, mens ikke et enkelt svar ble ansett som algebraisk. Blant tertiære studenter var det flere av dem, men fremkalt hovedsakelig og overfladisk ved pålegg. Det var også forskjeller mellom videregående og tertiære studenter i de oppnådde poengene, hvor tidligere score er omtrent to tredjedeler av sistnevnte i monotonicitet og totale elementer. Likevel er forskjellene i denne sammenhengen ikke overraskende, og de fleste er ikke signifikante.

Table of contents

1 Introduction

The aim of my thesis is to investigate the concept images of 26 Finnish upper secondary school students who have chosen the advanced syllabus in mathematics. It is partly done with comparison to university students of mathematics; I can utilize an extensive sample of 89 Finnish tertiary students whose conceptions on the same issue were studied by Tossavainen, Haukkanen and Pesonen (2013).

The notion of concept image is due to Tall and Vinner. They published their famous paper on concept images and concept definitions in 1981. It has been widely utilized and applied ever since they presented it. Basically, the term concept image describes the total cognitive structure that is associated with the concept while the concept definition is a form of words used to specify the concept.

As for monotonicity, it is a concept which may not be familiar to the man in the street. Even less so, if its meaning, as a mathematical term, is considered. However, there are numerous phenomena in everyday life and various branches of science which are characterized by monotonicity – in standard language terms, when one quantity increases, the other does not decrease/increase. The former quantity is often time. E.g., once 'life' was mentioned, life expectancy at birth in Finland (the number of years that a new-born would live provided that the rate of mortality remains unchanged) remained the same or increased throughout the years 1971-2015 according to Statistics Finland (men: from 66 to 79 years, women: from 74 to 84 years). In the concept of monotonicity of a function, some domain is always involved at least implicitly.

Once the significance of the concept is apparent, it is relevant to ask how and to what extent it is taken into account in upper secondary school studies in Finland. The answer may be somewhat astonishing. If the word *monotonicity* is searched for in the National Core Curriculum for General Upper Secondary Schools 2003 (Opetushallitus 2003), it is found only one time in advanced mathematics, i.e., in connection with the course MAA8 *Radical and logarithm functions*, as one objective 'to study the inverses of strictly monotonic functions'. In the corresponding curriculum 2015 (Opetushallitus 2015) the position of the term is not better – again it was found once, in MAA13 *Advanced course in calculus*, one objective being the same as above. The terms *increasing* and *decreasing* do appear but only with respect to exponential functions. In spite of the facts above, monotonicity is studied as an application of the derivative and the terms increasing, decreasing and monotonic are presented. (At my school time in the 1980's the position of monotonicity was parallel to the scheme presented above.) Consequently, it is interesting to ask what kind of tasks the upper secondary school students have the ability for and what possibly could be better. What is the cognitive structure of students like and what aspects are overriding in it?

The issue of the thesis, being restricted to the understanding of monotone functions, is not explored extensively. The main articles on the issue are those of Rasslan & Vinner (1998) and Tossavainen, Haukkanen & Pesonen (2013). Instead, concept images of matters like limits, continuity, tangent, derivative, function, equation, area etc. have been studied after the famous paper of Tall and Vinner (1981) – there are over 2000 references to it! Other interesting articles are presented e.g. by Bingolbali & Monaghan (2008), Tossavainen, Attorps & Väisänen (2011), Viirman, Attorps & Tossavainen (2010).

My thesis starts from the premises above. I have taught mathematics at the university level, yet nothing about monotonicity and only seldom calculus and have no experience of teaching at upper secondary schools. Consequently, there is a charm of novelty in the theme.

The disposition of the thesis is as follows. The theoretical framework and some specific studies on conceptions of monotonicity are presented as a basis. Next, the methodological issues together with the implementation in practice are presented and discussed. The survey of 26 upper secondary school students was carried out using a questionnaire. It consisted of eight exercises (tasks as a synonym), of which four dealt with monotonicity and four other topics were related to calculus as well. The results are presented task by task, and the foci are on the criteria for classifying the responses, the distribution of concept image classes, the performance and the illustrative examples of representative responses, which are commented. For comparison to the university students, descriptive statistics and the t-test are used. In the discussion, the offerings are dealt with and some critical remarks done. The thesis ends with conclusions which summarize the work, present the limitations and implications and outline a few ideas for future prospects.

As the comparative aspect is also present, it is naturally necessary for the writer and most advisable for the reader, to be familiar with the contents of the article by Tossavainen et al. (2013). They state in the abstract *inter alia*

> *[I]n this paper, we investigate which aspects are overriding in the concept images of monotonicity of Finnish tertiary students, i.e., on which aspects of monotonicity they base their argument in different types of exercises related to that concept. … Our findings indicate that a mathematics student's conception about monotone functions is often restricted to continuous or differentiable functions and the algebraic aspect – the nearest one to the formal definition – is rare.* (ibid., 1117)

The theme for research is restricted by the following decisions:

In the study, the relationship between the aspects in the concept images and the success in solving the exercises is not considered. The data is too small for the purpose and there is no reason to challenge any previous findings.

The study is mostly quantitative, aiming to objectivity, not judging the writer as a part of the research process, still not denying that the responses are not out of the context. The aim is not to analyse the character of the study further, to position it nor describe the other possible approaches for investigating this kind of issue. In my view, the thesis itself and its results can be reviewed almost independently of the notions concerning the philosophy of science.

The statistical theory is not in the focus either. The strategy is to play safe, it means especially the criteria on when the t-test is appropriate for use.

Even though the curricula are referred to, their contents or the contents of textbooks, are not laid emphasis on.

It is natural to follow most of the guidelines in the study of Tossavainen et al. and hence the research questions are set as follows:

(1) What kind of aspects of monotonicity are present and, possibly, dominant in the upper secondary school students' (advanced level) concept images?
The hypothesis is that the students solve the problems by using differentiation mainly. Hardly any new concept image classes will be recognized.

(2) What are the most typical misconceptions and deficiencies in mathematical reasoning that the participants have concerning monotonicity?
It is hypothesized that the participants have readiness to deal with continuous functions merely. About other deficiencies, the writer has no preconceptions of.

(3) To what extent do the results and answers to (1) and (2) compare with the results of university students?
Most likely the performance of the upper secondary school students will be somewhat worse. There will probably be differences in relative frequencies of some concept image classes.

2 Theoretical framework and literature review

In this chapter I first give a concise account of the crucial mathematical concepts occurring in this study. Otherwise, the knowledge of real analysis, to the same extent as in mathematics courses in upper secondary schools in many countries, is required and it suffices the reader. The article of Sfard (1991) was familiar from my earlier studies and it seemed appropriate to contribute to the topics I deal with, although my survey does not directly comment on what has lead the students to response in the way they did, but it allows to investigate whether there are any signs of structural knowledge. The studies of Tall (and Vinner) have the significance which has been and still is a carrying force for other researchers in the field of concept images. (The reader may find a parallel, if recalling the statement: the history of Western philosophy is nothing but marginal remarks on Plato's works.) Concept images related to monotone functions have not been widely explored. This chapter closes with presenting some previous studies on the issue.

2.1 The mathematical background

In analysis of one real variable, a function $f: A \longrightarrow \mathbf{R}$ is said to be *increasing*, if

$x < y \Rightarrow f(x) \leq f(y)$ for all $x, y \in A$

and *decreasing*, if

$x < y \Rightarrow f(x) \geq f(y)$ for all $x, y \in A$.

The function f is said to be *monotone (monotonic)*, if it is either increasing or decreasing. If the equality is excluded in the conditions, then f is *strictly increasing/strictly decreasing/strictly monotone*.

It should be noted that in the mathematical literature, the terminology is sometimes used in a different way: what is here called strictly monotone, is called just monotone. The reader should always be aware of that. The same is true with publications in mathematical education.

Quite often in the courses of analysis, the domain A is an interval, but in general, the domain A need not be an interval; it is just enough that it is an *ordered set*:

The relation < defined on a set E is a total order, if

1. For all $x, y \in E$ exactly one of the following conditions holds: either $x < y$, $x = y$ or $y < x$.
2. If $x < y$ and $y < z$, then $x < z$, i.e. relation is transitive.

A set on which a total order relation is defined, is called an (totally) *ordered set*.

If one wishes to generalize the concept of monotonic function (or monotone function), the range does not need the set of the real numbers. More general, a monotonic function is a function between ordered sets that preserves or reverses the order. This concept historically arose in calculus, and was later generalized to the more abstract setting of order theory.

From the definition above one can see that monotonicity is a non-analytic concept, but in the upper secondary school it is very often presented as an application of the derivative, e.g.

For what values of a the function $f(x) = x^3 + ax^2 + 3x + 10$ is increasing everywhere?

The essential character of monotone functions is that it always preserves or reverses the order. Though the concept of order is rigorously defined in mathematics, it is also an everyday concept of which even small children have an intuition (Yli-Luoma 1995, 61).

2.2 On the dual nature of mathematical conceptions

My study deals with monotonicity which is a property of functions. How are functions perceived by students? In this paragraph, I make account of acquisition of new mathematical conceptions on the basis of the article by Anna Sfard (Sfard, 1991).

The conceptions of the function concept can be classified as *pre-operational*, *operational* and *structural*: A student with pre-operational conception has a rudimentary and inconsistent concept image (for definition see paragraph 2.3) whereas it is operational if he/she can view a function as a process, and structural if he/she is able to view the function as an object in its own right. Some sketches of criteria for classification of conceptions of monotonicity are presented later in connection with "degrees of structuralization".

Sfard explains this, as a result of the historical development of the notion of number, as follows:

> *(1) the preconceptual stage, at which mathematicians were getting used to certain operations on the already known numbers (or, as in the case of counting – on concrete objects); at this point, the routine manipulations were treated as they were: as processes, and nothing else (there was no need for new objects, since all the computations were still restricted to those procedures which produce the previously accepted numbers).*
>
> *(2) a long period of predominantly operational approach, during which a new kind of number begun to emerge out of the familiar processes (what triggered this shift were certain uncommon operations, previously regarded as totally forbidden, but now accepted as useful, if strange); at this stage, the just introduced name of the new number served as a cryptonym for certain operations rather than as a signifier of any "real" object; the idea of a new abstract construct, although already in wide use, would still evoke strong objections and heated philosophical discussions;*
>
> *(3)* the structural *phase, when the number in question has eventually been recognized as a fully-fledged mathematical object. From now on, different processes would be performed on this new number, thus giving birth to even more advanced kinds of numbers.* (Sfard, 1991, 13-14)

Developmental priority of operational conceptions over structural is empirically demonstrable despite the common practice of introducing new concepts by help of structural definitions.

As stated, this formation of structural conception arose from the development of general knowledge on numbers. Sfard (ibid., 16-23) presents as a preliminary problem, whether the suggested concept formation model is valid also in the case of individual learning? Or, is it true that when one gets familiar with a new mathematical notion, the operational conception usually develops first? Sfard suggests that the scheme constructed on the basis of historical examples offers a way to describe learning processes, too. Thus,

> *according to our scheme of historical development, three steps can be distinguished in the process of concept formation. These three stages correspond to three "degrees of structuralization" which may be named on the grounds of purely theoretical analysis of the relationship between processes and objects. In the light of the same analysis, our model of learning can be refined along similar lines: if the conjecture on operational origins of mathematical objects is true, then first there must be a process performed on the already familiar objects, then the idea of turning this process into an autonomous entity should emerge, and finally the ability to see this new entity as an integrated, object-like whole must be acquired. We shall call these three stages in concept development* interiorization, condensation *and* reification*, respectively.* (ibid., 18)

Sfard notes that investigating these stages directly is methodologically difficult, because one deals with student's *implicit* beliefs about the nature of mathematical objects. This forces one to describe *external* characteristics such as students' behaviours, attitudes and skills.

In interiorization students get acquainted with the processes which will finally bring about a new concept. These processes are operations performed on mathematical objects of lower level. What could this mean when monotonic functions are concerned? Maybe a vague sense that something changes in function's value when argument decreases or increases. Or just a thought that the graph of monotonic function behaves in a certain way.

The phase of condensation is a period of squeezing lengthy sequences of operations into more manageable units (ibid., 19). Even though deciding whether a given function is monotonic is not a lengthy process itself, it may be very non-trivial for a student. At this stage the student becomes more capable of seeing a process in question as a whole, without an urge to go into details. It may consist of figuring the function, deciding a criterion for its eventual monotonicity, applying the criterion with possible further arguments, and giving a result. Combining the process with other processes, making comparisons and generalizations become easier – thanks to condensation. A progress in condensation would manifest itself also in growing easiness to treat or converse between different representations of the concept. This is especially true with functions which have several different representations: formulae, tables, graphs, verbal descriptions.

This phase of condensation lasts as long as a new entity remains firmly connected to a certain process (ibid., 19). Only when a person becomes capable of conceiving the notion (concept) as a fully-fledged object, it has been reified. Reification is an ontological shift, where one becomes able to see something familiar in a new light. Thus, whereas interiorization and condensation are gradual, quantitative more than qualitative processes, reification is a leap – a process solidifies into object, a static structure. Sfard (ibid., 22) deals with reification thoroughly and she also presents a model, where the object A is a result of processes on concrete objects through interiorization, condensation and reification. Then there will be processes on A and through interiorization, condensation and reification the object B is born etc.

For my purposes, the general model of Sfard seems to be too heavy machinery, because observing such development is not possible for temporal reasons. But it is worth considering the meaning of reification when monotonic functions are in question. To see the monotonicity as a whole, as an object of higher level, is not possible to my mind, if the definition and its role and significance is not perceived by the student. Of course, this perception requires understanding the function concept and the concept of ordered set as well. It is striking to note that if someone has an analytic concept image of monotonicity, then there are, in fact, much more prerequisites: the derivative, the difference quotient and the limit, whose definition causes problems for students in advanced courses of mathematics, namely

Let f be a real-valued function defined on a subset D of the real numbers. Let c be a limit point of D and let L be a real number. We say that $\lim_{x \to c} f(x) = L$ if for every $\varepsilon > 0$ there exists a δ

such that, for all $x \in D$, if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$

2.3 On the concept image and concept definition

In the 1970s, empirical research emphasized that individuals build up their mental imagery of a concept in a way that may not always be coherent and consistent (Tall 1998, 37). That means, according to my interpretation, that there may be elements which do not necessarily have anything to do with each other or elements which are contradictory to each other. An interesting account of the coherence of concept images is given by Viholainen (2008). He mentions some criteria for a high level of coherence of a concept image:

> *1. An individual has a clear conception about the concept.*
> *2. All conceptions, cognitive representations and mental images concerning the concept are connected to each other.*
> *3. A concept image does not include internal contradictions, like contradictory conceptions about the concept.*
> *4. A concept image does not include conceptions which are in contradiction with the formal axiomatic system of mathematics.* (ibid., 235)

The literature lists a lot of cases of which many are related to analysis, especially to the limit of a function and a limit of a sequence. Many students have a misbelief that in $\lim_{n \to \infty} s_n = s$ the general term s_n can never equal s (Schwarzenberger and Tall, 1978). Another common misbelief is that in the definition of the limit of a function in certain point x_0 depends on the value of the function in that point. Real analysis has an infinite character which is a source of many erroneous conceptions.

The way in which the human brain works is often at variance with the logic of mathematics. For understanding the processes which sometimes result in successful, sometimes in failed conceptions, a distinction between mathematical concepts and cognitive processes must be made. Tall & Vinner (1981, 152) note that during the mental processes of recalling and manipulating a concept, many associated processes are brought into play, consciously and unconsciously affecting the meaning and usage. Likely the same is true when a certain concept is encountered for the very first time.

As a curiosity, a former definition of concept image is presented. Vinner and Hershkowitz (1980, 177) suggest the following:

> Let C denote a concept and let P denote a certain person. The P's mental picture of C is the set of all pictures that have ever been associated with C in P's mind. Besides the mental picture of a concept there might be a set of properties associated with the concept (in the mind of our person P … This set of properties together with the mental picture will be called by us the concept image …

It should be noted that in the definition above, there may be incorrect elements involved. For example, that an altitude should always fall inside the triangle. Or, from the topic of this thesis, that the concept of monotonicity makes sense only with continuous functions.

It is also an interesting finding, that correct/incorrect concept images can be born just out of very simple facts like the position of a triangle (ibid., 181). Right-angled triangles were recognized better, if the sides were horizontal and vertical.

The still current and widely cited definition of the concept image comes from Tall and Vinner (1981, 152). It considers several aspects of the concept image:

> We shall use the term *concept image* to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures.

There are many noteworthy facts related to the definition. The concept image varies not only globally but also during the time as the understanding of an individual develops. So, it is a matter of a process during which radical changes can happen. This is the case for example when new sets of numbers are introduced (natural numbers, whole numbers, rational numbers, real numbers, complex numbers). Then certain properties do change: subtracting two natural numbers does not necessarily yield a natural number, as it is the case with whole numbers. Among the real numbers one cannot take the square root of negative numbers; with complex numbers one can do that, and so forth.

There are, of course, many other reasons for a concept image to change, besides widening the set of objects as above. In my study, there are no possibilities to investigate such changes.

Tall and Vinner (ibid., 152) also give the following definition:

> We shall regard the *concept definition* to be a form of words used to specify that concept.

Even though many mathematical concepts are generally introduced in the same way in teaching, a *personal* concept definition can differ from a *formal* definition which is widely accepted by the mathematical community. A good account of the role of mathematical definitions is given by Edwards and Ward (2008). They describe mathematical definitions being of fundamental importance in the axiomatic structure that characterises mathematics which is certainly the case. The enculturation of college mathematics students includes their acceptance and understanding of the role of mathematical definitions, that the words of the formal definition embody the essence of and completely specify the concept being defined. They also play a role in the students' experiences in mathematics courses themselves in the sense that they are a means to a deeper understanding of a given concept.

Edwards and Ward see that the problems related to applying mathematical definitions have two possible reasons. Firstly, a student could have an incomplete or faulty understanding of the content of particular definition. Secondly, he could have a mathematically incorrect understanding of the role of nature of mathematical definitions in general (ibid., 226)

According to Tall and Vinner, for each individual a concept definition generates its own concept image, "concept definition image" which is a part of the concept image. For example, one may just have, for the algebraic structure of the ring, a personal concept definition 'a set in which two operations are defined such that the result essentially resembles the set of whole numbers with the addition and multiplication operations'.

Tall and Vinner (1981, 153) call the part of the concept image or concept definition which may conflict with another part of the concept image or concept definition, a *potential conflict factor*. Such factors need never be evoked in circumstances which cause actual cognitive conflict but if they are so evoked the factors concerned will then be called *cognitive conflict factors*.

It is also particularly interesting that different aspects may be emphasized in certain situations – then the portion of the concept image which is activated at a particular time is called the *evoked concept image* (Tall and Vinner, 1981, 152). A good example of that is the function concept which is one of the fundamental concepts in mathematics and by its character offers many ways to build up a concept image. Roughly speaking, one can consider a function as something which 'is', some other or even the same person in another situation as something which 'does'. Here I see these verbs as much as a property of a function and perception. A detailed account of that is given by Viirman, Attorps and Tossavainen (2010). They make use of five categories to get preliminary understanding of mathematics students' defining ideas of the function concept:

1. ***Correspondence/dependence relation.*** *A function is any correspondence or dependence relation between two sets that assigns to each element in the first set* exactly *one element in the other set. Domain and range may or may not be mentioned.*

2. ***Machine.*** *A function is a "machine" or one or more operations that transform variables into new variables. In this case no explicit mention of domain and range is made.*

3. ***Rule/formula.*** *A function is a rule, a formula or an algebraic expression. Compared to the second category, the difference is that now regular behaviour is expected whereas the machine could conceivably perform totally different transformations of different elements.*

4. ***Representation.*** *The function is identified with one of its representations.*

5. ***Nonsense.*** *A meaningless answer or no answer at all.* (ibid., 11-12)

In the same article, Viirman et al. (2010) found that out of the three conceptions of the function concept, pre-operational, operational and structural, the structural conception among 34 Swedish university students appeared to be rare, only three out of 34. Another result was that even though the concept images seemed to agree quite well with their concept definitions, they were not very rich.

2.4 Some specific studies related to concept images related to monotonicity

For my study, the foremost guideline is the article by Tossavainen, Haukkanen and Pesonen (2013). They studied which aspects are overriding in the concept images of monotonicity among Finnish tertiary mathematics students. They also investigated how the imposition of exercises affects the arguments the students give. They found out that mathematics students' conceptions are often restricted to continuous functions and such conceptions which are related to the formal definition of monotonicity are rare. This is the most natural reference for me – allowing reasonable comparison of my data with a meaningful control group. The findings of Tossavainen et al. are worthy of further consideration, this is done in the results chapter, aiming to give the reader a better chance to observe the differences between the groups of upper secondary and tertiary students.

The study of Rasslan and Vinner (1998) is also very relevant for my study. They succeeded in demonstrating that just knowing the definition of increasing (decreasing) function may not be enough. They studied 180 Israeli Arab high school students and their concept definitions and concept images on the increasing/decreasing function concept. One result of this study was that although 68 percent of the students could state the definition, only 36 percent of the students applied the definition successfully and correctly. Another 28 percent of the students applied the definition with varying level of success.

Rasslan and Vinner present (1998, also in their former studies) the terms *pseudo-conceptual behaviour* and the *concept substitute phenomenon*.

> The *pseudo-conceptual* behaviour might give the impression that such behaviour is based on conceptual thinking but, in fact, it is not. ... The ***criterion*** as a concept substitute is a special case of the concept substitute phenomenon. The concept substitute is a common tendency in many students to avoid concepts. It is a typical class situation in which students have to face concepts. (ibid. 34)

Another interesting result they achieved is related to the classification of the tasks they gave, especially that of the question 4: In your opinion, what is an "increasing function in a certain domain"?
The definition categories are the following:

> ***I*** *An algebraic definition (with the use of the universal quantifiers) (1 %)*
>
> ***II*** *A condition about the function derivative (3 %)*
>
> ***III*** *Increasing means a positive slope (2 %)*
>
> ***IV*** *If x increases y increases (41 %)*
>
> ***V*** *A combination of a definition with a concept substitute (21 %)*
>
> ***VI*** *Incorrect answers based on pseudo-conceptual mode of thinking (26 %)* (ibid., 35-36)

From this categorization, it turns out that at least 68 % of the students knew the definition of the increasing function which is a good result. But it is striking that only 1 % of the sample knew the formal definition and those in categories II and III confused definitions with criteria for these definitions. Moreover, the students in the category IV used expressions which do not necessarily indicate satisfactory knowledge of the concept. This makes the whole picture not so encouraging.

An important property of monotone functions, namely the local property (function maybe increasing on one interval and decreasing on another) was not well understood. That can be concluded from the fact that 36 % expressed their answer such that it showed understanding of monotonicity being a local property, not global (here the function $f: \mathbf{R} \longrightarrow \mathbf{R}, f(x) = x^6$ was considered). The correct responses were distributed among the whole sample such that 8 % gave an argument related to the function derivative and 21 % an argument based on the visual aspect of increasing/decreasing concept while 7 % reasoned on the general property of functions of type $y = x^n$, n even (ibid., 38-39).

Rasslan and Vinner (ibid., 33-34) also noticed that there is a risk of some type of functions becoming a prototype of increasing functions (odd power functions) and even worse, they can become also a concept substitute. Therefore, a rich manifold of example functions should be provided in the teaching sequence. In general, Rasslan and Vinner doubt whether increasing/decreasing functions should be taught by this “special case” approach.

There are not many studies about mathematics students’ understanding of monotone functions. Bardelle and Ferrari (2011) investigated the consequences of a teaching experiment in which learning was supported in an example-based way. The plan of the course included a wide use of graphs in standard lectures, tutoring sessions and examinations. They used examples of concepts to clarify the definitions given, e.g. examples and non-examples of monotonic functions in order to clarify the boundaries of monotonicity. So, they suggested innovative teaching paths, emphasizing semiotic aspects which are mostly omitted in this study. Still, a few interesting findings of theirs should be mentioned here. They describe the students’ four main ideas of increasing function in answering various questions, namely:

(1) the proper definition;
(2) a function f is increasing in $[a, b]$ if $f(a) < f(b)$; students just compare the values of the function at the end of the interval;
(3) a function is increasing when the increasing pieces of a graph are predominant compared to the decreasing ones;
(4) as in (1), but applied to connected portions only of graphs of discontinuous functions. (ibid., 237)

The following figure demonstrates that wrong concept images (2) - (4) can lead to wrong results when the task asks to tell which of the graphs below do not represent an increasing function on the given interval (with explanation):

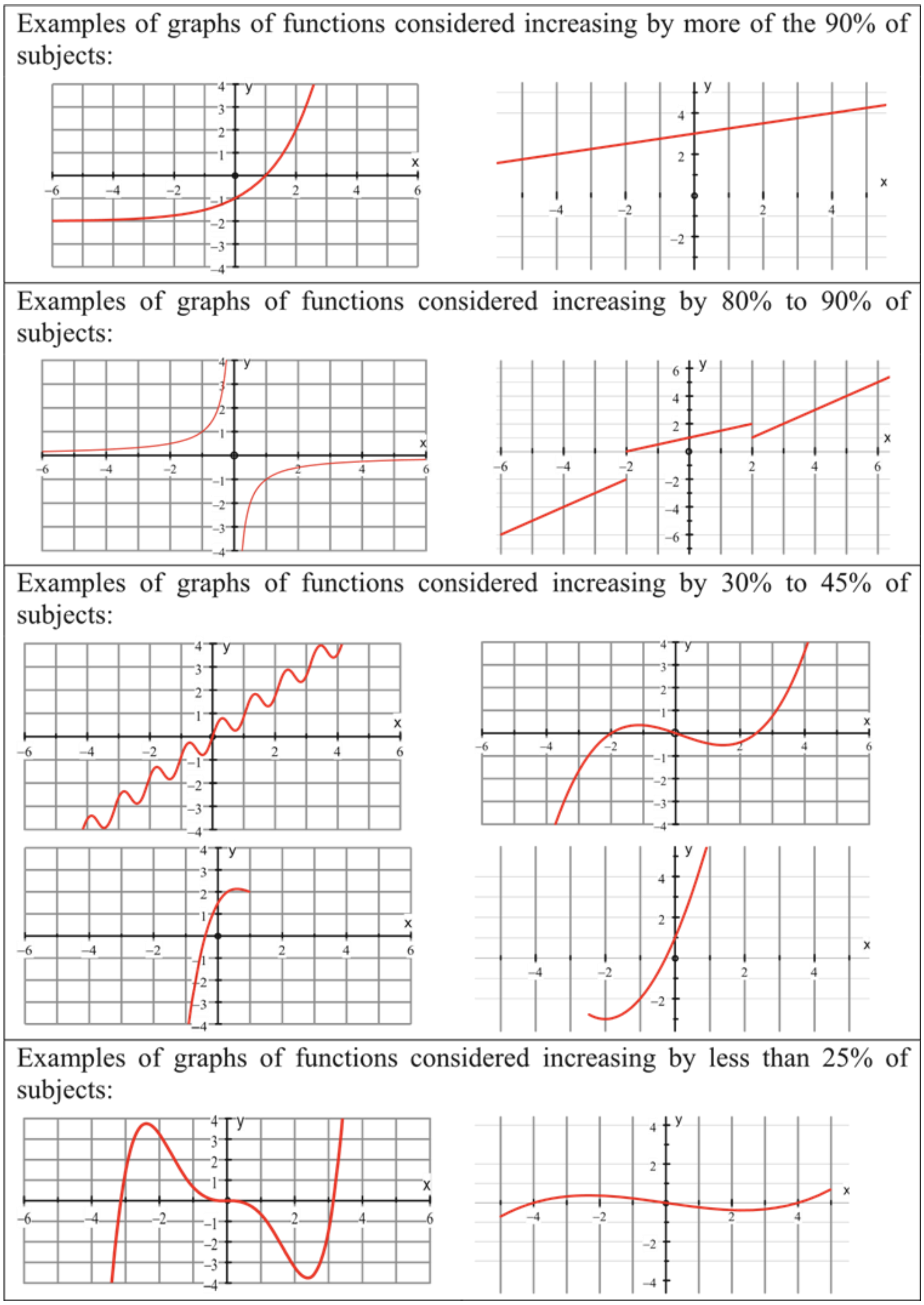


Figure 1. Graphs of functions considered increasing by different percentage of subjects by Bardelle and Ferrari (2011).

Also, Bardelle and Ferrari (2011) found that almost all students showed poor understanding of the standard definition or they had difficulties in its application. They also conclude that if mathematical terms have a colloquial meaning too, students often use them instead of the mathematical ones.

A striking result, in my opinion, is related to the following: Question 3 (ibid., 236), which has been included in the last test only, is

> *Mark any of the sentences below that correspond or are equivalent to the definition of increasing function in the interval [a, b]. Justify your answer.*
>
> *(a)* *For any* $x_1, x_2 \in [a, b]$, $f(x_1) < f(x_2)$ *holds.*
> *(b)* *For any* $x_1, x_2 \in [a, b]$, *such that* $x_1 < x_2$, $f(x_1) < f(x_2)$ *holds.*
> *(c)* *There exist* $x_1, x_2 \in [a, b]$, *such that if* $x_1 < x_2$, *then* $f(x_1) < f(x_2)$ *holds.*
> *(d)* $f(a) < f(b)$.
> *(e)* *There exist* $x_1, x_2 \in [a, b]$, *such that* $f(x_1) < f(x_2)$ *holds.*

The aim of this question was to understand possible connections with the knowledge of the standard definition of monotonicity. Less than a quarter of the sample chose the right answer, even though books and notes were allowed in the examination. This definitely shows poor or unstable knowledge of the definition.

3. Methodology

The survey was carried out on 20th April 2016 at the University of Tampere Teacher Training School, which is administratively a part of the Tampere University School of Education. In all other respects, the school enjoys great independence. There are 900 students in the Teacher Training School, and every year around 300 student teachers complete their training at this institution.

I had been in touch with the mathematics teachers Mr Jukka Männistö and Mr Tero Luodeslampi as early as in December 2015. At that time, we agreed on a rough timetable for spring 2016, which was then finalized in March. Two days before the survey I visited the nominated class introducing myself and the contents and aim of the study. The class consisted of 26 upper secondary school students, aged 17 to 18 years old. To have enough time (the length of a normal lesson is 75 minutes) it was possible to lengthen the time up to 90 minutes by using the 15-minute break as well. No technical devices were allowed. At the time of the survey, the ninth course of the national curriculum in advanced mathematics (called MAA9 *Trigonometric functions and number sequences*) was being taught. During the school year 2015-2016 there were ten obligatory courses (Opetushallitus (Finnish National Agency for Education), 2003) in advanced mathematics. An important fact is, that based on their studies up to that date, the students had in theory the ability to answer the test questions (possible exceptions are mentioned). Naturally, it would have been even better, if all the courses in advanced mathematics had been completed by the time of the survey, but due to the character of the Teacher Training School, a large number of surveys are conducted by student teachers, and so unfortunately it is not possible for everyone to have optimal arrangements because of schedule and other reasons.

As stated in the introduction, a similar study has been done earlier among 89 mathematics students at two Finnish universities (Tossavainen & al., 2013). Thus, it was well justified to use the same questionnaire as in the previous study. The questionnaire was used as such with the exception of omitting the questions inquiring the participants' background, which were irrelevant in this case unlike the original study where the background information was more versatile. The questionnaire (Appendix 1) contains eight (8) test exercises related to certain fundamental concepts of calculus of one real variable. Four of the exercises (1, 3c, 3d, 5) dealt right with the monotonicity of a function. They were formulated so that one emphasized the algebraic aspect, one the geometric aspect and one both the verbal and illustrative aspects, but the participants were free to choose and use any strategy. The fourth of these exercises was the only one where 'monotonicity' was explicitly mentioned and in it none of the above-mentioned aspects were emphasized. The concept image of each response was categorized according the guidelines given in each task separately in chapter 4. The categorizing scale (algebraic, analytical, geometric, experimental, erroneous, blank answers) is the same as in the previous study, where the scale was a result of a considerate, careful multistage process (Tossavainen & al., 2013, 1120). I applied it also for one more exercise (2). In my opinion, the classification is reasonable and often (but not always) quite easy to use. For a comment on the statistical methods, see the end of this chapter. There were responses in which none of the classes seemed to match and I felt a desire to introduce a new one, 'unclear verbal' or other like that, but did not do so. In any case, it is easy to point out many such responses. In three exercises (3a, 3b, 4) the above-mentioned categorization turned out being unsatisfactory and irrelevant. Therefore, I used other means to classify and describe the responses.

Because the study is partly a comparative one, it is important to ask whether and to what sense and to what amount my classification is comparable to the classification of the others? On the 16th of December 2016, I met with Mr Tossavainen and Mr Haukkanen. Together we went through exercise 5, which is the indicative one for classification of concept images. We agreed on almost all the classifications I had done. Moreover, all answers from one student were considered as a sample for checking the sufficient unanimity, I only made some minor changes then. After this stage unanimity was good but not perfect. Moreover, I have met with Mr Haukkanen several times afterwards as well as requested further criteria from Mr Tossavainen by email correspondence.

The responses have been analysed through the content analysis. Krippendorff (2004, 19) mentions three essentially different traditions in the methodology of content analysis. The first supposes that the content is inherent in the text. What else there may possibly be present in the text, is not considered. So, I read and analysed the responses 'as they are'. If there were exceptions to this principle they are clearly and separately mentioned at the appropriate point later in this thesis.

Furthermore, the evaluation of the tasks follows the guidelines of Tossavainen et al. (2013). The evaluation/scoring scale is: 3 = a correct answer with a reasonable explanation, 2 = a correct conclusion with one or two minor errors in the explanation, 1 = a correct conclusion with several minor errors or with one major error/a wrong conclusion due to one or several minor flaws in the reasoning, 0 = a wrong conclusion with no explanation or with several serious defects in the explanation/a missing answer. What was earlier mentioned about the discussion for unanimity with Mr Haukkanen and Mr Tossavainen, applies also to the scoring which is customized to a certain extent to the expected level of performance. The scoring scale is not very fine, but it is sufficient for the purpose. Clearly the scale is at least an ordinal scale. Whether it can be considered as an interval scale, meaning that the difference of level between responses assessed e.g. 0 and 1 points is (at least approximately) equal to the difference between responses assessed 2 and 3 points, can be disputed. However, such disputes in general go beyond the scope of the thesis. Many statistical tests are done, in general and in practice, assuming the measurement of variables being done in the interval scale. This is generally acceptable, when it serves the purpose of the research and presenting it. Still, if statistical methods are used in a 'flexible' way, it is useful to be aware of whether the requirements of a certain method are fulfilled or not.

The quantitative methods I applied in this survey consist of the use of descriptive statistics and Student's *t*-test. The reader should note that unlike Tossavainen et al. (2013) I have *not* considered the concept image classes being an ordinal scale, even though there would be reasons to do so. This, of course, disables the use of many statistical tests and methods, including the Spearman correlation coefficient. The cross tabulation of concept image classes and evaluated points would be possible, but it would still be impossible to analyse to what extent the obvious differences are due to a more advanced mathematical maturity of those having 'better' concept images (in the sense of Tossavainen et al. (2013)) and to what extent simply to the fact that not all approaches can produce full three points in all the exercises. On the other hand, I cannot see any reason to deny the existing differences and whatever the mechanisms behind the differences are, they apparently will be manifested in greater groups than investigated just in one study (ibid., 1125).

4. Results

In this chapter the main results are presented. After the general descriptive statistics, each task is dealt separately, the main emphasis being on the research questions and secondary students, the study concerning the tertiary students (Tossavainen et al. 2013) is referred to for comparison. The references to the tertiary students in this chapter are mentioned separately.

Table 1. Descriptive statistics of points (secondary students).

	N	Minimum	Maximum	Mean	Std. Deviation
Task1	26	0	2	,96	,871
Task2	26	0	2	,62	,852
Task3a	26	0	3	1,65	1,018
Task3b	26	0	1	,15	,368
Task3c	26	0	3	1,00	,894
Task3d	26	0	2	1,08	,796
Task4	26	0	3	1,04	,871
Task5	26	0	3	1,19	1,167
Monotonicity items	26	1	9	4,23	2,355
All items	26	3	15	7,69	3,415
Valid N (listwise)	26				

Table 2. Descriptive statistics of points (tertiary students).

	N	Minimum	Maximum	Mean	Std. Deviation
Task1	89	0	3	1,73	1,312
Task2	89	0	3	1,72	1,348
Task3a	89	0	3	2,02	1,158
Task3b	89	0	3	1,17	,980
Task3c	89	0	3	1,85	,899
Task3d	89	0	3	1,18	,912
Task4	89	0	3	1,43	1,086
Task5	89	0	3	1,34	1,279
Monotonicity items	89	0	12	6,10	3,244
All items	89	1	24	12,42	5,768
Valid N (listwise)	89				

The output of independent samples (secondary and tertiary students) test (Levene's test for equality of variables, t-test for equality of means) carried out with SPSS is presented in Appendix 3. Please note that the test requires equal variances. This means that the results are presented here if and only if equal variances are assumed. For the other cases, please see Appendix 3.

4.1 Task 5

Task 5. Examine the monotonicity of the function

$f: (0, \infty) \to (0, \infty),\ f(x) = x^2 - 5.$

(The task is not emphasizing any specific aspect of monotonicity. In the former study by Tossavainen et al. (2013) the category of concept image was determined by this task after which it was examined whether there were any differences in total success between the categories.)

Here it should be noted that the formula does not match with the given domain (or image). This was to see whether the participants pay attention to this, but that was not the case. After the descriptive statistical charts, each type of solution is presented and discussed. Note that the charts are also given from the data of the study of Tossavainen et al. (2013) for comparison.

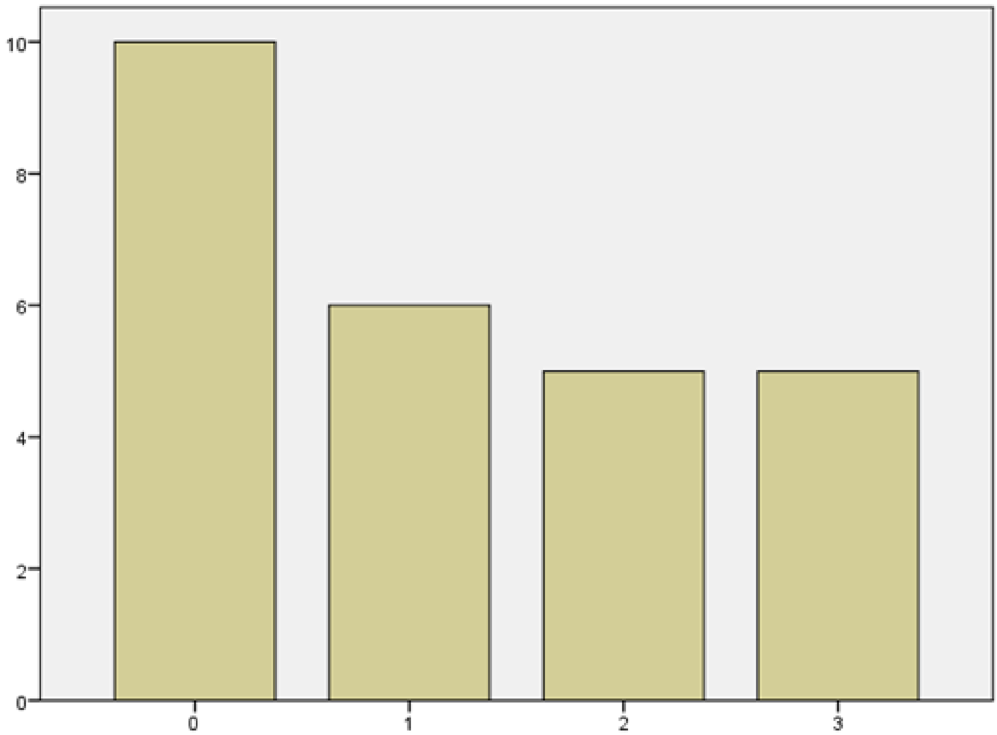


Figure 2. Distribution of points in task 5 (secondary students).

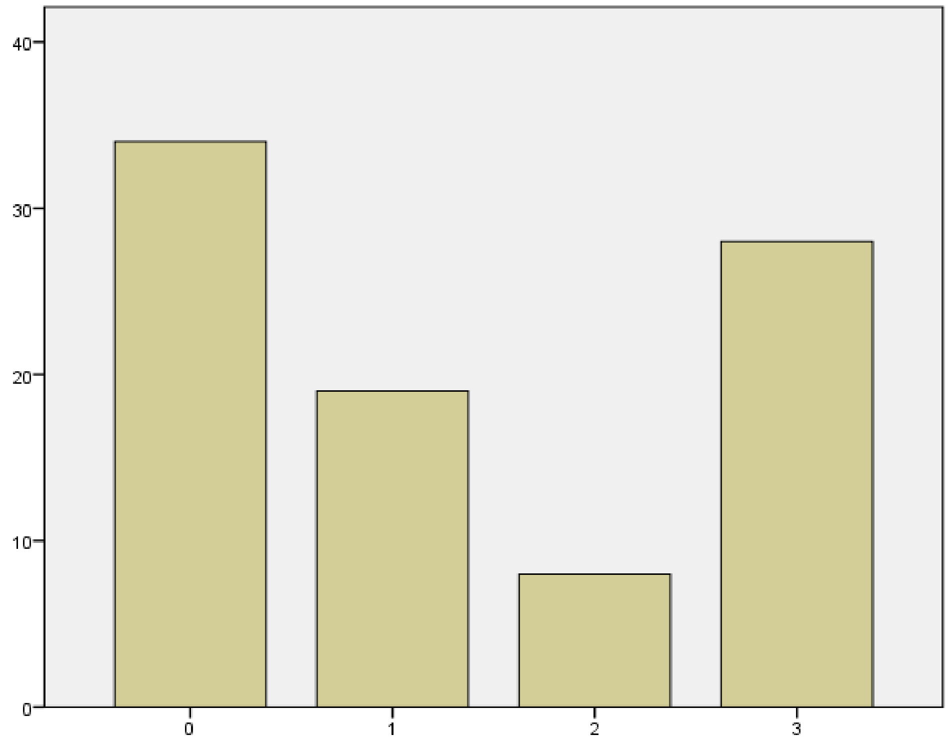


Figure 3. Distribution of points in task 5 (tertiary students).

The t-test does not give reason to exclude the hypothesis of equal means in both groups: $t(113) = 0.52$, $p = 0.61$.

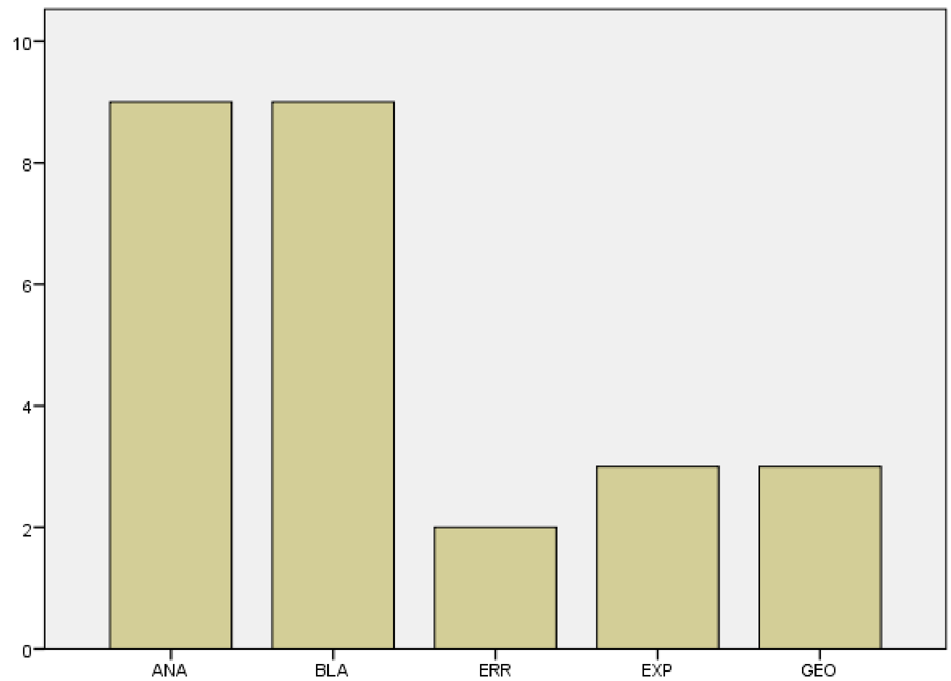


Figure 4. Distribution of concept images in task 5 (secondary students).

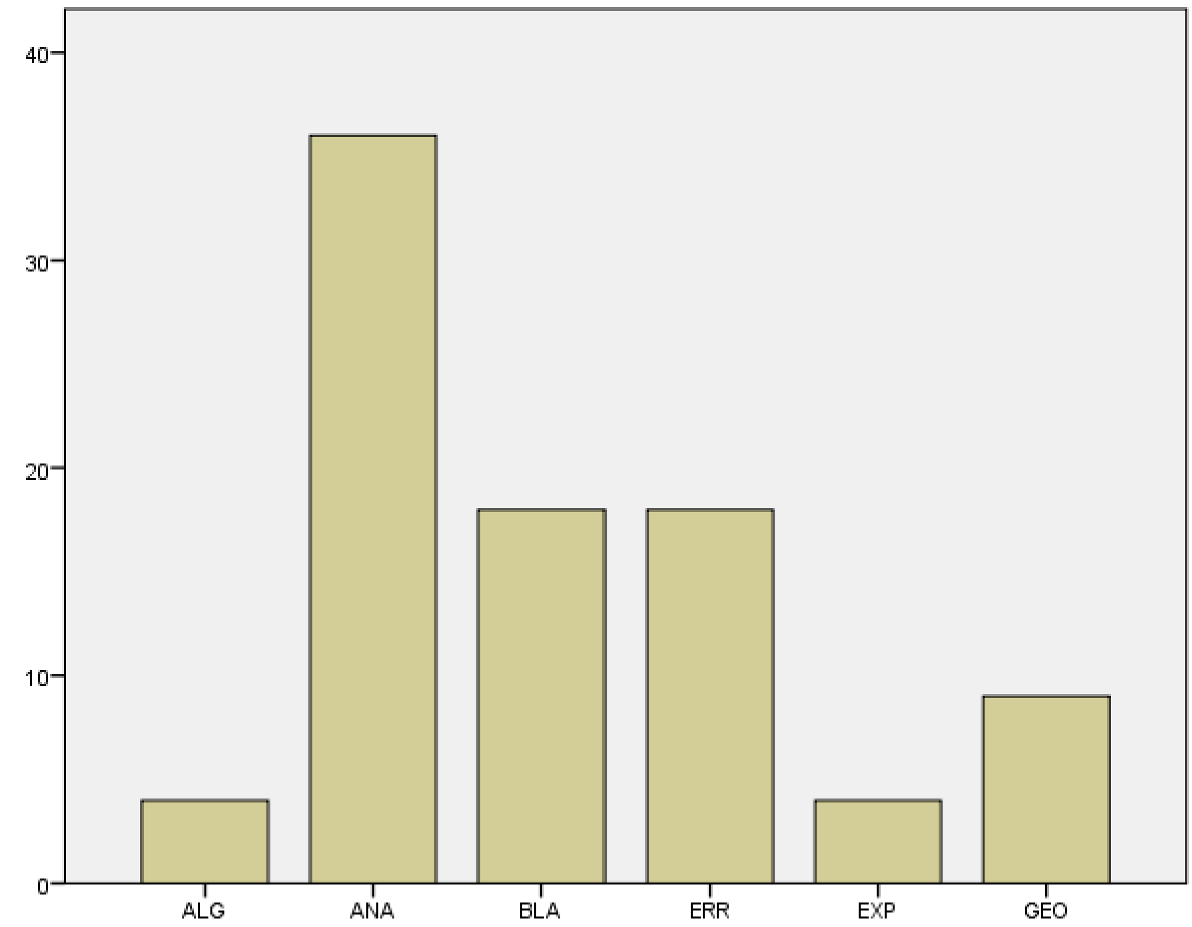


Figure 5. Distribution of concept images in task 5 (tertiary students).

4.1.1. Algebraic concept image

Here it is observed that no such answer was found which can be classified to be algebraic. Algebraic means here, for example showing that if $0 < x < y$ then $x^2 - 5 < y^2 - 5$. That is why such one is borrowed from Tossavainen et al. (2013):

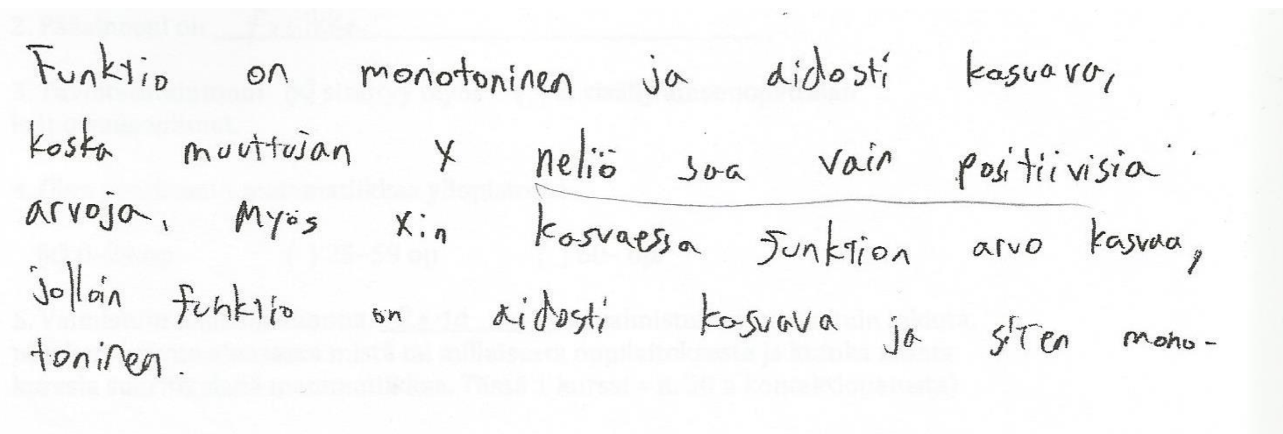


Figure 6. A solution representing the algebraic aspect to monotonicity.

In figure 6 the translated text is*: "The function is monotonic and strictly increasing, because the square of the variable x gets only positive values. Also, when x increases, the value of the function increases, whereupon the function is strictly increasing and thereby monotonic."*

Even if the answer is unprecise, redundant, clumsy and not totally correct, it contains the basic idea of the definition of increasing function. Let us recall that in the article of Tossavainen et al. (2013), only four students out of 89 showed the algebraic aspect of monotonicity here. A mature and correct example is presented by Tossavainen et al. (2013, 1121).

4.1.2 Analytic concept image

Here the idea of analytical examination is to differentiate the given function and show that the derivative is positive. As elsewhere, the personal style varies and it does not have an effect on the classification like the differentiation does.

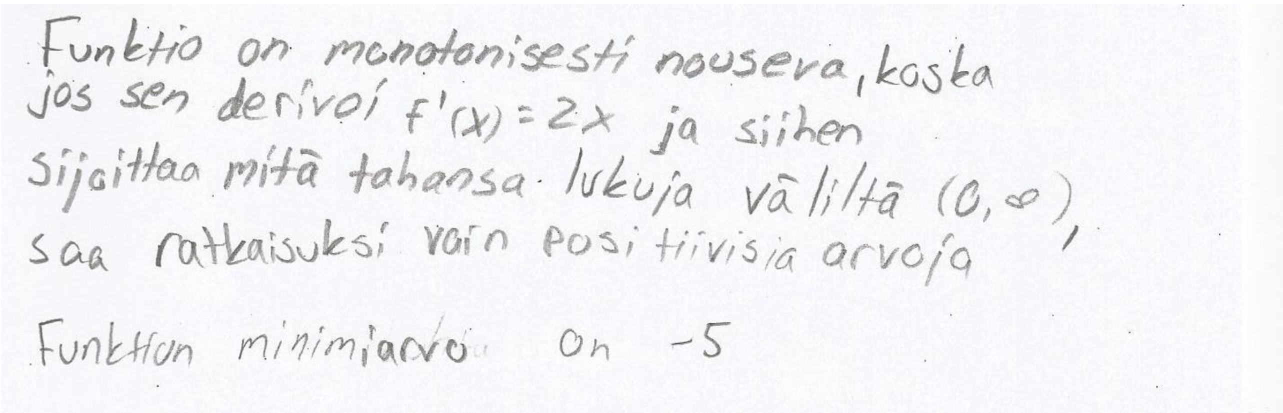


Figure 7. A solution representing the analytic aspect to monotonicity.

In figure 7 the translated text is: *"The function is monotonically ascending, because if it is differentiated $f'(x) = 2x$ and whatever numbers from the interval (0, ∞) are inserted to it, only positive values are obtained for solution The minimum value of the function is -5."*

The student works at the operational level which is, in my opinion, always the case when a student studies monotonicity correctly this way.

4.1.3 Geometric concept image

This concept image manifests itself by invoking the graph of the function. Then one concludes monotonicity by looking at the graph. Nothing else is needed for the classification, and if elements of any other type of concept image are involved, then the classification is done case by case.

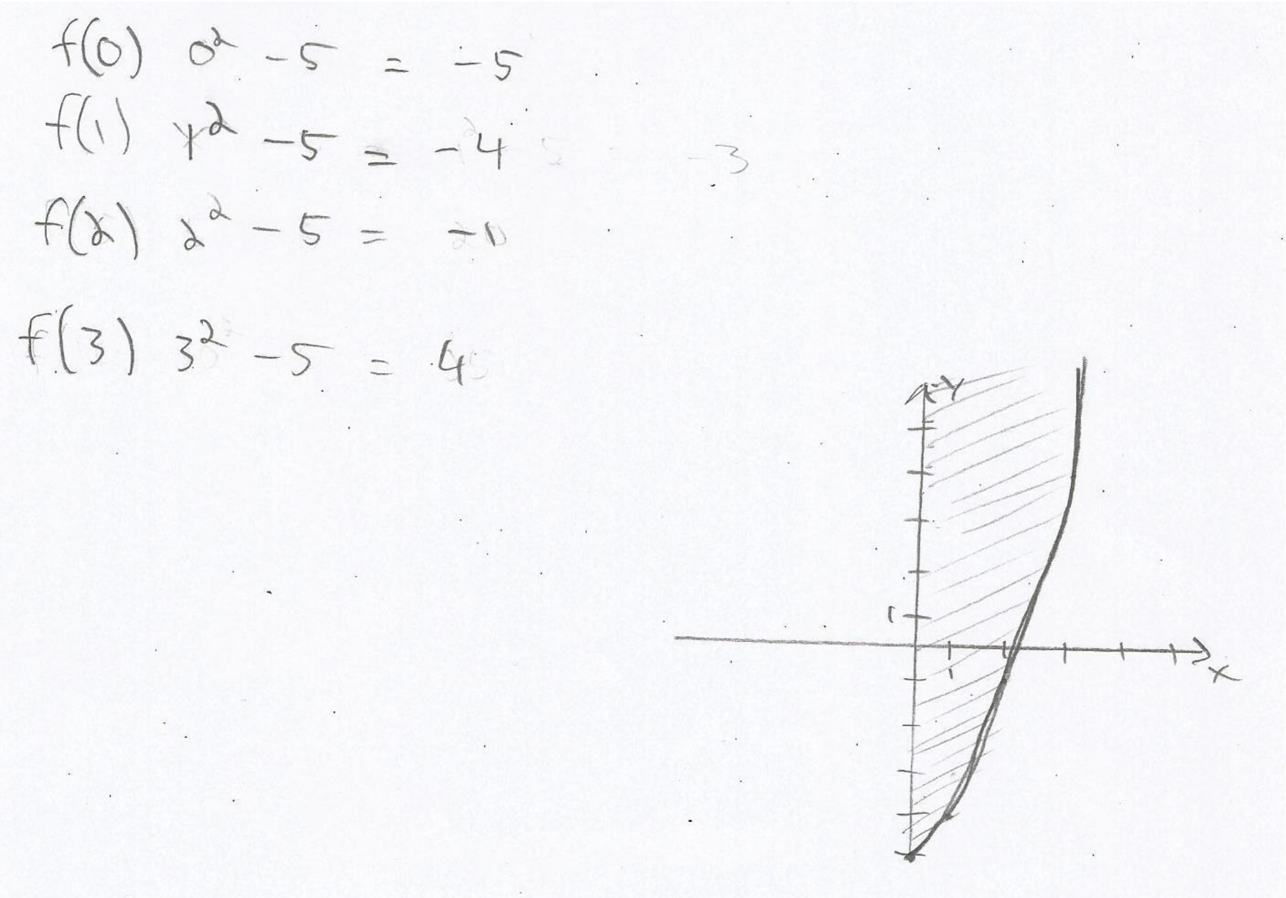
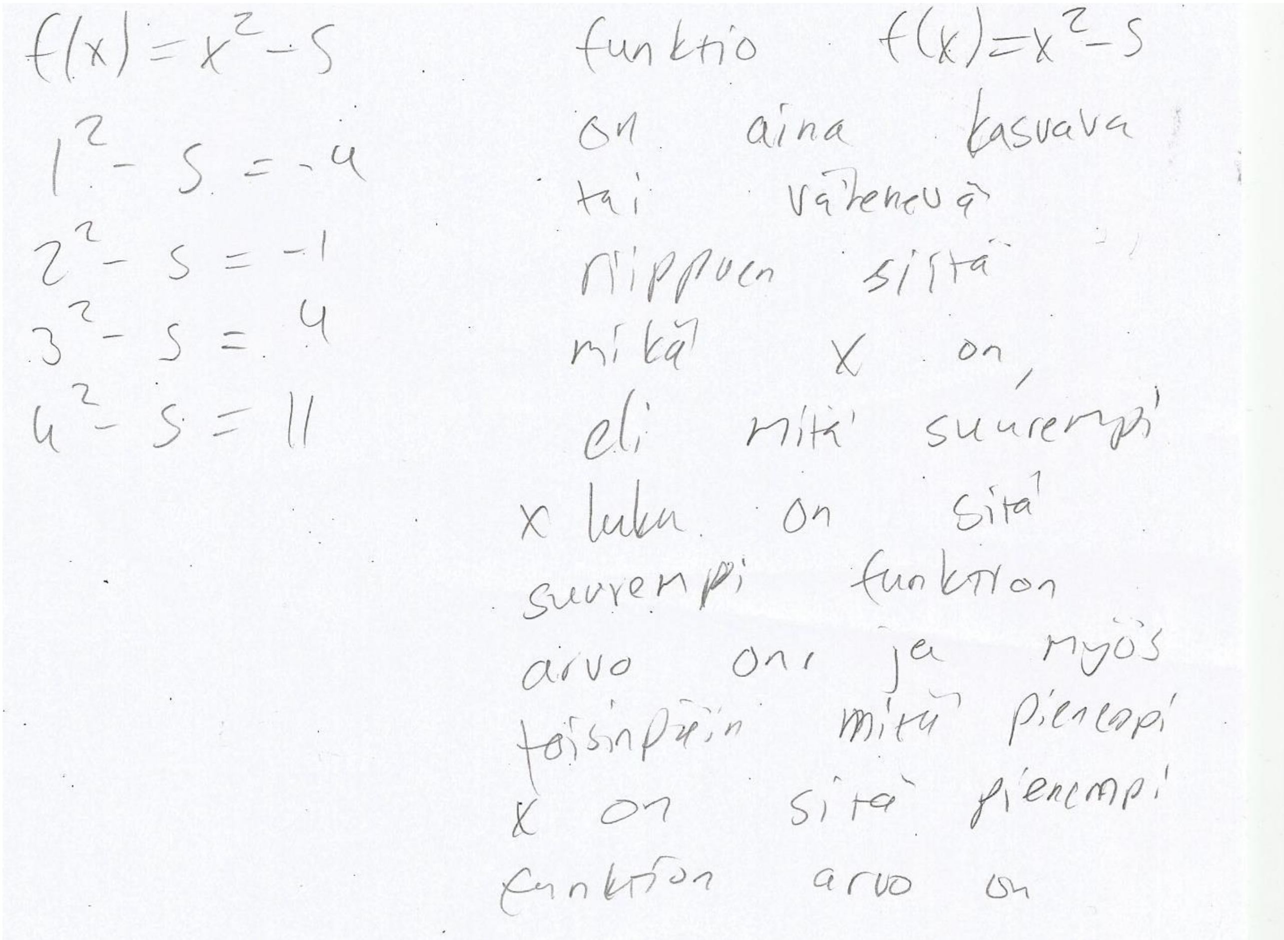


Figure 8. A solution representing the geometric aspect to monotonicity.

I have interpreted this solution to represent geometric concept image, for certain values of the function have been calculated to enable drawing the graph and certainly not the other way around. Recall that it was not allowed to use any kind of technical devices, especially a graphic calculator.

4.1.4 Experimental concept image

This means comparisons between values of the function for some discrete values of the variable even if the conclusion is not always stated. In the light of the previous figure 8 a typical response is not of great interest, but the following scan demonstrates the vagueness of the perception.


$f(x) = x^2 - 5$

$1^2 - 5 = -4$

$2^2 - 5 = -1$

$3^2 - 5 = 4$

$4^2 - 5 = 11$

funktio $f(x) = x^2 - 5$ on aina kasvava tai vähenevä riippuen sitä mikä x on, eli mitä suurempi x luku on sitä suurempi funktion arvo on, ja myös toisinpäin mitä pienempi x on sitä pienempi funktion arvo on

Figure 9. A solution representing the experimental aspect to monotonicity.

In figure 9 the translated text is "*the function* $f(x) = x^2 - 5$ *is always increasing or decreasing depending on what* x *is, i.e. the greater* x *number is, the greater the value of the function is, and the opposite the smaller* x *is, the smaller the value of the function is*". It is not clear whether the student understands monotonicity being a property related to an interval, not to the variable x. On one hand, the definition of monotonicity is somehow present but on the other it is clearly induced by experiment.

4.1.5 Erroneous concept image

In the erroneous solutions, I met with the equation $x^2 - 5 = 0$ solved and this has nothing to do with monotonicity. This is, however, a non-typical response in my data. The erroneous responses to this exercise among tertiary students are more versatile (e.g. considering the wrong function or not knowing the concept of monotonicity). For a more detailed account of this, see Tossavainen et al. (2013, 1123).

4.2 Task 1

Task 1. Show that for all real numbers holds $0 < x < y \rightarrow 0 < x^2 < y^2$.
(The task is emphasizing the algebraic aspect.)

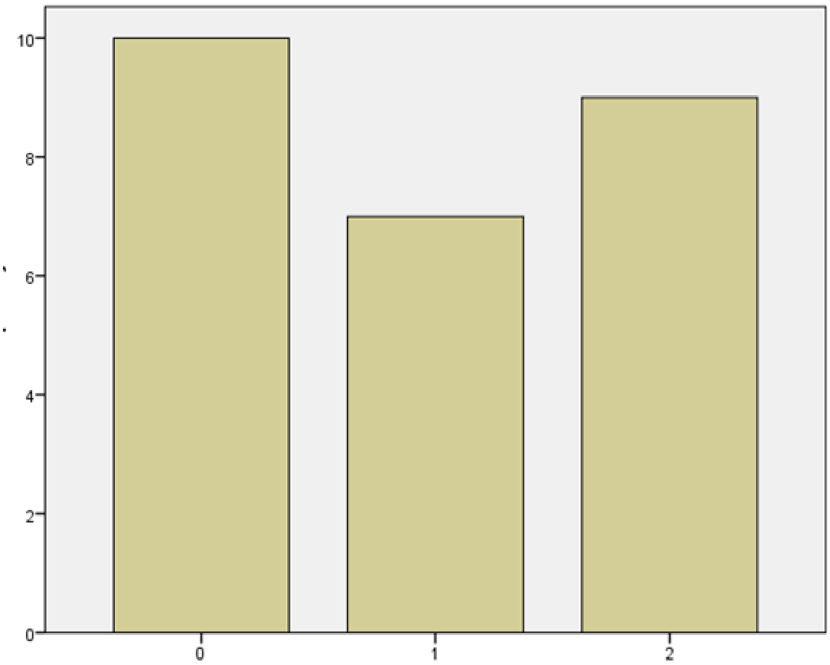


Figure 10. Distribution of points in task 1 (secondary students).

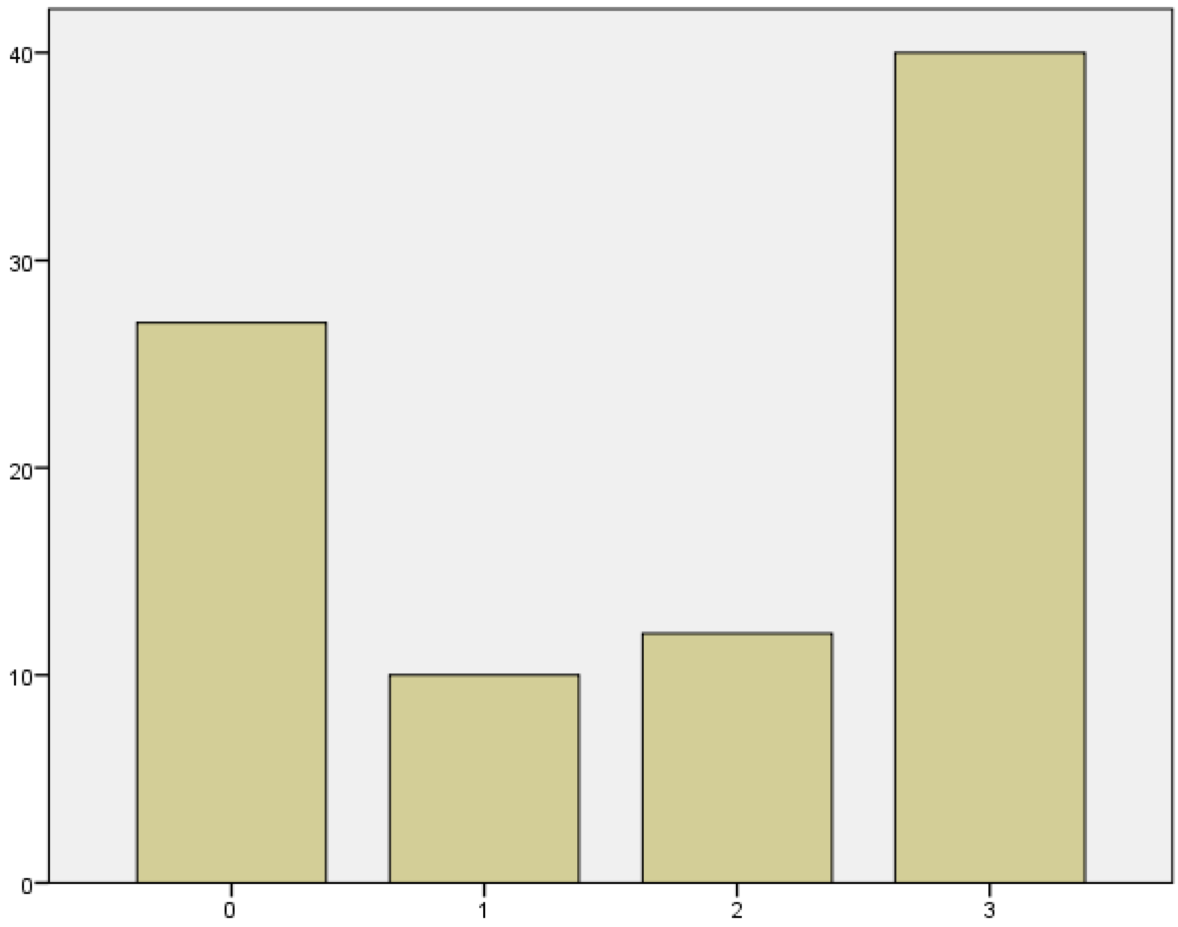


Figure 11. Distribution of points in task 1 (tertiary students).

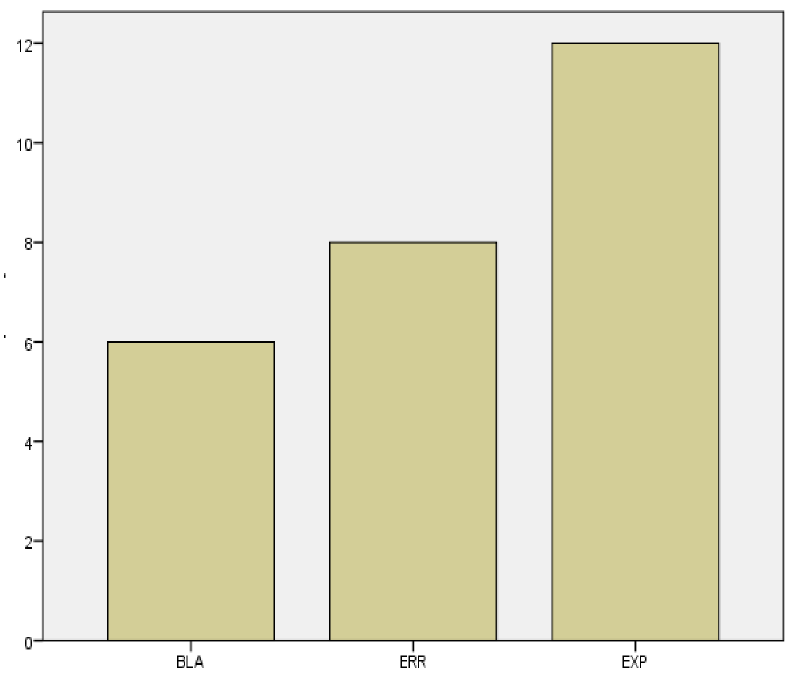


Figure 12. Distribution of concept images in task 1 (secondary students).

Tossavainen et al. (2013, 1125) found that in exercise 1 an algebraic method was used by as many participants as 63 (out of 89), while five were analytic, seven erroneous and 14 blanks. This immediately seems striking in the light of my data and first it is reasonable to search for the explanation for the difference. Is it possible that the difference is due to their different way of classification? That cannot be excluded. I have all their questionnaires available and almost all their statistics, but not the classification of each questionnaire of theirs. Because my primary intention is not to analyse their data, I did not reclassify their answers. But having had a thorough look at all of them, it is fair to say that I probably would not have classified that many as algebraic but rather erroneous. And there were responses some of which I would have classified experimental as well. In this study, I do not consider an answer to be algebraic if a student just repeats the proposition (or its equivalent form or expression of some kind) without arguments or the argument(s) do not make sense. It is just the crucial idea in the exercise to *show* that squaring preserves the inequality when the positive real numbers are concerned. So, the dilemma can be formulated also like this: If and when a student encounters a task emphasizing the algebraic aspect as in exercise 1, starts working with it and ends with no reasonable solution, is his or her concept image algebraic then? Tossavainen et al. (2013) answer that the algebraic aspect of monotonicity is implicitly contained in many students' concept images of monotonicity, evoked only due to formulation. Anyway, had I classified the answers in my survey with solely explicit mentioning of squaring as algebraic, not much would have changed, because there were only two such cases in my data. It is also plausible that experimental solutions are more common among upper secondary school students, for the sake of immaturity and accessibility to less mathematical means.

A few solutions are presented. The first two of them are from the study of Tossavainen et al. (2013) and they are presented first because there are no such solutions in my data and second because they represent the high quality which is rare in their data, too.

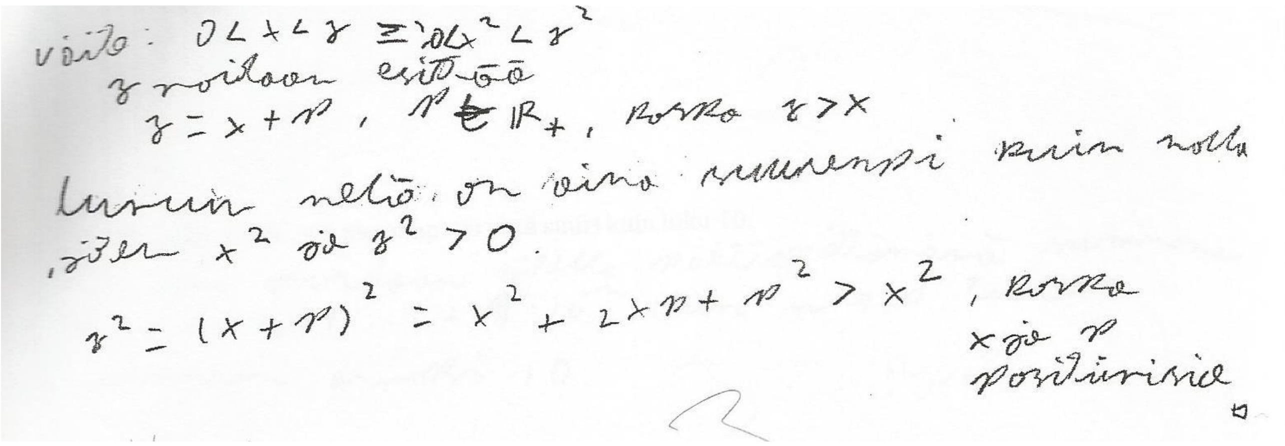


Figure 13. A solution representing the algebraic aspect to monotonicity.

The translated text: "*The assertion:* $0 < x < y \;\rightarrow\; 0 < x^2 < y^2$
y can be presented in the form of $y = x + p, p \in R_+$ *because* $y > x$
The square of a number is always greater than zero, hence x^2 *and* $y^2 > 0$.
$y^2 = (x + p)^2 = x^2 + 2xp + p^2 > x^2$, *because x and y positive."*

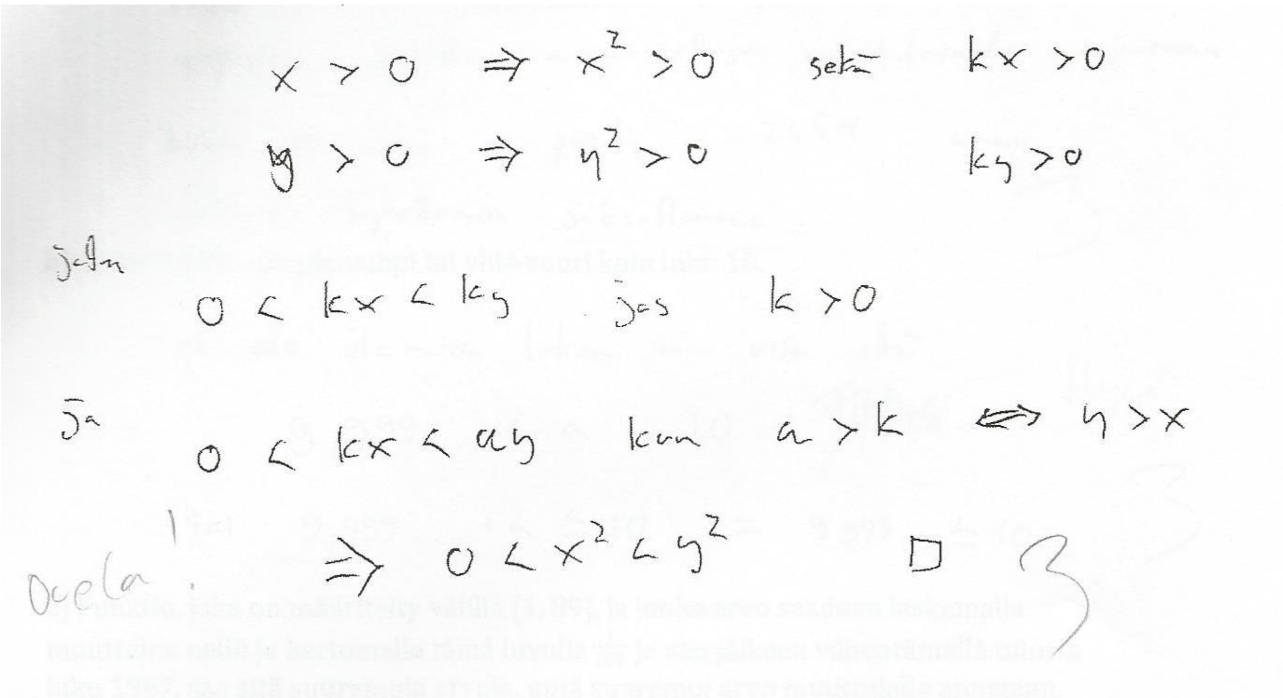


Figure 14. Another solution representing the algebraic aspect to monotonicity.

The translation of the words: sekä/and, joten/so, jos /if, ja/and, kun/when.
Here the student uses the fact that multiplying an inequation by a positive number is correct.

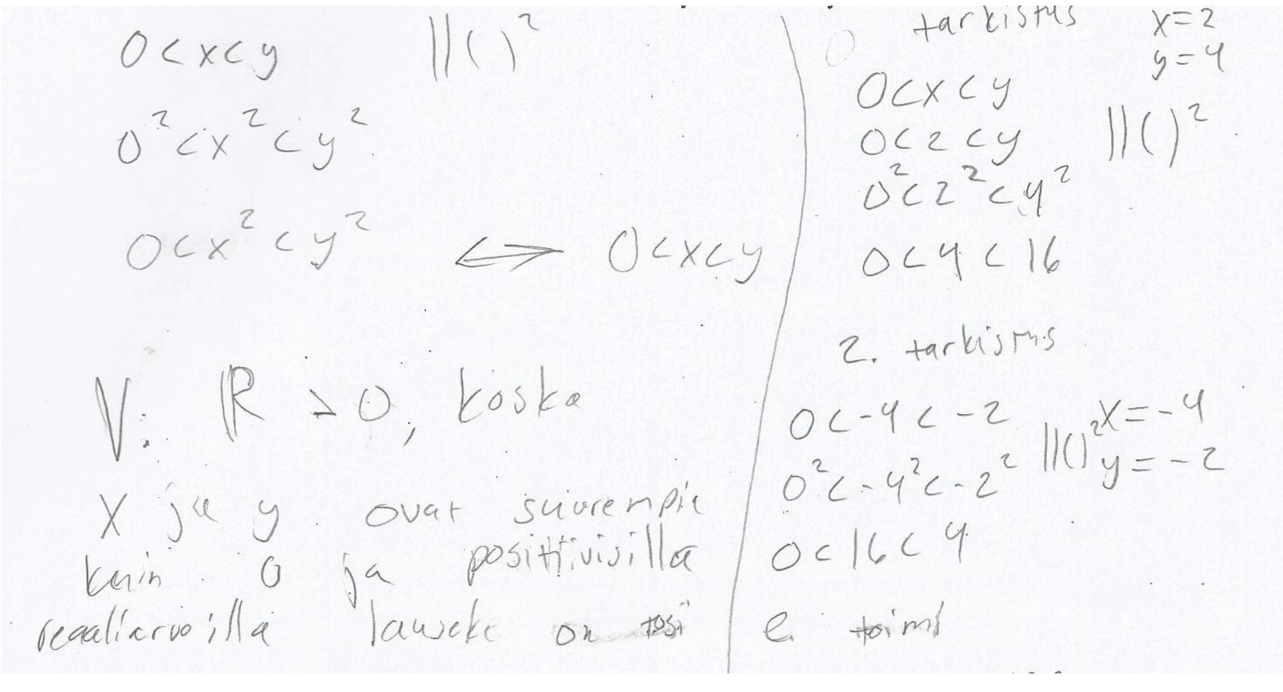


Figure 15. Still another solution possibly representing the algebraic aspect to monotonicity, but now not as clearly as in figures 13 and 14.

Here the squaring of $0 < x < y$ is performed correctly, but without explanation. Then the conclusion translated is: *“* $\mathbf{R} > 0$ [sic!], *because x and y are greater than 0 and with positive real values the expression is true”.* Note the careless use of the equivalence symbol. On the right-hand side, there are some verifications and on negative values *“it doesn’t work*”. As considered above, one could call this an algebraic approach, too.

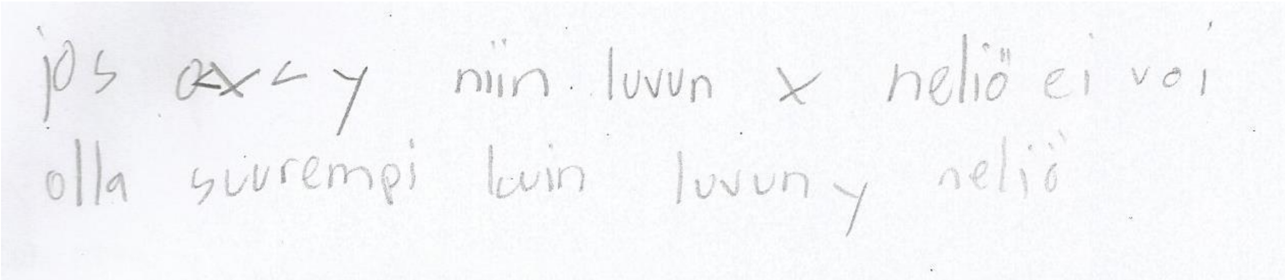


Figure 16. A solution without a valid argument.

The translated text is: “*If* $0 < x < y$ *then the square of the number* x *cannot be greater than the square of the number y*”. The argument is missing. Other reformulations of the assertion as were also to be found.

4.3 Task 3c

3.c) Comment the truthfulness of the following statement and justify.

A function which is defined on the interval $[1, 89]$ and whose value is got by squaring the variable and multiplying this by the number $\frac{1}{107}$ and after that subtracting the number 1987, has the property: the greater the variable’s values the greater the values of the function.
(The task is given in a verbal, descriptive way.)

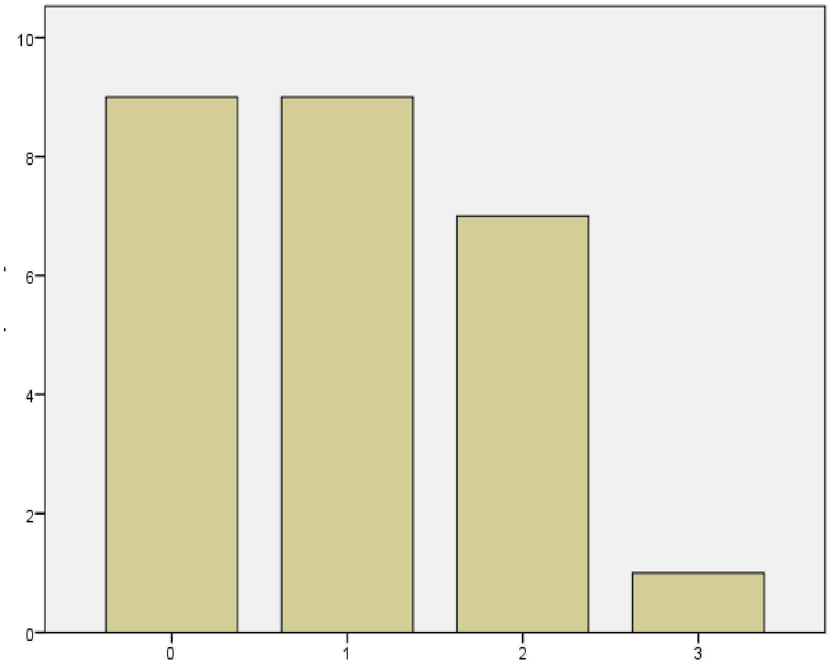


Figure 17. Distribution of points in task 3c (secondary students).

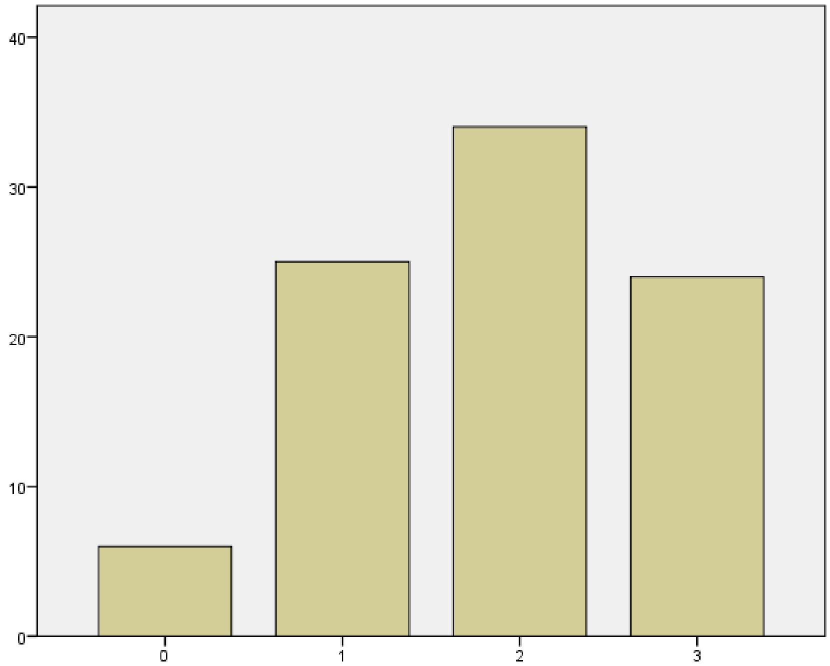


Figure 18. Distribution of points in task 3c (tertiary students).

The t-test does give reason to exclude the hypothesis of equal means in both groups: $t(113) = 4.27$, $p < 0.01$.

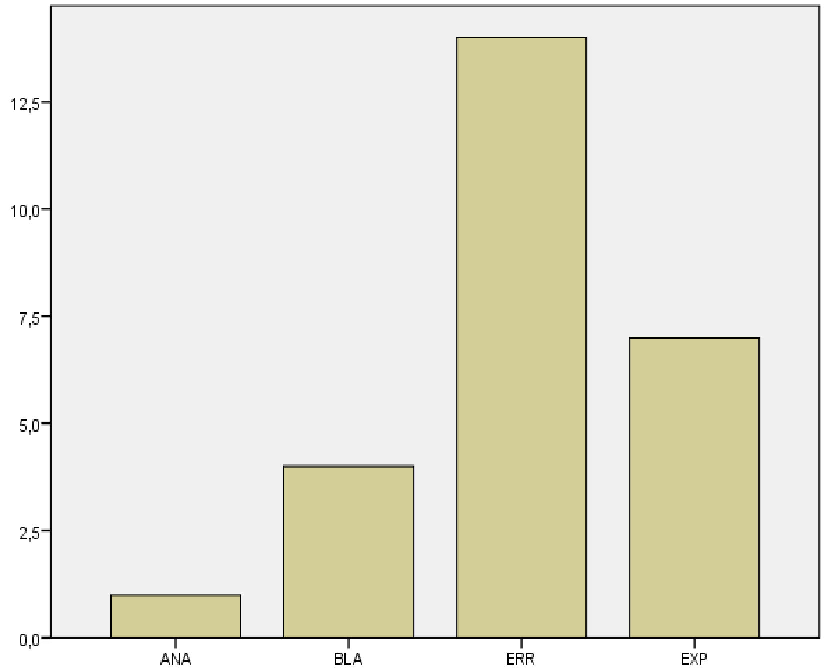


Figure 19. Distribution of concept images in task 3c (secondary students).

Originally (Tossavainen & al. 2013, 1126) this task was designed to inspect what aspects of monotonicity are evoked when the task is given in a verbal, descriptive style. Secondly it is meant to examine to what extent, if any, the heavier cognitive load of the imposition causes problems in solving.

Recall that a descriptive solution does not present a proper aspect to monotonicity (ibid., 1126) and is not therefore represented in figure 19. Still it is correct to say that the verbal imposition of the task evoked many verbal answers (how many exactly is a matter of definition, but the majority of the erroneous answers are characterised by a verbal, descriptive way of expression). In this sense the difference between the tasks 1 and 3c is clear: while 3c evokes clearly verbal concept images, 1 evokes algebraic concept images only implicitly. The more is demanded for classifying an answer as algebraic, the greater the is difference.

There are some answers trying to hint that the operations performed make the function to increase, but often the reader must read between the lines. Some of the arguments are so vague that they are difficult to translate into English. There is not a single answer explicitly noting that the function is a *composition* of first squaring, then dividing by 107 and subtracting 1987 and their preserving effect on the order. Instead there are expressions like *'increases due to the power'*.

Besides the fault of considering an incorrect function, there are mistakes in pure experimental calculation and some cases where the domain is omitted or neglected, leading to a faulty result.

After that discussion, an analytic solution (the only one) is presented in figure 20.

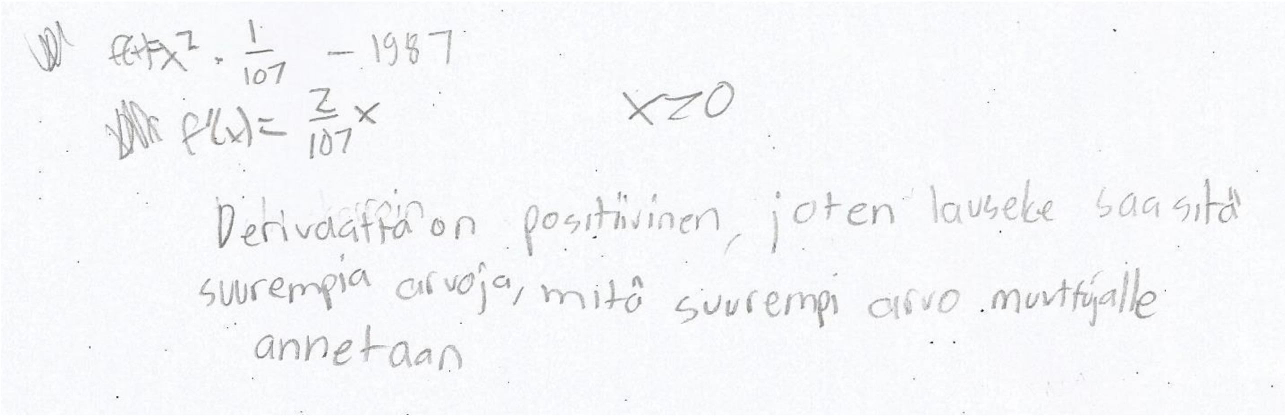


Figure 20. A solution representing the analytic aspect to monotonicity.

The text translated is: *"The derivative is positive, so the expression gets the greater values the greater value to the variable is given"*. −This is a firm and concise solution which includes the essential and nothing else. It is astonishing that only one solution of this type occurred, for, as mentioned before, monotonicity is mentioned in the context of calculus in the curriculum. So, why is calculus not more widely made use of?

4.4 Task 3d

Comment the truthfulness of the following statement and justify.

3.d) The graph of a strictly decreasing function is a descending line or other descending curve.
(The task is emphasizing the geometric aspect.)

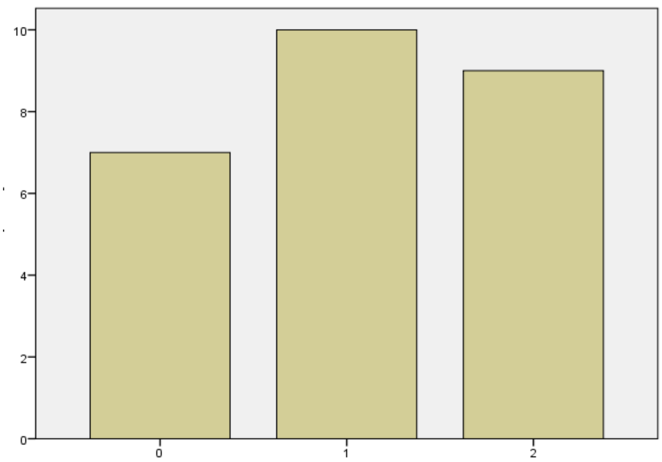


Figure 21. Distribution of points in task 3d (secondary students).

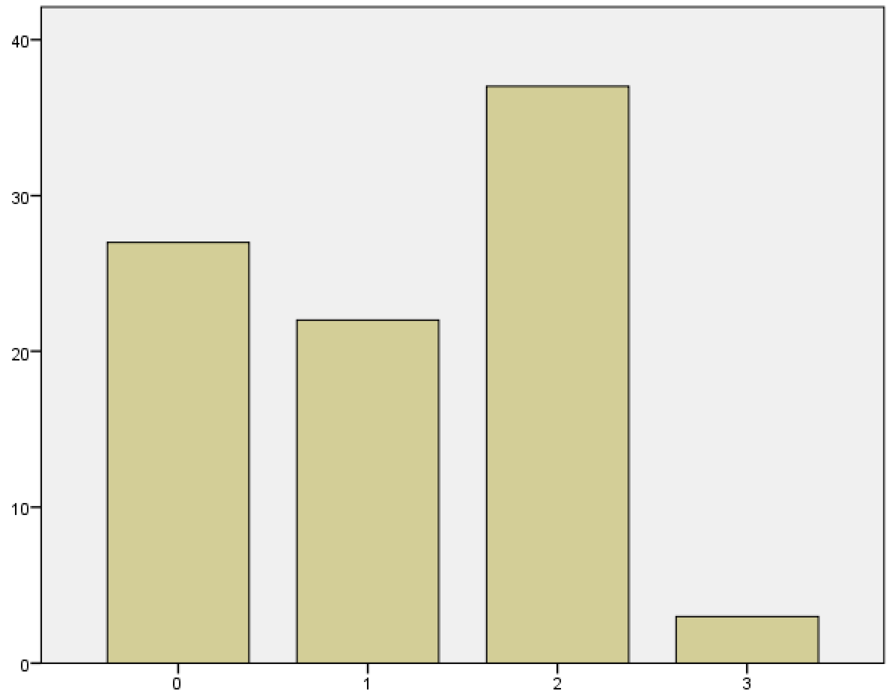


Figure 22. Distribution of points in task 3d (tertiary students).

The t-test does not give a reason to exclude the hypothesis of equal means in both groups: $t(113) = 0.52$, $p = 0.60$.

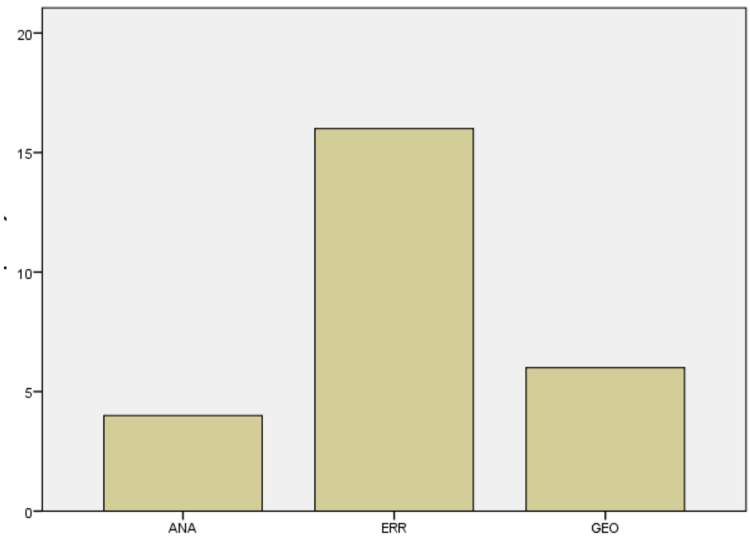


Figure 23. Distribution of concept images in task 3d (secondary students).

After seeing the results of Tossavainen et al. (2013) concerning this task, the expectations were not high. Among the university students, the task was generally carried out worse than the previous ones (with the means 1.73, 1.85, 1.18, 1.34 for 1, 3c, 3d, 5, respectively). It is also true that in my study the proportion of erroneous solutions is higher in 3d than in the other three tasks (see figures 4, 12 and 19).

As before, the analytic approach is related to the derivative. One student notes that "*a function containing variables of first degree descents straight or the derivative is the same, but with variables of higher degree the curve's derivative can change, but the sign of the derivative remains negative*". (The Finnish expression was also clumsy.) Another student states "*true as far as the derivative of the curve does not change to positive anywhere".*

In order to have a geometric concept image of a strictly decreasing function, it is natural to think of sketching a figure, but it does not have to be so. "*True, at least in the x-y -system of coordinates. The slope of a strictly decreasing function in the x-y system of coordinates has to be less than or equal to zero to be strictly decreasing*." Another student put it as follows: "*The proposition is true. The greater the values of the variable, the smaller the values the function got. When the values are placed into the system of coordinates, they form a descending line or other curve*".

In fact, this task is the only one out of the four which allows to reveal the fact that the student never considers other functions than continuous ones when asked to examine monotonicity. This defect occurred in all the answers. On the other hand, it is not astonishing, for according to Tossavainen et al. (2013) there were only three correct responses. The phenomenon can be explained just by them lacking the qualifications to consider other kind of functions as well. In the university, they should already have the qualifications, in the upper secondary school not necessarily, if they are never taught the definition.

In the responses to exercise 3d there are errors of almost all kinds. To give a thorough account of all of them would go beyond the scope. So, I just mention the most essential ones briefly, also those occurring in other categories than erroneous (not in any specific order):

- "*True*. O*nly one zero*"
- "*Right, for then a function gets negative values*"
- "*There might be only a certain place in a function, where it is strictly decreasing, so the graph of the function is not necessarily descendent*" (illogical)
- "*The graph of a strictly decreasing function is a descending line, because a strictly decreasing line must descend all the time*" (considering only lines)
- "*It is a descending line and it can make curves too, but rarely both at same time*"
- "*The graph of a strictly decreasing function can be a descending line or other descending curve as far as it is continuously decreasing and does not increase or stop ever*"
- "*The graph can be descending, but still not always strictly decreasing*"
- "*True, for a strictly decreasing function cannot get greater value than that at previous* [?] *point*"
- "*A function is always strictly decreasing when a graph is descending so the assertion is true*" (illogic)
- Perception that strictly decreasing functions are characterised by x tending to infinity, y tending to minus infinity
- Not understood that the derivative can equal to zero at single points
- Confusion with positive and negative direction of the abscissa

4.5 Results related to other exercises

4.5.1 Task 2

Examine the sequence $\left(\frac{7n-2}{3n}\right)$ when $n \to \infty$.

This is a relevant question, because monotonicity can be a property of discrete (*arithmetical*) functions as well.

The responses were classified also in this exercise, resulting in one analytic, eight experimental, nine erroneous and eight blank responses.

The exercise of this type (considering a sequence and its limit) was not probably known to the students, for they had not encountered sequences in the curriculum yet at the time of the carrying out of the test. Still, it gives a reason to ask, whether they are capable of handling the function as a continuous one and obtaining the result that way, i.e. considering the limit $\lim_{x\to\infty}\left(\frac{7x-2}{3x}\right)$. Secondly, it is interesting to see how the students reacted when they bumped into the question.

As said, there was only one student who handled the exercise in an analytical way, that is to say that he differentiated the function. But doing so, he differentiated the function wrongly when he calculated the derivative of $(3n)^{-1}$, leading to a false result at this stage. Still the response can be further analysed. This reveals other faults and misconceptions. He seemed to think that it is a matter of finding the smallest and greatest values – not the limit. He concludes that "*there is not the greatest value*" which is true, arguing obviously by the sign of derivative, but the response makes one doubt that he does not understand that 'increasing' does not generally mean a non-existent limit.

Experimental approaches are done by giving a few (one to five) values (often very small, 100,000 at the largest) for n. There are occasional minor flaws, but they are unessential. More important is that no one succeeds to state that the values *approach* $\frac{7}{3}$. Three out of eight say explicitly that the values increase, the best formulation being "*when* $n \to \infty$*, the sequence* $\left(\frac{7n-2}{3n}\right)$ *increases, but not much*".

Among errors, there are four cases where the infinity symbol is operated on as if it were a real number. A few students try to find the zero of the expression and the rest of this category are too cryptical to be explained in any sensible way.

4.5.2 Task 3a

Comment the truthfulness of the following statement 3.a) and justify.

3.a) The number π is irrational and its value is 3.141592654.

The exercise is related to monotonicity such that the sequence 3, 3.1, 3.14,.. is monotone. There are, however, no reasons to expect that the students would consider it in such a way, because it is beyond the scope of the curriculum. Instead, it measures the general knowledge of mathematics. (It was Lambert who first proved the irrationality of π in 1761. In my master's thesis on mathematics I dealt with this, among other properties of π and *e*.)

Here it should be mentioned that the formulation of the statement, containing two propositions, can be criticised for two reasons: First investigating the logical understanding of the connective 'and' could be done in other studies. Second, the formulation causes extra difficulties in interpreting and evaluating the responses which are often quite concise. Though aware of the facts, for the sake of comparability, the questionnaire was used.

The categorisation used elsewhere in this study does not necessarily make sense in 3a. Instead of it, I analysed the task in two parts (I 'the number π is irrational, II its value is 3.141592654) and in doing so, I was looking for general characteristics in common which the responses share with other responses and found such.

I The positive result is that more than a half (14 students) knew the irrationality of the number π, expressing it clearly meanwhile three said that π is rational. Three referred to the geometric definition of the number (one correctly, one wrongly, one imprecisely), four did not comment on this (3a I). Compared to the other exercises, 3a was easy to response – there was only one blank in the whole task. One response, whether yes or no, was impossible to interpret.

II Even more of the students recognized that the given value 3.141592654 is (only) an approximate value – 16 of them expressed this, and the majority expressed the term *approximate value* explicitly. Among this group there were only a few mistakes. One student mentioned "*endless number, in other words irrational number*" and three wrote about π something like "*... its exact value has never been totally defined ...*" showing that he/she is unaware that it is not a matter of time at all. Five responses were false, one did not comment on this (3a II). Again (naturally) there were three geometric expressions, which did not meet the idea of the task.

4.5.3 Task 3b

Comment the truthfulness of the following statement 3.b) and justify.

The number $9.999\ldots$ is less than or equal to the number 10.

This exercise is related to monotonicity such that the sequence 9, 9.9, 9.99,… is strictly increasing, though this is not directly asked the students. Like 3a, it measures also the general knowledge and ability to understand infinitesimal terms. It may be asked whether any potential conflict occurs and if so, does it manifest itself though the formulation, being of yes or no type with justification, does not offer a good chance to reveal any conflict. Tall and Vinner (1981, 156-159) have made a thorough account of the issue and it is hard to get results not already familiar. I would just like to note that because geometric series (not being presented to the students at the time of the data collection) cannot probably be a source of a conflict, a conflict could be evoked, in principle, if a student reasoned: if $9.999\ldots < 10$, what is the arithmetic mean of the numbers then?

Again, the classification of concept images used elsewhere is not adequate here. I ended up in a grouping after the first analyse on the responses. It seems possible that some of the students do not understand that the relation 'less than' is included in 'less than or equal to', but they may consider them exclusive. As said in the Methodology section, it is assumed that the content of a response is inherent in a text. As an exception, I have interpreted 'not equal to 10' to be 'less than 10' because of the imposition and the context.

The main result here is straightforward. In practice, none of the students fully understood the essential character of 9.99…: whatever the approximate is, it is still unequal and the equality comes only out of an infinite number of decimals. No one showed a sign of a cognitive conflict. 23 students clearly indicated that $9.999\ldots < 10$ and three of them even said explicitly that the statement of the imposition ('less than or equal to') is wrong and thus made a logical blunder. Of the rest of the three students, one said '*partially true*', another '*depends on whether an approximative value or number rounded off is wanted*' and still another '*equal because rounding off*'. Even though the understanding is poor (in 3b the points are absolutely the lowest among all the tasks), there are, however, a few indications to the correct direction: *"…but because there in an infinite amount of decimals and the number* $9.99\ldots$ *approaches the number* 10 *all the time …*". The arguments of seven students dealt with rounding off explicitly, of two students implicitly "*… easier to denote equal to the number* 10".

4.5.4 Task 4

4. Examine the continuity of the function $f\colon \mathbf{R} \to \mathbf{R}, f(x) = \frac{x^4-1}{x-1}$ at $x = 1$.

This exercise has no direct connection to monotonicity (except that it is monotone everywhere in the intervals $(-\infty, 1)$ and $(1, \infty)$). It is easy to see that the function is not defined at $x = 1$ and thus is not continuous at that point. So, nothing more is needed. Whether it is crucial to talk about continuity at points where the function is not defined, is another issue. My opinion is that the essential character of discontinuity is related to one of the two reasons: either $f(x)$ does not have a limit as x tends to c, or it has a limit, but the limit is not $f(c)$. This kind of approach may interfere with the clear fact that all the elementary functions are continuous in the domains.

12 students noted that $f(1)$ is of the form $\frac{0}{0}$ and thus f is not continuous at $x = 1$. (Not all the responses of this kind were so clear and concise.) One student concluded continuity at $x = 1$ from $\frac{0}{0}$. Four students introduced (unnecessarily) the concept of limit, one calculated $f(1) = 4$ and one calculated $f(1) = 0$. Two participants thought that the examination is done by differentiation. The number of blank answers was five.

Only two participants gained three points. The best understanding is shown by the following response: "*When into the function in question is inserted the variable value* $x = 1$, *the function does not follow the mathematical rule, where denominator is unequal to* 0. *So, the function is not defined at the point* $f(1)$, *when the function is not continuous*."

In general, the responses are fragmentary, but, as said, almost a half of the respondents succeeded in fulfilling the minimum requirement. Often the responses are unclear and this is especially true with the logical structure. Definite and explicit answers are missing − sole $\frac{0}{0}$ is not such.

5 Discussion

In this chapter I first discuss the answers to the research questions. Then other central themes at a more general level such as conflict factors, internal contradictions, internal references, validity and reliability etc. are dealt with. Also, the advantages and disadvantages of following the guidelines of the other study are reflected. Finally, a research frame for possible future studies is described.

First, the concept images in my study seem to be quite fragmentary. The concept images were classified in five exercises in all, thus producing 130 items. If they are listed by the rate of the occurrence, the erroneous ones were most common (49), followed by experimental (30), blank (27), analytical (15) and geometric concept images (9). It should be noted that not a single response in my study was classified as algebraic. There may be slight differences between me and the previous authors Tossavainen et al. (2013) in the criteria for classifying a concept image to be algebraic; this was discussed in 4.2. However, it was clearly seen that the respondents lacked the means, especially the algebraic means, to prove that in the set of the positive real numbers, the squaring preserves the order. The best responses in my data in this exercise (1) were only experimental.

After this account, my results comply with Tossavainen et al. (2013) on that the analytical and geometric aspects turned out to be dominant in most students' concept images of monotonicity. But it is true only provided that the experimental and erroneous concept images are not considered in this context in my study. Hence, the conclusion is that the hypothesis related to the research question (1) was too strongly formulated, and the numbers given in the previous paragraph are a part of the answer to (1). The numbers do not reveal a tendency to give verbal and descriptive answers, especially to exercise 3c, and there is the temptation to introduce a new concept image class for them, but such response would not represent a proper aspect to monotonicity (ibid., 1126). Moreover, the demarcation would be volatile, particularly if one and only one class is looked for per response.

The differences in the scores between the upper secondary and the tertiary students are presented in figures 1 and 2 (see page 17). The performance of the former in the monotonicity items and in all the items, measured by points, turned up to be approximately two thirds of the latter, the means being 4.23 vs. 6.10 and 7.69 vs. 12.42, respectively. Note that I have regarded the use of the t-test as appropriate only when equal variances are assumed. That was the case, by Levene's test, with exercises 3a, 3c, 3d and 5. Of these four exercises, only in 3c the t-test showed statistically significant differences between the groups. The tests themselves require quite a lot of evidence to reject the hypothesis of equality of means, which configuration one should be aware of. Nonetheless, two possible, non-obvious reasons can explain the differences to some extent. Firstly, I cannot guarantee the motivation of the students to be the same as at the tertiary level. Secondly, the participants were, at the time of the survey, participating the course MAA9 *Trigonometric functions and number sequences* but had not yet encountered the sequences. In practice this means that concerning task 2, they were underdogs. The same may be true, partly for similar reasons, with 3a and 3b, where the sequences can be considered to be present – to irrationality there is no reference in the National Core Curriculum for General Upper Secondary Schools 2003. This leads to answer the research question (3) that the performance of the upper secondary school students was less successful than that of the university students; this is seen from descriptive statistics. But the statistical significance is present in one exercise only.

I sketched some, possibly vague, criteria for conceptions of monotonicity to be pre-operational, operational or structural in connection with the theory of Sfard (1991). The responses do not give any reason to revoke them. Among the upper secondary school students, the performance and the conceptions seem to be at the operational level as its best. This appears mostly by use of the derivative of the function in a correct way to investigate monotonicity while mistakes in differentiating, other mistakes and uncertainty with terminology refer to the direction of pre-operational conceptions. As the variety of errors and misconceptions is so extensive, a few specific examples would not exhaust the set of deficiencies at all. Let the very frequent underlying conception 'monotonicity is a relevant term only with continuous functions' be presented as an example of a more general kind and this is often true among university students, too. As said before, in my way of thinking, to be a structural, a conception of monotonicity must include elements order preserving/reversing in some form. This idea may be included in the geometric concept image of monotonicity, but for a researcher, it is impossible to recognize it solely by graphs. In general, the students should be spurred to equip the graphs with sufficient reasons or comments. −It is not clear whether all the respondents in my study really understood the term monotonicity which is explicitly mentioned in the tasks only once. It is true that the word is seldom used in the mathematical textbooks for upper secondary students, compared to words increasing and decreasing. But only two students equipped the term with a question mark, and no one asked for its meaning during the test nor presented any other oral questions. As a summary of answering the research question (2), the clearest and the most common misconception is that conceptions about monotonic functions are restricted to continuous functions. It is concluded from the non-existence of antithetical examples.

Evoked concept image is one of the most central terms in Tall and Vinner (1981). Tossavainen has also dealt with it in many of his articles (e.g. Tossavainen et al. 2011, Viirman et al. 2010). Although the term is not explicitly mentioned in my research questions, it is implicitly present all the time. I have been observing the relation between the task imposition and the overriding aspects of the responses in concept images. Without denying any of the results of other authors I note that in my data, the conclusions on this are not straightforward. At quite a general level, I am inclined to say that when the respondents were mathematically unable to cope with the task (1), they ended up in the experimental approach. When the participants encountered the tasks given in a verbal way only (3c, 3d), it possibly led to a heavier cognitive load or embarrassment, which manifested itself in erroneous responses. In (5), which is a basic, 'neutral' exercise of monotonicity, the analytic concept image was evoked to almost the same proportion as among tertiary students. These remarks are not by any means to be generalized.

Tall and Vinner (1981) also presented the terms *potential conflict factor* and *cognitive conflict factor*. In the questionnaire used in this study, the best task for investigating the occurrence of conflict factors, is 3b (to comment $9.999 \leq 10$). As a starting point, it is practically impossible to say anything new about the issue, not already included in the article (ibid., 156-159). Unfortunately, the formulation of the task does not ease the work. Partly regardless of the formulation, as a result one can note that conflict factor(s) remain potential, not actual. Cognitive conflict factor could be evoked, e.g., by considering the corresponding geometric series $9 + 9.9 + 9.99 + \cdots$. The responses do not offer information for explanation of erroneous concept images, because they are so minimal. It seems to be the case that the respondents do not understand the difference between infinite and finite number of decimals, as demonstrated by "$9.999 \ldots + 0.00 \ldots 1 = 10$" at best.

Tall and Vinner (1981), Tossavainen et al. (2013, 1128) mention internal contradictions in the students' concept images. They note the presence of this phenomenon in many responses and explain some mistakes being due to internal contradictions. Viholainen (2008) lists internal contradictions and contradictions with the formal axiomatic system of mathematics as different properties of a concept image. Not disputing the existence of internal contradictions, I would rather say that a large majority of mistakes in responses, concerning secondary students, is due to contradiction with theory of real analysis (or the mistakes due to immaturity) and the material does not provide evidence to reveal internal contradictions, cf. the paragraph right above.

Even though the exercises 1, 3c and 5 include the common characteristic, squaring the variable, there were no internal references at all, even though the students may have benefited from observations on the two other tasks. On the other hand, such references were rare among tertiary students, too.

It is worth discussing the advantages and disadvantages of following the guidelines of the study by Tossavainen et al. (2013). It should be emphasized that from the very beginning, when I was suggested the subject, my attitude towards it was and has been positive and enthusiastic. After that, at some point, a decision had to be made whether to use the same questionnaire as such. As a result of two advanced courses on methodology during the academic year 2015-2016 the decision making eased. (One was about quantitative research methods through SPSS, the other dealt with philosophy of science, especially education.) I ended up using the same questionnaire for the following reasons: comparison to tertiary students should be made, however so that the focus would be in my target group, upper secondary school students. Needless to say, it was most sensible to rely on the study published in *International Journal of Mathematical Education and Science*. Besides my choice enabled deeper intercourse and better support from scientific community. During any research process, critical thoughts are generally due to rise, my study not being an exception. Not until the survey was carried out, did I meet with more of those thoughts. Of course, I have been fully aware of the fact that reiterating any ideas of other authors does not increase the own contribution. About the questionnaire, some remarks are made. To put it more precisely, does the formulation of the test questions contain a few unessential/misleading elements, either deliberate or accidental? Instead of the formulations used I would have excluded logical operators 'and' (3a) and 'or' (3b) in the test, and presented 3a with two separated clauses and 3b with either 'less than' or 'equal to'. In 4, concerning continuity, a task where the function is defined at the point in question would, in my view, have been preferable. The incorrect range in 5 is disputable. −Now, with hindsight, I would have liked to ask the students directly, but implicitly, about concept images of monotonicity, for example: "How do you understand the term monotonicity?". Such an action would have enabled this research to give a distinct, independent description on the participants' concept images of monotonicity. On the grounds of the cited question together with "Draw a mind map about the term monotonicity" or of the map alone, in my view, the degree of structuralization could be estimated. −On the other hand, such inquiries could have affected the responses to other exercises.

In my view, it is clear that the four items (1, 3c, 3d, 5) in the questionnaire used, measure the intended knowledge on monotonicity – in each case, this was argued separately in the Results section. The other four items deal with other concepts of calculus of one real variable. Naturally, their subject matter could have been chosen differently. This does not change the fact that the chosen ones (2, 3a, 3b, 4) are valid indicators as well, in spite of the comments on 3a, 3b, 4, 5 above. Not directly related to the validity, in all fairness, it must be said that the participation was not necessarily voluntary or the students' motivation the best possible due to the character of the teacher training school (surveys are frequent). Concerning the reliability, I have been working mostly alone and not participating in teamwork to such an extent as the previous authors, however recalling the intercourse between me and Mr Haukkanen and Mr Tossavainen, presented in the Methodology section, let the reader decide whether he/she agrees on my view of sufficient reliability for the purpose.

About the fact, mentioned in the Methodology section, that when the ordered concept image category variable correlates very significantly with the participants' performance in calculus at a more general level (Tossavainen & el., 2013), I noted that the result is evidently true among upper secondary school students and had no cause to repeat the same procedures. But the need to explain the result further remains. If a participant ends up (consciously or subconsciously) in a certain category variable in the indicative exercise, it is likely that he makes the same choice in other parts of the test. Consequently, it will be difficult to exclude the effect of the fact that not all approaches (concept image categories) can produce full three points in all the exercises. So, let the following research frame be presented for possible future use.

The research frame is as follows: There is one indicative test question with a neutral imposition (or possibly two, if the consistency is wanted to be assured, cf. above). Then there are eight other exercises, of which in exactly two, the participants are ordered to make use of algebra, calculus, geometry and the natural language, respectively, to the greatest possible extent. These responses are assessed and the final concept image classes in the indicative task(s) determined. The differences between the classes in the total scores are analysed by performing a oneway analysis of variance.

6 Conclusions

Summary

The thesis investigates concept images of monotonicity of Finnish upper secondary school students and has the originating idea of having an extensive reference material from two Finnish universities available on which basis the article by Tossavainen, Haukkanen and Pesonen (2013) was published. My survey, being an essential part of the thesis, was carried out in Tampere in April 2016 by using the same questionnaire as in Tossavainen et al. (2013). For the sake of comparability, also the similar guidelines in scoring and categorization of concept images were principally followed in processing the responses of 26 students. The quantitative analysis of the data collected consists of descriptive statistics and Student's t-test and it was done by using SPSS.

It was found out that overall, there were so many faulty responses that erroneous concept images were most common, followed by experimental approaches, while not a single response was considered as algebraic. Among tertiary students there were more of them, but evoked mainly and superficially by imposition. There were also differences between secondary and tertiary students in the scores achieved, the formers' scores being approximately two thirds of the latters' in the monotonicity and the total items. Nevertheless, in this context the differences are not surprising and most of them are not significant.

Implications

It is known that the definition of monotonicity (see p. 1) is not explicitly mentioned in the advanced mathematics curriculum for the upper secondary schools and the term itself is mentioned there only once. That is unfortunate and may even be considered as a deficiency, for it is a part of general knowledge of mathematics. However, nothing restrains an individual teacher from presenting the definition. Whether this would temporarily be better done by first giving the definition (and preliminaries) and then a versatile set of examples and non-examples of monotone functions in versatile ways, or the other way around, I do not comment, as there are various approaches in the literature. But personally, I would prefer to first present examples of monotonicity from discrete mathematics and consequently not solely as an application of the derivative. So the students would come to understand that monotonicity is a non-analytic concept by the definition. Emphasizing the importance of the definition of monotonicity does not mean any underestimation of the significance of its use in calculus.

Limitations

The sample of my survey, consisting of 26 upper secondary school students, means that with generalizations one must be cautious. However, it became clear that no algebraic concept images occurred in the responses which most probably is the case among the upper secondary schools in the whole country – no indication showing the respondents differed from the respective average population exists. The sample size reduces the chances for getting statistically significant results in general. Still, the comparison between the data in this study and the one of 89 Finnish university mathematics students makes sense. But again, a larger sample size would possibly provide more differences between secondary and tertiary mathematics students and these results would have better statistical significance and generalizability.

Possible future research

The intention of this study was partly to reveal differences between upper secondary school and university mathematics students. Despite the limitations and any criticism the study may deserve, a further research on the very same topic, viz. finding out differences between the groups mentioned above, would not be of great interest and significance. What could then be done in the future associated with different aspects of monotonicity (or any other concept)? Accepting the results of Tossavainen et al. (2013, 1125), the question arose concerning the mechanisms lying behind the observed differences in measured level of performance in the four monotonicity items/all the eight items between students with different category variables. One possible means of approaching the issue was sketched out in the Discussion chapter.

List of references:

Bardelle C. & Ferrari PL. (2011) Definitions and examples in elementary calculus: the case of monotonicity of functions. *ZDM Mathematics Education.* 43:233–246.

Bingolbali E. & Monaghan J. (2008). Concept image revisited. *Educational Studies in Mathematics.* 68(1):19–35.

Edwards, B., & Ward, M. (2008). The Role of Mathematical Definitions in Mathematics and in Undergraduate Mathematics Courses. In Carlson M. & Rasmussen C. (Eds.), *Making the Connection: Research and Teaching in Undergraduate Mathematics Education* (pp. 223-232). Mathematical Association of America.

Krippendorff K. (2004). *Contents analysis – an introduction to its methodology*. London: Sage Publications.

Opetushallitus (2003). *Lukion opetussuunnitelman perusteet 2003*. Vammala: Vammalan kirjapaino Oy.

Opetushallitus (2015). *Lukion opetussuunnitelman perusteet 2015*. Helsinki: Next Print Oy.

Rasslan S. & Vinner S. (1998). Images and definitions for the concept of increasing/decreasing function. In Olivier A. & Newstead K., (Eds.). *Proceedings of the 22nd conference of the international group for the psychology of mathematics education* (Vol. 4); 1998 Jul 12–17; Stellenbosch: PME; pp. 33–40.

Schwarzenberger R. & Tall D. (1978). Conflict in the learning of real numbers and limits. *Mathematics Teaching*. 82: 44–9.

Sfard A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*. 22:1–36.

Tall D. & Vinner S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*. 12(2):151–169.

Tossavainen T., Attorps I. & Väisänen P. (2011). On mathematics students' understanding of the equation concept. *Far East Journal of Mathematical Education.* 6(2):127–147.

Tossavainen T. & Haukkanen P. (2012). Matematiikan opiskelijoiden käsitekuvia monotonisuudesta. Arkhimedes. 4:24–31.

Tossavainen T., Haukkanen P. & Pesonen M. (2013) Different aspects of the monotonicity of a function. *International Journal of Mathematical Education in Science and Technology*. 44(8): 1117–1130.

Viirman O., Attorps I. & Tossavainen T. (2010). Different views – some Swedish mathematics students' concept images of the function concept. *Nordic Studies in Mathematics Education*. 15(4):5–24.

Viholainen A. (2008). Incoherence of a concept image and erroneous conclusions in the case of differentiability. *The Montana Mathematics Enthusiast*. 5(2-3), 231-248.

Vinner, S. & Hershkowitz, R. (1980). Concept images and common cognitive paths in the development of some simple geometrical concepts. In Karplus R. (Ed.), *Proceedings of the fourth international conference for the psychology of mathematics education* (pp. 177-184). Berkeley: University of California, Lawrence Hall of Science.

Yli-Luoma P. (1995). *Meta-analytisk syntes av taluppfattningar hos småbarn* (Helsingin yliopiston kasvatustieteen laitoksen tutkimuksia 143). Helsinki: IMDL Oy.

Appendix 1 (questionnaire)

1. Show that for all real numbers holds $0 < x < y \rightarrow 0 < x^2 < y^2$.

2. Examine the sequence $\left(\frac{7n-2}{3n}\right)$ when $n \rightarrow \infty$.

3. Comment the truthiness of the following statements and justify.

a) The number π is irrational and its value is 3.141592654.

b) The number 9.999… is less than or equal to the number 10.

c) A function which is defined on the interval $[1, 89]$ and whose value is got by squaring the variable and multiplying this by the number $\frac{1}{107}$ and after that subtracting the number 1987, has the property: the greater the variable's values the greater the values of the function.

d) The graph of a strictly decreasing function is a descending line or other descending curve.

4. Examine the continuity of the function $f: \mathbf{R} \rightarrow \mathbf{R}, f(x) = \frac{x^4-1}{x-1}$ at $x = 1$.

5. Examine the monotonicity of the function $f: (0, \infty) \rightarrow (0, \infty),\ f(x) = x^2 - 5$.

Appendix 2 (the scoring and categorization of concept images)

	Task1	Task1 type	Task 2	Task3a	Task3b	Task3c	Task3c type	Task3d	Task3d type	Task4	Task5	Task5 type	Monotonicity points	Total points	Task2 type
1	1	ERR	0	2	0	3	ANA	0	ERR	0	3	ANA	7	9	BLA
2	2	EXP	1	3	0	2	ERR	2	ANA	1	3	ANA	9	14	ANA
3	2	EXP	0	2	0	1	ERR	1	ERR	1	1	EXP	5	8	BLA
4	0	ERR	0	0	0	1	BLA	1	ERR	1	0	BLA	2	3	BLA
5	0	BLA	0	2	0	1	ERR	1	ERR	0	0	BLA	2	4	ERR
6	1	EXP	1	2	0	0	ERR	1	GEO	1	1	ANA	3	7	EXP
7	1	ERR	0	0	0	0	ERR	2	ERR	0	1	ANA	4	4	ERR
8	0	BLA	0	2	0	2	ERR	2	ANA	1	0	BLA	4	7	BLA
9	1	EXP	0	1	0	0	BLA	0	ERR	1	0	BLA	1	3	BLA
10	1	ERR	2	1	0	0	BLA	0	ERR	0	0	BLA	1	4	ERR
11	0	BLA	2	1	0	0	ERR	1	ERR	2	1	GEO	2	7	EXP
12	2	EXP	0	2	0	1	EXP	1	ERR	2	2	EXP	6	10	ERR
13	2	EXP	0	2	0	0	EXP	0	ERR	2	3	ANA	5	9	ERR
14	2	EXP	0	2	0	1	ERR	1	GEO	1	0	BLA	4	7	ERR
15	0	ERR	0	3	0	0	EXP	1	ERR	0	0	BLA	1	4	ERR
16	0	BLA	0	2	0	0	BLA	2	ERR	2	0	BLA	2	6	BLA
17	2	EXP	2	3	0	2	ERR	0	ERR	1	2	GEO	6	12	EXP
18	2	EXP	2	3	1	1	ERR	2	ANA	1	3	ANA	8	15	EXP
19	0	ERR	0	3	0	1	ERR	1	ERR	1	3	ANA	5	9	BLA
20	0	BLA	1	2	0	1	EXP	0	ERR	0	1	ERR	2	5	EXP
21	2	EXP	2	1	1	1	EXP	1	GEO	3	0	ERR	4	11	EXP
22	1	EXP	2	0	0	2	EXP	2	ERR	1	2	GEO	7	10	EXP
23	0	ERR	0	0	1	0	ERR	0	GEO	1	2	ANA	2	4	ERR
24	0	BLA	1	0	1	2	EXP	2	GEO	1	1	EXP	5	8	EXP
25	2	EXP	0	2	0	2	ERR	2	GEO	3	2	ANA	8	13	ERR
26	1	ERR	0	2	0	2	ERR	2	ANA	0	0	BLA	5	7	BLA

Appendix 3 (Levene's test for equality of variables, t-test for equality of means)

Independent Samples Test

		Levene's Test for Equality of Variances		t-test for Equality of Means						
									95% Confidence Interval of the Difference	
		F	Sig.	t	df	Sig. (2-tailed)	Mean Difference	Std. Error Difference	Lower	Upper
Task1	Equal variances assumed	20,382	,000	2,808	113	,006	,769	,274	,226	1,311
	Equal variances not assumed			3,490	61,476	,001	,769	,220	,328	1,209
Task2	Equal variances assumed	24,742	,000	3,943	113	,000	1,104	,280	,549	1,658
	Equal variances not assumed			5,019	65,055	,000	1,104	,220	,665	1,543
Task3a	Equal variances assumed	,895	,346	1,466	113	,146	,369	,252	-,130	,867
	Equal variances not assumed			1,573	45,631	,123	,369	,234	-,103	,840
Task3b	Equal variances assumed	20,712	,000	5,162	113	,000	1,015	,197	,625	1,404
	Equal variances not assumed			8,024	106,282	,000	1,015	,126	,764	1,265
Task3c	Equal variances assumed	,102	,750	4,266	113	,000	,854	,200	,457	1,250
	Equal variances not assumed			4,278	40,917	,000	,854	,200	,451	1,257
Task3d	Equal variances assumed	3,011	,085	,520	113	,604	,103	,198	-,289	,495
	Equal variances not assumed			,560	45,903	,578	,103	,184	-,267	,472
Task4	Equal variances assumed	8,989	,003	1,672	113	,097	,389	,232	-,072	,849
	Equal variances not assumed			1,886	49,941	,065	,389	,206	-,025	,802
Task5	Equal variances assumed	1,885	,172	,518	113	,606	,145	,280	-,409	,699
	Equal variances not assumed			,544	44,077	,589	,145	,266	-,391	,681
Monotonicity_points	Equal variances assumed	5,986	,016	2,733	113	,007	1,87035	,68429	,51464	3,22606
	Equal variances not assumed			3,249	55,557	,002	1,87035	,57576	,71677	3,02394
Total_points	Equal variances assumed	12,565	,001	3,970	113	,000	4,72342	1,18987	2,36608	7,08077
	Equal variances not assumed			5,209	70,186	,000	4,72342	,90682	2,91491	6,53193