UNIVERSITETET I AGDER

Students’ perception of infinity

Perception, obstacles, conception

SOKRATIS THEODORIDIS

SUPERVISOR

John D. Monaghan

University of Agder, 2017

Faculty of Engineering and Science

Department of Mathematical Sciences

Summary

Several studies in mathematics education have shown that students face significant difficulties for understanding the concept of infinity. The purpose of this study is to suggest an explanation of how students understand the concept of infinity and identify the difficulties for understanding it. Explanation will be given through an investigation on the following aspects of understanding and the implementation of the corresponding theoretical background: i) primary perception in terms of Singer and Voica's(2008) categories and secondary perception in the context of comparison of infinite sets ii) obstacles in coming to understand the concept in terms of the Theory of Epistemological Obstacles and iii) the construction of understanding in terms of APOS theory. The empirical part of this study was conducted in Norway. The subjects are five 12th graders and data for this study were collected by means of a questionnaire and an interview. A precise overview of the aim and contents of the thesis is given in the introductory part. The thesis begins with a historical analysis of the concept of infinity, aiming to reveal its paradoxical and contradictory nature. Next, the theoretical background is presented along with the basis of each theory. There has been a qualitative analysis on the collected data, by methods found in the literature. The results have shown that most of the students perceived infinity as a process while specific difficulties related to the notion of actual infinity were noticed. The analysis of the structure of understanding, raised the importance of "encapsulation" as a way to understand its actual form. The thesis hopes to shed light on students' understanding and thus make a small contribution to mathematics education research.

Sammendrag

Flere tidligere studier matematikkdidaktikk har vist at elever har store vanskeligheter når det kommer til å forstå konseptet uendelighet. Hensikten med denne studien er å foreslå en forklaring på hvordan elever forstår konseptet uendelighet og å identifisere vanskeligheter knyttet til forståelsen. Forklaringen vil bli gitt gjennom en undersøkelse av følgende aspekter for forståelse og implementeringen av tilhørende teoretisk rammeverk: i) primær persepsjon etter Singer og Voica (2008) sine kategorier og sekundær persepsjon i konteksten av å sammenlikne uendelige mengder, ii) hindringer i å nå en forståelse av konseptet ved bruk av teorien om epistemologiske hindringer, og iii) konstruksjonen av forståelse ved bruk av APOS teorien. Innsamling av empiri til denne oppgaven ble gjort i Norge og informantene er fem VG2 elever. Data for studien er samlet inn ved hjelp av spørreskjema og intervju. Et detaljert overblikk av mål og innhold i studien blir gitt i innledningen. Studien begynner med å ta for seg en historisk analyse av konseptet uendelighet, med mål om å avdekke dets paradoksale og motstridende natur. Deretter blir det teoretiske rammeverket presentert sammen med grunnlaget til hver teori. Det har blitt utført en kvalitativ analyse på innsamlet data ved hjelp av metoder fra funnet i litteraturen. Resultatene har vist at de fleste elevene oppfattet uendelighet som en prosess, samtidig som spesifikke vanskeligheter knyttet til forestillingen av faktisk uendelighet ble oppdaget. Analysen av strukturen av forståelse viser viktigheten av «innkapsling» som en måte å forstå forståelses faktiske form. Studien håper å kaste lys på elevenes forståelse og dermed gjøre et lite bidrag til matematikkdidaktisk

Preface

Before you lies the thesis “Students’ understanding of infinity: perception, obstacles, conception”. It has been submitted in partial fulfillment of the requirements of the double degree of the Joint Nordic Master’s Programme in Didactics of Mathematics – NORDIMA. I was engaged in researching and writing this thesis from December 2016 to May 2017.

The basis for this research stemmed from my passion in mathematical counterintuitive facts and problems that push the mind to new areas and challenge the cognitive versatility. One of the hardest mathematical concepts responsible for intellectual struggle and controversies, is infinity. As a future mathematics teacher, the passion has turned into curiosity about infinity in a student’s mind. How do students understand infinity? I wrote this thesis in an attempt to find out.

The project was undertaken at the request of the Faculty of Engineering and Science, University of Agder. The research was not easy, but conducting extensive investigation has allowed me to formulate the research questions and to reflect on appropriate theory to answer them.

The contribution of my advisor, John Monaghan, had been critical though all the process. I would like to thank him for his guidance and support during the academic year 2016-17. I would also like to thank professor Reinhard Siegmund-Schultze for his inspiration on the topic of the thesis as well as his comments on the historical part.

I also wish to thank all the participants of the study, both students and teachers. A big thanks to my colleagues at UiA who contributed with their educational experience and their native Norwegian speaking skills. Last but not least, a note to friends and family: Thank you for being rational, thank you for being there.

I wish you a pleasant reading.

Sokratis Theodoridis

Kristiansand, May 2017

Table of Contents

CHAPTER 1: Introduction

Infinity is a topic that could be both interesting and problematic for students. It had always been an intriguing topic for mathematicians, philosophers and other scientists through history. The great philosopher Aristotle, distinguished between two notions of infinity, the potential and the actual infinity. The former could be understood as the infinite presented over time, while the latter is the infinite present at a moment in time, which is incomprehensible, because the underlying process of such an actuality would require the whole of time. Aristotle's *dichotomy* is maintained through the literature and is widely treated in mathematics education as a process and as an object for potential and actual infinity respectively(Tirosh,1999). This duality results to a challenge when it comes to introducing infinity in mathematics education and specifically in mathematics students.

Fischbein, Tirosh and Hesh (1979) explored the psychological background of the concept of infinity. Fischbein et al. (1979) distinguish between infinity as an unquestionable mathematical structure and infinity as a pure construct where it is impossible to decide for its acceptance or rejection. They refer to these aspects as the mathematical and the psychological reality of the concept respectively. They investigated the way that intuition works at geometrical tasks related to mathematical infinity. The study was conducted on 6^{th} to 8^{th} grade students and pointed out to the contradictory nature of infinity as an effect to intuition. The results showed that most of the contradictions were raised due to the students' thinking in terms of finite objects and events. Monaghan (2001), examines the young people's ideas and points out two problems when talking to children about infinity. The first, in line with Fischbein's conclusion, is that there are no referents for discourse on the infinite in the real world. The second lies in the use of language when talking to children about infinity. For example, when a teacher enters a classroom, he/she will make use of a language that has been built in a finite world. Tirosh and Tsamir (1996) also studied the students' intuitive thinking about infinity by investigating on the role of representations. The research was conducted on 10^{th} to 12^{th} graders and the students were given tasks that included different representations of infinite sets. Specifically, they investigated the effects of the following representations of infinite sets: numerical-horizontal, numerical-vertical, numerical-explicit or geometric. The results showed that different representationσ had indeed an impact on students' reasoning.

Singer and Voica (2008) analyze the relationship between perception and intuition for the concept of infinity. The authors distinguish between the primary perception, which is the process by which human beings interpret sensory information and the secondary perception, which is the processing of sensory information based on previous experience. The research was conducted on students in elementary, secondary and undergraduate education. They identified through their results three categories of primary perceptions: processional, topological and spiritual. The results also showed that students made use of these perceptions in their reasoning that leads to the formation of a secondary perception. Tirosh and Tsamir (1999) investigated the students' secondary perception, specifically through the comparison of infinite sets. They mention that several studies have indicated four criteria that students use to determine whether a given pair of infinite sets are equivalent, such as "all infinite sets have equal number of elements", "comparison of infinite sets is impossible", "a subset contains fewer elements than the whole set" and "there is one-to-one correspondence between the elements of the two sets". The use of more than one criteria lead the the students to contradictions during the comparison. However, the difficulties in understanding do not only lie in the perception of the concept. There are difficulties that rise due to students' previous knowledge and the nature of the mathematical topic itself.

These difficulties are named by Brousseau (1997) as the Epistemological Obstacles. The theory of epistemological obstacles is a theory of student thinking, which has its roots in the work of Bachelard (1938) and was integrated in mathematics education and the research of Guy Brousseau (1997). Cornu (1991) has given some examples of epistemological obstacles for the concept of limit, such as the failure to link geometry with numbers or the obstacle of that lies in the metaphysical aspect of the notion of limit. Sierpinska (1987) investigated the notion of epistemological obstacle with humanities students and their understanding of the limit concept. She further categorizes epistemological obstacles into heuristic and rigouristic by presenting a diagram which illustrates these obstacles. Some of the sources of epistemological obstacles that she identified related to limits is infinity, function and real number. Herscovics (1989), attributes the term "epistemological" for obstacles found in the historical conceptual development and uses the term "cognitive" for obstacles found in an individual's conceptual development. Moru (2007) reviews the previous works on and refers to the investigation of epistemological obstacles as complementary to mathematical teaching. In his PhD thesis, Moru uses the theory of Epistemological Obstacles to identify obstacles during the development of understanding of the limit concept of undergraduate students. For the case of infinity, Cihlar, Eisenmann and Kratka (2009) have obtained a set of obstacles to understanding infinity and refer to them as "uncoverable" (or unavoidable in Brousseau's terms). These include the previous knowledge of "finiteness", considering infinity as a process and the previous knowledge concerning the ordering of a set of natural numbers. The theory of Epistemological obstacles is based on Piaget's theory of developmental stages.

Piaget and Garcia (1968/1989) developed a constructivist approach to knowledge. Their central belief is that knowledge is determined by stages. Brousseau (1997) adapted this view on knowledge and defined an obstacle as a set of knowledge grounded in the knowledge structure of an individual that can be successfully used in one context or situation but in another this set gives wrong results.

In the context of Piaget's theory, Sfard (1991) calls on the dual nature of mathematical concepts and develops the notion of reification. Reification is the transition from a process conception stage to the object conception stage. For example, potential infinity corresponds to a process conception (thinking of "again and again"). Thus, this process conception can be developed to an object conception through reification. That is, to see infinity as a fully-fledged object, corresponding to the notion of actual infinity. For Sfard (1991), reification expands cognitive capacity when it comes to mathematics. In a similar manner, Gray and Tall (1994) developed the theory of procept. Infinity may be considered as a process and as a concept (in total) as well. Lakoff and Núñez(2000) have introduced the so-called Basic Metaphor of Infinity (BMI), which arises when one conceptualizes actual infinity as the result of an iterative process. Cotrill et al. (1996) have expanded the idea of processes and objects to include actions and schemas and develop the theory called APOS. Actions can be seen as transformations on an object to obtain other objects while schemas are coherent collections of actions, processes, objects (and even collection of other schemas). Dubinsky (1990), claims that in order to understand a mathematical object on must first understand its function in process. Only through "encapsulating" an idea within its process do we then come to know its product(object). Thus, the fundamental relationship: dual nature of the mathematical concept-development in the mind of the individual is inherent in the above theories.

The concept of infinity is a key concept in mathematics and its teaching. The fact that it is not a topic directly addressed in mathematics classrooms, provoking students to think about infinity offers a good opportunity to introspect their understanding and specifically their perception as it is basically a first-time interaction with tasks related to infinity. Furthermore, as Sierpinska (1997) stresses out, “overcoming an epistemological obstacle” and “understanding” can be two ways for speaking about the same thing, thus these obstacles play a critical role in the formation of knowledge. Sierpinska (1990) also mentions: *“A description of the acts of understanding a mathematical concept would thus contain a list of the epistemological obstacles related to the concept, providing us with information about its meaning” (*p.27). Hence, the investigation of the contradictory nature of infinity as it appears through mental structures and mechanisms or in other words through “acts of understanding”, attributes at an explanation of understanding.

Therefore, this thesis sets to answer the following questions for the research conducted on the upper secondary students:

- How did students perceive infinity?
- Which epistemological obstacles did students encountered in coming to understand the concept of infinity?
- How might an understanding of the concept of infinity be constructed by students?

The above questions will be answered with the assumption that knowledge can be acquired after the individuals’ thinking passes through the Piagetian stages, as they appear in APOS theory. A second assumption related to the epistemological obstacles needs to be made. That is the assumption that some of the obstacles found in the historical development of the concept of infinity can also be identified in the individual’s cognitive development which implies an acceptance of the “genetic principle”. (more information on the “genetic principle” will be given in Chapter 3).

Finally, by answering the questions on perception, obstacles and the stages of conception, I aim to suggest an explanation on how students understand the concept of infinity.

Consequently, an epistemological and cognitive approach to the understanding of the concept of infinity could reveal intriguing results that could indicate an improved design for pedagogy.

In the chapters that follow, there will be a historical account of mathematical infinity (Chapter 2). In Chapter 3, the theoretical background used to answer the research questions is presented. Chapter 4 includes detailed information on the methodology that have been followed for the research design. The results of the study conducted on the participating students are presented in Chapter 5 and are discussed in Chapter 6. Finally, my conclusions are included in the last Chapter 7.

The participants of the study are 5 upper secondary school students. The data collection was done in two sessions, a questionnaire session and an interview session. The results were transcribed and analyzed in terms of the theoretical framework. After the analysis, there has been an attempt to answer the research questions by reviewing the results and finally, draw some significant conclusions.

[This page is intentionally left blank]

We start our study by trying to look at the nature of infinity, as it appears in the human mind from the very beginning of its conception in mythology up to the theory of cardinal infinity. We will see the conceptual evolution of infinity, starting from the mythological doctrines to the philosophical mindset and the scientific attitude of ancient Greeks, moving on to the inertia of the Middle Ages and the Scientific Evolution and the paradoxes of the 17^{th} Century, to finally finish our time travel with Cantor's theory of infinity. The purpose is to have an insight on the nature of infinity, notice the difficulties in the conceptual development and observe the way through which several debates over the notion lead to great mathematical discoveries.

CHAPTER 2: From mythology to Cantor

The concept of infinity had always been a challenging notion to the intellectual and cultural history of the human kind. Its contradictory to the physical world nature, lead to its conceptualization through mental constructions occurring in mythology, theology, philosophy and mathematics. In the following section, we will follow the historical development of the concept, emphasizing on the mathematical infinity. Not only because this is a thesis on mathematics education but *"…above all, infinity is the mathematician's realm, for it is in mathematics that the concept has its deepest roots, where it has been shaped and reshaped innumerable times, and where it finally celebrated its greatest triumph"* (Maor,1991, p.2).

2.1 Mythology

In order to realize the earliest conceptions of infinity, we have to look back at the Hindu and Iranian mythology and the myths that are due to the Aryan tribes that invaded the Indus Valley around 1500-1200 BC. According to the myth, the demons Danavs battled the celestial gods Adityas, representing a battle between the constrained and the unbounded. The ancient Iranian creation story tell of a god named Ormazd who created light and heaven in the form of an egg whose center was the Earth and top reached the Infinite World (Vilenkin,1995, p.2-3). According to the Egyptian myth, the primordial God Heh personified infinity and eternity. Moreover, the figure of Heh with the arms upraised supporting the sky, represented the number one million in the hieroglyphic system. Such attempts for the manifestation of the notion of infinity by large numbers were also made by the ancient Hindus who told stories of battles that involved 10^{23} monkeys (Vilenkin, 1995, p.5). Infinite is also disclosed in the idea of "eternal recurrence" in the Hindu and Buddhist tradition, where the human souls reincarnate an infinite number of times through infinite space and time. While in the ancient Jewish cosmology and mystical tradition known as Kabballah, the world consists of ten spheres which emanate from the Infinite Light. In many ancient traditions, there is the appearance of ouroboros, a serpent or dragon eating its own tail representing the recreation of life through death by the universe. A similar appearance occurs in the Norse mythology with Jörmungandr or the World Serpent, a serpent so enormous that grasps his own tail and forms a circle around Midgard, the visible world.

As mythology tried to answer the most difficult and basic questions of the surrounding world, the same was the case for the aspect of ancient civilizations on the infinite. It seems that the notion was commonly related with cosmogony, unboundedness, recurrence and ideas of very large numbers. However, in the 6^{th} century BCE[1], a need to substitute mythological with rational explanations

[1] Before common era

about the surrounding world started to arise, explanations that would lead to the birth of philosophy and science in Greece.

2.2 Philosophy and science

Philosophy and science firstly arose in Greece around the 6th century BCE. We need though to point out that at the time, philosophy and science were closely related, in the sense that they were pursued by the same people (Vilenkin,1995, p.4). Villarmea (2001) expresses the status of science and philosophy in antiquity, by saying that: "*The first scientists were philosophers and…the first philosophers were scientists*" (p.5).

Chronologically, the first to introduce a new way of thinking for the natural world and the place of human beings in it were the pre-Socratic philosophers. Placed between the 6th and 5th century BCE, they challenged the traditional ways of thinking and paved the road for rationality and argumentation.

The Milesian School

According to Herodotus, the very first of the pre-Socratics philosophers, *Thales of Miletus* (c.620BCE-546BCE) being a mathematician and astronomer, predicted the eclipse of 585BCE marking the beginning of scientific methodology.

However, the first to introduce the notion of the *ἄπειρον* or *apeiron* (that which has no limit) in history of philosophy was Thales' student and master successor of the Milesian school, *Anaximander of Miletus* (c.609BCE-546BCE). In his philosophical poem *"Περί Φύσεως"* (On Nature) he said that the material cause and first element of things was the Infinite (Rioux,2004, p.14). Anaximander characterizes the limitless as the beginning of the world and the beginning of everything. He defines it as a substance self-defined, unchangeable, unable to die and indestructible, from which all things derive. Anaximander believed in an existence of an infinite amount of worlds in our universe, a remarkably similar approach to MWI (Many Worlds Interpretation) of quantum theory. It seems that Anaximander used the word apeiron for a more abstract meaning similar to that of the mathematical infinity (Theodosiou ,2010, p.165).

Pythagoras, Pythagoreans and the "irrational"

While Anaximander had an optimistic and fearless view on the "boundless", on the other hand *Pythagoras*(c.580BCE-c.500BC) thought of infinity as something evil. In the Pythagorean table of opposites, infinity is associated with the bad and finitude with the good. The Greek philosopher, mathematician and astronomer founded his school or his society in Croton of southern Italy. It was a society governed by a set of rules and secrecy (e.g. restrictions in a matter of diet were required while the members recognized one another by means of secret signs). Information about the Pythagoreans as well as for Pythagoras are excerpted out of quotes and fragments by subsequent authors. Thus, it is impossible to prove what is credited to the Pythagoreans or Pythagoras himself.

The Pythagorean School was a society whose members believed that the universe could be understood in terms of whole numbers or ratio of whole numbers and worshipped them as such. One is created out of two elements, the even and the odd, that is, for the Pythagoreans, the limited and the infinite respectively. Then One overflowed into the Two, then the Three, then the Four. They considered Ten as the perfect number or the holiest of all, a number composed by the sum of One, Two, Three and Four (Pesic,2003, p.10). Briefly, the order of creation is as follows: first, the

One is created out of the *apeiron* and the limited/definite[2], then numbers out of the One, and then the world out of numbers (Drozdek,2008, p.22). They also related music to the numbers (and thus believed that the universe is music) by observing lengths of chords and tones produced after shortening them in relation of ratios involving whole numbers up to four. Hence, one can easily understand the made up[3] slogan attributed to Pythagoras and his school: *"Πάντα κατ' αριθμόν γίγνονται"* (everything happens according to numbers).

But what shook the foundations of the Pythagorean Brotherhood and disturbed their belief that "All is number" was their own discovery of geometrical magnitudes that cannot be expressed by ratios of whole numbers (known today as the irrational numbers). The real identity and work of the discoverer is lost or remained a well-kept secret by the Pythagoreans, as something that would tear down their whole numerological philosophy and change Greek mathematics forever. In particular, what was in doubt was the idea of *commensurability*, that is: for any two magnitudes, one should always be able to find a fundamental unit that fits some whole number of times into each of them.

To have a sense of the discovery let us imagine a square with side length $\alpha \in \mathbb{N}$ and its diagonal of length $\beta \in \mathbb{N}$. Then the lengths α, β are incommensurable.

Proof: Let these lengths be *commensurable* (as Pythagoreans believed), that is there is a fundamental unit $\mu \in \mathbb{N}$ such that: $\alpha = \mu\kappa$ and $\beta = \mu\lambda$ for some $\kappa, \lambda \in \mathbb{N}$.

Now assume without loss of generality that:

either $\alpha \in A$ or (but not both) $\beta \in A$, *where* $A = \{n \in \mathbb{Z} | (\exists r \in \mathbb{Z})[n = 2r]\}$

Applying Pythagoras' theorem[4] in one of the rectangular isosceles triangles composing the square, one has: $\alpha^2 + \alpha^2 = \beta^2 \Rightarrow 2\alpha^2 = \beta^2$

Substituting for α, β we get: $2(\mu\kappa)^2 = (\mu\lambda)^2 \Rightarrow 2\mu^2\kappa^2 = \mu^2\lambda^2$

We know that[5] $\mu \neq 0$, hence we can simplify and get $2\kappa^2 = \lambda^2$ which means that λ^2 is even, thus λ is even *(if λ was odd, i.e. $\lambda = 2m + 1, m \in \mathbb{Z}$,then we get $\lambda^2 = 2(4m + 2) + 1$ meaning that λ^2 is odd, hence λ is even)*

Since λ is even, it can be expressed as follows: $\lambda = 2w, w \in \mathbb{Z}$.

Substituting the latter expression in $2\kappa^2 = \lambda^2$ we get: $2\kappa^2 = (2w)^2 \Rightarrow 2\kappa^2 = 4w^2 \Rightarrow \kappa^2 = 2w$, meaning that $\kappa^2$is even, which means (reasoning as before) that κ must be also even.

Finally, we get that: $\alpha = 2\mu t$ and $\beta = 2\mu w$ meaning that:

both α and β belong to the set $A = \{n \in \mathbb{Z} | (\exists r \in \mathbb{Z})[n = 2r]\}$

which contradicts our initial assumption that either $\alpha \in A$ or (but not both) $\beta \in A$

We are then lead to conclude that there is no common unit $\mu \in \mathbb{N}$ such that: $\alpha = \mu\kappa$ and $\beta = \mu\lambda$ for some $\kappa, \lambda \in \mathbb{N}$, hence α, β are *incommensurable* ■[6]

Clegg (2003) indicates the above result as *"…frankly devastating if you believe that the universe is driven by pure whole numbers"* (p.62). The discovery of irrational magnitudes is definitely a

---

[2] Pythagoreans assumed the existence of *apeiron* and the *limiters*(limited/definite) as the two principles of material cause of the world (Drozdek,2003, p.23)

[3] From sources like Aristotle's comments: *"[The Pythagoreans] hold that things themselves are numbers…"*(Aristotle, n.d./1953)

[4] Alternatively, one could use the method mentioned in Plato's *Meno,* where Socrates teaches the slave Meno how to double the area of a square of side α and diagonal β, i.e. how to show that $\beta^2 = 2\alpha^2$

[5] Ancient Greeks worked with numbers as geometrical magnitudes and since 0 is not a geometrical magnitude at all, the case of $\mu = 0$ is trivial.

[6] A generalization of Euclid's proof of the irrationality of $\sqrt{2}$

pivotal moment for the history of mathematics and is sometimes attributed to *Hippasus of Metapontum* (c.5th century BCE). Nevertheless, there is no historical evidence that it was him who found out about this new kind of numbers. There is the assumption though, that Hippasus was the one who spread the word of his discovery outside of the Pythagorean School because he had already expressed a different belief on the creation of the universe. He considered as the beginning of everything, the "fire" and not the "numbers" as the Pythagoreans believed. An anecdote tells of Hippasus discovering irrational numbers on a boat, and his colleagues were so horrified, that they threw him overboard where he drowned. Regardless of this being just a story or a fact, it is the irrational thinking to murder someone because of his discovery of irrationals that shows us the radicalness of *incommensurability*.

The initial and challenging idea of *incommensurability* might have started to rise due to the method of the subdivision of a regular pentagon (Sondheimer & Rogerson,1981, p.34). One can draw the diagonals of the pentagon ABCDE and then observe that the intersection points $A'B'C'D'E'$ form another pentagon and so on[*Figure1*]. This process could go on indefinitely but there is no "smallest pentagon" whose side could serve as an ultimate unit of measure. Hence, the ratio (which is equal to $\varphi = \frac{1+\sqrt{5}}{2}$,also known as the *golden ratio*) of a diagonal to a side in a regular pentagon cannot be rational.

Nevertheless, Pythagoreans could not accept the fact that the certain ration was not rational, thus it was impossible for them to confront irrational numbers and especially irrational ratios of geometrical magnitudes.

Finishing this glimpse on the secret Pythagoreans' sect, we find in Plato's dialogue *Theaetetus*, a dialogue concerning the nature of knowledge, references to two more contributors in the struggle with the irrational. These are *Theaetetus of Athens*(c.417BCE-319BCE) and Theaetetus's teacher, *Theodorus of Cyrene*(465BCE-398BCE) who was Plato's mathematical tutor. In the dialogue, Theaetetus explains to Socrates his discovery that there are degrees of irrationality. His theorem in modern language:

Theorem: The square root of any (positive whole) number that is not a perfect square (of whole numbers) is irrational. The cube root of any (whole) number that is not a perfect cube (of whole numbers) is irrational. (Mazur,2007, p.241)

The other participant in Plato's dialogue is Theodorus, who had proved the irrationality of $\sqrt{2}$,$\sqrt{3}$,$\sqrt{5}$,… up to $\sqrt{17}$ where for some reason he stopped[*Figure2*].

Yet it would be remiss if we did not acknowledge the considerable contribution of Pythagoras and his followers to the mathematics world. Apart from the famous Pythagoras' Theorem, they explored the principles of mathematics, the concept of mathematical figures and most of all the idea of proof. But even the philosophical crisis caused from an insider is considered as a milestone in mathematics. Sondheimer & Rogerson (1981) reasonably mention for this crisis: *"The discovery of irrational* [ratios of] *magnitudes may be regarded as the beginning of theoretical 'pure' mathematics"* (p.32).

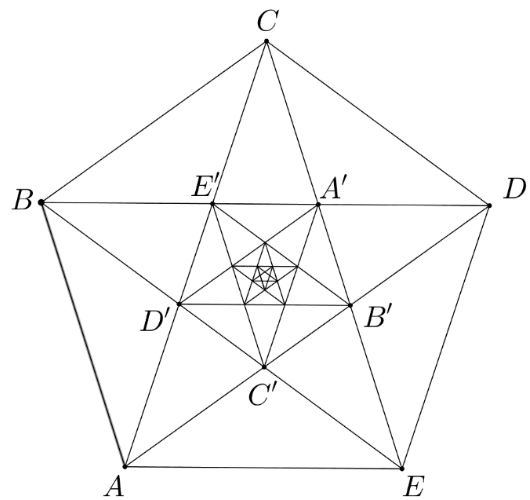


Figure 1. *The regular pentagon ABCDE.*

$\frac{AC}{AB} = \varphi = \frac{1+\sqrt{5}}{2}$

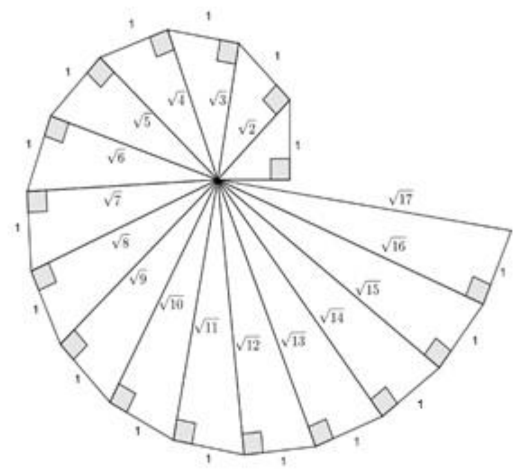


Figure 2. *Thedorus's spiral: starts with an isosceles right triangle with both legs of length 1. More right triangles are added, one leg the hypotenuse of the previous triangle, the other, outside leg always of length 1. the hypotenuses of these triangles have lengths* $\sqrt{2}, \sqrt{3}, \ldots, \sqrt{17}$

Eleatics

Often mentioned as the rival school of the Pythagoreans, the Eleatics had contradictory beliefs to the Pythagorean philosophy. They believed in continuous magnitudes but also in infinite divisibility contrary to the Pythagoreans' point-unit-atom aspect. *Parmenides of Elea* (c. 515 - 450 B.C.), known as the founder of the Eleatic School, developed his philosophy on monism and timelessness. That is, the universe is a permanent single whole in an unchanging reality while time is infinite, without beginning, end or middle.

One of the most dedicated defenders of Parmenides' philosophy, was his student, *Zeno of Elea* (c. 490 - 430 B.C.). Most of the information that we know about Zeno, have survived through Plato's, Aristotle's, Simplicus's and Proclus's writings. Zeno wrote a book of paradoxes defending the Eleatic philosophy by logical means. This book has not survived but all his arguments are paraphrased by Aristotle. These paradoxes were counterintuitive to the concepts of "line" and "point" which were being used unrestrictedly in Greek geometry from Thales up to Zeno's time (Davis&Hersh,1981, p.226). He was the first to show through his arguments that a line segment can be decomposed into infinitely many parts of nonzero length (Vilenkin,1995, p.7). Among the 40 arguments attributed to Zeno, there are the following most famous four paradoxes dealing with continuous space and time: the *Dichotomy* paradox, *Achilles and the tortoise* paradox, the *Arrow* paradox and the *Stade* paradox.

Dichotomy: A moving object will never reach any given point because that which is moved must arrive at the middle before it arrives at the end, and so on *ad infinitum*. Therefore, the object can never reach the end of any given distance.

Achilles and the tortoise: The slower will never be overtaken by the quicker, for that which is pursuing must first reach the point from which that which is fleeing started, so that the slower must always be some distance ahead.

Arrow: If everything is either at rest or moving when it occupies a space equal to itself, while the object moved is always in the instant, a moving arrow is unmoved.

Stade: Consider two rows of bodies, each composed of an equal number of bodies of equal size. They pass each other as they travel with equal velocity in opposite directions. Thus, half a time is equal to the whole time ("Zeno and the Paradox of Motion", n.d., par.5).

The paradoxes are usually interpreted as Zeno's arguments on the impossibility of continuous motion in "infinitely divisible space and time". Moreover, Zeno reveals through the paradoxical nature of his arguments the conflict and the chasm between the discrete and the continuous. For example, in the case of the *Arrow*: Suppose that space is made out of points and an arrow that flies from the bow to the target. Then the flight can be decomposed in infinitely many moments where the tip of the arrow successively occupies every point between the bow and the target. The problem is that the arrow at any one fixed point is motionless and in between the points there is nothing so "How can the flight of the arrow be a sequence of motionless stills? Where did the motion go?" (Rucker,1982, p.81.). Obviously for Zeno being himself a monist, space is an undivided whole which cannot be broken into parts.

Similar view on space is shared by C. S. Peirce and perhaps Kurt Gödel. Gödel distinguishes between the intuitive continuous line and the set theoretical notion of the set of points: *"According to this intuitive concept, summing up all the points, we still do not get the line; rather the points form some kind of scaffold on the line"* (Gödel as cited in Rucker,1982). Peirce elaborated further on the continuous line by saying that there is no conceivable set, no matter how large that could exhaust the line but if we call the cardinal of the universe of sets **Ω** then there are **Ω** points on the line (Peirce,1893 /1992, p.47)

Zeno managed to challenge the simple view of "apeiron" as a cosmological or divine entity by proposing his paradoxes that arise from the cardinality of points on a line, which line is infinitely divisible. It was no more just a boundless thing but something that could be used and manipulated mathematically (Heath,2014, p.66). We could spend hundreds of pages for the resolution of Zeno's paradoxes by means of physics, philosophy and mathematics. Researches intrigued from the paradoxes, have been continuously conducted from the ancient Greeks up to nowadays. For the imperishable scientific value, we resort to Bertrand Russell who mentions that Zeno's arguments *"…have afforded grounds for almost all theories of space and time and infinity which have been constructed from his time to our own"* (Russell,1914, p.183). It would be at Zeno's time, when the first signs to grasp the mathematical characteristics of *apeiron* were coming on surface and the Western civilization was getting closer to the roots of infinity. Not to forget the fact that Dowden(n.d.) stressed out:

> *Awareness of Zeno's paradoxes made Greek and all later Western intellectuals more aware that mistakes can be made when thinking about infinity, continuity, and the structure of space and time, and it made them wary of any claim that a continuous magnitude could be made of discrete parts.* (para.6)

It would be now Anaxagoras' and Aristotle's turn to join Zeno for the quest for infinity.

Anaxagoras vs Zeno - the "smallness"

Anaxagoras of Clazomenae(c.500-428BCE), in opposition with Zeno, maintained that matter is infinitely divisible. Being a contemporary of Zeno, he was probably aware of his works as well as the Eleatic philosophy. Anaxagoras did not conceive any paradoxical situation in the arguments of Zeno and he claimed: *"There is no least among small things; there is always something smaller. For that which exists cannot cease to exist as a result of division regardless of how far the latter* *continues".* (Anaxagoras as cited in Vilenkin,1995, p.6). What is remarkable here is that the "apeiron*"* is related to "smallness", another evidence that Anaxagoras was thinking towards the mathematical complexity of the infinite or as we would say more explicitly nowadays, the nature of the continuum. However, it is not only the continuum that is inherent to Anaxagoras' reasoning for the infinite. In a probable reaction against Zeno's Achilles paradox he quoted: *"The sum of all things is not a bit smaller nor greater, for it is not practicable that there should be more than all, but the sum total is always equal to itself"* (Anaxagoras as cited in Heath,2014, p.66). To have an insight to the previous quote, we explain below the *Dichotomy* paradox in terms of elementary Calculus:

Let us imagine a man that has to cross a distance d. At first, he has to cross a distance of $\frac{d}{2}$, next an additional of $\frac{d}{4}$ and so on *ad infinitum.* In this way, no matter how far he goes on, he will never be able to cross the distance according to Zeno. Now according to Anaxagoras, $d = \frac{d}{2} + \frac{d}{4} + \frac{d}{6} + \cdots$ or equivalently $1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots$. Consider now the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$...we already know that the series converges[7] and sums up to 1 indeed (i.e. the sum is finite), hence Achilles has to cross a finite distance. Assuming that Anaximander implicitly thought in terms of convergence, then he is right. The sum is indeed equal to itself.

Could this be the solution to the Achilles and the Tortoise paradox? Unfortunately, (or fortunately for the unsettling minds) the answer is no. The reason is summarized in the following rhetorical question: how an infinite series of acts can be performed in a finite time?

Aristotelian infinity

Aristotle(384-322BCE) objected to the ambiguity of the use of the notion of "apeiron" in Zeno's arguments while he would place infinity outside of the real world.

Aristotle was born in Stagirus of northern Greece and at the age of 17 joined Plato's academy[8] for the next twenty years. It was while he was still at Plato's academy that he wrote *Physics*, where he tries to clear the air for "apeiron" after Zeno's arguments. For the vagueness of the notion, Aristotle warns the reader that "…the topic entails walking on very shaky ground" (as cited in Vilenkin,1995, p.8). Even though he places the infinite outside of the real world, he further gives five reasons, usually thought to support its existence: i) the infinity of time, ii) the division of magnitudes in mathematics iii) constant generation and destruction, iv) the fact that there is always something beyond limit, and v) the belief that numbers, geometrical magnitudes and the space outside the cosmos are infinite because they never give out in thought (Kouremenos,1995, p.31)

Since the philosopher has given his five arguments on why one should believe in the existence of the infinite, he continues by saying that the infinite can't be a thing and that there is no thing such an infinite body. His definition of a body is that it is something that "is bounded by surface" hence if it was infinite then such a body does not exist. But then Aristotle as a philosopher, poses the vital question, what if there were no infinity? Then obviously, time has a beginning and an end, there is no such thing as infinite divisibility of the line and numbers must stop. He responds to his counter question that: *"…clearly there is a sense in which the infinite exists and another in which it does not"* and concludes that: "*The infinite does not actually exist as an infinite solid or magnitude*

[7] It would be almost 2000 years later than Anaximander when great mathematicians like Sir Isaac Newton and Augustin-Louis Cauchy would perfect the idea of convergent series

[8] Remarkably the word "academy" has its origins in Plato's garden where he taught. The trees of the garden were believed to belong to Academos, a mythological hero, thus the name "academy" which is used up to present

apprehended by the senses…The infinite exists potentially, the infinite is motion" (Aristotle, n.d./1930), ruling out the existence of actual infinity. But why all the fuss about Aristotle's deep philosophical arguments?

It is the importance to the mathematical world of Aristotle's philosophy that lead him to the distinction between the *actual* and *potential* infinity. Clegg (2001) mentions that this distinction: *"…would keep mathematicians happy all the way up to the nineteenth century…whether infinity was real or unreal…Aristotle's move of infinity into the virtual world of the potential made the mathematics work*" (p.32).

Aristotle introduced two procedures on his argumentation for the finite and the infinite, that is a summative and a divisive procedure, corresponding to the modern notions of *extensive* and *intensive* infinity. For the summative procedure, suppose "*a finite magnitude* [where one] *takes always a limited amount in the same proportion and adds that*" (Aristotle, n.d./1930). Following this procedure, say we take $\frac{3}{4}$ of the "*finite magnitude*" and add $\frac{3}{8}$, then add $\frac{3}{16}$ (*"same proportion"* $=\frac{1}{2}$), then for Aristotle, we will not be able to exceed the amount of the finite magnitude. However, if we keep adding the same amount, no matter how small, we can exceed any finite magnitude. Recall here, that for Aristotle there is no infinite body (infinitely large), hence for him the latter procedure could not go on forever, unlike the first which can be carried as long as we please (an idea that supported his acceptance of only the potential infinity). In order to clarify this, Aristotle separates the case of magnitudes from that of numbers.

According to him, there is a limit in the diminishing direction of numbers but there is no limit in the direction of more, where it is always possible to exceed every plurality. For the case of magnitudes, he holds for the opposite, that is: in the diminishing direction, it is possible to exceed every magnitude but not in the direction of more.

We are lead then to the divisive procedure and the idea of infinite divisibility of continuous magnitudes. The divisive procedure is something that exists potentially as a never-ending process. It is not hard to see that the procedures of the summative and the divisive are closely related in a sense that: *"…the infinite in respect of division, is "in sense the same" as divisibility in the direction of the inverse addition, or the infinite in respect of addition"* (Bowin,2007, p.242). Through the idea of infinite divisibility, comes the idea of potentially whole and complete magnitude. We can always keep adding material parts which are produced by divisions, consequently we have a magnitude which is filled by a potentially infinite number of material parts. With this reasoning, Aristotle rejects the paradoxical arguments of the Eleatics, meaning that the distance is continuous, thus infinitely divisible, thus composed by infinite potentially material parts which could be added potentially to make the potentially whole.

To recap, what Aristotle would say for the set of natural numbers, is that it is potentially infinite but since it is a set with no end, the set cannot actually be infinite. Extending his opinion for the magnitudes, he believes that the space of the universe is limited and bounded forming a vast sphere, while his response on what is outside of that sphere would be: *"what is limited, is not limited in reference to something that surrounds it"* (Aristotle, n.d./1930). The Aristotelian conception of infinity would echo up to the first half of the 19^{th} century, when Bertrand Bolzano introduced the notion of the set. Nevertheless, the acceptance of the potential nature of infinity, resulted to great results for mathematics, among them Eudoxus's works on Geometry and Number Theory.

Eudoxus of Cnidus

Eudoxus(c.395-342BCE) is considered as one of the greatest mathematicians and astronomers of Greek antiquity, known for his complete theory of proportions[9] and the famous method of exhaustion. He managed to develop a theory for comparing lengths and other geometrical quantities, a comparison that failed to work for the Pythagoreans for lines of lengths 1 and $\sqrt{2}$. It is known that the Greeks could not accept irrational ratios of geometrical magnitudes but accepted irrational geometric magnitudes since they obviously existed, such as the diagonal of the unit square. Hence, Eudoxus used purely geometrical means without assigning any numbers to the magnitudes, thus he avoided irrational numbers. Precisely, what Eudoxus meant in modern notation is that,

$\frac{a}{b} = \frac{c}{d}$, if and only if $\forall (m, n) \in \mathbb{N}$ one of the following holds:

i. if $ma < nb$ then $mc < nd$, and vice versa
ii. if $ma = nb$ then $mc = nd$, and vice versa
iii. if $ma > nb$ then $mc > nd$, *where a, b, c, d arbitrary geometrical magnitudes which are pairwise of the same kind (i.e. possibly irrationals)*

Eudoxus perfected his theory by carrying out a limit process which came to be known as the *method of exhaustion*. (Zippin,1962, p.40). The natural numbers m and n are arbitrary and thus infinitely many, Eudoxus's definition amounts to an indirect consideration of a limit. We know today that every real number can be expressed as the limit of a sequence of rational numbers. Even though Eudoxus did not use an explicit theory of limits, the method is considered by many scholars as the beginning of calculus. It was one of the many attempts after the discovery of the irrationals, to find a rational number μ for the relation $\alpha = \mu\beta$, so for the difference $\alpha - \mu\beta$ to become as small as possible. It should be noted at this point that Eudoxus and the later Greek mathematicians never thought of the method of exhaustion as a process carried out in an infinite number of steps. In their mind, there was no such thing as an infinitely small magnitude but a magnitude that could be made as small as possible by repeating division (Giannakoulias,2005, p.11). More than a century later, Archimedes would attribute to Eudoxus proofs of the theorems that the volume of a pyramid is one-third the volume of the prism having the same base and equal height; and the volume of a cone is one-third the volume of the cylinder having the same base and height.

2.3 Hellenistic Period

The Hellenistic Period is usually accepted to begin in 323 BC with Alexander's death and ends in 31 BC with the conquest of the last Hellenistic kingdom by Rome. This period is also mentioned sometimes as the *"golden age"* of Greek mathematics, as new ideas and works appeared in mathematics that are still being used today. Some of the famous mathematicians of the Hellenistic Period include Aristarchus, Apollonius, Hipparchus and the greatest mathematician of antiquity, Archimedes.

Archimedes-the greatest of antiquity

Archimedes(287-211BCE) used the *method of exhaustion* to prove a remarkable collection of theorems related to areas and volumes. All of his works are characterized by rigorous proofs, strong

---

[9] "The theory of proportions was so successful that it delayed the development of theories for real numbers for 2000 years" (Ji,2010/2010)

originality and remarkable creativity. In fact, he avoided the use of the notion of infinity and infinitesimals. However, his methods and approaches to infinite processes for solving mathematical problems would consist the fundaments of what is known today as the Integral Calculus. Archimedes observed the following:

Theorem: The ratio of any circle to its diameter is constant.

Furthermore, he gave an accurate approximation for this constant. In his book, *Κύκλου Μέτρησις* (On the Measurement of the Circle) (c.3rd Century BCE/1897), Archimedes approximated the ratio of the circumference of a circle to its diameter by inscribing and circumscribing regular polygons to the circle. Thus, reaching consecutively at the case of the regular 96-*gon* he concludes that the ratio of the circumference of a circle to its diameter is equal to π, where $3\frac{10}{71} < \pi < 3\frac{1}{7}$, which is $3{,}1409 < \pi < 3{,}1429$. For the uniqueness of π, Archimedes cut the circle area into equal radial sectors and regrouped the sectors such that they became approximately a rectangle with the base being the length of the circumference P and the height r. He then showed starting at the simple example of hexagons inscribed and circumscribed to the circle that this figure (approximate rectangle) can always be enclosed in two rectangles which approach the circle area A from above and from below with sufficiently fine partitioning of the circle in radial sectors. Thus, it follows the relationship $r\frac{P}{2} = A$ for any circle and thus from $P = 2\pi r$ follows $A = \pi r^2$ and conversely(Siegmund-Schultze,2016).

In his work, *Τετραγωνισμός Παραβολής* (On the Quadrature of the Parabola), we find one of the most concrete applications of the method of exhaustion. Specifically, Archimedes proved the

Theorem: *The area of any parabolic segment is four-thirds the area of its vertex triangle.*

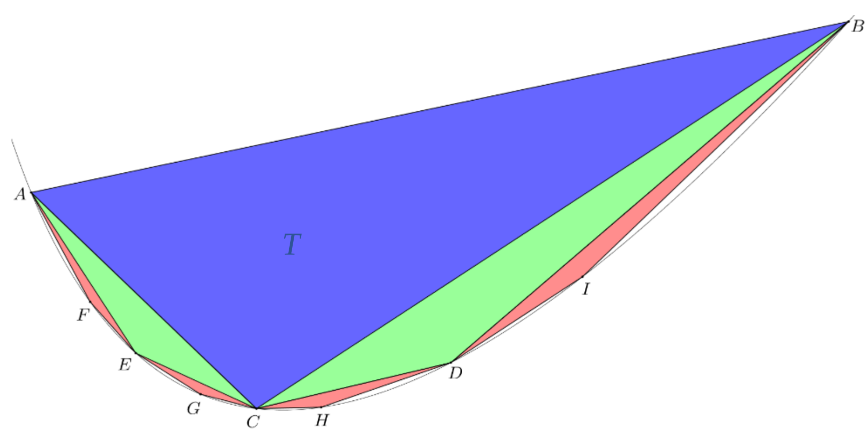


Figure 3. *Archimedes proved that the area of each green triangle is one eighth of the area of the blue triangle, each of the red triangles has one eighth the area of a yellow triangle. Finally, he concluded that Area of the Parabola* $= \frac{4}{3}T$.

His main insight is that when we remove the vertex triangle ABC from the parabolic segment we are left with two smaller parabolic segments which themselves have vertex triangles ACE and BCD. Removing this layer of two triangles we get four even smaller parabolic segments, whose four vertex triangles AF E, EGC, CHD, DIB form the next layer, and so on. Archimedes proved that the sum of the triangular areas could be made as small as one pleases by constructing a large enough number of triangles.

2.4 The Decline of Greek Mathematics

After Apollonius of Perga(247-205BCE), known for his treatise on Conic sections, began the decline of Greek mathematics. The geometry of conics didn't develop until *Apollonius' problem* stimulated Descartes' work in the middle of the 17^{th} century, while the method of exhaustion remained unmodified up to the 17^{th} century when Cavalieri fully developed his method of indivisibles. Development of mathematics would now follow a different direction, specifically that of trigonometry due to the influence of the needs of astronomy and later on the direction of Number Theory with Diophantus as the main contributor (Giannakoulias,2005, p.14). Heron, for example, found the famous formula for the area of the triangle (also known as Heron's formula) while Ptolemy created the table of chords[10] but also gave explanations for astronomical phenomena that were the standard for 1400 years.

The last period of antique society is that of Roman domination. Romans destroyed Corinth after the battle of 146CE and the Christian dogma started its domination upon philosophy and learning. Scientific reasoning and critical thinking were banished by fanatical behavior and superstitions. This resulted to the murder of *Hypatia of Alexandria*(c.470-415CE) in 415CE, a great mathematician, astronomer and leader of the Platonic school in Egypt, by a mob of Christian fanatics. Furthermore, one of the most representative and unfortunate event that indicates the end of the ancient culture is the burning of the Alexandrian library.

While ancient Greek mathematics were falling into decline and the Roman empire was being established**,** western knowledge would diffuse to the East, where mathematical research would reach new heights. Arabs were being taught the Greek knowledge by Greeks who inhabited at the conquered areas of the Byzantine empire. Being themselves great mathematicians and great translators, the Arabs would translate works of Euclid, Apollonius, Diophantus and more by the 9^{th} and 10^{th} century CE. Hence, they became keen with the ancient Greek heritage while the Babylonian methods for solving arithmetical problems were already known to them. All these facts would pave the ground for the creation of Algebra in the early 19^{th} Century. Arabs were also influenced by Hindu mathematicians, who invented the decimal numerical system (the so-called Hindu-Arabic system). Hindu mathematicians treated rational and irrational quantities indiscriminately while the problems of incommensurability were of little importance to them (Boyer&Merzbach, 1968, p.61-62)

It is argued among academics, such as Murty (2013), that infinity as a mathematical concept has its roots in India and its discovery is credited to the Kerala (or Madhava) school of mathematics and astronomy. The problem of infinity as a mathematical idea appears in Brahmagupta's *Brahmashputasiddhanta* in which he raises the question of what is the value of $\frac{1}{0}$. This question is answered in the 12^{th} century by Bhaskaracharaya who correctly deduces that it is infinity by an ingenious limit process (p.43).

2.5 The Middle Ages

Meanwhile in Europe and during the Middle Ages (5^{th}-15^{th} Century CE), science and mathematics evolution stagnated and philosophy was reduced to the role of a servant of theology. As a consequence: "*The infinite also ended up in the theological sphere- it became an attribute to God*"

[10] An extensive trigonometric table used for practical purposes, mainly in Astronomy

(Vilenkin,2013, p.10). It was at the 15th century when the philosophy of Plato and Aristotle would start gaining ground again due to the cosmological and theological controversies and debates.

An initial breakthrough against the old dogmas was made by *Nicholas of Cusa* (1401-1464). Being influenced by Plato and Neoplatonic thinkers such as Plotinus and Proclus, Cusa had a heliocentric view on the relationship of the earth and the sun, well before Copernicus and Galileo. About the Infinite, he maintained that our finite minds cannot know the Infinite. To illustrate the notion of infinity, he considered circles of larger and larger diameters. As the circles increase in size, a given length of the circumference is less curved and more similar to a straight line. A segment of the infinite circle would therefore coincide with a straight line. However, the rational mind, cannot comprehend such an actualization of the Infinite but can only be seen through a mystical insight. Dealing with the mathematics of infinity, Nicholas used a similar approach to Archimedes for counting the volume of a sphere by chopping it up into thin slivers. Remarkably, as Archimedes did, he was being careful, using only finite processes. It was a use of form of *indivisibles*, that is for the case of a circle, a really small radial sector would never quite become triangle or cannot be divided infinitely.

Despite the fact that the medieval period added little to the Greek works in geometry or to the theory of algebra, Boyer and Merzbach (1968) mention that the contributions of the period were:

> *…chiefly in the form of speculations, largely from the philosophical point of view, on the infinite, the infinitesimal, and continuity…Such disquisitions were to play a not significant part in the development of the methods and concepts of calculus, for they were to lead the early founders of the subject to associate with the static geometry of the Greeks the graphical representation of variables and the idea of functionality".* (p.94)

2.6 17th Century-An explosion of mathematical ideas

The seventeenth century saw some of the most important discoveries in mathematical science. There has been an explosion of ideas, not only in mathematics but also in astronomy and science in general, leading this period to be named as the Scientific Revolution. Namely, some of the key figures and their mathematical developments were Pierre de Fermat (1601-1665) who is credited the invention of modern number theory, Blaise Pascal (1623-1662) known for his contribution to probability theory, Girard Desargues (1591-1661) and his early development of projective geometry, not to forget John Wallis (1616-1703) who contributed extensively in the origins of Calculus and also introduced the symbol ∞ for infinity.

For the concept of the infinite, the 17th Century provides us with many paradoxes that would later constitute the topic for Bolzano's *Paradoxien des Unendlichen* (Paradoxes of Infinity) (1851/. These paradoxes were a result of foundational discussions of two main topics investigated at the time: i) the theory of indivisibles and ii) the theory of space (i.e. investigation of geometrical volumes and surfaces) (Mancosu,1999, p.118). These ideas would set the ground for Newton and Leibniz to co-invent Calculus.

The theory of indivisibles

Galileo Galilei(1564-1642), the Italian astronomer, physicist, engineer, philosopher, and mathematician introduced in his work *"Discorsi e Dimostrazioni Matematiche, intorno a due nuove scienze*"(Discourses and Mathematical Demonstrations Relating to Two New Sciences)(1638/1954)the notion of indivisibles. Galileo developed the notion by means of four mathematical examples: 1. The paradox of the Aristotelian wheel 2. The equality of certain circular rings and areas of circles which leads to the equality of the circumference of a circle with a point (also known as the bowl paradox) 3.a comparison between the sets of the natural and the square numbers 4.the construction of a hyperbolical point system which leads to the special case of a circle with infinite radius, which degenerates into a line.

In order to get an insight to Galileo's notion of indivisibles, we should look on the wheel paradox. The paradox deals with the question why two connected concentric circles, one of which rolls along a straight line, during one revolution cover equally long straight line in spite of their different circumferences (Knobloch,1999, p.88). Galileo thought of the circle as a polygon with infinitely many sides. This thinking led him to resolve the paradox by analyzing the motion of concentric polygons. Hence, he concluded that the smaller circle leaps along the way (Mancosu,1999, p.235). For his mathematical analyses, he set a correspondence between the finite number of divisible sides(*quanta*) of the polygons and the infinitely many indivisible sides of the circle(*non-quanta*). For Knobloch (1999), the key to the understanding of Galileo's theory of the infinite lies in the pair of *"quanti/non quanti"* (p.90).

In the third mathematical example mentioned above, Galileo points out that the square numbers $(1,4,9,16,\dots)$ are clearly fewer than the natural numbers $(1,2,3,4,\dots)$. He continues by setting a one-to-one correspondence, that is each square with its root and finally comes on surface Galileo's paradox that the two collections are at one equal and unequal. To be more specific, according to Parker (2009), the paradox lies at the conflict of the following two principles

Euclid's principle: The whole is greater than the part

Hume's principle: Two collections are equal in numerosity if and only if their members can be put in one-to-one correspondence (p.18).

Through his paradox, Galileo (1638/1954) concludes that the notion of numerosity simply does not apply to the infinite: *"*[N]*either is the number of squares less than the totality of numbers, nor the latter greater than the former"* (as cited in Parker,2009, p.17).

Another participant in the foundational discussion on the indivisibles was Galileo's contemporary, the mathematician *Bonaventura Cavalieri* (1598-1647) who was inspired by Kepler's work (*Nova stereometria doliorum variorum)* (1615) and wrote his *Geometria Indivisibilibus Continuorum Nova Quadam Ratione Promota* (1635) and *Exercitationes Geometricae Sex* (1647/1980) (where he fully developed his theory of indivisibles). Cavalieri's central idea was that a line is made up of an infinite number of points (a point is the *"indivisible"* of a line) while a plane was made up of an infinite number of lines (a line is the *"indivisible"* of a plane). According to Stergiou (2009), Cavalieri's view on infinity differed from the Aristotelian conception of potential infinity, he used infinity as an auxiliary notion… (p.50). Mancosu (1999) mentions that Cavalieri's theory: *"... is an attempt to provide a measure for infinite collections of indivisibles"* (p.120). These infinite collections implied the notion of a set or even better the first appearance in history of mathematics of an infinite set (Jullien,2005, p.35). Moreover, it is a fact that the method forms a part of differential and integral calculus. These days, Cavalieri's method of indivisibles has been implemented in geometry as the Cavalieri's principle[11]: *"If, in two solids of equal altitude, the sections made by planes parallel to and at the same distance from their respective bases are always equal, then the volumes of the two solids are equal"* (Vialar,2015, p.484). A representative example of applying the principle is to show that two stack of coins, forming a right circular cylinder and an oblique circular cylinder, are of the same volume.

[11] Cavalieri's principle had already been used (c.260 CE) by Chinese mathematicians such as Liu Hui for finding the volume of spheres.

The theory of space-Torricelli's trumpet

Evangelista Torricelli (1608-1647), the Italian physicist and mathematician, was a friend of Cavalieri and a student of Galileo. By 1641, his studies extended to the Torricelli's Trumpet (also known as Gabriel's horn) which was a result of an extension of Cavalieri's method of indivisibles to cover curved indivisibles (O'Connor and Robertson, n.d.). He demonstrated a solid which is infinite in length and has finite volume. Such a paradox would attract the attention of many geometers and philosophers of the time.

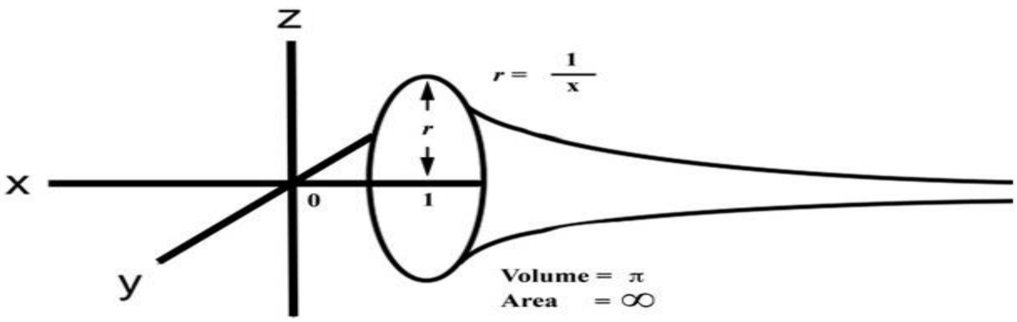


Figure 4. *Torricelli's trumpet*. Adapted from C. Cooper, *Torricelli's trumpet. Retrieved from http://www.coopertoons.com/caricatures/torricellistrumpet.html*

The solid is generated by rotating $f(x) = \frac{1}{x}$ about the x-axis between $x = 1$ and $x = \infty$.What it is now to be found is the surface area of a cross sectional slice and the volume of the solid.
To determine the volume V of this object, we'll have to integrate the cross-sectional area πr^2 for $[1, \infty]$.

$$\mathrm{V} = \int_1^\infty \pi \left(\frac{1}{x}\right)^2 dx = \pi \int_1^\infty \frac{1}{x^2} dx = \pi \left(\lim_{\alpha\to\infty} \int_1^\alpha x^{-2} dx\right) = \pi \lim_{\alpha\to\infty} \left[-\frac{1}{x} |_1^\alpha\right] =$$

$$= \pi \lim_{\alpha\to\infty} [-\tfrac{1}{a} - \left(-\tfrac{1}{1}\right)] = \pi$$

To determine the surface area S, we make use of the surface area formula for a rotation about the x-axis, that is:

$$\mathrm{S} = 2\pi \int_1^\infty \frac{1}{x} \sqrt{1 + \left(\frac{f(x)}{dx}\right)^2}\, dx =$$

$$= 2\pi \int_1^\infty \frac{1}{x} \sqrt{1 + \frac{1}{x^4}}\, dx$$

$$2\pi \int_1^\infty \frac{1}{x} \sqrt{1 + \frac{1}{x^4}}\, dx \geq \int_1^\infty 2\pi \frac{1}{x} dx$$

Thus, the comparison test[12], by $2\pi \int_1^{\infty} \frac{1}{x} \sqrt{1 + \frac{1}{x^4}}\, dx$ *diverges to infinity* (i.e. the surface S is infinite)

In the previous calculations, we see that indivisibles were used implicitly on what is called the disk method.

Torricelli's paradox shows how mathematics may appear to prove something – but in reality, it fails. For example, let us think of Torricelli's trumpet as a bucket that can hold π gallons of paint but still you can never have enough paint to paint it. This proof caused a lot of mathematicians and philosophers of that time to think that there was something wrong with the idea of infinity. However, as Jago (2013) mentions: *"…infinity works just fine in mathematics. But we have sometimes to change our ideas how the world works to fit in"*.

Geometrical results such as Torricelli's, motivated many thinkers of the time to revise the issue of our knowledge of infinity and contribute even more to the Scientific Revolution. Among such thinkers were Thomas Hobbes, John Wallis, Gilles de Roberval and Isaac Barrow. René Descartes was also aware of Torricelli's hyperbolic solid. Descartes with his work "*Discours de la méthode"* (1866/2012) and along with Pierre de Fermat became the father of analytical geometry. Even if Descartes used infinite series to solve Zeno's Achilles paradox, he defended his view that we, as finite minds, cannot fully grasp the idea of infinity (Schechtman,2014, p.15). In 1644, one year after Toricelli's result, he declared: "*Since we are finite, it would be absurd to determine anything concerning the infinite; for this would be to attempt to limit it and grasp it."* On the other hand, Pierre de Fermat (1601-1665) developed a method (cf *method of adequality*) for determining maxima, minima and tangents to various curves that was essentially equivalent to differentiation. For Bell (2005), in Fermat's work on maxima and minima we have the first appearance of the idea of *infinitesimals* (p.77). Furthermore, Fermat was one of many to notice the inverse relationship between integrals and derivatives but not the importance of this relationship.

Definition: An infinitesimal is a number whose magnitude exceeds zero but somehow fails to exceed any finite, positive number. (Tropp,2002, p.viii)

Infinitesimals had been a strongly controversial concept at the time. It would shake the grounds of religious and scientific beliefs. Nevertheless, the use of infinitely small and infinitely large magnitudes would lead to great developments for mathematics, especially at the great works of Leibniz.

Newton and Leibniz

Gottfried Wilhelm von Leibniz (1646-1716) viewed calculus in terms of sums and differences. He is credited the ingenious notation of Calculus which is even used nowadays, such as the notation $\int x\, dx$, where $\int$ was an elongated representation of the first letter of the Latin word *"summa"*[13], meaning summation, and d was the first letter of the Latin word *differentia*, meaning differential (infinitesimal distance). For example, x changes by an infinitesimal quantity dx. However, Leibniz did not give a precise definition for the notion of infinitesimals or explanation for the way he used but *"…the jiggery-pokery which resulted from the application of his unexplained rules was enormously fruitful; and his marvelously suggestive notation of 'differentials' is still very much with us"* (Gardiner,2012, p.16). By using a differential triangle to discover the slope of a tangent

---

[12] *If* $f(x) \geq g(x) \geq 0$ *on the interval* $[\alpha, \infty]$ *then, if* $\int_{\alpha}^{\infty} g(x)dx$ *diverges then so does* $\int_{\alpha}^{\infty} f(x)dx$

[13] The word itself betrays the origin of the integral in the theory of indivisibles

line to a curve, he was able to derive the power, product, quotient and chain rules. Gardiner adds for Leibniz's rules: "…[they] *tamed the infinitesimally small so successful, that even the ordinary user could harness its potential with relatively little fear of going astray"* (2012, p.17).

Unlike Galileo, Leibniz could not accept the "non-quanta" or "no-quantity" view on indivisibles. He gave an exact definition of indivisibles and had a clear idea of what they are. For Leibniz, indivisibles where infinitely small positive quantities which are smaller than any quantity given, while infinitely large quantities are quantities larger than any quantity given. It does not matter whether they appear in nature or not, because they allow abbreviations for speaking, for thinking, for discovering and for proving (Knobloch,1999, p.95).

Leibniz advocated a really sophisticated and useful view on actual infinity. The excerpt of his letter to the philosopher Foucher (1686/2010), gives us a broader sight on his view on infinity:

> *I am so in favor of the actual infinite that instead of admitting that Nature abhors it, as is commonly said, I hold that Nature makes frequent use of it everywhere, in order to show more effectively the perfections of its Author. Thus, I believe that there is no part of matter which is not, I do not say divisible, but actually divided; and consequently, the least particle ought to be considered as a world full of an infinity of different creatures.* (as cited in Anstey,2010, p.219)

Leibniz held that matter is infinitely divided consists of infinitely many monads or an actual infinity of monads. But what makes his thesis complicated is the fact that he rejected actual infinity in mathematics. Particularly, in "*Accessio*" (1672), he identifies an infinite number (under the condition that there is such a number) with nothing or zero. Galileo's paradox for example, occurs when someone assumes the existence of an infinite number, hence for Leibniz such a number had to be identified with nothing.

While Leibniz viewed his calculus in terms of sums and differences in Germany, at the same time *Sir Isaac Newton (1643-1727)* thought of calculus in terms of motion. He studied at the Cambridge University under another great contributor of calculus, Isaac Barrow. Through the years 1665-1666 he developed[14] his Treatise of Fluxions, in an attempt to comprehend the surrounding world in terms of calculus. Not to forget that this period, Newton developed his fundamental ideas on universal gravitation, as well as the law of the composition of light. L.T. More remarks: *"There are no other examples of achievement in the history of science to compare with that of Newton during those two golden years"* (1934, p.41).

In 1671(1966), Newton writes the monograph under the title *Methodus Fluxionum et Serierum Infinitarum,* where he makes use of the so-called *fluxions* and *fluents.* Newton thinks of the variables as a result of the continuous movement of points, lines and planes and not as sums of an infinite number of infinitesimals. (Giannakoulias,2004, p. 41)

Definition: A dependent on time fluxion $\dot{x}$ of a quantity x(*fluent),* is the velocity of which the variable x is increased through the motion that created this fluctuation. (Stergiou,2009, p.80)

Definition: The *moment* of the fluent is the amount it increases in an *"infinitely small"* interval of time o,denoted by $\dot{x}o$. (Ben-Menahem,2009, p.1143)

[14] Newton discovered his general method during the years 1665-66 when he stayed at his birthplace in the country to escape from the plague which infested Cambridge

Obviously, these notions bring us in mind the notion of the derivative. In the following example, we will see how Newton would find the fluxion(derivative) of $y = x^2 - x$.

Example: Let pass an infinitely small interval o. Then x is changed into $x + \dot{x}o$ and y into $y + \dot{y}o$. Since the point $(x + \dot{x}o, y + \dot{y}o)$ is still a point of the parabola y then:
$y + \dot{y}o = (x + \dot{x}o)^2 - (x + \dot{x}o) \Rightarrow \dot{y}o = 2x\dot{x}o + (\dot{x}o)^2 - \dot{x}o$
Then Newton, intuitively thinking, would "cast out" the terms that contain o in a power greater than 1 and would get $\dot{y}o = 2x\dot{x}o - \dot{x}o$. He would divide by $\dot{x}o$ to finally get $\frac{\dot{y}o}{\dot{x}o} = 2x$ or in the modern notation $\frac{dy}{dx} = 2x$.

We observe that Newton thought of infinitesimals as variables that could arbitrarily approach to zero but also as seen in the previous example, could be equal to zero. Inherently, Newton's method raised serious objections, as to the simplification of the terms seen above. As a reaction, Newton wrote *De quadrature curvarum* (1963/2008), where he changed some of his notation and used a method for the derivative, almost identical to the one we use it today. In his scientific masterpiece "*Principia*" he moved one step away from infinitesimals and one step closer to the notion of the limit by introducing his theory of *"prime and ultimate"* ratios. As Newton (1687/2014) writes in Principia:

> *Those ultimate ratios with which quantities vanish are not truly the ratios of ultimate quantities…but limits towards which the ratios of quantities decreasing without limit do always converge, and to which they approach nearer than by any given difference, but never go beyond, nor in effect attain to, till the quantities are diminished in infinitum (as cited in Struik,2014, p.300)*

Newton's as well as Leibniz's calculus faced some serious criticism, mainly on the philosophical aspect of infinitesimals and the lack of rigor caused by their use. One could say that it was a reasonable reaction since "*Newton himself admitted* [that]*…his method is 'shortly' explained, rather than accurately demonstrated*" (Boyer,1949, p.193). As for Leibniz, Mancosu (1999) mentions: *"Leibniz does not explain how he arrived at his equations and leaves the reader totally in the dark as to the heuristics and formal proofs of the results therein presented"* (p.151). Bishop George Berkeley of the Church of England raised serious concerns about the efficacy of calculus and made a fierce critique on the notion of Newton's fluxions while referring to infinitesimals as "evanescent Increments": *"And what are these Fluxions? The Velocities of evanescent Increments? They are neither finite Quantities nor Quantities infinitely small, not yet nothing"*. By the end of 17^{th} Century a debate on infinitesimals raged. Great minds of the mathematical science, such as Gauss and Bolzano would react to the unrestricted use of infinitesimals after Newton's and Leibniz's works bridging the mathematics of 18^{th} and 19^{th} Century.

2.7 18^{th} and 19^{th} Century, the appearance of "*menge*"

With the background of Calculus, having been set by Newton and Leibniz, mathematicians of the 18^{th} and 19^{th} Century would work on more complex notions that involved the idea of infinity. Gauss's brilliant mind and Bolzano's insightful intuition would lead to discoveries of theorems and notion that would bring infinity one step closer to its mathematization.

Gauss, the "prince" of mathematicians

Carl Friedrich Gauss (1777-1855) had a new view of infinitesimals. The status of infinitesimals in the 18^{th} Century reached a point where they behaved similarly to real numbers. However, Gauss raised caution when one uses using infinite quantities and thought of the only legitimate use as a limit.

This view reflects on his letter (1831) to his student and astronomer H. Schumacher:

> *... first of all, I must protest against the use of an infinite magnitude as a completed quantity, which is never allowed in mathematics. The Infinite is just a manner of speaking, in which one is really talking in terms of limits, which certain ratios may approach as close as one wishes, while others may be allowed to increase without restriction*. (as cited in rjlipton,2014)

It seems from the above excerpt that Gauss opposed to the notion of actual infinity. However, Gauss established results in Euclidean geometry and analysis by examining the behavior of mathematical entities at infinity. In his 1812, published paper *Disquisitiones Generales Circa Seriem Infinitam* (General Investigations of Infinite series) he deals with the convergence of the hypergeometric series. It was the first important and rigorous investigation of convergence of infinite series (Rassias,1991, p.5). Thus, he developed an exact criterion for the convergence of the hypergeometric series, known today as the Gauss's criterion. Vilenkin (2013, p.15), is not exaggerating at all when he refers to Gauss as the "prince of mathematicians". Gauss started showing his amazing mathematical skills by the age of 8 when his teacher asked his class to add together all the numbers from 1 up to 100. What Gauss actually did, was to think of the formula $1+2+3+\cdots+n=\frac{n(n+1)}{2}$ to find out that the sum was equal to 5050. At the age of 15 he would discover that the number of primes $\pi(v)$ which are less or equal to a natural number v is approximately equal to $\frac{v}{lnv}$. Moreover, in a dissertation (1799) written as a degree requirement, he gave the first complete proof of *"The Fundamental Theorem of Algebra"*[15].

This brought up once again the "good old" topic of irrational numbers and forced Gauss to accept irrational numbers as solutions to equations[16]. For example: $x^2-2=0$ has two solutions $\sqrt{2}$ and $-\sqrt{2}$ in $\mathbb{R}$, which are both irrational numbers. Gauss's proof of the Fundamental theorem of Algebra was criticized by Bertrand Bolzano as to the impurity of its geometrical nature.

Bertrand Bolzano

Bertrand Bolzano (1781-1848) gave a satisfying definition for the continuous function, defined the notion of the derivative and that of the limit of a sequence. For the first time, the notion of continuity would be connected to that of limit. In 1834, he found a continuous function that is nowhere differentiable ("non-differentiable Bolzano function"), defying this way Newton's and Leibniz's principle that "every continuous function is differentiable" (Giannakoulias,2004, p.50).

Bolzano viewed the notion of infinity in terms of the abstract notion of set(*menge*). This conception led him to break the traditional mathematical view of the infinite (Ewald&Ewald, 2005, p.249). In the *"Paradoxes of Infinity"*, he considered directly the points that had concerned Galileo.

[15] Every polynomial equation with one unknown has at least one solution.

[16] In 1824, Niels Henrik Abel would provide the first proof of the impossibility of obtaining radical solutions for general equations beyond the fourth degree

He looked further in the nature of infinite sets and for the case of all fractions (i.e. rational numbers) between 0 and 1, he showed that there was a one-to-one correspondence with the infinity of fractions between 0 and 2. He did this using the function $f(x) = 2x$,and found that $\forall x \in [0,1]$ he got a unique number of the interval $[0,2]$. This process could also be reversed for $f(x) = \frac{1}{2}x, \forall x \in [0,2]$.This insight was also to be highly significant to Cantor's work. (Clegg,2013, p.134-135). However, the property of finite sets, the whole is greater than the part, lead Bolzano to conclude that the results of the comparison of infinite sets remained paradoxical. Hence, his attempt to arithmetize infinity failed. The paradoxical situation can be observed in the paragraph of Bolzano's own words:

> *As I am far from denying, an air of paradox clings to these assertions; but its sole origin is to be sought in the circumstance that the above and oft-mentioned relation between two sets, as specified in terms of couples, really does suffice, in the case of finite sets, to establish their perfect equimultiplicity in members.* […] *The illusion is therefore created that this ought to hold when the sets are no longer finite, but infinite instead*" (1851/1950, p. 98)

Bolzano's published work *Paradoxien des Unendlichen* (The Paradoxes of the Infinite) (1851) would be later acknowledged by mathematicians such as Charles Sanders Peirce, Georg Cantor, and Richard Dedekind. Cantor himself pointed out that Bolzano lacked both a precise definition of the cardinality of a set and a precise definition of an ordinal number. For his works *Beyträge zu einer begründeteren Darstellung der Mathematik* (1810), *Der binomische Lehrsatz* (1816) and *Rein analytischer Beweis* (1817). Bolzano in these works attempts to free Calculus from the infinitesimals and refers to them as *"...a sample of a new way of developing analysis"* (as cited in O'Connor & Robertson, n.d, par.14). However, the rigorous foundations of calculus would come in the 19th Century by works of Cauchy and Weierstrass.

Cauchy and Weierstrass towards rigor in analysis

Grabiner (2012) distinguishes between two facts that made successful the rigorization of calculus, by the French mathematician *Louis Augustine Cauchy* (1789-1857) (p.5). First, the fact that Cauchy understood the 18th limit concept in terms of inequalities (ε-δ definition). Second, that all of Calculus could be based on limits, transforming previous results on continuous functions, infinite series derivatives and integrals into theorems. Cauchy defined a limit as follows:
"When the successive values attributed to a variable approach indefinitely a fixed value so as to end by differing form it by as little as one wishes, this last is called the limit of all others" (as cited in Sondheimer & Rogerson,1981, p.139)
In *Cours d'analyse* (1821), Cauchy stated that both infinity and infinitesimals were variable quantities. Thus, to him, an infinitesimal was something that got smaller and smaller while never actually reaching zero (meaning that its limit is equal to zero). Cauchy energetically rejected the notion of actual infinity and defined the irrationals as the limit of a sequence of rational numbers (Nunez, p.312,2010). For example, e can be defined as $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.However, the weakness of Cauchy's method for defining the irrationals is that he had no proof of the existence of the limit of the sequence. Secondly, his notion of limit was based on the real numbers which means that we cannot define the notion of a number through that of a sequence (Giannakoulias, p.52,2004). Furthermore, Cauchy realized the inverse relationship between the derivative and the integral.

Unlike many of his predecessors as Leibniz, Cauchy thought of the integral as the limit of a sum and not as a sum, a thought which lead him to prove the *Fundamental Theorem of Calculus*:

Theorem: Let $f(x)$ be a continuous function and $F(x) = \int_{x0}^{x} f(t)dt$, then $F'(x) = f(x)$.

Several ambiguities at the attempt to establish a rigorous Calculus would lead Karl Weierstrass (1815-1897) to develop his ideas on a pure arithmetical basis. As Archimedes avoided infinitesimals in his method of exhaustion, Weierstrass drove infinitesimals out of analysis. Hilbert (1925) lucidly refers to Weierstrass' contribution:

> *"…he removed the defects which were still found in the infinitesimal calculus, rid it of all confused notions about the infinitesimal, and thereby completely removed the difficulties which stem from that concept…this happy state of affairs is due primarily to Weierstrass's scientific work."* (as cited in Benaceraff & Putnam,1983, p.183)

Weierstrass supported the view that notions such as the limit, continuity, convergence, derivative and integral should be defined in a close relation with the system of the real numbers. Thus, he restructured mathematical analysis by creating an arithmetical theory of real numbers. He defined real numbers in terms of series of rational numbers and also gave a theory of irrational numbers, around 1860. By using Bolzano's techniques, he proved that every bounded infinite set of points has a limit point, that is, a point such that every interval around it contains infinitely many members of the set. Weierstrass's result is now called the Bolzano-Weierstrass theorem. The old infinity of infinitesimal would at the time be replaced by the new infinity of infinitely large collections (Lavine,2009, p.35).

2.8 Hilbert's Paradox and Dedekind's "cuts"

20^{th} Century began with the International Congress of Mathematicians in Paris on 8 August 1900. Hilbert presented at the congress 23 problems, among them the *Grand Hotel Paradox* which illustrates the counter-intuitive properties of infinite sets. By then, the logical foundation of Calculus was achieved by Dedekind's theory on irrational numbers.

The Grand Hotel Paradox

David Hilbert (1862-1943), in his lecture for the congress of 1900, would illustrate the counterintuitive properties of infinite sets by presenting an ingenious problem, the so-called *"Hilbert's Grand Hotel Paradox".*

Imagine a grand hotel where there are infinitely many rooms. One night a guest arrived asking for a room but the hotel was full-each room was occupied by one person. Hence, the manager requested the guest in *Room 1* to move to *Room 2*, the guest in *Room 2* to move to *Room 3*, the guest in *Room 3* to move to *Room 4*, the guest in *Room n* to move to *Room n + 1*. Since the hotel had infinitely many rooms, there was no problem in moving, there was always a room to move to. This left *Room 1* vacant, and therefore, the guest was accommodated.

The next night, a bus of 60 passengers arrived and they asked for one room for each passenger. The same thing happened. The manager requested the guest in *Room 1* to move to *Room 61*, the guest in *Room 2* to move to *Room 62*, the guest in *Room n* to move to Room *n + 60*. Since the hotel had infinitely many rooms, there was no problem in moving, there was always a room to move to. This left 60 rooms vacant and therefore the hotel accommodated the 60 new guests.

The next night a bus infinitely long with an infinite number of passengers arrived. The manager requested the guest from *Room 1* to move to *Room 2*, the guest from *Room 2* to move to *Room 4*, the guest from *Room 3* to move to *Room 6* and the guest in *Room n* to move to *Room 2n*. This left all the rooms with odd numbers vacant and therefore the infinite number of passengers were accommodated.

The next night, an infinite number of buses arrived, each of which had an infinite number of passengers. The manager assigned all the guests in the hotel to the prime number 2. He requested the guest in *Room 1* to move to Room 2^1 or *Room 2*, the guest in *Room 2* to move to Room 2^2 or *Room 4*, the guest in *Room 3* to move Room 2^3 or *Room 8*, the guest in Room n to move to Room 2^n.

Next, he assigned *Bus 1* to the second prime number which is 3, *Bus 2* to the 3rd prime number which is 5, *Bus 3* to the 4th prime number, and Bus n to the $(n + 1)$-th prime number. Possibly, the manager thought of Euclid's proof that there is an infinite quantity of prime numbers and decided that all the buses can be assigned in one-to-one correspondence with the prime numbers.

Now, each guest in each bus was assigned to the room number which is a power of prime, the prime number in which the bus is assigned to. For example, Passenger 4 of Bus 3 would be assigned to the Room 5^4. This means that each bus had a corresponding prime number and each passenger number had a corresponding power of prime. This means that each passenger in all the buses had a room in the hotel.

The paradox lies in the result that a fully occupied hotel with infinitely many rooms may still accommodate additional guests, even infinitely many of them, and that this process may be repeated infinitely often. While Hilbert tried to explain the properties of infinity through his paradox, he would mention 25 years later at the Westphalian Congress of Mathematicians(1925):*" ... the meaning of the infinite, as that concept is used in mathematics, has never been completely clarified…"*and comments on Weierstrass's analysis: *"... the infinite still appears in the infinite numerical series which defines the real numbers and in the concept of the real number system which is thought of as a completed totality existing all at once…"*(as cited in Benaceraff & Putnam,1984,p.183)

Weierstrass along with Dedekind and Cantor provided an arithmetic rather than a geometric ground as a foundation for calculus. Analysis would be shown to depend logically only on the properties of the natural numbers, what would be called by Felix Klein (1895), the *"arithmetization of analysis"* (as cited in Kleiner,2012, p.255). But despite the establishment of a rigorous calculus, the mystery of irrationals would remain open. Up to 1850, real numbers would be categorized as rationals and irrationals, algebraic[17] and transcendental. Irrationals would be defined as "not rationals" while transcendental numbers would be defined as "not algebraic". We see that irrationals and transcendentals were defined through the ambiguous definition of something that "is not" (rational-algebraic) while properties of these numbers were not clear.

Dedekind Cuts

Richard Dedekind (1831-1916) threw light on the properties of irrationals by using the concept of continuity. What Dedekind was searching for was the difference between a rational and an irrational on the real number line. Consider a number line where the set of rational numbers is placed in an ordered system. We can always find another rational number which is in between two

[17] *Definition:* Algebraic number is any complex number that is a solution of some polynomial equation whose coefficient are all integers

rational numbers. However, this line is not continuous because of the existence of the irrationals, creating "gaps" between the rational numbers. Dedekind in 1872, attempted to fill these "gaps" by looking at "cuts". In modern terminology, he wanted the rational numbers to be "dense" among the real numbers (we know in modern mathematics that the rational numbers possess the property of "denseness" but yet do not constitute a continuum).

Definition: A Dedekind cut is a subset C of the rational numbers Q with the following properties:

1.$C \neq \emptyset$ and $C \neq Q$

2.if $p \in C$ and $q < p$, then $q \in C$

3.if $p \in C$, then there is some $r \in C$ such that $r > p$ (i.e. C has no maximal element)

An intuitive explanation on a Dedekind cut(*schnitt*) is given by Mankiewicz (2000):

> *Imagine the line of numbers as a solid tube of finite length, filled with ordered rational numbers. A cut of the tube will give us two portions, A and B, and will reveal two cross sections (the edges of A and B). Seeing those exposed sides, we can read the numbers show us (one or the other). If they do no show us any number, then the intersection has become on an irrational* (p.150)

For example, we can think of $\sqrt{2}$ as two sets of rational numbers. Let $L = \{\alpha \in Q: a^2 < 2\}$ and $R = \{\beta \in Q: \beta^2 > 2\}$. Both L and R can be entirely defined and described within the rational number system. The pair of sets $\{L, R\}$ then defines what we call the Dedekind cut and
$\{L, R\} = \sqrt{2}$.

The entire set of reals can be constructed by taking all possible pair of subsets $\{L, R\}$ of the rational numbers, where L and R must satisfy certain conditions.

Richard Dedekind redefined the term "infinity" for use in set theory. With his new idea of "cuts" he managed to *"...define the real numbers in terms of infinite sets of rational numbers. In this way, additional rigor was given to the concepts of mathematics, and it encouraged more mathematicians to accept the notion of actually infinite sets"* (Dowden, n.d., par.1.b).

2.9 The triumph

Despite the fact that mathematics would reach a high and rigorous status up to the end of the 18th Century, infinity as a mathematical notion was lacking of precision. Galileo's paradoxes of the one-to-one correspondence between all the natural numbers and the squares of all the natural numbers to infinity or the paradox of the co-centric circles remained under the shadow of Galileo's facile conclusions. He concluded that concepts like less, equals and greater could only be applied to finite sets of numbers, and not to infinite sets. However, *Georg Cantor* (1845-1918) was not satisfied with Galileo's explanations.

Cantor, usually mentioned as "the creator of Set Theory", investigated the properties of actually infinite sets. For two finite sets, it is clear that a one-to-one correspondence between them can be set up if and only if the two sets have the same number of elements. This is not the case for infinite sets and Cantor showed that the set of points on the real line constitutes a higher infinity than the set of all natural numbers, that is, the astonishing fact that there are degrees of infinity. Cantor realized that he could pair up all the fractions (or rational numbers) with all the whole numbers (in the same way such as the natural with the even numbers etc.), thus showing that rational numbers were also the same sort of infinity as the natural numbers, pointing out this way that the properties of infinite sets are counter-intuitive (intuition says that there should be more fractions than whole numbers). Hence, the infinity of rational numbers is of the same size as the infinity of naturals and the set of rational numbers can be listed (i.e. the elements of Q can be in correspondence with N).

Cantor conceived an ingenious method for proving this time that real numbers cannot be listed or equivalently that ℝ is uncountable.

Theorem: ℝ is uncountable

Proof: The proof is based on contradiction, so we suppose that ℝ is countable. We accept that every real number x has a decimal expansion, $x = N.x_1x_2 \ldots$ In order to ensure uniqueness for the representation, we choose the convention: one agrees never to terminate the expansion of an infinite string of 9's. Otherwise, we cannot ensure uniqueness. For example, $N.4999 \ldots = N.5000 \ldots$ which are two different representations of one real number. As we assumed that ℝ is countable, we can make a list of all the elements of ℝ along with their decimal expansions:

$$\begin{aligned} x_1 &= N_1.x_{11}x_{12}x_{13} \ldots \\ x_2 &= N_2.x_{21}x_{22}x_{23} \ldots \\ x_3 &= N_3.x_{31}x_{32}x_{33} \ldots \\ &\ldots \end{aligned}$$

We now consider the real number $0.y_1y_2y_3\ldots$which is defined by $y_i = 1$ if $x_{ii} \neq 1$ and $y_i = 2$

$$y_i = \begin{cases} 1, \textit{if } x_{ii} \neq 1 \\ 2, \textit{otherwise} \end{cases}$$

This number differs from every x_i in the list in at least the i-*th* position. Thus, y_i is not in our list.

■

What Cantor proved, is that even an infinite set of numbers (in the previous case, the set ℝ) cannot contain all possible numbers.

He drove his theory even further, by introducing the word "transfinite" and the notion of transfinite *cardinal* and *ordinal* numbers in order to distinguish between the different degrees of infinity. He made use of the Hebrew letter aleph (ℵ) to denote with $\aleph_0$ the "transfinite" cardinality of the countable infinite set of natural numbers and with $\aleph_1$ the next larger cardinality, that of the uncountable set of *ordinal numbers*. Ordinal numbers were also introduced by Cantor in 1883.Simply put, an ordinal number is an adjective which describes the numerical position of an object, e.g., first, second, third, etc. For example, ω is defined as the lowest transfinite ordinal number and is the "order type" of the natural numbers. Cantor hypothesized that $\aleph_1$ is the cardinality of the set of real numbers, an assumption that lead Cantor to the idea that there is not a third kind of infinity between that of natural and that of real numbers.

That was the famous *continuum hypothesis* (1879) which Cantor had not been able to prove.

The continuum hypothesis would be included in the famous Hilbert' presentation of his 23 problems. Kurt Gödel in 1940, demonstrated that the hypothesis is consistent with the Zermelo-Fraenkel axiomatic set theory. Paul Cohen, an American mathematician, would prove in the early 1960's that the Continuum hypothesis is independent of the ZF axiomatic theory, i.e. one cannot prove if the hypothesis is true or false within the given axiomatic set theory. Cohen's method of proving his result remained controvertible until Gödel gave his stamp of approval in 1963.

As in many radical theories through the historical development of the notion of infinity, Cantor's theory had faced severe critique, especially by his old professor L. Kronecker and the French mathematician H. Poincare. In order to realize the tension of the controversy, we refer to Poincare's own words (1908) for Set Theory: *"Later generations will regard Mengenlehre [set theory] as a disease from which one has recovered"* (as cited in Kleiner, p.192, Cantor refused to be intimidated:

> *"My theory stands as firm as a rock; every arrow directed against it will return quickly to its archer. How do I know this? Because I have studied it from all sides for many years; because I have examined all objections which have ever been made against the infinite numbers; and above all, because I have followed its roots, so to speak, to the first infallible cause of all created things"* (as cited in Dauben,1990, p.298)

As Cantor aged, he suffered from mental illnesses (beginning about 1884), a fact which many authors would ascribe to *"his constant contemplation of such complex, abstract and paradoxical concepts"* (history). Cantor spent his last years in the Halle sanatorium, recovering from attacks of manic depression and paranoia, until he finally died in 1918.

2.10 End of the journey

We will stop at this point our journey through the historical development history of infinity, as we have reached what Hilbert called "Cantor's paradise". Through this review, we had the chance to realize the vastness of the notion of infinity as it appeared in mythological doctrines, philosophical discussions and scientific studies. It is no wonder that more than 2000 years had to pass for infinity to be established mathematically. However, in this finite world, the struggle to comprehend and conquer the notion of infinity remains vivid.

CHAPTER 3: Theoretical Framework

The historical review of the development of the concept of infinity in the previous chapter, helps us to realize not only the difficulty of understanding that lies in the nature of the concept itself, but also the many different ways that it had been perceived and conceptualized to finally reach an original mathematical formation. In this chapter, in an analogy with the different approaches taken through history to understand infinity, I will start by first investigating the nature of infinity and its contradictory attribute. Next, there will be an examination on the perception as a concept and as it appears in the cases where students deal with tasks involving infinity. Finally, I will present two theories that will give us the chance to have an insight in the mental processes and obstacles in understanding the concept.

3.1 On the nature of infinity

3.1.1 *What is infinity?*

An attempt to answer the question in a philosophical or physics context could fill hundreds of pages and still the answer may not be satisfactory. Nevertheless, we have already seen that the concept has reached the status of a mathematical object before going through a philosophical-scientific development.

Let us now have a look on the roles that the word "infinity" had in Greek culture:

1.As a *noun*, "infinity" was related to mythological, theological and metaphysical beliefs, like those attributed to the realm of the gods

2.As an *adjective*, describing the noun in terms of the absolute, like the universe, the being, space or time.

3.As an *adverb*, used to describe processes that were considered to be continued indefinitely, like the processes of extending, subdividing, adding etc. (Luis, Moreno&Waldegg, 1991, p.212)

It will be best to remain within the limits of a mathematics education research and give an answer combining the mathematical and cognitive aspects of the notion, independent from the "noun" role. We will see how the "adverb" and the "adjective" role is related to the potential and actual character of infinity respectively. That is to define *potential* and *actual* infinity by means of someone's understanding of the behavior of mathematical entities inherent to the concept of infinity.

The notion of potential infinity arises when someone realizes unending processes in mathematics. For example, the infinite process when someone starts from 1 and adds one in each step indefinitely, without stopping. Other examples are the indefinite extension of a line segment or creating polygons with more and more sides. We have also seen previously an evident appearance of the potential infinity where someone can create indefinitely regular pentagons by joining the intersection points of the diagonals of a regular pentagon. However, potential infinity plays a fundamental role in Calculus. As Luis et al. (1991) mention: *"...potential infinity subsists within mathematics as the modus operandi which constitutes the operatory nucleus of standard calculus"* (p.213). To be even more specific, we should have a look at some cases involving the concept of limit. One understands infinity in a potential way if he/she thinks of $0.\bar{9}$ as the $\lim_{n\to\infty} \sum_{n=1}^{\infty} \frac{9}{10^n}$. That is, we will keep adding 9's to 0.9 indefinitely, reaching arbitrarily close to 1. Potential infinity is inseparable from the notion of limit and there is an implicit use of it when someone for example attempts to find the $\lim_{x\to 0} \frac{\sin x}{x}$. While x is getting closer and closer to 0 but never equals to zero, then $f(x) = \frac{\sin x}{x}$ gets closer and closer to 1 or $\lim_{x \to 0} \frac{\sin x}{x} = 1$(for more on limiting processes c.f. Tall,1980).

According to the idea of actual infinity, infinity is often realized as an *object*. For Lakoff and Núñez (2000) *"…the interesting cases of infinity in modern mathematics are cases of actual infinity"* (p.158). For example, cases that include infinite sets, points at infinity, a transfinite number or the sum of an infinite series. In particular, someone understands the actuality of infinity if he/she thinks of the natural numbers as a set. In order to make a rough distinction between the actual and the potential idea, we have to think of the sequence of numbers which have no end. Then we are lead to think of potential infinity. While thinking that there are infinitely many numbers, we are thinking of actual infinity. As we have seen through history, establishing the idea of actual infinity had not been an easy task. Bolzano (1851/1950), in order to give answers to paradoxes of infinity, felt the need to refer to infinity as an attribute of collections(para.13). Several years later, Cantor (1874) would develop the concept of actual infinity through the transfinite cardinal number theory. Despite the Cantor's triumphant idea of the existence of different types of infinity, the contradiction within the notion remains. Fischbein, Tirosh and Hess (1979) noticed a remarkable fact: *"The world of* $\aleph_0, \aleph_1, \aleph_2$*...composed of actual infinities represents a potential, not an actual form of infinity…The contradictory nature of infinity can be pushed to higher levels but cannot be completely eliminated"* (p.4).

We see that even at the higher levels of thinking, such as Cantor's, contradictions *"cannot be completely eliminated"* (Fischbein et al.,1979, p.4). Reasoning from numerous historical debates and Fischbein's et al. remark, we can now safely attribute to the nature of infinity the term "contradictory". On the other hand, one could refer to the nature of infinity as "contradictory" stemming from the counter-intuitive properties of sets. In the next paragraph, I will be discussing this attribute and how it intervenes in intuition.

3.1.2 The contradictory nature of infinity

Fischbein et al. (1979) refer to the term "intuition" as the form of knowledge that is direct, formal and self-evident. For the case of infinity, the same authors make a clear distinction: *"Accepting definitions, theorems and logical proofs is one thing. Using the concept of infinity in various real, psychological contexts in the process of thinking and interpreting is another"* (p.3).

When Galileo presented his paradox of counting square numbers, in fact he argued that a "smaller" subset of an infinite set can, itself, be infinite. Of course, one can simply overcome this paradox by thinking in terms of the Aristotelian philosophy and thus, reject the existence of an actual infinity. Fortunately, Cantor had another opinion. He argued that there is no need to reject the actual nature of infinity. He made comparisons between infinite sets meaningful by saying that there are indeed infinite sets of the same size and other infinite sets that are larger than others. However, in both cases, we have the counter-intuitive property of infinite sets, that is, the whole can equal to its parts or what is known as the "part-whole" relationship.

The "part-whole" relationship is the reason that Galileo's case was characterized as a paradox. While Cantor has given a brilliant answer to Galileo, nowadays a student would not accept easily properties such as the "part-whole" relationship. Obviously, confusion is caused due to the every-day experience in a finite world. Luis et al. (1991) explain in a lucid way the "root" of this conflict by saying that: *"…*[it] *lies in the fact that the intellectual schemes of the individual stem from daily experience where it is obvious that the whole is always bigger than any of its parts"* (p.219). In the same line of thought, Monaghan (2001) distinguishes between two problems talking to children about the infinite. These are: a) the real world is finite and there are no real referents for discourse on the infinite and b) the language used within this finite world to talk to children about the infinite (p.240). These being said, the question raised by Núñez (2005) seems completely reasonable: *"How do we grasp the infinite if, after all, our bodies are finite, and so are our experiences and everything we encounter with our bodies?"* (p.1). Monaghan (2001) points out another reason for confusion and contradiction while comparing the cardinality of infinite sets. That is the fact of the dual nature of infinity or equivalently the "process-object duality". According to him, the "part-whole" relationship is object related (p.245). A student could easily conclude that since the set of even numbers is smaller (i.e. there are gaps between them) than the set of natural numbers. On the other hand, another possible answer could be that since both of the sets are infinite (i.e. we can keep counting natural or even numbers for ever) then it is not possible to compare them. A similar conclusion to Galileo's answer to his own paradox, who saw the sets of natural and square numbers as totalities, hence to him, a comparison was not applicable.

We have seen that the "part-whole" relation as a counter-intuitive result and the dual nature of infinity as the roots of the conflict in the process of understanding the concept of infinity. Thus, since mathematical infinity is a fundamental concept of mathematics and appears in several foundational concepts that are taught in a mathematics classroom, we will need to have a further insight at the student's thinking. This I will try to achieve by i) identifying students' perceptions related to the notion of infinity and adapting ii) the Piagetian Theory of Genetic Epistemology as a basis for the framework of APOS theory, iii) the idea of Psychological Recapitulation as a basis of the Theory of Epistemological Obstacles.

3.2 Identification of Students' Perceptions

The term "perception" is defined in Merriam-Webster dictionary as follows:

a: *awareness of the elements of environment through physical sensation*

b: *physical sensation interpreted in the light of experience* (perception.2017)

For example, an individual sees objects moving on four wheels and is aware that these objects are cars. On the other hand, when he is asked to draw a car, he will recall his past experience of watching cars to draw one. In line with the linguistic definition, Singer and Voica (2008) have transferred the notion of perception in their studies by making the distinction between *primary* and *secondary* perception. In an analogy, both perception and intuition aim at producing meaningful interpretations of the world. Perceptual representations give plausible explanations that can or cannot be contradicted by further experiences. Intuition, on the other hand, certifies and functions based on beliefs. Keeping in mind this distinction for the rest of the research, I will proceed at the definitions of the two kinds of perception, as given in Singer and Voica (2008).

Primary perception: is an active and spontaneous process by which human beings organize and interpret sensory information, independently of any instruction. For example, when one looks at a painting, he/she could only feel euphoria or other emotions. However, if the viewer is a painter, he/she will perceive the process of painting, the materials that have been used, the technique etc. The latter kind of perception is called secondary.

Secondary Perception: is a filter of selection, interpretation and representation of information, which is created by successive experiences, generated inclusively by systematical educational interventions.

In this study, I will be examining the students' primary perceptions by looking for spontaneous answers to the questions related to infinity (e.g. asking what does infinity mean to each student). Specifically, primary perceptions can be identified as *processional*, *topological* and *spiritual.*

Processional Perception: this perception corresponds to the potential infinity and functions as a modality to understand this nature of infinity (Singer&Voica,2008; Fischbein,2001; Fischbein et al.,1979; Monaghan,2001; Tsamir&Tirosh,1999). Fischbein (1987) relates the processional perception to *"dynamic/potential infinity"* and distinguishes between two dimensions of the perception (p.91).
-a *temporal* dimension: it is related to the perception that infinity is something with no end and impossible to be measured. Some possible expressions that point to a temporal perception are: "infinity is unending", "infinity is something that never ends"
-a *spatial rhythmic* dimension: it is related to the perception that infinity is something unending, something that keeps rising. Some possible expressions e.g. "infinity is something that never ends and keeps growing"." the closer you get to the stars the further they go"
Topological perception: this perception is connected to the conceptualization of infinity as a big entity, bigger than anything else. We consider that a topological perception manifests when the student evokes properties and transformations that are invariant to the change of shape. Singer and Voica (2008) refer to the representation of infinity by the number-line as a topological type of representation (p.196). Some possible answers that indicate a topological perception are: "*infinity is something huge*"," *enormous*", *"unlimited"*.
Spiritual perception: is the perception which is affected by feelings and emotions. As Lakoff & Nunez (2000) emphasized, infinity is the highest entity that encompasses all the other categories and is naturally extended to nature or religion. Some possible answers that indicate a spiritual perception are: *"Infinite is the love for my parents", "Infinity is something that no one can grasp", "Only God can reach infinity" "Infinity means absolute"*.

Apart from the primary perception, examining students' perceptions in depth might reveal a secondary perception of the notion of infinity. A secondary perception can be expressed in the case of the comparison of infinite sets. According to Tsamir (2001, p.290), students tend to apply certain criteria in their responses to different comparison-of-infinite set. These criteria are:

i) *the part-whole criterion*: a proper subset of a given set contains fewer elements than the set itself
ii) *the single infinity criterion*: all infinite sets have the same number of elements, since there is only one infinity
iii) *the "infinite quantities-are-incomparable" criterion*: two infinite sets cannot be compared
iv) *the one-to-one correspondence criterion*: a simplified version of the bijection criterion

Investigating both kind of students' perceptions is the first step for revealing an understanding of the notion of mathematical infinity. The next steps that should be considered can be taken by examining further in learning processes or structures. This necessity is apparent in Singer and Voica's (2003) conclusion: *"If we take into consideration recent researches in mind and brain, there is a close interrelationship between predispositions-intuitions and the learning process, which rebuild connections and structures* (p.6). Thus, we should be looking for an insight in the mental structures and mechanisms through which knowledge is built. For this, we should resort to the Theory of Genetic Epistemology.

3.3 The Theory of Genetic Epistemology

The Theory of Genetic Epistemology is a theory established by Jean Piaget (1896-1980) which studies the origins of knowledge. Ho (2008) puts it simply and refers to this theory as the development theory of knowledge acquisition (p.13). Piaget in his theory, thinks of knowledge in terms of stages and processes through which knowledge is formed. In Piaget's own words (as cited in Bringuier,1980): "*The study of such transformations of knowledge, the progressive adjustment of knowledge, is what I call genetic epistemology"* (p.7).

Piaget and Garcia (1962/1989), having a sophisticated view on knowledge, in the *"Psychogenèse et histoire des sciences"*(Psychogenesis and the History of Science), have elaborated the concept of genetic development. What the authors did in their work, is to identify the mechanisms of passage from a cognitive stage to another in order for the individual to be lead to the acquisition of knowledge. Piaget & Garcia refer to these mechanisms as the "transitional mechanisms". One of these mechanisms is what Piaget called *reflective abstraction.* It is the main mechanism for the mental constructions in the development of thought. Piaget wrote for the development of thought: *"The development of cognitive structures is due to reflective abstraction…"* while for mathematics he wrote: *"…it [reflective abstraction] alone supports and animates the immense edifice of logico-mathematical construction"* (Piaget,1985, p.143;1980, p.90). But what how does reflective abstraction work in mathematics? Piaget (as cited in Arnon et al.,2014), provide us with an example for the case of functions:

> *They are first constructed as operations that transform elements in a set, called the domain, into elements in a set, called the range. Then, at a higher stage, as elements of a function space, functions become content on which new operations are constructed. Integers are another example. At one stage, an integer is an operation or process of forming units (objects that are identical to each other) into a set, counting these objects and ordering them. At a higher stage, integers become objects to which new operations, e.g., those of arithmetic, are applied*. (p.6)

Such kind of examples lead researchers of mathematics education to the belief that reflective abstraction can become a tool in describing the mental development of more advance mathematical concepts. Specifically, for mathematics education, the most inherent theory to Piaget's Genetic Epistemology and the notion of reflective abstraction is the APOS theory.

3.3.1 APOS Theory

We have seen previously the correspondence between the nature of infinity and the conception of this dual nature. That is, the conception of potential infinity as a process and the conception of actual infinity as an object. APOS Theory will help us to understand the distinction between the potential and the actual, not only in the students' thinking but also in the historical development of the concept of infinity.

It is a constructivist theory of how learning a mathematical concept might take place. Arnon et al. (2014) call on Piaget's concept of reflective abstraction in children's learning as their main inspiration of the development of the theory (p.5). In fact, APOS Theory reformulates Piaget's ideas to fit the context of cognitive development in the level of pre-graduate and university mathematics.

Many researchers for several years had discussed concepts in mathematics as both processes and objects. In 1991 for example, Anna Sfard wrote an article called *"On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin"*. For Sfard, only when a process has been changed into an object can it in turn be operated on by other processes. Thus, one has to look on the learner's understanding of the concept of infinity.

Dubinsky & McDonald (2001) extended this process-object relation by adding two more levels of understanding, calling this theory APOS.APOS Theory assumes in total four mental structures called *Action*, *Process*, *Object* and *Schema* and the two basic mental mechanisms of *interiorization* and *encapsulation.* These levels constitute the acronym APOS. APOS theory postulates that a mathematical concept develops as one tries to transform existing physical or mental objects. Below I will explain the mental structures (Dubinsky & McDonald,2001, p.2-3):

Action: This level is characterized by the individual having an essential external perception of the mathematical concept. He/she is required to recall from memory or to follow step-by-step instructions to carry out a transformation. For example, in the case of the derivative of the function $f(x) = x^5$. A student requires the general expression $f'(x) = nx^{n-1}$ and can do little more than perform the action $f'(x) = 5x^4$. This student is considered to have an action understanding of the derivative.

Process: A student with a process understanding, repeats the action and then reflects upon it. He/she can make an internal mental construction called a process which will include performance of the same kind of action. Then we say that the action has been *interiorized* into a mental process. For example, in the case of the derivative of the function $g(x) = (x^5 + 1)^2$, the student will think of squaring the binomial $x^5 + 1$ and then that the derivative of $g(x)$ is the sum of $(x^{10})'$, $[2(x^5)]'$,$1'$. Then we say that the student has a process understanding of the derivative.

Object: An object is constructed from a process when the student becomes aware of the process as a totality and realizes that transformations can act on it. Then we say that the student has *encapsulated* the process into a cognitive object. For example, in the case of the derivative of the function $h(x) = (x^5 + 1)^7$, the student confronts a situation where he/she has to think of $h(x)$ as the composition of $f(x) = x^7$ and $g(x) = x^5 + 1$ by applying the process or action of composition of functions (depending on the level of understanding of the composition of functions). Then h(x) should be conceptualized as an object which arises from the composition of two functions. Now, the process understanding for finding derivatives must be encapsulated in the context of the chain rule to find the derivative $\mathrm{h}'(\mathrm{x})$.

Schema: According to Cotrill et al. (1996) a schema is *"a coherent collection of actions, processes, objects, and other schemas that are linked in some way"* (p.172). A mathematical topic often involves many actions, processes and objects that need to be organized and linked into coherent framework, which is called *schema*. It is coherent in the sense that it provides an individual with a way of deciding whether the schema applies in dealing with a mathematical situation. For example, it is the schema structure of the derivative that is used to determine the local extrema of a function, say $h(x) = (x^5 + 1)^7$. The coherence lies in understanding that to determine the local extrema of $h(x)$, one has to find: the derivative $h'(x)$, the critical points of $h(x)$ when $h'(x) = 0$.Then use these critical points to construct the sign diagram of $h(x)$ and finally determine the nature of the extrema of $h(x)$.

Moving on to describe further the mental mechanisms of *interiorization* and *encapsulation*.

interiorization: It is the mechanism that makes the mental shift from Action to Process. Interiorization permits one to be conscious of an action, to reflect on it and to combine it with other actions. (Dubinsky,1991, p.107).

encapsulation: encapsulation occurs when an individual applies an Action to a Process, that is, sees a dynamic structure (Process) as a static structure to which Actions can be applied. Dubinsky et al. give the following explanation: If one becomes aware of the process as a totality, realizes that transformations (explicitly or in one's imagination), then we say the individual has encapsulated the process into a cognitive object) (Dubinsky, Weller, McDonald, & Brown,2005, p. 339)

3.4 Psychological Recapitulation

What is of much interest to us in this study, is Piaget's idea to include in his theory another version of the biogenetic law of recapitulation, namely, the law of *psychological recapitulation*. What is psychological recapitulation? It is the belief that the students' intellectual development, traverses more or less the same stages as mankind one did. For Piaget, the mechanisms that function within the intellectual development are considered: *"as invariable, not only in time but also in space…they do not change from place to place or from time to time. They are exactly the same, regardless of the period in history and the place of individuals"* (Furinghetti & Radford,2008, p.630). Clearly, Piaget has adapted the idea of biogenetic recapitulation by the claim that the individual's cognitive development passes through the same stages as the cognitive development of great minds did through certain historical periods. In the following scheme, we see the way that the biogenetic law is being implemented in Piaget's Genetic Epistemology, as the law of Psychological Recapitulation.

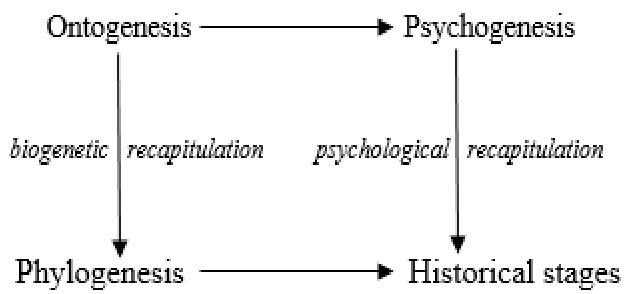


Figure 5. *The transposition of the law of biological recapitulation to the law of psychological recapitulation*

The very first starting point of Genetic Epistemology is the assumption that ontogeny recapitulates phylogeny, that is, an organism's fetal development follows the species 'previous evolutionary forms. This assumption had been extended to the field of cognition and mental activities as the law of Psychological Recapitulation. It is the idea that the development of mental functions, also known as psychogenesis, recapitulates the conceptual development of the mankind over several historical stages. As previously mentioned, Piaget & Garcia envisioned knowledge in terms of intellectual instruments and mechanisms. Adapting the law of Psychological recapitulation, they claim that the transitional mechanisms from one historical period to the next are analogous to those of one psychogenetic stage to the next. The Theory of Epistemological obstacles had been developed in line with this constructivistic approach to knowledge.

3.4.1 The Theory of Epistemological Obstacles

The concept of epistemological obstacle first emerged in Gaston Bachelard's article: *"La formation de l'esprit scientifique"* (The formation of the scientific mind) (1938). He supported the view that "*the problem of scientific knowledge must be posed in terms of obstacles*" (Bachelard 1938/2002, p.25) and grouped under the name of epistemological obstacles the limits which restrain the previous knowledge and which must be overcome and replaced by another form of knowledge. In other words, what we already know prevents us from discovering something new. Bachelard (1972) examined the idea of the epistemological obstacle and objected that the notion is not acknowledged within the field of education:

> *I have often been struck by the fact that the professors of sciences, more than others, do not comprehend that what they teach is not understood. They think that the scientific reasoning begins a lesson, that they can make a demonstration to be understood by repeating it point by point…* [in a physics class] *…it is not about gaining an experimental culture, but about changing it, about overcoming the obstacles accumulated from the daily life.* (as cited in Sălăvăstru,2014, p.34)

All of the above content of Bachelard's quote becomes clear when Hercsovics (1989) provides us with three epistemological obstacles, found from Bachelard's work:

- the tendency to rely on deceptive intuitive experiences
- the obstacles caused by natural language
- the tendency to generalize (as cited in Moru,2007, p.35)

Bachelard's quote is really rich, in the sense that brings back the "finite world" obstacle or the "real world experience" mentioned previously. One of the most concrete examples that showed us that intuition ran against learning was the result of Gabriel's horn (see 2.6) A deceptive intuitive experience, such as the experience of painting a horn, Bachelard (1938) would call as the obstacle of the *"first experience"* in learning, an obstacle that occurs by an experience which has not undergone a rational critique or observation. (as cited in Chimisso,2013, p.202). Sălăvăstru (2014) mentions for the "first experience": *"It would be delusional to build the learning process without taking into account the previous knowledge of the students, knowledge more or less correct, often contaminated by the imagination, affectivity, environment and so on"* (p.35).

As for the obstacles caused by natural language, we will have to look back to Monaghan's reasoning on why talking to children about infinity, becomes problematic. I will not be mentioning again the two problems but one could completely grasp the "natural language" problem if he/she thinks of the mathematics teacher entering a classroom and starting to teach in a language created in the finite world.

Finally, I would like to provide an example out of the mathematics education research limits but within Bachelard's philosophy of science for the "tendency to generalize". Simply, one cannot understand a foreign culture by meeting a madman and then conclude "Yes, I know, everyone is crazy in this country!".

Almost 45 years after Bachelard's work, Guy Brousseau (1933-) would integrate the idea of epistemological obstacles into his Theory of Didactical Situations (1970) in mathematics. The Theory of Didactical Situations(TDS) examines the relationships that occur in the triadic relationship of *teacher-student-content* (Sriraman, & Törner,2008, p.669).

Brousseau (1997) describes a general apparent obstacle as follows:

> *Errors are not only the effect of ignorance, of uncertainty, of chance, as espoused by empiricist or behaviorist learning theories, but the effect of a previous piece of knowledge which was interesting and successful, but which is now revealed as false or simply unadapted. Errors of this type are not erratic and unexpected, they constitute obstacles. As much in the teacher's functioning as in that of the student, the error is a component of the meaning of the acquired piece of knowledge* (p.82)

Thus, for Brousseau, an obstacle as well as a piece of knowledge is the result of the interaction between the three-way schema mentioned previously. An obstacle is of the same nature as knowledge and behaves in the same manner. To overcome an obstacle is to work in the same way as applying knowledge. It requires repeated interaction and a discourse between the student and the object of knowledge-the obstacle. Brousseau (1983) classifies the sources of obstacles as follows:

(1) an *ontogenetic source* (related to the student's own cognitive capacities, according to their development i.e. their age)
(2) a *didactic source* (related to the teaching choices)
(3) an *epistemological source* (related to the target knowledge) (as cited in Radford,1997, p.29)

In this study, I will make use of obstacles of an epistemological source, namely the *epistemological obstacles* while I will refer to the other two if necessary. In his widely-quoted paragraph, Brousseau (1997) throws more light on the notion: *"The obstacles that are intrinsically epistemological are those that cannot and should not be avoided, precisely because of their constitutive role in the knowledge aimed at. One can recognize them in the history of the concept themselves"* (p.87). Hence, an epistemological obstacle is strongly inherent to the concept to be taught while has its roots to the nature of the concept and what I mean by "nature of the concept" is the structure of the mathematical concept through its historical development.

We can list now the two essential characteristics of epistemological obstacles:

- epistemological obstacles occur both in the historical development of scientific thought and educational practice
- they are unavoidable and essential for the acquaintance of the target knowledge

For the first point, consider the difficulty of the conceptualization of non-natural numbers as members of the same family as natural numbers which had been apparent in history and was further extended in the conceptualization of irrationals. In relation with the second point, such an obstacle arises when understanding certain mathematical concept interferes with the understanding of a more complex one. For example, understandings of natural numbers interfere with the understanding of fractions (cf. Cortina, Visnovska, & Zuniga ,2014). To make things clearer and for the sake of preciseness on the notion of an epistemological obstacle, I will be choosing the safe way of referring to examples already given by Cornu (1991). Cornu has given the following examples of epistemological obstacles of the past related to the history of the limit concept:

- the failure to link geometry with numbers
- the notion of the infinitely large and infinitely small
- the metaphysical aspect of the notion of limit
- is the limit attained or not? (p.159-162)

Having a look at the historical development, the above obstacles had been indeed the stepping stones for mathematics until Cauchy and Weierstrass would introduce the limit concept. We have already seen the debates and the confusion caused by notions such as the indivisibles or infinitesimals while the question *"is the limit attained or not"* still echoes in a mathematics classroom.

3.4.1.1 The function of history in the Theory of Epistemological Obstacles

Identifying epistemological obstacles is far from being an easy task. Nevertheless, it is not always the case that a difficulty for conceptualization found through history will also be constituting a difficulty in a student's thought. Fischbein, Jehaim and Cohen (1994), conducted a research on the irrationals and their possible corresponding epistemological obstacles by the following assumption: the concept of irrational numbers encounters obstacles which render difficult their understanding and acceptance as it happened in the history. Indeed, such obstacles exist but are not of epistemological nature. It seemed that students were not disturbed intuitively by the idea of incommensurability, thus, his assumption that the obstacles are of a primitive nature (i.e. epistemological) was brought down.

It is appropriate here to make a fundamental distinction between a *fact* and an *obstacle* (Bachelard 1938/2002, p.27). For a historian of science, it is a fact that up to Cantor, the "part-whole" relationship constituted a paradox. From an educator's point of view, the same relationship had been an obstacle to thought, until the development of the transfinite numbers in Cantor's thought. After my attempt to give the way that history functions in an epistemological framework, I should proceed at Brousseau's (1997) suggested guidelines for a research on epistemological obstacles:

From the outset, therefore, researchers should
a) find recurrent errors, and show that they are grouped around conceptions
b) find obstacles in the history of mathematics
c) compare historical obstacles with obstacles to learning and establish their epistemological character (p.99)

What Brousseau suggested in other words (Vamvakoussi, Vosniadou, & Van Dooren,2013) (and this will be done for this research), is to trace down obstacles in the historical development of the notion of infinity and compare them with obstacles in learning manifested through recurrent errors (p.314).

The idea that some of the difficulties in the historical development of a concept are met also in the individual's cognitive development implies in a broad sense, the acceptance of the biogenetic law. In a psychological version, we can say that the idea of epistemological obstacles implies the acceptance of the law of psychological recapitulation. We observe at this point, that the Piagetian view on knowledge is also a part of Brousseau's view on obstacles, mentioned often in the literature as the Piagetian side of Brousseau's epistemological obstacles. One can notice this in the following quote: *"…in each progression, what gets surpassed is always integrated with the new…"* (Piaget&Garcia,1962/1989, p.28). In conclusion, we find two points where Piaget's theory is implicitly used in the Theory of Epistemological Obstacles: a) epistemological obstacles are both found in the historical and cognitive development of a concept, b) an obstacle in learning is a piece of knowledge, obtained by the function of certain mechanisms. In other words, we see a constructivist approach to learning, in both of the theories.

3.5 Refining and implementing the theory

Furinghetti and Radford (2008) mention that the study of the development of student's thinking belongs to the psychological domain while the conceptual development belongs to the historical domain. In this study, the part of the students' primary and secondary perception of mathematical infinity, has been investigated by means of interviews and questions. The results will be further analyzed in the context of APOS Theory. For the latter part of the historical domain, historical records on the conceptual development of infinity, are the only material for study. These two parts can be connected by the law of psychological recapitulation, which as we have seen previously is a transposition of the biological law of recapitulation. However, I will not be looking for parallels between student thinking and historical thinking about the concept of infinity. This would be a really simplistic view on the law of Psychological Recapitulation without any positive implications for mathematics education. What will be done, is to relate epistemological obstacles to student's errors and obstacles as proposed previously, in Brousseau's theory. The development of the theory and its implementation in my study, is given in a nutshell in the next diagram:

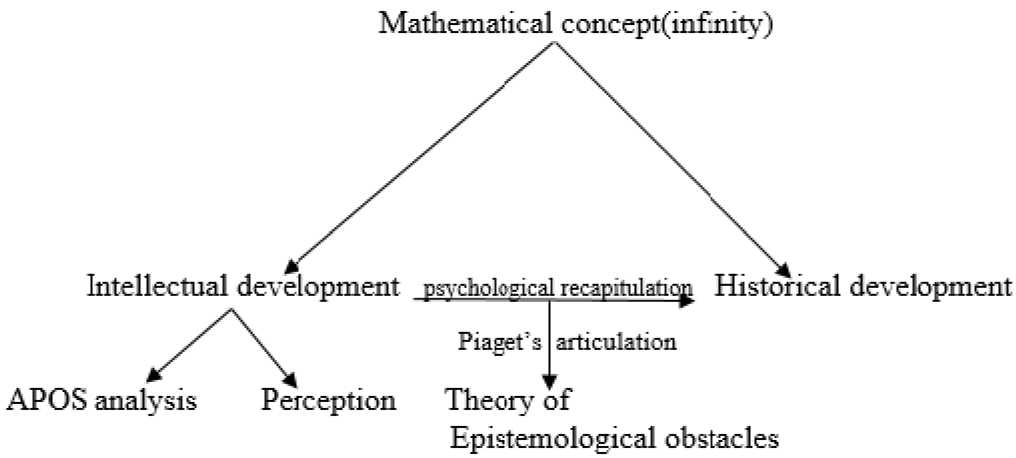


Figure 6. *Implementation of the theoretical framework*

As seen in [Figure 6], the concept of infinity has been decomposed according to the theoretical framework of this study. Firstly, the concept is broken down in two domains in relation with students' understanding. That is, the psychological domain of the concept, which corresponds to the individual's intellectual development and the historical domain which corresponds to the historical development of the concept. The two domains are connected through the law of Psychological Recapitulation as explained previously. However, as Radford et al. mention*:" …the theoretical framework has to ensure a fruitful articulation of the historical and psychological domains as well as to support a coherent and fecund methodology…"* (p.143). Thus, the two domains, have been articulated through Piaget's genetic epistemology, which functions as the basis for the Theory of Epistemological Obstacles. On the left-hand side of the diagram, the intellectual development is being examined through mental structures and mechanisms (APOS analysis) as well as through identifications of students' perceptions (in context of the section 3.2). The examination of students' understanding of the mathematical concept, lies in the basis of the above diagram which constitute the theoretical framework of the study and supports the methodology, in the way described in Chapter 4.

[This page is intentionally left blank]

CHAPTER 4: Methodology

The current chapter of this qualitative research will describe in depth the research methods that have been chosen as the most appropriate. Several topics are analyzed, such as actions that took place during the conduct of the research, methods found through literature and instruments for data processing-analyzing and ethical issues. Furthermore, external and internal factors that may affect the credibility of the study are examined.

4.1 Research Design

In an attempt to keep my research within a strictly scientific framework, I have resorted in researches on the methodology of conducting a high-quality mathematics education research. Trying to find a methodology that would result in a structurally and functionally well-defined thesis, I have followed concrete guide lines given by academic mathematics educators (Battista, Smith&Boerst,2009; Cobb;2007; Harel;2006; Lester,2005). Such guiding lines or principles are given by Harel (2006) on his review of Lester's paper (2005), described as: *"Three guiding principles for researchers to think about the purpose and nature of a mathematics education research*" (p.60). Below, I mention these principles and describe the corresponding ways that these principles have been followed for this study:

1.The *goals of mathematics education research(MER) are to understand fundamental problems concerning the learning and teaching of mathematics and to utilize this understanding to investigate existing products and develop new ones that would potentially advance the quality of mathematics research*: The research was designed in order to investigate the students' perception on the concept of infinity and infinite sets, that is to investigate students' understanding of mathematical infinity. Next, the research moves on to investigate the obstacles in this understanding by looking at their nature and thus, to find epistemological obstacles in coming to understand the concept of mathematical infinity. Finally, this understanding is investigated by the implementation of APOS theory, by looking on the mental structures and mechanisms that appear on specific mathematical problems related to the concept. In the conclusion part, the research questions will be answered with the purpose of contributing in the broader field of mathematics research.

2.*To achieve these goals, MER must be theory based, which means studies in MER must be oriented within research frameworks*: The study is oriented within the theories mentioned in Chapter 3 of the Theoretical Framework and makes use of these theories in order to give answers to the research questions.

3.*The research framework's argued-for concepts and their interrelationships must be defined and demonstrated in context, which must include mathematical context*: The theoretical concepts and their interrelationships have been analyzed in Chapter 3 of the Theoretical Framework. In this chapter, the concept of perception is demonstrated in the context of the nature of mathematical infinity and that of infinite sets. Furthermore, I have presented four examples of epistemological obstacles related to the limit concept as given by Cornu (1991, p.159-162), aiming to demonstrate the theory of Epistemological Obstacles in a mathematics context. Finally, the mental stages of APOS theory are explained in the context of the derivative of a function.

The research dissertation can be characterized as *Qualitative Cognition-Focused Research* (Battista et al.,2009, p.219).

For Battista et al. such research:

- describes the nature of student's conceptions
- pinpoints mechanisms for learning
- pinpoints causes for mislearning

The three points mentioned above are covered within the theoretical framework and its application on my data analysis at the discussion part of this thesis. The mental stages of APOS theory describe the nature of student's conceptions, the mechanisms of interiorization and encapsulation are mental mechanisms that function in a learning progression while epistemological obstacles could be "causes for mislearning". Moreover, this qualitative-cognition study focuses on students' learning of a particular mathematical topic, that of the mathematical infinity. Thus, it can be also characterized as a *"Topic-specific cognitive study"* (TSCS) (for more on TSCS cf. Cobb,2007).

4.2 Pilot Study

After the presentation of the topic of my thesis in Week 41,2016, I have written the "project outline", as required from the Department of Mathematical Sciences. The purpose of the project outline is two-folded, it is meant 1. to convince the school of the viability of the thesis and whether or not all of the requirements regarding the topic`s theme, problem and your theoretic and methodical approach have been covered, 2. to function as a guiding tool for the further development of the thesis ("Project Description", n.d., par.3)

Later on, in week 4,2017 I have given a presentation of my research done up to the day of the presentation in order to get feedback and possible comments. The audience consisted of professors of the department as well as fellow master's students. I presented also a questionnaire (Appendix A), presented as an initial thought for an instrument of survey. The initial questionnaire was given to fellow master's students of Mathematics Education, in order to get a general feedback on the relevance to the topic in question and a specific feedback according to the school curriculum. That is, feedback on whether students are aware of notions used in the questionnaire. A common remark was that the term "cardinality" of a set would probably be unknown to the upper secondary school students who participated, hence it was changed to "number of elements". I have also been advised that I should better use the Norwegian word *"mangekant"* for referring to the polygon, since "polygon" would probably be an unknown word to most of the students. Hence, the word "polygon" was replaced by "mangekant". As also suggested by a fellow master student, I have moved TASK 1 to the end of the questionnaire, as TASK 1 possibly would affect the students' next responses. For the shake of preciseness in the tasks, I have also made some amendments at the presentation of each task, thus the questionnaire was delivered as seen in Appendix B[18].

4.3 Setting and participants

The group that was studied consists of five upper secondary school students. Specifically, the participants are *VG2* graders (i.e. 12th graders) of the Norwegian *Videregående Skole* (ages16-19). Moreover, the students have chosen to follow the course *"Matematikk R1"* (Mathematics R1) which is considered to be the most advanced course of the VG2 level. I have been informed by the teacher of the class that the students had a good use of the English language and that they were high achievers in mathematics. Therefore, the class was chosen as an "information rich" case.

[18] The blank space given in the questionnaire for the student to write his answer has been shortened in the appendix B, for the sake of brevity.

Information rich cases are those from which one can learn a great deal about issues of central importance to the purpose of the research (Patton, p.169,1990)

In particular, there have been a use of the technique of *Random Purposeful Sampling* for choosing out of an available sampling frame. This sampling takes random subset of participants from a population of interest, and lends credibility to a study (Purposeful Sampling, n.d.). That is, the 5 upper secondary students have been selected after the classroom teacher's inquiry for five volunteers. Thus, the size of the sample reflects on the purpose of investigating the students' understanding of infinity.

I should note here that after the presentation at week 4 and after a discussion with my advisor I was aiming for a sample size of 15 students (5 upper secondary,5 1st year bachelor,5 3rd year bachelor students). In my inquiry for university volunteers out of a 50 students pool sample, there has been no response. This resulted to a modification of the scope and the title of the thesis.

4.4 Instrumentation

Two instruments were used as data gathering tools while my data were gathered in two sessions. In the first session, participants were asked to fill a questionnaire designed by the researcher while in the second session participants were interviewed in a duration of approximately 10-15minutes.

4.4.1 *The Questionnaire*

The questionnaire consisted of several "tasks" and questions taken out of previous research related to the topic of the thesis. The questionnaire was comprised of four tasks that identify perceptions related to the concept of infinity. Below I will provide the subject of investigation of each task, the aim of each question as well as the sources of the tasks and questions.

TASK 1: Task design for this task was informed by considering and using parts of tasks in Fischbein,2001; Dubinsky et. al, 2005; Makri,2015. The task consisted of four questions related to the understanding of the density of rational numbers, one question on the cardinality N1 and a question on the irrational number π. Students were asked how many numbers are there in $(1,2), (1,3)$ and then compare the cardinality of these intervals. Next came the question of how many numbers are there in (0.8,1.1) for a possible reconsideration of their previous answers. Then they were asked to mention the number that is closest to 2. The purpose was to bring up the issue of $1.999 ... = 2$ for a further discussion in the interview. Finally comes the question on π and its infinite decimal expansion. The students are asked if they agree that π has infinite decimal places. The question aims at the examination of a student's perception both on the transcendence of π but also on infinity as it appears on the decimal expansion of π.

TASK 2: Task design for this task was informed by considering and using parts of tasks in Dubinsky et al.,2005; Duval,1983; Kattou, Michael, & Kontoyianni,2009; Tsamir & Tirosh, 1996. The task consisted of two questions and aimed to shed light on the comparison and understanding of infinite sets. That is to investigate the possible use of the criteria of "part-whole" relationship and "one-to-one" correspondence. In the first question, students were asked to do five comparisons of infinite sets:

a) between the set of natural and even numbers, as the two sets were represented verbally
b) between the set $\{1,2,3,4 ...\}$ and the set $\{1, \frac{1}{2}, \frac{1}{3}, ...\}$ in a horizontal representation of the sets
c) between the set $\{1,2,3, ...\}$ and the set $\{1,3,5,7, ...\}$ in a vertical representation
d) between the set of squares with sides $1cm, 2cm, 3cm$ ... and the set $\{1^2, 2^2, 3^2, ...\}$ in a combination of horizontal-geometric representation

e) between the set of natural numbers and the set of real numbers as they were represented verbally. The comparison was put on position e) on purpose. After students would compare equivalent sets, then came the comparison of the natural and real numbers for examining the intuition behind the “smaller” and “bigger” infinity.

The different representations were used to examine the impact in the selection of a criterion to determine the equivalence of infinite sets.

In the second question of the task, the students were asked to determine the number of elements of the set $\{-3, -2, -1, 0, \{1,2,3 \ldots\}\}$ or in other words to investigate whether an infinite set can be conceived as a single entity.

TASK 3: Task design for this task was informed by considering and using parts of tasks in Fischbein et al.,1979; Makri,2015. The task consisted of four questions related to the contradictory nature of infinity as it appears on geometrical representations. In particular, at the first question is asked whether a line segment could be divided indefinitely. At the second questions, students are asked to imagine the result of a regular polygon with as many sides as possible, implicitly bringing up the examination of the conception of limit and infinitesimals. That is, as the sides of the regular polygon increase it approaches the shape of the circle by limit, while the circle will consist of an infinite amount of infinitely small sides. The last two questions were related to the notion of actual infinity. The students were asked on how many lines can they draw through a line while the last one, was a geometrical representation of infinite points. The students in the last question were asked to compare the points of two circles, one of larger circumference than the other.

TASK 4: Task design for this task was informed by considering and using parts of tasks in Pehkonen, Hannula, & Maijala, 2006; Makri,2015. The task consisted of two questions related to infinity as a notion and two questions related to the understanding of infinitely large numbers and infinitesimals. In particular, the students were asked what infinity means to them (independent of a mathematical concept) and to make a sentence with the word infinity. On the other two questions, the participants were asked to mention what is the biggest and the smallest number that they know. The four questions aimed to categorize students’ perceptions of infinity and reveal a possible view of infinity as a number. I should mention that the latter questions brought up an ambiguity concerning the “smallest” number notion. Even if the use of the word “smallest” was used inadvertently, it gave me the opportunity to emphasize on the importance of language the size of a number (the issue will be described briefly in the Discussion session).

4.4.2 *The Interview*

The interview functioned as a follow-up session to the first session of the questionnaire. After the document analysis, the interview session took place approximately one week after the completion of the questionnaire. The interview provided an opportunity to investigate the participants’ perception further and to gather data which have not been obtained by the analysis of the questionnaire (Sharma,2013, p.51). The interviews were of *semi-structured* character: the interviewer followed an “interview guide” which contained questions and topics that needed to be covered and which had not been covered in the questionnaire. When the answers given pointed to obtaining further information, the conversation strayed from the guide for the sake of new information (Cohen & Crabtree,2006). I have taken under consideration Sharma’s (2013) view on an interview: “*The interview method takes the form of a dialogue in which the researcher seeks to elicit information form the subject about how the latter thinks”* (p.51). Furthermore, I have tried to include some important aspects: a) to maintain a relaxed manner b) to ask clear questions c) to keep notes d) to establish trust e) to keep track of the responses (Sharma,2013, p.51).

In particular, for the two written questions related to APOS theory (1.5,2.2), I have designed the follow-up questions of the interview session according to Arnon et al. (2014). I have created an outline of questions, keeping a semi-structured character of the interview. The authors mention: *"Interviews are the most important means by which data is gathered in APOS-based research."* (p.95) As usually happens in an APOS-based research, students were asked to clarify their responses and/or to expand on them. If the questions failed to elicit sufficient responses, the interviewer took a more didactical route and gave hints to observe in which particular mental construction the student is. Further details will be given in the Results part, where we will be able to see the functionality of the interview instrument within the components of an APOS based study, as well as to see the didactical route that sometimes had to be taken during an interview.

4.5 Procedure

Before starting collecting data, I have submitted within the predetermined time, a report-form (Appendix C) with information about my research study and data collection to the Norwegian personal security commission(NSD). The form was followed by a *Change Request Form* (Appendix D) for changes that were subject to notification. The report-form was approved(no.52487) and I moved on with my data-collection. The previous actions are compatible with the Law on Personal Information.

For the group of 5 upper secondary students, my advisor contacted the Head of the school as well as the students' mathematics teacher. As decided with the teacher, there have been an inquiry by the teacher for volunteers. The volunteers were given a form of consent attached to the *Information Letter* (Appendix E). The form of consent was handed to them by the teacher. In consultation with the teacher about the time and place for the students to fill the questionnaire, I have visited the school and delivered the questionnaire. The questionnaire was delivered face-to-face to the upper secondary students and was filled under a typical supervision by the researcher. The interview took place one week later, within the school building and was audio recorded with a smartphone using the Samsung Voice Recorder (2017) application. The participants were interviewed individually in an approximately 12 minutes' time for each, in a classroom indicated by the teacher.

4.6 Data Processing and Analysis

Being myself an amateur researcher in the field of mathematics education, I searched for concrete definitions of data processing and analysis to have an initial view on the actions and processes involved. As found in the website of the University of Tartu:

-Data processing**:** *A series of actions or steps performed on data to verify, organize, transform, integrate, and extract data in an appropriate output form for subsequent use. Methods of processing must be rigorously documented to ensure the utility and integrity of the data.* ("Research Data Management", n.d., par.2)

The data processing has been done in two steps, corresponding to the data gathered by the written answers of the students and the interview responses. Once I had distinguished firmly the questionnaires student wise I proceeded at the examination of the answers given. Firstly, the students' mathematical errors were highlighted, in order to be used at the search of epistemological obstacles. Then, I have proceeded at looking for answers difficult to understand or difficult to tell as the answers were hand written. Most important is that I have prepared questions for the interview related to the written answers given, aiming to clarify some of the written answers and that would point to certain concepts of the theory.

The audio data were filed in my personal laptop and for their audition I have made use of headphones. I have tried to pinpoint key words that would relate students' responses to the theoretical framework. For this reason, I kept notes during the audition of the data and repeated whenever needed.

For the questions related to APOS theory (1.5,2.2), I used the written answers in the design of interview questions because of their ability to reveal student difficulties that require further analysis. The audio recordings were converted to high quality audio files and the responses were transcribed carefully, including the sounds that students made, intervals of silence and words that cannot be heard clearly (Arnon et al.,2014, p.96)

- Data Analysis: *Data Analysis involves actions and methods performed on data that help describe facts, detect patterns, develop explanations and test hypotheses. This includes data quality assurance, statistical data analysis, modeling, and interpretation of results*. ("Research Data Management", n.d., par.2)

The data analysis was carried out according to the theoretical framework used. Next, I will present each data analysis related to the three research questions and their corresponding means of data analysis.

➢ Data analysis related to the first research question

The theory that I used concerning the students' perception, pointed out to circular process for the qualitative analysis. Hence, I have resorted to Dey (2003) and the circle of qualitative data analysis (p.32). Day describes three points on the circle of the analysis process. The points as implemented in the qualitative analysis through the lens of perception are described:

Describing: The description vertex lies in the description of the categories and criteria that point to a students' perception. Meaning that the theoretical concepts describe the phenomenon of the occurrence of infinity as a concept in a students' mind.

Classifying: The students' answers have been classified according to their perception on the nature of infinity and on the infinite sets. Specifically, the students' perceptions were classified according to the answers that pointed to Singer & Voica's categories of perception. For the infinite sets, the answers have been classified according to the criteria used for their comparison(Tsamir,2001).

Connecting: At this point, the students' answers are related to the concepts of theory, meaning the interrelationships, for example between the potential nature of infinity and a processional perception or the potential nature of infinity and the use of the single infinity criterion.

The above points will be highlighted in each part of the data analysis for the first research question. Below the circle of a qualitative data analysis is presented:

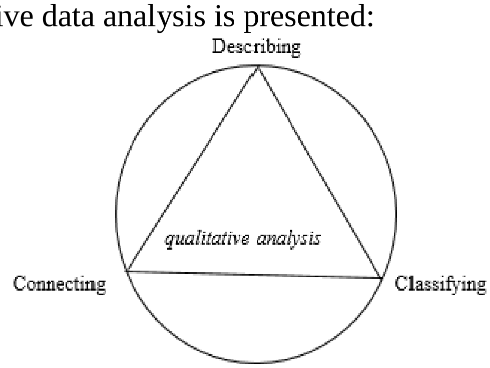


Figure 7. "*The core of qualitative analysis lies in these related processes of describing phenomena, classifying it, and seeing how our concepts interconnect."* (Dey, p.31,2003) Adapted from Qualitative *data analysis: A user friendly guide for social scientists* (p.32), by I. Dey,2003, London: Routledge. Copyright [1993] by Ian Dey.

➢ Data analysis related to the second research question

The theory used in this part of the data analysis is the theory of Epistemological Obstacles. As mentioned previously, in order to analyze my data for finding epistemological obstacles I will be using Brousseau's method provided for researchers. Hence the actions for analyzing the data were in the following sequence:

i) Find the students' errors and group them in relation to the conception that might have provoked these errors

ii) Resort to the historical analysis of Chapter 1 to find obstacles in the conceptual development of infinity through history

iii) Find out whether an obstacle is epistemological, looking at the nature of the obstacle as it appears in history and in understanding

➢ Data analysis related to the third research question

As this analysis will be done through the lens of APOS theory, I will be following the instructions that are given within the theory. The main guide for the analysis will be a theoretical analysis of the mathematical concept in question (in the context of APOS theory this analysis is called *genetic decomposition*) as an assessment of whether the student's understanding have followed the stages of the decomposition. As Arnon et al. (2014) mention: *"The data…analysis phase is crucial for APOS-based research, since without empirical evidence, a genetic decomposition remains merely a hypothesis."* (p.95) The way that the genetic decomposition functions in the data analysis is further described in the Results part.

4.7 Ethical Considerations

Permission to involve the participants in the study was sought from both the participants' teacher and the participants themselves. As mentioned earlier, the participants signed the form of consent, given to them by their teacher. The questionnaire data collected from the participants is kept in files and the only person who has access to these files is the researcher. The interview audio files are kept in a secured personal computer that can only be accessed by the researcher. For the sake of anonymity, the participants' names are imaginary. By the completion of the project, data will be destroyed.

Work that is not original (i.e. researcher's work) has been referenced and in-text cited in *American Psychological Association* 6th edition style (2009). For the creation of the reference list, I have used EndNote™ X8(2016) as shown in an one hour lecture during the autumn semester-2016.

4.8 Reliability and Validity

Credibility of the research studies rests on the reliability of their data, methods of data collection and also on the validity of their findings (Lecompte & Preissle, 1993; Seale,1999; Cohen et al.,2000; Silverman,2001; Moru,2006)

4.8.1 Reliability

✓ *External Reliability*

External reliability concerns the replicability of scientific findings. That is, in what degree will a researcher using the same research methods will obtain the same results. No study is able to attain perfect external reliability though (LeCompte &Goetz,1982; LeCompte & Preissle,1993; Moru,2006).

In order to enhance the external reliability of this study, I will examine the five threats to external reliability as given by LeCompte &Preissle (1993, p.334-335):

Researcher status position (*"To what extend are researchers members of the groups being studied and what positions do they hold?"*): in the current study, the researcher is a master's student, not a member of the group of upper secondary school students, hence the researcher's status position is not a factor that affects external reliability

Informant choices: a researcher who would like to replicate the results of this study should be contact individuals similar to those who participated in this study. Information on the participants' and how they were selected can be found in the previous sections of the Methodology art.

Social situations and conditions: The content of the data could be influenced by the social context within which they are gathered. As Becker, Geer & Hughes (1961) demonstrate in their study that data differentiated between the data they gathered while alone with participants and what they acquired from participants in groups. The fact that the participants worked on the questionnaire and were interviewed individually, might have influenced their answers. The conditions under which the participants filled the questionnaire were not that different from those of an ordinary exam or test. The interview was held in a relaxed and friendly manner, maintaining characteristics of a discussion, which probably resulted to honest and reliable to investigate answers.

Analytic constructs and premises: Constructs used in the study were developed throughout Chapter 3. Such constructs include: epistemological obstacles, perception, actions, processes, objects, schema, the infinity and infinite set concepts etc. The reference of these constructs in the study shows how the researcher had conceptualized them. The interpretations given to these constructs though, is not universal and is influenced by factors such as experiences, beliefs, prior knowledge, etc. For example, to avoid misinterpretation of the biological term genetic, the term genetic decomposition has been defined in the context of APOS theory. In a mathematical context, the cases of infinite sets are discussed with the naive set theoretical language. The results of the study are not considered reliable for replication for a reader who would use the formal logic language of axiomatic set theory on infinite sets. Thus, I have tried to give clear definitions and hypotheses, aiming to maintain external reliability in a high level.

Methods of data collection and analysis: Reliability related to this factor depends on the potential for subsequent researchers to reconstruct data collection and analysis strategies. For this reason, I have resorted to other researchers, whose methods have already been replicated (implying external reliability) and used their methods as an operating manual.

✓ *Internal Reliability*

Internal reliability concerns the degree to which researchers applying similar constructs would match these to data the same way as the original researchers (LeCompte&Pressle,1993; Seale,1999; Silverman,2001).

To reduce threats to internal reliability, I have made use of the strategy of *Low-inference descriptors* (LeCompte & Preissle, 1993). This involves: *"Verbatim accounts of participant conversations, descriptions phrased concretely and precisely as possible from field notes or recordings of observations, and such other raw data as direct quotations from documents…"*. In this study, the strategy of low inference descriptors was followed by audio recording all face to face interviews, carefully transcribing the recordings, and presenting extracts of episodes in reporting the results. Moreover, some written answers have been directly quoted in the Results section. This way, the person who wishes to duplicate the research, gets a further insight of the topics investigated.

4.8.2 Validity

✓ *External validity*

External validity is the extent to which the results of the study can reflect outcomes elsewhere, and can be generalized to other populations or situations (Diether, n.d.).

The external validity of the research results was enhanced using the methodological triangulation. This involves studying the nature of the problem from a variety of viewpoints in order to expand the understanding of the phenomenon under study (Burns & Grove,1993; Moru,2006). If various methods correspond the researcher becomes confident about the findings (Cohen & Manion,1994). In this study, the triangulation was achieved by using questionnaires and interviews to complement the data obtained. The tasks of the questionnaire correspond to tasks found in the literature through already conducted research. The fact that further results have been established through these tasks, increases the external validity of the current study.

✓ *Internal validity*

Internal validity raises the problem of whether conceptual categories understood to have mutual meanings between the participants and the observer are shared (LeCompte & Preissle,1993, p.342).

To assure internal validity, I have piloted the questionnaire (as mentioned in the Pilot Study paragraph) to fellow mathematics education master's students. This enabled the researcher to make modifications to the questionnaire, according to the remarks given during the pilot testing. For the research instrument of interview, I have been informed by the teacher that the participants were good users of the English language. However, as English is their second language, there might have been some loss of preciseness in their answers. For example, I have been asked by a student, what is the English word for "unending". I have asked individuals with a certain experience at teaching in Norwegian schools as well as the teacher of the participants for information on the mathematics curriculum up to the VG2 level, to be sure that the mathematical concepts involved in the questionnaire and interview, have already been taught.

4.9 Summary

The purpose and nature of the research were formed by following guiding principles found in the literature related to mathematics education. It has a cognitive approach to the topic of understanding mathematical infinity thus, it is characterized as a qualitative-cognition focused research. The five participants were chosen randomly and purposefully, out of a sample population that was considered as "information rich". For the data collection, two instruments have been used: the questionnaire and the interview. The data were analyzed in relation with the research questions, meaning that there have been three ways of data analysis. In line with the notification form for data collection, I have considered some ethical issues, such as securing the data and after the completion of the thesis, destroying the data. Finally, the factors that could affect reliability and validity were examined. In order to achieve reliability and validity, several qualitative strategies have been followed. Concluding, absolute validity and reliability could not be achieved from the findings of the discussion at the sections 3.8.1 and 3.8.2.

[This page is intentionally left blank]

CHAPTER 5: Results

The results will be presented and analyzed in three sections. For the section 5.1, I have followed the circle of qualitative data analysis. For the section 5.2, the data analysis and presentation has been carried out by Brousseau's "guidelines" for finding epistemological obstacles. Finally, for the last section, there is an APOS based study of the results, hence, analysis and presentation is done according to the APOS framework. These analyses will be used to give answers to my research questions (presented in Chapter 1). In the interview transcripts, the letter "R" is used to represent the researcher's questions-comments while the initial letter of the interviewee's name is used to represent the interviewee's answers-comments.

5.1 Perception*(Describing)*

The participants' answers have been classified according to their primary and secondary perception of the concept of infinity, as seen in Singer & Voica (2008, p.191). The word perception will be used as defined by the authors: *"an active process of selecting, organizing and interpreting information brought to the brain by the senses"* (p.189). Next, let us recall each of the categories of primary perception and give the corresponding examples out of the students' written and oral responses.

Processional Perception: this perception corresponds to the potential infinity and functions as a modality to understand this nature of infinity (Singer&Voica,2008; Fischbein,2001; Fischbein et al.,1979; Monaghan,2001; Tsamir&Tirosh,1999). The processional perception has two dimensions:

-a *temporal* dimension: it is related to the perception that infinity is something with no end and impossible to be measured. e.g. Bern: *Infinity is an amount we cannot define, because it goes on forever*.

-a *spatial rhythmic* dimension: it is related to the perception that infinity is something unending, something that keeps rising. e.g. Mikael: *...a pattern which can be followed and contains no limitations*

Topological perception: this perception is connected to the conceptualization of infinity as a big entity, bigger than anything else. e.g. Henrik: *something that always is bigger than anything else.*

Spiritual perception: is the perception which is affected by feelings and emotions. e.g. Bern: *...it doesn't make sense that a number can go on forever*

These categories as they appeared in the students' answers, can be seen in the following diagram.

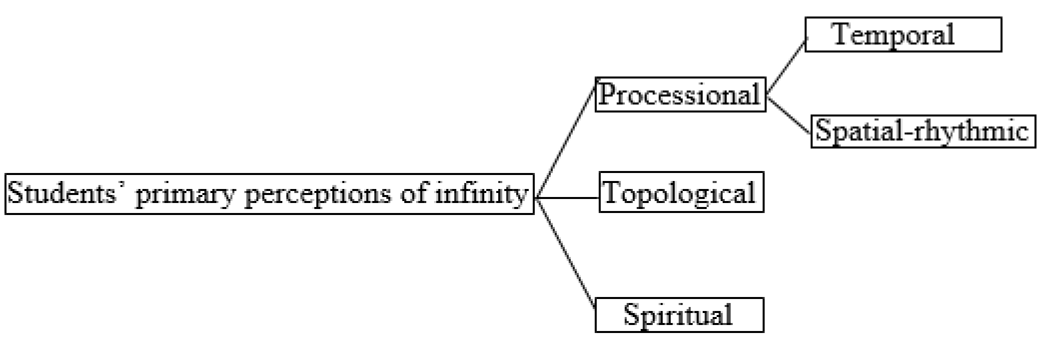


Figure 8. Classification of the perceptions of infinity. Singer&Voica (2008,p.191)

The questions that were related to comparison of infinite sets, aim to reveal a secondary perception of the students. For this reason, I will be using Tsamir's (2001, p.290) criteria, as they have been used by the students that participated in the study to determine whether a given pair of sets are equivalent (p.290). These criteria will help in a methodical examination of a secondary perception. In this study, students made use of three out of the four criteria. Next, I present these three criteria and recall their description.

Part-whole criterion: a proper subset of a given set contains fewer elements than the set itself.

Single infinity criterion: all infinite sets have the same number of elements, since there is only one infinity.

One-to-one correspondence criterion: in other words, a simplistic use of the bijection criterion.

The above criteria are concentrated and presented in the following diagram.

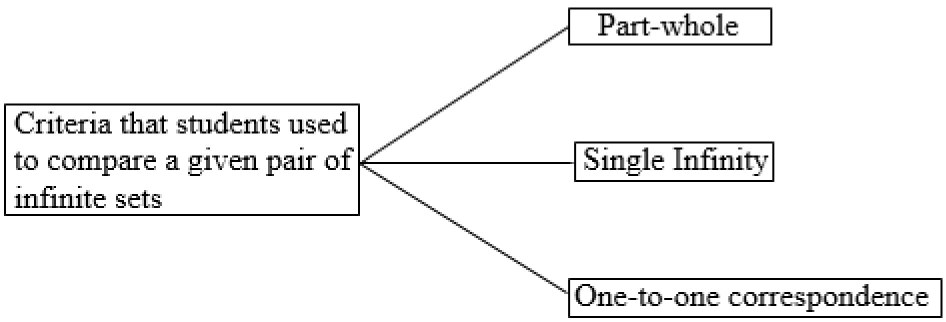


Figure 9. The three criteria that students used for the comparison of infinite sets (Tsamir, p.290)

We proceed at this point to analyze the responses student wise both for the primary and secondary perception(*Classifying).*

✓ *Aline*

➢ Primary perception. Aline answered the question 4.1. What does infinity mean to you?: "*To me infinity means something that never ends"*. It seems like she has a processional-temporal perception. This view was reflected in more of her answers. For example, on the questions 1.5 What is the biggest number yet smaller than 2, she answered "1,99(9)- infinite many 9 after 1". During the interview session, the following discussion took place:

R: *we have 1,9 so then we add another 9 and then we add…*

A: *yes, it's closer and closer to 2*

R: *yes, so if it goes on forever, don't you think that 1,999… will be equal to 2?*

A: *yes, I think so*

We see that Aline's processional perception, indeed functioned as a modality to grasp the potential nature of infinity and thus, the notion of the limit process. This perception helps the specific student to avoid the false conception of infinity as a number as in the questions 4.2-4.3 What is the biggest-smallest number, she answered: (for the biggest)- *I don't know because I can always write some zeros in the end of a number and it'll be bigger than it was before.*

(for the smallest)- *A number can always get smaller, so I don't know what is the smallest number I know.*

As for the case of the divisibility of the line segment (question 3.1), Aline maintained her processional perception which lead her to grasp the notion of infinite divisibility, a notion that goes against the intuition of every-day life.

Question 3.1: *...Do you think that we will arrive in a situation that the segments will be so small that will be unable to divide them?* Aline's written answer: "*No, I don't think so because you can always divide a number, for example number* $10^{-999999999}$ *is very little but you can divide it and get smaller number."*

During the interview:

R: *What do you mean by little?*
A: *That it is very, very small*
R: *So, could the* $10^{-999999999\ldots}$ *be the smallest number that you know?*
A: *No, because you can always write another number*
R: *You mean you can add 9's?*
A: *Yes*
R: *How many 9's do you think we can add?*
A: *Infinitely many*
R: *So, we can keep dividing?*
A: *For how long do you think we can do it?*
R: *Forever*

We observe that the student has a good conception both of the actual and the potential infinity. She maintained that the smallest number is unknown to her, by "infinitely many" she viewed the amount of 9's as a totality while the spatial-rhythmic perception through infinite divisibility suggests that there is a primary perception of infinity (Singer &Voica,2008, p.191).

- Secondary perception. Aline gave some early signs of usage of the *one-to-one* correspondence criterion. Specifically, when she was asked to compare the set of natural numbers with the set of even numbers (2.1.a) she wrote: *"They have the same number of elements. You can write* $\infty = 2\infty$ *and it's true (or I think it's true)*". Aline, uses the one-to-one correspondence criterion but also gives a sign that she makes some thoughts on different kinds of infinity (though she's not sure). For the case (2.1.b) of the sets $\{1,2,3,4,\ldots\}$ and $\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\ldots\}$ she makes use of two criteria: the single infinity criterion: *"they have equal number of elements, because they never end, they have infinite many elements"* and the one-to-one correspondence criterion:"$\infty = 1^{-\infty}$", probably meaning $\infty = n^{-\infty}$. We observe that while keeping the processual perception, when the student uses the single infinity criterion, implicitly thinks in the "topological" way: *"they have infinite many elements"*. This lead to the false answer on the comparison on the set of natural numbers and the set of real numbers by writing that: *"both of them have infinite many elements"*. The one-to-one correspondence "works" for the case of the comparison between the points of the two circles (3.4), Aline writes: *"There are infinite many points in both circles. Say that you have* ∞ *many points in the smaller circle and* 4∞ *in the biggest one. Each circle has the same number of points because* 4∞ *is the same as* ∞*"*. When the student was asked to make a sentence that contained the word "infinity" (4.1) the possibility that she had an idea of different infinities was confirmed: *"There is bigger infinity between 2 and 4 than it is between 2 and 3*.

In order to have a further insight in the student's response, the following dialogue occurred:

R: *Which numbers do you think there are between 2 and 4?*
A: 3*…and many others*

While Aline has written in previous questions (1.1,1.2,1.3) that there are infinite many numbers between 1 and 2 and that there are not more numbers between 1 and 3 and 1 and 2, she thinks in “part-whole” terms. Aline gave her initial answer based on the part-whole criterion, since she thought that the numbers contained in the interval [2,4] are more than the numbers in [2,3]. Spontaneously, when she was asked for the numbers between 2 and 4, she first answered 3 which means she was thinking in terms of sets of natural numbers or finitistic terms.

The interviewee was asked to give her opinion on the questions:

A: *“I think that they were a little bit…like a surprise for me because I didn’t think about infinity in this way”,*

Her commend indicates that the student had not been involved in situations with different representations of mathematical infinity.

✓ Bern

➢ <Primary perception> Bern’s primary perceptions were deeply grounded in the finite world. In question 1.5 he answered: *“I agree* [that π has infinite decimal places]*, but it does not make sense”* maintaining a spiritual perception. When Bern was asked to comment on this, he made his spiritual perception more obvious and gave an answer affected by his feelings:

B:*…in mathematics it doesn’t make sense that a number can just go on forever…I feel like everything in mathematics should have an end*

Bern’s thinking starts from relating a finite entity, like a *“number”* to something that “goes on forever”, which leads him to deny the existence of potential infinity. On the question 4.1 What does infinity mean to you, Bern wrote: *“An amount we cannot define, because it goes on forever”*. During the interview session:

R: *Is it an amount or something that goes on forever? Because 2 kilos are considered as an amount…*
B: *“hmmm…pretty much…it’s kind of impossible to define it as something, you can’t say infinite kilos, it’s just like infinite…I think”*

We see that Bern is confused due to the dual nature of infinity. He gives an answer through a temporal-processual perception but refers to infinity both as an undefinable amount and as a process that goes on forever. For the case of the question on the division of the segment (3.1), Bern writes:” The *value will be so tiny that we perceive it as* ≈ 0 *, but it will still be divided”.* He thinks in finitistic terms, as he thinks that there will be a value in the end (≈ 0) but, in the sense of potential infinity it will still be divided.

➢ <Secondary perception> For the comparison of infinite sets(TASK 2), Bern made use of the single infinity criterion, by writing the abbreviation *“eq”*, meaning that every set has equal number of elements. The same was for the comparison of the set of natural numbers and the set of real numbers. During the interview, Bern was asked:

R: *Let's say you have the interval* [0,1] *with real numbers inside and the set of natural numbers {1,2, 3,…}. Which one do you think has more numbers?*
B: *It makes more sense that {1,2,3,…} has more numbers but still you could infinitely…like create more numbers between 0 and 1. But still it kind of doesn't make sense… If you think like…that infinity cannot be defined then both are equal"*

In his answer, we can see that it is hard for Bern to think counter-intuitively to his real-world experiences. Initially, he can see that there is possibly a "bigger" infinity between [0,1] but cannot accept it: "it *kind of doesn't make sense"*. In his last sentence, he sees infinity as one thing (i.e. a single infinity that cannot be defined and concludes that $[0,1]$ and $\{1,2,3,\ldots\}$ are of the same cardinality.

✓ Henrik

➢ <Primary perception> Henrik in his written answer on 4.1 What does infinity mean to you? highlights the processional view on infinity by a topological one: *"Something that always is bigger (or smaller) than anything"* Using the world *"always"*, attributes a "time" perspective in his view, specifically a processional-temporal perspective. On the other hand, the part "*bigger (or smaller) than anything"* clearly shows Henrik's topological perception. The same perception appears in his written answer on 2.1,2.2: What is the biggest, smallest number that you know?
"∞, if you can call it a number". In this answer, appears a strong topological perception for infinity, seen as something really big or really. Remarkably though, this perspective interferes with the interviewee's doubt on whether infinity is a number or not. His written answer for How many numbers are there between 1 and 2,1 and 3, 0.8 and 1.1, was the same: *"Unlimited"*. The use of this word gives a *"diffused"* (Singer & Voica,2008, p.192) attribute to the student's perception which is one of the main characteristics of the topological perception. Henrik appears to have a spiritual perception for the question 1.5: Do you agree or disagree with the statement that: "π has infinite decimal places?"
"Agree, because π is something we have found in the nature. And no number or length or anything found in the nature can be 100% accurate". For the case of the divisibility of the line segment (3.1) Do you think that we will arrive in a situation that the segments will be so small that we will be unable to divide them? Henrik made a clear distinction between the real-world and the mathematical world and answered:*" Practically yes, theoretically no. Any length can always be divided in two"* During the interview I asked for a further explanation:

R:*" What do you think we will reach if we keep dividing practically?*
H: "*After certain number of times…it is not possible because you will reach the atom…in mathematics you can just smaller and smaller and smaller…"*.

Henrik's spatial rhythmic perception lead him to the distinction between the possible infinite divisibility in mathematics in contrast with the real-world infinite divisibility

➢ <Secondary perception> For the comparison of the sets of TASK 2 Henrik used the single infinity criterion and with a processional-spatial rhythmic perception of infinity as it appears at the concept of infinite sets. His reasoning: *"The same elements because you can find a new element in both of the sets"*. We can see also that Henrik has reached close to using the one-to-one correspondence as he uses the word "both" and the expression "a new element". This reasoning though, lead him to the conclusion that the set of natural numbers have the same number of elements as the set of real numbers. While Henrik attempts to compare the points of the two given circles (3.4) he writes: *"Unlimited in both, no matter how close you zoom into a circle, it will always curve and there will always be more points".* It is visible once again that the topological view of continuous nature: *"…no matter how close you zoom"* is highlighted by the processional-spatial rhythmic perception. Henrik uses the continuous process of "zooming", hence he concludes that there are "unlimited" points in both. The student has used the single infinity criterion to conclude for the equal number of elements. He has used in his reasoning the word "both", meaning that he considers one kind of infinity. This criterion lead him to answer that set of natural numbers is of the same cardinality as that of the real numbers.

✓ Irina

➢ <Primary perception> Irina has replied to the question 4.1What does infinity mean to you? by simply writing *"Never ending"*. This expression is a characteristic one of a processional-temporal perception. The same processional perception appears with a rhythmic spatial attribute on her answers at the 4.1,4.2 questions: What is the biggest-smaller number that you know?
"infinity, since we can always ad-subtract a number to an exact number and get a bigger number"
In the same manner of a processional-spatial rhythmic perception, Irina sees the infinity of the points of a circle (3.4): *"you can make the gap between the points shorter and shorter"*. For the question 1.5, Irina's answer was: 1,999 .... For the 1.5 question on π's infinite decimal places, Irina wrote: *"I believe that π can't be infinite, we just don't know the ending"*. Then the following discussion related to these responses took place in the interview session:

R:*" You said for π that we don't know the ending of the digits, so do you think that also for 1.999… we don't know the ending?*
I: "*Hmmm, I think we know the ending but we don't know…, it's sort of infinite nines, since if we just stop at a place that won't be the closest to 2 as we just add the number nine and we can continue after that for infinity. Well, I think with π is a number that perhaps, will at a time end?"*

At this answer, there is a conflict between a topological and a processional perception. Irina thinks in topological terms when she refers to *"infinite nines",* as an entity that "stops at a place" but then again, she thinks in a potential way since we can continue after that place for infinity. However, it is remarkable the fact that she implicitly thinks of two stages of infinity: the stage of "infinite nines" and the next stage where "we continue after that for infinity". Lastly, she has a clear processional-spatial rhythmic perception when she concludes that a circle has infinite number of points. This can be seen in her following written answer: *"…you can make the gap between the points shorter and shorter".* In 3.1(divisibility of the line segment), Irina also makes a clear distinction between the real-life and mathematics: *"Theoretically speaking we will be able to divide forever, however this division will be quite hard to actually do in real life"*. The same rational is used to answer for the lines that one can make through a point: *Theoretically infinite*

➢ <Secondary perception> Irina has responded to all of the comparison cases of TASK 2 with the answer: *"both are infinite"*, without giving a justification, possibly because she used the "easy to use" single infinity criterion. The same criterion is used for the comparison of the set of natural and real numbers(2.1a) and the comparison of the points of the circle (3.4), where the given answer is:*" both are infinite"*.

✓ Mikjel

➢ <Primary perception> Mikjel in his answer to the question 4.1What does infinity mean to you? refers to infinity as a "*system"* or as a *"pattern"*. This means that he has a processional perception and more specifically a spatial rhythmic perception. His written answer: *"A system or a pattern which can be followed and contains no limitations".* The key words to realize Mikjel's perception are the words *"system"," pattern"* which point to a non-static process that *"can be followed"*. The same perception appears when Mikjel justifies the infinity of the decimal numbers of π (1.5):*" the decimals follow a pattern that has no end"*. Maintaining a processional thinking, Mikjel wrote for the number closest to 2:*" There is no biggest number smaller than two. You can get closer by adding infinite nines to 1.9999…"*. Mikjel during the interview session interrupted me when I tried to comment on his written answer:

R: *You wrote that the number is 1.9999…what do you think…*
M:*" No, no! There is no biggest number beneath 2, but I mean…you can only get closer to 2 by adding nines. There is always a new 9 that can be added on, so nines will never stop, there is no number that is the biggest number under 2"*.

Mikjel's strong processional-spatial rhythmic perception is closely related to the notion of limit, getting arbitrarily close to 2… (1.5):

R: *Do you think there is a difference between 1.999… and 2?*
M: *Yes, yes there is a difference but the difference gets smaller when you add nines*
R: *So, this way at some point you will reach 2 if you add more nines?*
M: *No, you will never reach 2*

➢ <Secondary perception> Some even more interesting results came out when investigating the answers related to the comparison of infinite sets. Mikjel made use of the part-whole criterion and concluded that the set of natural numbers has more elements than the set of even numbers (2.1.a):

"Natural numbers because they always contain the amount of even numbers". While in the next question on the comparison of the sets $\{1,2,3,4,\ldots\}$ and $\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots\}$ made the use of the one-to-one correspondence. Mikjel wrote: *"Equal. Because for each number, there is a number* n^{-1}. In question 2.1.c. and the comparison of $\{1,2,3,4,\ldots\}$ and $\{1,3,5,7,\ldots\}$ he wrote:*" Equal. Because there is infinite in both. Set B will always be in front as you count".* To make things clearer, there was a discussion on the interview session on his answer:

R: *"You said set B will be in front…"*
M: *"I mean for every of these naturals numbers there will always be one of these numbers, so you can match them one by one…like infinite…*
R: *"Correspondence let's say."*
M: *"Yes! Correspondence."*

Then, trying to examine Mikjel's intuition on rate of convergence, since he used the expression *"will always be in front as you count"* I asked:

R: *"Do you think that set B will reach first infinity first? As you said it will be in front as you count…"*
M: *"Hmmm, you can't…No! No! Because I don't feel you can reach infinity, it doesn't actually count with the speed but…it's just infinite numbers or…yes infinite numbers in both of the sets".*

This didn't seem as a sign that Mikjel was intuitively thinking of the notion of convergence. In addition, he used the single infinity criterion for his reasoning, by saying that the both sets are just infinite. Surprisingly, in question 2.1.e. Mikjel gave an answer that points straight forward to his suspicion that there is a "bigger" infinity for the set of real numbers, he wrote:

" There are more real numbers, because the real numbers contain all the natural numbers plus many more in between. The real numbers will at every point have more numbers even though both goes into infinity".

The student has reached really close to Cantor's countability-uncountability argument. He referred to the density of rational numbers, while he goes deeper by saying that *"the real numbers will at every point have more numbers"*. One could consider his latter part of the answer as a primary conception of Cantor's diagonalization argument. In the end of his answer though, he uses the single infinity criterion as something that contradicts (*"even though")* his answer.

5.2. Epistemological obstacles

As said in the first paragraph of the results section, in order to identify epistemological obstacles in the student's understanding of infinity, I will be following Brousseau's method (2006, p.99). A further description of the method and the actions that will be utilized in the search of epistemological obstacles, is given below.

- Find recurrent errors, and show that they are grouped around conceptions
  -Identification of errors in students' responses (both written and oral) to the questions
  -Relating the errors to the conception that may have given rise to them
- Find obstacles in the history of mathematics
  -Finding obstacles in the history of the conceptual development of infinity will be done by looking at the historical analysis presented at the beginning of this study.
- Compare historical obstacles with obstacles to learning and establish their epistemological character
  -Explain why the identified conception is an epistemological obstacle

In the next table, I present the errors as they appeared in the students' written and oral responses and as they are grouped around the related conceptions.

Errors	**Conceptions**
Infinity is an amount	Infinity is a number
Infinity is the biggest number	
Minus infinity is the smallest number	
$1^{-\infty}=\infty$	
There is a bigger infinity between 2 and 4 than it is between 2 and 3	A proper subset of a given set contains fewer elements than the set itself
The set of natural numbers has more elements than the set of even numbers	
The set {-3, -2, -1, N} has infinite amount of elements	Infinity is a process
The set of real numbers has the same cardinality with the set if natural numbers	
There is no "greater" infinity	
1.999… is the biggest number smaller than 2	Repeating decimals and irrational numbers as totalities
π can't be infinite	
π has an end	
π has unique representation "π"	Irrational numbers can only be represented numerically
Infinity makes no sense	Metaphysical aspect of the notion of infinity

Table 1.Errors as they appeared in students' written answers, grouped around conceptions

Infinity is a number: Conceiving infinity as an amount or a number lead students to also manipulate infinity as they would do naturally with a number. There was a use of the signs as thinking in terms of the number line, hence since infinity is the biggest number then its opposite would be minus infinity. This use of infinity also brought up the indeterminate form of 1^{∞}.

A proper subset of a given set contains fewer elements than the set itself: The expression "bigger" infinity was raised by Tall's extrapolation of measuring properties. Still, the part-whole relationship (more elements in (2,4) than (2,3)) preserved the contradiction between different sizes and cardinal infinities. The interference of the part-whole relationship is best given in Mikjel's written answer: *"The set of natural numbers has more elements because they always contain the amount of even numbers"*

Infinity is a process: Reasoning that the set N is infinite because you can keep counting, add always one more etc. is a result of the conception of infinity as a process. Consequently, it would be hard for someone then to conceive it as a totality and give the answer that the cardinality of the given set is 5.

In this study, as well as in Monaghan's study, the dominant answer on set comparisons was *"both are equal"*. Students maintained a processional view on the infinite sets, which lead them to think that all the sets contain an infinite amount of elements, hence they are of equal cardinality. In the same manner, maintaining a view on infinity as a process, it is difficult for someone to grasp the notion of cardinal infinities (e.g. $\aleph_0$, $\aleph_1$) as it is a notion strongly related to actual infinity.

Repeating decimals and irrational numbers as totalities: When 1.999 ... is seen as a totality then the expecting answer would be that 1.999 ... is the bigger number closest to 2(Tall,1981). On the other hand, other answers indicated that when 1.999 ... is conceived as a process (by adding nines we get closer to 2) then the student was lead to the correct conclusion that there is no biggest number closest to 2. In other words, students that implicitly used the notion of the limit process have concluded that such a number does not exist. In the same sense, since π has not repeating decimals to lead to a concrete process conception, then π is conceived as a totality with end, that cannot be infinite.

Irrational numbers can only be represented numerically: It is the conception that since π has infinite decimals, then it cannot be written in a numerical form as it would be impossible. Hence, the only representation possible is by using the letter π.

Metaphysical nature of infinity: It is a conception rooted in the real world, a finite world with no real referents for discourse of the infinite (Monaghan,2001, p.240). Seen through a real-world lens, then of course infinity would make no sense.

In the next table, I will be referring to the conceptions of *[Table2]* as obstacles in learning infinity, in order to follow Brousseau's terminology and relate them to the corresponding obstacles in history.

Obstacles in learning	**Obstacles in history**
Infinity is a number	Controversies in the 17th Century concerning the infinitely small and infinitely large
A proper subset of a given set contains fewer elements than the set itself	Galileo's comparison of the sets of natural and square numbers is considered as a paradox
Infinity is a process	Rejection of actual infinity and Cantor's cardinal theory
Repeating decimals and irrational numbers as totalities	Pythagoreans' idea that everything can be expressed in ratios of whole numbers (i.e. a number cannot have infinite decimal places)
Irrational numbers can only be represented numerically	Incommensurability of the diagonal of the unit square
Metaphysical nature of infinity	Zeno's Paradoxes of Motion

Table 2.Comparing obstacles in learning with obstacles in history

We can see in the table that the obstacles in learning have appeared in the context of conception in the historical development. I will not go through a historical analysis, as this has been done in Chapter 1 but I will refer to the previous table as an evidence that the obstacles in learning are inherent to the nature of the concept of mathematical infinity. This is in line with one of the two main characteristics of a possible epistemological obstacle.

At this point, I will point out the epistemological obstacles as the obstacle in learning that occur due to previous knowledge and as they appeared in this study:

- *Infinity is a process*[19]: an epistemological obstacle rooted in the individual's knowledge of counting of the natural numbers. Specifically, in the results of this study, most of the students that thought of infinity as a process gave the answer: *both sets are equal because we can always add more.* This reasoning resulted to the "flattening" phenomenon, previously, mentioned for the case of infinite sets
- *A proper subset of a given set contains fewer element than the set itself*: an obstacle stemming from the properties of finite sets. Hence, the knowledge of the properties of finite sets stands as an unavoidable obstacle, in the sense that one will use the properties of finite sets to conclude that the natural numbers contain more elements than the set of even numbers.
- *Metaphysical nature of infinity*: an epistemological obstacle that lead a student to characterize infinity as an absurd concept. The knowledge built in a finite world leads someone to reject counter-intuitive appearances of infinity in mathematics, a rejection based on real-world experiences.

The rest of the obstacles in learning mentioned on table, fall in the categories of ontogenetic and didactic obstacles. I will not be referring to them analytically but it would be difficult for an 11th grader to grasp that π for example, can be represented as a continued fraction, which points to an ontological nature. Furthermore, there could be an agreement in a classroom and a teacher to notate $.999\ldots$ as $.\dot{9}$, that is to use a notation that leads the student to think of the repeating decimal as a totality.

5.3 APOS analysis

5.3.1 How will the APOS theory be used?

For the implementation of APOS theory in this section of the study I will be following the research *paradigm*[20]as provided in Arnon et al. (2014, p.94) An APOS based study involves three components: theoretical analysis, design and implementation of instruction, and collection and analysis of data. The following figure shows how these components are related.

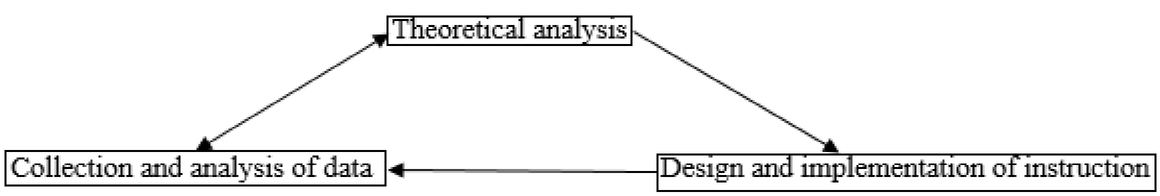


Figure 10. *Relation of the components of an APOS based study.* Adapted from *APOS Theory A Framework for Research Curriculum Development in Mathematics Education* (p.94), by I. Arnon et al., 2014, New York: Springer Science & Business Media. Copyright 2014

[19] *"There is no actual infinity; and when we speak of an infinite collection, we understand a collection to which we can add new elements unceasingly."*, Poincare as cited in Weller et. al (2004)

[20] a philosophical and theoretical framework of a scientific school or discipline within which theories, laws, and generalizations and the experiments performed in support of them are formulated(paradigm.2011)

A theoretical analysis corresponds to a *genetic decomposition* of a mathematical concept. That is, a description of the mental constructions and mental mechanisms that an individual might make in constructing her or his understanding of a mathematical concept (Arnon et al.,2014, p.94). In other words, this decomposition reflects the researcher's knowledge and understanding of the mathematical concept in question. For a researcher *"The genetic decomposition becomes the working hypothesis that is used to evaluate the degree to which learning has taken place".* The role of the instructional treatment is to get students to make proposed mental constructions and use them to construct an understanding of the concept as well as apply it in mathematical situations. For example, presenting the set of natural numbers as $\{1,2,3, \dots\}$ encourages the student to think in the process level. While presenting the set of natural numbers as N, then the researcher encourages the student to encapsulate the set of natural numbers as an object. As seen in the figure, there is a reciprocal relationship between theoretical analysis and collection of data. This means that in an interview session, the researcher assesses students' answers and relates them to the genetic decomposition. If the student does not make the mental constructions called for by the genetic decomposition then the instruction is reconsidered and revised. In other words, the cycle continues until the empirical evidence and genetic decomposition point towards the same mental construction.

I have chosen to analyze two questions out of the questionnaire that will produce significant insight in students' mental construction: a) (1.4) *What is the biggest number yet smaller than 2*

b) (2.2) *How many elements are in the set*

{-3, -2, -1,0, {1,2,3…}}

These examples have been chosen as the most "rich" questions where the process and object structures appear concretely. Similar topics have been investigated in the literature related to mental structures, thus we will have the chance to relate the results to the ones already provided. The written answers that have indicated a fruitful implementation of APOS analysis in specific cases, are given and analyzed below.

5.3.2 The issue of $1.999 \dots = 2$

Tall (1981) in his research has shown that do not accept the equation $.999 \dots = 1$ as true, but also gave the answer that $0.999 \dots$ is the number closest to 2. The same results were found in my data as students were asked during the interview session whether $1.999 \dots = 1$: *"there is always something in between"," no matter how close we get we never reach 2"* etc.

I start working on the case, by the genetic decomposition of the concept of the infinite repeating decimals.

Action: A student recites an initial sequence of digits, which may be seen as the beginning of a repeating decimal expansion.

Process: Forming sequences of digits of indeterminate length that is extended to form an infinite string. The student grasps the idea that from some point on the decimal repeats forever to form an infinite string.

Object: The process of forming an infinite string may be encapsulated into a mental object when the student reflects on the process of forming an infinite string and begins to see an infinite string as an entity (Arnon et al.,2014, p.76). The genetic decomposition is summarized in the following diagram

Schema: The above collection of conception stages and mechanisms constitute the infinite repeating decimal schema.

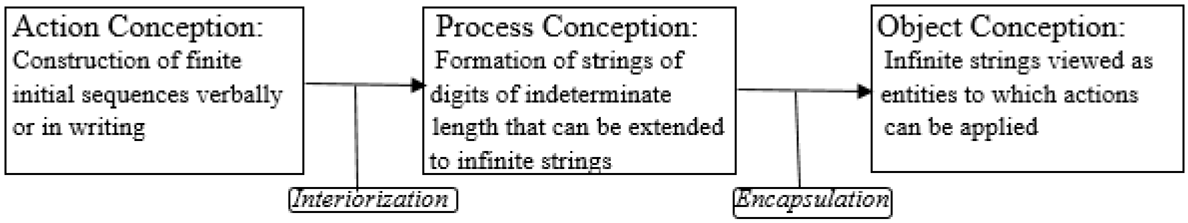


Figure 11. Genetic Decomposition of an infinite decimal. Adapted from *APOS Theory A Framework for Research Curriculum Development in Mathematics Education* (p.94), by I. Arnon et al., 2014, New York: Springer Science & Business Media. Copyright 2014

Irina on the issue $1.999\ldots = 2$

Irina's wrote as an answer to (1.4): *"$1.\overline{9}$"* constructing a finite initial sequence of 9's verbally with $\overline{9}$ being an indication. I asked about her answer

R: *"You wrote $1.\overline{9}$…"*
I: *"One point nine, in…nine nine nine sort of…"*

constructing finite initial sequence of nines and showing indications of a process conception. Up to this point, Irina's conception is at the action level.

R: *"So you think…"*
I: *"Yeah, I took that over sort of to…show that it was an infinite number of nine…I don't know if that's…the way I supposed to do it"*

The student has interiorized her action conception and moved to the process conception by extending the string of 9's of indeterminate length ("$\overline{9}$", "*nine nine nine*") to an infinite string of 9's ("*infinite number of nine")*. Keeping that in mind, I moved on an attempt to determine whether a complete process conception had been constructed:

R: *"Do you think…like you said for π later, that we don't know the ending of digits after π, so do you think also for 1.9 that we don't know the ending?"*
I: *"I think we know the ending but we don't know…it's sort of infinite nines…since if we stop at a place that won't be the closest to 2 as we just add the number 9 and we can continue after that for infinity. Well I think for π it's a number that perhaps will at a time end?"*

Irina process conception is not complete. She actually conceived 1.999…. as consisting of a finite 9's but of indeterminate length. With an incomplete construction of the process level, Irina encapsulates 1.999… as an object, that is as a finite string of 9's. Then she applies actions on this object:" just *add the number 9 and we can continue after that for infinity"*. As the student diverged from the genetic decomposition, I resorted to a revision of the instruction:

R: *"Let's say you start with 1.9 then we add another 9, another 9 and so on…so if we keep adding 9's don't you think we will reach…like we are getting closer to 2? Let's say at an abstract point, if we keep adding 9's we will get really close to 2 that 1.99999… will be equal to 2?*
I:*" …uhmmm…I…you know…will never be equal to 2…uhmmm…but of course it will forever become closer and closer and closer but it will never reach 2?"*

It seemed that my attempt to complete Irina's process level, apply action on the object (*"at an abstract point"*) and finally get an answer that *1.999…=2* failed.

R:*" So you don't agree that it is equal to 2?"*
I:*" Almost equal"*

For Dubinsky et al. (2010, p.262), students who have not yet constructed a complete process conception of the infinite decimals, usually will think of the relation $1.999 \ldots = 2$ as false. This is confirmed in Irina's case.

Finally, I have attempted to convince the student for the equality, by using the naive proof:

$$10x = 19.999 \ldots$$
$$\underline{-x = 1.999 \ldots}$$
$$9x = 18 \Rightarrow x = 2$$

Surprisingly, the student presented a second regularly used proof for the equality $0.999 \ldots = 1$:

I:*" Can I borrow the pen? I've seen:* $0.333 \ldots = \frac{1}{3}$

$3 \times 0.333 \ldots = 3 \times \frac{1}{3} \Rightarrow 0.999 \ldots = 1$*"*

R: *"Even if we prove it, either my way or your way, you still don't believe that it's…"*

I: *"I mean it is not the same as 2*[or 1] *I guess, because it's an abstract number, I don't…at least I don't understand it, I think we have a hard time understanding it!"*

R: *"Yes, that's true"*

As in the case study of Edwards (2007), the student might have been limited to seeing both parts of the equality as the same process. Although, what is apparent is that Irina refers to $1.999 \ldots$ as an abstract number. This abstraction stems from Irina's encapsulation of $1.999 \ldots$ as a non-static object (incomplete process structure), while on the other hand the number 2 is conceived as a static object (Dubinsky et al.,2005, p.261). In this case, we say that the encapsulation is the *transcendent object* (Brown, MacDonald, & Weller,2010). The name was chosen to indicate that this object must be understood as not being produced by the process itself, but instead as transcending the process (p.62). For example, the appearance of the transcendent object would have been avoided if the student had gone through a complete process stage (i.e. to see 1.999… as a static object) and then to make the comparison to the static object number 2.

Aline on the issue $1.999 \ldots = 2$

Aline wrote as an answer to the question 1.4: *"1.99(9) – infinite many 9 after 1".* Her written answer shows her action conception. She has written a finite initial sequence of 9's with the "(9)" indicating a process conception. of However, Aline during the interview was convinced that 1.999…*"at some point will be equal to 2"*:

R: *"About the nines, do you think that they stop somewhere that we don't know?"*

A: *"No, I don't think so, I just think they go on forever"*

Aline had also extended 1.99(9) to an infinite string of 9's. I went on an attempt for Aline to encapsulate the process by seeing the infinite as a totality and act on it:

R: *"So, let's say we have 1.9 and then we add another 9 and then we add…"*

A: *"Yes and it's closer and closer to 2"*

R: *"Yes, so if it goes on forever, don't you think that…let's say it goes on and on and on… don't you think that at some point this will be equal to 2?"*

A: *"Yes, I think so!"*

R: *"You think so?"*

A: *"Yes!"*

However, when I wrote down Aline's conclusion:

R:*" So you think this is right…* [1.999…=2]*"*
A:*" Hmmm, I'm not sure, because...eh, but it's not…eh…there is difference between them…so that maybe…?...because we don't know how long it goes"*

Dubinsky et al. (2010, p.262) have given an explanation for the latter situation by saying that a student has not yet structured a complete process conception of the infinite decimal. This is also possible in Aline's process construction. Aline might have imagined a finite string of 9's, as I wrote down the number 1.999…, but then again thought of this string as of indeterminate length. In accordance with Dubinsky et al. (2010) who added in the explanation: *"Conceptions such as infinitely small differences with 1*[2] *could exist without conflict in this situation."*, Aline said that there is a difference between the two numbers. A second explanation given by Dubinsky et al. (2010, p.261) for a similar to Aline's situation, is that the student might be limited to a process conception, without having encapsulated the process. Then correctly the individual sees that 2 is not directly produced by the process. In other words, a student without having encapsulated the process, correctly concludes that just by adding 9's, even forever, 1.999…. cannot be equal to 2.

5.3.3 The cardinality of {-3, -2, -1,0, {1,2, 3,…}}

Firstly, we begin to work on the mathematical problem by creating a genetic decomposition of the concept of the element of a set. It is appropriate to do so because in the above problem, the set $\{1,2,3,...\}$ appears as an element of the given set. However, I will not proceed on describing the interaction of the genetic decomposition with the data or the instruction part of APOS methodology, as the students seemed to have a good understanding on the properties of finite sets and finite sets is not the concept in question in this study. Below a diagram of the genetic decomposition is presented and then follows a brief description of the conception levels.

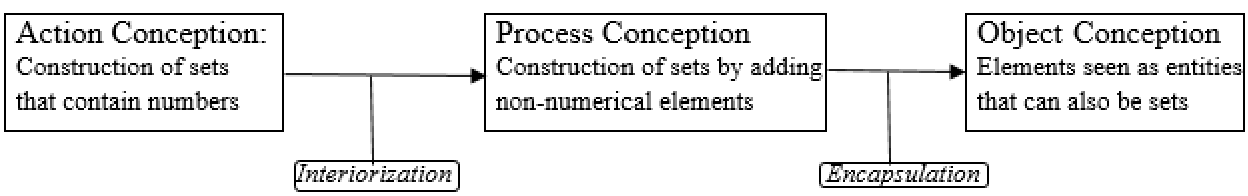


Figure 12. Genetic Decomposition of an element of a set

Action: A student constructs sets that contain only numbers.
Process: At the level of a process conception, a student can reflect on the number as an element of a set and add more elements like functions, geometrical objects, letters etc.
Object: The student has encapsulated an element and now conceives it as an object that can be also a set.

Since the schema of the element of a set has been given and the students had a good understanding of the fact that an element of a set can also be another set, I will proceed now to the genetic decomposition of the set of natural numbers in the same manner as I did for the concept of the element of a set. The decomposition is created by looking partly at Dubinsky's et al. (2010, p.261) construction of the set of natural numbers.

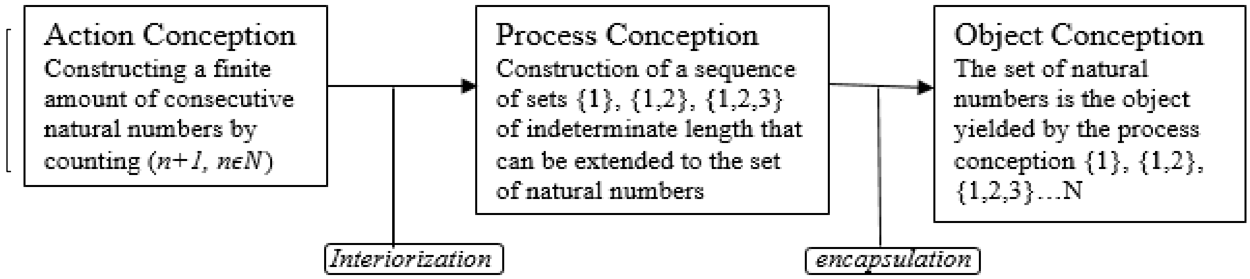


Figure 13.Genetic decomposition of the set of natural numbers

Action: At this level, the student acts on the natural numbers by the operation of addition. He/she constructs a finite amount of numbers, verbally or in writing by the rule of $n + 1$.This is done by recalling from memory the act of counting.
Process: At this level, a student reflects upon the action of counting and constructs the results of counting in the context of finite sets. Then takes place the internal mental construction of the sequence $\{1\}, \{1,2\}, \{1,2,3\} \ldots$ and can be extended up to the set of natural numbers.
Object: Then the process is encapsulated to yield the object of the set of natural numbers. The extension can be represented as $\{1\}, \{1,2\}, \{1,2,3\} \ldots \mathbb{N}$.

After the encapsulation of the set of natural numbers as an object, the above schema interacts with the schema of the element of a set, thus a student can see that the set $\{-3, -2, -1, 0, \{1, 2, 3, \ldots\}\}$ consists of 5 elements. 4 students have given the answer "5", while 1 of them did not gave an answer at all. As it seemed that the students viewed the set of natural numbers as a totality, the questions of the interview were based on the different representation of the set of natural numbers. That is, by using curly brackets encourages a student to conceive that $\{1,2,3, \ldots\}$ is a single object. However, I examined the case when $\{1,2,3, \ldots\}$ is represented as N(with no curly brackets. Below we will have an insight at a specific student's mental constructions and mechanisms as they appeared in his written and oral responses.
Schema: The collection of the above mental

Mikjel on the cardinality of {-3, -2, -1,0, {1,2, 3,…}}

The purpose of the task is not to examine students' understanding of properties of sets or subsets. Obviously, as the most student answered, the answer is that the above set has five elements. However, the point was to alternate the representation of the set as $\{1, 2, 3, \ldots\}$ and $\mathbb{N}$, in order to examine the mental structures that would take place and a possible different answer.

2.2 *How many elements are in the set {-3, -2, -1,0, {1,2, 3,…}}?*

Mikjel wrote: *"5. Because no matter how many numbers you add inside the fifth element, it still is five elements".*

His answer indicates that Mikjel has a complete schema of the element of a set (even if we don't know if he followed an APOS construction). He is able to see that an element of a set could be anything, even if it is an infinite set. During the interview session:

R: *"What if I told you that {1,2, 3…} is constructed by {1}, adding one and get {1,2} …*
M: *Yes…*
R: *… it's like plus one every time, yes?*
M: *Yes*
R: *…so this way you said, we have 5*
(continues in the next page)

M: *You will always have 5 elements because {1,2}, {1,2,3}, {1,2,3,4} and this* [{1,2, 3,…}] *is always one element"*

Since the initial answer was correct, I have designed my instructions during the interview session so the student would think as close as possible to the genetic decomposition. As seen in the previous dialogue, I have indicated the action level for the construction of the set of natural numbers. Furthermore, I have given indications to the student so he could interiorize the counting to the sequence of sets as presented in the dialogue. Finally, the student himself encapsulated the set of natural numbers as an object by extending his view on the {1,2}, {1,2,3} etc. as single objects to the set of natural numbers.

Next, I felt the need to examine whether or not the representation of the set of natural numbers would have an impact on the student's object conception. During the interview:

M: *"what if I told you these* [{1,2, 3,…}] *are the natural numbers…because they are…so we can write it like this, with N instead of 1,2, 3,… how many elements do you think we have now?*

R: *"How many elements? ehmmm…so this is all the natural numbers?"*

R: *"Yes, this set*[N]*"*

M: *"Oh yes yes! But I mean no, it will always be five elements, I mean this is only one element, isn't it?"*

R:" *Yes, yes…"*

M:*" No matter how many numbers inside it will be still one element…as long as it is inside that box it will still be one element"*

It seems that Mikjel maintained his object conception, despite the different representation of the set of natural numbers. Moreover, the use of curly brackets reinforced his "totality" view on the set of natural numbers, meaning that even though we keep adding numbers inside the brackets, it will still be one element.

Remark

Irina, even though she wrote "5" as an answer, she was not completely sure after representing the set differently. She also stressed out the use of the brackets. The next dialogue took place, after I presented a construction of the set of natural numbers, indicating only the action level.

R:*" If I write {-3, -2, -1,0, {1,2, 3,…}} as {-3,-2,-1,0, ℕ } , how many elements do you think we have now?"*

I:*" Then we would have infinite"*

R:*" But you wrote 5"*

I: *"Yeah, I thought even though it's infinite elements within that, I though the entire set…"*

R:*" But it is still one, isn't it?"*

I:*"Yeah, I guess!*

R: *Ok"*

I: *"But then that is infinite numbers since natural numbers will never stop, but I thought when they were inside of these*[{}]*…, yeah that…that they would count as one"*

We see that while Irina goes somewhere in between the answer of infinite and 5 elements, finally the use of brackets reinforces her object conception as an object. The conclusion that the incomplete implementation of the APOS circle in Irina's case, created her doubt on the cardinality of the given set, would be risky since our sample is small. I would not refer to this as a conclusion but as an indication.

[This page is intentionally left blank]

CHAPTER 6: Discussion

The aim of this dissertation was to examine the students' understanding of mathematical infinity by investigating the aspects of perception, obstacles in understanding and mental structures and mechanisms. These aspects have been investigated by looking at primary and secondary perceptions, by finding epistemological obstacles in understanding and by means of APOS analysis.

6.1 Discussion on perception*(connecting)*

The study addressed what Monaghan (2001) calls underlying conceptions (p.244). That is, the study was not concerned on student's wrong or correct answers (as this is something that will be addressed next by the use of the Theory of Epistemological Obstacles) but examined the student's perceptions of infinity as a concept and as related to mathematical entities (i.e. infinite sets, limits, infinitesimals). There has not been a use of advanced mathematical concepts as I was aiming to get concrete evidence that would lead to the categorization of the student's perception, according to Singer&Voica (2008). The infinite sets and the geometrical representation of infinite sets as geometrical objects(circles) were used in the context of comparison and were presented in a way to encourage students to use Tsamir's (2001) criteria.

As a first result, we saw that the context in which every task was presented seemed to interfere with the student's perception [numeric/geometric (Monaghan,2001, p.244) and verbal]. For example, Henrik wrote that *"infinity is the smallest and the biggest number"* while at a point in the interview uses the processional key expression *"smaller and smaller…"*. The same student very often in his answer used *"unlimited"* as a synonym for infinity. This comes in contrast with the concept of infinity of points in a segment, a segment being limited though constituted of the infinite points (Sbaragli as cited in Spagnolo,2004, p.2). Aline has a processional perception when she is asked for the divisibility of the line segment but on the other hand perceives the number of points of the circle as a totality *("infinitely many points")*. Irina refers to the potentiality of dividing the line while on another question she refers to infinity as the biggest-smallest number. Many cases of spiritual perceptions appeared at the "make a sentence" question (4.4). Irina wrote that: *"Humans are not able to completely understand the concept of infinity"* and Bern let his feelings guide him to say that:*" infinity makes no sense"*. Of course, the other perceptions of these participants functioned to work on the rest of the tasks. The above come in agreement with other studies that have pointed out the existence and persistence of alternative perceptions (preconceptions, intuitions, Tsamir) which are not in line with the accepted mathematical definitions and methodologies.

Moreover, it seems that most of the students that participated, had the trend in most of the tasks to consider infinity as a process rather than an object. According to Monaghan, this has an impact on accepting the belief of the infinitesimally small. However, Aline's has used the notion of the infinitesimal as a useful fiction:

Question 3.2: What is the regular polygon*(mangekant)* with the most number of sides you can imagine?

"A circle, I think that a circle has so many sides that you can't even see them because they are so small"

Aline has thought in terms of infinitesimals (*"you can't even see them because they are so small")* and imagined a circle that consists of *"so many sides"*. In other words, she realized that the circumference of a regular polygon can reach the circumference of a circle by limit, as the sides of the polygon are increased.

For the "comparison of infinite sets" case through which we examined the secondary perception of the students, the results indicated that most of the students made use of the single infinity criterion. That lead to the consideration of all infinite sets having the same cardinality, a phenomenon that Arrigo and D'Amore (1999) call *"flattening"* (as cited in Sbaragli,2004, p.62). However, the different representations(Tsamir,2001) affected Mikjel's answers. When the sets of natural and even numbers were represented verbally Mikjel made use of the part-whole criterion. At the next question, where there was a numerical representation ($\{1,2,3,4,\dots\},\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\dots\}$) Mikjel made use of the one-to-one criterion. Apart from Mikjel's case, the different representations of the infinite sets [horizontal, vertical, numeric-explicit, geometric (Tirosh&Tsamir,1996), and verbal] did not seem to have a strong impact on students' answers. For example, Aline uses the one-to-one correspondence criterion on a horizontal representation of infinite sets but at the case of the vertical representation (where it was supposed to encourage the use of one-to-one criterion) she answered that the sets never end so they are of equal cardinality. Another important remark, is that at the tasks of different sized circles and the comparison of the real numbers and natural numbers lead to an intuition of different sizes of infinities but still this intuition was contrary to the cardinal infinity. In Tall's terms (1980,1981), there was an *extrapolation* of the measuring properties.

Finally, I would like to raise the importance of language when talking about infinity(Monaghan,2001). Specifically, I will refer to the case where the concept of the convergence of a sequence appeared. In my question to Mikjel I have used the word "reach" which corresponds to something that is reachable, as an object. Instead of "reach" I could have used the expression "goes to infinity" which would be compatible to Mikjel's processional view on infinity.

These results on perception describe the students' sensory thoughts on infinity. It was necessary to have an insight of this aspect of understanding, since perception is the first mental "image" that students' have before proceeding deeper in understanding a mathematical concept. Therefore, the observations of students' perception are important in the sense that perception underlies the understanding of the concept of infinity.

6.2 Discussion on the epistemological obstacles

The inquiry into the epistemological obstacles aimed at finding obstacles in coming to understand the concept of mathematical infinity. These obstacles are unavoidable as they are rooted in previous knowledge as well as in the nature of mathematical infinity.

The study demonstrated three epistemological obstacles. As observed in most of the tasks, students maintain a "process" view on infinity. This is mainly connected to the previous knowledge of counting, i.e. counting is an indefinite and unending process. The same obstacle accounts for the view that by adding more elements in a set results to an infinite set. Specifically, the obstacle lies in the exclusive process view on infinity. That is, actual infinity is far from being conceived by the students, hence the answer that all infinite sets are of the same cardinality is then reasonable. Remarkably, the same mindset appears in the historical development of the concept, since the rejection or the exclusiveness of potential infinity, was leading the mathematical world to paradoxes.

The second epistemological obstacle as found in this study, stems from the previous knowledge of the properties of infinite sets. Specifically, the property that a proper subset of a given set contains fewer element than the set itself. In other words, the part-whole relationship but as it appears in finite sets. As mathematical infinity appears in infinite sets, it brings up the counter-intuitive property that the part could be equal to the whole. Thus, we see that the obstacle lies in the previous knowledge of finite sets but also in the counter-intuitive nature of infinite sets. Furthermore, the obstacle is reinforced by real finite world experiences, meaning that the finite set properties are compatible with real world situations. The same obstacle has made its appearance during history, in mathematicians' first attempts to arithmetize infinity. Galileo for example, even though he established a one-to-one correspondence between the natural numbers and the square numbers, he concluded that infinite quantities could not be compared as both sets are infinite.

Finally, the counter-intuitive nature of infinity or what could be called its metaphysical nature, stood as an obstacle in understanding infinity. It was noticed during the data analysis that infinity was characterized as a senseless concept. This characterization is due to the general knowledge acquaintance in terms of the finite world. This could be also characterized as a fundamental epistemological obstacle in coming to understand mathematical infinity, as the metaphysical aspect of the notion of infinity leads a student far from an understanding process.

The epistemological obstacles found are broadly consistent with the examples that Herscovics(1989) had found from the work of Bachelard(1938). Indeed, there is a tendency of students to rely on deceptive intuitive experiences. This tendency appears specifically in the third epistemological obstacle. The tendency to generalize appeared as "flattening" the cardinalities of infinite sets. That is, since there is only one infinity then all infinite sets are equal. Alternatively, we could refer to the epistemological obstacles found as "cognitive obstacles". This is a Herscovics's definition who distinguished between epistemological obstacles found in history and in present. Thus, we can refer to the three obstacles found as cognitive obstacles. In line with the Brousseaunian notion of epistemological obstacle, the obstacles found are those *"…from which one neither can nor should escape, because to their formative role…"* in the knowledge acquisition for the notion of mathematical infinity (1997, p.87). Indeed, one has to have a view of the potential infinity, as potential and actual infinity are the two sides of the same coin. Furthermore, the part-whole relationship is an unavoidable obstacle as it appears both in finite and infinite sets. However, the formative role of the relationship lies in the realization that it runs counter-intuitively in the case of infinite sets. The same for the obstacle of the metaphysical aspect of infinity. One should not avoid this aspect, as it can lead him/her to be aware that the mind needs to extend to fields of thinking outside of a finite world.

In this study, the search for epistemological obstacles took place by viewing these obstacles as a "functional necessity" (Bachelard as cited in Herscovics,1989, p.61). Meaning that, epistemological obstacles are not viewed in this study as points of stagnation but as interactive obstacles that lead further to understanding a mathematical concept.

6.3 Discussion on APOS analysis

The investigation of students' mental mechanisms and structures set out to assess the impact of a genetic decomposition in students' understanding of a mathematical concept. That is, after a theoretical analysis of mathematical infinity in context of mental stages, there have been an assessment on what degree knowledge is acquired when the student follows these stages.

The next findings are considered as indications and not as conclusions, as the studied sample was small and the APOS analysis was conducted individually. The study on the two mathematical problems of the repeating infinite decimals and the infinite set of natural numbers, showed that encouraging the student to follow the genetic decomposition stages could lead to the target knowledge. Though, I should raise caution during the passages from one stage to another since an incomplete conception (e.g. a process conception) could lead to an incomplete understanding. The final transition from the process conception to the object conception seemed the most difficult of the mental mechanisms. I refer to the mechanism of encapsulation, which functions as the mechanism of passage from the process stage to the object stage. Applying an APOS methodology during the interview, showed that students do not necessarily follow the four mental stages in their thinking. Nevertheless, by giving hints for following the sequence of the mental stages, indicated that can lead to knowledge or to be more specific to the formation of a mental schema. These hints could also include different representations that can act radically on the mental constructions. See for example the case of

The complexity of the transition from the process to the object conception has been also pointe by Sfard. Sfard (1991) wrote about the "inherent difficulty of reification", which is similar to encapsulation. Specifically, she mentions the difficulty of seeing something familiar in a completely new way (p.30). In our case, the difficulty lies in the familiar notion of potential infinity to be seen as an object or equivalently to comprehend the actual nature of infinity. The process-object transition is not only to be the most difficult but also the most important.

Thus, the results point to the importance of the mechanism of encapsulation. It has been noticed in the data analysis, that when the concept of the infinite decimal was encapsulated, then a student can act on the object and finally conclude for example that $.999\ldots = 1$, which was the knowledge to be acquired through a task of the questionnaire. Tall et al. (1999) comment on the scope of the transition, as they refer to the Object as a product: *"Once the possibility is conceded that the process construction can be conceived as an "object", the flood-gates open. By 'acting upon' such an object, the process-object construction can be used again and again"* (p.5). By product the authors refer to the product that comes out of the process and then is encapsulated in an object. Then, this object can be used for other schemas, again and again.

CHAPTER 7: Conclusion

Investigating the three aspects of the students' understanding, gave us the opportunity to have an overview of how perception, obstacles and mental stages function within a development of understanding. Through the data analysis it is noticed that the dual nature of infinity is one of the main reason for bringing up contradictions. Thus, the obstacles in understanding arose due to the contradictory nature of infinity. However, encouraging a student to follow certain mental structures could lead to responses that indicate an understanding of problematic situations related to infinite repeating decimals or the set of natural numbers.

7.1 Implications

Most of the students perceived infinity as a process, as something we "keep doing" or something that "keeps going". This primary perception is maintained to a secondary perception and leads to the "flattening" phenomenon where all infinite sets are considered of equal cardinality. Although, by "measuring infinity", in the sense that mathematical infinity is presented by geometrical shapes of different size or emphasizing in the density of rational numbers by asking for a comparison of an interval of real numbers to the set of natural numbers, could lead to a very first thought for the existence of different kinds of infinity. I should also emphasize at this point, the importance of the distinction between the real-world and the mathematical realm, when talking to students about mathematical infinity. Maintaining a finitistic view on mathematical infinity, brings up feelings of rejection of an anyway precisely defined mathematical notion. Thus, when teaching mathematical concepts related to infinity, one should first raise cautiousness on the fact that sometimes mathematical entities behave counter-intuitively, so the mind needs to stray from real world experiences. Only then, actual infinity could be accepted as a sensible mathematical entity.

Some of the obstacles in understanding were identified as epistemological. These obstacles are not only unavoidable but moreover one should intentionally look for them. As they constitute a piece of knowledge and an inherent part of the process of understanding, they should not be viewed as obstacles to avoid but as critical moments for further cognitive development. By looking at the epistemological obstacles found in this study, lead me to draw some further conclusions. A main difficulty lies in the potential nature of infinity, due to which most of the students' errors were generated. Hence, the understanding of the actual nature of infinity once again appears as a necessity for avoiding errors. This understanding of the actual form of infinity underlies also the acceptance of the counter-intuitive functioning of the part-whole relationship between infinite sets. However, distinguishing between the mathematical properties and the psychological aspect of infinity, the ground should be first set by understanding that mathematics is a reality itself, sometimes not compatible with the real-world experiences. Only then, a student would understand the "metaphysical" aspect of infinity, accept and appreciate the behavior of infinity in a mathematical context.

The main contribution that we obtained from an analysis in terms of mental structures and mechanisms is an insight to student's thought, especially when it comes to infinite processes. This insight pointed to the crucial mechanisms of interiorization and encapsulation. That is, to help students to interiorize repeated actions without end, to reflect on seeing an infinite process as a completed totality and encapsulate then the process to construct an understanding object understanding of infinity. As previously mentioned, an object understanding of infinity accounts for an understanding of its actual nature, the nature that appeared difficult to grasp by the students. Moreover, a guidance for the student to follow the developmental stages is crucial. A guidance by giving hints through the genetic decomposition can be effective, as it can throw light in the construction of understanding and finally result to knowledge acquisition.

Finally, the aspects of understanding examined in this study, appeared to be related. Primary and secondary perceptions can indicate the causes for recurrent errors. In terms of our theory, perception can help in the categorization of errors, which in turn can help indicate epistemological obstacles. Furthermore, epistemological obstacles could be identified in the encapsulation or interiorization step. Whether the epistemological obstacles found in this study are the same that appear in the stages of APOS theory, is a question that requires further research.

7.2 Limitations

Although we managed to give answers to the research questions, there were some unavoidable limitations. First, because of the unexpected reluctance of bachelor students to participate in the study, the research was conducted on a small size of population who were upper secondary students. Therefore, a generalization of the study requires a larger group of participants. Furthermore, the cultural factor should be taken into account, meaning that the research was conducted on students who are educated within the Norwegian Education system. Even though the cultural effect on students' cognitive development has been found debatable through the literature (cf. Vygotsky), we consider it as a limitation of this study. Finally, the use of English language might have affected preciseness of students' answers.

7.3 Possible further research

Pedagogical strategies based on our previous analysis of students' understanding of infinity could be developed. For example, further research could be conducted in the implementation of a genetic decomposition of a concept related to infinity in a ACE Teaching Cycle (Arnon et al.,2014). The cycle consists of cooperative mathematical activities, instructor-led class discussion on the completed activities and homework exercises. Moreover, as previously mentioned, further research could be done on epistemological obstacles as they appear in the APOS stages and especially the encapsulation step, in order to shed light on the relation of the obstacles and the stages of cognitive development.

"We must know. We will know."

Hilbert, D.

REFERENCES

Archimedes, *The Works of Archimedes, edited in Modern Notation with Introductory Chapters by T.L. Heath* (Cambridge: at the University Press, 1897) Aristotle, Ross, S. D., Hardie, R. P., Gaye, R. K., Stocks, J. L., & Joachim, H. H. (1930). *The Works of Aristotle: Physica; De Caelo; De Generatione Et Corruptione*. Oxford University Press.

American Psychological Association. (2009). *Publication manual of the American Psychological Association.* Washington, D.C.: American Psychological Association.

Anstey, P., & Jalobeanu, D. (Eds.). (2010). *Vanishing Matter and the Laws of Nature: Descartes and Beyond*. Routledge

Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Fuentes, S. R., Trigueros, M., & Weller, K. (2014). *APOS theory: A framework for research and curriculum development in mathematics education*. Springer Science & Business Media.

Arrigo, G., D'Amore, B. (1999). *'Lo vedo ma non ci credo...'. Ostacoli epistemologici e didattici al processo di comprensione di un teorema di Georg Cantor che coinvolge l'infinito attuale*.['I see it but I don't believe it'.Epistemological and didactical obstacles in the process of understanding Georg Cantor's theorem on the actual infinity].*Teaching of Mathematics and integrated sciences* 22B, 5,465-494

Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Fuentes, S. R., Trigueros, M., & Weller, K. (2014). *APOS theory: A framework for research and curriculum development in mathematics education*. Springer Science & Business Media.

Bachelard, G. (1972). L'engagement rationaliste

Bachelard, G., & Jones, M. M. A. (2002). *The Formation of the Scientific Mind: A Contribution to a Psychoanalysis of Objective Knowledge:* Clinamen

Battista, M., Smith, M. S., Boerst, T., Sutton, J., Confrey, J., White, D., ... & Quander, J. (2009). Research in Mathematics Education: Multiple Methods for Multiple Uses. *Journal for Research in Mathematics Education*, *40*(3), 216-240.

Becker, H. S. (Ed.). (2002). *Boys in white: Student culture in medical school*. Transaction publishers.

Bell, J. L. (2005). *The continuous and the infinitesimal in mathematics and philosophy*. Polimetrica sas.

Benacerraf, P., & Putnam, H. (1983). *Philosophy of mathematics: Selected readings*. Cambridge University Press.

Ben-Menahem, A. (2009). *Historical encyclopedia of natural and mathematical sciences* (Vol. 1). Springer Science & Business Media.

Bowin, J. (2007). Aristotelian infinity. *Oxford Studies in Ancient Philosophy, 32*, 233-250

Boyer Carl, B., & Merzbach Uta, C. (1968). A history of mathematics. *Wiley*, *7*, 1.

Bolzano, B. *Paradoxien des Undenlichen.* [Paradoxes of the infinite] Translated by Steele, D. A., & Příhonský, F. (1950). Paradoxes of the Infinite by Dr Bernard Bolzano.

Burns, N., & Grove, S. K. (1993). The practice of nursing research. *Conduct, critique & utilization*, *4*.

Bringuier, J. C. (1980). Conversations with jean Piaget. *Society*, *17*(3), 56-61.

Brousseau, G. (2006). *Theory of didactical situations in mathematics: Didactique des mathématiques, 1970–1990* (Vol. 19). Springer Science & Business Media.

Brown, A., McDonald, M., & Weller, K. (2010). Step by step: Infinite iterative processes and actual infinity. *Research in Collegiate Mathematics Education VII*, *7*, 115.

Brousseau, G. (1997). Theory of didactical situations in mathematics (Edited and translated by N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield). *Dordrecht, NL: Kluwer*.

Cavalieri, B. (1980). 1647, Exercitationes geometricae sex, Bologna. *Cavendish, Charles, British Libra Descartes, René*, *1637*.

Chimisso, C. (2013). *Gaston Bachelard: Critic of science and the imagination*. Routledge.

Cihlar,J., Eisenmann, P., Kratka, M., & Vopenka, P. (2009) Cognitive conflict as a tool of overcoming obstacles in understanding infinity.*Teaching Mathematics and Computer Science* (pp. 279-295). Charles University of Prague,2009

Clegg, B. (2013). *A brief history of infinity: the quest to think the unthinkable*. Hachette UK.

Cobb, P. (2007). Putting philosophy to work. *Second handbook of research on mathematics teaching and learning: a project of the National Council of Teachers of Mathematics*, *1*(1).

Cohen D, Crabtree B. "Qualitative Research Guidelines Project." July 2006. http://www.qualres.org/HomeSemi-3629.html

Cooper, C., (n.d.) Toricelli's Trumpet.Retrieved from http://www.coopertoons.com/caricatures/torricellistrumpet.html

Cornu, B. (1991). Limits. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 153-166). Dordrecht: Springer Netherlands.
Cohen, L., Manion, L., & Morrison, K. (2000). Action research. *Research methods in education*, *5*, 226-244
Cortina, J. L., Visnovska, J., & Zuniga, C. (2014). Unit fractions in the context of proportionality: supporting students' reasoning about the inverse order relationship. *Mathematics Education Research Journal*, *26*(1), 79-99.
Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process scheme. *The Journal of Mathematical Behavior*, *15*(2), 167-192.
Davis, P. & *Hersh*, R. (1981). *The mathematical experience*. Boston: Birkhäuser. *Davis*, P. & *Hersh*
Dey, I. (2003). *Qualitative data analysis: A user friendly guide for social scientists*. Routledge.
Descartes, R., & Gilson, E. (2012). *Discours de la méthode*. Vrin.
Diether, B.(n.d.),Writing Chapter 3:Methodology.Retrieved from: http://www.bibme.org/citation-guide/apa/website/
Dowden,B.,(n.d.).The infinite. Retrieved from http://www.iep.utm.edu/infinite/#H5
Downden, B.(n.d.). *Zeno's Paradoxes*. Retrieved from: http://www.iep.utm.edu/zeno-par/
Drozdek, A. (2008). In the Beginning was the. *Apeiron: Infinity in Greek Philosophy, Stuttgart*.
Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. *Advanced mathematical thinking*, 95-126.
Dubinsky, E., & McDonald, M. A. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education research. In *The teaching and learning of mathematics at university level* (pp. 275-282). Springer Netherlands.
Dubinsky, E., Weller, K., Mcdonald, M. A., & Brown, A. (2005). Some historical issues and paradoxes regarding the concept of infinity: An Apos-Based analysis: Part 1. *Educational Studies in Mathematics*, *58*(3), 335-359.
Dubinsky, E., Weller, K., McDonald, M. A., & Brown, A. (2005). Some historical issues and paradoxes regarding the concept of infinity: An APOS analysis: Part 2. *Educational Studies in Mathematics*, *60*(2), 253-266.
Duval, R. (1983). L'obstacle du dedoublement des objets mathematiques.[The duality obstacle of mathematical objects] *Educational Studies in Mathematics*, *14*(4), 385-414.
Edwards, B. (1997). An undergraduate student's understanding and use of mathematical definitions in real analysis. In *Proceedings of the 19th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 17-22).
EndNote™ X8 [Computer Software],(2016). Clarivate Analytics.Retrieved from: http://endnote.com/downloads/30-day-trial
Ewald, W., & Ewald, W. B. (2005). *From Kant to Hilbert Volume 1: A Source Book in the Foundations of Mathematics* (Vol. 1). Oxford University Press on Demand.
Fischbein, E., Tirosh, D., & Hess, P. (1979). The intuition of infinity. *Educational studies in mathematics*, *10*(1), 3-40.
Fischbein, H. (1987). *Intuition in science and mathematics: An educational approach* (Vol. 5). Springer Science & Business Media
Fischbein, E., Jehaim, R., & Cohen, D. (1994). The irrational numbers and the corresponding epistemological obstacles. In J.P. Ponte & J.F. Matos (Eds.), *Proceedingsof the 18th PME International Conference, 2, 352-259*
Fischbein, E. (2001). Tacit models and infinity. *Educational Studies in Mathematics*, *48*(2), 309-329
Furinghetti, F., & Radford, L. (2008). Contrasts and oblique connections between historical conceptual developments and classroom learning in mathematics. *Handbook of international research in mathematics education*, 626-655.
Galilei, G. (1954). Discourses and mathematical demonstrations relating to two new sciences. *Leiden (1638)*.
Gardiner, A. (2012). *Infinite processes: background to analysis*. Springer Science & Business Media.
Gauss, C. F. (1813). *Disquisitiones generales circa seriem infinitam*: Apud H. Dieterich.
Giannakoulias, E. (2004). *"Sintomi Istoriki Anaskopisi tis Ekseliksis tou Apeirostikou Logismou"* [A Brief Historical Review of Infinitesimal Calculus], Athens: National and Kapodistrian University of Athens
Grabiner, J. V. (2012). *The origins of Cauchy's rigorous calculus*. Courier Corporation.
Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A" proceptual" view of simple arithmetic. *Journal for research in Mathematics Education*, 116-140
Harel, G. (2006). Mathematics education research, its nature, and its purpose: A discussion of Lester's paper. *ZDM*, *38*(1), 58-62
Heath, A. (2014). Mathematical Infinity and the Presocratic Apeiron. *Stance: An International Undergraduate Philosophy Journal*, *7*, 59-68.

Herscovics, N. (1989). Cognitive obstacles encountered in the learning of algebra. *Research issues in the learning and teaching of algebra*, 60-86.

Ho, W. K. (2008). Using history of mathematics in the teaching and learning of mathematics in Singapore. *1st RICE, Singapore: Raffles Junior College*.

Jago, M. [Numberphile] (2013, July). *Infinity Paradoxes – Numberphile*. Retrieved from https://www.youtube.com/watch?v=dDl7g_2x74Q&t=222s

Ji, S. (2010). *Lecture 8.Eudoxus of Cnidus Who Avoided a Fundamental Conflict*, Retrieved from https://www.coursehero.com/file/10191794/Fhw-s2a/

Jullien, V. (Ed.). (2015). *Seventeenth-Century Indivisibles Revisited* (Vol. 49). Birkhäuser.

Kattou, M., Thanasia, M. I. C. H. A. E. L., Kontoyianni, K., Constantinos, C. H. R. I. S. T. O. U., & George, P. (2009). Teachers' perceptions about infinity: A process or an object. In *Proceedings of CERME* (Vol. 6, pp. 1771-1780).

Knobloch, E. (1999). Galileo and Leibniz: Different approaches to infinity. *Archive for history of exact sciences*, *54*(2), 87-99.

Kleiner, I. (2012). *Excursions in the History of Mathematics* (Vol. 178). Springer Science & Business Media.

Kouremenos, T. (1995). *Aristotle on mathematical infinity* (Vol. 58). Franz Steiner Verlag.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic books

Lavine, S. (2009). *Understanding the infinite*. Harvard University Press.

LeCompte, M. D., & Goetz, J. P. (1982). Problems of reliability and validity in ethnographic research. *Review of educational research*, *52*(1), 31-60.

LeCompte, M. D., Tesch, R., & Goetz, J. P. (1993). *Ethnography and qualitative design in educational research*. Academic Press

Lester, F. K. (2005). On the theoretical, conceptual, and philosophical foundations for research in mathematics education. *ZDM*, *37*(6), 457-467

Luis, E., Moreno, A., & Waldegg, G. (1991). The conceptual evolution of actual mathematical infinity. *Educational Studies in Mathematics*, *22*(3), 211-231.

Makri, G.C. *"Antilipseis Mathiton Gia To Apeiro"*[Students' perception of infinity].,2015,Athens, National and Kapodistrian University of Athens

Mankiewicz, R. (2000). *The story of mathematics*. Cassell.

Mancosu, P. (1999). *Philosophy of mathematics and mathematical practice in the seventeenth century*. Oxford University Press on Demand

Maor, E. (1986). *To infinity and beyond: A cultural history of the infinite*. Boston: Birkhäuser. Chicago (Author-Date, 15th ed.) Maor, Eli. 1986

Mazur, B. (2007). How did Theaetetus prove his theorem? *The Envisioned Life: Essays in Honor of Eva Brann*, 235-253

Monaghan, J. (2001). Young peoples' ideas of infinity. *Educational studies in Mathematics*, *48*(2), 239-257.

More, L. T. (1934). *Isaac Newton: a biography* (Vol. 579). Dover publications.

Moru, E. K. (2007). Talking with the literature on epistemological obstacles. *For the Learning of Mathematics*, *27*(3), 34-37.

Moru, E. K. (2006). *Epistemological obstacles in coming to understand the limit concept at undergraduate level: a case of the National University of Lesotho* (Doctoral dissertation, University of the Western Cape).

Murty, M.R. (2013). Book Review [Review of the book PASSAGE TO INFINITY: Medieval Indian Mathematics from Kerala and its impact, by George, J.G.]. Hardy-Ramanujan Journal 36(2013),43-46

Newton,I. (1669). *Methodus Fluxiorum et Serierum Infinitarum*, manuscript of 1671; facsimile reprint Ed. Blanchard, A., Paris,1966

Newton, I. (2008). *The mathematical papers of Isaac Newton* (Vol. 7). Cambridge University Press.

Nuñez, R. (2003). Conceptual Metaphor and the Cognitive Foundations of Mathematics: Actual Infinity and Human Imagination. *Metaphor and Contemporary Science*, 49-72.

Núñez, R. E. (2010). Enacting infinity: Bringing transfinite cardinals into being. *Enaction: Toward a new paradigm for cognitive science*, 307-333.

Paradigm.2017. In *Merriam-Webster.com* Retrieved May 03,2017 from https://www.merriam-webster.com/dictionary/perception

Parker, M. (2009). Philosophical method and Galileo's paradox of infinity. *New Perspectives on Mathematical Practices. Hackensack, NJ: World Scientific*, 76-113.

Patton, M. Q. (1990). *Qualitative evaluation and research methods*. SAGE Publications, inc.

Pehkonen, E., Hannula, M. S., Maijala, H., & Soro, R. (2006). Infinity of numbers: How students understand it. *International Group for the Psychology of Mathematics Education*, *4*, 345.

Peirce, C. S. (1992). Reasoning and the Logic of things: The Cambridge Conferences Lectures 1898. edited by Kenneth Laine Ketner; with an introduction by Kenneth Laine Ketner and Hilary Putnam

perception.2017. In *Merriam-Webster.com* Retrieved May 09,2017 from https://www.merriam-webster.com/dictionary/perception

Pesic, P. (2003). *Abel's Proof: An Essay on the Sources and Meaning of Mathematical Unsolvability*. London; England: MIT Press. Retrieved from http://www.jstor.org/stable/j.ctt1dnnc74

Piaget, J., & Garcia, R. (1989). *Psychogenesis and the history of science*. Columbia University Press.

Piaget, J., (1980).*Adaptation and intelligence: Organic selection and phenocopy*. Chicago: University of Chicago Press, 1980.

Piaget, J. (1985). *The equilibration of cognitive structures: The central problem of intellectual development*. University of Chicago Press.

Project description(n.d),University of Agder, Retrieved from http://www.uia.no/en/about-uia/faculties/faculty-of-engineering-and-science/department-of-mathematical-sciences/about-the-master-thesis/project-description

Radford, L. (1997). On psychology, historical epistemology, and the teaching of mathematics: Towards a socio-cultural history of mathematics. *For the learning of mathematics*, *17*(1), 26-33.

Radford, L., Bartolini Bussi, M. G., Bekken, O., Boero, P., Dorier, J. L., Katz, V., ... & Vasco, C. (2002). Historical formation and student understanding of mathematics. *History in mathematics education*, 143-170.

Mathematics, *18*(4), 371-397.

Rassias, G. M. (Ed.). (1991). *The mathematical heritage of CF Gauss*. World scientific.

Research Data Management(n.d.), University of Tartu, Retrieved from https://sisu.ut.ee/rdm_course1/data-collection-processing-and-analysis

Rioux, J. W. (2004). Nature, the Soul, and God: An Introduction to Natural Philosophy: Cascade Books.

Rjlipton (2014). Diagonalization Without Sets. Retrieved from https://rjlipton.wordpress.com/2014/08/03/diagonalization-without-sets/

Rucker, R. (1982). Infinity and the Mind: The science and philosophy of the infinite.

Russell, B. (1914). *Our knowledge of the external world as a field for scientific method in philosophy*. Open Court.

Samsung Voice Recorder. (2017) Samsung Electronics Co., Ltd. (Version 20.1.83-89) [Mobile Application Software]. Retrieved from https://play.google.com/store/apps/details?id=com.sec.android.app.voicenote&hl=el

Sbaragli S.: *Teachers' convictions on mathematical infinity.* Doctoral thesis, FMFI UK, Bratislava 2004

Sondheimer, E. & A. Rogerson (1981). *Numbers and Infinity. A Historical Account of Mathematical Concepts*. Cambridge, MA: Cambridge University Press.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational studies in mathematics*, *22*(1), 1-36.

Sălăvăstru, D. (2014, January). Obstacles and Errors in the Appropriation of the Psychological Language. In *Argumentum: Journal the Seminar of Discursive Logic, Argumentation Theory & Rhetoric* (Vol. 12, No. 1).

Schechtman, A. (2014). Descartes's Argument for the Existence of the Idea of an Infinite Being. *Journal of the History of Philosophy*, *52*(3), 487-517.

Seale, C. (1999). Quality in qualitative research. *Qualitative inquiry*, *5*(4), 465-478.

Sharma, S. (2013). Qualitative approaches in mathematics education research: Challenges and possible solutions. *Education Journal*, *2*(2), 50-57.

Siegmund-Schultze,2016. *Small Questions for Peculiarity of Mathematics* [PDF Lecture Notes]. Retrieved from https://fronter.com/uia/main.phtml

Sierpińska, A. (1987). Humanities students and epistemological obstacles related to limits. *Educational studies in Mathematics*, *18*(4), 371-397. Silverman, D. (2001). *Interpreting qualitative data: Methods for analyzing talk, text and interaction.* London: Sage Publishers.

Singer, M., & Voica, C. (2003, September). Perception of infinity: does it really help in problem solving. In *The Mathematics Education into the 21st Century Project Proceedings of the International Conference*.

Singer, F. M., & Voica, C. (2008). Between perception and intuition: Learning about infinity. *The Journal of Mathematical Behavior*, *27*(3), 188-205.

Spagnolo F., *Opponent Report:Teacher 's Conviction On Mathematical Infinity*.Palermo, Retrieved from unibo.it: http://www.dm.unibo.it/rsddm/it/articoli/sbaragli/Spagnolo.pdf

Sriraman, B., & Törner, G. (2008). Political union. Mathematical Education Disunion: Building Bridges in European Didactic Traditions. In L. English et al (Eds) The Handbook of International Research in Mathematics Education, 660-694.
Stergiou, V.A. (2009). "*Istoriki Ekseliksi, Erminies kai Didaktikes Proseggiseis tis Ennoias tou Apeirostou"* [Historical evolution, Interpretations and Didactical Approaches of the Notion of Infinitesimal" [PhD dissertation], Patra: University of Patras
Struik, D. J. (2014). *A source book in mathematics, 1200-1800*. Princeton University Press.
Tall, D. (1980). The notion of infinite measuring number and its relevance in the intuition of infinity. *Educational Studies in Mathematics*, *11*(3), 271-284
Tall, D. (1980). Mathematical intuition, with special reference to limiting processes. In *Proceedings of the Fourth International Conference for the Psychology of Mathematics Education* (pp. 170-176).
Tall, D. (1981). Intuitions of infinity. *Mathematics in School*, *10*(3), 30-33.
Tall, D., Thomas, M., Davis, G., Gray, E., & Simpson, A. (1999). What is the object of the encapsulation of a process?. *The Journal of Mathematical Behavior*, *18*(2), 223-241.
Theodosiou, S. (2010) *"Simpantikes Enoraseis: Astronomia kai Poiisi"* [Cosmic Insights: Astronomy and Poetry]. Athens, March 2010: Diavlos.
Tirosh, D., & Tsamir, P. (1996). The role of representations in students' intuitive thinking about infinity. *International journal of mathematical education in science and technology*, *27*(1), 33-40.
Tropp, J. A. (2002). Infinitesimals: History & Application. *University of Texas, Texas, Austin*.
Tsamir, P., & Tirosh, D. (1999). Consistency and representations: The case of actual infinity. *Journal for Research in Mathematics Education*, 213-219.
Tsamir, P. (2001). When The Same is not perceived as such: The case of infinite sets. *Educational studies in mathematics*, *48*(2-3), 289-307.
Vialar T. Handbook of mathematics. Ed. Books on Demand,Paris, 2015.
Vilenkin, N. Y. (2013). *In search of infinity*. Springer Science & Business Media.
Villarmea Requejo, S. (2011). The revolution of thought: from mythology to Hellenistic science.
Vamvakoussi, X., Vosniadou, S., & Van Dooren, W. (2013). The framework theory approach applied to mathematics learning.Routledge
Weller, K., Brown, A., Dubinsky, E., McDonald, M., & Stenger, C. (2004). Intimations of infinity. *Notices of the AMS*, *51*(7), 741-750.
Zeno's Paradox of Motion. (n.d.). Retrieved from http://www.mathpages.com/rr/s3-07/3-07.htm
Zippin, L. (1962*). Uses of Infinity*. Mathematical Association of America. doi:10.5948/UPO9780883859247

[This page is intentionally left blank]

Appendix A

<u>TASK 1</u> (daily experience)

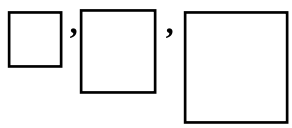
What does infinity mean to you?
What is the biggest number you know?
What is the smallest number you know?
Please make a sentence that contains the word "infinity"

<u>TASK 2</u> (understanding of sets)

Which of the following sets has the biggest cardinality? Justify your answer

a) **The set of natural numbers or the set of even numbers**
b) **The set {1,2,3,4,…} or the set B={1,1/2,1/3,1/4,…}**
c) **The set A= {1,2,3,4,…} or the set B= {1,3,5,7,…}**

1cm 2cm 3cm …

d)The set of squares A={  **,..} or the set $\{1^2, 2^2, 3^2, \ldots\}$**

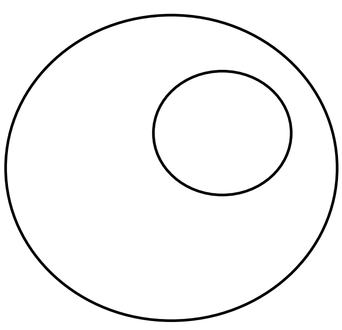
e) The set of natural numbers and the set of real numbers

How many elements are there in the set S= {-3, -2, -1, 0, {1, 2, 3,…}}?

<u>TASK 3</u> (real numbers/rational numbers, more real than rationals)

Are there any numbers between 1 and 2? If yes, what is the closest number to 2?
Are there any numbers between 1 and 3? If yes, how many?
If any, are there more numbers between 1 and 2 than 1 and 3?
Do you agree or disagree with the statement that: "π has infinite decimal places"?

<u>TASK 4</u> (geometrical perspective-limit)



How many points are there in each circle? Can you compare the number of points between the two circles?
How many lines can you make through a point?
What is the polygon with the most number of interior angles you can imagine?

Appendix B

Participant:

<u>TASK 1</u>

1.1. How many numbers are there between 1 and 2?
1.2. How many numbers are there between 1 and 3?
1.3. Are there more numbers between 1 and 3 than 1 and 2?
1.4. How many numbers are there between numbers 0.8 and 1.1?
1.5. What is the biggest number yet smaller than 2?
1.6. Do you agree or disagree with the statement that: "π has infinite decimal places"? Justify your answer

<u>TASK 2</u>

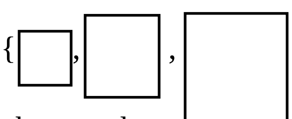
2.1. Which of the following sets has more elements? Please justify your answer for each case

a) The set of natural numbers or the set of even numbers

b) The set A= {1,2,3, 4,...} or the set B={1,$\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, ...}

c) The set A= {1,2,3, 4,...}
or the set B= {1,3,5, 7,...}

d)The set of squares A={ 1cm, 2cm, 3cm



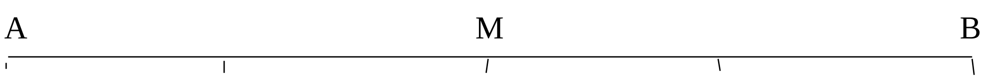
,...} or the set B={1^2,2^2,3^2,...}

e) The set of natural numbers or the set of real numbers

2.2 How many elements are in the set $\{-\mathbf{3}, -\mathbf{2}, -\mathbf{1}, \mathbf{0}, \{\mathbf{1}, \mathbf{2}, \mathbf{3} \ldots\}\}$

<u>TASK 3</u>

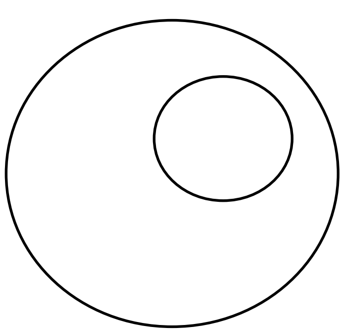
3.1. *We divide AB into two equal segments AM and MB. Then we divide AM to AD=DM and MB to MG=GB. We continue dividing in the same manner. Do you think that we will arrive in a situation that the segments will be so small that we will be unable to divide them?* Please justify your answer



3.2. What is the regular polygon(*mangekant*) with the most number of sides you can imagine?

3.3. How many lines can you make through a point?

3.4.



How many points are there in each circle? Can you compare the number of points between the two circles? Please justify your answer

TASK 4

4.1. What does infinity mean to you?
4.2. What is the biggest number you know?
4.3. What is the smallest number you know?
4.4. Please make a sentence that contains the word "infinity"

Tusen takk!

Appendix C

NOTIFICATION FORM

Notification form (version 1.4) for student and research projects subject to notification or license
(cf. the Personal Data Act, the Personal Health Data Filing System Act and associated Regulations).

1. Intro		
Will directly identifiable personal data be collected?	Yes ● No ○	A person will be directly identifiable through name, social security number, or other uniquely personal characteristics. Read more about personal data. NB! Even though information is to be anonymised in the final thesis/report, check the box if identifying personal data is to be collected/recorded in connection with the project.
If yes, please specify	■ Name □ Social security number □ Address □ E-mail □ Phone number □ Other	
If other, please specify		
Will directly identifiable personal data be linked to the data (e.g. through a reference number which refers to a separate list of names)?	Yes ○ No ●	Note that the project will be subject to notification even if you cannot access the link to the name list/register, as the procedure often is when using a data processor
Will there be collected background information that may identify individuals (indirectly identifiable personal data)?	Yes ● No ○	A person will be indirectly identifiable if it is possible to identify a person through a combination of background information (such as place of residence or workplace/school, combined with information such as age, gender, occupation, etc.). NB! In order for a voice to be considered as identifiable, it must be registered in combination with other background information, in such a way that a person can be recognised.
If yes, please specify	There will be a reference during the data analysis on the schools-year of study that the questionnaire+interview takes place.	
Will there be registered personal data (directly/indirectly/via IP or email address, etc.) using online surveys?	Yes ○ No ●	Read more about online surveys.
Will there be registered personal data using digital photo or video files?	Yes ○ No ●	Photo/video recordings of faces will be regarded as identifiable personal data.
Have you applied for an assessment from REC regarding whether the project should be considered health research?	Yes ○ No ●	NB! If REC (Regional Committees for Medical and Health Research Ethics) has assessed the project as health research, you do not have to submit a notification form to the Data Protection Official. (NB! This does not apply to projects using data from pseudonymous health registries.) If you have not received a reply from REC, we recommend that you await filling out a notification form until you have received a reply.
2. Project Title		
Project Title	History in the service of teaching: students' conception of infinity	Please state the project title NB! This cannot be "Master's thesis" or the like, the must describe the content or aim of the project.
3. Responsible Institution		
Institution Section/Faculty Department	Universitetet i Agder Fakultet for teknologi og realfag Institutt for matematiske fag	Select the institution to which you are affiliated. All administrative levels must be specified. If it is a student project, select the institution to which the student is affiliated. If your institution is not listed, please contact the institution.
4. Project Leader (Researcher, Supervisor, Research Fellow)		

First name	John	Fill in the name of the person who will have the day-to-day responsibility for the project. In a studentproject, this will usually be the student's supervisor.
Surname	Monaghan	The student and the supervisor should usually be affiliated with the same institution. If the student has an external supervisor, the assistant supervisor at the student's place of study should be registered as the project leader.
Position	Professor	
Telephone	38	
Mobile	141750	Place of work must be affiliated with the responsible institution, e.g. a department, institute or section.
Email	john.monaghan@uia.no	
Alternative email	J.D.Monaghan@education.leeds.ac.uk	Please notify us if you change your e-mail address.
Place of work	Universitetet I Agder,Campus Kristiansand	
Address (work)	Gimlemoen 25	
Postcode/city (work)	4630 Kristiansand S	

5. Student (master, bachelor)

Student project	Yes • No ○	If the project will be carried out by more than one student, please choose one as contact person. Remaining students can be added under question 10.
First name	Sokratis	
Surname	Theodoridis	
Telephone	+47	
Mobile	46719239	
Email	sokrat16@student.uia.no	
Alternative email	msoctheod@yahoo.com	
Address (home)	Kasernevelen 12	
Postcode/city (work)	4630 Kristiansand S	
Specify project type	• Master's thesis ○ Bachelors' thesis ○ Semester paper ○ Other	

6. Objective

What is the purpose of the project?	The aim of this thesis is to investigate and identify the epistemological obstacles in coming to understand the infinity concept and the interactions between history and mathematics education.According to Brousseau's didactical theory of obstacles, they are visible when one looks into the historical development of the mathematical concept in question.Hence, there will be a historical analysis of the concept of infinity while seeking for the mechanisms (Piaget and Garcia,1983) that led to the evolution of the notion of infinity, in order to relate these mechanisms to the cognitive development of the individual,in an attempt to investigate even further into students' thinking about the concept of infinity	Briefly describe the purpose or theme of the project and/or the research question.

7. Sample

Please specify your sample	□ Children attending day-care institutions □ School children □ Patients □ Users/clients/customers □ Employees □ Children connected to child welfare □ Teachers □ Health/medical personnel □ Asylum seekers ■ Other	
Describe the sample/participants	University of Agder 1st and 3rd year bachelor students International Baccalaureate® International School students	The sample refers to those who participate in the study or whom you collect information about.

How will personal data be registered and stored/processed?	☐ On a server in a network belonging to the institution ☐ An isolated computer belonging to the institution (i.e. with no connection to other computers or networks, internally or externally) ☐ A computer in a network with internet access belonging to the institution ■ Private computer ☐ Video recordings/photographs ■ Audio recordings ■ Manually/on paper ■ Portable storage device (laptop, USB , memory card, CD, external hard drive, mobile phone etc.) ☐ Other	Please specify each of the different ways the data will be registered/processed. You may check more than one box if applicable. By "institution" we mean the institution responsible for the project. NB! As a general rule, personal data should be stored on a research server belonging to the responsible institution. Using other media for storing - such as private computer, mobile phone, USB, external server - is less secure, and must therefore be given account for. Such storing must also be clarified with the responsible institution, and personal data should be encrypted.
Other, please specify		
How will the data be protected from unauthorized access?	Protected computer password, during data procession computer will never be left unattended. Paper data will be scanned and stored in the computer.Audio recordings will also be transferred from my mobile phone(protected) to the computer.	For instance, will the computer be password protected, will the computer be kept in a locked room, how will portable units, printouts, recordings etc. be protected from unauthorized access?
Will data be gathered/processed by an external processor?	Yes ○ No ●	External processor refers to someone who gathers or in other ways processes personal data on behalf of the person/institution responsible for the project e.g. supplier of electronic questionship, transcription service provider or interpreter. These assignments must be regulated by a contract.
If yes, please specify		
Will personal data be gathered or transferred through e-mail/the internet?	Yes ○ No ●	For instance by transferring data to a collaborator, data processor etc. If personal data is to be sent by email, the information should be adequately encrypted. We do not recommend not storing personal data on cloud services. If cloud services are used, a data processor agreement with the supplier of the service must be signed.
If yes, please specify		
Will there be others working on the project, in addition to the project leader/student, who will have access to the data?	Yes ○ No ●	
If yes, who? (name and place of work)		
Will personal data be shared with national or international institutions?	● No ○ Yes, national institutions ○ Yes, institutions in other countries	I.e. in national or international multicenter studies where personal data is shared.
11. Assessment/approval by other regulating bodies		
Will your project require a dispensation from the duty of confidentiality in order to gain access to data?	Yes ○ No ●	In order to gain access to information which is subject to duty of confidentiality, e.g. data from hospitals, the Labour and Welfare Administration (NAV), or other public institutions, you must apply for a dispensation from the duty of confidentiality. A dispensation is normally granted by the relevant government department.
If yes, please specify		
Will your project require assessments/approval from other regulating bodies?	Yes ○ No ●	For instance, the registry owner may need to grant access to data, or permission may be needed from the management before a company can be used as a subject in a research project.
If yes, please specify		
12. Period for processing of personal data		
Start of project End of project	06.03.2017 31.03.2017	Start of project: The date when the sample will be contacted and/or when the collection of data will begin. End of project: The date when the data will be anonymised, deleted, or filed in order to be included in a follow-up project.
Will personal data be published (directly or indirectly)?	☐ Yes, directly (name etc.) ■ Yes, indirectly (identifiable background information) ☐ No, anonymous	NB! If personally identifiable information is to be published, explicit consent must be collected from each person, and they should be given the opportunity to read through and approve of any quotes.
What will happen to the data when the project is completed?	■ The data will be anonymised ☐ The data will be filed with personal identification	NB! Here we mean the data material, not the publication. Even if personal data is to be published, the remaining data is usually to be anonymised.Data which is anonymised can no longer be traced back to individuals. Read more about anonymising data.

13. Finance		
How will the project be financed?	Expenses for stationery will be paid by me	
14. Additional Information		
Please add any additional relevant information		

Appendix D

Change Request Form

For changes made in student and research projects that are subject to notification or license

(cf. The Personal Data Act and the Personal Health Data Filing System Act with associated regulations)

Form to be sent by email to personvernombudet@nsd.uib.no

1. PROJECT	
Project leader/supervisor: John Monaghan Student: Sokratis Theodoridis	Project number: 52487

2. CHANGE(S)	
New project leader/supervisor:	*When changing project leader, a confirmation from both former and new project leader must be enclosed. If the project leader no longer works at the institution, a confirmation from the department can be enclosed.*
New date of anonymisation: Start of Project-**01.03.2017** End of Project-**31.12.2017** Will there be given new information to participants? Yes: ____ No: __x__ If no, please explain why: date of anonymization is not extended for more than one year	*If the date of anonymisation will be extended for more than one year, new information should be given to the participants.*
Additional method(s);	*Fill inn which methods will be used, for instance interviews, questionnaires, observation, registries, etc.*
Additional sample/participants: "Please specify your sample": Other(meaning University students**)+School Children** "Sample age": Adults **+Adolescents (16-17 years old)**	*In case of small changes in the number of participants, a Change Request Form may not be necessary. If in doubt, contact us before submitting the form.*
Other changes: "12.Period for processing of personal data-Will personal data be published(directly or indirectly)?: **No,anonymous** (*instead of Yes,indirectly*)	

Appendix E

Request for participation in research project "History in the service of teaching: students' conception of infinity"

Background and Purpose The purpose of the project is to investigate students' conceptions of infinity and the origins of some possible misconceptions. Briefly, there will be a historical approach to the subject in question by investigating the transitional mechanisms of the individual's and the historic cognitive development. This project is a Master's thesis for the Department of Mathematical Sciences at the University of Agder. Students of your institution and your year of study, have been selected in order to have an overview on the conception of infinity at the senior high school level.

What does participation in the project imply?

All that is required for the participants is that their answers are given based on their own knowledge and intuition without the use of external sources. Participants may ask on the context of each question but not on information that could indicate a possible answer. Data will be collected at two sessions: 1) a writing session-answering the questions (durations 30 minutes 2) an oral session(interview-discussion for each participant-10 minutes approximately). Questions will concern the several aspects of the main research topic. Data will be collected by paper and audio recordings.

What will happen to the information about you?

All personal data will be treated confidentially. Access to personal data will have only the writer of the thesis. Data will be stored in a personalized secured laptop.

Participants will not be recognizable on the publication.

The estimated end date of the project is 31.12.2017. All collected data will be made anonymous while audio recordings and paper sheets will be destroyed by project completion.

Voluntary participation

It is voluntary to participate in the project, and you can at any time choose to withdraw your consent without stating any reason. If you decide to withdraw, all your personal data will be made anonymous.

If you would like to participate or if you have any questions concerning the project, please contact: Sokratis Theodoridis-tel +4746719239 / John Monaghan-tel 38 141750

The study has been notified to the Data Protection Official for Research, NSD - Norwegian Centre for Research Data.

Consent for participation in the study

I have received information about the project and am willing to participate

----

(Signed by participant, date)