



UNIVERSITY OF AGDER

Laboratory mathematics: learning equations with a double-winch as a tool

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This master's thesis is carried out as a part of the education at the University of Agder and is therefore approved as a part of this education. However, this does not imply that the University answers for the methods that are used or the conclusions that are drawn.

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Preface

I would like to express my sincere gratitude to the people who made this study possible.

First of all, I am indebted to Professor Francisca Pauline Vos, who supported and nourished my interest to didactics research and in particular, to the subject of Laboratory Mathematics. Pauline's help can not be overestimated and went beyond scientific mentoring and supervision, stretched past advising on the appropriate literature. Her constant encouragement and support throughout the study was an important part of my confidence in the project that I had undertaken. The freedom she allowed in this study contributed greatly to my growth as a researcher and instructor. At last, her erudition and positive disposition to students made it a pleasure to discuss a variety of subjects with her, ranging from science to everyday life.

I would also like to thank the academic and administrative staff at the University of Agder who contributed greatly to the productive, creative and supportive atmosphere at the University and were always available to help in my work as well as in other issues of the University life.

It is to be mentioned, that my research would have been impossible without the technologically advanced, extraordinary library of the University that gave me access to the widest selection of literature on my subject.

My work on the thesis would have been much harder without the help of my colleagues and friends from Russia – the teachers from my secondary school, the colleagues of my undergraduate Moscow Pedagogical University and the wide circle of my friends who supported, encouraged me, and were curious and inquisitive about my project.

My sincere gratitude is due to the schoolchildren who participated in the study as well as to their parents. The enthusiasm they had shown fueled the study from the beginning and kept the research momentum going to this final stage.

I would like to personally acknowledge a mathematics teacher Valentina Samutina from my secondary school, who awakened an interest and love to mathematics and, through her enthusiasm for teaching, evoke a desire to teach mathematics.

A special “thank you” goes to my children, who have shown patience, understanding, empathy and sympathy to the fact that their mother had to accomplish such a significant endeavor. I hope they will be as proud of me as I am of them.

Thank you all very much. I hope that my work will be useful and of interest to the readers.

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Sammendrag

Flere studier viser at elever oppfatter algebra som et abstrakt og vanskelig forståelig emne, nesten uten forbindelse til hverdagslivet. Motivasjonen som ligger til grunn for denne forskningen var derfor å finne ut hva som bidrar til å gjøre det abstrakte emnet algebra mer meningsfullt for elever, og hvordan vi kan hjelpe dem til å oppdage at algebra kan være nyttig for å etablere sammenhenger mellom ulike variabler.

Å benytte mekaniske verktøy med par av vekter som beveger seg med forskjellig hastighet, kan være nyttig med tanke på undervisningsformål. Først av alt, - de forskjellige hastighetene på vektene gjør variabler og deres sammenheng håndgripelig. For det andre, kjenner elevene til verktøyets fysiske egenskaper fra ikke-matematiske erfaringer i livet, og dette kan bidra til deres læring av algebra. For det tredje, vekker enhetens ukjente egenskaper nysgjerrighet hos elevene og dermed bidrar til å holde dem motiverte. Til slutt, - elevene kan være aktivt engasjert ved å undersøke noe selv.

Hines (2002) og Izsak (2004) har beskrevet bruken av et spolesystem, et sett av koaksialsylindere av varierende diameter, for å fremme læring av lineære funksjoner. Inspirert av disse studiene, laget jeg et flerkasus-studie med en dobbel mekanisk vinsj for å utforske bruken av denne enheten med tanker på matematikk undervisning. Enheten gjorde det mulig å visualisere begrepet lineær ligning.

Forskningsspørsmålet er; hvilke muligheter for å lære begrepet lineær ligning og hvilke aspekter av dette begrepet, er eksponert i et dobbelvinsj-laboratorium?

En rekke oppgaver ble utviklet (utforsking av verktøyet, noe som bidrar til prediksjoner). En rekke økter ble gjennomført med fem russiske elever (6. og 8. trinn). Øktene ble videofilmet. Transkripsjonene ble analysert med hensyn til elevenes forklaringer på forholdet mellom variablene og deres bruk av representasjoner.

Elevene på 6. trinn ga muntlig uttrykk for sammenhengen mellom variablene høyde og antall omdreininger og beskrev høyde per omgang, uten behov for symboler. En ikke-standard måte å skrive den lineære ligningen på, ble lagt fram av dem. Elevene på 8. trinn var fortrolig med standard symbol-ligninger og brukte disse i sine prediksjoner. Alle elevene foretrakk å sette inn sine målinger i tabeller, ikke i grafer. Elevene var aktive, dynamiske og trygge, og de brukte enheten til å kontrollere sine formler og beregninger.

Summary

Many studies show that algebra is perceived by pupils as an abstract, difficult to understand subject with almost no connections to every-day reality. Therefore, the motivation for this research was to find out what helps to make the abstract topic of algebra more meaningful to pupils and how we can assist them in discovering that algebra can be useful for establishing connections between different variables.

Using mechanical tools with pairs of weights that move with different velocities could be promising for instructional purposes. First of all, the different velocities of the weights make variables and their relationship tangible. Secondly, pupils know some physical properties of the tool from non-mathematical life experiences, and this can support their learning of algebra. Thirdly, the unknown properties of the device arouse curiosity thus helping to keep the pupils motivated. Finally the pupils can be actively engaged by investigating something themselves.

Hines (2002) and Izsak (2004) described the use of a spool system - a set of coaxial cylinders of varying diameters - to promote the learning of linear functions. Inspired by these studies, I designed a multiple case-study with a mechanical double winch for exploring the use of this device for mathematics instruction. The device allowed visualization of the concept of linear equation.

The research question was: what opportunities for learning the concept of linear equation and what aspects of that concept are exposed in a double-winch laboratory?

A series of tasks were developed (exploring the tool, making predictions). A series of sessions were conducted with five Russian pupils (grade 6 and 8). The sessions were videotaped. Transcriptions were analyzed with respect to students' explanations of relations between variables and their use of representations. The grade 6 pupils expressed the relationships between the variables (height and number of turns) verbally in terms of the situation (height per turn), without a need for symbols. Nonstandard way of writing of the linear equation was offered by them. The grade 8 pupils were at ease with standard symbolical equations using these for their predictions. All pupils preferred to put their measurements in tables, and not in graphs. Pupils were active, dynamic and confident, and they used the device to verify their equations and calculations.

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1 Introduction

Few years ago I moved to Norway and was surprised by the perception of physical sciences and mathematics as of something obscure and very difficult to learn. I observed that mathematics was regarded as something special that belongs only to books having nothing to do with life, something of purely theoretical significance. Perhaps it is the combination of the two factors, a fear of difficulty and a false perception of uselessness contributes to the less than suitable performance that Norwegian pupils show in international tests such as Pisa and TIMSS. These tests investigated the mathematical knowledge of pupils in different countries and confirmed that Norwegian pupils perform below average compared to other European countries and far worse in algebra (Grønmo et al., 2012).

Although I recognize the difficulties and intellectual challenges, associated with mathematics and physical sciences, I believe that they should not appear obscure and moreover, they should not be perceived as “useless”. The difficulty of the subjects could be overcome and the beauty of them can be exposed.

I think and hope that I could be useful in this area. I wish to show to the pupils that mathematics is not something that is boring and complicated, but is a subject that can be very useful, helpful and, surprisingly, enjoyable and interesting. My great hope and aspiration is to motivate pupils to such an extent that they will regard mathematics not as “difficult” but as “challenging”. Moreover, I want the pupils to realize that mathematics is not just a good training for the brain providing a good thinking discipline, but a skill that can be useful in real life.

Building a positive disposition towards mathematics and algebra in particular takes skills from the instructor. I recall my experience of learning mathematics in middle and high school and recognize that learning mathematics was reduced to purely theoretical studies combined with solving great many problems for the text book. The experience was often exhausting and tedious. I could not see any practical value in school mathematics except, probably, algebraic text problems, where practical quantitative problems could be solved quickly with algebra. However, in order to enjoy solving difficult problems, mathematics had to be practiced on a huge number of routine problems to acquire “a mathematical flair”. The meaning of equations and functions remained an obscure enigma to me whose purpose was exclusively limited to “brain gymnastics”.

My school-time perception of mathematics is paralleled in research findings on mathematics didactics showing that algebra is viewed as an abstract subject that is difficult to understand. Moreover, pupils do not see much connection of algebra with reality and sometimes even do not see any usefulness in it, arguing that a productive, fulfilling life is possible without any knowledge of algebra (Usishkin, 1995). Pupils overlook the conceptual power of algebra as a generalization of arithmetic or as a general way to study relationships between quantities (Usiskin, 1988). Exposing these aspects of algebra is definitely an educational opportunity that should not be missed.

When I started studying physics, I began to understand the meaning and purpose of mathematical functions much better and clearer. Studying physics was accompanied by laboratory exercises in mechanics, hydrodynamics, electromagnetism and optics with a very direct and unambiguous connection to the reality and the outside world, further reinforcing my interest to the subject. Therefore, when I was to decide what subject to choose for further specialization, I chose physics.

I studied at the Moscow Pedagogical University for five years and obtained the Masters of Science degree as a teacher of physics and computer science. The laboratory experience was an important part of my studies, allowing better understanding, consolidating and retaining the theoretical knowledge presented in books and lectures. Later on, when studying mathematical didactics at University of Agder, I wondered, if there existed laboratory exercises for teaching mathematics and if they did, what kind of “laboratory” could be used to teach mathematics? My literature and internet search yielded scarce results. Therefore, I was delighted to learn that Paulina Vos, a professor in University of Agder, worked in this area and I gladly accepted her offer to conduct research on the subject.

Laboratory mathematics is something that can connect mathematics with real life. Mathematical concepts become visible and touchable. Using mechanical tools could be very promising. First of all, mathematical laboratory visualizes the process, makes the mathematics behind the observed process real. Secondly, the physical properties of the tool are known from the previous non-mathematical life experience, which can support learning situation. Finally, the best way to study is to let pupils investigate something themselves based on the knowledge they have from the past.

My goals and challenges in using mathematical laboratory were to demonstrate that algebra is useful and motivate school pupils to learn it. Other goals were to perform research on mathematics didactics – to observe how the learning occurs, to observe the effects of learning, thus expanding the didactics research base for teachers and researchers.

2 Theoretical framework

2.1 Realistic teaching: teaching mathematics through modeling and applications

2.1.1 Discovery learning and creating the knowledge

This study is conducted in the frameworks of exploring one of the recent trends in mathematical didactics. It's often called "realistic teaching" or "learning by doing". The origin of this trend is in a wide-spread consensus that one of the difficulties in teaching mathematics is in the fact that the classical "lecture-type" approach leads to a considerable gap between the mathematical knowledge and practical environment. Pupils see, for example, algebra, as an abstract subject without connection with reality.

A way to overcome this disconnection is through making the instruction more "interactive" and connected to "reality". It is expected that innovating the instruction methods, abandoning the old lecture-type "talk-and-chalk" approach will not only make the instruction more efficient, but also "liberate" the creativity of students and pupils. (Alsina, 2007; Pollak, 1969, 2007). Thus, teaching should be understood broader than just passively accumulating knowledge. The goal of any instruction or learning should be not only in "instilling" the knowledge and concepts already discovered by the culture or society, but in making the learner capable of generating new knowledge and ideas.

This goal has been recognized long time ago. According to Jerome Seymour Bruner, the goal of education is not in simply conveying knowledge, but in teaching how to work with the information learned and apply it for problem solving (Bruner, 1966). He maintained that to achieve this goal the learner should learn through an active process by making "sense" of the environment and the environment self-manifestation. As a result, the knowledge is produced during the instruction process, not simply distributed. This premise is still widely recognized even by modern researchers (Jacobs, Hurley, & Unite, 2008). Bruner emphasized that students are active learners and learning necessarily involves constructing their own knowledge: he named it "discovery learning" (Bruner, 1966).

The cognition of the laboratory algebra experiment involves three modes of data representation: enactive, iconic and symbolic (Bruner, 1966):

- Enactive (actions, doing) – often comes first and is difficult to codify otherwise than through action.
- Iconic (picture, image) – knowledge, stored visually in the form of an image. A famous saying "a picture is worth a thousand words" represents this concept that any diagrams or pictures are often useful to accompany a verbal description.
- Symbolic (formulas, numbers). – almost always comes last and is the hardest to be enacted. At the same time it is often times the most efficient and adaptive form of representation. Symbolic representation has advantage that the symbols can be manipulated, classified, be adapted to a variety of tasks including abstract concepts.

These modes of representation essentially codify the ways in which knowledge is absorbed and stored in the memory of the learner. They are not clearly determined; the boundaries between these modes are often washed out. In setting up our research sessions, we paid attention to observing the 3 modes of representation and the interaction between them. The reason for that was that Bruner emphasized that the cognition occurs most effectively when cognitive representation follows the described order: Enactive → Iconic → Symbolic. Thus, a

proper organization of concept representation is a necessary and sufficient condition for efficient instruction. In other words, the learning of any concept, even a “difficult” one is greatly facilitated if the presentation of the knowledge is properly organized, according to the Enactive → Iconic → Symbolic flow chart. Bruner emphasized that students are active learners and learning necessarily involve constructing their own knowledge – discovery learning (Bruner 1961).

The following features of Bruner’s theory are relevant to my study:

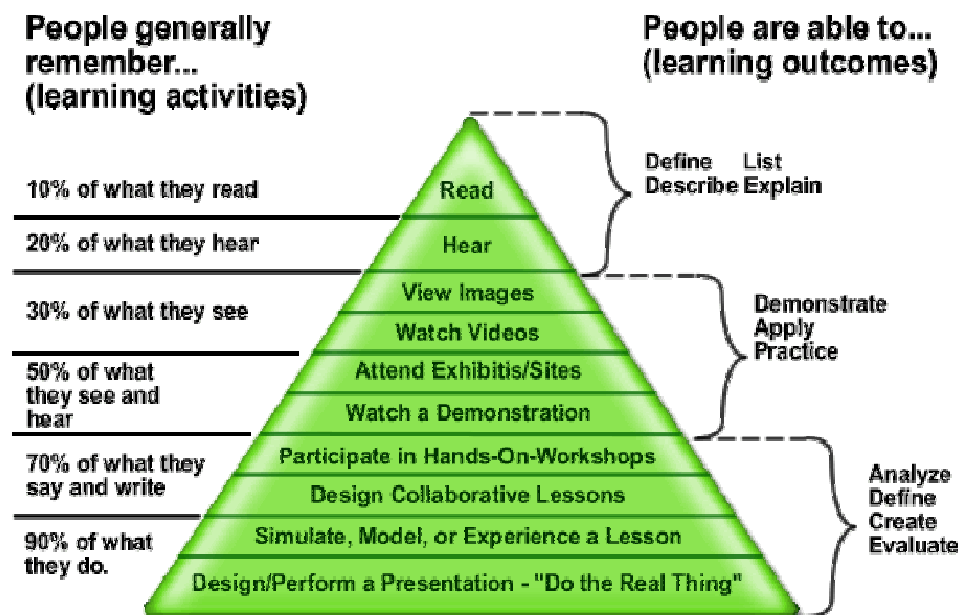
- a) Active approach to learning.
- b) Knowledge should be produced.

The dynamic nature of knowledge was also recognized by Jean Piaget. He regarded knowledge as a perpetual unfolding process, not a static response to the environment or instruction. “Scientific thought, then, is not momentary; it is not a static instance; it is a process. More specifically, it is a process of continual construction and reorganization” (Piaget (1971) in Brooks (1993)).

Thus, according to Piaget, knowledge consists of more than just perception or concepts. Piaget considered perception as static and extremely limited; he had exhibited a strong commitment to knowledge arising from action (Piaget, 2000). This strong conviction motivated me further to conduct a study on teaching mathematics with the means of “laboratory algebra” where abstract algebraic concepts could be represented with some real-world activities.

The importance of interactive context for cognitive development and effective instruction is emphasized in Edgar Dale’s “Cone of Experience” model (Dale, 1969). Edgar Dale proposed that information retention happens more effectively through doing rather than through “listening”, “reading” or “observing”. To illustrate his model, Edgar Dale even constructed a visual representation of the Cone of Experience that has been reproduced in various forms in the literature.

Figure 2.1.1: Dale’s “Cone of Experience”



Briefly, according to Dale, people remember only 10% of what they read as opposed to 90% of what they do. Moreover, the qualitative outcome or learning is also different. In learning through reading people become capable of defining and describing, while learning through doing shifts the outcome to analyzing, creating, designing. Although Dale himself warned about a possible overuse of his concept, one could acknowledge that the most effective methods of instruction involves direct, purposeful, hand-on learning experiences. The underlying premise is that effective learning is achieved when knowledge is produced, re-discovered rather than distributed or “instilled”.

Interestingly, this notion of learning through doing has been present in the human culture for centuries in one form or another, with various degrees of sophistication and detail. A curious example can be found in the ancient Chinese philosophy in an aphorism ascribed to Confucius: “I hear and forget, I see and remember, I do and understand”.

2.1.2 Usefulness of mathematics presented through modeling and applications

Although it is regarded as a recent innovation, the use of modeling and applications in mathematics instruction has a considerable history. It has been known since late sixties – early seventies from proponents of the so-called New Math movement (Pollak, 1969), (Pollak, 2007).

One of the reason for the emergence of the New Math movement was an attempt to overcome the disconnect on how mathematics is viewed by instructors and the mathematics students. Most often, mathematics is taught by mathematicians, the people who studied mathematics and worked in mathematics professionally. For mathematicians, the “beauty” of mathematics and the intellectual stimulation it provides is a sufficient reason to study it.

However, it is not the case on the other end of the professor-student pair. The students taking mathematics, the pupils in schools, often represent a broader range of attitudes towards mathematics. It is not enough to motivate them by simply showing the intellectual beauty of mathematics. The most frequent question asked by the students of mathematics is “Why do I have to know that?” This is the background on which the idea of modeling in mathematics instruction appeared. The idea was to show the usefulness of mathematics. As Henry Pollak, one of the pioneers of the modeling approach said: “ My feeling was that when we turned towards teaching the usefulness of mathematics, teaching the modeling was needed in order to make sense of the application, and that combined with making sense of the mathematics would then produce a first rate curriculum.” (Pollak, 2007, page 114)

But, the use of modeling and applications in mathematics instruction had other sources, in addition to initiating an interest in mathematics (Pollak, 2007):

- Keeping the students/pupils interested in mathematics. If this happens, they could learn more and more advanced concepts, opening ways to advancements in other sciences from physics to biology. “We have been working... and finding out how successful they [samples of modeling curricula] are with how many students and how well it works in motivating kids and keeping them interested” (Pollak, 2007, page 119).
- Providing inter-science connections. Allowing to naturally bringing together various scientific subjects and mathematics. The hope was to make the instruction more complete, not restrictive.

- The hope was that modeling and applications will lead to a better mathematical instruction, better conceptualization of mathematical ideas, and better retention of mathematical raining.

Several authors advocated a deviation from the classical “lecture-type”, “talk-and-chalk” methods in favor of introducing modeling and applications, more context, more actions in the curriculum (Alsina, 2007; Pollak, 1969, 2007).

Another effect of modeling and applications in mathematical instruction has been discussed (Burkhardt, 2006). It was suggested that in addition to improving the mathematical instruction through the effects described above, a synthetic effect will be achieved – the student of mathematics will be more creative, a better problem solver.

In this respect, modeling and applications were compared to the classical mathematical word problems. Often, word problems are designed for a particular mathematical skill. The content of a problem and its relation to reality is distilled for the student, and often it is distilled several times. The pupil is not required to formulate a problem. However, it is often stated that the ability to formulate the problem, to separate important from unimportant is essential for real problem solving. Recognizing a problem is a step to solving it. One citation illustrates this idea well:

“We would like to quote a pen portrait of the typical mathematics graduate: 'He is good at solving problems, but not so hot at formulating. The graduate is not particularly good at planning his work not at making a critical evaluation of it when completed.' The evidence from schools, universities and the employers leads one to the conclusion that we must look again at the teaching of applications of mathematics at all levels.” (Pollak, 2007, page 118, chapter 7).

I consider the laboratory algebra experience as a modeling experience in applying algebraic concepts in the context of a physical device. I hope that although the pupils will work with problems that were already pre-formulated for them, the problem solving process will be brought back from abstraction to reality and will allow establishing an inter-disciplinary instructional experience.

2.1.3 Concrete materials in the classroom

The use of concrete materials in mathematical didactics has a long history and goes back to the times as early as the time of Pythagoras who is credited with the discovery of regular polyhedrons – Platonic Solids. Although it is not known currently to what extent 3D models of the platonic solids were used in instruction in the ancient times, concrete materials were used to count objects and to represent various mathematical concepts such as, for example, “square numbers”.

The use of concrete materials in mathematical didactics proceeded from the antiquity to modern times. The reasons for using concrete materials received various theoretical justifications over the centuries, but all on them involved the desire to engage the senses of the person instructed.

Oftentimes, the motivation for the use of concrete was a practical one, based on the notion that abstract mathematical concepts had to be related to practice. This motivation was sometimes expressed rather radically. Charles Laisant, the editor of the journal “Mathematical

Education”, wrote in his work “Initiation Mathematique”, around 1909: “Examine the laboratory of the school: the weights, the measures and surveying instruments. If you do not find such tools, run away and do not return” (Szendrei, 1996).

However, it has been also pointed out that demonstration tools were the answer to the challenge of mass education: “... by having these tools manipulated by the teacher, it was possible to exhibit mathematical properties, illustrate mathematical concepts and procedures and to constitute a “learning environment.” (Szendrei, 1996, page 418). Demonstration tools have been used to represent concepts ranging from the number system to the law of reflection.

As of today, several types of instructional demonstration tools are recognized (Szendrei, 1996):

- a) Real-life tools in the classroom.
- b) Tools designed specifically for instruction.
- c) Games.

The selection of the particular type of educational tools largely and primarily depends on the teaching philosophy.

There are pedagogical schools (for example Genoa Group Projects described by Szendrei) that defend strongly the use of real life tools in instruction as opposed to the use of tools specifically designed for instruction. Their arguments are very diverse and sometimes even extravagant including such points as:

- a) “Positive feedback” between the instruction room and real life.
- b) Less time wasted for educational materials that are not real-life tools. The time spent working with tools designed for instruction is essentially wasted because the learner will never use educational tools in real life.

The proponents of the tools designed specifically for instruction argue that such tools often illustrate the didactic point better, thereby reducing educational “noise” associated with not needing to take into account the factors not relevant to the didactic issue (Szendrei, 1996).

Despite the difference in the educational philosophies and approaches, many educators agree that the use of concrete materials remains a critical issue in didactics. Moreover, it is emphasized by Szendrei that concrete educational materials are not panaceas, not “miracle drugs” of education.

The use and selection of educational materials, concrete didactic devices have to be carefully selected according to the didactic goals of the instruction plan, the educational philosophy of the school, the resources and educational goals of the society, etc. Moreover, the teacher must carefully plan the use of educational devices and concrete materials, paying attention to a proper selection of instructional tasks. A feedback form from the pupils/students expressed as evaluations and suggestions are of crucial importance.

I have undertaken this research study to investigate the role that concrete materials, namely mechanical devices, can play in the teaching of algebra concepts. I chose to design a device specifically to illustrate the didactic point relevant to the concept of linear/affine equation. I hoped to establish a positive connection between the algebraic concepts and their physical manifestation in the functioning of the designed device.

2.2 Wheel algebra

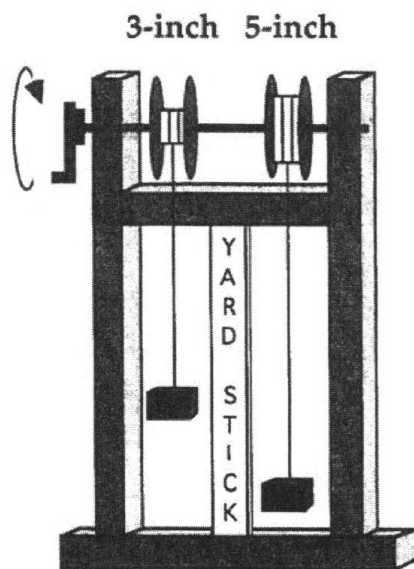
2.2.1 Dynamic physical models and understanding

Since my study addresses the learning of the concept of equation by school pupils using a mechanical device, I studied research done before in the field of using mechanical devices for the learning of algebra.

For example, Greer (1992) reported that "...a physical system of a bucket attached to a rotational handle ... could promote ... views of multiplication away from that of an isolated relationship between multiplier, multiplicand and the product..." (Greer, 1992).

Andrew Iszak described the use of a winch with two spools system to analyze how students introduced and refined initial, faulty algebraic representations of the winch into an unconventional yet sound equation (Iszak, 2000).

Figure 2.2.1: Winch with two spools, take from Iszak (2000)



Two methods of formalizing the experimental knowledge into algebra are described by Iszak (2000):

- Notational variations - experimenting with and adjusting algebraic symbol patterns to match computations and physical observations. In other words, the selection of a proper algebraic expression to represent the observed phenomenon is not straightforward.
- Mapping variations - adjusting correspondence between algebraic symbols and physical system attributes. In other words – understanding what part of the equation corresponds to

what particular part of the system. Thus the ability to symbolically relate the physical properties of the observed system is not straight forward.

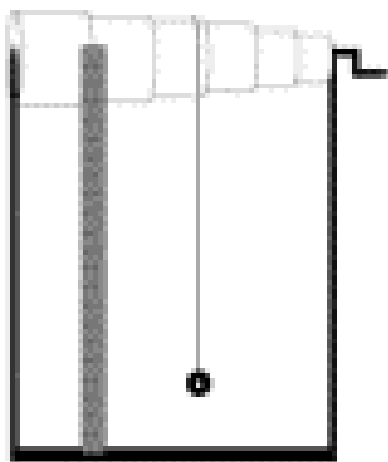
According to Iszak (2000), the ability of students to make translations between representations and the physical device is dependent on their previous exposure to mathematics in a way that is not very obvious: "... learners' previous knowledge of mathematics & physical system both supported and constrained their equation writing efforts". Moreover, "... much of the research ... is based upon data analyzes that do not consider a learner's prior knowledge of the physical world, particularly ... experiences with dynamic physical models, and little research explains how students link such prior knowledge to representing equations" (Izsak, 2000).

Another investigation of Iszak concerned the same device but with the question on how can students generate algebraic models without direct instruction from more experienced others (Izsak, 2004). The results were the following:

- students can use criteria to regulate their problem-solving activity, for example to decide when one algebraic expression is better than another or what type of approach to choose for problem solving. For example, a graphical solution may be compared to a purely algebraic solution and one may be preferred to the other from several standpoints, such as ease of use, accuracy etc.
- constructing knowledge for modeling with algebra can require students to coordinate criteria for algebraic representations with several other types of knowledge. This could involve the understanding (or a lack of it) of the physical and numerical details and their understanding on how to verify the results.

Another researcher, Ellen Hines, described the use of a set of coaxial cylinders of varying diameters, to promote the learning of multiplication and the learning of linear functions (Hines, 2002).

Figure 2.2.2: Set of coaxial cylinders of varying diameters, taken from Hines (2002)



Hines also used the “Etch a Sketch” (a graphical representation) to teach the concepts of co-variation of two variables, of multiplication and of repetitive addition. The study claims that the students were able to differentiate the multiplication from repetitive addition, generalize and abstract their observations of the physical device to generate correct algebraic representation of their observations.

Referring to the study of Pirie and Kirien (1992), Hines notes that “...learners may develop connections between representations as they progress from informal modes of actions to formal reasoning levels...” This is an important skill. An algebraic function is a way of linking the observed variables in a general, formal way. An abstraction from individual actions and trends to a general pattern is an important step in understanding functions. Thus, the ability to formalize the observations of a physical device, and express these through equations has to be brought into the process.

However, in my opinion, the pupil in the Hines study still took a considerable amount of efforts to understand that a function involves a co-variation of two quantities and also to develop a picture of a rate of change. Translating their experiences to a symbolic form was still an extra-effort.

In my study I tried to create the tasks which allowed pupils to verbalize an equation and to write the equation in symbolic form. What I took from the Iszak (2000) and Hines (2002) studies, is that I created tasks, which allowed pupils to use different representations: verbal and symbolical. What I added from Iszak is the graphical representation.

2.2.2. Predict knowledge

A mastery of any mathematical concept involves the ability to apply the concept flexibly and creatively in various situations, including non-standard ones. The ability to creatively use mathematical knowledge is built on a good conceptual understanding of mathematics, where various mathematical concepts exist not as isolated items, but as a system of interconnected concepts. Using prediction questions and problems has been described as a powerful method of mathematical instruction that allows one to build a good conceptual understanding of mathematics.

Using prediction questions in mathematical instruction, requiring the students of mathematics to solve prediction questions is an effective way to bring the mathematical knowledge into action. In some respect, solving prediction questions can be regarded as “learning by doing” on the intellectual level.

It has been shown that using predictions greatly improves the quality of instruction. For example, Kasmer et al. (2012) conducted a study among two groups of middle-school students of approximately the same level of mathematical experience and proficiency. The target material was the linear and exponential functions in introductory algebra. The authors showed that the group with prediction practices utilized in the instruction consistently outperformed the other group at the end of the course (Kasmer et al., 2012).

Moreover, according to the authors, the use of prediction practices activated the following two aspects of the learning process:

- Prediction creates a learning opportunity by helping students visualize what is happening in the problem and what the results might be.

- Prediction provides a learning opportunity to build connections among mathematical ideas.

In my opinion, the didactical power of prediction practices can be very effectively and uniquely utilized in the environment of laboratory algebra. Iszak (Izsak, 2004) and Hines (Hines, 2002) devote a significant attention to the power of the prediction exercises.

First of all, obviously, environment of laboratory algebra allows a direct visualization of the mathematical concepts explored. The visualization process can be quite comprehensive in the laboratory algebra.

The study by Presmeg (2006) mentioned in Kasmer et al. (2012), describes several modes of visualization that can be utilized by students:

- a) Concrete imagery, where a student mentally pictures the problem, the context, or a plausible solution.
- b) Pattern imagery, in which relationships are depicted in a visual–spatial scheme (e.g., graphs and tables)”.

The laboratory algebra environment engages both visualization types:

The direct observation of the physical device will allow the participants to create a mental image of an algebraic concept through the functioning of the device. For example, the angular coefficient can be related to the functioning of the pulleys that lift the weight. This is the concrete imagery. The pattern imagery is realized through representing the observations in forms of graphs, tables and formulas that will be involved in the process of problem solving. Thus, the concrete imagery will be formalized through the pattern imagery.

I hope that the direct activation of the various visualization types will positively contribute to the learning of the algebraic concepts represented in the laboratory session.

In addition, I hope that using prediction questions in the laboratory algebra will allow the participants to inter-connect their mathematical knowledge by actively using it. During the session, a school pupil will be put in a non-standard situation where a practical problem will have to be solved and the pupil will be given the freedom to select the method for solving the problem with an opportunity to directly verify the solution. Thus the pupil will have to creatively explore the previous knowledge with an opportunity to establish additional connections between the concepts learned previously.

Thus, I would like to include prediction questions in every session. I would like the predict questions to have the following characteristics:

- a) Be relatively challenging.
- b) Illustrate, directly or indirectly, the meaning of the algebraic concepts involved in the study – angular coefficient as speed of change, the offset as a starting point.
- c) An opportunity should be present to utilize all the methods of representing the function – formula, graph or table.
- d) A verification of the predicted result should be required.

2.3 Cognitive aspects of realistic teaching

2.3.1 Translation between data representations

The proposed research project in “laboratory algebra” involves tasks on problem solving.

A very important and even essential part of problem solving in mathematics is the ability to make translation between various representations (Clement, 1980; Janvier, 1987).

Translation is a process in which constructs of one mathematical representation are “mapped” or converted between each other (Janvier, 1987). Translation usually involves transitioning from one symbolic or verbal representation in which the problem is given (for example, word problem) to another, more appropriate for the study (graph, algebraic equation, table).

Several classification schemes have been developed to describe and systematize the translations and to elucidate possible relationships between the representations. Janvier presented his classification in a table to represent the possible scenarios involved in translations between various representations of a function, Fig 2.3.1.1:

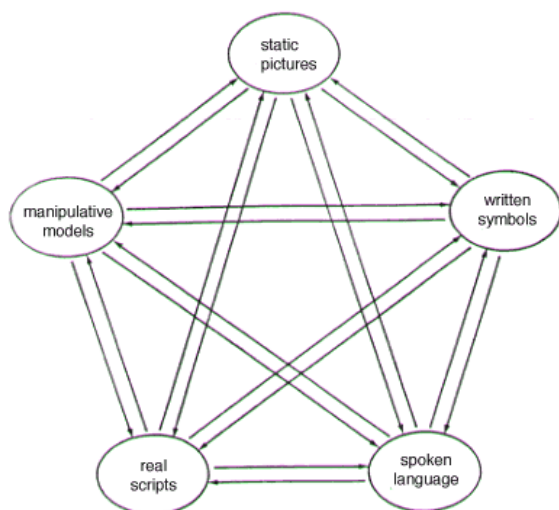
Figure 2.3.1.1: “Translation process” according to Janvier (1987)

TRANSLATION PROCESSES

To From	Situations, Verbal Description	Tables	Graphs	Formulae
Situations, Verbal Description		Measuring	Sketching	Modelling
Tables	Reading		Plotting	Fitting
Graphs	Interpretation	Reading off		Curve fitting
Formulae	Parameter Recognition	Computing	Sketching	

An alternative picture has been developed by Lesh and co-authors. They defined five distinct types of representation systems that occur in mathematics learning and problem solving (Lesh, 1987). As described by Lesh, they are, Fig 2.3.1.2:

Figure 2.3.1.2: Translation representation, taken from Lesh (1987)



- a) Experience-based "scripts"-in which knowledge is organized around "real world" events.
- b) Manipulative models - like arithmetic blocks, fractions, etc., where the "elements" in the system have little meaning per se, but the "built in" relationships and operations fit many everyday situations.
- c) Pictures or diagrams-static figural models that, like manipulatable models, can be internalized as "images".
- d) Spoken languages-including specialized sub languages related to domains like logic, etc.
- e) Written symbols-which, like spoken languages, can involve specialized sentences and phrases ($x + 3 = 7$, $A \cup B = (A \cap B)$) as well as normal sentences and phrases.

The knowledge of these representations is relevant to “laboratory algebra” and the setup of the sessions in laboratory algebra, because the laboratory algebra experiments involve an active use of various representations and switching between them. I tried to setup the sessions to intensely involve several representations in the problem solving activities.

2.3.2 Conceptual understanding

The ability to make translations between mathematical representations is perceived as being part of “conceptual understanding”. This is explained by Kilpatrick et al. (2002) as follows. They first define the five strands of mathematical proficiency, in which “conceptual understanding” is one of the strands.

The five strands of mathematical proficiency interwoven and interdependent in the development of proficiency in mathematics; therefore, helping children acquire mathematical proficiency calls for instructional programs that address all its strands.

Conceptual understanding – the ability to understand mathematical concepts, operations and relations and convey them to others. Being able to represent mathematical situations in different ways and knowing how different representations can be useful for different purposes.

Procedural fluency – the ability to carry out mathematical procedures accurately, efficiently and appropriately for the situation.

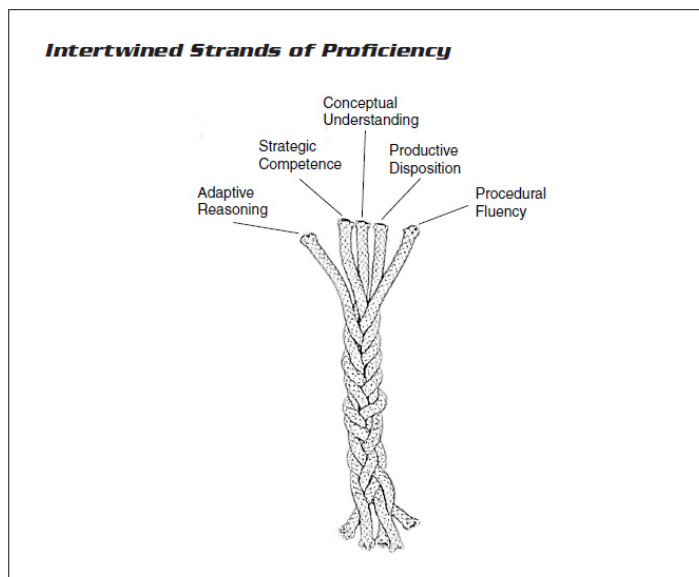
Strategic competence – the ability to solve mathematical problems.

Adaptive reasoning – logical thought, explanation, verification and justification.

Productive disposition – the inclination to see mathematics as a sensible, useful subject and apply mathematics willingly and freely.

The five strands of mathematical proficiency are shown in the figure 2.3.2.

Figure 2.3.2: Strands of mathematical proficiency, take from Kilpatrick et al. (2002)



Fundamental to the goal of successful instruction is the way the knowledge is represented, stored and inter-connected. It is claimed that conceptual learning, learning with understanding is more powerful than just memorizing, because it provides better knowledge retention and better flexibility in the ability to apply knowledge (Kilpatrick et al., 2002). Knowledge that has been learned with understanding provides the basis for generating new knowledge and for solving new and unfamiliar problems (Kilpatrick et al., 2002). For the laboratory algebra experiments, it is important to realize that if the conceptual understanding is combined with skills (procedural fluency); the outcome of learning could be really powerful. Moreover, since these two aspects of mathematical proficiency are interwoven, interdependent, they may not be possible without each other.

2.4. Learning in a social context: scaffolding and social interaction

Teaching of any subject, including mathematics, necessarily involves social interactions. In this respect, I want to mention Bruner's enactment of instruction/cognition. Not only should a teacher identify the mode of representation, but also to effectively provide so-called "scaffolding" (Bruner, 1978). "Scaffolding" is understood as an intelligent, structured interaction between the learner and the instructor, taking place within an appropriate social framework.

The concept of scaffolding has been elaborated to great details since its introduction. Following Foley (1994) and Applebee (1986), I would like to outline five criteria of effective scaffolding, since we find it very relevant to the environment that "laboratory algebra" provides.

Criteria of effective scaffolding:

- a) Student ownership of the learning event.
The instructional task should allow the students to make their own contributions as the activity evolves.
- b) Appropriateness of the instructional task.
The task should be based on the knowledge and skills the students already have, but should be difficult enough for learning to occur.
- c) Structured learning environment.
Provide a "natural sequence of thoughts and language", thereby presenting the student with useful strategies for problem solving.
- d) Shared responsibility.
The tasks are solved jointly, thus the role of the teacher is more collaborative and evaluative rather than strictly didactical.
- e) Transfer of control.
As the students internalize new ideas and procedures, they should take greater responsibility of how the instruction proceeds.

On the other hand I want to refer to the instructive Vygotskian-Luria theory of Higher Mental Functions (HMFs) regarding social interaction.

Higher Mental Functions can be loosely described as complex socio-biological functions of the brain responsible, in part, for cognition and cognitive development. An important essence of HMF, as emphasized by the Vygotsky-Luria school is the complex interplay between the social and biological factors. According to this theory, an important component of HMFs is their social genesis. In other words, effective cognition requires a properly supporting social environment. "The higher human mental functions are complex self-regulated processes, social in origin, mediated in structure and conscious and voluntary in their mode of function" (Akhutina et al., 2011).

I find these concepts important for teaching, because learning almost never takes place as a self-directed self-study. Most often, it involves studying in groups and interaction with the teacher. This idea could even be considered from a broader perspective.

The successful instruction should involve a wide socio-cultural background. This socio-cultural didactic background should include also the favorable interaction between learners as peers, as well as cultural environment in the society that encourages and appreciates learning.

I shall try to illustrate this thought by introducing the Vygotskian idea of the Zone of Proximal Development (ZPD), as I find the idea of ZPD very relevant to the study undertaken.

The Zone of Proximal Development is an important concept that relates to the difference between what a child can achieve independently and what a child can achieve with guidance and encouragement from a skilled partner. Advanced concepts or concepts that are difficult to master on one's own, can indeed be acquired with guidance and encouragement in cooperative learning. A less competent child can learn more through cooperation with more skilled peers (partners). When a learner is on the zone of proximal development, providing the appropriate assistance for the task will greatly facilitate the process of learning. Thus, the interaction with a "more knowledgeable other" –peer, instructor – will lead to noticeable breakthroughs that otherwise would be difficult in self-study.

The idea of ZPD is going to be influencing the design of the research sessions. Most of the research sessions were set up in groups of two and it was keeping the contact between pupils and researcher to allow the possibility scaffolding (Bruner, 1978) and ZPD (Akhtutina et al., 2011).

2.5 Summary and research question

My goal is to make mathematics easier to perceive by school pupils, so that they would not be afraid of it, but would like it. To achieve this, I would like to show that mathematics is not simply an abstract and obscure subject, but a subject of great relevance to the every-day reality; that the mathematical equations could be seen, touched and even played with.

The pupils will get an opportunity to unfold their curiosity by exploring the unknown properties of the physical device and study the device properties with the help of their previous experience, mathematical and non-mathematical.

In addition, I would like to study how children learn in a social environment, when the learning occurs through discussions and interactions between the members of the group and under the supervision of an instructor, who supports the learning, keeps the pupils thinking and motivates them. Observing this learning process and the effects of learning thus expanding the didactics research base for teachers and researchers is an important direction of research on mathematics didactics

I decided to exploit the illustrative nature of mechanical devices to represent algebra. By linking together the physical world and algebraic concepts that might be found somewhat abstract, I hoped to enhance the efficiency of mathematical didactics and to investigate the cognitive processes of mathematical didactics with physical artifacts.

For this I designed a tool, see Chapter 3. From the literature I will take how pupils will work in an enactive mode, based on Bruner, (1966), and how they will represent the observed phenomena: graphically, tabulated or symbolically, based on Janvier, (1987).

This also connects to Pollak (2007) on the usefulness of mathematics and to the learning on social contexts according Bruner (1978) and Vygotskian (Akhtutina et al., 2011).

Exploring linear and affine relations is the most natural and simplest research topics accessible with device constructed. Therefore, I decided to concentrate on didactic aspects of teaching the concept of linearity with physical devices and investigate the cognitive aspects of learning the linear/affine relations with the mechanical devices.

I formulated the following research question for the study: What opportunities for learning the concept of linear equation and what aspects of that concept are exposed in a double-winch laboratory?

3 Methods

Because the current state of the project was exploratory, it was kept at a relatively small scale. Quantitative methods was not considered as very useful and reliable (Bryman, 2008). Thus, I opted for a micro ethnographic qualitative method.

3.1 Preparation study

In this paragraph I explain how I prepared for the sessions. The preparation study contained the following:

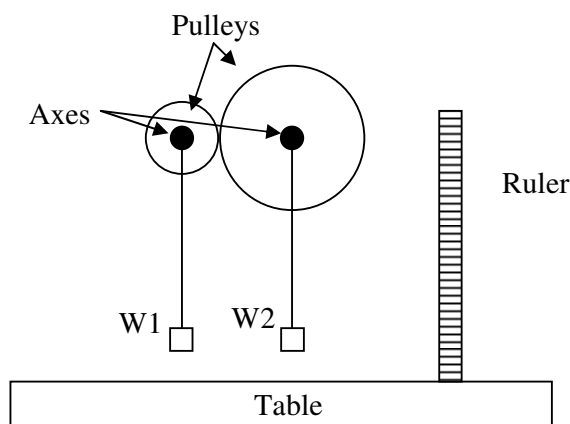
- Elaboration of the winch, including winch design and creation,
- Selecting mathematical and algebraic concepts to be presented and studied,
- Creating tasks,
- Designing research sessions.

3.1.1 Mechanical tool design

My research was inspired by the rich research opportunities that stemmed from the use of mechanical devices described in the studies of Hines (2002), Iszak (2000) and Iszak (2004). Thus, I developed a mechanical dynamic system.

The suggested mechanical tool is called a “coupled winch” which consists of two pulleys of different diameters that are touching at the circumferences of one another and thus can transmit rotational motion between each other. We start by suggesting that the “large” pulley has a diameter approximately twice that of the “small” pulley. $D_L=2D_S$. The idea behind this construction is that it can be much more flexible than the “classical’ winch arrangement and that a far richer spectrum of algebraic concepts can be taught (see Figure 3.1.1):

Figure 3.1.1: Mechanical tool design



The pulleys can be rotated on the corresponding axes, the two thin cords will be winding on the related axes sides and lift the target weights – W1 and W2. The heights of the weights can be measured with the ruler.

Several sources of the described winch system have been identified and are briefly described below:

- a) Building the system from scratch in a machine shop. Advantages: the system can be customized to the proper design and function requirements. Disadvantages: the process is lengthy and costly, Extensive design efforts are needed.
- b) Purchasing industrial grade gear-boxes from a commercial supplier of machine components.

Example 1: <http://www.activepowersports.com/great-planes-electrify-gearbox-s280-53a1-ratio-std-gpmg0200/>

Example 2: <http://www.ondrivesus.com/gearboxes.htm>

Example 3: <http://www.mcmaster.com/#gear-boxes/=n9o5fu>

Advantages: Reliable, robust design.
Significantly reduced cost if a routine, stock item is found.

Disadvantages: The inner mechanics will be hidden, the operation not very visual. Will probably have to buy from abroad and pay import taxes, delivery may be not available.

- c) Get a toy construction set: Quercetti Georello Kaleido Gears set http://www.amazon.com/gp/product/B00000J048/ref=oh_details_o00_s00_i01?ie=UTF8&psc=1 and Quercetti Georello Tech http://www.amazon.com/gp/product/B0002CYT80/ref=oh_details_o00_s00_i00?ie=UTF8&psc=1

Advantages: Cheap, very visual.

Disadvantages: Still design efforts are required to put the system together. No guarantee that the intended system functionality will be achieved. May appear “childish” for the pupils of grade 8 and higher.

Based on the consideration above, Quercetti Georello Kaleido Gears set and Quercetti Georello Tech set were selected for constructing the double-winch mechanical tool.

3.1.2 Mechanical tool creation

The following difficulties have been experienced during the device construction:

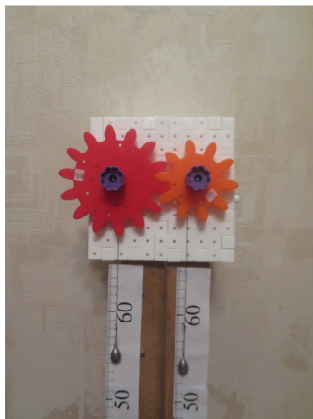
First of all, it was necessary to increase the friction between the pulley and its shaft to prevent the weight from spontaneously falling down under the gravity. The increased friction would

allow holding the weight in place after each pulley turn. Thus, the Quercetti Georello Tech construction set could not be used, even though it offered more flexibility in mechanical construction and had a better appearance. The implementation of a stopping mechanism for pulleys was not exactly obvious and required extra design and construction efforts. In addition, it wasn't very obvious how to attach a ruler to a construction built from the Quercetti Georello Tech set.

These problems have been overcome in the Quercetti Georello Kaleido Gears. The pulley was rotating on the bolt, set in the construction base. Tightening the bolt allowed to adjust the friction exerted on the pulley thereby preventing the weight from going down. Of course, with this design, there was a possibility for the bolt to become loose, but this never happened, perhaps due to a relatively small number of rotations, performed in the experiments.

Another problem was the following: the fishing line holding the weight was attached to winding around to a cylindrical axis. Since the diameter of the axis was small (comparable to the diameter of the fishing line), the effective diameter of the axis" was increasing with each rotation due to the overlap of the fishing line chunks wound up on the axis. Moreover, the fishing line was being wound up on the axis not very regularly. As a consequence, certain irreproducibility in the system performance occurred. However, this did not seriously deteriorate the observed results and did not affect the conclusions drawn from the experiments. Moreover, the intrinsic imperfection of the system provided an interesting dimension to the experiments, bringing up issues of accuracy and "exactness" in measurements. The completed system is shown in Figure 3.1.2:

Figure 3.1.2: Mechanical tool creation



3.1.3 Mathematical and algebraic concept selection

The main mathematical concepts explored in the experiments were those of the linear equation:

$$y = a * x + b ,$$

with its aspects:

Proportional in the form: $y = a * x$

Affine in the form: $y = a * x + b ,$ with $b \neq 0$

Simultaneously, the following relevant aspects of the linear equation have been expected:

- a) the unknown
- b) the variable
- c) the constant

The expected ways of representing the aspects:

- a) enactive mode
- b) verbal representation
- c) tabular representation
- d) graphic representation
- e) symbolic representation

3.1.4 Creation of the tasks

An important part of this project was the elaboration of algebra tasks. A series of tasks was created for two target groups of different age: with or without algebra exposure.

Task design was determined by the nature of the research questions. Since we wanted to investigate what opportunities for learning the concept of linear equation and what aspects of that concept are exposed in a double-winch laboratory, I constructed tasks to have the following characteristics:

1. The problems should look simple, easy to understand and be not overloaded with data.
2. They should allow different kind of representation (verbal, table, graphs, symbols).
3. Four task blocks with three tasks Task 1, 2 and 3 each were designed. Task 1, 2 and 3 should be arranged in ascending order of difficulty:
 - a) Task 1 should, first of all, inspire curiosity and motivate the participant to proceed to the next tasks. The solution could be found by taking a measurement on the device or by making an educated or intuitive guess drawn from the characteristics of the system.
 - b) Task 2 should call for a possibility to create and write down a linear equation. That is, it should contain the known, the unknown and the required constants. This included possibility to make and fill out a table, plot a graph and create an equation.
 - c) Task 3 should be a predict question. It requires predicting the behavior of the system after a change in the system initial conditions.
4. All age groups should solve the same task, so it could be possible to compare how they were solving them.

It was created four following blocks of three tasks each block.

Task block 1

One pulley and one weight

Equation in the proportional form

- Task 1.1 Set up red weight on “0” position and rotate the pulley.
What should be the height after 5 turns?
- Task 1.2 What is the relationship between the height and the quantity of turns?
Say by words or write an expression that allows one to find the height if the number of turns is known.
- Task 1.3 If we lift the pulley very high, e.g. to the ceiling, can you predict the how many turns will be required to lift the weight to the ceiling? The ceiling height is $2.5\text{m} = 2500\text{cm}$. The weight starts moving from the floor that is 0 cm .

Task block 2

Two pulleys and two weights

Equation in the proportional form

- Task 2.1 Inspect the whole system. What is your expectation of how it works?
What weight rises faster?
- Task 2.2 Set up the red and grey weights at “0” position. Rotate the red wheel and register (measure) the heights of the red and the grey weights. How do the heights of the red and the grey weights depend on handle rotation of the red pulley? Write a mathematical expression.
- Task 2.3 We place the system very high, for example at the ceiling.
The weights start from the floor.
What would be the height difference after 25 turns?

Task block 3

Two pulleys and two weights

Equation in the affine form

- Task 3.1 Set up the red weight at the “10 cm” position and the grey one at “0cm”.
Rotate the red wheel and register the heights of the red and the grey weights.
- Task 3.2 At what height did the weights meet?
- Task 3.3 We place the system very high, for example at the ceiling
The red weight starts from the table level (70cm) and the grey one from the floor. What is the height at which they meet?

Task block 4

Two pulleys and two weights

Equation in the proportional form

- Task 4.1 Set up the red and grey weights at the “0” position.
Rotate the red wheel and register (measure) the heights of the red and grey weights.
- Task 4.2 What is relationship between the height and quantity of turns for a given weight?
Write a mathematical expression.
- Task 4.3 Try to predict the position of the grey weight if you know position of the red.

3.1.5 Participants and venue

The participants of the study consist of two groups: those with almost no algebra experience and those with algebra experience.

The first group consisted of 6th grade pupils. I will call them “juniors” in my report.

The second group consists of 8th grade pupils. I will call them “seniors”.

Participants: The pupils were from Troitsk Moscow, Russia
The juniors were named Anna, Andrew and Oleg
The seniors were named Roman and Dmitry

Place of research session: Private flat in Troitsk Moscow, Russia

Data collection planning:

The data were collected in the following sequence:

- Pre-session instruction to the participants
- Video-taping or audio-taping during the sessions
- Workbooks of the participants
- Instructor observations and dialog with participants put down in log notes book
- Post-session questionnaire

3.2 Data collection

The implementation part should include:

- Observation of dialogues with the pupils following the study.
- Video and audio recording.
- Post-session questionnaire to follow the research session (have the pupils answer a set of post-session questions, evaluate the session and offer suggestions).

3.2.1 Planning of the data collection

The session was expected to take approximately one hour. I anticipated that the juniors would need more time. The planned organisation for the sessions with the juniors and with the seniors is like this:

Session with junior pupils:

1. Introductory instructions for the pupils immediately after arriving,	
introduction to the device	about 5 min
2. Task Block 1	about 20 min
3. Task Block 2 or 3 or 4	about 30 min
4. Final questionnaire	about 5 min
<hr/>	
Total:	60 min.

Session with the senior pupils:

1. Introductory instructions for the pupils immediately after arriving,	
introduction to the device	about 5 min
2. Task block 1	about 15 min
3. Task Block 2 or 4	about 20 min
4. Task Block 3	about 20 min
5. Final questionnaire	about 5 min
<hr/>	
Total:	65 min.

As can be seen, the seniors were expected and did, in practice, cover more tasks within an hour. Therefore, I tried to organise several sessions with the juniors to cover approximately equal range of tasks.

3.2.2 Actual implementation of the sessions

Table 3.2.2, lists the actual implementation of the sessions held:

Table No 3.2.2: Actual implementation of the sessions

Date	Session No	Participants	Names	Task block	Time
05.08.2013	1	Juniors	Anna	1 and 3	45 min
07.08.2013	2	Juniors	Andrew and Oleg	1 and 2	50 min
08.08.2013	3	Juniors	Andrew	3	45 min
10.08.2013	4	Seniors	Roman and Dmitry	1, 2, 4 (just task 4.3) and 3	65 min

Verbal permission for tape- and video-recording was obtained prior to the section. Session with Anna was audio-recorded and the other sessions were video-recorded. The tasks designed for the section were distributed and explained.

The pupils were monitored closely during the session and were engaged in a dialogue if they required assistance or appeared not to understand the question or had difficulties expressing their thoughts. If a solution method had been selected and implemented by the pupils, they would still be asked to attempt other possible methods and comment on their preference for the method selection.

During the session I observed the discussion of the pupils and offered hints that could direct the pupils to the solution. I was constantly motivating the pupils to think, maintaining the dialog if necessary. Engaging the participant pupils in the dialog proved to be very instructive in elucidating the through process and in understanding how the pupils handle the concepts under the study.

Pupils answered some questions after the session. The sessions were timed by observing the watch and also automatically by the video recorder.

3.3 Data analysis

All the video- and audio- recordings were transcribed. Only those parts that were considered important for the report were translated into English for further analysis and presentation.

The time stamps on the video recordings were used to analyze and verify the time spent for problem solving. Attention was paid to the amount of time spent for solving each of the tasks offered. The time spent for solving the problems did not differ significantly between the juniors and the seniors. However, the seniors started faster at the beginning and had enough time to solve an additional prediction problem of block 4 in the time interim between the second and third blocks.

The empirical material collected was transcribed in protocols, analyzed and interpreted using qualitative methods.

The preliminary findings were presented in a University of Agder's Monday's research seminar in November 2013, open to all colleagues interested in this topic.

Further analysis of the data allowed classification and analysis of the findings. The most important findings relevant to the research question were numbered 1 - 16 and listed below as well as in the session describing the findings in details. A correspondence between each particular finding and a session is established, proper references are given on the relation of each particular finding and the corresponding research session.

Going through transcripts putting numbers:

- | | |
|----------|--|
| 1 | Proportion with numbers and symbols. |
| 1 linear | Using linear (affine) functions implicitly through proportional relations. |
| 2 | Repeated addition. |
| 3 | Verbal representation of relations. |
| 4 | Representation of unknowns. |
| 4 | Distinctions between the concepts of known, unknown and variable. |
| 5 | Table representation of relations. |
| 6 | Graphic representation of relations. |
| 7 | Symbolic representation of relations. |
| 8 | Different ways of writing equations. |
| 9 | Accuracy and perdition. |
| 10 | Average value (arithmetic mean). |
| 11 | Activities in the enacted mode. |
| 12 | Mental calculation. |
| 13 | Height difference (pupils approached the variables and the parameters). |
| 14 | Confidence. |
| 15 | “Intuitive” solutions. |
| 16 | Verifying the validity of the results. |

3.4. Ethical consideration

An informal agreement for session participation was obtained from the parents of the pupils and from the pupils themselves. No monetary compensation for session participation has been provided for the pupils, the participation was voluntary. Verbal permission to perform tape- and video-recording was obtained from the school pupils as well as from their parents prior to the session.

All participants' names have been changed to protect the privacy of the participants.

4 Findings and analysis

4.1 List of tasks completed during the sessions

The actual assignment of the tasks proceeded as follows:

In the first session, the problem on the affine function proved to be too complex for the juniors and was completed with my help. Therefore, for the second session with the juniors, I decided to offer one or two blocks of the problems on the linear proportional function. Since the juniors of the second session had solved the offered problems quite confidently, I decided to administer an additional session on the affine function to one of the participants of the session 2 to find out, how effectively the concepts of the preceding session had been assimilated. The fourth session, conducted with the seniors, was designed for three blocks – two on the linear proportional function and one on the affine function. However, the seniors proceeded at a fast pace and were able to complete a prediction problem of the fourth block before proceeding to the problem on the affine function.

The following table summarizes the tasks assigned and accomplished during the sessions:

Table 4.1: Summary of session assignments

Session No		Participants	Age	Block of the tasks			
				1	2	3	4
1	Junior	Anna	12				
2	Juniors	Andrew, Oleg	12, 12				
3	Junior	Andrew	12				
4	Seniors	Roman, Dmitry	14,15				

4.2 Chronological report of the sessions

4.2.1 Session No 1 dated 05/08/2013

This session included only one participant, a 6th grade girl called Anna.

In task 1.1 with one pulley, a trend was spontaneously established: every pulley turn raised the weight by 4cm.

03.06 Anna: At every rotation of the wheel in that (clockwise) direction, the weight is raised by 4 cm.

Later, the “predict question” problem, Task 1.3, was solved spontaneously by a proportion.

In Task 3.1 with two pulleys the question of the instructor, asking to predict what pulley will raise the weight faster, was answered spontaneously:

08.23 Anna: The fastest, most probably, will be the weight with the smaller wheel, because the first (red) wheel is larger and the turn is larger, but the small wheel will be rotating more.

The pupil could not provide a more coherent explanation, despite her persistent attempts.

14.23 Anna: I understand it, but can not explain.

As a result, the following answer was written down: “The gray weight, attached to the smaller pulley, will rise faster.”

In Task 3.1 that involved two pulleys (equation in the affine form), Anna was asked to set the red weight at 10 cm and the grey weight at zero. She was expected to rotate the red pulley and find the height of each height after each turn.

Anna was my first pupil, and I was still inexperienced in carrying out the research. So I did not ask her at this stage to make the table, and she solved the Task 3.2 and found the height where the weights meet by rotating (enactive mode). So, Anna did not do the measurements in the order of the task and did not fill the table.

Then, Anna moved to the predict question of Task 3.3, where one was expected to determine the height at which the weights will meet if the red started at 70 cm from the floor and the gray weight from the floor. Spontaneously, a solution involving a proportion was proposed, analogously to the tasks of block 1.

19.41 Anna: So, this could be solved using a similar scheme. Thus the red weight... Let's find the relationship... This one is 70, and the gray weight is at 0.

She made no attempt was made to carry out a proportion calculation. After thinking for some time, she asked, with a laugh, for a ruler to take a physical measurement of the height where the weights meet.

After realizing that the pupil was “stuck”, I suggested her to fill out a table containing the data relating the raising of the weight and the pulley turns. The table was filled out very meticulously taking the data as precisely as possible - both the pulley turns and the position of the weight on the ruler were observed very closely. In spite of the routine nature of the measurements, the work was accomplished with enthusiasm and great satisfaction, as if Anna was waiting for the answer at the end.

A small confusion occurred when the weights met at non-integer number of turns. After the table was filled out, no attempt was made to generalize on the data.

34.05 Anna: I am confused about what is to be calculated.

Then, I suggested plotting a graph from the data in the table. Once the graph axes were explained by the instructor (horizontal axis – number of turns, vertical axis – height), the graph was plotted confidently and quite precisely.

Again, it was verbalized that “the height of the red weight is every time increased by five and that of the gray – by six”. A question on if there was an error in the measurements was answered with “Aha, a small one”.

The point of intersection of the two graphs was found graphically for the situation when the red weight started at 10 cm and the gray weight– from zero.

Together, we summarized that we had found the graphical description for the motions of both red and gray weights.

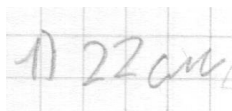
- 43.31 Instructor: Thus, we have found the trend. The graphs are the required trends.
- 43.37 Anna: I can...
- 43.39 Instructor: This is the trend for the red and that one is for the gray.
- 43.45 Anna: I could...
- 43.51 Instructor: There we have started at 10cm, and can start at 70cm.
- 44.02 Anna: This means that I could plot the same thing, it looks similar. That means it will be... We need to start from 70, not from 10.

So, Anna spontaneously stated that graph should start from 70 cm and not from 10 cm. Then, Anna adopted that the graph starting from 70 cm will be paralleled to the graph describing starting from 10cm. Anna spontaneously suggested continuing the line for the gray-weight graph further, to intersect the graph for the red weight with the start at 70 cm. Therefore, my suggestion to write down an expression based on the graphical data (translate between graph and formula) was not followed upon.

4.2.2 Session No 2 dated 07/08/2013

In this session, two boys Andrew and Oleg, 6th graders, were participating.

In the Task 1.1 with one pulley the pupils were expected to find the height of the weight after 5 turns. They were acting stepwise (enactive representation), i.e. rotating the red pulley, counting turns and then, determined the weight position after 5 turns by reading the instrument scale. The data the students read from the instrument scale was written down in the prepared paper notebook (symbolic representation).



A photograph of a handwritten note on a grid background. The text is written in blue ink and reads "1) 22 cm".

In the Task 1.2 they were expected to find a relation between two variables, i.e. between quantity of turns of one red pulley and the height of the weight which is attached to that pulley.

They read a task and Andrew said:

- 02.56 Andrew: One turn (incomprehensibly) cm.

So, Andrew inaudibly verbalized the measurement for one turn.

Then I asked a series of questions:

- 03.08 Instructor: Can a mathematical expression be written? If we know the quantity of turns, how can we calculate the height?
- 03.12 Andrew: It is necessary to divide centimeters to the quantity of turns.
- 03.18 Instructor: What should we find then?
- 03.21 Andrew: Height of one turn.
- 03.28 Instructor: So, how could we calculate the height of the weight?
- 03.33 Andrew: It is necessary to multiply the number of turns to the height of one turn.

So, he found that relation with units (turns, height) without difficulties and used a verbal representation.

Then, an attempt was made to determine the distance travelled per one turn by dividing the measured height by the observed quantity of turns. The attempt was carried through to completion.

Handwritten calculation on grid paper: $2) 22 : 5 =$

Then it appeared that in trying to deduce the mathematical equation (most likely, he was not familiar or confident with the concept of mathematical expression, but he tried his best) he attempted to predict the height of the weight, using his knowledge of the height per turn, that was, as a matter of fact, determined very imprecisely.

He used the method of symbolic representation, and used a proportion relation, correctly composed.

Handwritten notes on grid paper showing a proportion and calculation:

$100 \text{ оборот} \sim 35,5 \text{ см}$
 $500 \text{ оборот} - ? \text{ см}$
 $3,5 \text{ см} \cdot 5 = 17,5 \text{ см} - \text{высота груза}$

(1 turn ~ 3 cm, 5 turns - ? cm)

$3,5 \text{ cm} * 5 = 17,5 \text{ cm}$ – height of the weight)

The calculated result differed from the experimentally found value because of the height for one turn was found with a considerable error.

The task with the prediction question (Task 1.3) was solved independently without much of my comments. He used proportion again

3) Высота = 2,5 м (2500 см) = 25000 мм
 количество оборотов - ?
 1 оборот - 3,5 см 35 мм

1) $2500 : 3,5 = 714,2(8)$

Ответ: 714,2(8)

$$\begin{array}{r}
 25000,35 \\
 - 245 \quad 714,2(8) \\
 \hline
 50 \\
 - 35 \\
 \hline
 150 \\
 - 140 \\
 \hline
 100 \\
 - 70 \\
 \hline
 300 \\
 - 270 \\
 \hline
 300
 \end{array}$$

Here is a transcript of the presented solution:

Height = 2,5m (2500 cm) = 25000mm
 Quantity of turns - ?
 1 turn - 3,5 cm 35mm
 $2500 : 3,5 = 714,21(8)$
 Answer: 714,21(8).

The required number of turns was calculated arithmetically using the height for one turn previously determined.

I should note that most of the times, the calculations were performed without a calculator. It was observed that when they could use a calculator or not, I advised them to use calculator, but the pupils refused it. It seems that their teacher is not welcoming in using calculator during lessons and they got used to not using it.

Then I asked a question:

- 15.32 Instructor: By the way, for example we do not know height and do not know quantity of turns. Can we see any regularity in the process?
- 15.40 Andrew: To lift the weight to the ceiling?
- 15.57 Instructor: Not just to the ceiling, in general... lift it to an airplane in the sky, for example. Can we find any regularity in the motion?
- 16.05 Andrew: No.
- 16.08 Instructor: So, we can predict the motion to the ceiling, but not to the plane?
- 16.13 Andrew: If we do not know height and quantity of turns, we can't.

Thus we observe and find that although the mathematical pattern had been already verbalized by the pupil, the concept of variable has not been fully understood and mastered by the pupil.

The pupil did not propose to substitute a numeric variable by a letter. A transition from numerical values to letter variables is still in development and is still difficult.

We moved to the Task Block 2 with two pulleys. The boys tried to predict which weight was rising quicker - they rotated the red pulley and found that the weight on the orange (smaller) pulley was rising quicker.

Then they moved to solving Task 2.2 and were recording the position of the weight after each turn. They were rotating and trying to remember the height of each weight after each turn. So, I offered them a table. The boys looked a little bit skeptic, but started filling in the table. They were rotating the red wheel, measuring heights of each weight after each turn and translated data into the table:

Column headings: Turns, Red-weight height, cm and Gray-weight height, cm

Количество оборотов	Высота красного грузика, см	Высота серого грузика, см
0	0	0
1	3,5	6
2	8,8	12,5
3	12,3	19,5
4	17,1	26,6
5	22	32

When they finished, I asked again:

20.36 Instructor: Can you offer a mathematical expression describing how both of the weights go up in height?

20.44 Andrew: You mean their difference in height?

An attempt was made to find the required relationship through the height difference per each turn.

20.50 Instructor: Well... some coefficient of height change for each weight. If we know this coefficient, we can write down mathematical expression.

20.57 Andrew: Yes.

21.01 Instructor: The dependence of height on turns.

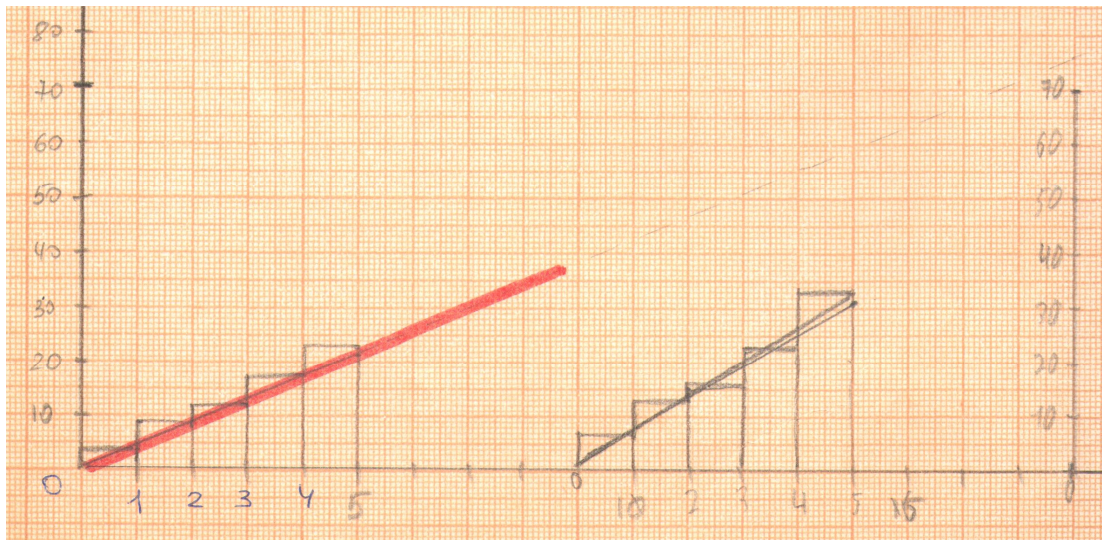
21.06 Andrew: Yes.

21.10 Instructor: So, the task is to find a coefficient.

- 21.16 Andrew: Exactly or approximately?
- 21.19 Instructor: Approximately, of course.
- 21.25 Andrew: Well, it will be approximately 3.5 here and 6 here.
- 21.28 Instructor: A little bit more, I suppose.
- 21.33 Andrew: Aha, 3 here and...3 and 9 and here...

The height per turn was correctly associated with the required coefficient. However, although the data were already in the table, the children could not decide how to utilize the data.

Then I offered them to plot the graph. The tabulated data were transformed to a graph. They plotted a graph; I helped them to draw the line more carefully.



Finally they divided one of the heights they took from the graph to the corresponding quantity of rotations and got the correct answer for the coefficient for that session.

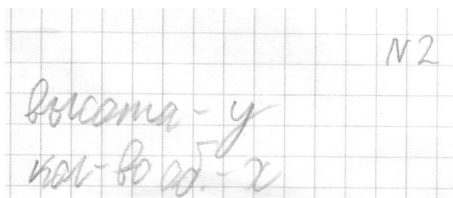
Then I wanted them to transform data from the graph to the formula and write equation.

I asked again:

- 27.38 Instructor: Let us think, how we can describe the position at any time for any height point...
- 27.44 Andrew: At any point...
- 27.49 Instructor: If we denote the height by some letter, say "y", then any of these heights (I am pointing to the height scale) can be "y". And any one of these (pointing to the scale of turns) is represented by "x". How can we compose a mathematical dependence?

- 28.04 Andrew: Just a second... “x” is the number of turns, “y” is the height... the ‘height’ must be divided by “turns”...
- 28.12 Instructor: And what are we getting by that?
- 28.21 Andrew: We get the height by one turn.
- 21.25 Instructor: Write it down.
- 28.31 Instructor: And can you express the height through the turns and a coefficient?
- 28.36 Andrew: Yes, the height divided by the coefficient.
- 28.38 Instructor: How to find the height if you know the quantity of turns and the coefficient?
- 28.41 Andrew: Multiply the turns by the coefficient.
- 28.43 Instructor: Write it down.
- 28.48 Andrew: Expression?
- 28.54 Instructor: Yes, expression. Height is “y”, correct?

The boy wrote the following:



Height – y,
number of turns – x

- 29.18 Instructor: Let’s write down the coefficient. What did we get for the coefficient? 4.2. Let’s write it down. (I’m writing) 4.2
- 29.37 Andrew: Y divided by X [He is writing $y : x = 4.2$].
- 29.43 Instructor: Correct. There is another way to represent it [I am writing $y = 4.2 * x$]. It’s even simpler. Now, from knowing the number of turns, we can obtain height.

The boy expressed a relationship between the height and the number of turns verbally and then symbolically by using letters instead of numbers.

$$4,2$$
$$y : x = 4,2$$
$$y = 4,2 \cdot x$$

Then I asked to write down a mathematical expression for the “gray” weight by analogy.

Using the analogy, the boy had determined and written down the coefficient for the second wheel, expressed the relation in the form comfortable to him:

$$y : x = 6 .$$

Is what he came up with, he divided the height by the number of turns. The division is more acceptable for him in this situation.

By the analogously to the previous problem, he wrote down an equation in the form proposed by me:

$$y = 6 * x .$$

$$6$$
$$y : x = 6$$
$$y = 6 \cdot x$$

These observations still suggests that the pupil have difficulty deviating from the known forms of mathematical expression, in this case – the proportional relationship, to the equation. The conceptualization of the linear equation is still forming for him in this situation.

The predict question (Task 2.4) asked to find the height difference after 25 turns. Calculation was performed without using the linear function, by simple calculation. He multiplied quantity of rotations to the height of one rotation for the first (red) pulley, then did the same with second (orange) one. Then he found heights difference (see below).

$$25 \cdot 4,2 = 9700 \text{ cm}$$

$$\begin{array}{r} 250 \\ 42 \\ \hline 500 \\ 02 \\ \hline 9700 \end{array}$$

$$\begin{array}{r} 3 \\ 25 \\ \hline 0 \\ \hline 210 \end{array}$$

$$\begin{array}{r} -210 \\ 37 \\ \hline 113 \end{array}$$

This session is interesting as it illustrates how children with some minimal introductory algebra experience get accustomed to using letters instead of numbers for solving algebraic problems. The idea of using a variable in the process of solving a problem is gradually becoming more and more familiar and convenient. In principle, the idea of using a letter to represent a number is not so foreign; however, the pupils are still not very confident with it. The “dynamic” understanding of the concept of the variable is still not fully developed in this stage.

4.2.3 Session No 3 dated 08/08/2013

This session was conducted with the boy from the second session - Andrew, 6th grade.

In Task 3.1 that involved two pulleys (equation in the affine form) the boy was asked to set the red weight at 10 cm, the grey one at zero. Then he was expected to rotate the red pulley and find the height of each weight after each turn. The data was collected through tabulation, presented below:

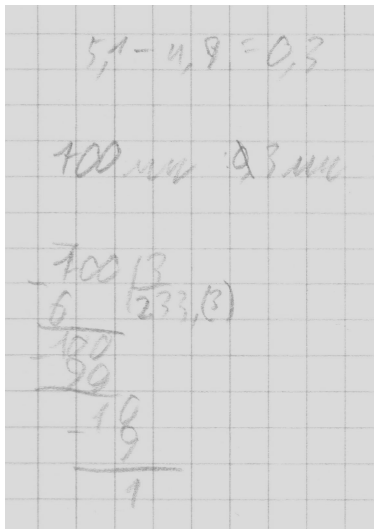
Table columns: Number of turns / height of red weight, cm / height of grey weight, cm

Количество оборотов	Высота красного грузика, см	Высота серого грузика, см
0	14,7	5,1
2	19,5	10
3	24,2	15,3
4	28,9	21,2
5	33,6	27,7
6	38,5	34,5
7	43,4	41
8	48,2	47,5
9	53	54

Finally, the height per turn was approximated from his measurements; no precise calculations have been performed.

Having found the approximate height gain per turn for the red and the gray weights, he estimated the height difference per one turn and estimated the number of turns when the gray weight catches up with the red one.

- 18.05 Andrew: The gray overtakes the red at tree-tenth. What is left is to calculate this value in millimeters: 233 and 333 in infinity.
- 19.06 Instructor: Good! Now, please elaborate your calculations. What did you divide by what?
- 19.13 Andrew: 700 by 3 – that’s what it overtakes by. And we got [number of turns]...



So, he used a proportion based on a system of equations differences. The difference of the initial heights corresponded to the difference of the free terms and the difference in angular coefficients was the “height difference per turns”

Interpretation for the solution made by Andrew:

$$d - b = \text{height difference,}$$

$$a - c = \text{rotation difference}$$

$$x = (d - b) / (a - c).$$

The same solution could come from the system of linear equation:

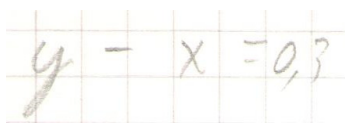
$$a * x + b = c * x + d$$

$$x = (d - b) / (a - c).$$

Thus, a linear equation (in the affine form) was used implicitly.

My request to write out an equation describing the motion of the two weights leads to the following answer:

20.25 Andrew: Suppose red is y and gray is x , the $y - x$ equals three-tenth (0,3)



A photograph of a handwritten equation $y - x = 0,3$ written in brown ink on a light-colored grid background. The equation is written in a cursive style.

A repeated request to relate the height and the number of turns was answered with a statement that one still needs more time to evaluate the height at which the weights meet.

That is instead of generalizing and formalizing the observations the pupil was still focused on obtaining the “answer” for the problem - evaluating height of the meeting point. The equation of motion had not been completed.

My previous suggestion of solving the problem graphically was refused as being too time consuming.

11.53 Instructor: Can a graph be presented here?

11.58 Andrew: Potentially, it can be.

12.01 Instructor: Will the solution be simpler with the graph?

12.05 Andrew: Too, long....

Building on the graphical method of solving the problem, I suggested to think that the graph would be useful, as a starting point, the graph made for the previous task by Andrew.

26.20 Instructor: Will this graph be useful, what do you think?

26.22 Andrew: The graph?

26.24 Instructor: Yes, can you use that graph for this task? Remember the task we started from zero [and you made a graph].

26.26 Andrew: Yes.

26.28 Instructor: How will the graph go in this problem?

26.30 Andrew: In this. One... ... From 70.

26.32 Instructor: Good. How will develop?

26.40 Andrew: The red will start from 70, the gray-from zero.

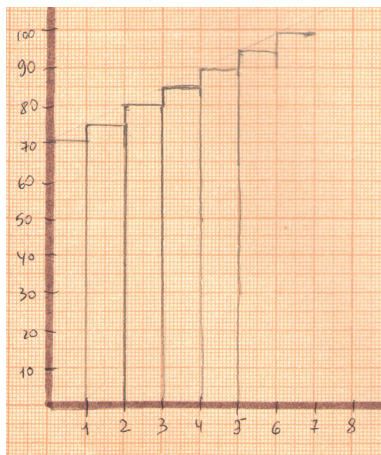
26.43 Instructor: Aha. And the slope will be the same?

- 26.46 Andrew: Slope? [thinking]
- 27.01 Instructor: Will the slope change or not?
- 27.08 Andrew: Will probably change [hesitantly].... However.... No it will not change [remains uncertain]

Thus, an intuitive analogy to the graph of the previous task has been drawn and, intuitively, the problem was solved correctly.

I suggested him plotting a graph, but he objected, arguing that it would take too much time. And then he gestured and talked about a graph, but without drawing. He showed where the graph would start for the red and grey weights, and, with some hesitation, reasoned that the slope would be equal.

Therefore, he started to plot it, but our tie was over. So, he would have been able to find solution graphically, if time had allowed. The graph was plotted as shown below.



4.2.4 Session No 4 dated 10/08/2013

Two 8th graders, Roman and Dmitry, were taking part in this session.

In the Task 1.1 with one pulley the pupils were asked to find the height of one of the weights after 5 turns. They started rotating and counting without my additional instructions and even without reading the task carefully.

They just proceeded immediately to measurements. Having calculated the height at 5 turns, i.e. having obtained the result by enactive mode of activity, they independently attempted solving a more complex task. An attempt to theoretically verify the obtained result was made.

- 1) Verbally, the concept of “travel per turn” was introduced and verbally, the unknown x was introduced to represent the number of turns in the subsequent calculations.

- 2) An additional pulley turn was quantified in terms of “travel per turn” , the obtained data were entered into the work table, the difference in the weight positions between the 5th and 6th turns was determined.
- 3) Then, the difference was multiplied (mentally) by 5 and the multiplication result was found. It was somewhat different from the previously obtained value of travel after 5 turns.

- 00.44 Roman: OK, 5 iterations...1,2,3,4 and 5
- 00.59 Dmitry: At what heights?
- 01.03 Roman: Should be at 27
- 01.11 Dmitry: Yes
- 01.20 Roman: Ok that is... how many centimeters per turn?
- 01.33 Dmitry: x turns [unclear]
- 01.36 Roman: Wait, I haven't read the task yet (reading task No 1.1)
Now, 5 turns (pause) So, we are to characterize the x ...
- 02.06 Dmitry: Yes and then we are to find the number of turns and simply calculate out and ...done...
- 02.12 Roman: Let's make another turn - we got 6 turns, 32 cm. We write down:
5 turns equal 27, 6 turns equal 32.
The height difference between the turns equals 5. 5 by 5 is 25; so it turns out to be very approximate, It's not precise [unsatisfied].

5 - 27
6 - 32

They read Task 1.2 and attempted to establish a generalized relationship between the number of turns and the height.

Dmitry said:

- 03.01 Dmitry We are to calculate ... how much is to be added
- 03.08 Roman Let's look from zero. That's easy without accounting for errors.
- 03.28 Dmitry Just one rotation first. Then write down how many turns it takes

The pupils continued filling out their own version of the table:

5 - 27
 6 - 32
 1 = 4
 2 =

I offered them my version, more convenient in my opinion. They agreed to work with my version of the table, copied their data into it and continued working with the table I had offered. Column headings: Number of turns, pcs. Height, cm.:

Количество оборотов, шт.	Высота, см
1	4
2	8 8,5
3 3	13,5 14
4	18,5
5	23,5 23

They quantified the rise of the weight after 5 turns, tabulating the height with each pulley turn. Then, they repeated the measurements to verify the preceding measurement series. They were attempting to reduce the error in their measurement of the height increase per one turn.

The measurement errors worried the pupils.

04.27 Dmitry: Is your error 1 cm?

04.34 Roman: No.

The pupils were fascinated by the concept of measurement accuracy. The height gain with each turn was determined by mental calculation, by subtracting the heights measured at two subsequent turns. They concluded that the height increase with each turn is between 4 and 5 cm.

08.48 Roman: It's probably like that – one has to calculate the “arithmetic general”, so to say.

Verbally, the concept of the “arithmetic mean” was presented, although named with an unconventional term - “arithmetic general”. The non-standard terminology did not adversely influence the solution. The concept of the arithmetic mean was well understood, just referenced with a term that is not generally accepted.

Note an unconventional method for arithmetic mean calculation:

The arithmetic mean was calculated by the subsequent addition of the heights after 1, 2, 3, 4 and 5 turns, followed by the division of the sum by the sum of turns ($1+2+3+4+5=15$). The calculation was performed without a calculator.

$$68 : 15 = 4,5 \dots$$

10.46 Roman: So it's 4.5 ... x by 4.5...[writing]

$$68 : 15 = 4,5 \dots$$

$$x \cdot 4,5 =$$

Then, with the equation obtained, a mental calculation for the case of two turns was performed, the result was again verified – the height after two turns was measured experimentally on the device. The calculated and experimental results almost coincided.

- 11.06 Roman: Let's see... 4.5 multiplied by 2, most likely will be 9... (Checking of the device)
- 11.21 Dmitry: It seems like you still do have an error...
- 11.25 Roman: Really? Why? The arithmetic general [the arithmetic mean] ... 4.5. Good enough? [asking to Instructor and smiling]
- 11.59 Instructor: What have you just found?
- 12.01 Roman: The "arithmetic general" [the arithmetic mean].
- 12.03 Instructor: The arithmetic mean. Have you found the heights per turn?
- 12.05 Roman: Yes.
- 12.08 Instructor: So have you answered the first question?
- 12.10 Roman: Yes.
- 12.12 Instructor: After some number of turns, in x turns, what will the height be?
- 12.15 Roman: In x turns, the heights will be $4.5 \cdot x$.

Once again, the equation describing the height dependence is verbally presented with a more precise coefficient.

In the Task 1.2. they were expected to find a relation between two variables, i.e. between quantity of turns of one red pulley and the height of the weight which is attached to that pulley.

This task was completed quickly.

- 12.41 Instructor: The second task. It's quite similar to the first one. One just needs to generalize the first problem.
- 12.46 Roman: Surely... One turn contains 4.5 cm.
- 12.48 Instructor: Go ahead and write that down. One turn...
- 12.52 Dmitry: Is 4.5 cm.
- 12.55 Instructor: And write down an equation by which to calculate the height from the number of turns.
- 13.00 Roman: So, suppose in x we have a turn, multiplied by 4.5, it will be... is this right? [thinking].
- 13.04 Instructor: Probably the height is to be indicated somehow?
- 13.10 Roman: Suppose the height is h .

In the result, the dependence was expressed both verbally and in writing:

$$\begin{aligned} \text{1 оборот} &= 4,5 \text{ см.} \\ x \cdot 4,5 &= h \end{aligned}$$

"1 turn = 4.5 cm"

The task with the prediction question (Task 1.3) was solved using proportion again.

- 13.40 Instructor: How are you going to solve that?
- 13.42 (both): By dividing.
- 13.44 Instructor: Go to the ceiling and start rotating?
- 13.46 Dmitry: Simply by dividing.
- 13.48 Roman: We'll divide 2500 by 4.5.

Verbally, a correct solution was presented. However, when writing the solution down in the notebook, a mistake was made in converting meters to cm; as a result, an erroneous result was obtained. No attempt to verify the result was made, contrary to previous exercises.

$$- 25000/45 \\ \underline{555.5(5)}$$

We moved to the Task block 2 with two pulleys.

First, the boys were asked to predict which weight was rising quicker.

- 15.16 Dmitry: The right weight is rising faster
- 15.18 Instructor: The Right-side one?
- 15.23 Dmitry: It was rising and I noticed that it was rising faster.
- 15.25 Instructor: So, the question is answered? [smiling]
- 15.28 Dmitry: [to Roman] See it for yourself. However, it might have been higher initially.. I am not sure. Lift it up.
- 15.35 Instructor: Rotate clockwise... [They rotated the red pulley and found that weight on orange (smaller) pulley was rising quicker].
- 15.53 Dmitry: Looks like I guessed correctly. The right one is faster.
- 15.55 Roman: Because it has a smaller radius, so it has to make more turns.
- 15.59 Instructor: Correct.
- 16.04 Roman: How is that... The bigger the wheel, the longer distance it covers in one turn. How do we describe that in writing?
- 16.17 Dmitry: And what weight will be rising faster?
- 16.21 Roman: Naturally, the one on the smaller pulley.

The explanation is drawn from the previous non-mathematical life experience (rotating wheel and the distance traveled).

Then they moved to solving Task 2.2 and were recording the position of the weight after each turn. The weights start from 0.

- 16.59 Roman: OK, It's just one turn. Here, as we have already understood, it's 4.5 approximately. OK, 4, let it be 4. This is for the red one, And for the gray... we'll have 5.

A correct analogy to the previous task has been drawn. The height increase per one turn for the red weight has not changed and is still 4.5 cm. However, too crude rounding to 4cm has been performed.

The pupils were filling out their version of a table to represent the heights of the red and gray weights as functions of the turns.

Handwritten calculations showing the relationship between turns and heights for red and grey weights:

- 1 turn: K. 4, C. 5
- 2 turns: K. 9, C. 10
- 3 turns: K. 14, C. 16
- 4 turns: K. ~~18~~ 19, C. 22,8
- 5 turns: K. 23, C. 28

Then they filled up my version of the table and used my version for the future calculations. Column headings: Number of turns, height of red weight, height of grey weight.

Количество оборотов	Высота красного грузика	Высота серого грузика
1	4	5
2	8,5	10
3	14	16
4	18,5	22
5	23	28

17.29 Dmitry: The more turns, the bigger the height difference between the weights

The height difference between the red and grey weights is perceived as an important value and is actively being used in the subsequent calculations.

- 18.06 Roman: Question... Here in the third task, we are to write down the difference in the heights of the weights that is to find out how the height difference is changing?
- Instructor: You can select. You either can consider each single weight or one relative to the other, whatever is more comfortable.

I was trying to steer them away from calculating the difference in weight heights towards linear equations for each of the weights. But my steering did not help.

The predict question Task 2.3 asked to find the height difference after 23 turns. The timer was at zero as a new video file has started.

- 00.00 Roman: $10+16+22+28=81$ (laughs) Dividing 81 by ...what here... 6-10-15. Dividing by 15, getting 5,4 and then, multiply by 23.
- 00.34 Dmitry: Understood.
- 00.37 Roman: What will be? 124.2 .
- 00.40 Dmitry: What is for the red?
- 00.43 Roman: 103.5 for the red and for the gray.. well approximately, what did we get 124.2 . The difference is 21.3 .

The final calculation was performed without using the linear function, by simple calculation. They multiplied the number of rotations by the height of one rotation for the red (K) pulley with the red weight, and then they did the same for the orange pulley with the grey (C) weight. Then they found the height difference.

$$\begin{aligned}
 K &= 103,5 \\
 C &= 124,2 \\
 & (K) \cdot 103,5 - (C) \cdot 124,2 = 21,3
 \end{aligned}$$

(K = red, C = grey)

The height for one turn for the orange pulley was performed using “общее арифметическое» (the arithmetic “general”) again.

- 01.44 Instructor: And how did you calculate that? Show once more again.
- 01.48 Dmitry: We have again calculated for the gray, as in the first time.
- 01.57 Instructor: And what did you get for the gray?
- 02.01 Dmitry: 5.4 .
- 02.04 Instructor: Where have you written that? I don't see that.

- 02.08 Dmitry: Here [append to the paper writing].
- 02.11 Instructor: The result is correct; it's just not clear how you have obtained that.
- 02.17 Roman: 81 is the arithmetic mean for the gray, 81 divided by 15 equals...
- 02.25 Instructor: 81 is the mean? The total?
- 02.38 Roman: It's the total, in cm, approx. ...5.4 .
- 02.46 Instructor: 5.4 is what?
- 02.47 Dmitry: It's the Arithmetic general [arithmetic mean].
- 02.48 Roman: Arithmetic mean.
- 02.49 Dmitry: The height of the gray.
- 02.50 Roman: Per one turn.

$$81 : 15 = 5.4 \text{ cm}$$

- 03.02 Instructor: And how did you proceed next.
- 03.05 Roman: Next, we multiplied 5.4 by 23 and naturally, 4.5 by 23 and subtracted the two.
- 03.09 Instructor: OK. Good, the answer is accepted.

The next task was Task 4.3 from the Task Block 4 with two pulleys. Pupils should predict position of grey weight knowing position of the red one.

- 03.32 Instructor: Here you can use the data already obtained.
- 03.40 Roman: "Predict the position of the gray if the red is known..." Here one must know the difference for that.
- 03.46 Dmitry: Difference per turn.

Again, the difference per turn serves as a starting point in their solution.

- 04.06 Roman: So, we get height difference per turn will then be in the "arithmetic general" 4.5 cm minus 5.4 cm, equals 0.9cm. This is the difference per turn. Then we first multiply it by 4...
- 04.29 Roman: So, suppose we have x as the number of turns, multiply that by 4.5 plus x , multiplied by 0.9. This will be the h of the gray. Good?

$$4,5(\text{cm}) - 5,4(\text{cm}) = 0,9$$

$$x \cdot 4,5 + x \cdot 0,9 = h(\text{cm})$$

(C is for the grey weight).

Extra interpretation for the solution made by Roman:

Given: $a = 4.5$

$c = 5.4$

$y_1 = x * a =$ position of the red weight

Find: $y_2 = x * c =$ position of the grey weight

The following parameters have been introduced by the pupils for the solution.

a) $|a - c| =$ height difference per single rotation = 0.9 .

Note that the absolute value (modulus) was proposed as a parameter. The pupils subtracted 5.4 from 4.5 and obtained a positive value. No comments has been made of the sign selection. No comments have been offered to justify or explain the use of the positive difference. Obviously, the positive value of the height difference can be used specifically to describe the motion of the faster-raising weight, but no specific comments were offered on this issue.

b) $x * |a - c| =$ height difference per x turns.

c) y_2 , the faster-raising weight can be described by the following:

$$y_1 + x * |a - c| = y_2$$

By transforming the left-side of the above equation, we can verify the identity of the result:

$$x * a + x * |a - c| = x * a + x * (c - a) = x * c \equiv y_2$$

Strictly speaking, the pupils did not complete the task, because the solution they offered requires the knowledge of several parameters:

- the height of the weight – input to the problem
- the height difference per turn – system property
- the number of turns completed

However, the pupils did not offer a specific way to determine the number of turns that had to be performed to raise the weight to the given height. Although a question of semantics, the formulation of the problem suggests, that only the height of the first weight and the system parameters could be known.

The same solution could be obtained a little bit more rigorously from the following system of linear equations:

$$y_1 = x * a + b$$

$$y_2 = x * c + d$$

$$b = d = 0$$

$$y_2 = y_1 + x * c - x * a = y_1 + x * (c - a) = y_1 + (y_1 / a) * (c - a) = y_1 * c / a$$

It can be seen that the ratio of the heights per turn is a more convenient parameter for this problem.

Further, a verification of the obtained result was conducted twice: first by entering the given number of turns into the equation and calculating, then by measuring on the device and comparing the results.

- 04.48 Dmitry: Somehow you haven't completely calculated, but that's OK.
- 04.52 Roman: Suppose, there will be 4 turns or 5 .
- 04.58 Instructor: It could be verified, by plugging ...
- 05.01 Roman: Yes, now. Let's make it easy.
- 05.05 Dmitry: Let's select any number of turns.
- 05.08 Roman: Let's do two... 9... That will be 2 by 0,9 gives 1.8.. thus 9+1.8=10.8 we have the gray on the second [pulley]...

$$\begin{array}{r} x \cdot 4,5 + x \cdot 0,9 = h(c) \\ \hline 9 + 1,8 = \underline{10,8} (c) \end{array}$$

(C is for the grey weight).

- 05.15 Roman: Let me see (rotating the wheel) No, not zero yet...
- 05.25 Instructor: The origins coincide?
- 05.29 Roman: Aha, one and two.
- 05.40 Dmitry: And what did you get?
- 05.45 Roman: 10.8, Ok 10.5- roughly there.
- 05.49 Instructor: OK the checking is done.

In this task, the pupils already recognized the importance of reducing the measurement errors (for example by starting the motion strictly from 0) for obtaining practical results of higher precision.

This problem proved to be difficult and challenging for the pupils, however, I still decided to include this problem in the session, because the pupils had already accumulated the necessary data for it and managed to solve the previous problems very fast.

In the Task Block 3 with two pulleys the pupils avoided solving part 1 (Task 3.1) where they were asked to determine the heights of the red and grey pulley after each turn, if the red one started from 10 cm and grey one started from zero. They didn't collect the data to the new table and jumped to Task 3.2 where they were asked to find the height at which the weights meet. Dmitry told that the height can be calculated, but couldn't explain how.

They rotated the red pulley few times and experimentally determined the height where the weights met.

Then they moved to Task 3.3. in which there were asked to predict the height where the weights meet if the red starts from the table surface (70 cm above the floor) and the grey starts from the floor.

The strategy they adopted for solving the problem relied on determining how much the distance between the two weights decreases per turn and then, on calculating the number of turns required for the gray weight to completely "catch up" with the red one.

Initially, they calculated the number of turns required for the grey weight to reach the table surface. Then, they determined how far the red weight "ran away from the grey during these turns. They repeated these measurements, but found that with this strategy, it was too difficult to "locate" the meeting point reliably.

Then, the pupils attempted to solve the problem independently from each other.

Roman attempted to calculate the distance change between the two weights after a given, pre-determined number of turns, while keeping in mind the distances per turn, obtained previously – 5.4 and 4.5 cm.

Dmitry started a new series of measurements to determine the number of turns of the red wheel per a single turn of the gray wheel.

Both methods required a considerable amount of time (14 min for Roman and 20 min for Dmitry) and turned out to be quite imprecise. Both pupils admitted the imprecision in their solutions.

Dmitry (after the session): "I determined that for two turns of one weight the other one gives three and than I was incrementing one "wheel" by two, the other – by three and was waiting. That was very confusing."

Roman (after the session): "What I got was that the distance between the weights shortens by 9cm per two turns."

As a matter of fact, Roman, while solving the problem, expresses an idea of solving the problem with formula (equation) or graphically, however, neither of the two pupils attempts implementing this idea. So, a need for formula arises:

- 13.57 Roman: So, we got 11 turns. Thus, for red, 11 by 4,5... Using a formula is simpler and in general better. It's OK.... Equals to 49,5...
- 14.15 Roman: Thus with every ... the red at this time will make already...will go up by... In principle one could utilize a graph here...

4.3 Aspects of the concept of linear equation

In this paragraph I answer the first part of the research question:

“What aspects of the concept of linear equation are exposed in a double-winch laboratory”?

4.3.1 Activities in the enactive mode

The pupils were very active and enthusiastic in working with the device – rotating pulleys, taking measurements. The senior pupils even started taking measurements without my instructions and even without reading the problem statement fully. They were following a sequence of actions: rotating pulleys, counting turns and determining the weight position by reading the instrumental scale. Thus, the abstract concept of variable “became touchable” through the observation of the variable parts of the system. This is one of the many aspects of linear equation that were exposed in my research.

Taking a measurement from the device was the activity, to which the pupils resorted in difficult situations. For example, in session 3, in which one had to find the height where the weights meet, the pupil completed the table but could not complete calculating the average value of the height gain per turn. The pupil solved the problem “experimentally” by rotating the pulley once and measuring the height increase directly on the device, without performing precise calculations.

Similarly, in the session with the senior-grade pupils (session No 4), the data obtained in calculations were verified in the enactive mode by taking measurement physically on the device.

They did this by their own initiative, without being prompted.

4.3.2 Selecting a method for solving the problem

The seniors show an extended set of skills and abilities, compared to the juniors. Most of the things that juniors can do, the seniors can also do with several additional abilities. Both the seniors and the juniors used a wide set of skills. They measured physically (enactive mode), they calculated proportions and differences, used repeated addition, wrote equations, made graphs and so forth.

The most complicated problem involving the affine function (Task 3.3) was approached by seniors with the most direct, but labor-intensive and time-consuming method: the repeated addition.

Proportionality was utilized most commonly in many problems. It appears that the pupils had a good understanding of how to use proportion and therefore, used this method widely.

The findings are shown in a table below.

Table 4.3.2: Methods used by the pupils in problem solving

“S”	spontaneous – did not require interaction with the instructor
“A”	adopted – required a discussion or even guidance from the instructor
“Space”	no task given
“-“	the method not used

Problem	Method	Juniors	Seniors
Single-pulley system			
The height of the weight after 5 turns	Enactive	S	S
	Proportionality	S	S
	Equation	-	S
Relationship between height and number of turns	Proportionality	S	S
	Equation	-	S
Predict the number of turns to elevate the weight to the height of the ceiling	Proportionality	S	S
The system of coupled pulleys:			
Relationship between the heights of weights and the number of turns of a given pulley	Proportionality	S	S
	Repeated addition	-	S
Predict height difference after 25 turns (juniors) and 23 turns (seniors)	Proportionality	S	S
Predict the position of a given weight (gray) if the position of the other one (red) is known	Angular coefficients* difference		S
	Equation		S
Find height where weights meet if red one started at 10cm and grey one from zero.	Enactive	S	S
Predict the height where weights meet if red one started from the table level and the grey one from the floor	Angular coefficients* difference	S	-
	Proportionality	S	S
	Graph	A	-
	Repeated addition	-	S
	Equation	-	A

* Russian word for gradient

The data from the table show a similarity between the age groups in the “characteristics” (details, features) of the solution process:

- Both age groups were similarly active in solving the given tasks.
- Both age groups showed great creativity.
- Both groups used proportionality a lot.

- Many tasks were solved “spontaneously” – without the instructor’s help
- An “adopted” method, which emerged after prompting by the instructor, was used for subsequent tasks.

4.3.3 Verbal representation of relations

The junior pupils used verbal representations to express relations between the height for one turn, number of turns and total height gained. They could verbalize the relationships well.

When prompted to use letters (e.g. y and x), the junior pupils adopted these but each time found it necessary to attach a verbal comment explaining the relation.

When writing mathematical expressions, they proceeded in a similar manner – letter variables had to be accompanied by a verbal comment, before a junior pupil could start writing equations. The seniors were doing both ways: writing equation and verbalize them later, and verbalize first and then writing equations.

The following Table shows details of variables that were represented verbally by the pupils. It is also indicated whether the pupils used the words spontaneously or adopted them. They also invented their own words.

Table 4.3.3: Verbal representation of relations

“S” spontaneous – did not require interaction with the instructor (researcher)

“A” adopted – required a discussion or even guidance from the instructor

“-“ not used

Concept	Word used	Juniors	Seniors
Height	Height	S	S
Turn	Turn	S	S
Height of one turn	Height of one turn	S	-
	Travel per turn	-	S
Height difference between the weights	Height difference	S	S
Height difference between the weights per turn	Height difference per turn	S	S
Slope	Slope	A	A
Angular coefficient (slope coefficient)*	Coefficient	A	-
Mathematical expression	Mathematical expression	A	A
Linear equation	Equation	A	S
Table	Table	A	S
Graph	Graph	A	S
Infinity	Infinity	S	S
Approximately	Approximately	S	S
Exactly	Exactly	S	S
Arithmetic mean	Arithmetic mean	A	A
	Arithmetic general	-	S
Errors	Errors	S	S

* Russian word for gradient

It appears that linear equations are not yet self-explanatory for juniors because they did not come spontaneously. Senior pupils introduced verbally the concept of “travel per turn” spontaneously. Equations were verbalized before they were written.

The concept of arithmetic mean was adopted and presented verbally by words “arithmetic mean” for juniors. But the seniors spontaneously invented some special word “arithmetic general” and adopted the right words later.

4.3.4 Table representation of relations

Tabulation was routinely used for recording data. The senior-grade pupils created their versions of tables and started filling them out independently, spontaneously, without the instructor’s help. The junior pupils started filling out data tables after the instructor suggestion. However, using the analogy, the junior pupils created and filled out their own tables independently in the subsequent tasks in the next sessions. Attempts were made to use the tabulated data to calculate the arithmetic mean of the height gain per turn. The tabulated data were utilized for plotting graphs in the sessions with junior pupils. The table below details how pupils used tables for representation of their measurements.

Table No 4.3.4: Table representation of relations

“S” spontaneous
 “A” adopted
 “_” not used

Problem	Juniors	Seniors
The raising of the weight after 5 turns	-	S
Relationship between height and number of turns	-	S, A**
Relationship between height and number of turns for two connected pulleys	A	S, A**
Predict the height where weights meet if red one started from the table level and the grey one from the floor	S*	S

* Once adopted, the other was spontaneous

** Constructed their own version of table spontaneously and adopted the instructor’s version.

4.3.5 Graphic representation of relations

The juniors were not very enthusiastic about my suggestion to solve the problem graphically. However, after considerable efforts, a graph was plotted and answers were obtained.

Nonetheless, in the next session, the pupil refused to attempt a graphical solution, claiming that the graphical solution would be too time consuming. At the same time, he correctly answered my question to show a graphical representation of a solution in the next task on the affine function. He could indicate the starting point and knew that the slope would not be different from the linear situation.

The seniors also avoided graphs and my suggestion to attempt a graphical solution was not followed. No attempt was made to “sketch” a graph. However, after multiple attempts to solve the affine-function problem using the repeated addition method, an idea of a graphical

solution was suggested. During the task on the affine function, Roman told the following: “In principle one could utilize a graph here...”

The Table below shows how pupils used graphs in a problem solving.

Table No 4.3.5: Graphic representation of relations

“S” spontaneous
 “A” adopted
 “_“ not used

Problem	Juniors	Seniors
Relationship between height and number of turns for two connected pulleys	A	-
Predict the height where weights meet if red one started from the table level and the grey one from the floor	A	S

4.3.6 Symbolic representation of relations

4.3.6.1 Representation of the unknown

Of all symbols available, the juniors were mainly operating with numbers. They occasionally used words (verbal expressions) and the “?” sign. After suggesting the use of letters as substitutes for numbers, they did so simply “mechanically” to represent the unknown.

The seniors were more confident in using letters/numbers to write expressions for un-coupled motion. Spontaneously, letters “ x ” and “ h ” were introduced in mathematical expressions.

The symbols were also used to describe and distinguish the weights: “K”- for “krasniy” (Russian for red) and “C” for the “seriy” (Russian for grey). In simple problems, both approaches were used: only numbers and numbers substituted by letters.

The following table shows how pupils used symbols to represent an unknown.

Table 4.3.6.1: Symbolic representation, representation of unknown

“S” spontaneous
 “A” adopted
 “_“ no introduction

Concept and artifact	Introduced as	Juniors	Seniors
The height	Number	S	S
	“?”	S	-
	“y”	A	-
	“h”	-	S
The weight	Letters (“(R)” and “(G)”)	-	S

Number of turns	Number	S	S
	“?”	S	-
	“x”	A	S
Height per turn	Number	S	S
	”y” and ”x”	S	-
Height difference between weights	Number	S	S

4.3.6.2 Distinction between the concepts of known, unknown and variable

In session No 2, the juniors described a relationship between the known and unknown correctly. The unknown, denoted as “?”, was found with the help of a proportional relation. However, when presented with a problem of finding the equation describing the motion of the weight in a system that was lifted to a big height – “the height of an airplane” - the pupil answered that without the knowledge of the height and the number of turns, one could not predict the motion of the weight.

Having received a suggestion to denote the height and number of turns by letters, he was able to write down a mathematical equation with these letters. However, for that pupil, a letter was used as a substitution for a particular, given number he measured. He didn’t show understanding that a letter can represent a variable.

This lack of variable conceptualization exhibited itself when the pupil “mechanically” substituted a number by letter in the expression for the difference of angular coefficients. They wrote:

$$y - x = 0.3 ,$$

where “x” and “y” stood for “height per turn” and were constant.

The senior-grade pupils used “x” spontaneously to describe the unknown, right from the first task. At that stage the “x” was used for writing an equation. However, the data do not show clearly whether they understood that “x” represents the unknown or the variable.

4.3.6.3 Different ways of writing equations

4.3.6.3.1 Relations expressed by division

The division operation was used primarily by the juniors. While solving the problem for finding a mathematical expression relating height and number of turns, the pupils would usually begin by expressing verbally: ”It is necessary to divide centimeters to the number of turns“, and then proceeded with a symbolic expression, adapting the symbols “x” and “y”. So, they wrote spontaneously expression by division as:

$$y : x = 4.2 ,$$

and later they adopted the standard way of writing a linear equation in the form:

$$y = 4.2 x .$$

Remarkably, the division operation comes naturally in this problem as a distance per turn, although division is usually perceived as more difficult than the multiplication operation.

4.3.6.3.2 The standard way of writing equations

Although the juniors started writing equations using the division operation, the standard way of writing equations was later adopted by the junior pupils. The standard way was initially presented by me as simply a more convenient way of presenting the equation that the pupils obtained.

Then, for the next problem, the junior spontaneously utilized the division-based form of equations, but then independently, spontaneously started using the standard form.

The seniors used the standard approach with multiplication from the beginning. They also verbalized the equation first and then composed the equation in the standard form using symbolic notation. The senior pupils described the height as a function of turns for one weight:

$$x * 4.5 = h ,$$

and in the coupled problem they described symbolically the height of one weight if the position of the other is known:

$$x * 4.5 + x * 0.9 = h .$$

However, only relatively simple problems were solved using equations. In more complicated problems they used repeated addition and proportionality. But, after getting confused repeatedly in using the repeated addition method, I suggested to the seniors to use an equation but this idea was not followed upon. During the task on the affine function, Roman said: "...So, we got 11 turns. Thus, for red, 11 by 4.5... Using a formula is simpler and in general better....".

4.3.6.4 Approaching the variables x and y and the parameters a and b in the linear equation $y = a * x + b$)

Both the juniors and the seniors spontaneously named the variables in connection to the situation, as height, number of turns, height increase for one weight after some turns and height difference between two weights after some turns and effectively used these situated terms for problem solving. They often looked at stepwise differences and increases. In algebraic terms, not used by the pupils, they verbally described:

$$y_c - y_k \text{ and } y_{k2} - y_{k1} ,$$

where:

y_c height of the grey weight

y_k height of the red weight

y_{k1} initial height of the red weight

y_{k2} height of the red weight after some turns

The following table shows which variables and constants pupils were choose in problem solving.

Table 4.3.6.4: Variables and parameters (constants) used by the pupils

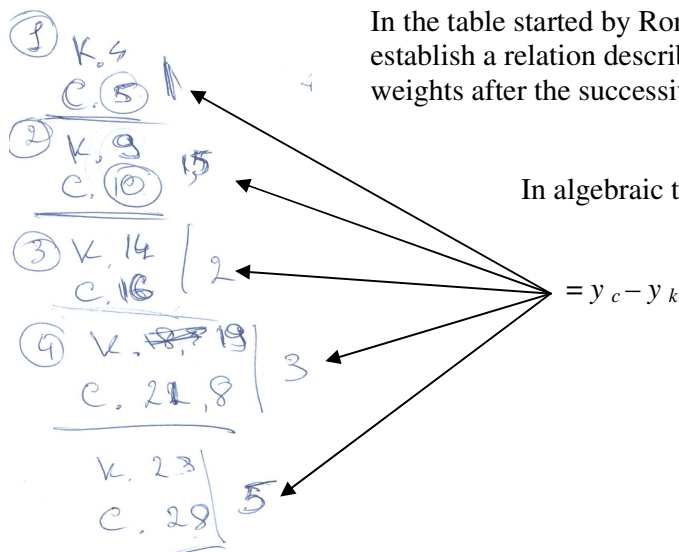
“S”	spontaneous
“A”	adopted
“-“	not used

Concept	Juniors	Seniors
<i>Variables</i>		
1. The height of the weight	S	S
2. Number of turns	S	S
3. The height difference between the initial height of one of the weights and that after few turns (in algebraic terms: $y_{k2} - y_{k1}$)	-	S
4. The height difference for the two weights after several turns (in algebraic terms: $y_c - y_k$)	-	S
<i>Coefficients</i>		
1. The height of the one weight after one turn (the height increase per turn for one weight)	S	S
2. The height difference between two weights after one turn (That is, the difference of the angular coefficients)	S	S
3. The free term of the affine function, constant	A	A

Subsequently, in solving the problems on determining the position of one of the weights while the position of the other was known, the seniors successfully used the term “height increase per turn” or the term “height difference per turn” when two weights were involved. When written mathematically it seems more difficult ($x_2 - x_1$ or Δx), but for the pupils, the stepwise difference appeared to be practical.

The juniors successfully used the term of the “height difference between two weights after one turn” to solve the problem on the affine function.

The seniors, however, attempted to evaluate the relation between the height differences of the weights after the number of turns. This attempt is evident from the data table they spontaneously created and from the verbalization of the trend, offered by Dmitry: “The more turns, the bigger the height difference between the weights”



In the table started by Roman, an attempt is evident to establish a relation describing the distance between the weights after the successive number of turns.

In algebraic terms:

$$= y_c - y_k$$

Moreover, in solving the affine-function problem, the seniors again attempted to find a function of the relation between the distance between the weights and the number of turns. However, they did not succeed. The seniors used the repeated addition approach for solving the affine-function problem. The resulting solution was too cumbersome to be completed in the allotted time.

Thus the quantities that could be physically measured on the device, such as: the number of turns, the raising height of the weight, the height increase per turn, the height difference of the two weights per turn, the height difference for a given number of turns, provided a rich ground for many spontaneous reflections on constants and variables that constitute a linear equation, as well as other variables and constants that do not constitute the classical linear equation, but could be regarded as valuable aspects that come spontaneously to pupils.

4.3.6.5 Making connections between representations

Throughout the sessions, the pupils were frequently transiting between activities in the enactive mode and using verbal and symbolic representations of the measurements.

The following table (Table 4.3.6.5) shows how the pupils were translating between representations.

This table is to be understood as an account of actions performed by the pupils. For example, in item 4: the pupils spontaneously took measurements in enactive mode, tabulated the data in the table of their design and spontaneously created an equation to describe the trend.

Item 8: From the formula that they had established, they calculated the height of the weight from the number of turns; then, they went back to the device and verified the calculated result by rotating the pulley and measuring the height.

Table 4.3.6.5: Translations between representations

“S” spontaneous
 “A” adopted
 “_“ not used

		Enactive mode	Table	Graph	Formula
Juniors	1)	S→	---	---	S
	2)	S→	A→	---	S(A)
	3)	S→	A→	A→	---
	4)	S→	S→	---	S
	5)	S→	S→	A→	---
Seniors	6)	S→	---	---	S
	7)	S→	S(A)→	---	S
	8)	S←	---	---	← S
	9)	S→	S→	A→	A→

Most commonly, the activities would start in the enactive mode that would develop in one of the following routes:

- Almost immediate solution (finding the required results)
- Enactive mode with verbalization, data tabulation, followed by equation derivation
- Enactive mode with verbalization, data tabulation followed by plotting a graph and attempting to derive an equation.

With seniors, a spontaneous attempt was noticed to use the device to verify the result obtained from formula calculations, i.e. a return to the enactive mode from formula manipulations (represented by the backward arrow pointing from “formula” to “enactive mode”, the second-from-bottom row in the table).

4.3.7 Revisiting the proportionality aspect

4.3.7.1 Proportion with numbers and symbols

The juniors often used the proportionality, with numbers for the known constants and symbols, for example the question mark, for the unknown.

The seniors used the proportion represented numerically for simple problems, and linear equations for more complicated problems. The affine-function problem was also solved with proportionality, but the results were too approximate and inaccurate.

4.3.7.2 Using linear (affine) equations implicitly through proportional relations

A linear equation was used implicitly in the problem about the height difference between the two weights after 25 turns by finding the heights after 25 turns for both the red and the gray weights and then subtracting the two. The juniors and seniors solved this problem similarly.

An equation in the affine form was used implicitly for the problem of finding the meeting point of two weights. Interestingly, this was spontaneously done by a junior pupil.

4.4 Learning opportunities

In this paragraph I answer the remaining part of the research question: “What opportunities for learning the concept of linear equation are exposed in a double-winch laboratory”.

This study revealed that the double-winch laboratory allowed reviewing the pre-knowledge as evidenced by the spontaneous responses and spontaneous solutions generated by the pupils. The experiment has shown that many concepts were spontaneously enacted. For example, the juniors spontaneously used the question mark “?” to denote the unknown in the proportional relation. Then, symbolic forms of the unknown were adopted, using the letters y and x . However, this transition was still very “mechanical” in nature, lacking the mature understanding, as evidenced by the subsequent use of the same letters for constants, not variables. Thus, the full and complete adaptation of the concept of variables, as denoted by letters, wasn’t yet completed.

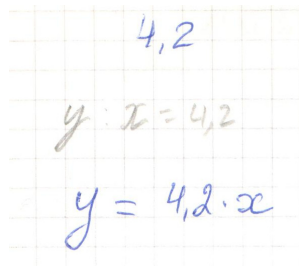
At the same time, the double-winch laboratory offers opportunities for more effective instruction, particularly when the researcher or session instructor is actively involved in the process. When the session supervisor, instead of being a passive observer, is actively involved in the process and stimulates the pupils’ intellectual activity by proper questions and guidance, new concepts or ideas can be presented to the pupils so that the pupils can efficiently adopt them into their learning process or chose to ignore the incorporation of new concepts, depending on how well the creation of the Vygotskian Zone of Proximal Development proceeds.

Therefore, the instruction proceeds through the following steps:

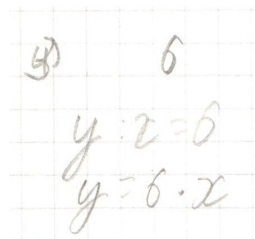
- | | |
|---|---|
| Recapitulating on the previous concepts, → reviewing the known. | Spontaneously generated solutions and answers |
| Generating an educated guess → | Spontaneously generated response based on the previous experience |
| Learning new concepts → | The knowledge generated and adopted during the experiment |

During the course of experiments, the juniors exhibited the process of assimilation (adaptation) of the concepts of “mathematical expression” and “equation”. For example, when asked to establish a mathematical expression describing the relation between the raising height of a weight and the number of pulley rotations, the juniors would always start from verbalizing the relation.

A suggestion to symbolically represent the elements of the relation (the height, number of turns, height per turn) symbolically would usually be adopted and followed by a spontaneous creation of the equation using the division operation.


$$4,2$$
$$y : x = 4,2$$
$$y = 4,2 \cdot x$$

In the subsequent experiments, the more standard way of writing linear equations was adopted and implemented. Subsequently, when solving a similar height vs. number of turns problem, requiring the creation of an equation, a junior spontaneously used his “original” version of the equation based on the division operation and spontaneously preceded to the more standard form of linear function that had been adopted previously.


$$6$$
$$y : x = 6$$
$$y = 6 \cdot x$$

Although the seniors initially started with their own version of the table for data tabulation, they, as well as the juniors, had transitioned to the table-form initially offered by the supervisor and continued the instructor’s table template in subsequent tasks.

The graphical approach for solving linear equations or certain systems of linear equations was also adopted (assimilated) during the study sessions. Perhaps, the graphical method was already known to the pupils, at least they did not deny the familiarity with the method. However, the implementation of the method proceeded without much enthusiasm, perhaps due to considerable expected time cost. The pupils were attempting to find the most efficient solution and perceived the graphical method as lengthy and cumbersome. The concept of the slope of the linear graph was adopted and they related it to the concept of angular coefficient (Russian term for gradient) of the equation.

Additionally, the concept of the free term in the affine function was articulated during the process of solving the affine problem graphically. However, the adaptation of this concept for this particular case may have been unreliable and needs further work. I did not have an opportunity to verify if the pupils could spontaneously use this concept later in problem solving.

The term “coefficient” was adapted to denote the increasing height of the weight after one pulley rotation and simultaneously associated with the angular coefficient of the linear equation.

The juniors adopted the concept of the “arithmetic mean”, while the seniors spontaneously enacted this concept, although with a slight change in the concept name.

Many concepts, once adopted and adapted, were used repeatedly and spontaneously, for example the “standard” form of the linear equation or the “convenient” form of the data table from the Instructor, thus implying that the process of learning was indeed happening and new knowledge was constructed.

The general atmosphere of the sessions is also worth noting. The children showed a lot of curiosity, were active and enthusiastic, and were asking questions. Particularly, this was true for the seniors who proceeded through the session in a good pace and could complete much more than I had planned. The post-session interview showed that the children were very satisfied with the laboratory experience and could recommend the laboratory to their friends. The seniors even continued discussing the problems offered in the laboratory after the session was over. I only hope that they were sincere and were not trying to pay a tribute to politeness. All these facts of course do not imply that the children learned something substantial, but they do reveal the favorable environment that the laboratory experiment had created thereby potentially leading to promising learning opportunities.

4.5 Additional results of the sessions

In this paragraph I describe observations that do not answer the research question, but that I consider also of interest to mathematical education researcher and teachers.

The first thing that is worth noticing is that the physical device evoked curiosity. It stimulated the pupils to pursue the tasks. A consistent pattern was observed:

First, the pupils worked in the enactive mode experimentally and they put their measurement into a table, acting stepwise (enactive representation). The focus of the activity on the first stage of problem solving was acquiring the data on the device, tabulating the acquired data followed by, and data processing.

The measurements were conducted enthusiastically, particularly by the seniors who took a good momentum and could complete all tasks that I offered to them.

Another notable observation is that all pupils were able to indicate which of the weights moved faster. Through use of intuitive, informal methods and the enacted mode, they could all solve the tasks. The age/training difference manifested itself in the fact that the juniors had to be prompted, while the seniors came to a formal solution themselves.

The juniors comprehended intuitively a linear relation by finding the average value of height per turn with a certain error while the seniors offered a formalization of their understanding through writing an equation. The pupils offered a number of intuitive solutions. One of them, and the most common one, was guessing. For example, guessing was the first approach, when determining which of the weights was the fastest. The process proceeded merrily and energized the ensuing experiments. Seniors, in addition, offered the reliable explanation of the behavior of the weights.

Another intuitive method was shown in the problem on the affine function, the junior refused to plot a graph; explaining that it was because of the duration of the procedure. However, from the graph constructed in the previous problem involving the linear function, a correct assumption was drawn about both the expected shift in the Y-direction and on the expected slope.

The pupils were confident with the device. The juniors usually required a few encouragements, while the seniors did not need any and started working without my assistance. They started tabulating data in their own version of a table and acted confidently.

Regarding accuracy and estimation, the juniors appeared to be well aware of the fact that a certain value obtained in the measurements (height per turn) can be approximate with various degrees of precision. One of the juniors even asked me what value was expected: “exact” or “approximate”.

The seniors were fully aware that in this experiment, they were dealing with approximate values and thus, they were trying to minimize the measurement errors by checking each other's calculations. The word "approximate" was used very often in their conversations. They were trying to conduct their measurements in a manner that reduces the measurement error.

The concept of average manifested itself in plotting the graph by junior students during the second session. During the third session, a junior-grade pupil attempted to find the arithmetic mean for the average of the height increase per turn. However, he selected an incorrect method for calculating the arithmetic mean. Initially, using the tabulated data, he summed up all the heights that he measured for ten subsequent turns of the pulley and divided the sum by the total number of turns. At a second approach, again using the tabulated measurements, he attempted to find the height increase per each turn, then sum up the determined values and divide the sum by the number of turns. In these measurements the number of turns coincided with the number of measurements, but no explicit remarks of this have been made. However, because of the cumbersome nature of the measurements and unaccounted accumulated errors in each measurement, the result was far from the real value.

The seniors were more successful in calculating the arithmetic mean, but used an incorrect and non-existing terminology for the result and word "arithmetic general". It, however, did not adversely affect the content of the concept to be expressed and did not obscure the perception of the concept. Lately, they adopted and used the correct name of the term "arithmetic mean".

I would also like to mention the attempts that the pupils made to verify the validity of results – "checking the answer". The juniors made no attempts to verify the results mathematically but the seniors made occasional attempts to verify formal equations by the enactive approach or by the use of proportion.

More formal solutions, involving equations, they proceeded as follows:

- Writing down the equation,
- Substituting the data in the equation (for example 2 turns)
- Calculating the result mathematically
- Verifying the result by enactive mode on the device
- Comparing the results.

All this was done in a good pace without my involvement. The activity proceeded as an enacted mode rather than a formal theoretical solution.

I also observed that most of the calculations were performed mentally. The use of the calculator was very infrequent. It appeared that the ability to perform complex mental calculations was perceived by the pupils as a sign of mental "agility" that was taken with pride. Moreover, although not verbally expressed, the pupils implied by their behavior that the ability to perform quick mental calculation was simply convenient, because quick mental calculations could be done "on the fly" without deferring the pupils from working the device.

From the post-session questionnaire and feedback of the session from the pupils, I saw that all the participants liked the problems offered. The problem that required predicting the weight positions was unanimously recognized as the most difficult. Interestingly, one of the pupils emphasized the problem was also interesting: "The problem was difficult but by no means boring". The simplest problem was that of finding the height after a given number of turns or that of finding the height difference. Four out of five would recommend the problems to friends. One of the pupils even utilized an exclamation mark to mark the answer "yes!" Overall, the pupils, particularly the seniors, liked the sessions. While leaving the venue,

senior-pupils asserted confidently that the sessions were interesting, entertaining and merry. They were so involved in the process of solving the problem that they could not stop discussing the problem on the way out.

5 Discussions

5.1 The potentials of “laboratory algebra” to represent algebraic concepts in a realistic, touchable way

As Pollak (2007) described, the motivation to study or continue to study mathematics varies between the professors and the students. The motivation among educators is often quite idealistic. They could appreciate the beauty of certain parts in a mathematical expression and they may have an idea of how equations are used for solving real life problems. The motivation for the students is less aloft. Most of the time, students just ignore or do not see any practical value in mathematics and regard it as a means of torture.

Laboratory algebra is intended to bridge not only the gap between the instructor and the student, but also the theory-practice gap. A practical implementation of the somewhat abstract concepts, the ability to visualize, predict and verify turns mathematics into a very important part of the cultural background of the child.

Concrete supplementary materials to the learning of mathematics have been recognized as a power by Szendrei (1996). They have been means of instruction from antiquity to modern times. At the same time, proponents of the use of concrete instructional materials warn against using them blindly. For a successful use of concrete materials, a careful planning of their use is advised with many details taken into account. Both research and didactical goals dictated the selection of the particular instructional device for my study. Szendrei (1996) in a review on the use of concrete materials for instruction distinguishes two major types of supplemental materials:

- Practical tools and devices, designed for a different purpose but adapted for instruction
- Special tools specifically designed for educational purposes

Since the goal of my study involved both research and instruction, I opted for the use of special tools and decided to design a device - the double-winch. The following motivations contributed to the selection of the specially designed device:

Flexibility and efficiency

The device was designed with specific research and /or instructional goals. The proposed research question involved investigating on didactical details associated with several aspects of the concept of linear equation. Using an instructional device specifically designed for this purpose appeared to be logical and most efficient. The device was designed so, that all the research aspects could be easily accessed on a single device. Selecting a practical, real-world tool for that purpose seemed less promising and more difficult. At least, I was unable to easily identify a tool that would allow me to access all the research tasks on a single device.

Possibility of device customization

A commercially-available construction set was used to build the device. The modular nature of the construction set allowed for an easy modification of the device design. The modular, customizable nature of the device may be beneficial for adapting the device for future research with new research questions. The device and the construction set could be regarded as an investment in future research.

Laboratory algebra has a potential to represent algebraic concepts in a realistic, touchable way. The school pupils had an opportunity to experience a real-world functioning of the linear function and its attributes. In some cases, the correspondence was quite direct. For example, the free term in the affine function (the offset) was represented as the height of a weight from the floor. The rate of change (in Russian: angular coefficient) is usually interpreted as a “speed” with which the function changes. With the double winch the pupils could actually see that correspondence as a speed with which the weights were raising. They could also see correspondence between quantities of turns as a “time”. The real physical experience helped the pupils build an inner feeling of how algebra works. The ability to “feel” the linear function and its aspects contributed to what is called “discovery learning”. The algebra in my study was built in the experience, embodied in practice.

Moreover, the pupils had an opportunity to practically verify the theoretical solutions they obtained. They could see immediately on the device if their approach was reasonable and if their theoretical predictions made sense.

5.2 “Laboratory algebra” potential for didactical purposes

The laboratory algebra sessions described in this thesis were carried out not only for research, but also for didactical purposes. I hoped that my laboratory algebra sessions would provide not only research material, but would also present opportunities to teach.

The active involvement of the pupils in working with linear algebra, the participation of the pupils in problem solving allowed them to explore their knowledge in an unconventional, non-standard way. The pupils were allowed to exhibit a free approach to problem solving. This resulted in what could be called “creation of knowledge”. In our particular experience, the “creation of knowledge” manifested itself through several ways that the pupils used to represent the linear equation. For example, the following representation of the linear equation was created (or invented) by a junior pupil:

$$y : x = a ,$$

where a was the rate of change (in Russian: angular coefficient).

This equation is not the classical way to represent a linear equation, but still, it correctly represents the concept. In fact, representing the linear equation in this form opens additional didactic opportunities that could be explored. For example, the idea of an equation as a co-variation of two variables (Hines, 2000) could be represented with this notation. One could argue that the variables y and x are varying together with the same rate.

Another didactic possibility is in how pupils understand linearity. Obviously, mental pictures of linearity are different among school pupils. The linear equation of the form:

$$y = a * x ,$$

is most likely, the first equation the pupils will encounter in mathematics.

Later on, an inverse equation of the type:

$$y = 1 / x ,$$

and a power equation of the type:

$$y = x^n ,$$

can be introduced to the pupils.

This could lead to a false impression that the linear equation is the one with the simplest appearance, when there is nothing in the denominator and there is no exponential.

The notation:

$$y : x = a \quad (y/x = a)$$

may open a way for some pupils to look beyond the appearance of the equation and look into its behavior. For example, in the notation:

$$y / x = a ,$$

a very important aspect of linearity is emphasized: when an argument changes by a certain amount, the result changes similarly by the same amount: “ x changes by 2 => y changes by 2”.

In this way, new knowledge and understanding could be created.

Introducing the concept of the linear (affine) equation was not the major didactical goal of the sessions. Indeed, I anticipated that the participating pupils to have some idea of linear equations. The investigational as well as didactical goals were to build upon the understanding that the pupils had. Based on literature accounts describing the power of prediction questions in mathematical instruction (e.g. Kasmer & Kim, 2012) I designed activities in which the pupils had to make predictions. According to the literature, as well as personal experience in mathematics, making predictions is quite challenging. School pupils, attempting a prediction task, should have not only a sufficient mastery of the mathematical subject, but also have sufficient confidence in the skills they have. In the framework of laboratory algebra, the pupil attempting a prediction, should also have a sufficient familiarity with the system. The laboratory algebra system should not appear as something totally “new” but something that the pupils has already understood to a certain degree.

Therefore, I shifted the prediction question towards the end of the session. I also tried to make the predict question relatively challenging, but manageable, not exceedingly challenging.

According to the theoretical basis of prediction activities, described in the theoretical chapter part, I wanted the predict question to have some illustrative power. The predict question was intended to illustrate some particular aspect of the algebraic concept investigated in the session. I decided to construct the predict questions so that they involve working with the rate of change of the linear (affine) equation. It has been shown by Bezuidenhout (1998) that the concept of the rate of change is not as intuitive as one might think. The concept of the rate of change is often misunderstood even by students of the introductory calculus courses (Bezuidenhout, 1998). I constructed the predict questions so, that the framework of laboratory algebra would allow visualization of this concept. So, the predict questions involved observing weights lifted by the device and noticing the speed with which the weights moved. I hoped that the direct observation of the weight moving from the pulleys of different diameters would provide a mental image of the rate of change that would be linked to the tangent (in Russian: angular coefficient) as the property of the rate of change. This was the first layer of visualization.

The second layer of visualization was in requiring the pupils to represent their understanding mathematically – through a system of equations or graphs that the pupils would use while solving the predict question. Thus I hoped that the predict question will create an instructional feedback system: The rate of change or offset would be visualized in the reality and observed on the device. And then, they would be “condensed” through a mathematical representation by a graph, table or formula.

Additionally, the presence of the physical device provided an opportunity for verifying the answer obtained. Verification of the answer to the prediction question was encouraged during sessions.

Most commonly, at least two representations were used in parallel. Very often, a formula or equation was paralleled by an expanded verbal description. It appeared that the pupils needed several “layers” to fully describe the models that they were trying to represent. It could be hypothesized that the confident use of a symbolic representation takes time to develop and master, it’s not an instantaneous process. A picture of symbolic scaffolding can be drawn – a learner of a certain symbolic representation mode uses a scaffolding of another, more familiar symbolic representation, until a confidence in the new symbolic representation is achieved. At this point the choice of a symbolic representation becomes a question of choice. To illustrate this, one could say that when a school pupil is in the process of mastering the use of equations to symbolize the data, the pupil parallels the equations with extended verbal description. Once the mastery is achieved, the verbal and formulary description can be used interchangeably.

It my study, which was relatively short in scope and duration, it was not possible to embrace and investigate all the components of mathematical proficiency described by other researchers (see, for example, Kilpatrick 2002). However, I hoped that the proposed sessions in laboratory algebra will allow the pupils to improve on all aspects of mathematical proficiency, described above.

It is obvious that the nature of the problems offered affects predominantly the conceptual understanding related to the concepts of linear equation. I hoped that conceptual understanding will be improved through relating the linear equation to something the pupils can observe on a physical device. This can add an additional dimension to their mental picture of linear equation.

I also hoped that solving problems about a physical device will allow the participating pupils to improve on their procedural fluency, and not by simply practicing mathematical transformations once more. I hoped that the practical realization of algebra in a physical device will allow the pupils to look beyond the algebraic transformations involved in the problem solving. However, I still maintain and hope that participating in the laboratory algebra experiments will allow the pupils to engage and involve all the remaining aspects of mathematical proficiency, such as using adaptive reasoning and acquiring a productive disposition. Interestingly, the use of a physical device promoted procedural fluency in an unexpected direction – mental calculations.

I should note that most of the times, the calculations were performed without a calculator. It was interesting for me to observe that when I advised them to use calculator, the pupils refused it. It seems that their teacher was not welcoming the use of a calculator during lessons and they got used to not using it. This observation does not seem to be important because it seems to be unrelated to the development process of conceptual understanding of the linear equation. However, I have often observed that the ability to perform numerical estimates quickly in mind is quite useful and often gives the person an advantage. In other words,

people who are quick with numbers often better understand mathematics and natural sciences such as physics, because they can think quantitatively rather than qualitatively.

The social environment of the study also contributed to the selection of the tool. The research was conducted among children that previously had shown some inclination to creative learning. Some of the participants were technically “inclined”, some artistically, but all of them were quite aware of practical tools and artifacts. It was difficult or even near-impossible to surprise them with some practical tool. Attempting to use the practical tool for teaching mathematics, was even less promising in that particular environment. My concern was that bringing a practical tool into the study and then, using the tool improperly, beyond the original design intended of the tool might create unnecessary confusion.

Another important social consideration of the study is the organization of the study groups. Most of the pupils that participated in the study were organized in groups. The intent of such organization was to encourage a supportive teamwork and to observe the manifestation of the teamwork. Most of the research sessions were set up in groups of two, except one, very first session that consisted of only one pupil. The motivation of the group work was the Vygotskian idea of the Zone of Proximal Development – the ZPD, where interaction between the group members would lead to a cumulative effect that increases the efficiency of learning.

To me it was interesting to observe if the group interaction really enhances the learning efficiency or the group study simply leads to a situation when one of the participants pulls most of the load and the other is simply “enjoying the ride”. My research sessions were not designed to study specifically this question systematically and in depth. Therefore, all my remarks on this issue will be mainly observational, speculative. A series of special study session might be required to investigate the cognitive aspects of working in pairs in the framework of laboratory algebra. However, a few preliminary remarks could be made.

The first observation is that working in pairs created a positive psychological background. The pupils simply felt themselves more confident. In addition, a complex mixture of mutual support and mutual competition introduced a certain dynamics into the sessions. The group session proceeded at a faster pace, compared to the session with one pupil only.

The second observation - working in pairs creates an opportunity for discussing and exchanging ideas. The possibility to verbally express an idea led to a better formulation and better understanding of the idea and in turn that resulted in a better performance of the group.

However, a drawback of working in groups has also been noticed. Occasionally, a better motivated and probably more skilled pupil took all the responsibility for the task and directed the problem-solving activity. The other group member was just passively following the lead.

Thus, to achieve the positive effect of working in pairs, the group work has to be properly directed and the pair has to be properly formed. A few additional questions arise after observing the pupils studying in pairs. Is there a best size of the study group? Intuitively, it seems that increasing the size of the study group is only efficient up to a certain point after which the usefulness of the group drops. It also appears that a proper organization and setup of a study group might be required to maximize efficiency. These are all interesting subjects potentially worthy of the whole separate venue of investigation.

But even from the observations performed in this study, some important trends can be revealed: the teacher input might enhance the effect of the group study. The teacher input does not have to be in only lecturing, instructing or “pushing the pupils forward”. If the pupils are motivated and interested in the project, “pushing” or “pressuring” the pupils is not

needed. The best role of the teacher in the laboratory environment is helping the pupils when they are confused and can not proceed. The best form of help is indirect help. I found that the best form of help I could provide was indirect help, the best influence I could have was in stimulating the pupils to think and think creatively. Pupils continue thinking under indirect input from the teacher. For example, answering a question with a question can offer the pupil with another perspective from which to look at the problem. In fact any problem solving activity usually involves the ability to ask proper questions at the right point of investigation.

5.3 Advantages and disadvantages

The system allowed making the variables and constants of the linear equation to become “touchable” and even make these aspects of equations to be “reversible”. For example, the most common variable in physics is “time”, which is not reversible. What is gone is gone and we can not (at least yet!) travel back in time.

In our case of the wheel algebra, the number of turns can be reversed; thus the function can be studied at separate points, and pupils can go back and forth between different points. Also, the motion of the wheels can be arranged in such a way, that they are moving in opposite directions (with one weight going up while the other goes down) – a possibility for inverse functions is there as well.

The device encourages creativity. In the session with the senior pupils, while solving the predict question on the affine equation, one of the participants unexpectedly started solving the predict question in the enacted mode on the device. At the same time, the other participant continued solving the predict question by repeated addition. The results in both cases were inexact, but obvious is the creative approach to problem solving shown by both the participants.

The device design, unfortunately, allowed for too large experimental errors. I make some suggestions in Recommendation chapter.

5.4 New research questions that are emerging from my research

1. What opportunities for learning the concept of inverse function and what aspects of that concept are exposed in a double-wheel laboratory?
2. To what extent does the nature of the device as a non-idealized system help? Does the physical nature of the device in the “laboratory algebra” environment promotes or hampers the application of the existing knowledge?
3. Can a formula be used as a model for “wheel algebra” lesson?

5.5 Recommendations

Some recommendations are already mentioned above. I just suggest the following additional recommendations.

Session Time:

It should be one hour maximum. In sessions lasting more than an hour, pupils will get tired and the session becomes inefficient.

Participants:

I would organize the participants in groups of two or three. The session format would designate a role for each group member, for example: “measurement taker”, “measurement quality controller”, “generator of ideas”, “analyst”, “result vs. reality verifier” and so on. I would be interested in exploring how the group works, in exploring the inner mechanics of the functioning of the group. Correspondingly, the theoretical background devoted to problem solving in groups would need to be reviewed.

Sequences of the tasks in blocks:

Task1:

I would leave the first task unchanged: taking measurements and predicting the device behavior to evoke curiosity and motivate the participants.

Task 2:

I would modify the second task by adding a question requiring the participant to “map out” the trend as it emerges. For example, I would ask: “While rotating the pulley, do you see the trend in the motion of the weight? Can you describe the trend verbally or graphically?”

In the task on the affine equation, I would emphasize the non-zero starting point for one of the weights. The task would be formulated as follows: “How does the height of the weight depend on the number of turns of the pulley? Establish a mathematical relationship between the two, given the fact that the starting point of one of the weights is non-zero”.

The affine equation question is absent in block 3. This is simply because a similar question had been presented in the previous blocks. However, the question about a mathematical relationship describing the weight position, was supposed to motivate the pupils to take and tabulate the data.

While the seniors had no problems taking and tabulating data, the juniors (Anna) did not initiate taking data without specific instructions to do so.

Task 3

Predict question should be present always, because it will allow investigating the level of proficiency of the pupils and allow the pupils to show their knowledge in a creative way.

Algebraic concepts exposed in the tasks

The range of tasks can be expanded significantly to include not only the linear and affine equation, but also inverse functions. The weights could be arranged on the pulleys to move in the opposite directions – when one weight goes up, the other goes down. This situation could potentially model inverse functions. The data will be taken first from the step-wise rotation of pulley 1 (red) and then, from the step-wise rotation of pulley 2 (orange). The data will be required to be tabulated.

An interesting predict question task would be in asking the participants to plot and approximate prediction sketch of the weight position. For example, one could supply a graph, describing the motion of the weight and ask to approximate and plot a similar graph at a different set of initial conditions, for example at different starting heights of the two weights. Another possibility would be in asking the pupils to predict and sketch the motion of the weight with a different pulley diameter, different direction of the pulley rotation and so on.

This task on “predict a graph” will, in my opinion, allow the pupils to become more familiar with graphs and remove the perception of graphs as time-consuming and illustrate the power of graphical methods for problem solving.

Device:

In addition to the conveniently-read scale, arrange a dial or a something similar, that the pupils could run against or under the weight. This would allow measuring and reading the device scale reproducibly in a consistent manner.

Make two corresponding marks on the pulley and on the device base that would allow determining precisely the start and the end of a pulley turn, thus reducing the error of measurements.

Use the same color for the weight and the pulley to avoid confusion in referring to pulley/weight.

The fishing line holding the weight was occasionally getting into the gap between the pulley and the pulley shaft, thus contributing to errors of measurements. Make the pulley and the shaft one body, for example, by gluing the two together.

One could make the system more complex, for example create a system of three and more engaging pulleys that are arranged in a line, and investigate the motion of the weight attached to the terminal pulleys.

Create a system with engaging pulleys that have shafts of noticeably different diameters and investigate how the gradient will change as compared to the system with similar diameters of the shafts of the pulleys.

6 Conclusions

The “big picture” of this study was investigating the cognitive processes that occur as school children translate the somewhat abstract knowledge of algebra to model mechanical artifacts and back. I refer to this approach as to “laboratory algebra”.

In this particular thesis, the study was reduced to a investigating the “laboratory algebra” of a very specific algebraic concept – that of a linear function. We focused our attention on what particular aspects of the concept of linear function can be exposed in studies with mechanical artifacts and what opportunities for learning the concepts can be exposed.

The framework of the double-winch system designed for this study proved to be very prolific in generating insights into school-pupil cognition as they applied algebra for mathematical problem solving in laboratory situations.

Below, I briefly outline the conclusions that I have drawn:

6.1 Mathematical and algebraic concepts, taught during the sessions

The essence of the concept of the function is relating two sets of variables in a predictable way.

The students obtained exposure to the following aspects, relevant to the concept of linear/affine function:

- The constituents of the linear function that in our case are the raising height of the weight, the number of turns and the height per turn.
- Various ways to represent a function such as: verbal, symbolic, tabular, graphic.
- The concept of proportionality.
- Proportionality as an implicit linear function.
- Slope (gradient) and its relation to rate of change.
- The free term and its influence on initial conditions.

In addition to the concepts pertinent to linear/affine function, the school pupils got exposure to a more general but important mathematical problem-solving skills:

- Transiting between various ways to represent a function.
- Making connections between representations.
- Different way of writing equations, selecting a most convenient way to represent a function for a given task.
- The concepts of known, unknown and variable.
- The accuracy and estimation.
- The average value (arithmetic mean).
- “Intuitive” solutions.
- Interpreting and verifying the obtained results, analyzing the validity of the “answer”.

However, the idea of function as a co-variation of two variables or as a one-to-one mapping between two sets data has not been conceptualized. At least, there were no attempts to explicitly express this understanding. It appeared to me that the pupils limited their approach for solving the presented problems to applying, with different degrees of skills, the proportional relationship between two quantities. The “proportional thinking” seems to be representing “static thinking” that describes the two fixed quantities that are not necessarily changing.

6.2 Learning modes and opportunities

Most of the problems presented to the pupils were successfully resolved without apparent difficulties. Pupils were creative within their limitations.

The pupils were very enthusiastic about participating in the research sessions, showed curiosity in the design and operation of the mechanical construct.

The pupils also appeared confident in approaching the suggested problems and adept in applying mathematics to solve the presented problems.

Due to the enactive mode of learning an abstract concept of the variable “became touchable” through the observation of the variable parts of the system.

The observed school pupils between the 6th and 8th grades have good mathematical techniques that they apply skillfully. The pupils are taught to solve problems. All of them tried to avoid using a calculator.

What was found not fully established yet was a more conceptual realization of the mathematical concepts used, for example slope as a rate of change. Perhaps, this conceptual development is reserved for senior school years or even for starting university years. This observation further emphasizes the role of the school teacher, the instructor on presenting the concepts in addition to teaching the mathematical skills

6.3 Didactical implications

Perhaps, the most immediate one is that “laboratory algebra” is a natural way to encourage collaborative studies, to create the Vygotskian Zone of Proximal Development that fosters effective learning. In addition, the learning-by-doing setting of the laboratory algebra environment can potentially help learners to translate between the abstract nature of mathematical concepts and the correct application of these concepts for modeling real-world artifacts.

However, what seems to remain a problem is the proper conceptualization of the mathematical concepts. By “proper conceptualization” I mean the understanding of the mathematical implications and the ability to apply them for problem solving effectively. Although the pupils had taken experimental data, observed two parameters varying in a wide range of values and even could solve the problem given, no attempts were initiated by the pupils to conceptualize their observations and to take their understanding on a higher level, to explain and understand why their selected method will or will not work. The pupils were

capable of making this transition, their cognitive abilities were developed enough for that task, but a help from the instructor was needed.

Similar difficulties were noted in how the pupils conveyed their results to the others. When the pupils were asked to “generalize” their observations or to express the observations in a “condensed” form, an idea of a graph or a “general formula” was used by them. With help and hints the pupils were able to complete a graph or write an equation but they would have to be asked to do so. After completing a graph or a formula, the pupils were able to look at their understanding of the function more conceptually and even were able to recognize that graphs and formulas can be powerful tools. However, none of the students observed was able or willing to independently perform this transition to conceptualization – a help from the instructor was needed. At this point it is too premature to generalize this observation – a greater pool of school pupils is needed for future observation.

At this point I have to recognize that “laboratory algebra” is not a panacea for everything. A successful development of a mathematical proficiency depends not exclusively on a didactic method, but also on a proper interaction with colleagues, instructors that would allow developing a greater mathematical culture. In other words – too much doing without discussing is an educational opportunity missed!

Another important observation is the proper design of the artifacts. During the preparation, I have developed and tested several prototypes for the device, each with advantages and disadvantages. I selected only one of them for the sessions. The most obvious of the problems of working with mechanical systems is the proper organization of the measurements and the proper interpretation of the results. Essentially, working with a mechanical artifact involves taking a set of measurements. Each measurement is affected by random errors that can not be avoided in any real situation. The random errors unavoidable in real measurements should be properly treated for meaningful conclusions to be made. For example, a set of data, that would ideally *map a line*, in real life, would more likely be *scattered around* the expected line. Although it has not been an issue in my sessions, this issue can obscure the presentation of the subject and draw the attention of the pupil away from the subject. A careful design of experimental apparatus as well as a proper design of instruction tasks is very important.

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Appendixes

Appendix 1: Task Block 1

Задание No 1

Одно колесико и один грузик

1. Установите грузик в положение «0» и поднимайте, вращая колесико. На какой высоте грузик будет находиться через 5 оборотов?
2. Как связаны высота грузика и количество оборотов колесика? Расскажите словами или запишите математическое выражение с помощью которого, зная количество оборотов можно вычислить высоту.
3. Если мы поднимем колесико очень высоко, например к потолку, можешь ты предсказать сколько оборотов колесика потребуется, чтобы поднять грузик? Высота потолка 2,5 метра или 2500 см, грузик начнет движение от пола, т.е от 0 см.

Appendix 2: Task Block 2

Задание No 2

Два колесика и два грузика.

1. Какие у Вас ожидания о том, как данная система будет работать?
Какой из грузиков поднимается быстрее?

2. Установите красный и серый грузики в положение «0 см».
Вращая красное колесико фиксируйте высоту красного и серого грузиков.

Как высота грузиков зависит от количества оборотов колесика?
Запишите математические выражения зависимости.

3. Представьте, что мы расположим систему очень высоко,
например на высоту потолка. Грузики начинают двигаться от пола.
Какова будет разница высот после 25 оборотов?

Appendix 3: Task Block 3

Задание No 3

Два колесика и два грузика

1. Установите красный грузик в положение «10 см», а серый в «0 см». Вращая красное колесико фиксируйте высоту красного и серого грузиков.
2. На какой высоте грузики встретились?
3. Представьте, что мы расположим систему очень высоко, например на высоту потолка. Красный грузик начнет двигаться от уровня стола (70 см), а серый от уровня пола. На какой высоте они встретятся?

Задание No 4

Два колесика и два грузика

1. Установите грузики в положение «0». Вращайте красное колесико по часовой стрелке фиксируйте высоту красного и серого грузиков.
2. Как высота грузиков зависит от количества оборотов колесика? Запишите математические выражения зависимости.
3. Попробуйте предсказать положение серого грузика, если Вы знаете положение красного.

12 лет.

① 1 полный оборот - 4 см

2500 см - высота до потолка

②

$$\begin{array}{r} 2500 \text{ см} \quad | \quad 4 \\ - 24 \quad \quad | \quad 625 \\ \hline 10 \quad \quad \quad \\ - 8 \quad \quad \quad \\ \hline 20 \quad \quad \quad \\ - 20 \quad \quad \quad \\ \hline 0 \end{array}$$

Ответ: Нужно прокрутить колёсико 625 раз

①. Быстрее серебряный грузик - тот, который прикреплен к меньшему колёсику.

②
Жр. грузик - 10 см
Сер. грузик - 0 см

Ответ: Грузики встретятся на высоте ^{34,5}~~32~~ см (+/- 1 мм)

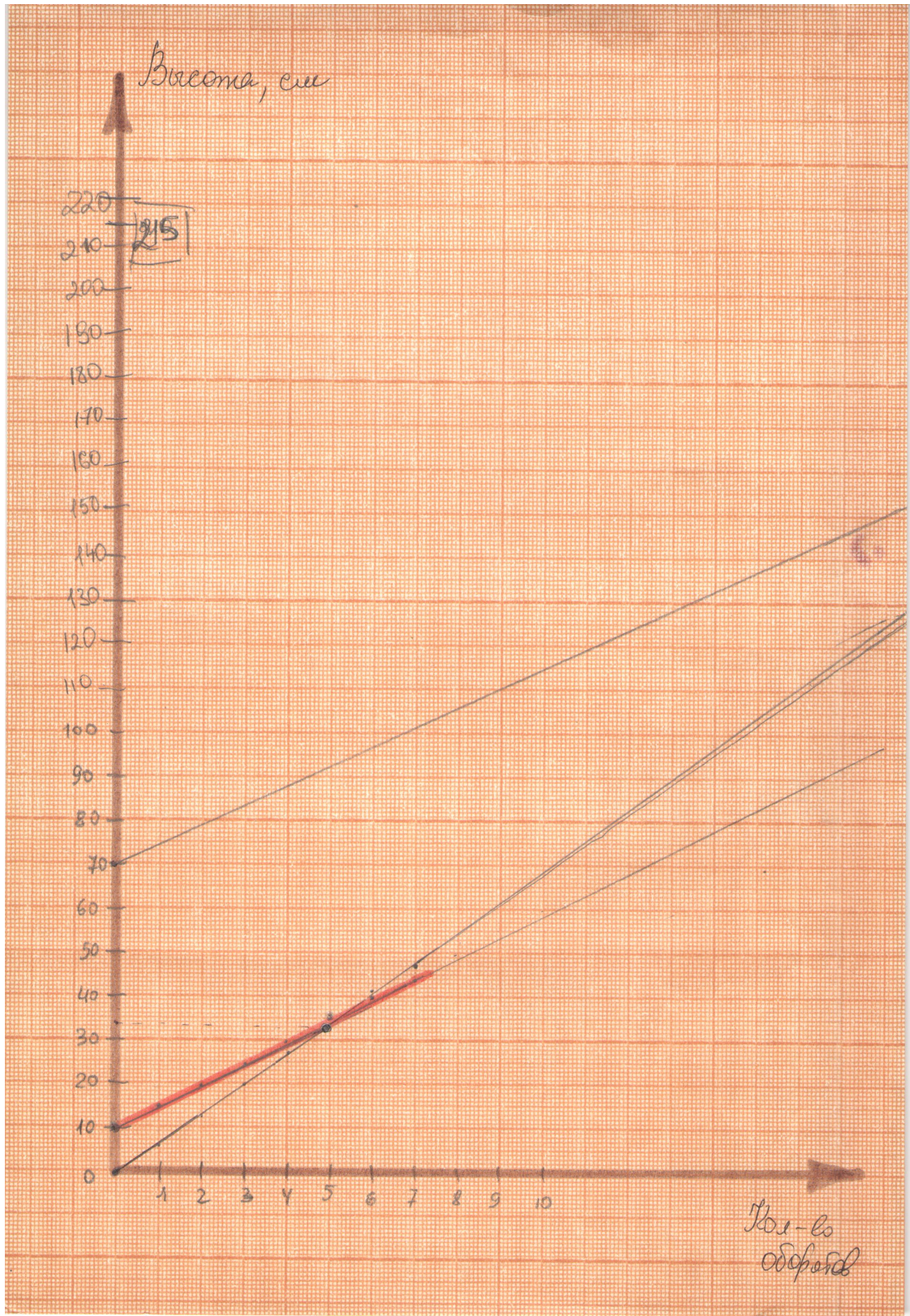
③
Жр. грузик - 70 см
Сер. грузик - 0 см

Ответ: Они пересекутся на высоте 215 см

Appendix 6: Solutions of the Session No 1, Table

Количество оборотов	Высота красного грузика см	Высота серого грузика см
0	10	0
1	14,8	6,5
2	19,5	12,8
3	24 2 ,3	19,9
4	29,3	27
5	34,3	33,8
6	39	41
7	43,7	47,5

Appendix 7: Solutions of the Session No 1, Graph



Appendix 8: Solutions of the Session No 2 (Andrew), page 1

N1

12 сем

1) 22 см

2) $22 : 5 =$

100 оборот - $\approx 3 \text{ см}, 5 \text{ см}$

500 оборот - ? см

$3,5 \text{ см} \cdot 5 = 17,5 \text{ см}$ - Высота прыжка

3) Высота = $2,5 \text{ м} (2500 \text{ см}) = 25000 \text{ мм}$

кол-во оборот - ?

100 оборот - $3,5 \text{ см} \quad 35 \text{ мм}$

1) $2500 : 3,5 = 714,2(8)$

Ответ: $714,2(8)$

$$\begin{array}{r} 25000,35 \\ 245 \quad 714,2(8) \\ \hline 50 \\ -35 \\ \hline 150 \\ -140 \\ \hline 100 \\ -10 \\ \hline 300 \\ -270 \\ \hline 300 \end{array}$$

N2

Высота - y
кол-во об. - x

Appendix 9: Solutions of the Session No 2, Andrew, page 2

4,2

$$y : x = 4,2$$

$$y = 4,2 \cdot x$$

g) 6

$$y : x = 6$$

$$y = 6 \cdot x$$

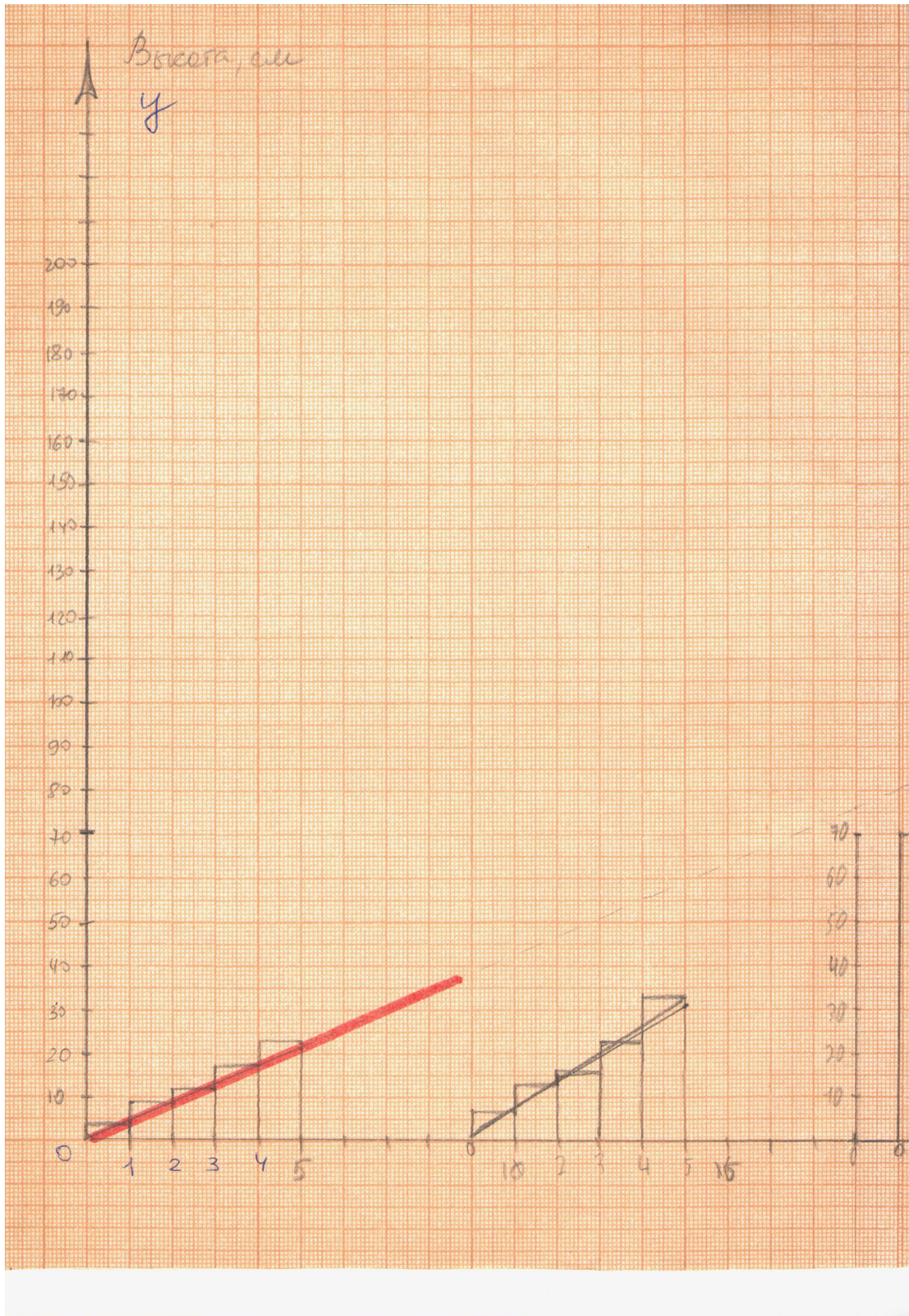
$$25 \cdot 4,2 = 9700 \text{ cm}$$

$$\begin{array}{r} 250 \\ 47 \\ \hline 500 \\ 92 \\ \hline 9700 \end{array}$$

$$\begin{array}{r} 3 \\ 25 \\ \hline 6 \\ \hline 210 \end{array}$$

$$\begin{array}{r} 210 \\ - 97 \\ \hline 113 \end{array}$$

Appendix 11: Solutions of the Session No 2, Graph



Appendix 12: Solutions of the Session No 2 (Oleg), page 1

√1

12 сем

1) 22

√2

$$1) 22 : 5 = \frac{22}{5}$$

$$2) \frac{22}{5} \cdot 5 = \frac{110}{5} = 55 \text{ (мм)} = 5 \text{ сантиметров}$$

$$3) 55 : 5 = 11 \text{ (мм)} = 1 \text{ сантиметр}$$

√3

$$1) 2500 : 35 = 71 \text{ и } 15 = 2500 \overset{35}{\underset{15}{\div}} = 71 \text{ (мм)} = 71 \text{ миллиметров и } 15$$

√4

$$1) 25 \cdot 4,2 = 250 \cdot 42$$

$$\begin{array}{r} 250 \\ \cdot 42 \\ \hline 500 \\ 1000 \\ \hline 10500 \text{ (мм)} \end{array}$$

$$\begin{array}{r} 2500 \overset{35}{\underset{15}{\div}} \\ \underline{11} \quad 2 \quad 35 \\ 2500 \overset{35}{\underset{15}{\div}} \\ \underline{245} \quad 7 \quad 14 \quad 0 \\ 50 \\ \underline{35} \quad 3 \\ 150 \\ \underline{140} \\ 100 \\ \underline{70} \\ 300 \end{array}$$

Appendix 13: Solutions of the Session No 3 (Andrew), page 1

4,9

1) 10cm 0cm 5,4

1) ~~Dimensi. 10a. Bercome 51cm~~

3) ~~$5,1 + 10 + 15,3 + 21,2 + 27,7 + 34,5 + 41 + 47,5 + 54$~~

~~$10,1 + 30,4 + 51,6 + 79,3 + 113,8 + 154,8 + 208,8 + 47,5 = 250,3$~~

$\begin{array}{r} +208,8 \\ +47,5 \\ \hline 256,3 \end{array}$	$\begin{array}{r} 256,3 \\ 18 \\ \hline 76 \\ 12 \\ \hline 43 \end{array}$	5,1
--	--	-----

4,9	$\begin{array}{r} 21,2 \\ 10,1 \\ \hline 10,8 \end{array}$	$\begin{array}{r} 24,7 \\ 10,2 \\ \hline 10,9 \end{array}$	$\begin{array}{r} 34,5 \\ 16,9 \\ \hline 17,6 \end{array}$	$\begin{array}{r} 74,9 \\ 4,9 \\ \hline 10,0 \\ 4,9 \\ \hline 94,7 \\ 4,9 \\ \hline 99,6 \\ 4,9 \\ \hline 10,5 \\ 4,9 \\ \hline 59,4 \end{array}$
10,4	$\begin{array}{r} 41,0 \\ 17,6 \\ \hline 23,4 \end{array}$	$\begin{array}{r} 54,0 \\ 24,1 \\ \hline 29,9 \end{array}$	$\begin{array}{r} 29,9 \\ 26,1 \\ \hline 0 \end{array}$	$\begin{array}{r} 20,5 \\ 4,9 \\ \hline 59,4 \end{array}$
10,8	$\begin{array}{r} 47,5 \\ 23,4 \\ \hline 24,1 \end{array}$			
16,9				
17,6				
23,4				

24,1	$\begin{array}{r} 29,9 \\ 24,1 \\ \hline 20 \end{array}$	$\begin{array}{r} 18 \\ 20 \end{array}$
29,9	$\begin{array}{r} 24,1 \\ 20 \\ \hline 24,1 \\ 20 \end{array}$	

Appendix 14: Solutions of the Session No 3 (Andrew), page 2

$$4,8$$

$$5,1 - 4,8 = 0,3$$

$$100 \text{ mm} \cdot 0,3 \text{ mm} =$$

$$\begin{array}{r} 100 \text{ B} \\ 6 \overline{) 233,8} \\ \underline{180} \\ 53 \\ \underline{48} \\ 5 \end{array}$$

$$22$$

$$22 \quad 31$$

Appendix 15: Solutions of the Session No 3 (Andrew), page 3

$$y - x = 0,3$$

$$49 - x = 70x$$

$$5,4 - x = 70$$

$$x = \frac{5,4 \cdot 100}{49}$$

$$x = \frac{71800}{49}$$

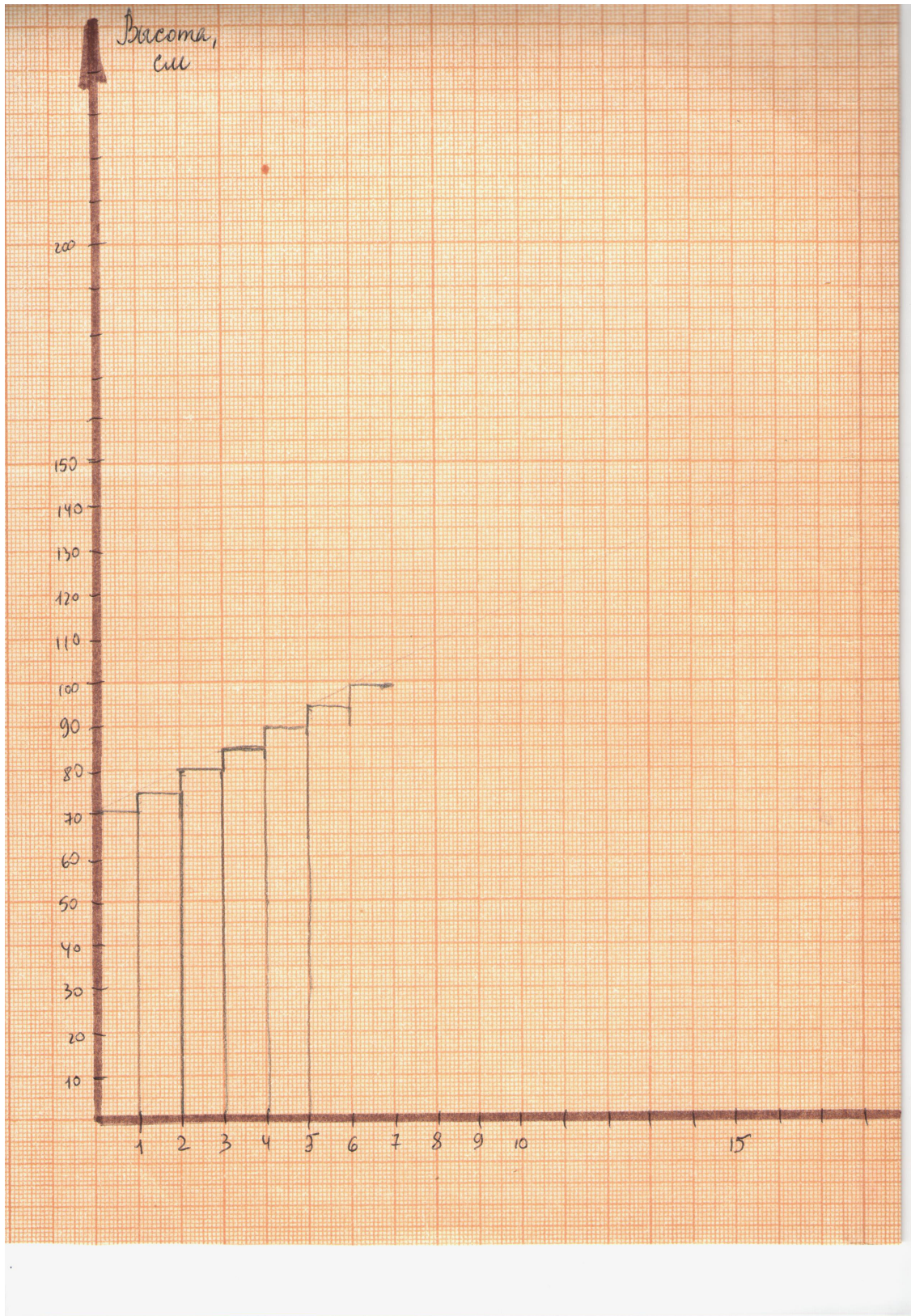
$$x = \frac{110 \cdot 100}{54}$$

$$x = \frac{34300}{54}$$

$$\begin{array}{r} 54 \\ \cdot 100 \\ \hline 37800 \end{array} \bigg| 49$$
$$\begin{array}{r} 211,4 \end{array}$$

$$\begin{array}{r} 49 \\ \cdot 100 \\ \hline 34300 \end{array}$$

Appendix 17: Solutions of the Session No 3 (Andrew), Graph



Appendix 18: Solutions of the Session No 4 (Roman), page 1

14. Арм.

5 - 27

6 - 32

1 = 4

2 =

$\sqrt{1}$

~~4,5~~

88 : 15 = 4,5...

$x \cdot 4,5 =$

2

108000 = 4,5 cm.

$x \cdot 4,5 = h$

3

$25000/45$
 $\sqrt{555.5(5)}$

1) K, 4
 C, 5

2) K, 9
 C, 10

3) K, 14
 C, 16

4) K, ~~18~~ 19
 C, 21, 8

K, 23
 C, 28

$\sqrt{2}$

~~103,5~~

K = 103,5

C = 124,2

~~(K.)~~ 103,5 - (C.) 124,2 = 21,3

81 : 15 = 5,4 cm

~~29~~

Appendix 19: Solutions of the Session No 4 (Roman), page 2

$$\begin{aligned} & \sqrt{4} \\ 4,5(\text{cm}) - 5,4(\text{cm}) &= 0,9 \\ x \cdot 4,5 + x \cdot 0,9 &= h(\text{cm}) \\ \hline 9 + 1,8 &= \underline{10,8}(\text{cm}) \end{aligned}$$

№3

встретимся на 35-и см.

1

К. 4,5
С. 5,4

Сергей за 13 оборотов достигнет уровня стола.

Красной - 58,5 см (над уровнем стола)

С. - 11 оборотов (58,5 над ур. стола)

К. - поднимется на 49,5 см

С. - 9 (49,5)

К. - (40,5)

7,7 ~~15,4~~

$$35(\text{см}) \cdot 70 = 2450$$

$$35 \cdot 70 =$$

$$35 \cdot 10 = 70 \cdot x$$

$$\frac{10}{3} = 4,5$$

$x = \text{кол-во оборотов}$

$y = h$

$$y = 10 \text{ см}$$

$$70 + 4,5x$$

$$y = 5,4x$$

$$70 + 4,5x = 5,4x$$

$$0,9x = 70$$

$$x = 77,7$$

151.

N1.

1) 2) 5 одр 3 2 # cu.
3
5) 6-32

2) Подбором - 45 cu.

3) 555(5)

N2.

1) септ 2 десемпел.

Подр.

3) K. 4

C. 5

2 одором K 9

C 10

3) одр K 4

C 16

4) K 18,5 9

C 2 21,8

5) K 13.

C 28

5,4.

K 103,5

C 124,2.

P 21,3

Appendix 22: Solutions of the Session No 4 (Dmitry), page 2

Comp 35.

⊙ K P 500p.
C 11.

$\frac{2}{-3}$

$\sqrt{3}$

9
47
22
25, 7, 9
369
195

~~302488~~
~~2248~~
9
7
1842

~~3~~
48

~~1879213579~~ 188

~~336, 9, 17, 15, 18, 27~~ 300

~~30 34~~

~~284 27~~

~~34~~

Appendix 23: Solutions of the Session No 4, Table No 1

Количество оборотов, шт.	Высота, см
1	4
2	9 8,5
3	13,5 14
4	18,5
5	21,5 23
1	5
2	10
3	16
4	

Appendix 24: Solutions of the Session No 4, Table No 2

Количество оборотов	Высота красного грузика	Высота серого грузика
1	4	5
2	8,5	10
3	14	16
4	18,5	22
5	23	28

ВПЕЧАТЛЕНИЯ

1. Тебе понравились задания?

Конечно! Очень интересно и занимательно!

2. Что было для тебя наиболее трудным?

3-е задание с двумя фузиками
Это достаточно трудное, но никто не
скучиле задание

3. Что было для тебя наиболее легким?

Задания с 1-м колёсиком

4. Ты могла бы порекомендовать эти упражнения своим друзьям
или подругам?

Я бы порекомендовала бы, если бы их
встретила!

5. Что бы ты хотела изменить в заданиях?

Все задания очень хорошие, но только не
удобно измерять на какой высоте находятся
фузики

Appendix 26: Questionnaire (Andrew)

ВПЕЧАТЛЕНИЯ

1. Вам понравились задания?

Да

2. Что было для Вас наиболее трудным?

3

3. Что было для Вас наиболее легким?

Выяснить разницу между (NЧ)

4. Вы могли бы порекомендовать эти упражнения своим друзьям?

Да

5. Что бы Вы хотели изменить в заданиях?

Ничего

ВПЕЧАТЛЕНИЯ

1. Вам понравились задания?

Да

2. Что было для Вас наиболее трудным?

~~№3, №4~~ №2 - 3

3. Что было для Вас наиболее легким?

№1 - 1

4. Вы могли бы порекомендовать эти упражнения своим друзьям?

нет

5. Что бы Вы хотели изменить в заданиях?

Ничего

ВПЕЧАТЛЕНИЯ

1. Тебе понравились задания?

Да

2. Что было для тебя наиболее трудным?

№3(3)

3. Что было для тебя наиболее легким?

№1

4. Ты мог бы порекомендовать эти упражнения своим друзьям
или подругам?

Да!

5. Что бы ты хотел изменить в заданиях?

Нет.

Appendix 29: Questionnaire (Dmitry)

ВПЕЧАТЛЕНИЯ

1. Тебе понравились задания?

Да

2. Что было для тебя наиболее трудным?

3,3

3. Что было для тебя наиболее легким?

1,1

4. Ты мог бы порекомендовать эти упражнения своим друзьям
или подругам?

Да

5. Что бы ты хотел изменить в заданиях?