

Developing Algebraic Thinking in a Community of Inquiry

Collaboration between Three Teachers and a Didactician

Claire Vaugelade Berg

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Inquiry

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*A mes parents
Jeannine Martin et
Bernard Vaugelade*

Preface

This doctoral study was financed by the University of Agder, with the allowance from the Ministry of Education and Research. I was given a four years doctoral scholarship which included teaching duty. I am grateful that this study allowed me to deepen my two main interests, concerning algebraic thinking and working collaboratively with teachers.

Before undertaking this study, I had some experience as a mathematics teacher both at lower and upper secondary school. I see this experience as especially useful while researching the processes behind collaboration between teachers and didacticians.

First of all, I want to express my deep gratitude to the three teachers who agreed to give from their own time (during the evenings) in order to make this collaboration possible. Without their willingness and enthusiasm this research would not have happened.

I also want to thank my colleagues at The University of Agder, and other colleagues both in Norway and France, for their interest, support, and encouragement, especially in the final phase of my research. The Nordic Graduate School of Mathematics Education provided support which enabled me to attend several courses and summer schools, for this I am most thankful.

I am also very grateful to Anne Watson for her comments and insights that were of great value in the finalisation of this thesis. To my supervisors, Barbara Jaworski and Hans Erik Borgersen, I want to express the deepest gratitude and thankfulness for sharing with me their knowledge and competence, and for all the encouraging and challenging comments. If I had to characterise these years as a doctoral student, I would say that I was entering gradually into “l’Ecole de la rigueur scientifique”.

Finally, I want to thank especially my family who supported and encouraged me during this doctoral work. My deepest thanks to my husband, Gudmund, and to our children Catherine, Christian, Mathias, Elisabeth and Nicolas. Without you, none of this work would have been possible.

Claire Vaugelade Berg
Kristiansand, Norway
February, 2009

Abstract

In this thesis I report from a study of the development of algebraic thinking of three teachers, from lower secondary school, and a didactician from a university in Norway (myself). The thesis offers an account of the relationship between the participants' development of algebraic thinking and the processes related to the creation and development of a community of inquiry. In addition, the thesis presents elements of the relationship between the teachers' development of algebraic thinking and their thinking in relation to their teaching practice.

My theoretical framework was elaborated according to the criteria of relevance and coherence. In order to conceptualise the participants' development of algebraic thinking within the community of inquiry, I started from Wenger's theory of community of practice and expanded it in order to include both the dimension of inquiry and Karpov's ideas of cognitive and metacognitive mediation.

Methodologically, I understand my study as a case study, within a developmental research paradigm, addressing the development of algebraic thinking within a community of inquiry consisting of three teachers and a didactician. The collaboration between the teachers and the didactician was organised through regular mathematical workshops, and interviews with each teacher both before and after classroom observations. During the workshops, the participants engaged with some mathematical tasks which were offered by the didactician.

The results of this study indicate that the participants' development of algebraic thinking is deeply interwoven with the processes related to the creation and development of the community of inquiry. It seems that the participants' confidence in the community was developing gradually while the confidence in the subject-matter was related to the nature of the mathematical tasks with which the participants engaged. In addition, the study shows how the teachers engaged in a process of both looking critically into their own teaching practice as a consequence of their collaborative engagement within the community of inquiry, and of envisaging possible implications for their future teaching practice.

Furthermore, I offer insights into my own development both as a didactician and as a researcher and how these relate to research outcomes.

Overall, the thesis contributes to a better understanding of issues related to collaboration between in-service teachers and a didactician from a university, while focusing on the development of algebraic thinking. Implications are also suggested concerning the way algebra could be addressed in schools.

Sammendrag

I denne avhandlingen rapporterer jeg fra en studie av utviklingen av algebraisk tenkning hos tre ungdomsskolelærere og en didaktiker fra et universitet i Norge (meg selv). Avhandlingen beskriver sammenhengen mellom deltagerens utvikling av algebraisk tenkning og prosessene relatert til dannelsen og utviklingen av et utforskende fellesskap (community of inquiry). I tillegg presenterer avhandlingen elementer fra sammenhengen mellom lærernes utvikling av algebraisk tenkning og deres tenkning relatert til deres undervisningspraksis.

Min teoretiske ramme ble utarbeidet i henhold til kriteriene for relevans og koherens. For å konseptualisere deltagerens utvikling av algebraisk tenkning innenfor det utforskende fellesskapet tok jeg utgangspunkt i Wengers teori om praksisfellesskap (community of practice) og videreutviklet den til å inkludere både den utforskende dimensjonen og Karpovs ideer om kognitiv og metakognitiv formidling.

Fra et metodologisk synspunkt, ser jeg min studie som en case-studie innenfor utviklingsforskningsparadigmet (developmental research paradigm). Den omhandler utviklingen av algebraisk tenkning innenfor et utforskende fellesskap bestående av tre lærere og en didaktiker. Samarbeidet mellom lærerne og didaktikeren ble organisert gjennom jevnlig matematiske verksteder og intervjuer med hver lærer både før og etter klasseromsobservasjoner. På verkstedene utforsket deltagerne oppgaver som didaktikeren hadde valgt.

Resultatene av denne studien indikerer at deltagerens utvikling av algebraisk tenkning er dypt sammenvevet med prosessene relatert til dannelsen og utviklingen av et utforskende fellesskap. Det synes som at deltagerens tillit til fellesskapet utviklet seg gradvis, mens tilliten til faget var knyttet til de matematiske oppgavens natur. I tillegg viser studien hvordan lærerne gikk inn i en prosess hvor de både så kritisk på sin egen undervisningspraksis, som en følge av samarbeidet innen det utforskende fellesskapet, og hvor de tenkte gjennom mulige implikasjoner for sin fremtidige undervisningspraksis.

Videre gir avhandlingen innsikt i min egen utvikling, både som didaktiker og som forsker, og hvordan denne innsikten settes i sammenheng med forskningsresultatene.

Mer overordnet bidrar avhandlingen til en bedre forståelse av spørsmål relatert til samarbeid mellom lærere og en didaktiker fra et universitet, mens fokus holdes på utviklingen av algebraisk tenkning. Videre foreslås implikasjoner for måten algebra kunne bli presentert på i skolen.

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1 Introduction

1.1 Overture

The research reported in this thesis addresses the development of algebraic thinking within a community of inquiry consisting of mathematics teachers in lower secondary schools and a didactician from a university college. My particular focus is on the processes related to the development of algebraic thinking. A major emphasis of the study is on exploring the creation and the development of the community of inquiry.

The research was conducted with a group of three mathematics teachers currently¹ working in lower secondary school and myself, acting both as a didactician and as a researcher. Our collaboration started during Summer 2004 and ended one school year later. All the three teachers and I were involved in the research throughout the period.

In this section, I introduce the research. In the second section, I introduce my professional background and my own motivation for conducting this research. The third section addresses the process of engaging in research, and in the fourth section I present briefly the research setting. The fifth section outlines the aims of the research and gives a short introduction to the main themes, as presented in the title of the thesis. The sixth section describes the way the collaboration between the teachers and myself was organized through workshops, interviews and classroom observations. In addition, I introduce and explain the elaboration of a six-step developmental and analytical framework. Finally, the last section gives an outline of the content of the thesis.

1.2 My professional background and motivation

I can remember from my own time at both lower and upper secondary school (in France) that I always enjoyed working especially on tasks related to algebra, it was like entering another exciting world and this fascination developed further during my studies at University level with the culmination when working towards Galois theory in an algebra course. The historical aspect of Galois and the development of the notion of “group” was the theme of my Master study, and through this work I could combine my interest in modern Galois theory with the approach required for taking an historical perspective and reading Galois’ first *Mémoire*.

In that sense, my interest for algebra and algebraic thinking has been present for many years. Through my experience as a teacher in mathematics and physics both at lower and upper secondary school I have also

¹ During the school year 2004-2005

experienced the variety of difficulties students meet when engaging with tasks related to algebra. This growing awareness developed as part of a desire to “do something” involving the teaching and learning of algebra but “what to do” was not clear at that time.

I also want to point to a particular course at Master level that, I can see today, was important in my own development and in the articulation of “what was possible to do” in relation to the teaching and learning of algebra. This course was based on John Mason and Joy Davis’ (1991) book *Fostering and Sustaining Mathematics Thinking through Problem Solving*, and I can remember that the way of working introduced through this book was really unusual for me. The purpose of the course was to engage and work on some mathematical tasks and, at the same time, to reflect on what we, a small group of students, were doing. Everything, both the mathematical task and our own reflections had to be written in a logbook. At the beginning, I couldn’t see the point of reflecting on my own work and having to write about my own reflections, which did disturb me from doing mathematics. By the end of the course, I had completely changed my view and it was difficult *not* to reflect on what I was doing.

Before engaging within a doctoral study, I had the opportunity to teach in a school (primary school which was extending into both primary and lower secondary). Here my aim was to look for possibilities to teach mathematics differently in order to increase pupils’ engagement. These ideas resulted in a project, called the Mathias-project², which ran for three years, involving teachers both at primary and secondary level. In the Mathias-project, my role was to look critically into the way mathematics was presented to pupils and to engage teachers in discussions and reflections concerning possibilities to develop their own teaching by looking critically into their own practices. Through that project, I had the opportunity to refine and reformulate my own request, from asking “what is it possible to do” in relation to the teaching and learning of algebra into “what does it mean to work with teachers”. I do not claim that these questions were research questions, they were not, but they helped me to re-articulate and re-formulate my search at that time. They might be considered as pre-research questions which create a basis for research.

In a sense, this way of ‘working with teachers’, and I use these terms in a loose way on purpose, developed during the Mathias-project could be seen as related to Donald Schön’s (1983, 1987) “reflection-on-action” in order to be able to “reflect-in-action”. However, I can see now that the Mathias-project was a *developmental work* with teachers, it was *not* research on the developmental work because, as I mentioned above, I

² “Mathias” stands for *Mathematics in Arendal* at St. Franciskus school.

had not thought about and articulated some research questions. Nevertheless, I found the whole process of working and reflecting together with teachers really interesting and challenging. The fact of being accepted within the doctoral program gave me the opportunity to address, study and deepen, in a scientific way through conducting research, what it means to work with teachers and to look more specifically on algebra and algebraic thinking.

This short description of my background gives an overview of my starting point as a doctoral student. Thereby, I entered the research process with a desire to explore further issues related to algebra and algebraic thinking, and ways of working with teachers.

1.3 Engaging in research

Doing research is very much about development, and I agree with Michael Polanyi (1958) arguing that development is a transformational process. Being engaged in research means that the researcher has to select a particular (researchable) issue, to articulate and gradually refine adequate research questions, and to elaborate or choose a coherent and relevant theoretical framework which allows her to address the purpose of the study. Furthermore, it implies that the researcher chooses or locates a suitable methodological approach, conducts the analysis of relevant data, and engages in the analytical process of writing which helps focusing and refining her own thoughts. Finally the researcher presents her views to public scrutiny and criticism, and eventually the findings need to be re-formulated according to the feedback from other peers, as they critique the researcher's work. This transformational process has many levels and layers which are deeply interconnected and interrelated, and these aspects reflect the complexity and the challenges one has to face while entering into "the world of research".

I experienced this transition as moving from "being interested in algebra and working with teachers" into a research process in which I had to give a precise definition of my research goals and a formulation of research questions. This transition mirrors the difference between having opinions and being able to produce and present evidence which is grounded in research. I can see now that the formulation of research questions requires a process of re-finishing and re-formulating (Stake, 1995). According to Michael Bassey (1999), their formulation should set the immediate agenda for the research, they should enable data to be collected and analysis to be started, and finally they should clearly define the boundaries both in time and in space within which they will operate.

In my case I had to define what aspects of algebra I wanted to concentrate on and what I meant by "what does it mean to work with mathematics teachers?"

1.4 The research setting

In order to conduct my research, I contacted several mathematics teachers at lower secondary school, asking them if they would agree to work in collaboration with me during a school year. In addition, I indicated that I was interested in algebra and algebraic thinking. Among these teachers, I contacted Mary and Paul, since I had been working at their school as a mathematics and French teacher, and knew them as colleagues. Particularly, I knew Mary quite well, as she was my daughter's mathematics teacher during three years at lower secondary school. Mary and Paul agreed to work with me and I discussed with them the possibility to have another teacher in our group and asked them if they knew about somebody else who could be interested in working with us. They talked to a colleague, and her husband, John, answered positively to the proposition. I also knew John as a colleague, since I was a French teacher in his school some years ago. I started with these informal contacts during spring 2004, and Mary, Paul, and John agreed to participate in the research for one school year and to work collaboratively with me.

These three mathematics teachers are all teachers in lower secondary school (pupils aged 13-16 years), teaching mathematics along with other subjects. The names of the teachers have been changed to preserve their anonymity. However, I use my own name in the research. Therefore I refer to our group as Mary, Paul, John, and Claire. Mary and Paul are colleagues, working in the same school. John works in another school, just a few kilometres away from Mary's and Paul's school. Both schools are situated in a small town in the South of Norway.

The three teachers are experienced teachers with several years of practice. Mary has been working for five years in lower secondary school and has also had experience from teaching in primary school. Now, she teaches mathematics in Grade 8 (ages 13-14 years) and sciences in Grade 10 (ages 15-16 years) during spring 2004. Paul has been working in lower and upper secondary school since 1980. During autumn 2004, Mary and Paul shared with each other the responsibility of their respective classes in mathematics, being sometimes both present during a teaching period. John has been working in lower secondary school since 1982, and taught mathematics and sciences in Grade 9 and 10 (spring 2004).

Both Mary and Paul took courses at University. Mary described herself as a plant physiologist with a little mathematics, while Paul defined himself as a zoologist with no education in mathematics. John described himself as a teacher who had studied four years in teacher education, with a little mathematics from his teacher education. He also has mathematics, physics, chemistry, and biology from early upper secondary school.

During our first group meeting (Workshop I, 16.06.04), I asked the teachers the reasons why they agreed to collaborate with me. Mary mentioned the opportunity to reflect a little more about what she was doing and also to look for alternative ways of teaching; she talked about methods for organising her teaching. Paul considered these meetings as a possibility to learn more, both in relation to the subject-matter and to methods. He also remarked that this collaboration could be useful, not only in relation to teaching methods, but also as a means to work with mathematics as a subject-matter profit. John pointed to having the possibility to sit down and talk about mathematics, in contrast to the discussions usually related to every-day problems. His hope was to have the possibility to reflect on mathematics, to the different problems he and other teachers met, and to have a “methodological-pedagogical-mathematical break”. He also emphasized that these possibilities were rare in the very limited mathematical milieu in his school.

1.5 The aims of the research

The main aim of my research is to explore the way a community of inquiry addresses and develops algebraic thinking and, through participation, shows evidence of learning. Following this perspective, my research questions are:

- a) In what ways is the development of algebraic thinking related to the development of our community of inquiry?*
- b) What relationships can be discerned between teachers developing algebraic thinking during the workshops and their thinking in relation to their practice in the classroom?*

I consider that the way my research questions are formulated reflects the exploratory nature of my study. I elaborate further on this issue later (see Section 3. 1). The first research question addresses both the processes related to the creation and the development of a community of inquiry focusing on algebraic thinking. Further, it seeks to elaborate a theoretical frame enabling the description of the development of algebraic thinking within a community of inquiry consisting of three teachers and a didactician/researcher. I use the term “didactician” as a teacher who is largely experienced in research, and who is able to collect and analyse data systematically in order to address clearly formulated research questions (Jaworski, 2007).

The second research question addresses the possible link between the teachers’ development of algebraic thinking, as it emerges from our community of inquiry, and their thinking related to their own practice. These concerns reflect the fact that, by adopting Etienne Wenger’s (1998) theory, it is possible to consider the teachers as evolving within

several communities of practice and to explore the relations between these different communities.

The central themes, as presented in the title of the thesis, are *community of inquiry* and *developing algebraic thinking*. The research follows a situated cognition perspective, as presented by Jean Lave and Etienne Wenger (1991), in which the focus is placed on the ‘person-in-the-world’ and where learning is understood as participation in the social world. The term *community of inquiry* derives from Wenger’s (1998) idea of *community of practice* where learning is understood as social participation. I want to emphasise the fact that the dimension of inquiry emerged as a result from the process of analyzing data and was *not* present from the beginning. In that sense, the term *community of inquiry* captures one of the results of my study: our community can be characterized as a community of inquiry. Furthermore, in this study, *algebraic thinking* is understood from a Vygotskian perspective, in which the role played by psychological tools as mediating agents in relation to scientific concepts is crucial. These issues are explored in more depth in Chapter 2 concerning the theoretical framework elaborated in this thesis.

1.6 The nature of the collaboration between the teachers and myself: the conceptualization of a six-step framework

Drawing on the experiences both from the course inspired by Mason and Davis’ (1991) book *Fostering and Sustaining Mathematics Thinking through Problem Solving*, from the Mathias-project, and from my study of the research literature, I developed a framework for working collaboratively with teachers in order to address issues related to algebraic thinking. Central elements which this framework had to address were: engaging collaboratively, the teachers and myself, with some mathematical tasks, addressing in some way algebra and algebraic thinking, and discussing and thinking about what we were doing. In addition, I wanted to follow the teachers’ thinking in relation to their practice in the classroom.

Starting from these building blocks, I gradually elaborated, during spring 2004, a developmental and analytical framework consisting of six steps. Three of these steps (the first, the second, and the sixth) are related to our monthly meetings. During these, I proposed meeting as a group at Mary’s and Paul’s school. We met in the evening (usually from 7 p.m. to 9 p.m.) and worked for about two hours. These meetings were referred to as “workshops”. During the remaining steps (the third, the fourth, and the fifth), I planned to follow individually each teacher in his/her class, observing their teaching and interviewing them both just before and just

after the teaching period. These steps are referred to as “observation steps”.

The steps in the framework can be described in the following way:

1. At the first step, the researcher (myself) presents to the teachers some mathematical tasks related to algebra and studies the way the whole group engages collaboratively and cooperates in undertaking the proposed task.
2. During the second step, our group engages in discussions while we share with each other our thinking concerning the proposed task. Here my focus, as a researcher, is on the way algebraic thinking is addressed, how we interact and collaborate with each other, and on the developmental nature of this collaboration.
3. In the third step, the researcher interviews each teacher separately just before a teaching period. My purpose is to get insights into what the teacher plans to address during his/her class, how this goal will be achieved, from the teacher’s perspective, and finally why the teacher chose that particular aim.
4. In the fourth step, the researcher follows each teacher into his/her class and observes his/her practice. These classroom observations are meant as a basis for the interviews in the next step.
5. During the fifth step, the researcher interviews again the teacher right after the teaching period, seeking for a kind of evaluation of it. In this way, we (the teacher and I) can compare what was the aim for the teaching period with what has been achieved, both from the teacher’s and my own perspective.
6. Finally, during the last step, we all meet again during a workshop and each teacher has the possibility to share with the other participants his/her experiences related to the observation in class.

These features are summarized in the following table:

Table 1: The six-step framework

Step	Participants	Kind of activity
1	Mary, Paul, John, and Claire	Mathematical workshop
2	Mary, Paul, John, and Claire	Mathematical workshop
3	Mary and Claire Paul and Claire John and Claire	Interviews and classroom observations
4	Mary and Claire Paul and Claire John and Claire	Interviews and classroom observations
5	Mary and Claire Paul and Claire John and Claire	Interviews and classroom observations
6	Mary, Paul, John, and Claire	Mathematical workshop

During the school year of our collaboration, nine workshops were organized. I also had the possibility to follow John four times, Mary twice, and Paul once, observing in their classes. For practical reasons, it was not possible to follow Mary and Paul several times.

1.7 The structure of the thesis

In this section I outline the structure of the remaining chapters of this thesis. In Chapter 2, I present my theoretical framework. My criteria for the elaboration of this chapter are the following: I seek to define or elaborate a *relevant* and *coherent* framework. The criterion of relevance requires that my theoretical frame allows me to address *learning in a mathematical context*. The criterion of coherence requires that the epistemological assumptions related to learning as a social practice and learning in a mathematical context are *compatible*. As mentioned earlier, I consider that a theoretical framework rooted in and based on Wenger's (1998) and Lev Vygotsky's (1978, 1986) ideas offers me the possibility to conduct my research. Central notions, which were briefly introduced in the first chapter, are further developed in this chapter. The notion of *community of inquiry* derives from the work of Wenger (1998) and the relation between learning and development is considered from a Vygotskian perspective. Furthermore Wenger's central concepts of meaning, practice, community, and identity are presented. In addition, the idea of confidence (Graven, 2004) is introduced and the centrality of inquiry (Jaworski, 2005a, 2006; Wells, 1999) is underlined.

In Chapter 3, I present an elaboration of the methodology of my research. In this chapter, I take seriously into account Leone Burton's (2002) advice to address "The Why" and not only "The How" of my research. This means to make visible my underlying assumptions and to discuss and justify the choice of the research approach. Central features of design-based research, developmental research, and action research are presented and discussed, and my own research approach is explained. I continue with a detailed description of the way the research was elaborated with the six-step framework including both mathematical workshops, interviews and classroom observations. Finally, I address the way the analysis of data was conducted and explain how inquiry emerged from it.

In Chapter 4, I present a synthesis of the findings emerging from the analysis of the data. This chapter follows the chronological order of the workshops and an in-depth analysis of each workshop is presented. Issues concerning the development of our community of inquiry are related to the development of algebraic thinking within our group, and the idea of confidence, both in our community of inquiry and in the subject-

matter, is explicitly addressed and clearly articulated. In addition, the analysis of the teachers' evaluation of our collaboration during the school year, and their reflections in relation to their own teaching practice is presented. An emerging aspect of the analysis is the crucial role played by the mathematical tasks, which were proposed to the teachers, as a means both to develop our community of inquiry and to address algebraic thinking. This aspect is developed further in this chapter. Finally, I offer a conceptualisation of the idea of participation within our community of inquiry.

In Chapter 5, I present an outline of my own learning process through the whole thesis and what it meant for me to engage in this research. The chapter is organized according to the following three dimensions: addressing the way the theoretical framework has been elaborated, addressing the way algebraic thinking is understood, and addressing elements of my own learning as these emerged from Chapter 4.

In Chapter 6, I discuss conclusions from the findings presented in this thesis and make explicit how the study illuminates the research questions and how the theoretical framework which I elaborated, based on Wenger (1998) and Vygotsky (1978, 1986), allows me to address learning in a mathematical context. Finally, I indicate possible implications of this study for educational research and how questions which are discussed in this work might be further elaborated. Issues concerning reliability, validity, and generality are also addressed.

2 Theoretical framework and literature reviews

My thesis is about developing algebraic thinking within a community of inquiry consisting of three teachers and a didactician. I examine the way algebraic thinking is addressed within our community and, by looking into the way the participants engage in and act during the mathematical workshops, I trace the development of our community of inquiry. Therefore, the key elements in my study are *algebraic thinking*, *inquiry*, *development and learning*, and the *community* consisting of the three teachers and myself. In order to elaborate a coherent theoretical framework addressing these central elements, I present first the way in which *learning* and *development* have been conceptualized and how the link between these concepts has been addressed. Then I present the way in which algebraic thinking and inquiry are understood and underline the importance of inquiry as a means to explore and develop algebraic thinking.

In order to locate a situated approach to learning, based on Jean Lave and Etienne Wenger (1991) and Etienne Wenger (1998), theoretically, I refer briefly both to the constructivist and socio-cultural perspectives on learning. The notion of *community* is developed by referring mainly to the work of Wenger (1998). Besides addressing ontological and epistemological considerations about learning and development, I explain how I understand *algebraic thinking* by referring to Lev Vygotsky's (1986) idea of *scientific concepts*. Finally, in the last section of the chapter, I present the elaboration of a theoretical framework which allows me to comprehend holistically the central elements in my study.

2.1 Theoretical perspectives on learning and development

Adopting Wenger's (1998) theoretical framework in my study means conceiving learning as social participation, and more precisely following a perspective placing "learning in the context of our lived experience of participation in the world" (p.3). In that sense, Wenger follows Lave and Wenger (1991) who propose to view learning as "an integral and inseparable aspect of social practice" (p.31). This conceptualisation of learning is embedded in a theoretical perspective, referred to as *situated learning*, which aims to bridge previous views on human psychological development. On the one hand, we have a Piagetian perspective according to which analytic primacy is given to individual, cognitive functioning, and where social processes are assumed to be derivative; on the other hand, a Vygotskian perspective in which social practice and processes are given

analytic primacy, and where individual, psychological functioning is seen as derivative with learning being one of its characteristics (Lave & Wenger, 1991; Wertsch et al., 1996). Furthermore, Lave and Wenger put emphasis on the fact that, in a situated perspective on learning, learning is not merely situated in practice, but learning is understood as “an integral part of generative social practice in the lived-in world” (p.35). These different approaches in conceptualizing learning are reflected in the way each perspective chooses the unit of analysis, where a ‘unit of analysis’ could be defined as a minimal unit preserving the properties of the whole (Davydov et al., 1985). According to the Piagetian view, the unit of analysis is the individual, cognitive functioning, with focus on the “mental schemes” as generalized patterns of action. Here, the social processes are understood from the perspective of their influence in generating challenges and disequilibria in the individual’s thought. The psychological processes emerging from these disequilibria are referred to as “assimilation” and “accommodation”.

Contrary to the Piagetian view, the Vygotskian perspective, in which analytic primacy is given to social processes, takes “mediated action” as the basic unit of analysis (Wertsch, 1985, 1991; Wertsch et al., 1996). Here the individual’s higher mental functioning is seen to derive from social life, with human action mediated by tools and signs.

Vygotsky distinguished between three major classes of mediating agents: the material tools, the psychological tools (language, signs, symbols, texts, formulae), and the human mediator. In contrast to material tools, which serve as a mediator between the human hand and the object of action, psychological tools are internally oriented, as these transform the unmediated interaction of the human being with the world into mediated interaction. In that sense, psychological tools transform the inner, natural psychological processes of perception, attention, memory, into higher mental functions, or “new cultural forms of psychological functions” (Kozulin, 1998, p.4). The influence of the cultural context is fundamental in the Vygotskian perspective as “the type of these transformations depends on the type of symbolic tools available in a given culture and the conditions under which the appropriation of these tools by individuals is taking place” (Kozulin, 1998, p.4).

I consider Vygotsky’s emphasis on the importance of the type of symbolic tools in relation to the transformations of psychological functions as relevant for my study since the introduction and use of algebraic symbolism is crucial to the development of algebraic thinking. I develop this aspect further in Section 2. 5.

In an attempt to develop the conventional notions of ‘learning *in situ*’ or ‘learning by doing’ further, Lave and Wenger (1991) present the *situated perspective on learning* as a transitory conceptualization, “a bridge”, be-

tween a Piagetian and a Vygotskian frame. In their approach, learning is understood as being an integral and inseparable aspect of social practice. According to these authors, the unit of analysis is presented as “person-in-the-world”, and it suggests an explicit focus on the person, not in isolation, but as a member of a sociocultural community. This definition aims to underline participation as based on situated negotiation and tries to dissolve “dichotomies between cerebral and embodied activity, between contemplation and involvement, between abstraction and experience: persons, actions, and the world are implicated in all thought, speech, knowing, and learning” (p.52). In contrast to Piagetian theory in which intellectual development precedes learning (Orton (1987) talks about *readiness*), a Vygotskian perspective proposes a strong relationship between learning and development and underlines the sociocultural nature of both. As Alex Kozulin, Boris Gindis, Vladimir Ageyev, and Suzanne Miller (2003) describe it:

Learning in its systematic, organized, and intentional form appears in sociocultural theory as a driving force of development, as a consequence rather than a premise of learning experiences. (p.5)

My interpretation of Kozulin et al.’s quotation is that, adopting a Vygotskian perspective on learning enables me, as a researcher, to understand development as a consequence of learning, in a sense the individual’s learning is evidenced by her development. The consequences for my study are such that following the participants’ development within our community of inquiry enables me to deduce learning processes for our group. I develop further how I understand the participants’ development within our community in the next section.

In this section I have introduced situated learning as a social theory of learning. The link to Piagetian and Vygotskian theoretical perspectives and the different conceptualizations of the relation between learning and development have also been addressed. In the next section I present central ideas from Wenger’s (1998) theory.

2.2 Learning as social participation

There is a growing body of research showing an interest in the social elements involved in teaching and learning mathematics, and many educational psychologists have adopted this perspective (e.g., Boaler & Greeno, 2000; Brown, Collins & Duguid, 1989; Greeno, 1997). This position, referred to as situated theory, or social practice theory, seeks to take into account the fundamentally social nature of cognition and learning, and to see meaning, thinking and reasoning as products of social activities. The fundamental ideas of the situated perspective are that knowledge is co-produced in settings, and thereby should not be considered as the preserve of individual minds (Boaler, 2000) and, that partici-

pation in social practices is what learning mathematics is (Greeno & MMAP, 1998). Thereby, an individual's development is understood through his/her *participation* and *interactions* with broader social systems, including a process of *enculturation*.

The emergence of the situated approach to cognition and learning in mathematics education becomes visible through the publication of several texts which have become significant for the development of this new perspective (Carraher et al., 1985; Cobb, 1994, 1995, 2000; Jaworski, 2000, 2005a, 2006; Lave, 1996; Lave & Wenger, 1991; Lerman, 2000, 2001; Nunes et al., 1993; Watson, 1998; Wenger, 1998). Jean Lave's book (1988), *Cognition in Practice*, offers studies which give a description of the mathematical practices of grocery shoppers and dieters. In her research, fundamental questions concerning the relation between mathematical practices out-of-school and the practices in a school situation are addressed. According to the traditional individual psychological view, these "out-of-school" practices have been considered as merely the application of school techniques. In many senses, this new theoretical position, which proposes that learning is a social phenomenon constituted in the world, stands in direct contrast to the constructivist approach which views learning through the metaphor of construction in teaching and learning (Boaler, 2000; Kirshner, 2002; Lave, 1997; Sfard, 1998). Thereby, learning is seen as a process of enculturation and refining practice rather than one of acquiring knowledge. However, the idea of situated learning implies more than that learning merely takes place in contexts. As Lave and Wenger (1991) underline:

In our view, learning is not merely situated in practice – as if it were some independently reifiable process that just happened to be located somewhere: learning is an integral part of generative social practice in the lived-in-world. (p.35)

This perspective on learning, as an integral part of social practice, challenges current pedagogic practice. According to Terezinha Nunes, Analucia Schliemann and David Carraher (1993) and Lave and Wenger (1991), learning might be conceived as located in particular forms of situated experience, and not merely in mental schemes. This approach allows for understanding knowledge as relational by nature, between individuals and settings (Lerman, 2000). Especially, it is through Lave's studies of tailors in Liberia that Lave and Wenger (1991) introduced the idea of learning through apprenticeship, challenging with this research the previous established theoretical assumptions concerning a two-sided formal/informal educational model. The implication is that "there is no learning which is not situated and therefore the fundamental assumption that formal learning, which is meant to refer to schooling, is characterised by decontextualised knowledge is not viable" (Lerman, 1998a,

p.33). According to Lave (1996), these claims might also be relevant to learning in school settings:

Learning is an aspect of changing participation in changing “communities of practice” everywhere. Wherever people engage for substantial period of time, day by day, in doing things in which their ongoing activities are interdependent, learning is part of their changing participation in changing practices. This characterization fits schools as well as tailor shops. There are no distinguishable “modes” of learning, from this perspective, because however educational enterprises differ, learning is a facet of the communities of practice of which they are composed. (p. 150)

However, Lave’s central argument, concerning the fact that there are no distinguishable “modes” of learning, is challenged by Stephen Lerman (1998a). His argumentation is threefold: the first aspect concerns the recognition of a clear distinction between “voluntary” life-long situations (work practices, cultural groups) and “non-voluntary” temporary situations (schools, hospitals). Second, the intentionality of the schoolteacher plays a central role which can not be ignored. Finally, even though Lave’s perspective on learning is closer to Vygotsky’s cultural-historical theories than to Piaget’s individualistic cognitive theory, the processes related to *how* children and adults become participants in the social practices in which they act have to be clarified. However, while constructivist theories do not address issues related to how students develop identities as learners, one of the issues that representations of learning as trajectories of participation within community of practice leave unexplained relates to the subjectivity and regulation of individuals within those practices (Boaler, 2000; Lerman, 2000).

I argue that this current research offers an example of the processes of becoming a participant in a community of inquiry³, and that the way I elaborated my theoretical framework, by combining and elaborating further Wenger’s theory with Vygotsky’s notions of mediation and scientific concepts, enables me, as a researcher, to describe in details how learning and becoming occur in the particular social setting described in this research.

An issue that I need to take account of relates to *transfer theory* (Anderson, Greeno, Reder & Simon, 2000; Anderson, Reder & Simon, 1996, 1997; Greeno, 1997; Kirshner & Whitson, 1997; 1998). Lave (1988) challenges cognitivism and transfer theory in mathematics learning, where the notion of transfer of knowledge is understood as the transfer from one situation to another of “decontextualized mental objects in the minds of individuals” (Lerman, 2000, p.26). As Lave (1988) remarks:

³ The meaning of the terms “community of inquiry” is explained in Sections 2. 2. 3, and 2. 2. 6.

It is puzzling that learning transfer has lasted for so long as a key conceptual bridge without critical challenge. The lack of stable, robust results in learning transfer experiments as well as accumulating evidence from cross-situational research on everyday practice, raises a number of questions about the assumptions on which transfer theory is based. (p.19)

Transfer theory is of relevance in this study since it addresses how the three teachers and I are gaining understandings and developing awareness concerning algebraic thinking during the workshops situation and to what extent the teachers are able to bring with them these experiences while planning their own teaching. Here transfer theory is addressed in terms of transfer of (algebraic) knowledge between socially very different environments.

According to James Greeno (1997), cognitive research strategies are committed to a factoring assumption, which entails that learning in one situation results in some knowledge that the learner has then acquired. In a sense, some acquired knowledge has become the property of an individual. Cognitive researchers study then the individual's behaviour in a range of situations and can inform conclusions on the generality of the acquired knowledge. This again implies the consideration that the individual and knowledge are two separate entities (Ernest, 1998a). The situated approach to learning does not follow such a factoring assumption, and therefore the issue of generality of knowledge involves the study of situations in which the individual's learning allows him/her to become a more effective participant. In Lave and Wenger's (1991) words: "the generality of any form of knowledge always lies in the power to renegotiate the meaning of the past and future in constructing the meaning of present circumstances" (p.34). I believe that Lave and Wenger's (1991) quotation enables me to reformulate issues related to transfer of knowledge, as these occur in this research, in terms of offering to the teachers the opportunity to re-consider previous teaching experiences and to imagine future teaching situations in the light of what is happening during our workshops.

Furthermore, Jo Boaler (2000) argues that when addressing the idea of transfer of knowledge, it is necessary to take into account the social setting within which it is addressed:

What is fundamental to the situated perspective is an idea that knowledge is co-produced in settings, and is not the preserve of individual minds. Situated perspectives suggest that when people develop and use knowledge, they do so through their interactions with broader social systems. ... The different activities in which learners engage co-produce their knowledge, so that when students learn algorithms through the manipulation of abstract procedures, they do not only learn the algorithms, they learn a particular set of practices and associated beliefs. ... Situated perspectives turn attention away from individual minds and cognitive schemata, so that success is not focused on individual attributes, but on the ways in which those attributes play out in interaction with the world. This is

not to say that knowledge cannot be transferred to new situations, only that *it is inadequate to focus on knowledge alone, outside of the practices of its production and use.* (p.3, my emphasis)

Thereby, an understanding of transfer of knowledge, within a situated perspective, is integrated in relation to a broader social analysis, an aspect which is addressed in this study. However, this process is complex and there are still issues that need to be addressed, such as subjectivity and the processes concerning the regulation of individuals within practices (Lerman, 2000). Paul Ernest (1998a) proposes an in-depth analysis of the ontological and epistemological views underpinning the idea of transferability of knowledge and warns against reducing these differing perspectives on knowledge application to a fixed dichotomy.

It seems, from the literature, that even though several authors recognize the complexity of the learning processes, some argue that “perhaps learning is, after all, not a unitary phenomenon, and thus not amenable to one all-embracing theory” (Adler, 1998, p.176). On the other hand, Julian Williams and Liora Linchevski (1998) warn against neglecting individual and psychological aspects of learning and advocate for “learning theories to incorporate the psychological with the social, and for the metaphor of concepts as mental objects to coexist with the metaphors of learning as ‘participation’ in social processes and in communities of practice” (p.155). Anna Sfard (1998) summarizes this position by referring to the recognition of “a patchwork of metaphors rather than a unified, homogeneous theory of learning” (p.12).

John Anderson, James Greeno, Lynne Reder and Herbert Simon (2000), in an attempt to reconcile proponents of cognitive and situated approaches to learning, propose, with high priority, to engage in research aiming to unify the different perspectives. The goal is to articulate “a more inclusive and unified view of human activity in which dichotomies such as individual versus social, thinking versus action, and cognitive versus situative will cease to be terms of contention and, instead, figure in coherent explanatory accounts of behavior and in useful design principles for resources and activities of productive learning” (p.13).

I consider that the challenge of offering a more holistic theoretical perspective on learning is addressed in Lave and Wenger (1991), with the introduction of the idea of participation, where “participation is always based on situated negotiation and renegotiation of meaning in the world” (p.51). Thereby, they claim that:

The notion of participation thus dissolves dichotomies between cerebral and embodied activity, between contemplation and involvement, between abstraction and experience: persons, actions, and the world are implicated in all thought, speech, knowing and learning. (p.52)

Furthermore, Lave and Wenger (1991) introduce the idea of *community of practice* as “a set of relations among persons, activity, and world, over

time” (p.98). Here learning is understood as an increasing participation in communities of practice, and concerns the whole person. Thereby, learning implies becoming a different person, and through viewing learning as *legitimate peripheral participation*, it becomes an evolving form of membership of a particular community. For example, Kirsti Hemmi (2006) described how students encountered proof in a community of mathematicians at university level. There is a clear shift in this perspective, from considering learning and thinking as taking place in the individual mind, to re-placing these dimensions in their sociocultural settings. Learning is now described in terms of being an *integral part of generative social practice in the lived-in world* where the idea of *participation* is central.

By conducting my research within Lave and Wenger’s (1991) and Wenger’s (1998) frames, I am able to address and follow the *development of algebraic thinking within our community of inquiry* in terms of the social activities, the mathematical workshops and the interviews with the teachers before and after class. Thereby, the knowledge emerging from our collaboration is to be considered as relational, between individuals (the three teachers and myself) and the setting (workshops and interviews) within which it is situated. However, I want to emphasize the fact that even though the title of my thesis is *Developing Algebraic Thinking in a Community of Inquiry*, I did *not* set up, from the beginning, to develop a community of inquiry. The dimension of inquiry appeared through the analytical process and emerged as a fundamental aspect of our community. In that sense, I introduce already one of the findings of the research in the title.

In order to be consistent with this approach, I propose to define the unit of analysis for my study as *the growth of knowledge, in relation to algebraic thinking, within our community of inquiry*. Here I follow Vasily Davydov and L. Radzikhovskii (1985) and their definition of unit of analysis as “the minimal unit that preserves the properties of the whole” (p.50). This implies that I have to search for the minimal unit which preserves the properties of a social theory of learning.

According to Barbara Rogoff (1995), the observation of development within a sociocultural approach might be conducted following three planes of analysis which describe the personal (apprenticeship), the interpersonal (guided participation), and the community processes (participatory appropriation). However, I choose to follow Lerman (1998b) and focus ‘the zoom of a lens’ on the knowledge as emerging from our community of inquiry. Thereby, my analysis is on the learning processes as situated within our community of inquiry and on the knowledge resulting from the activities within that particular setting. In Rogoff’s

terms, I adopt the plane of analysis corresponding to participatory appropriation.

Lave and Wenger's (1991) perspective on situated learning is developed further in Wenger's (1998) theory which addresses learning as social participation. According to Wenger, the components of a social theory of learning are: *meaning, practice, community, and identity*, and each of these components are presented in Sections 2. 2. 1 to 2. 2. 4.

In addition, I introduce, in Section 2. 2. 5, Mellony Graven's (2004) idea of *confidence* in relation to our community and to the subject-matter. As mentioned earlier in this present section, inquiry plays a crucial role in my study, and, in Section 2. 2. 6, I introduce the way the idea of *inquiry* is presented more widely in research literature (Elliott, 2005; Jaworski, 2005a, 2006; Lindfors, 1999; Wells, 1999) and how it is conceptualized and understood in my current study. Finally, Section 2. 2. 7 presents Wenger's (1998) concept of *boundary objects* and its importance in relation to the connections between our community of inquiry and the teachers' school communities as well as my own community at University of Agder⁴.

2.2.1 Meaning

In the introduction to his book, Wenger (1998) defines meaning as “a way of talking about our (changing) ability – individually and collectively – to experience our life and the world as meaningful” (p.5). Thereby, meaning might be understood as a way of describing our relationship with the world around us, having the characteristics of variability and meaningfulness. But meaning does not exist as a kind of decontextualized entity. It can be experienced in our everyday life through a process called *negotiation of meaning*, defined as “the process by which we experience the world and our engagement in it as meaningful” (p.53), which involves the dimensions of *participation* and *reification*. According to Wenger, human engagement can be addressed as a *continual experience of negotiation of meaning*, since “living is a *constant process* of negotiation of meaning” (p.53). This process of negotiation of meaning includes our social relations as factors but does not necessarily involve language or direct interaction with other human beings. In the concept of negotiation of meaning, the term *negotiation* aims to indicate “a flavour of continuous interaction, of gradual achievement, and of give-and-take” (p.53). Thereby, the negotiation of meaning is a process whose nature is both historical and dynamic, contextual and unique. Furthermore, there is a relationship of mutual constitution between the process of negotiation of meaning and the multiple elements affecting this process, as this negotiation continually changes the situations to which it gives meaning

⁴ Agder University College (AUC) became University of Agder (UiA) since 01.09.2007

and as such affects all participants. In that sense, negotiating meaning entails the dimensions of interpretation and action. These aspects are summarized as “meaning exists neither in us, nor in the world, but in the dynamic relation of living in the world” (p.54). Concerning the two dimensions of negotiation of meaning, *participation* and *reification*, the former suggests both action and connection and in that sense entails a personal and social aspect. The latter refers to “the process of giving form to our experience by producing objects that congeal this experience into “thingness”. In so doing we create points of focus around which the negotiation of meaning becomes organized” (p.58). Reification might include various aspects such as representing, naming, describing, as well as perceiving, interpreting, or recasting. Its essence consists of giving form to a certain understanding.

In our community of inquiry, the participants’ experience might be congealed in various points of focus, such as in words in dialogue (shared repertoire), in experiencing engaging in the same tasks (the previous workshops also become part of the shared repertoire), and in the notes on our notepads which allow us to create points from which we can negotiate meaning.

I consider the idea of negotiation of meaning as fundamental for my study as it allows me to focus on the process which is taking place during our workshops. I referred before to learning as involving the transformation of participation in collaborative endeavour, but the process enabling the transformation of participation was *not* addressed. I understand the negotiation of meaning, as realized within our workshops, as a continuous interaction between the participants, consisting of discussion, argumentation, and disagreement as we engage in the mathematical tasks, and through these address issues related to algebraic thinking. In that sense, Wenger’s criteria concerning the historical, dynamical, contextual and uniqueness aspects of negotiating of meaning are fulfilled. In addition, I consider this way of describing the process of negotiation of meaning as closely related to Gordon Wells’ (1999) notion of *inquiry* as indicating “a stance toward experiences and ideas – a willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them” (p.121). This dimension of *negotiation of meaning through inquiry* allows me to study the *process* by which our community develops algebraic thinking through the transformation of our participation during the school year. This idea is developed further in Section 2. 2. 6.

2.2.2 Practice

According to Wenger (1998), *practice* is related to meaning as an experience of everyday life, it is “a process by which we can experience the world and our engagement with it as meaningful” (p.51). It entails doing, not only in itself, but doing as embedded in a social and historical context which gives structure and meaning to the activity. Furthermore, Wenger, pointing to the traditional dichotomy between acting and knowing, manual from mental, and concrete from abstract, claims that the process of engaging in practice involves the whole person and therefore acting and knowing are inseparable.

How is the practice defined in my study? Three elements are fundamental in Wenger’s definition of practice: what people are actually doing, the social context, and the historical context. The doing, for our community, refers to the mathematical workshops and the interviews of each teacher before and after class. These activities are embedded in a social context consisting of the teachers’ respective school culture and myself coming from a university, and more generally to the Norwegian school system and the Norwegian society. The historical context refers both to the participants and to the focus of our engagement. It addresses the participants in relation to the teachers’ situation in their own school, in terms of how many years experience they have and their responsibilities, and to my own position as a researcher/didactician. Furthermore, the historical context addresses also algebraic thinking through its historical dimension in which the development of algebraic symbolism is fundamental (see Section 2. 5). The different aspects related to the elaboration of the practice of our community of inquiry are illustrated in Figure 1.

I consider the practice of our community of inquiry as emerging from the six-step framework which has been introduced in the introduction (see Section 1. 6). This model might be thought of as a methodological tool enabling the development of algebraic thinking through the creation and the development of the practice. It allows me to study what is happening within our community of inquiry, and to address my research questions. The nature of the practice is twofold: it consists of mathematical workshops involving all the four participants, and of interviews and classroom observations with each of the teachers. It is through the participation in the community of practice that the processes related to the development of algebraic thinking are realizable. However, the way the (actual) practice has been realized during the school year our collaboration lasted differs from the (thought) practice, as emerging from the six-step framework. This point is developed further in Section 3. 2. 1.

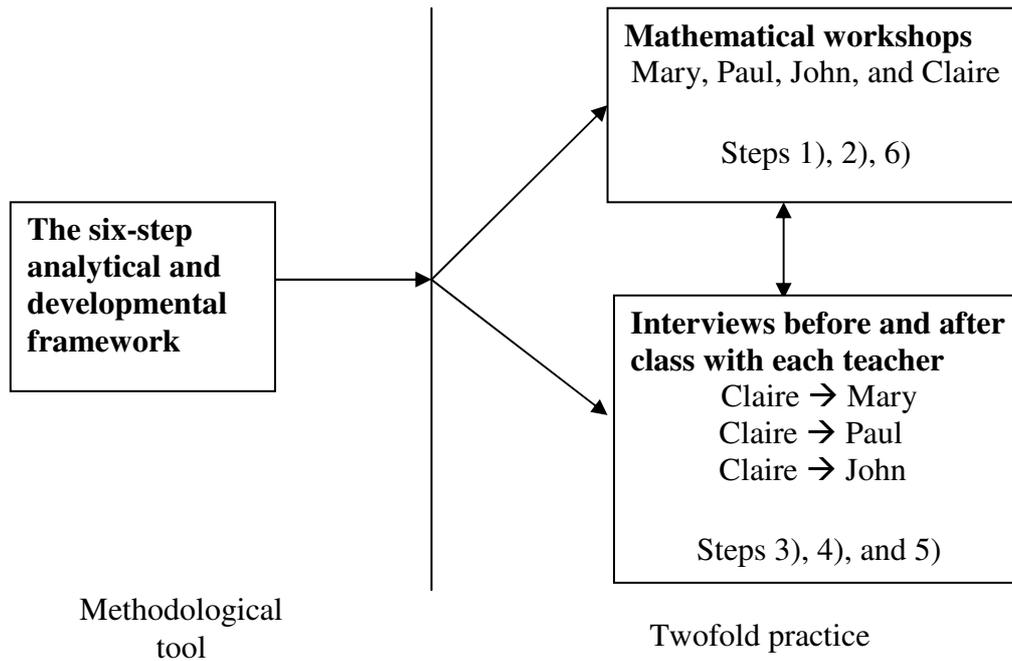


Figure 1: The practice, as emerging from the six-step framework

2.2.3 Community

Until now I used “our community” without saying clearly what I mean by this term. Wenger (1998) describes community as “a way of talking about the social configurations in which our enterprises are defined as worth pursuing and our participation is recognizable as competence” (p.5). As such, “community” does not refer merely to a group of persons: when sitting in a bus I do not feel as if I am a member of a community, here the dimensions of ‘worth pursuing’ and ‘recognition’ are central. In my study, the social configurations are the meetings we have regularly in the teachers’ school involving the three teachers from lower secondary schools and myself, a researcher/didactician from University of Agder.

Furthermore, I understand the ‘worth pursuing’ enterprises as related to both the workshops and the interviews, as we all are interested in the development of algebraic thinking, and in which our participation is recognized as valuable and appreciated by the other participants. I proposed in the introduction of this chapter the key elements of *algebraic thinking, inquiry, development and learning, and community*. In the title of my thesis I used the term *community of inquiry*, and as mentioned before (see Section 1. 5), the dimension of inquiry emerged from the analysis of data. How are the dimensions of learning and development related to the idea of community of inquiry? Barbara Rogoff, Eugene Matusov, and Cynthia White (1996) define learning as involving “transformation of participation in collaborative endeavour” (p.388). The concept of

transformation of participation is based on the idea that “learning and development occur as people participate in the sociocultural activities of their community, transforming their understanding, roles, and responsibilities as they participate” (p.390). In our community of inquiry, we address algebraic thinking and by making our understanding visible and accessible to the other participants, and by tracing the transformation of these understandings and roles, I can have access into the transformation of participation in our community of inquiry. In my study, I recognize both the individuals, each participant in our community, and the whole group as a community. In that sense, I want to avoid an individual – social dichotomy, as I consider the process of their mutual constitution as central. According to Wenger (1998), a community of practice is characterized by three dimensions which act as sources of community coherence. These are *mutual engagement*, *joint enterprise*, and *shared repertoire*. I present these three dimensions in the following.

Mutual engagement

The first dimension, *mutual engagement*, is related to people’s engagement “in actions whose meanings they negotiate with one another” (Wenger, 1998, p.73). As such, the community is defined by the mutual engagement of its participants; being engaged in the practice of a community means being engaged in what matters. Being engaged in our community means that all participants engaged in what matters for us, which is addressing algebra and algebraic thinking. In order to achieve this mutual engagement, we need to share some common goals, or to aim for common goals. This aspect is underlined by Wenger explaining that “what makes engagement in practice possible and productive is as much a matter of diversity as it is a matter of homogeneity” (p.75). The dimension of homogeneity is reflected in the fact that we all share interest in mathematics, in algebra and algebraic thinking, and in the teaching of mathematics, but still it might be a diversity concerning the aims we hope to achieve. Each one of us has goals, expectations, and assumptions related to their own communities more widely. The teachers might be interested in some implications for their own practice as mathematics teachers, my aim is to conduct research. Nevertheless, we agreed to engage mutually in a project focusing on developing algebraic thinking.

Joint enterprise

Wenger proposes three points in relation to joint enterprise. It results from a collective process of negotiation, it is defined by the participants in the very process of pursuing it, and it is not just a stated goal, it creates among participants relations of mutual accountability. What is the joint enterprise of our learning community? We all engage into the mathematical tasks and we all agree to share our thinking, both concerning issues related to teaching and issues emerging from the engagement

within mathematical tasks. In addition each of the teachers agreed that I could come and observe their teaching. Furthermore, building on the dimension of diversity from mutual engagement, Wenger explains that “because mutual engagement does not require homogeneity, a joint enterprise does not mean agreement in any simple sense” (p.78). In our joint enterprise we address algebraic thinking. This does *not* imply that we all share the same view concerning algebraic thinking and these differences might be seen as productive part of the enterprise, as they might be conceptualized as productive sources for our discussions and enhancing our thinking.

Shared repertoire

Wenger talks about the development of a shared repertoire as “over time, the joint pursuit of an enterprise creates resources for negotiating meaning” (p.82). Its nature might be heterogeneous and include routines, words, tools, etc. These have been produced or developed during the course of the existence of the community and have been gradually included in its practice. What is the shared repertoire which has been developed within our community of inquiry? Concerning routines, I see the practice, as emerging from the six-step framework, as part of our shared repertoire in the sense that, over time, we all know in advance what is going to happen during our workshops. The terminology we use might develop during the year our community lasts, since in the beginning each participant’s repertoire relates to their own community more widely. The mathematical tasks which were presented during the workshops also become gradually part of the shared repertoire, since we could make reference to one of the tasks because all the participants shared the same history related to that particular task. As mentioned earlier, I consider these aspects as examples of reification of the process of negotiation of meaning.

2.2.4 Identity

In the previous sections I presented three of the four components of a social theory of learning, and I consider that these components are fundamental to my work. The fourth component, identity, is introduced in this section. Although I recognise the importance of identity as a constituent element of a social theory of learning, I put less emphasis on it, since, according to my unit of analysis, which is the growth of knowledge within our community of inquiry, I chose to adjust the zoom of a lens (Lerman, 1998b) focusing on our group and how knowledge concerning algebraic thinking emerged from the processes of negotiation of meaning as situated in our particular setting consisting of the mathematical workshops. Thereby, my focus is not on the identity development for each of the participants, that is, I will not present a description of

Mary's, Paul's, or of John's individual identity development within our community of inquiry.

According to Wenger (1998), using the concept of identity allows focusing on the person, but without assuming the individual as a point of departure. Wenger's use of the concept of identity acts as a pivot between the individual and the social, thereby avoiding a simplistic dichotomy, but still addressing the distinction. In Wenger's words: "building an identity consists of negotiating the meanings of our experience of membership in social communities" (p.145). The resulting perspective offers the possibility to articulate "the lived experience of identity while recognizing its social character – it is the social, the cultural, the historical with a human face" (p.145). The profound connection between identity and practice is made visible since engaging in a practice entails the negotiation of ways of being a person in that situation and context. I consider that my choice of unit of analysis, as focusing on the growth of knowledge emerging from the development of our community of inquiry, allows me to address both the individual and the social aspects of my study in terms of recognizing the way the three teachers and I have experienced ourselves as mathematics teachers and both didactician and researcher, making visible the various communities we are members of (team of mathematics teachers, colleagues within a school or a university in a Norwegian context), and by addressing our professional background we make visible our own learning trajectory. In that sense, I understand the process of developing a community of inquiry as related to sharing histories of learning while recognizing its social character. From a researcher perspective, I consider the participants in our community as persons acting in our particular setting consisting of the mathematical workshops and interviews before and after class. But similarly I cannot think about this setting as disembodied from the cultures in which participants function more widely.

2.2.5 Introducing confidence as a fifth component

Confidence is not a component in Wenger's social theory of learning. It was introduced by Graven (2004) through her study of an in-service mathematics teacher education program aiming at enhancing participation in a community of practice in relation to South African curriculum change. In her paper, she argues for the necessity of taking into account and adding to Wenger's framework *confidence* as a fifth component in the social theory of learning. According to Graven, confidence, considered both as a process and a product of the mathematics teacher's learning, is related to the teachers' level of mastery in the practice of being professional mathematics teachers. Mastery involves confidence in relation to several domains. As Graven explains:

Mastery involved: confidence in what teachers had learnt and the meanings they formed in relation to changing developments in their profession; confidence in their ability to participate in the various practices (and communities) of the profession of mathematics teaching; confidence in their ability to access resources to supplement their learning; confidence in their identities as professional competent mathematics educators; confident acceptance that there was still much to learn and a willingness and confidence to be a life-long learner in the profession of being (and becoming) a mathematics teacher. (p.206)

I consider all these aspects of confidence as important and relevant to my study, as different levels of confidence (or lack in confidence) in relation to algebraic thinking and the use of symbolic notation might emerge through the analysis of data. However, looking at any particular school community, or community consisting of teachers and didacticians, *how* would we, as researchers, recognize the development of confidence with respect to the mathematics or/and to the community? I argue that my study gives evidence of processes related to the development of confidence with respect both to our community of inquiry and to algebraic thinking (see Chapter 4).

However, contrary to Graven (2004), I do not address issues of confidence as a “movement from the periphery of various overlapping mathematics and/or education communities towards more central participation, identification and belonging within these communities” (p.179). This description corresponds to Lave and Wenger’s (1991) *legitimate peripheral participation*, but is not a possible conceptualization in relation to my study. I address this point further in Section 2. 3.

2.2.6 Inquiry

As mentioned in the previous section, Graven (2004) considers that increased confidence might be understood both as a product of teachers’ learning and as a process, a kind of explanation for teachers’ learning. Nevertheless, this conceptualization begs the following question: what is the nature of the means, used by teachers and participants, in order to achieve or develop confidence? How can we, as researchers, capture and observe this development? I consider that *inquiry* plays a crucial role as a means to develop and achieve confidence both in the mathematics and in the community. In the following, I present briefly how the idea of inquiry is addressed more widely in the research literature, then I explain the way I understand and operationalize this concept in my research.

Matthew Lipman (2003), considering the role of ‘community of inquiry’ in education, claims that a necessary condition for the existence of inquiry is “some doubt that all is well, some recognition that one’s situation contains troubling difficulties and is somehow problematic” (p.94). Thereby, he argues that inquiry aims at a “self-correcting investigation” and involves “questioning, more narrowly a quest for trust, more broadly a quest for meaning” (p.95). According to Lipman’s understanding of

inquiry, as a “self-correcting practice” (p.178), the products of inquiry are judgments. However, I consider that Lipman’s notion of “self-correcting practice” implies a kind of norms, values, or judgments which needs to be critically addressed.

A different approach to inquiry is presented in Barbara Jaworski (2005a, 2006) and Gordon Wells (1999) in which both recognize the importance of a meta-level, called “metacognitive awareness” in Jaworski (2006) or “metaknowing” in Wells (1999). Furthermore, Wells (1999) distinguishes communities of inquiry from communities of practice by emphasising the importance of “metaknowing through reflecting on what is being or has been contributed and on the tools and practices involved in the process” (p.124). According to him, dialogic⁵ inquiry “indicates a stance towards experiences and ideas – a willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them” (p.121). This conceptualization of inquiry as indicating “a stance towards experience and ideas” is developed further in Jaworski (2006), proposing “*inquiry* as a fundamental theoretical principle and position” (p.187), and considering inquiry both as a tool for developing practice, and as a way of being in practice. According to her, a *community of inquiry* might address inquiry as operating at three levels: inquiry in mathematics, inquiry in teaching mathematics, and inquiry in developing the teaching of mathematics (Jaworski, 2005a).

Judith Lindfors (1999) proposes a different conceptualization of the different purposes of inquiry. Studying children’s inquiry, she explains that “inquiry acts are purposeful communication acts: they serve the speaker’s purpose of getting another’s help in her attempt to (further) understand” (p.26-27). She refers to two different types of inquiry acts: *information-seeking* and *wondering*. The former includes facts, clarifications, justifications, explanations, confirmations. Information-seeking is product oriented. It is oriented toward *what* one is wanting to know. The purpose of wondering is different. Wondering utterances hold the discourse open, tend to be playful. The goal is engaging in the process itself, exploring possible worlds, testing hypothesis. In wondering utterances, the speaker’s purpose is to engage another in playing with possibilities, reflecting, considering, exploring.

While Jaworski (2005a) and Lindfors (1999) address the different purposes of inquiry, Rebekah Elliott (2005), elaborating a framework for examining the professional development of professional developers, proposes the following three dimensions characterizing an inquiring stance: the *linguistic*, the *normative*, and the *contextual* dimension. Ac-

⁵ I recognise the importance of the “dialogic” dimension. However, in order to limit the scope of this thesis, I will not develop further on the idea of dialogism.

According to Elliott, the first dimension involves the development and use of a specific language with syntactical rules, and vocabulary aiming to examine practice and identifying problems. This specific language is “guided by the normative dimension of inquiry that lend[s] meaning to the words and phrases that are used to frame analysis within a stance of inquiry” (p.7). The normative dimension involves considering an idea or a problem from multiple perspectives, framing and re-framing ideas in order to examine and explore them deeply (inquiry as an iterative process). I understand Elliott’s normative dimension as closely related to the notion of negotiation of meaning through inquiry, as explained earlier (see Section 2. 2. 1), since the focus is on the process of addressing, exploring and inquiring ideas through continuous interaction.

In addition, Elliott emphasizes that discussion does not necessarily bring one solution to a problem, but appreciates the value of reasoned deliberation. The contextual dimension addresses the structure in which inquiry takes place. The term “structure” refers to “the *types of problems* that become objects of study when cultivating an inquiry stance, and the collective nature of who is involved in the inquiry into practice” (p.8, my emphasis). Thereby, Elliott’s frame gives me, as a researcher, directions to look for in order to identify a stance of inquiry, focusing both on the vocabulary used in the community (the linguistic dimension), the iterative nature of inquiry and the recognition of reasoned deliberation (the normative dimension), and the nature of both the proposed problems or tasks and the participants involved in inquiry processes.

In my study, I build on Jaworski and Wells’ definitions, that is, I understand inquiry as a stance towards experiences and ideas operating at two levels: inquiry in mathematics and inquiry in teaching mathematics. Furthermore, I consider that the third level in Jaworski’s characterisation of a community of inquiry (researching the processes of using inquiry in mathematics and in the teaching and learning of mathematics) was only addressed by me, since I was acting both as a didactician and as a researcher during my study. However, as mentioned earlier, the dimension of inquiry was not part of the design of my study, it emerged during the analysis of data. In order to build on and to develop further Jaworski’s characterization and to specify the different purposes of inquiry, as these emerged in my research, I propose the following six aspects: *inquiry into a mathematical task, inquiry into community building, inquiry into the other participants’ understanding of a mathematical task or into their own practice of teaching, and inquiry into Claire’s didactical and pedagogical aims*. The identification of these six different purposes for inquiry begs the following question: How are these purposes addressed within our community of inquiry? And how do we, as researchers, recognize when the participants are addressing these different purposes? In

order to engage with these questions, I propose to build on and to develop further Lindfors' (1999) characterization of the different types of inquiry acts. According to Lindfors, inquiry acts "are purposeful communication acts: they serve the speaker's purpose of getting another's help in her attempt to (further) understand" (p.27).

I consider that the theoretical tools related to inquiry, as presented above, enable me to deepen and to elaborate further on the idea of negotiation of meaning through inquiry. Using Lerman's (1998b) metaphor of "the zoom of a lens", I am now in a position to zoom in and to focus and examine the processes behind negotiation of meaning. This is the reason why, during the process of analysing my data, I introduced the construct of 'inquiry move'. I consider that this construct helps me, as a researcher, to identify inquiry acts and I propose to consider *an inquiry act as consisting of a succession of (one or more) inquiry moves*. I claim that this definition takes clearly into account the iterative nature of inquiry (the normative dimension, Elliott, 2005). In addition, I consider *inquiry acts as fundamentally situated within the sociocultural setting of the activity*. As such, inquiry acts are deeply related and dependent on the workshops from which these emerged, and have to be understood as the participants' own attempts to go beyond their present understanding, as we are seeking information or explanation, and wondering about some specific issue in relation to the mathematical task.

As explained earlier in this section, Lindfors (1999) suggests considering two types of inquiry acts: *information-seeking* and *wondering*. I propose to add a third kind of inquiry act: *experience-sharing*. I consider experience-sharing inquiry acts as a succession of (one or more) inquiry moves (utterances) where a teacher shares her own experience with other participants resulting in developing her understanding of her own teaching practice further. I understand this kind of inquiry as related to Wenger's ideas of mutual engagement, joint enterprise, and shared repertoire since, while sharing with each other teaching experiences, the participants might elaborate a common basis for mutual engagement and joint enterprise while developing a shared repertoire.

During several workshops, the teachers have shared with each other their own experience of teaching algebra, and they have often mentioned the various difficulties pupils meet with engaging in tasks related to algebra. During Workshop IX, it is possible to find evidence for the inquiry nature of the experience-sharing utterances, as these enable the participants to develop awareness of the multiple perspectives on an idea or on a problem of practice. The fact that these issues have been addressed several times (iterative nature of inquiry), that the participants have different opinions and discuss these (multiple perspectives on an idea or problem, and discussion where reasoned deliberation is valued)

are characteristic of the inquiry nature of these experience-sharing utterances, and are related to the normative dimension of inquiry (Elliott, 2005).

In contrast to inquiry acts, I argue for introducing the idea of *didactical acts* and I propose to define these as *purposeful communication acts serving the speaker's purpose of encouraging the other participants to address, discuss, and explore a specific didactical goal*. As for inquiry acts, I consider a didactical act as consisting of (one or more) didactical moves. In that sense, iterativeness is a common feature of inquiry and didactical acts. However, while inquiry acts are rooted in genuine wondering or/and questioning, the nature of didactical acts is completely different: in a didactical move, the speaker *knows* the answer, but his/her aim is to inspire and stimulate the other to move toward a chosen didactical aim.

In my study, these didactical acts, mainly expressed by Claire during the first workshops, enable the teachers to move towards developing algebraic thinking (for example, the use of algebraic notation). I chose to introduce this idea in this section since the analysis of data revealed that didactical and inquiry acts are often deeply connected to each other.

2.2.7 Boundary objects

Until now, I addressed the different dimensions which describe the actions within our community. In that sense, I considered our community and its practice as a whole, but without taking into account the connections between our community of inquiry and the teachers' school communities and my own community at University of Agder. I argue that this aspect is relevant to my study as it illuminates the potential continuities across boundaries between communities. Recognizing the fact that the teachers and myself participate in several communities begs the following question: by which means can these different communities be linked together? Wenger (1998) proposes the idea of *boundary objects* referring to "artefacts, documents, terms, concepts, and other forms of reification around which communities of practice can organize their interconnections" (p.105). Furthermore, these boundary objects are introduced into another community through *brokering*, that is, through the "connections provided by people who can introduce elements of one practice into another" (p.105). I argue that, in my study, *the mathematical tasks* which I proposed to the teachers during the workshops can be considered as boundary objects as these enable the teachers to establish a link between the practice in our community of inquiry and their own teaching practice in the school community. Thereby, through the recognition of multimembership, they might bring some elements of one practice into another (brokering). According to Wenger (1998), this action of brokering is complex:

It involves processes of translation, coordination, and *alignment* between perspectives. It requires enough legitimacy to influence the development of a practice, mobilize attention, and address conflicting interests. (p.109, my emphasis)

I consider that through the discussions the teachers and I have during the workshops, we have the opportunity to address the processes of translation in the sense of transferring the mathematical tasks into different contexts (classroom), and adaptation of the tasks to the teachers' own teaching practice. Furthermore, I understand the idea of coordination between the perspectives of our community and the teachers' school community as referring to the teachers' evaluation of the mathematical tasks as relevant (or not) for their own practice in school. This aspect is developed further in Chapter 4.

Concerning the third aspect mentioned by Wenger, alignment, I argue that, since the nature of our community is rooted in the idea of inquiry, the teachers and myself are engaged in a form of *critical alignment* (Jaworski, 2006), where "through the exercise of imagination during engagement, alignment can be a critical process in which the individual questions the purposes and implications of alignment with norms of practice" (p.190). Through engaging in mathematical tasks focusing on algebra and algebraic thinking, our group is addressing critical questions and thereby the teachers have the opportunity to question their current teaching practices and I might question my own role as a didactician. Therefore, the development of awareness concerning algebraic thinking results in a process of critical alignment both for the teachers and for myself.

The last aspect mentioned in Wenger's quotation, above, concerning legitimacy is related to systemic components of the teachers' school community. I recognize the importance of these elements in relation to the sustainability of critical alignment, however, these considerations are not part of my study and will not be addressed here.

2.3 The six components of a social theory of learning

The aim of this section is twofold: first I propose a summary of the different theoretical positions which form my theoretical framework and second I indicate which aspects of these theoretical positions do not fit in my study.

The theoretical framework that I propose in order to address the aspects related to the conceptualization of the idea of *community of inquiry* is derived from Wenger's (1998) social theory of learning and Graven's (2004) focus on confidence. Within this frame, inquiry plays a fundamental role (Elliott, 2005; Jaworski, 2005a, 2006; Lindfors, 1999; Wells, 1999). Wenger's four components of a social theory of learning are: *meaning* in relation to learning as experience, *practice* in relation to

learning as doing, *community* in relation to learning as belonging, and *identity* in relation to learning as becoming. In addition I consider *learning as confidence* in relation to its development both within our community of inquiry and the subject-matter (Graven, 2004), and *learning through inquiry processes*, as this aspect gradually emerged from the analytical process and became fundamental in my study. It is important to comprehend these elements holistically, as they are deeply interconnected and mutually defining. In my study we, the three teachers and myself, learn through the creation of and by becoming participants in our community, we learn through what we do together in the workshops, which includes engaging in mathematical tasks and addressing algebraic thinking, we learn through experiencing our meetings as meaningful, and we learn by belonging to our community as its existence develops in time. All these aspects are in constant evolution as they develop and re-define each other over time.

Before addressing the way I understand algebraic thinking and what I mean by the notion, I want to summarize the advantages and limitations of Wenger's theory in relation to my own study. What aspects of my study might be illuminated by Wenger's theory?

I have described how Lave and Wenger's (1991) and Wenger's (1998) theories relate to my own study through the dimensions of *community*, *practice*, and the central elements of *mutual engagement*, *joint enterprise*, and *shared repertoire*. In addition, I developed the notion of *negotiation of meaning* and its relation to *inquiry* (Elliott, 2005; Jaworski, 2005a, 2006; Lindfors, 1999; Wells, 1999). However, there are some aspects in Lave and Wenger's (1991) and Wenger's (1998) theories which are not relevant for my work. Lave and Wenger (1991) introduce *legitimate peripheral participation* as a "descriptor of engagement in social practice that entails learning as an integral constituent" (p.35). Viewing learning as legitimate peripheral participation (LPP) implies that learning is seen as an evolving form of membership, from legitimate peripheral participant to old-timers. One fundamental assumption in introducing LPP is that there *already* exists a community of practice to enter as legitimate peripheral participant. This notion cannot help me with the conceptualization of the development of participation for the participants in our community. Our community of inquiry did not exist before we began to meet in June 2004, and it stopped existing at the end of our last meeting, one year later, in June 2005. Through my study I can address issues related to the *creation* and the *development* of a community of inquiry, and therefore the introduced notions of mutual engagement, joint enterprise, and shared repertoire have to be considered from this perspective.

The theoretical considerations, as presented above, allow me to conceptualize learning in terms of their sociocultural settings, and Wenger's frame could apply to a large context of learning. However, when engaging in the elaboration of the theoretical framework for my thesis, I considered the criteria of *relevance* and *coherence* as fundamental. The issue related to relevance implied that the chosen frame should enable me to address learning considered in a social setting with a particular focus on *mathematical* learning, or learning in a *mathematical context*. On the other hand, the demand of coherence required me to attend to the affordances and constraints of choosing to locate my research in a situated perspective on learning. These challenges are developed further in Chapter 5.

In this section, I have presented the central concepts in my theoretical framework and the way these are related to each other. This theoretical framework allows me, as a researcher, to study and analyze the collaboration between three teachers and a didactician in relation to the development of algebraic thinking. The approach followed in my research is one possible form of collaboration with teachers, and in order to locate my research in the wider research literature I propose, in the next section, an overview of the research literature concerning teachers' professional development where some of the reported research focuses on algebra.

2.4 Literature review on teachers' professional development

My work with the three teachers can be seen as a mode of professional development leading to teachers' learning in terms of change in participation in socially organized activities and growth of knowledge in terms of development of algebraic thinking. I am aware of other possibilities to theorize teachers' professional development (see for example Adler et al., 2005; Krainer, 1994, 1998, 2004; Rowland et al., 2004; Tirosh, 1999; Tzur, 1999, 2001). However, I chose to concentrate this literature review on research studies which follow the same theoretical approach to learning as my own.

As explained in this chapter, my theoretical framework is rooted in a sociocultural approach to learning, and is based on Wenger (1998) and Vygotsky's theories. Based on the ideas of Wenger's (1998) 'community of practice', researchers try to understand the factors influencing teachers' learning and development, as in McGraw, Arbaugh, Lynch and Brown's (2003) research, focusing on the practice of mathematics teaching. Following the perspective developed by Lave and Wenger (1991), this kind of research addresses learning in terms of changes in participation in socially organized activities, and how individual's use of knowl-

edge is seen as an aspect of their participation in social practices. A common feature for studies following this approach is that they define the unit of analysis as being the group in order to account for the teachers' participation in activities of professional development. Research on teachers' professional development reveals that teachers' learning is a complex process involving an interconnection of individual, social, and organizational factors.

Several researchers have used this kind of theoretical approach (Arbaugh, 2003; Grevholm, Berg & Johnsen, 2006; McGraw, Arbaugh, Lynch & Brown, 2003; Sztajn, Hackenberg, White & Allexaht-Snider, 2007; Valero & Jess, 2000), and these studies illuminate different aspects of Wenger's theory. For example, McGraw et al. (2003) present the professional development of mathematics teachers in relation to the development of communities of practice with a focus on the practice of mathematics teaching. Arbaugh's research (2003) has several features in common with my own research: she refers to the idea of "study group", and reports on the professional development of seven teachers and herself, acting as one of the group participants and as a researcher. The study group met regularly during a period of seven months with the aim to "come together on a regular basis to support each other as they work collaboratively both to develop professionally and to change their practice" (p.141). Even if my research does not address this last issue concerning the change of practice, I recognize that several aspects from her conclusion are in accord with the results from my own study, as she emphasizes the high value that the teachers placed on their participation in the study group. Furthermore, she reports on the following aspects of participation that the teachers found most useful to their professional growth: building community and relationships; making connections across theory and practice; supporting curriculum reform; and developing a sense of professionalism. In addition some teachers perceived an increased 'self-efficacy' resulting from their participation in the study group, where 'self-efficacy' is described as "confidence about their own teaching practices" (p.153).

This idea of increasing confidence about both mathematics (algebraic thinking) and teaching practices constitutes a central aspect in the results from my study, and it is also addressed in Sztajn, Hackenberg, White and Allexaht-Snider (2007) as they use the notion of "trust". In their research the collaboration of school-based elementary teachers and university-based mathematics educators is addressed and they see the idea of trust as connected to vulnerability and as "a vital element of well-functioning teachers' learning communities" (p.973).

I understand Koellner and Borko's (2004) research as exploring the way a community of teachers begins to evolve. They report on the establish-

ment of a community, during a summer course, in relation to professional development among middle school mathematics teachers. The course goals consisted of increasing teachers' algebra content knowledge and creating a teacher community or network. In the conclusion of their article, they pointed to several themes which characterize the evolution of the community. Particularly, they report on the role the mathematical tasks played in relation to community building: "it appeared that using tasks that provided access to all participants was important in the co-involvement of the whole group as they solved problems" (p.228). I support completely their conclusion since the recognition of the crucial role played by the mathematical tasks is emerging from my research. Furthermore, they pointed to data analysis indicating that "clarifying and explaining, building off each others ideas, persistence, admitting weaknesses and laughing together were all characteristics that appear to be the ways in which community began to evolve" (p.228). I recognize the importance of these characteristics in building confidence both in mathematics and in the community, and I want to argue that clarification, explanation, building on each other ideas, and persistence are related to acts of inquiry, as presented in Section 2. 2. 6. The centrality of inquiry or "shared mathematical inquiries" is also emphasized in Lachance and Confrey's (2003) research. In their research they report on the use of mathematical content explorations in professional settings as a means aiming mathematics teachers to build professional communities. They conclude by claiming that "shared mathematical inquiries in a professional development setting not only give teachers an opportunity to work with one another and build community, but they also give teachers a means to develop deeper mathematical understandings" (p.131-132).

The identification of the factors supporting the development of trust within the community is addressed in the research of Sztajn et al. (2007). According to these researchers, three aspects emerged from their study: the professionalism of the mathematics educators; the organization of the project; and the establishment of school-university relations. I consider that these three aspects address issues related to the systemic components of a research project and in Valero and Jess' research (2000) the importance of these aspects and the difficulties encountered by professional development initiatives while establishing a community of practice in school are addressed.

In Section 2. 2. 6, I presented the idea of inquiry and explained how it emerged from the analysis of my data. This notion of inquiry constitutes a fundamental aspect and lies at the heart of a project, *Learning Communities in Mathematics* (LCM), which address the cooperation between school and university and is designed to "build communities of inquiry involving teachers and didacticians to develop teaching and en-

hance learning of mathematics” (Jaworski, 2004, p.1-29). Agreements with 7 schools were established, involving a group of at least 3 teachers from each school. The research project included schools from primary schools to upper secondary. During a series of workshops, teachers and didacticians explored together into what inquiry looks like in mathematics learning. Furthermore, it was the teachers’ responsibility to build on these ideas and to develop them further in the school context. One of the research questions concerned the learning outcomes for all the participants: “How does the thinking of all of us develop through our joint activity?” (p.1-30). The LCM project has many features in common with my own project, as the collaboration between teachers and didactician and activities organized as workshops. However, I recognize that there are fundamental differences: as I mentioned above, the idea of inquiry lies at the heart of the LCM project as the aim is to build communities of inquiry, while this idea emerged from my research. Another difference concerns the issue related to the learning outcomes for the participants: In the LCM project the learning outcome of the didacticians is addressed from the beginning, while during my research I gradually recognized the importance of my own learning outcome.

The collaboration between mathematics teachers and mathematics teacher educators is also reported in Zaslavsky and Leikin’s (2004) research as they consider an in-service professional development program for junior and senior high school mathematics teachers. However, their research focus is on the processes encountered by the teacher educators and more specifically on the growth of mathematics teacher educators through their practice. In order to conceptualize the teacher educators’ development, they elaborated a three-layer model based on Jaworski’s (1992, 1994) teaching triad and Steinbring’s (1998) model of teaching and learning as autonomous systems. As part of their conclusion they mentioned the challenge consisting of designing specific mathematical tasks and learning activities which are equally applicable to the students as well as to the mathematics teachers. As mentioned earlier, this aspect was central in Koellner and Borko’s (2004) research. The importance of the role played by the mathematical tasks which are presented to the mathematics teachers is strongly emphasized in my study, as it seems that the nature of the mathematical task might influence the way the teachers participate in the workshops. This result is in accord with previous research, as emphasized in Zaslavsky, Chapman and Leikin (2003) and Leikin’s (2004) researches. In their conclusion, Zaslavsky et al. (2003) write:

The professional growth of mathematics educators – including teachers, teacher educators, and educators of teacher educators – is an ongoing lifelong process of a dual nature. It occurs through direct and indirect learning, often by reciprocally

switching from acting as a learner to facilitating learning for others. In this process tasks play a critical role. (p.912)

I agree with their conclusion and I find it relevant for my study since the professional growth of the three teachers and myself can be described as a process switching from acting as learners during the workshops to facilitating learning for others (for the teachers addressing their thinking in relation to their practice in the classroom, for me during the design of the tasks). In my study, as in the studies just reported above, the choice, the design, and the way we all engage in the tasks play a central and critical role. The central role played by the mathematical tasks is also emphasized in Paul Cobb's (2000) study, as one of four aspects of the classroom learning environment which are critical in supporting students' mathematical development. As Cobb remarks, in order to be consistent with a situated perspective, the students' activity and learning is addressed in terms of overall goal or motive for their activity, which was, in that case, to search for trends and patterns in data. There is another aspect with Cobb's research which is in common with my own work: he refers to a shift in theoretical orientation, from following a constructivist psychological approach to learning to adopting a situated approach to problems related to mathematical learning and teaching. Furthermore, he emphasizes that the theoretical framework which is used in the reported research (Cobb, 2000) was not decided in advance, in a kind of top-down manner. The choice of the situated approach was pragmatic as a possible frame to address questions related to classroom-based work. Similarly, I did not decide in advance to adopt a situated learning approach. As explained in the introduction, I developed further the ideas emerging from the Mathias-project and wanted to study, as a researcher, the processes behind our collaboration. I decided to locate my research within a situated perspective since I believed this frame would offer me suitable theoretical tools.

The main focus in my study is placed on the development of algebraic thinking, and other researchers have combined teachers' education with a particular focus on algebra. However, even if the reported studies below share the same focus as my own research, the theoretical frame is different.

Agudelo-Valderrama, Clarke and Bishop's (2007) research reports on secondary mathematics teachers' conceptions of beginning algebra and on their conceptions of their own teaching practices. The study reveals that the teachers did not consider their own conceptions of mathematics as the crucial determinant in relation to their teaching practices. Instead, they pointed to systemic considerations as the main constraints directing classroom activities and explaining the difficulties related to introducing changes in their teaching. As in Sztajn, Hackenberg, White and Alex-

saht-Snyder's (2007) research, the idea of 'trust' is mentioned in their conclusion as the establishment and deepening of relationships between the participants in relation to the development of mathematics content knowledge.

While in my study learning is addressed in terms of participation in socially organized activities, Hough, O'Rode and Terman (2007) use the idea of concept maps to assess change in teachers' understanding of algebra. During a two-week summer course, the participants, engaged in project-based mathematics activities and investigations. The goal for this course was to develop the teachers' mathematics and pedagogical knowledge, and to increase their capacity to teach mathematics. The authors claim that they observed change in the participants' understandings of algebra, as a result of their participation in the project's activities. They described these changes in subject matter knowledge in terms of breadth, depth, and connectivity.

I consider that a central issue in relation to the development of teachers' knowledge is to consider how development is related to ownership. According to Jaworski (1999), three actors play an important role in the various teacher education programmes: teachers as pupils, participants, and partners with didacticians in the developmental process. Arbaugh (2003) discusses this issue in relation to the use of study group with mathematics teachers. She questions the possibility for teachers to become self-directive in their professional development, and considering her own role as a group participant and researcher, she asks: "how important is the "expert other" in a study group to the teacher involved?" (p.160). As Jaworski (1999) remarks, ownership differs since "pupils have little ownership, participants some, partners a great deal" (p.117). In my study I do not address the implications of the teachers' development of algebraic thinking on pupils' learning. However, a way to compensate for these inequalities in ownership between participants and partners might be to allow for teachers and educators/didacticians to work collaboratively (Jaworski, 1999). I argue that my study offers an example of such collaboration between three teachers and a didactician where each contribute to and learn from the processes and practices involved in the mathematical workshops and interviews.

In this section I presented a literature review over mathematics teachers' knowledge development. The next section addresses the way I understand algebraic thinking and how it has been addressed in the research literature.

2.5 Addressing algebraic thinking through the Vygotskian ideas of mediation and scientific concepts

In the previous sections I introduced Wenger's work and explained in which ways the ideas of *meaning*, *practice*, *community*, and *identity* are conceptualized in my study. Furthermore, I addressed *learning as developing confidence*, and *learning through inquiry processes*. Against this theoretical background I am able, as a researcher, to address and analyse learning processes in general terms. However, my study concerns the development of algebraic thinking and therefore I consider that the theoretical framework, as presented above, does not go far enough to help me, as a researcher, to address the *specificity of mathematical learning* and more particularly the *development of algebraic thinking*. This is the reason why I argue for the necessity of elaborating further my theoretical frame in order to capture learning as it occurs in the particular social setting of our workshops. In that sense, my aim is to elaborate a theoretical frame which enables me to address the specificity of mathematical learning. In addition I need to take into account the criteria of *relevance* and *coherence*. In order to address these goals, I propose to go back to Vygotsky's work and to establish a link between the Vygotskian ideas of *mediation* and *scientific concepts*, and Wengers' notion of *negotiation of meaning*. I believe that by going back to Vygotsky's work, I am in a position to elaborate a theoretical framework which is both relevant and coherent, and which goes behind and expands Wenger's work. I claim that the proposed frame is *relevant* as I am now able to point to the specificity of *mathematical learning*. Furthermore, it is *coherent* since, even though Wenger does not refer explicitly to Vygotsky's work, his theoretical perspective is firmly rooted in a sociocultural perspective on learning, a theoretical perspective within which Vygotsky was one of the first to make sense of the social, cultural, and historical context of learning. I believe that by elaborating my theoretical framework in this way I am in a better position in order to provide a suitable theoretical framework for conceptualizing processes related to the development of algebraic thinking and mathematical learning within a sociocultural approach to learning.

In this section I introduce the idea of *mediation* and explain how this concept might be understood in an educational setting and its relation to *scientific concepts*. Further I explain the way I see the link between Vygotsky's idea of mediation and Wenger's notion of negotiation of meaning. Towards the end of this section I present how algebraic thinking is understood in this study.

2.5.1 Mediation and scientific concepts

As explained earlier (see Section 2. 1), the three basic themes in a Vygotskian sociocultural approach to mediated action are a reliance on genetic, or developmental analysis, the claim of the social origin of higher mental functioning, and the claim that human action is mediated by tools, including both psychological and technical tools (Wertsch, 1985, 1991; Wertsch et al., 1995). A significant feature concerning these mediational means is that they both shape the action in essential ways and depend on the milieu within which the action is carried out (Wertsch, 1991). The consequences for my study are the following: the choice and use of mediational means, such as language, mathematical tasks and algebraic symbolism, have an influence on the actions our group will perform during the different workshops, and their choice and use are deeply context related. Using James Wertsch's (1991) words: "it is meaningless to assert that individuals "have" a sign, or have mastered it, without addressing the ways in which they do or do not use it to mediate their own actions or those of others" (p.29). A consequence of Wertsch's quotation is that I, as a researcher, need to take into account both the kinds of algebraic symbolism the teachers introduce *and* the purpose with introducing these, that is, how the symbols act as mediational artefacts between the teachers and algebraic thinking. This aspect is particularly important in the analysis of Workshop II (see Section 4. 1. 2).

In his analysis of the social processes underlying individuals mental functioning, and especially in his account of intermental processes, Vygotsky focused on small group interaction as found in the adult-child dyad. However, I consider that his interest on the specific sociocultural setting of formal schooling is of particular interest for my study. Here Vygotsky distinguished between two forms of experience, giving rise to two different, but interrelated, kinds of concepts: the *spontaneous* or *everyday concepts*, and the *scientific* or *theoretical concepts*. Spontaneous concepts emerge from the child's own reflections on everyday personal experience and "the weak aspect of the child's use of spontaneous concepts lies in the child's inability to use these concepts freely and voluntarily and to form abstractions" (Vygotsky, 1986, p.148). Still, spontaneous concepts play an important role in the child's learning process as a foundation for the acquisition of scientific concepts. In fact these two processes are deeply interrelated as "the development of a spontaneous concept must have reached a certain level for the child to be able to absorb a related scientific concept" (1986, p.194). In contrast, scientific concepts are understood as "the generalization of the experience of humankind that is fixed in science, understood in the broadest sense of the term" (Karpov, 2003, p.66). Vygotsky (1986) emphasized the importance of

the role played by instruction in scientific concepts in the child's mental development as:

Scientific concepts, with their hierarchical system of interrelation, seem to be the medium within which awareness and mastery first develop, to be transferred later to other concepts and other areas of thought. Reflective consciousness comes to the child through the portals of scientific concepts. (p.171)

According to Vygotsky (1986), the appropriation of scientific concepts evolves "under the condition of a systematic cooperation between the child and the teacher" (p.148). Here the use of "precise verbal definitions" (Karpov, 2003, p.66) or "initial verbal definition" (Vygotsky, 1986, p.148) by the teacher is central in the learning process. Vygotsky (1986) himself also acknowledged that "the difficulty with scientific concepts lies in their *verbalism*, i.e., in their excessive abstractness and detachment from reality" (p.148-149). Algebraic thinking and mathematical thinking more generally are examples of scientific concepts which need precise verbal definitions. Here the concepts of mediation and psychological tools play a central role in relation to the process of appropriation. Through the concept of mediation the focus is on "the role played by human and symbolic intermediaries placed between the individual learner and the material to be learned" (Kozulin et al., 2003, p.3). Psychological tools are defined as "those symbolic systems specific for a given culture that when internalized by individual learners become their inner cognitive tools" (p.3). This point is developed further by Alex Kozulin (2003) explaining that "psychological tools are those symbolic artifacts – signs, symbols, texts, formulae, graphic organizers – that when internalized help individuals master their own natural psychological functions of perception, memory, attention, and so on" (p.15-16). As such, psychological tools function as a bridge between "individual acts of cognition and the symbolic sociocultural prerequisites of these acts" (Kozulin, 1998, p.1).

These ideas have been developed further by Vygotsky's followers whose main innovations have been related to the contention that appropriation of psychological tools, such as scientific concepts, requires not only the appropriation of a certain verbal knowledge, but also the mastery of relevant procedures. The next innovation of Vygotsky's followers concerns their elaboration of the differences between spontaneous and scientific concepts which emerge from different types of learning (Karpov, 1995, 2003; Sierpinska, 1993a). According to Yuriy Karpov and H. Carl Haywood (1998), Russian followers of Vygotsky have designed a large number of instructional programs based on the ideas of an approach called *theoretical learning approach*. This approach is based on:

... students' acquisition of methods for scientific analysis of objects or events in different subject domains. Each of these methods is aimed at selecting the essen-

tial characteristics of objects or events of a certain class and presenting these characteristics in the form of symbolic and graphic models. (Karpov, 2003, p.71)

Jean Schmittau (1993), an American researcher who studied Russian elementary schools students, reported on the fact that children understood mathematics concepts at their most abstract level and she advocated for the role played by *pedagogical mediation* as necessary for individual appropriation of scientific concepts. According to her, “pedagogical mediation must facilitate the appropriation⁶ of the scientific concept through a mode of presentation that reflects the objective content of the concept in its essential interrelationships” (p.34). I understand Karpov and Haywood’s (1998) notions of *metacognitive mediation* and *cognitive mediation*, in relation to the main mechanism of children’s learning and development, as a further elaboration of Schmittau’s (1993) idea of pedagogical mediation. The former refers to the acquisition of semiotic tools of self-regulation, such as self-planning, self-monitoring, self-checking, and self-evaluating, while the later refers to the acquisition of scientific concepts representing the core of some subject-matter.

I believe that this conceptualisation of the idea of mediation, in the specific sociocultural setting of education, is relevant for my study since this distinction offers me the opportunity to articulate, in theoretical terms, the different aspects which emerged through the analysis of my data. Thereby, the notions of metacognitive and cognitive mediation enable me to distinguish the processes related to *what* our group is engaged in (addressing different scientific concepts through *cognitive mediation*), and *how* our group is acting during these meetings (addressing the changes in the participants’ modes of participation through *metacognitive mediation*). An important feature of these two types of mediation, as they emerged from the analysis, is that these are deeply interrelated and interwoven within our group’s practice and it is important to comprehend these aspects holistically.

2.5.2 Theoretical learning in a community of learners

Even though I am able to recognize these two aspects (metacognitive and cognitive mediation) in my study it does not mean that they are present all the time. According to Karpov et al. (1998), these two aspects of mediation have been emphasized differently in various approaches to instruction. For example, American researchers have developed a medi-

⁶ Some researchers use the term *acquisition* (Karpov, 1995, 2003; Karpov et al., 1998), others use *appropriation* (Schmittau, 1993, 2005), and some use both terms (Kozulin, 2003). Leont’ev (1981) used appropriation in the following sense: “The child does not adapt itself to the world of human objects and phenomena around it, but makes it its own, i.e. appropriates it” (p.422). According to Sfard (1998), both acquisition and appropriation belong to the acquisition metaphor (talk about states), while the participation metaphor use rather terms like knowing, practice, communication (talk about activities). In her article, she argues strongly for using both metaphors.

ated-learning approach explicitly based on Vygotsky's ideas of metacognitive mediation, and have designed innovative programs for teaching academic subjects in school (Brown, Campione, Reeve, Ferrara, & Palincsar, 1991; Cobb, Yackel, & Wood, 1992; Cobb, Wood, & Yackel, 1993; Schoenfeld, 1985, 1992). On the other hand, the main innovations of Vygotsky's Russian followers have been the recognition that the acquisition of psychological tools, including scientific concepts, involves the acquisition of a certain verbal knowledge accompanied by the mastery of relevant procedures, and the elaboration of the idea concerning the differences between spontaneous and scientific concepts. Particularly, their research has shown that the acquisition of spontaneous concepts results from *empirical learning*, while scientific concepts result from *theoretical learning*.

By adopting a critical scrutiny of the different approaches followed by both American and Russian researchers, Karpov et al. (1998) and Karpov (2003) reveal the weakness of each of these. The former emphasized two principal ideas: the first idea is that the course of instruction should be organized as students' cooperative, shared activity. The second idea, referred to as guided discovery, concerns the fact that scientific knowledge should not be taught ready-made, but rather should be constructed by students themselves. According to Karpov et al. (1998), this approach is highly relevant to the Vygotskian idea of metacognitive mediation. However, following a guided discovery approach is in contrast to the Vygotskian principle of teaching scientific concepts as a means of cognitive mediation in school.

Concerning Russian researchers, the theoretical learning approach had been considered as the best alternative to traditional school instruction. The main idea is to develop a high level of mastery and maintenance of scientific concepts, and intentional use by students (Karpov, 1995; Schmittau, 1993). According to Karpov et al. (1998), theoretical learning is an effective way of mediating students' cognitive processes. However, it seems that this approach has underestimated the role of students' collaborative problem-solving activity, and thereby the Vygotskian idea of metacognitive mediation.

In order to compensate these weaknesses, Karpov et al. (1998) propose to develop an instructional procedure which combines the strong features of the American and the Russian approaches. This new procedure, which they labelled *theoretical learning in a community of learners*, is based on "supplying students with methods of scientific analysis in subject domains leading to development of relevant scientific knowledge" (p.34). Furthermore, "students would master these methods by using them for solving concrete problems in the course of collaborative activity in which they would take turns solving a problem and planning,

monitoring, checking, and evaluating the process of the problem solving” (p.34).

Even though the aim of my study is not to design a new instructional approach, I consider that the idea of *theoretical learning in a community of learners*, as an instructional approach, has some common features with my own study. My aim, with this research, is to study the development of algebraic thinking within a community of inquiry consisting of three teachers and a didactician. This implies that the participants will have to address, discuss and inquire into a variety of scientific concepts within algebraic thinking. Furthermore, our group does not follow a guided-discovery approach, but rather a kind of “instruction” conducted by a didactician, one of the participants, during each workshop. These aspects are consistent with the theoretical learning approach and cognitive mediation. In addition, the dimension of inquiry implies that this “instruction” related to algebraic thinking is conducted in a community of learners in which all participants are engaged in activities involving questioning, planning, checking, taking initiative, and evaluating the processes of problem solving during each workshop. The analysis of data shows clearly a development within our group concerning *how* the different participants engage in the inquiry during the year, and I consider these aspects as consistent with the idea of metacognitive mediation. Therefore I believe that Karpov et al.’s (1998) idea of *theoretical learning in a community of learners* is relevant for my study, as it enables me, as a researcher, to operationalize the Vygotskian concept of mediation within the specific sociocultural setting of my study.

2.5.3 Establishing a theoretical link between Vygotsky’s and Wenger’s theories

I introduced above the idea of theoretical learning in a community of learners and claimed that this notion was relevant to my study. However, this claim begs the following question: what is the link between the Vygotskian ideas of metacognitive and cognitive mediation and Wenger’s theory, as explained earlier? I believe that the notion of *mediation of meaning* plays a central role here. Using Kozulin’s (2003) terms: “Mediation of meaning is an essential moment in the acquisition of psychological tools, because symbolic tools derive their meaning only from the cultural conventions that engendered them.” (p.26). In other words, during the process of acquiring (or appropriating) psychological tools, the way in which their meaning is mediated plays a central role. In his quotation Kozulin emphasizes the cultural conventions which engendered the symbolic tools. In the case of algebraic symbolism, the cultural conventions refer to the historical development of algebraic notation and the way the community of mathematicians uses it. I want to argue for adding another aspect which, I believe, plays a central role in the process of me-

diation of meaning: the *social setting* within which psychological tools are addressed, discussed, and explored with the aim of acquiring (or appropriating) these. Thereby I propose to take into consideration *both* the importance of the cultural conventions from which the meaning of symbolic tools emerged *and* the social setting within which the symbolic tools are used. This acknowledgement implies for my study that the practice *and* the nature of the mathematical tasks are components of the social setting, and as such play a central role. This recognition is pragmatic as it emerged from the analysis of my data. It is in that sense that I understand Wertsch's (1991) quotation: "it is meaningless to assert that individuals "have" a sign, or have mastered it, without addressing *the ways in which they do or do not use it to mediate their own actions or those of others*" (p.29, my emphasis). I consider that the specific social setting within which actions, and the use of symbolic tools, are situated is addressed in the last part of Wertsch's quotation, and this aspect needs to be taken account of.

As a consequence of these considerations, I can refer in my study, on one hand, to the way algebraic thinking is mediated, during the year our collaboration lasted, through metacognitive and cognitive mediation. On the other hand, I can refer to what is happening during each workshop and to the way the meaning of each mathematical task is negotiated between the participants. Using Lerman's (1998b) metaphor of "the zoom of a lens", by *zooming in* on what is happening between the participants during each workshop as they inquire into the mathematical tasks, Wenger's ideas of *negotiation of meaning, practice, community, and identity* act as theoretical tools describing the particular social setting within which the activity is situated. By *zooming out* and looking at what is happening between the participants during the year, the Vygotskian ideas of *mediation of meaning, metacognitive and cognitive mediation* act as theoretical tools describing the general cultural conventions and social settings within which the activity is situated, the acquisition of semiotic tools of self-regulation, such as self-planning, self-monitoring, self-checking, and self-evaluating, and the acquisition of scientific concepts representing the core of some subject-matter. I consider that these two levels, addressed through Vygotsky and Wenger's theories, are deeply related to each other, and through my study, I try to highlight both.

In the beginning of this section I argued for the need to go behind and expand Wenger's theory in order to be able to elaborate a relevant and coherent theoretical framework which would enable me to address *mathematical* learning, and more specifically the development of algebraic thinking within our community of inquiry. By going back to the Vygotskian ideas of mediation and scientific concepts and by combining

these with Wenger's theory, as explained above, I am now in a better position to present a suitable theoretical framework for my study. I believe that the elaborated theoretical perspective provides a *relevant* and *coherent* frame for conceptualizing processes related to the development of algebraic thinking and mathematical learning within a sociocultural approach to learning.

2.5.4 Addressing algebraic thinking

I am now in the position to present what I mean by algebraic thinking. In order to do this I go back to Vygotsky's idea of psychological tools. According to Vygotsky (1978, 1986), psychological tools are those symbolic artefacts, which consist of signs, symbols, texts, formulae, and that help individuals to master their own natural psychological functions, as for example memory, perception, and attention. In that sense, psychological tools might be conceived as "a bridge between individual acts of cognition and the symbolic sociocultural prerequisites of these acts" (Kozulin, 1998, p.1). As mentioned earlier, Vygotsky's followers elaborated the differences between spontaneous and scientific concepts, and claimed that these emerged from two different types of learning. In relation to the process of appropriation of psychological tools, a process which requires a different learning paradigm from the acquisition of empirical knowledge, Kozulin (2003) explains:

This learning paradigm presupposes (a) a deliberate, rather than spontaneous character of the learning process; (b) systemic acquisition of symbolic tools, because they themselves are systemically organized; (c) emphasis on the *generalized* nature of symbolic tools and their application. (p.25, my emphasis)

I consider that the deliberate aspect of the learning process has been addressed earlier through the emphasis on cognitive mediation (Karpov et al., 1998; Karpov, 2003; Kozulin, 2003) as one aspect of theoretical learning in a community of learners.

I turn now to the last aspect emphasized by Kozulin: the generalized nature of symbolic tools and their application. In relation to the development of scientific concepts in childhood, Vygotsky (1986) remarks that "at the earlier stage certain aspects of objects had been abstracted and generalized into ideas of numbers. Algebraic concepts represent abstractions and generalizations of certain aspects of numbers, not objects, and thus signify a new departure – a new higher plane of thought." (p.202). The importance of abstractions and generalizations of certain aspects of numbers is also emphasized by Anna Sierpinska (1993a) as she claims that "algebraic thinking develops upon the arithmetic thinking and transcends it through generalization" (p.105-106). She develops further her view in the following quotation:

Algebraic thinking is based on the generalization of one's own arithmetical operations and thoughts and is, therefore, characterized by free acting in and on the arithmetical domain. In algebra, arithmetic expressions can be transformed,

combined according to the general laws of arithmetic operations and not just calculated, “executed” as in the frame of arithmetic thinking. Operations are independent from the particular arithmetic expressions they are involved in. For an arithmetically thinking schoolchild $2+3$ is 5, period. For the algebraically thinking adolescent $2+3$ is a particular case of $a+b$, where a, b are any real numbers. For the algebraically thinking adolescent, *arithmetical operations are special cases of the more general algebraic notions*. (p.106, my emphasis)

I recognize that my own understanding of algebraic thinking has evolved and developed during the research process (see Chapter 5) and, as a result, I am able to present a conceptualization of algebraic thinking which is in accord with Sierpinska’s claims. I use the idea of algebraic thinking in the following sense: *By addressing and developing algebraic thinking, I mean to focus on the need, the choice, the introduction, the use and the meaning attributed to algebraic symbolism and on the way these various components of algebraic thinking are addressed and negotiated within our community through inquiry acts.*

When I refer to ‘the need’ for algebraic symbolism, I want to emphasise the discovery, exploration and investigation of patterns, aiming to grasp and express some *structure*. I am in a position, today, where I can recognise the importance of these steps as part of algebraic thinking, and this recognition is part of my own development, both as a didactician and as a researcher (see Chapter 5). Thereby, I understand the idea of addressing and developing algebraic thinking as consisting of two steps: first an *exploratory step* aiming to inquire into a specific mathematical domain (numerical patterns, geometrical relations), and second a *recording step* within which one tries to express the observed *structure*. During this second step, one can use words or/and symbols (idiosyncratic or algebraic) in order to express the patterns or structure observed in the first step. This recognition implies that *one is actually engaged in algebraic thinking while searching for patterns and structure*. As such, my claim supports and expands Sierpinska’s (1993a) argumentation as she considers algebraic thinking as “based on the generalization of one’s own arithmetical operations and thoughts”.

As emphasized by Sierpinska (1993a), arithmetical operations are now considered as special cases of more general algebraic notions, and *not* as a previous and distinct step before the introduction of algebra. This view is in accord with Analucia Schliemann, David Carraher, and Barbara Brizuela (2007) as they advocate for rethinking the relationship between arithmetic and algebra, and for taking a radically different view concerning what arithmetic and elementary mathematics are about. According to these authors “the key idea behind this view is that arithmetic is a *part* of algebra” (p.xii). Therefore, it is possible to re-define arithmetic as the part of algebra which deals with particular numbers and meas-

ures and treat these as instances or generic examples for more general classes of mathematical objects.

However, according to Carolyn Kieran (1989a), in order to be able to recognize the algebraic nature of arithmetical operations, it is necessary to emphasize the central role played by algebraic symbolism. Kieran, while emphasizing the importance of the introduction of algebraic symbolism, argues that the activity of generalizing is *not* sufficient to characterize algebraic thinking:

I suggest that, for a meaningful characterization of algebraic thinking, it is not sufficient to see the general in the particular; one must also be able to express it algebraically. Otherwise we might only be describing the ability to generalize and not the ability to think algebraically. Generalization is neither equivalent to algebraic thinking, nor does it even require algebra. For algebraic thinking to be different from generalization, I propose that a *necessary* component is the use of algebraic symbolism to reason about and express that generalization. (p.165, my emphasis)

I argue that this study offers me, as a researcher, the opportunity to challenge Kieran's position. As explained earlier, I am in a position to recognise and argue for the importance of, what I called, the *exploratory step* as part of algebraic thinking, and, thereby, I do *not* agree with the requirement concerning the necessity of using algebraic symbolism. Here, I understand Kieran's use of 'algebraic symbolism' as referring to the standard algebraic notation used among the community of mathematicians. I rather follow Anna Sfard (1995) who defines algebra as "any kind of mathematical endeavour concerned with generalized computational processes, *whatever the tools used to convey this generality*" (p.18, my emphasis). I understand her definition as a recognition of what I called the *exploratory step* and the *recording step*. Thereby, I argue for moving the focus away from using (or not) standard algebraic notation into being able to grasp patterns and to perceive structure through exploring generic examples. I consider that my previous work on Evariste Galois (Berg, 2002a ; Berg 2002b) offers evidence which supports this understanding of algebraic thinking since Galois used idiosyncratic notation in his First Mémoire despite the fact that Augustin Louis Cauchy had managed to advance the theory of permutations, with the introduction of suitable notation. In my Master thesis, I argued for the possibility that Galois' idiosyncratic notation was one of the reasons why his work was not accepted by the French *Académie des Sciences*. Nevertheless, nobody would claim that Galois was not engaged in deep algebraic thinking and using Wussing's (1984) words "he [Galois] was aware that his tendency to avoid formalisms and computational schemes made it difficult to understand his papers.... Galois' own work stems from an explicit new methodology, and from his deliberate policy of *thinking in terms of structures*." (Wussing, 1984, p.102, my emphasis). Therefore, I

argue for considering the ability to express the observed structure by using standard algebraic notation as a *result* of algebraic thinking and not as a condition *sine qua non* for it.

Thereby, I consider that my research allows me to expand on Sfard's claim and to situate it within in a theoretical setting, where learning is understood as social participation, in the sense that the processes related to the need for algebraic symbols in order to express an observed structure, the negotiation and use of algebraic symbolism to reason about and express generalization are parts of algebraic thinking. Through this research, I am able to address the various aspects of algebraic thinking, and to offer a detailed description of how algebraic thinking is mediated during each of our workshops in terms of negotiation of meaning. In addition, by looking at what is happening between the participants during the year, mediation of algebraic thinking is addressed in terms of meta-cognitive and cognitive mediation.

A clear articulation of what I mean by algebraic thinking, as presented in Figure 2, is the result of a developmental process concerning my own learning through this research. During the activity of our community of inquiry, and during the writing process of my thesis, I have been able to synthesise and make sense of my own thinking about algebraic thinking, and now I am able to present and to link the different components of algebraic thinking. The details concerning my own learning processes are presented in Chapter 5. A crucial feature of my study is that this model concerning algebraic thinking was *not* available when I was designing the mathematical tasks and preparing for the workshops with the teachers in the beginning of my study. It appears much later in my study. In Figure 2, I present the different components of algebraic thinking, as how these are related to each other.

Starting with a mathematical task, our group can engage within it either by looking at numerical examples or a figure/diagram, or by using algebraic symbolism. In the first alternative, the aim of working with several numerical examples (*operational component*) is to look for some *generic example* in order to uncover some patterns and structure (Rowland, 2000), which “involves making explicit the reasons for the truth of an assertion by means of operations or transformations on an object that is not there in its own right, but as a characteristic representative of the class.” (Balacheff, 1988, in Rowland, 2000, p.39; Mason & Pimm, 1984). I refer to looking at a generic example “as a characteristic representative of the class” as the *structural component* of algebraic thinking. I consider that both the operational and the structural components belong to the *exploratory step*.

The next step consists of expressing the observed structure in algebraic terms (*recording step*), which, in my study means to decide for the

choice of symbols through the process of negotiation of meaning. Furthermore, the manipulation of the chosen symbols is addressed in the *formal syntactic component*, and the meaning which the algebraic symbols endorse is referred to as the *formal semantic component*.

In the second alternative, which is engaging within the task by directly choosing and using algebraic symbolism without looking first at some numerical examples, our group has to decide, through the negotiation of meaning, what symbols to use in order to convey the meaning perceived in the task. The next step consists of symbols manipulation (*formal syntactic component*) and interpretation (*formal semantic component*) of the results. It is possible to come back and reconsider the choice of symbols (for example, see the analysis of the student-professor task, Chapter 4).

The process of negotiation of meaning in relation to the choice of symbols is repeated during each workshop, and following this thread during the year, it is possible to trace how our group addresses the different scientific concepts within algebraic thinking (cognitive mediation) and the participants' acquisition of semiotic tools of self-regulation (metacognitive mediation).

As mentioned earlier in this section, Vygotsky (1986) and Kozulin (2003) emphasize the importance of the cultural conventions which give meaning and purpose to algebraic symbolism: "symbolic tools derive their meaning only from the cultural conventions that engendered them." (Kozulin, 2003, p.26). My aim, in the next section, is to offer a brief overview of the development of algebraic symbols by presenting the social-cultural-historical context within which symbolic notation has developed. The activity in our community of inquiry, during the workshop, is embedded in this wider context.

In addition, by offering this brief overview of the development of algebraic notation, my aim is to underline the complex and time demanding processes behind the development of algebraic symbolism. As mentioned in the introduction, I became aware of the importance of the choice of symbols and of their power through my Master thesis about Evariste Galois' first Memoir, and particularly through Galois' idiosyncratic use of permutation notation (Berg, 2002a, 2002b). In a sense, I wanted to make visible and share with the teachers the richness of the historical aspects and of the cultural conventions of algebraic symbolism.

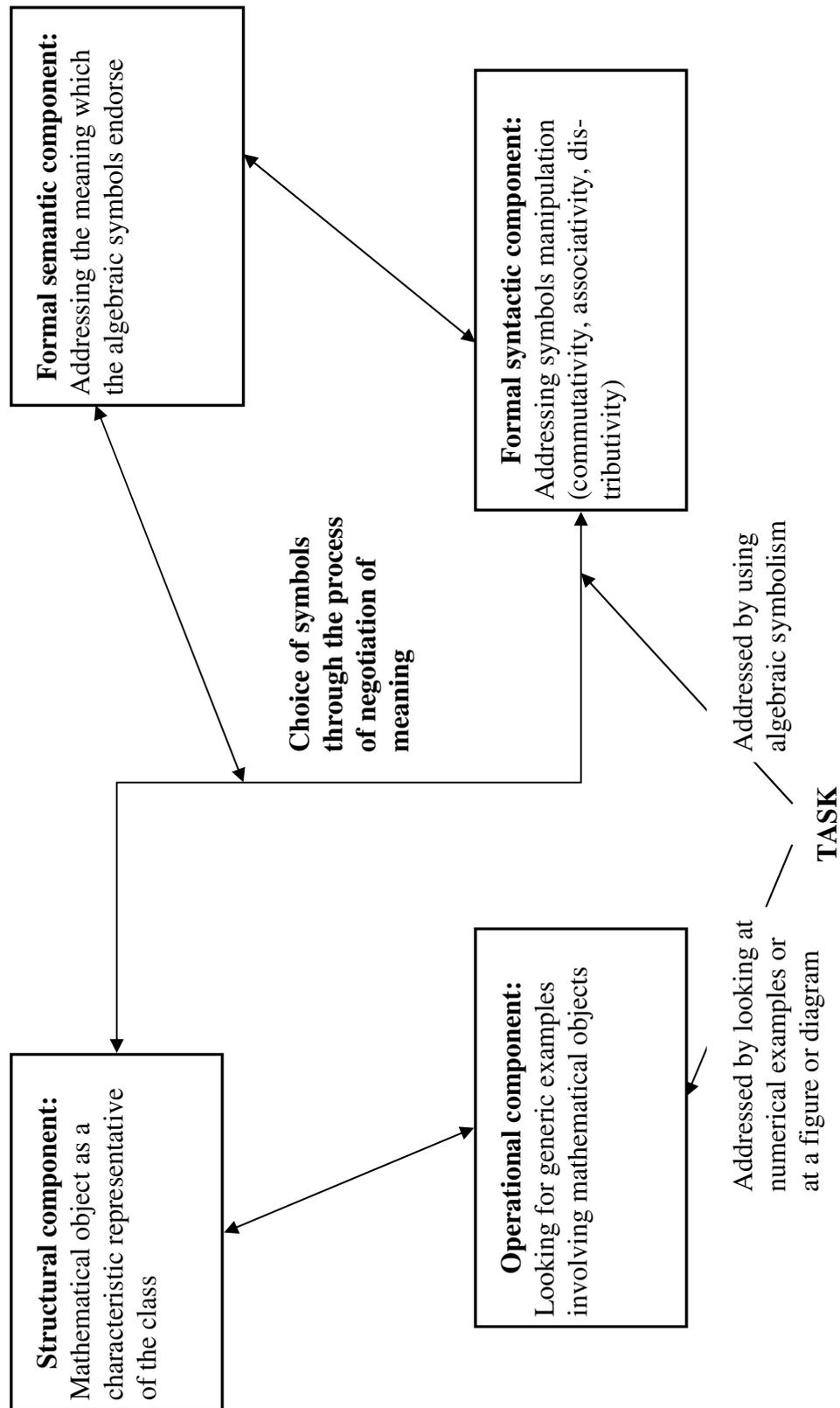


Figure 2: The different components of algebraic thinking

2.6 An historical perspective on the development of algebraic symbolism and literature review on algebra and algebraic thinking

It is generally accepted among historians of mathematics that the historical development of algebra⁷ has passed through the following three important stages: rhetorical, syncopated, and symbolic stages. *Rhetorical algebra* or early stage usually refers to the period before Diophantus (about 250 A.D.), in which all the arguments were written out fully in words and no symbols were available to represent unknowns. Concerning *syncopated algebra* or intermediate stage, is usually defined as the period from Diophantus, with his well-known *Arithmetica*, through to the end of the sixteenth century, and in which some abbreviations are adopted, as the use of letters for *unknown* quantities. Eon Harper (1987) gives the following description of the use of letters:

This procedure was first introduced into mathematics by Diophantus who solved equations in both one and two unknowns, whilst using just one symbol (the second was expressed as a linear combination of the first, e.g. x and $x+4$). Later, different letters were introduced for distinct unknowns. It is, however, important to recognise that the letters always represented *unknown* quantities, so that the algebraist's concern was exclusively that of discovering the true identity of the letter(s), as opposed to an attempt to express the general. (p.77)

It was during the third period of algebra, the *symbolic period* or final stage, that François Viète introduced the use of letters also for *given* quantities. Carl Boyer (1968) explains this breakthrough in these words, comparing Viète's novelty with previous work of Euclid and Diophantus:

Here Viète introduced a convention as simple as it was fruitful. He used a vowel to represent the quantity in algebra that was assumed to be unknown or undetermined and a consonant to represent a magnitude or number assumed to be known or given. Here we find for the first time in algebra a clear-cut distinction between the important concept of a parameter and the idea of an unknown quantity. (p.335)

We have to wait until 1637 and the publication of René Descartes' most celebrated treatise, the *Discours de la méthode pour bien conduire sa raison et chercher la vérité dans les sciences* ("Discourse on the method of reasoning well and seeking truth in the sciences"), to witness further development in symbolism. Presented as one of the three appendices to the *Discours de la méthode*, *La géométrie* is conceived as an illustration of his general philosophical method. Today, according to Boyer (1968), it is possible to follow Descartes' text without encountering difficulties in notation:

⁷ Since a fully history is beyond the scope of this thesis, this overview is based on Boyer (1968), Harper (1987), and Sfard (1995).

The Cartesian use of letters near the beginning of the alphabet for parameters and those near the end as unknown quantities, the adaptation of exponential notation to these, and the Germanic symbols + and −, all combined to make Descartes' algebraic notation look like ours, for of course we took ours from him. (p.371)

Luis Radford (2001) proposes to consider algebraic thinking as a metaphor for the false position method⁸ as in “the influence of false position methods in the emergence of algebraic ideas can be discerned through some important structural similarities between false position reasoning and early algebraic thinking” (p.16). I will come back to this point while discussing the criteria for choosing the mathematical tasks proposed to the teachers (see Chapter 3).

Most research about algebra learning has focused on students' success and failure, and it has provided important information on the way students perform at various ages and what kinds of errors and misinterpretations they typically make. These results are rooted in the traditional view of algebra instruction as following from arithmetic and “pre-algebra” as a transitional stage. Much of the research results concerning algebra learning have focused on students' errors in manipulating equations (Booth, 1984; Kieran, 1981, 1988). For example, the difficulties related to the belief that the equal sign (=) represents a unidirectional operator aiming to produce an output on the right side from the input as written on the left side of the equal sign. Furthermore, other researchers have considered the fact that students do not seem to be able to use symbolic expressions as tools for meaningful mathematical communication (Bednarz & Janvier, 1996; Kieran & Sfard, 1999), or do not comprehend the use of letters as generalized numbers or as variables (Booth, 1984, Küchemann, 1981).

Nadine Bednarz and Bernadette Janvier (1996) distinguished four principal trends in current research and curriculum development of school algebra: generalizing, problem solving, modelling and functions. These different views on algebra can be associated with the various ways the researchers conceive algebra, and which characteristics of algebraic thinking they believed should be developed by the students in order to find algebra meaningful. Zalman Usiskin (1988) proposed a slightly different categorization of perceptions on algebra: as generalized arithmetic, as a study of procedures for solving problems, as a study of relationships among quantities and as a study of structures. Connected to each of these approaches, Usiskin identified different roles of the letter symbols. A number of other characterizations of algebra can be found in the research literature. For example, the National Council of Teachers of

⁸ According to *Regula falsi* or rule of false position, one is to assume a certain value for the solution, perform the operations stated in the problem, and depending on the error in the answer, adjust the initial value using proportions.

Mathematics (1997) proposed four themes for school algebra: functions and relations, modelling, structure, and language and representations. Furthermore, James Kaput (1998, 1999) focusing on algebraic reasoning, identified the following characteristics: generalizing and formalizing, algebra as syntactically-guided manipulation, algebra as the study of structures, algebra as the study of functions, and algebra as a modelling language.

Widely, the research literature on algebra considers the teaching and learning of school algebra as being a major stumbling block in school mathematics. Many researchers have focused on various learning difficulties, and I offer in the following an overview of the main reported problems students meet.

Since my current study concerns algebra and algebraic thinking with teachers at lower secondary school, I concentrate on the early learning of algebra. The introduction of school mathematics usually involves the study of algebraic expressions, equations, equation solving, variables and formulae. Students' difficulties are mainly concerned with the meaning of letters, the change from arithmetical to algebraic conventions, and the recognition and use of structures. As part of the Concepts in Secondary Mathematics and Science (CSMS) project, Dietmar Küchemann (1981) reported on high school students' answers to an algebra test. These answers were classified according to the following six levels: letter evaluated, letter not used, letter used as an object, letter used as a specific unknown, letter used as a generalized number, and letter used as a variable. His research is conducted within a Piagetian framework, and building on Kevin Collis' (1974) research, he offers a hierarchical classification of the use of letter symbols. The learning difficulties associated with the different natures of arithmetic and algebra are addressed by Kieran (1989b) as the meaning of the equal sign (from an order to perform an operation to a relation of equivalence between both sides of the equal sign), and the "lack of closure" as students find difficult to consider expressions like $x + 3$ as a final answer (Collis, 1974).

The persistence of translation problem is addressed in John Clement (1982) and Nicolas Herscovics's (1989) research, and they suggest that this error is caused by the interference of natural language and structural aspects of algebra. Concatenation, which is the juxtaposition of two symbols, is another source of difficulty for the beginning algebra student. Matz (1979) observed that, in arithmetic, concatenation denoted implicit addition, while in algebra it implies multiplication. In relation to solving linear equations, the students' inability to operate with or on the unknown is referred to as the 'cognitive gap' (Herscovics & Linchevski, 1994) or 'didactical cut' (Fillooy & Rojano, 1989). Anna Sfard and Liora Linchevski (1994) consider the rupture between arithmetic and algebra

as an ontogenetic gap caused by the operational-structural duality of mathematical concepts. Sfard (1991) proposes the ‘theory of reification’ according to which the transition from computational operations to abstracts objects is a long and difficult process accomplished in three steps: interiorization, condensation, and reification. These phases form a hierarchical classification where processes on objects become mathematical objects in their own right, which can in turn be manipulated on and be part of a process at a higher level. Eddie Gray and David Tall (1994) have suggested the notion of ‘procept’ or ‘proceptual thinking’ as an intermediate phase between the operational and the structural level. The procept consists of three components: a *process* which produces a mathematical *object*, and a *symbol* to represent each of these.

According to Herscovics (1989), the goal for research on school algebra is to identify these cognitive obstacles and to prepare “teaching outlines, that is, sets of lessons aimed at overcoming specific obstacles.... Such teaching experiments will eventually provide teachers with alternatives presentations that teach to cognitive obstacles instead of ignoring them” (p.83).

For many years students’ difficulties with algebra were seen as a matter of cognitive development. However, recently, Julie Ryan and Julian Williams (2007) proposed to adopt a different perspective and to start from developing the theory and practice of teaching and aiming to elaborate “a theory of mathematics pedagogy, informed by understandings of the particular cultural and historical significance of mathematics” (p.7). Following on their perspective, pupils’ errors and misconceptions are not only inevitable, they are the result of the learner’s engagement in joint activity and “arise from an essential contradiction between the everyday-intuitive conceptions and more advanced mathematical conceptions” (p.153). According to Ryan and Williams, an important assumption about teaching and learning mathematics is that it occurs in classrooms developing inquiry dialogue which they characterise as communities of inquiry. I recognise that my research does not address pupils’ learning in classroom setting, however, I consider that I follow in my study a similar approach where the centrality of inquiry is highlighted.

Likewise, the Russian-based approach developed by Vasily Davydov and his colleagues (Davydov, Gorbov, Mikulina, & Savaleva, 1999) proposed a radically different approach to the teaching of algebra and argued for teaching algebra, including algebraic notation, from Grade 1. Jean Schmittau (2005), following this approach, reports on the implementation of Davydov’s Vygotskian-based elementary mathematics curriculum in the U.S. In her research, three essential characteristics of the Vygotskian approach to the development of algebraic thinking are explored and developed further: the initial development of algebraic con-

cepts from the most generalized conceptual base, the ascent from the abstract to the concrete, and the appropriation of psychological tools. Her approach is rooted in Vygotsky's perspective on algebraic thinking where "algebraic concepts represent abstractions and generalizations of certain aspects of numbers, not objects, and thus signify a new departure – a new, higher plan of thought" (Vygotsky, 1986, p. 202). From this perspective, algebra is no longer introduced as a generalization of arithmetic, but rather "as a generalization of the relationships between quantities and the properties of actions on quantities" (Schmittau, 2005, p. 18). A similar initiative, based on Vygotsky and Davydov's framework, is introduced to pupils at the beginning of primary school in Hawaii (Dougherty, 2001, 2004). The project called "Measure Up" is based on beginning with generalizations rather than specific instances. In this way, children can explore the concepts in action rather than trying to build the bigger picture from multiple specific examples, which is the case when a curriculum introduces first natural numbers. Rooted in Vygotsky's idea of scientific concepts, the project focuses "on real numbers in the large sense first, with specific cases found in natural, whole, rational, and irrational numbers at the same time" (Dougherty, 2004, p.91).

Thereby, Jean Schmittau (2005) and Barbara Dougherty's (2004) research showed what a Vygotskian approach to the learning of algebra means in terms of the pupils' learning. In my study, I do not try to implement a new curriculum approach to algebra based on a Vygotskian perspective, however I consider that the fundamental understanding of the development of algebraic thinking (ascent from the abstract to the concrete), as presented in Schmittau and Dougherty's research is relevant to my study. This consideration begs the following question: What would a Vygotskian approach look like, in terms of the teachers' learning? The answer to this question is directly related to the way I, as a didactician, organized the workshops and selected the mathematical tasks. I presented earlier in this section the different ways algebra and algebraic thinking have been characterized (Bednarz & Janvier, 1996; Kaput, 1998, 1999; National Council of Teachers of Mathematics, 1997; Usiskin, 1988), and I argue that in order to develop the participants' algebraic thinking in our community of inquiry it is crucial to adopt a holistic perspective, that is, to address all, or at least as many as possible, aspects which are relevant to lower secondary school algebra. Rather than referring to an ascent from the abstract to the concrete, I consider a development from the general characteristics of algebraic thinking to the particular aspects, as exemplified through the mathematical tasks I presented to the teachers. This is the view I had, in a more intuitive way, when I selected the mathematical tasks for the workshops. I develop further this aspect in Chapter 3. I claim that developing teachers' algebraic

thinking in this way might avoid a limited view on algebra, as evidenced when teachers tend to emphasise procedural knowledge for solving equations as the core of algebra (Bishop & Stump, 2000; Haimes, 1996; Menzel & Clarke, 1998).

2.7 Summary

In this chapter I have presented Wenger's ideas of meaning, practice, community and identity. In addition I introduce learning as developing confidence and learning through inquiry. I argued that the theoretical frame, as elaborated, enabled me to address learning in general terms, but it did not go far enough to help me to provide suitable theoretical tools in order to pinpoint the specificity of *mathematical* learning. This is the reason why I need to expand and go beyond Wenger's work. Therefore, I propose to go back to Vygotsky's work and more specifically to the ideas of mediation and scientific concepts. The theoretical links between these two theories are understood in the following way: using Lerman's (1998b) metaphor of "the zoom of a lens", I can refer to how algebraic thinking is mediated at a fine grain level, zooming *in* on what is happening during each workshop, through Wenger's ideas of negotiation of meaning, practice, community, and identity. I can also choose zooming *out* and taking a larger perspective while following how algebraic thinking is mediated, during the year of our collaboration, through Vygotsky's ideas of metacognitive and cognitive mediation. Thereby, I argue for considering *learning as transformation of participation in collaborative endeavour* and *inquiry moves/didactical moves* as two poles of a continuum where, by zooming *in*, I can, as a researcher, follow how transformation of participation is achieved through negotiation of meaning using inquiry. Thus I claim that Vygotsky's and Wenger's concepts are complementary, as a kind of two sides of a same coin, and I need both perspectives in order to elaborate a relevant and coherent theoretical framework. The relationship between the ideas of participation and metacognitive and cognitive mediation is developed further in Chapter 4, Section 4.3.

Furthermore, I argued for understanding *algebraic thinking* as consisting of two steps: first an *exploratory step* aiming to inquire into a specific mathematical domain (numerical patterns, geometrical relations), and second a *recording step* within which one tries to express the observed *structure*. During this second step, one can use words or/and symbols (idiosyncratic or algebraic) in order to express the patterns or structure observed in the first step. My aim, by introducing this perspective on algebraic thinking is to move the focus away from using (or not) standard algebraic notation into being able to grasp patterns and to perceive structure through exploring generic examples.

In this chapter, I presented the central ideas within my theoretical framework. My criteria for the elaboration of this chapter were the following: I sought to define and elaborate a *relevant* and *coherent* framework which enabled me to address *mathematical* learning. In addition, I presented my understanding of algebraic thinking. In the next chapter, I present some justifications for the choice of the adopted methodology, the kind of data which was collected in order to answer my research questions, and explain the way these were analysed.

3 Methodology

The purpose of this chapter is to explain and to offer some justification for the choice of the adopted methodology, and to describe the way the data have been collected and analysed. In addition, it aims to make visible my beliefs and underlying assumptions concerning my research project. The chapter is organised in the following way: in the first section I recall the aims of my study and my research questions. In the second section I present the rationale for the design of this study, the intended and the actual realisation of the design. Then I locate my research paradigm within the wider research literature concerning methodological issues, and in this way I seek to address “The Why” for my research (Burton, 2002). Finally, I present the kind of data which was collected in order to address my research questions and explain how these were analysed.

I choose to adopt this structure for the methodology chapter since I want to emphasise the fact that the way I designed my study was inspired by my previous work within the Mathias-project. In other words, I did *not* decide in advance to adopt a particular methodology and then to develop my framework from its characteristics. My approach was pragmatic, that is, the adopted design for my research was inspired by my previous experience, while working collaboratively with teachers, within a developmental project in a primary school.

3.1 Aims of the study

As explained in the introduction chapter (Section 1. 4), the aims of the research were to study the processes related to the development of algebraic thinking, as these emerged from the creation and development of our community of inquiry.

The research questions⁹ addressed in this study, as outlined in Chapter 1, Section 4, are:

- a) *In what ways is the development of algebraic thinking related to the development of our community of inquiry?*
- b) *What relationships can be discerned between teachers developing algebraic thinking during the workshops and their thinking in relation to their practice in the classroom?*

As I indicated earlier (see Section 1. 5), I consider this study as an exploratory research project, more specifically a *developmental research project*, in contrast to a study aiming to confirm or exemplify a particular theoretical perspective, and this view is reflected in the open-ended na-

⁹ The research questions evolved during my research. I offer an overview of this development in Chapter 5.

ture of the research questions. By using the term “exploratory research project” I seek to convey my search for developing an understanding of the relation between, on one hand, the processes behind the creation and development of a community of inquiry, and, on the other hand, the processes behind the development of algebraic thinking. By answering my first research question I am in a position of articulating and making visible *how* working within a community of inquiry enabled the participants to develop algebraic thinking. Furthermore, my aim, with this study, is not to provide definitive answers to these questions. I rather seek to contribute to the development of theoretical knowledge within the development of algebraic thinking and more widely within the area of mathematics teacher professional development.

3.2 Central features of my research

Given the exploratory nature of this research and my aim of studying the development of algebraic thinking, the choice of interpretive research paradigm with a qualitative research approach seemed the most appropriate, and I offer a justification for this choice later (see Section 3. 3. 1).

In this section I present the central features of my research and provide a rationale for the design of my study, the intended and the actual realisation of the design. As mentioned earlier, the design was inspired by my previous experience, as working collaboratively with primary teachers within the Mathias-project. It was not the result of choosing a particular methodological approach.

3.2.1 Rationale for the design of this study

My aim was to study the development of algebraic thinking. This aim could be achieved in different ways, as for example from a cognitive perspective on learning which focused on individual, psychological functioning. Since I wanted to build on my previous experience in working collaboratively with teachers in order to develop further the insights I got from Mathias-project, and especially what I meant by “what does it mean to work with mathematics teachers?” (see Section 1. 2), I elaborated a developmental and analytical framework during spring 2004. This framework, acting as a methodological tool, enabled the three teachers and myself to work collaboratively and to address algebraic thinking. In Chapter 2, I argued for considering our group as a community of inquiry within which I acted both as a didactician and as a researcher. As a didactician, I had didactical aims (I develop further this issue in Section 3. 2. 2), while as a researcher, my aim was to research on *both* the participants’ development of algebraic thinking *and* the development of our community of inquiry. I argue for considering these two aspects as fundamentally interdependent and mutually constituent. Originally, my plan was to study the *teachers’* development of algebraic

thinking, a reflection on my own development of algebraic thinking emerged gradually as a result of my research (see Chapter 5).

During the elaboration of the design of my study, my primary concern was to develop a framework that would enable me to work collaboratively with the teachers. In that sense, the elaboration of the developmental and analytical framework which was developed during spring 2004, was not guided by a particular methodological approach. Therefore, I have decided to present, in the following sections, the main characteristics of the design of my study first and then to identify and to locate my study within a methodological frame in relation to those widely discussed in the methodological literature.

In this section I presented the rationale for the design of my study. In the next section I present the intended framework and explain the reasons why the actual realisation was different.

3.2.2 The six-step developmental and analytical framework

The developmental and analytical framework which was briefly introduced in Section 1.6, became gradually elaborated during spring 2004, and was finished before I contacted the three teachers in May - June 2004. I refer to this framework as the *intended framework*. During the first months of our collaboration, the teachers and I adapted the framework in order to fit the constraints we met (see below). I refer to this framework as the *implemented framework*. I understand this process as related to design research (Wood & Berry, 2003), where the originally framework might be considered as a first prototype which “has been tested, implemented, reflected upon and revised through cycles of iterations.” (p.195).

During our first conversation, the three teachers and myself, I explained my situation as a researcher interested in working collaboratively with teachers, and having the development of algebraic thinking as a research aim. Thereby, I want to emphasise the fact that both the focus of my research (the development of algebraic thinking) and the six-step framework were decided by me *before* I contacted the three teachers. As mentioned above, I understand my research in terms of a *developmental research project*. This implies that in order to be able to address and research on development, I had to design a framework which would enable the participants to work collaboratively during the school year and develop algebraic thinking, and, in addition, which would enable me, as a researcher, to observe and analyse that development. Thereby, in an attempt to articulate how design research and developmental research are integrated in my research, I argue that the six-step framework emerged from engaging in design research and it provided me with a methodological tool which both enabled the participants to develop their algebraic thinking, and enabled me to research on that development. As such,

in my research, design research became part of developmental research, and I see these two aspects as deeply interwoven. In other words, since development does not occur in vacuum, I consider that in developmental research, there is a need for designing some kind of tool which enables development to happen, and, as such, I see design research as being integrated into developmental research in my own research. Therefore, I argue for considering my research as being conducted in a design/developmental paradigm.

In the following section, I present first the framework as it was presented to the teachers during spring 2004. Then I explain the reasons why the practice emerging from the framework differs from the anticipated realisation of that framework.

The intended framework

The different steps in the framework can be described in the following way:

1. At the first step, the researcher (myself) presents to the teachers some mathematical tasks related to algebra and studies the way the whole group engages collaboratively and cooperates in undertaking the proposed tasks.
2. During the second step, our group engages in discussions while we share with each other our thinking concerning the proposed task. Here my focus, both as a researcher and a didactician, is on the way algebraic thinking is addressed, how we interact and collaborate with each other, and on the developmental nature of this collaboration.
3. In the third step, the researcher interviews each teacher separately just before a teaching period. My purpose is to get insights into *what* the teacher plans to address during his/her class, *how* this goal will be achieved, from the teacher's perspective, and finally *why* the teacher chose that particularly goal.
4. In the fourth step, the researcher follows each teacher into his/her class and observes his/her practice.
5. During the fifth step, the researcher interviews again the teacher right after the teaching period, seeking for a kind of evaluation of it. In this way, we (the teacher and I) can compare what was the aim for the teaching period with what has been achieved, as described from the teacher's perspective.
6. Finally, during the last step, we all meet again during the next workshop and each teacher has the possibility to share with the other participants his/her experiences related to the observation in class.

The first, second, and sixth steps are referred to as “mathematical workshops”, while the third, fourth, and fifth steps are called “observation steps”. The plan was to follow the teachers during one year.

These different steps are represented in Figure 3.

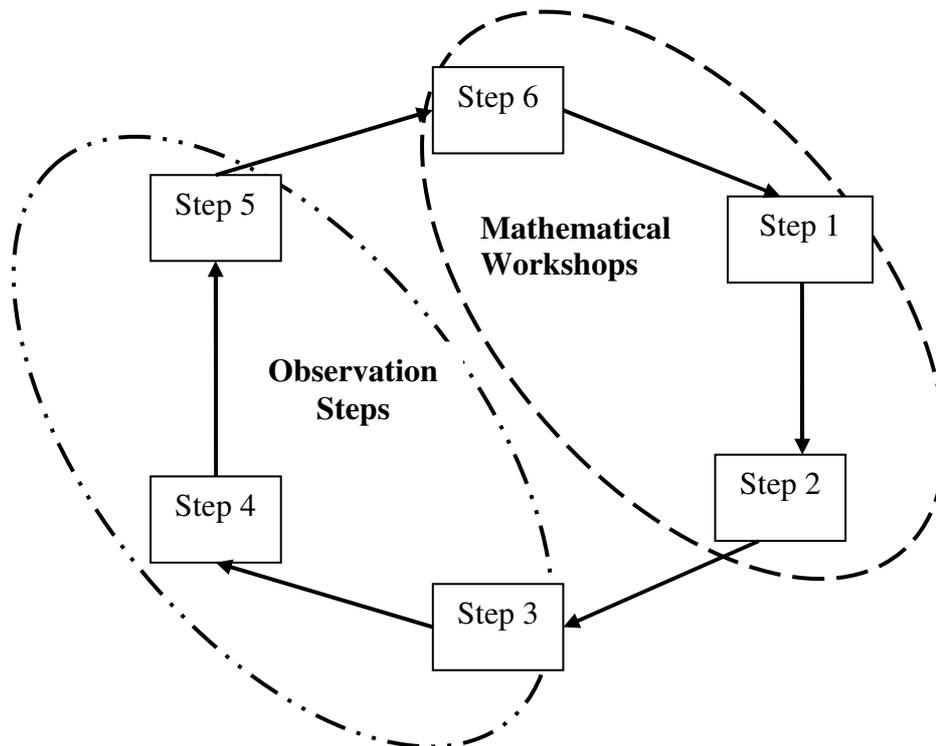


Figure 3: The practice as an anticipated realisation of the six-step framework

The actual realisation of the framework: the implemented framework
 As explained earlier, I elaborated the six-step framework before I started to work collaboratively with the three teachers, and during our first meeting (16.06.04) we discussed how to put these ideas into practice. We agreed to meet regularly for the workshops and decided that once a month was a reasonable schedule since we wanted some continuity in our work. Furthermore, it was during the first meeting that we discussed how to organize the “observation steps”, and we had to recognise that it was not possible to follow the cycle as planned, that is to move through each step and to complete the cycle within one month with organising a workshop and to follow each teacher in his/her class before the next workshop. It was difficult for practical reasons: either it was not possible for the teachers to let me observe in their classes or I was occupied in Kristiansand with doctoral courses¹⁰. Therefore, we decided to continue with the workshops even if we had not been through the observation steps. In that sense, we decided to give priority to the mathematical

¹⁰ During the school year 2004-2005, I was a doctoral student at Agder University College in Kristiansand and had to follow some of the doctoral courses.

workshops, and to meet regularly, once a month. Agreements concerning the “observation steps” were made either during a workshop, as one of the teachers mentioned the possibility for me to follow him/her in the class, or by contacting me by a phone call. As such, it was almost impossible to organise a schedule for the “observation steps” in advance, we had to agree in each case when it was possible both for the teacher and for myself.

These considerations show the difficulties I met as I tried to implement the intended design of my study and, as such, this process illustrates the constraints which a contextualisation of a framework within a particular social setting imposed on me, as a researcher. The recognition of the necessity to adapt the intended design to the actual social setting and the changes which result are part of engaging in design research, as described in Wood and Berry (2003).

Taking into consideration these aspects, the design resulted in a different practice from the intended one, which I call the *implemented framework* or *actual realisation* of the six-step framework. Figure 4 gives a representation of this practice, where the inner cycle, steps 1, 2, and 6 were performed nine times, while I was twice with Mary, once with Paul, and four times with John through the observation steps.

The practice as the actual realisation of the six-step framework consisted of nine workshops, starting in June 2004 and ending June 2005. At the end of each workshop we decided together which date was convenient for the next workshop. During the mathematical workshops, all four participants were always present. If one of the participants was not able to come during one of our meetings, we moved the workshop to another evening. All the workshops happened at Mary and Paul’s school, in the staff room. We were sitting around a table, and I had an audio recorder on the table. Our meetings lasted for approximately two hours, sometime less, usually from 7 p.m.

In the following I offer a description of the practice of our community of inquiry as consisting of ‘mathematical workshops’ and ‘observation steps’, and present its main characteristics.

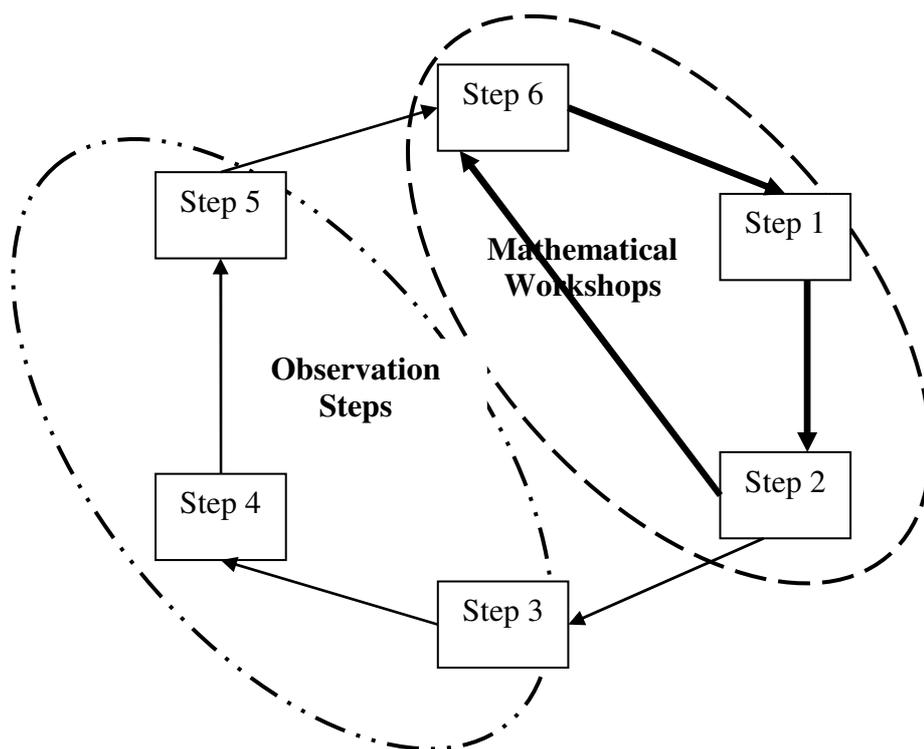


Figure 4: The practice as the actual realisation of the six-step framework

The mathematical workshops

During the workshops, we usually started by recalling and commenting on what we did during the previous workshop. If I had been following one teacher in his/her classroom I suggested that he/she could share some of his/her reflections with the other participants (Step 6). Then I introduced a mathematical task and we all engaged and explored it (Step 1). The last part consisted of sharing with each other our thinking and reflections concerning the offered task. Here my focus, as a researcher, was on the way algebraic thinking was addressed, how we interacted and collaborated with each other, and on the developmental nature of this collaboration (Step 2).

As described above, during the workshops I presented to the teachers a mathematical task within which our group engaged. Before presenting a particular task, I made an *a priori* analysis in order to be able to decide which tasks I could present to the teachers. I present the criteria for the *a priori* analysis later (see Section 3.4.1) and relate these to Koeno Gravemeijer's (1994b) idea of *thought experiment* (see Section 3.3.3). Likewise I made an intuitive and informal *a posteriori* analysis after each workshop, which consisted of writing down my own reflections and impressions on the way the workshop went on. These reflections consti-

tuted the background for the *a priori analysis* of the task for the next workshop. I want to emphasise the fact that the teachers were *not* involved either in the *a priori* or the *a posteriori* analysis of the mathematical tasks, these aspects were only on my own responsibility. I elaborate further on this issue in Section 3. 4. A more formal *a posteriori* analysis was conducted after finishing the process of data collection. The findings, as emerging from the analysis, are presented in Chapter 4.

The observation steps

During the observation steps, I interviewed the teacher just before the class, asking about *what* he/she planned to teach, *how* and *why*. During the teaching period, I was sitting back in the class, focusing on what the teacher did and taking field notes. In addition, I audio recorded what the teacher said during his/her teaching. Right after the teaching period, the teacher and I had an interview where we were able to discuss what happened in class in relation to what was planned, based on my observations and field notes. The interviews both before and after were short since the teachers had further teaching.

Since the aim of my study is *not* to provide an analysis of the teaching which I observed with each teacher I do not present an analysis from my observation in class. These classroom observations were used as a background for the interviews after a teaching period.

During the interviews, the teachers expressed some reflections in relation both to their own current practice in the classroom, and to how they envisage their future practice. These reflections are presented in Chapter 4.

The main characteristics of the practice of our community of inquiry

Considering the practice emerging from the actual realisation of the six-step framework, it is possible to identify a *cyclic* process, mainly between steps 1, 2, and 6, (the steps 3, 4, and 5 did not happen regularly) and referred to as the mathematical workshops where all four participants were involved. This process was *iterative* since we were able to organise nine workshops during the school year. In addition, the process consisting of an *a priori* and an *a posteriori* analysis of each mathematical task which I conducted, alone, before and after each workshop, is also cyclic and iterative since it repeated for each workshop. During this process I engaged both as a didactician, aiming to engage in the development of algebraic thinking, and as a researcher, since I was researching this development. I offer a further elaboration of the relation between development and research in Section 3. 3. 3.

Before addressing methodological considerations, I want to focus on an aspect of my research which I consider as important. I have been used to refer to “I, as a researcher” or “I, as a didactician” without offering a clear articulation of what I meant by these terms. I am now, while engag-

ing in the process of writing my thesis, in a position to look back critically to the complexity of my own role during the school year of our collaboration, and to discern these two roles. I consider that during the process of elaborating and refining the six-step framework (iterative cycles as part of design research) I was acting both as a researcher and as a didactician whose aims were to design a suitable methodological tool which would enable me to address the aim of my research and to work collaboratively with the teachers. Therefore, from the moment the framework was operative and the three teachers had agreed to work collaboratively with me, I was able to act both as a didactician (a developer) choosing pedagogical means in order to achieve didactical aims, and as a researcher researching the processes by which these didactical aims might be achieved. In other words, as a didactician I have a particular didactical aim which I seek to achieve. As a researcher, my aim is to collect and analyse data as a means to address my research questions. Chronologically speaking, I acted both as a researcher and as a didactician from the very beginning of my research, and I still act as a researcher, today, while writing my thesis, while my role as a didactician stopped when the collaboration with the three teachers ended. I offer a deeper articulation of my role as a didactician in Section 3.4.1. I recognise that I am, now, in a position where I can articulate these differences and I understand this recognition is a result of my own learning process, as the ability to get insights within the different layers in my research emerged gradually. In that sense, I did not have the same depth in understanding these different aspects while working collaboratively with the three teachers.

In this section I presented the intended six-step developmental and analytical framework and explained why and how the design developed through iterative cycles, following a design research approach (Wood & Berry, 2003). In addition I was able to characterise the design of my study both as cyclic and iterative. I also looked critically on my own role, acting both as a researcher and as a didactician.

In the next section I propose to look into the wider research literature on methodology in order to locate my own research design within the main educational research paradigms.

3.3 Addressing “The Why” as a justification for “The How”

In addressing methodological issues I wish to keep in mind the recommendations Leone Burton presented in her chapter called “Methodology and Methods in Mathematics Education: Where Is “The Why”?” (Burton, 2002). Methodological considerations are more than just describing the methods the researcher is going to use in conducting his/her research.

In addressing methodological issues, we, as researchers, have to make visible and give some justifications to the reader concerning what kind of theoretical paradigm we choose to work within. Following this approach requires the researcher to address and to look critically at epistemological issues. Engaging in research includes striving to make sense and to understand the world as personally or socially constructed. At the same time it is crucial to be aware that the process of making sense of the world is profoundly influenced by our beliefs about the nature of the reality (Bassey, 1999). This issue is also underlined by Susan Pirie (1998) who claims that “We need to clarify for the rest of our community the cultures from which we are coming and to make explicit the perspectives from which we are viewing the problems we tackle” (p.18). In my research the culture which I came from might be described as the community of mathematicians at Agder University College and more particularly my work with a Master thesis on the topic of Galois Theory. Thereby, my former education was very much focused on working with and reflecting on algebraic structures (groups, rings, fields) where algebraic symbolism was used as a means to represent these structures.

As mentioned before (Section 1. 3.) I described the process of engaging in research as a process consisting of several steps. One of these steps has to consider the way the research project might be located within a particular paradigm. The idea of research paradigm can be defined as “a network of coherent ideas about the nature of the world and of the functions of researchers which, adhered to by a group of researchers, conditions the patterns of their thinking and underpins their research actions” (Bassey, 1999, p.42), or as “a systematic set of beliefs, together with their accompanying methods” (Lincoln & Guba, 1985, p.15). Paul Ernest (1998b), drawing on Thomas Kuhn’s (1970) philosophical analysis of science, proposes the following definition:

With Kuhn’s conception, research is usually understood to take place within a recognized or unconsciously assumed overall theoretical perspective or paradigm. In education, and in social sciences in general, are found multiple research paradigms, each with its own assumptions about knowledge and coming to know (epistemology), about the world and existence (ontology), and about how knowledge is obtained (methodology). (Ernest, 1998b, p.32)

Thereby, following on Ernest’s (1998b) quotation, by adopting a particular research paradigm, the research also takes on the ontological, epistemological and methodological assumptions embedded in that position.

3.3.1 Locating my research paradigm: what are the alternatives?

As explained earlier (see Section 3. 2), I consider an interpretive research paradigm with a qualitative research approach as the most appropriate approach to conduct my research within. In order to capture and contrast the main features of the interpretive research paradigm with other relevant research paradigms, I propose, in this section, to situate this re-

search paradigm within a broader picture consisting of the main educational research paradigms, and to present the main characteristics of each of them.

Both Wilfred Carr and Stephen Kemmis (1986) and Ernest (1998b) refer to the work of Jürgen Habermas and distinguish between three main educational research paradigms: the natural scientific, interpretive (Ernest use the term “qualitative”), and the critical-theoretic research paradigm. Furthermore, these three research paradigms differentiate themselves from each other according to a particular type of interest which underpins the quest for knowledge: in the natural scientific paradigm the focus is on predicting and controlling the phenomena in question, in the interpretive paradigm the focus is on understanding and making sense of these phenomena, while the critical-theoretic paradigm focuses on achieving social justice through an understanding of the phenomena. In the following I present an outline of each paradigm and summarise the main features in Table 2.

Considering the *natural scientific research paradigm*, its central features are: its origin with the scientific methods as employed in the physical sciences or in experimental psychology, it is mainly concerned with objectivity, prediction, replicability, and the purpose of research is the discovery of scientific generalizations which describe the class of observed phenomena. Within this research paradigm the forms of inquiry which are used include survey, comparative experimental, and quasi-experimental methods. This position is based on a “positivist philosophy”, a term introduced by the French writer Auguste Comte, who justified the use of the word “positive” as an “opposition to any metaphysical or theological claims that some kind of non-sensorily apprehended experience could form the basis of valid knowledge” (Carr & Kemmis, 1986, p.61). A weakness of this particular paradigm is that this approach can be insensitive to the contextual setting of human situations. Referring to the methods devised within the scientific paradigm, Pirie (1998) claims that uncritical application of scientific methods may not necessarily produce research results that would be of interest or value to the research community of mathematics education.

The fundamental aspects of quantitative research such as representativity, replicability, and generalizability do not necessarily work in all areas of mathematics education. Furthermore it is important for the researcher not to ignore the affective and social aspects influencing the teachers and pupils. However, the specificity of our research field has to be clearly established and as Pirie (1998) points out:

Our interests lie in the realm of *mathematics* education, and we cannot disregard the influence and peculiar nature of the subject matter, namely, the mathematics, on the teaching and learning that concern us. (p.18)

This was the case in my study. I have recognised the specificity of the subject matter and striven hard in addressing it, through the elaboration of my theoretical framework within which the issues of *relevance* and *coherence* were central (see Chapters 2 and 5).

According to Carr and Kemmis (1986) and Ernest (1998b), the main characteristics of the *interpretive research paradigm* are: its origin in the methodology of sociology and social science research, including anthropology and ethnomethodology; a concern in recording phenomena in terms of participant understandings; and its use of various ethnographic, case study, and largely qualitative methods and forms of inquiry. One of the characteristics of the interpretive research paradigm is its use of case study, where the researcher explores and tries to make sense of the unique features and circumstances around a particular case. This does not mean that the researcher's aim is to value the uniqueness and particularities of a specific case. Rather, by considering a particular case as generic, the researcher makes an attempt to "explore the richness of a particular that may serve as an exemplar of something more general" (Ernest, 1998b, p.34). Thus research following an interpretive research paradigm can be characterised as adopting a bottom-up perspective, starting from an understanding of a specific and concrete case in order to be able to elaborate and formulate hypothesis about the general case. A weakness of this particular paradigm is the problem of how interpretive research findings can be validated (Lincoln & Guba, 1985). Michael Bassey (1999) suggests using the notion of *fuzzy generalization*, which is defined as "general statements with built-in uncertainty". This way of addressing generalization in relation to research based on case study stands in contrast to scientific generalization where there is no place for exceptions. As Bassey (1999) explains it:

It [fuzzy generalization] reports that something has happened in one place and that it may also happen elsewhere. There is a possibility but no surety. There is an invitation to 'try it and see if the same happens for you'. (p.52)

My intention with this current research is to present a "fuzzy generalization" by reporting and making clear to the reader that, based on the particular case of a community of inquiry which I studied, I am in a position to claim that these and these dimensions are central in the development of algebraic thinking. By phrasing the findings in this way, I recognise, as a researcher, that working within an interpretive research paradigm implies an element of uncertainty and I consider that this aspect puts emphasis on the contextualised nature of the research's findings.

Finally, the main characteristics of the *critical theoretic research paradigm* are: its origin in the work of Habermas and the Critical Theory of the Frankfurt School, and its concern to not only understand and describe the observed setting, but also to engage in social critique and to

promote change in order to improve or reform aspects of the social life. This research paradigm is often associated with the use of action research, in particular with “teacher-as-researcher”. A recognition of this paradigm is that difficulties in promoting changes might be due to hidden institutional sources of resistance to change, such as teachers’ and pupils’ ideologies, and research would seek to reveal such factors. Therefore it is crucial to acknowledge these potential difficulties and to address or at least to recognize the central role that these (hidden) dimensions might play during the research process. In that sense, the conclusion that energy and time invested in the research progress might not produce desired outcomes has to be taken in a holistic and critical way. The main features of these different paradigms are summarized in the following table:

Table 2: The main educational research paradigms

Component	Paradigms		
	Scientific	Interpretive	Critical theoretic
Ontology (existence)	Absolutist, objective knowledge	Subjective reality (personal meanings, existence established through social/personal acceptance)	Persons in society and social institutions (personal meanings, existence established through social/personal improvement and emancipation)
Epistemology (assumptions about knowledge)	Mainly quantitative and experimental, involving many subjects and contexts	Personal, constructed or socially constructed knowledge	Socially constructed knowledge through critical inquiry
Methodology		Mainly qualitative case studies of particular individuals and contexts	Mainly critical action research on social institutions
Intended outcome	Applicable knowledge and generalizations	Illuminative subjective understandings	Intervention for social reform, social justice
Interest	To comprehend and improve (through prediction and control) the world	To understand and make sense of the world	To achieve social justice, emancipation

The above-mentioned characteristics (developed from Carr and Kemmis, 1986 and Ernest, 1998b) are also found in Bassey's (1999) work where the distinctions made between *positivist* and *interpretive* paradigm corresponds to Ernest's (1998b) *scientific* and *qualitative* research paradigms. A comment has to be added concerning this presentation of these different research paradigms. It could be concluded that the distinctions between these are clearly established and that one can mainly talk about scientific versus interpretive research paradigm. Referring to the work of John Dewey (1916) and addressing the "false dualism" of educational research, Richard Pring (2000) warns:

My argument is that the opposition (not the distinction) between quantitative and qualitative research is mistaken. The "naïve realism" attributed to those who espouse the more quantitative methodology is not justified. How we conceive the world could be different and indeed, is different from social group to social group. Such "social constructions" are constantly reconstructed as new experiences force us to reshape how we understand things. Hence, the need for that interpretive and hermeneutic tradition in which we seek to understand the world from the perspective of the participants, or to understand a set of ideas from within the evolving tradition of which they are part. The qualitative investigation can clear the ground for the quantitative – and the quantitative be suggestive of differences to be explored in a more interpretive mode. (Pring, 2000, p.56)

Following on Pring's quotation, I do not consider the distinction between quantitative and qualitative research as a dichotomy, but rather as two different steps aiming to achieve a more holistic picture of a particular research issue. Following this perspective, a possible elaboration of my research might be to consider a larger group of teachers or different groups of teachers engaged in working collaboratively with a didactician or with each other, and to see if similar results might emerge from the analysis of data.

As an example of a research project adopting mixed methodologies, the work of Kirsti Hemmi (2006), focusing on proof in a mathematical practice, employs both quantitative and qualitative methods. From a quantitative basis, a smaller group of students was selected for a more detailed study. Hemmi underlines the complementarity of these two approaches as she explains:

With the help of quantitative inquiries, I could, for example by calculating percentages and correlations, get rough information about the aspects I was exploring. From focus group interviews I obtained data that were richer and shed more light on the uniqueness of individuals beyond the figures in the surveys. (p.69)

As mentioned above, I consider that, by underlying the complementarity of quantitative and qualitative research, these considerations might give me indications about ways to develop further my research project.

In the previous sections, I located my research paradigm within a broader picture of the main educational research paradigms, contrasting the interpretive research paradigm with the scientific and the critical theoretic research paradigm. In the next section I make explicit my own assumptions and, thereby, I offer a justification for choosing an interpretive research paradigm in my research.

3.3.2 What are my underlying assumptions?

As mentioned before (see Section 3. 3), Burton (2002) points out the need for a researcher to make visible and give some justifications concerning the choice of paradigm one chooses to work within. Therefore, my aim, in this section, is to address the “unconsciously” aspect and to make explicit my own position in relation to epistemological considerations.

I consider knowledge as personal or socially constructed and mainly understood through qualitative case studies of particular individuals and contexts. Thereby, since my study addresses the development of algebraic thinking, I understand this development as socially constructed through the creation and the development of our community of inquiry, and in that sense, drawing on sociohistorical precedents in millennia of mathematics. Furthermore the social aspect of knowledge construction is visible through the processes of ‘negotiation of meaning’ and ‘participation’, as explained earlier (see Chapter 2). On one hand, this position is in accord with the interpretive research paradigm as defined by Carr and Kemmis (1986). On the other hand, my interest in research is not only to understand and make sense of the world (in my case, looking at the teaching and learning of elementary algebra), it is also to look for ways to enhance teachers’ algebraic thinking. In other words, there is a dimension of development and change in my research, and this dimension, which is a characteristic of the critical theoretic paradigm (Carr & Kemmis, 1986), is crucial in my study. At the same time the intended outcome in this current research is not intervention for social reform or social justice. However, my research could be understood as seeking to enable the teachers to have access to acknowledged difficult areas of mathematics, and therefore our collaboration might be emancipatory. This perspective is addressed by Goodchild (2008), as drawing on Freire’s work, he argues that “teachers are ‘oppressed’ in their practice by historic, economic, social and cultural contradictions over which they have little or no control. However, through ‘critical alignment’ they may become aware of their situation and explore the possibilities they have to make things better for their pupils” (p.214). Thereby, there is a possibility to consider the idea of ‘critical alignment’ as a means to achieve emancipation. In terms of my research, by offering to the teachers the opportunity to work collaboratively within a commu-

nity of inquiry, they might reconsider the way they align with the practice at their respective school and, thereby, they might explore possibilities to change and improve their teaching. I address and develop this issue further in Chapter 4.

In locating my research within the wider methodological literature, the challenge consisted of finding and working within a research paradigm that included the mentioned dimensions and allowed me to evolve between these dimensions. I consider that, in making visible these issues, I am able to address Leone Burton's (2002) challenge:

In the discussion of methodology, I have incorporated a view on epistemology that is central to the approach that I am taking. How knowledge is constructed is a function of values and, indeed, is also about the community that can define those values and establish the gate keeping criteria for maintaining them. Inevitably, therefore, I see epistemology as interlocked with methodology. (Burton, 2002, p.6)

Following Burton's recommendation concerning the need to address explicitly epistemological issues, I consider that knowledge emerging from our community of inquiry is socially constructed through the interactions between the participants within our community of inquiry as these engage collaboratively in the workshops, and discuss during the interviews before and after classroom observations.

In my research I follow three teachers and myself during the school year our collaboration lasted, and thereby I consider that my research might be understood as a *case study*. By looking into the research literature in order to locate my approach, it is possible to differentiate between several types of case study. Lawrence Stenhouse (1988) proposes the following distinction: ethnographic, evaluative, educational, and action research case studies. I consider that his characterisation of *educational case study* fits with my own study:

Educational case study [is where] many researchers using case study methods are concerned neither with social theory nor with evaluative judgement, but rather with the understanding of educational action. ... They are concerned to enrich the thinking and discourse of educators either by the development of educational theory or by refinement of prudence through the systematic and reflective documentation of evidence. (Stenhouse, 1988, p. 50)

I understand Stenhouse's idea of researchers' concern with understanding of educational action as corresponding to my search for exploring and understanding the relation between the development of our community of inquiry and the development of algebraic thinking, and, as such, my research might contribute to enhance and develop insight in teachers' development of professional knowledge through establishing a community of inquiry in collaboration with a didactician. In my research I seek to develop an understanding of the nature of educational action by focusing on the development of algebraic thinking. In order to be able to achieve this aim, I engaged in design research which resulted in the

elaboration of the six-step framework. This framework acted as a methodological means enabling me, as a researcher, to engage in developmental research. According to Gravemeijer (1994b), the cyclic process consisting of thought experiment and feedback of practical experience lies at the core of Hans Freudenthal's concept of developmental research. This was the case in my research, as the cyclic process of thought experiments (*a priori* analysis) and feedback of practical experiences (informal *a posteriori* analysis) which I conducted before and after the nine workshops allowed me to engage in developmental research. In addition, "the cyclic process that Freudenthal discerns can also be seen as a learning process of the developer" (Gravemeijer, 1994b, p.113). It is in these terms that I consider the possibility for me, as a researcher, to develop understanding of educational action.

In this section I followed Burton's advice and made explicit my epistemological assumptions and the way this position is in accord with the methodology I chose to conduct my study. In addition, I argued for considering my research as an *educational case study* through which I seek to develop an understanding of a particular type of educational action, as contextualized in the social setting of our community of inquiry.

As explained earlier, see Section 3. 2. 2, I recognised both a *cyclic* and *iterative* nature in my research. In order to locate and characterise my research design, I propose, in the next section, to look into the wider research literature on methodological approaches in order to characterise my approach. I could see that the cyclic and iterative aspects were addressed both by an action research model and a design-based model, although neither of these seemed to fit exactly. In the following I present the main aspects of an action research and design-based model and make explicit the similarities and differences between these models and my study.

3.3.3 Identifying some aspects of action research model and design-based model

As mentioned above, I recognised both a *cyclic* and an *iterative* nature in my research: the cyclic aspect refers to the six-step framework within which steps 1, 2, and 6 belong to the mathematical workshop, while steps 3, 4, and 5 belong to the observation part. A detailed explanation of each step has been presented in Section 3. 2. 2.

The iterative aspect refers to the repetition, during the school year, of the cycle as presented above. However, as explained earlier (see Section 3. 2. 2), the actual realisation of the six-step framework was different from the intended framework, and therefore the iterative and developmental nature was mainly visible in relation to the workshops and not so much in relation to the observation part. During the school year, our group met nine times for mathematical workshops, while I had the op-

portunity to follow Paul once, Mary twice, and four times with John through the observation steps.

Action research model

According to Louis Cohen, Lawrence Manion, and Keith Morrison (2000), the action research approach is suitable for any setting involving people, tasks and procedures in which some kind of structural change results in an improved situation. In order to obtain the desired outcome, a six-step research device combines a straightforward cycle of 1) identifying a problem, planning an intervention, implementing the intervention, evaluating the outcome, 2) reflective practice, 3) political emancipation, 4) critical theory, 5) professional development; and 6) participatory practitioner research. In that sense, action researchers use systematic and critical enquiry in attempts to improve their own practical situation (Bassey, 1999).

Considering the process of action research, its essence might be conceptualized as a spiral of cycles involving the following steps (Carr and Kemmis, 1986; Wellington, 2000):

PLANNING → ACTING → OBSERVING/EVALUATING →
REFLECTING → RE-PLANNING

A more detailed action research process is presented in Jean McNiff (2002). However, it is important to remember that each element of this model needs a careful examination. For example, it might be highly problematic, during the planning phase, to identify which aspects of one's own practical situation need to be improved, and why.

Furthermore, Carr and Kemmis (1986) underline two essential aims in all action research approaches. These are concerned with *involvement* and *improvement*. They write:

Action research aims at improvement in three areas: firstly, the improvement of a *practice*; secondly, the improvement of the *understanding* of the practice by its practitioners; and thirdly, the improvement of the *situation* in which the practice takes place. The aim of *involvement* stands shoulder to shoulder with the aim of *improvement*. (p.165)

An important aspect of this involvement is further underlined as “Those involved in the practice being considered are to be involved in the action research process *in all its phases* of planning, acting, observing and reflecting” (Carr & Kemmis, 1986, p. 165, my emphasis).

Some aspects of the action research process are clearly in common with my own research: first of all, the emphasis on the cyclic and iterative nature of the research. Second, the ideas of involvement and improvement, as the three teachers and myself are involved in the six-step framework with the aim to improve algebraic thinking.

However, I consider that my research differs from action research on the following aspects: first concerning the involvement of all the partici-

pants in *all the phases* of the action research cycles. Looking back to the six-step framework, the teachers are *neither* involved in the planning of the workshops and therefore in the *a priori* analysis *nor* in the *a posteriori* analysis (see Section 3. 2. 2). Furthermore, they are not engaged in researching their own practice, neither am I, at least not by intention (see Chapter 5, concerning my own learning). The fact that emphasis was placed on algebraic thinking is my own decision and emerged from my own preferences, as explained in Section 1. 2. In addition the teachers and I were not looking critically into our own practice in an attempt to identify problematic issues.

I recognize that although my research shares some aspects in common with action research, there are some major differences, and therefore I consider that this particular methodology does not enable me to articulate all aspects of my research.

Design-based research model

The cyclic and iterative nature of the design of research is also addressed by The Design-Based Research Collective (2003). Design-based research was introduced as an answer to the critiques concerning the detachment of research from practice (Schoenfeld, 1999; Woods, 1986), which is referred to as the “credibility gap” (Levin & O’Donnell, 1999). Anthony Kelly (2003) describes Design research as an emerging research dialect whose operative grammar is both generative and transformative (p. 3). It is both generative by creating new thinking and ideas, and transformative by influencing practices. This research approach addresses problems of practice and leads to the development of usable knowledge (The Design-Based Research Collective, 2003).

According to Terry Wood and Betsy Berry (2003), design research can be characterized as a process consisting of five steps:

First, a physical or theoretical artifact or product is created. For the researcher/teacher educator the product being developed and tested is the professional development model itself. Second, the product is tested implemented, reflected upon and revised through cycles of iterations. The model is dynamic and emergent as the process progresses. Third, multiple models and theories are called upon in the design and revision of products. Fourth, design research of this nature is situated soundly in the contextual setting of the mathematics teachers’ day-to-day environment, but results should be shareable and generalizable across a broader scope. Fifth, the teacher educator/researcher is an interventionist rather than a participant observer in a collaborative, reflective relationship with the teacher(s) as the professional development model evolves and is tested and revised. (Wood & Berry, 2003, pp.195-196)

As underlined by Wood and Berry (2003), design research is not *only* about the development of physical/theoretical artefact, for the research or teacher educator the focus is placed on the professional development model itself.

In my research, the aim is *not* the development of a special type of mathematical tasks, even though the importance played by these must be recognized. The tasks proposed to our community of inquiry during the workshops have to be considered as tools whose purpose is to provoke, enhance, and give the opportunity for deepening the participants' thinking concerning algebraic thinking. This point is underlined by Jaworski (2005b):

However, design research talks particularly of a *product* emerging from the design research process, and sometimes it is hard, in a teaching development context, to identify what is the product of this developmental process. We might therefore talk rather of *developmental research*, where the tools of development form the basis of what is studied and the outcomes of the research process constitute a combination of development and of better understandings of the developmental process and its use of tools. (Jaworski, 2005b, p.360-361)

In my study I consider the “tools of development” as mediating the developmental process and these are exemplified through the mathematical tasks proposed to our community of inquiry during the workshops. Concerning the “outcomes of the research process”, I consider the results of this study as offering both a developmental framework (the six-step developmental and analytical framework) and a better understanding of the developmental process (in my case the participants' development of algebraic thinking). The analysis also addresses the use of mathematical tasks and the role these play within the framework. Before elaborating further on Jaworski's perspective of ‘developmental research’, I want to address the issue of visibility and invisibility in relation to the use of mathematical tasks. Originally, this issue was introduced by Adler (1999) who refers to the difficulties of teaching mathematics in multilingual classrooms. More specifically, she refers to “seeing and seeing through talk”, that is to understand talk, and to have the ability to see the mathematics through talk. I consider this issue as relevant in relation to the mathematical tasks which our community of inquiry engaged within. By visibility of the mathematical tasks, I refer to our engagement into mathematics, as we all engaged collaboratively in inquiring into the task, negotiating the meaning of it, and as the teachers were envisaging possibilities of implementing the task, or part of it in their respective teaching. In other words, the mathematical tasks enabled us *to do mathematics*. On the other hand, by seeing through the mathematical tasks, I refer to the participants' development of algebraic thinking. In that perspective, the mathematical tasks became *tools in the participants' development*, and by following the way we participated in our community of inquiry during the year, I was able to trace both the teachers' development of algebraic thinking (see Chapter 4) and my own (see Chapter 5). According to Lave and Wenger (1991), “this interplay [between visibility and invisibility] of conflict and synergy is central to all aspects of learning in

practice: it makes the design of supportive artifacts a matter of providing a good balance between these two interacting requirements.” (p.103). It was the case in my study since I had the responsibility, both as a didactician and as a researcher, to offer the teachers mathematical tasks which had the potential to address mathematics as a subject-matter and to enhance algebraic thinking. I deepen this issue further in Chapter 4 (see Section 4. 2).

Developmental research

An overview of the main aspects of developmental research is provided by Koeno Gravemeijer (1994a), Jan van den Akker (1999), and Simon Goodchild (2008). One of the fundamental characteristics of developmental research is the existence of a cyclical process between development and research. Goodchild (2008) describes the *developmental research cycle* in these terms: “Theory and evidence from prior research leads to an envisaging of development, this leads to actions which are evaluated and fed back into a new cycle of envisaging and action” (p.7).

Within the *developmental cycle*, there is a cyclical process between *thought experiment* and *practical experiment*. Here I understand the term *thought experiment* as referring to an envisaged teaching-learning process, while *practical experiment* as referring to the actual implementation of the thought experiment in the relevant social setting. Gravemeijer (1994b), referring to Freudenthal’s work, underlines the importance of thoughts experiments:

The developer will envision how the teaching-learning processes will proceed, and afterwards he or she will try to find evidence in a teaching experiment that shows whether the expectations were right or wrong. The feedback of practical experience into (new) thought experiments induces an iteration of development and research. (p.112)

In my research I consider the developmental cycle as consisting mainly of the nine workshops the teachers and myself had since, as explained earlier, the observation steps did not happen regularly. Thereby, I can talk about ‘the developer’ as myself, as I was able to envision how the three teachers and myself would respond to the different mathematical tasks which were proposed during the school year. Similarly, after the workshops, I engaged in an intuitive and informal evaluation of the way the participants engaged within the tasks, and this feedback constituted the basis for new thoughts experiments. Thereby, it was only me, and not the three teachers, who was engaged in ‘thought experiments’ or *a priori* analysis and in *a posteriori* analysis of each mathematical workshop. A more formal analysis of data was conducted when I engaged in the process of writing my thesis and where evidence for my findings was presented. However, I was not only engaged in development, I was also *re-*

searching on the development of algebraic thinking. This aspect is addressed in the research cycle.

Within the *research cycle*, there is a cyclical process between *global theories* and *local theories*. This means that “global theory is concretized in local theories. Vice versa, the more general theory can be reconstructed by analyzing local theories” (Gravemeijer, 1994a, p.451). In the process of elaborating my theoretical framework (see Chapter 2), I started by considering Wenger’s (1998) theory and argued that, because of considerations related to relevance and coherence, I needed to go back to Vygotsky’s (1978, 1986) ideas of scientific concepts. Therefore, I consider these two theories as ‘global theories’ and they constitute the starting point for the elaboration of my theoretical framework. I understand Gravemeijer’s ‘local theories’ as referring to the particular theoretical framework which I developed in order to conduct my research. As explained in Section 2. 5, I believe that by going back to Vygotsky’s work, I am in a position to elaborate a theoretical framework which is both relevant and coherent, and which goes behind and expands Wenger’s work. In addition, by elaborating my theoretical framework in this way I am in a better position in order to provide a suitable theoretical framework for conceptualizing processes related to the development of algebraic thinking and mathematical learning within a sociocultural approach to learning. Furthermore, according to Gravemeijer, one of the outcomes of my research might be that the more general theory concerning the *specificity of mathematical learning* can be reconstructed and enriched, by providing new insights and more details, as these emerged from the process of analysing the local theory which I used (see Chapter 4). Using Gravemeijer’s (1994a) words: “global basic theory is elaborated and refined in local theories” (p.452).

Goodchild (2008) offers the following representation of the different aspects of the developmental research cycle (Figure 5).

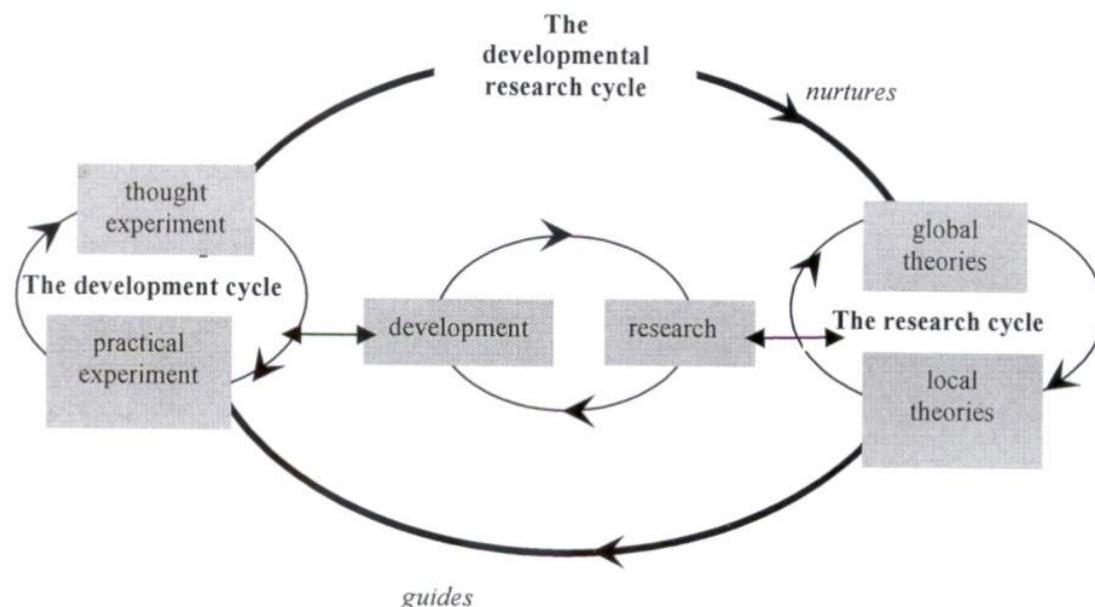


Figure 5. The developmental research cycle (from Goodchild, 2008)

An issue that I need to take account of relates to *legitimization* of new knowledge. Gravemeijer refers to Freudenthal’s (1991) recognition of “one of the most important differences between physics and social sciences is the possibility or impossibility, respectively, of replication. In educational development, replication in a strict sense is impossible” (p.452). As a consequence “new knowledge will have to be legitimized by the process by which this new knowledge was gained” (p.452).

In my study, the process by which this new knowledge was gained is addressed both through this chapter and the next chapter concerning the results of my analysis. In addition I present, in Chapter 5, a reflection over my own learning process during these years as a doctoral student. Therefore I consider that, by offering these insights to the reader, I am in the position to meet Freudenthal’s (1991) demand for “an attitude of self-examination on the part of the developmental researcher: a state of permanent reflection” (p.161).

As a result of these methodological considerations, I recognise that my research study shows some aspects in common both with an action research model and a design-based research model. However, I prefer to characterize my research as following a developmental research approach, as proposed by Jaworski (2005b) and Goodchild (2008), since the developmental research cycle enables me to make visible and to address the different steps within my research design.

In the next section, I focus on the mathematical workshops and more precisely on the elaboration of the mathematical tasks presented to the teachers. As mentioned earlier (Section 1. 5), these mathematical tasks were created or found by Claire, acting *both* as a didactician and as a researcher, and they played a fundamental role in this study. I develop this issue further in the next section.

3.4 Elaboration of the mathematical workshops

As explained in Section 2. 2. 2, the practice resulting from the actual realisation of the six-step framework consisted of nine mathematical workshops, which happened once a month, and some classroom observations, which happened more randomly. During the workshops, I prepared and presented to the three teachers mathematical tasks and, as explained earlier, I conducted both an *a priori* or ‘thought experiment’ (Gravemeijer, 1994b) and an informal *a posteriori* analysis of each task. As such, the mathematical tasks played a central role in my study, acting both as a means to provoke the participants’ reflections concerning algebra and algebraic thinking, and at the same time, as a means to enable our community to work together. In other words, the tasks were instruments both in the development of algebraic thinking and in the building of the community. Considering the *a priori* and *a posteriori* analysis, I had to elaborate some criteria in relation to both the choice of the different tasks, and the evaluation of each task after our workshops were finished. Furthermore, the criteria concerning the choice of the tasks had to reflect both aspects of the tasks: acting as a means both to develop algebraic thinking and to develop our community of inquiry.

It is also important to emphasise the fact that the mathematical tasks presented to the teachers were *not* decided in advance. It is as a consequence of an informal *a posteriori* analysis after each workshop that I decided which task I would propose to the teachers for the next workshop.

3.4.1 Key criteria for the *a priori* analysis

In this section I offer and explain the key criteria for the *a priori* analysis in relation to developing algebraic thinking, choosing didactical aim and pedagogical means, and establishing and developing a community of inquiry.

Developing algebraic thinking

What does it mean to address algebraic thinking as situated with respect to the development of our community of inquiry in which it takes place? I agree with Bell (1996) and Lee (1996) who characterize algebra and algebraic thought as consisting of the following aspects: language and communication, way of thinking, activity, tool, and generalized arithmetic.

As an attempt to conceptualize these different aspects of algebra and algebraic thinking, I propose to differentiate between the operational component, the structural component, the formal syntactic component, and the formal semantic component (see Chapter 2). The *operational component* addresses working with numerical examples and searching for patterns. It might involve the recognition of generic examples (Rowland, 2000) where not only the result is important but also the *manner* it has been established. This leads to the *structural component* which allows one to study in depth and to understand the inner structure of a generic example. In that sense the mathematical objects used in the generic example are, in the structural component, regarded as characteristic representatives of the class of objects (Balacheff, 1988), as for example, 15 which might be written as $14 + 1$ is representative for the class of odd numbers.

The *formal syntactic component* is characterized by the introduction of algebraic symbolism. By algebraic symbolism I mean the algebraic notation which is widely recognized and accepted by the community of mathematicians. The difficulties related to the choice of algebraic symbolism is largely reported in the research literature, mainly addressed from a cognitive perspective and linked to epistemological and didactical obstacles (Chaiklin, 1989; Filloy and Rojano, 1989; Herscovics, 1989; Herscovics and Linchevski, 1994; Kieran, 1989b, 1992; Küchemann, 1981).

In my study the issues related to the choice of algebraic symbolism are addressed in terms of *negotiation of meaning*. In that sense, I am interested in the way the people in our group, while engaging in mathematical tasks, question, discuss, argue, and choose symbols. The last aspect, the *formal semantic component*, addresses the meaning which the algebraic symbols endorse. Collis (1974), Harper (1987), and Küchemann (1981) describe the results of research on students' views of algebraic symbols in hierarchical terms. The implicit assumption on which this kind of research is grounded is that pupils need to reach the stage of formal operation in order to perform satisfactorily on certain algebraic tasks. This view is strongly reflected in the work of Herscovics (1989) as he argues that:

From a Piagetian perspective, the acquisition of knowledge is a process involving a constant interaction between the learning subject and his or her environment. This process of equilibration involves not only *assimilation* – the integration of the things to be known onto some existing cognitive structure – but also *accommodation* – changes in the learner's cognitive structure necessitated by the acquisition of new knowledge. However, the learner's existing cognitive structures are difficult to change significantly, their very existence becoming cognitive obstacles in the construction of new structures. (p.62)

Using the constructs which I introduced in my theoretical framework (see Chapter 2), Herscovics' quotation might be reworked as:

From a Vygotskian and Wenger's perspective, the transformation of participation in collaborative endeavour is a process involving negotiation of meaning between the participants within a community of inquiry. This process of negotiation of meaning involves different layers of inquiry addressing both the subject-matter and different aspects of the community. However, the participants' previous engagement in various communities has to be recognised and addressed while defining current mutual engagement, joint enterprise, and developing shared repertoire.

I recognise the historical significance of articulating the process of acquisition of knowledge using Piagetian terminology, since, as mentioned earlier, this epistemological approach had a strong dominance in educational research at the time Herscovics was writing. However, since my research is rooted in a sociocultural perspective on learning, I propose to articulate the participants' development of algebraic thinking using concepts taken from a sociocultural frame, where I rather talk about transformation of participation through metacognitive and cognitive mediation (see Section 4. 3).

This attempt is inspired from the result of my research and is pragmatic (see Chapter 4). However, I argue that it is a worthwhile enterprise, since it can help me, as a researcher, to grasp the epistemological differences between these two perspectives on learning.

To summarise, in my research I define the idea of "algebraic thinking" as a process consisting of the following four components: operational, structural, formal syntactic, and formal semantic. As explained above, the situated character of learning is visible in the way the choice of symbols is articulated, as this choice is understood through the process of negotiation of meaning, including questioning, discussing, arguing, and choosing symbols. These aspects are represented in Figure 6.

Through my study, I was acting both as a *didactician*, developing and organising the learning processes within our community of inquiry and as a *researcher*, researching the processes related to the participants' development of algebraic thinking. Part of the organisation of the learning processes was to decide which knowledge target I wanted our group to address during the mathematical workshops, and I referred to the choice of a particular area or knowledge target within a subject matter as to a *didactical aim*, and I want to emphasise the fact that this choice was also part of a developmental process (see *a priori* analysis of each workshop in Chapter 4). In addition I had to decide which *pedagogical means* to use in order to address the chosen didactical aim. I develop further on these issues below.

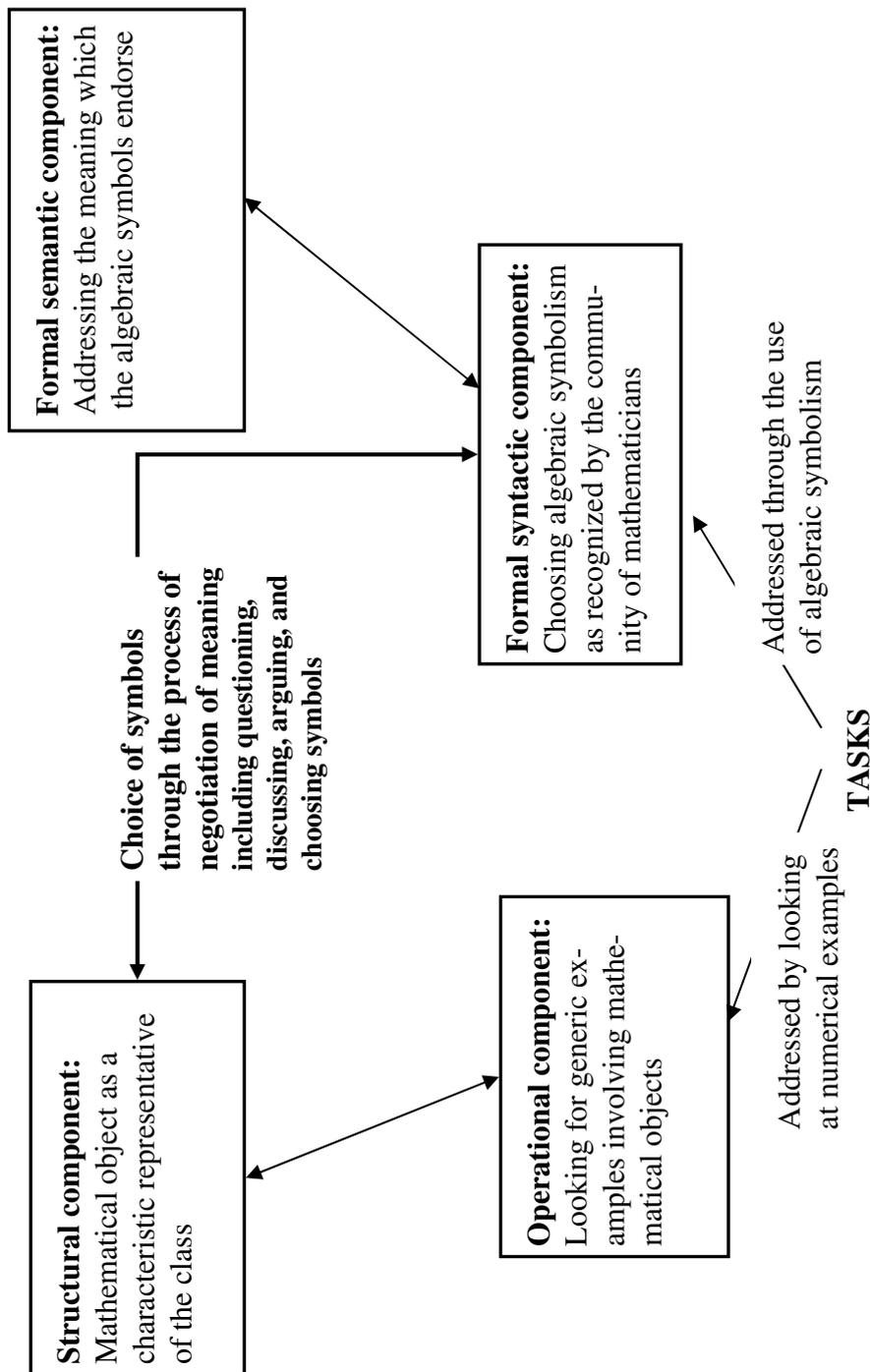


Figure 6: The different components of algebraic thinking: Focus on the situated character of learning through negotiation of meaning

Didactical aim and pedagogical means

The aim of my study was to study the processes related to the development of algebraic thinking, as these emerged from the creation and development of our community of inquiry. As mentioned earlier, the mathematical tasks, which I presented to the teachers, played a central role in my study, acting both as a means to provoke the participants' re-

flections concerning algebra and algebraic thinking, and at the same time, as a means to enable our community to work together. Furthermore, as part of the *a priori* analysis I had to decide which *didactical aim* I wanted to address, and which task to use in order to address the chosen didactical aim. My use of the term ‘didactical’ is inspired by Freudenthal’s (1991) definition of ‘didactics’:

Didactics of a subject area means the organisation of the teaching/learning processes relevant to this area. Didacticians are organisers: educational developers, textbook authors, teachers of any sort, maybe even students, who organise their individual or group learning processes. (p.45)

In the different mathematical tasks which I proposed during the workshops, I addressed the following didactical aims: the choice and use of algebraic symbols, the power of algebraic notation, and the meaning of symbols (the translation from natural language to algebraic notation). The two first didactical aims were inspired by my former experience working with a Master thesis on the topic of Galois Theory and within the Mathias project. The last one emerged from classroom observations (see Chapter 4).

Once a didactical aim was chosen, I had to decide which problem was suitable for our group to engage within. Here I want to make the distinction between a problem and a task: I understand a *mathematical task as a contextualised problem*. This implies that by developing and adapting a particular problem into a task acting as pedagogical means in order to address a specific didactical aim, one has to contextualise and adapt a problem to a particular social setting within which the task will be introduced. In other words, I consider *a task as what people actually do within the context of a specific social setting*. I want to put emphasis on the developmental aspect of this process since choosing, developing and adapting a particular problem into a task is highly dependent of the stage the community has achieved in terms of confidence both in mathematics and in the community (see *a priori* analysis of the each workshop in Chapter 4).

For example, during Workshop II, I chose “the choice and use of algebraic notation” as didactical aim. I am sure that it is possible to find many problems addressing this knowledge target with the research literature. I chose to consider a problem related to addition of even and odd numbers, and through the way it was contextualised and adapted (the way I presented it to the teachers, and the kind of questions I asked, the introduction of manipulatives to illustrate geometrical properties of even and odd numbers), the problem became the mathematical task which was introduced during Workshop II. In addition, my choice of introducing manipulatives was influenced by my previous experience during Workshop I. Therefore, I consider that the mathematical task, as it was proposed during Workshop II, acted as a *pedagogical means* to achieve the

chosen didactical aim. Thereby, I understand the idea of pedagogy as related to the process of contextualising a particular problem and, as such, as part of the organisation of the teaching/learning processes, as described in Freudenthal's (1991) quotation above. As such, I consider that by presenting a particular task within a specific social setting, a didactician creates a *mathematical environment* whose characteristics depends *both* on the mathematical task and on the social setting. For example, by presenting a task related to even and odd numbers to our community of inquiry during Workshop II, I created a numerical environment within which our group engaged.

Establishing and developing a community of inquiry

As mentioned before, the mathematical tasks which I presented to the teachers during the workshops acted not only as means to develop algebraic thinking, but also as means to establish the community. The following criteria were relevant in relation to establishing and developing our community:

- A. The task is accessible in order to motivate, engage, and provoke discussions among all participants in our community, addressing issues concerning both the “becoming” a member and later the “belonging” to the learning community (Wenger, 1998). By accessible I mean, in relation to the nature of the mathematical objects, the nature of the operation(s) between these objects, and the nature of the question(s).
- B. The task can be explored and solved using different approaches (numerical, geometrical, algebraic) aiming to provoke and enhance the participants' thinking in relation to algebra,
- C. The task offers the opportunity for widening and deepening of mathematical understanding with focus on algebra and algebraic symbolism, aiming to develop the participants' algebraic thinking (Sierpiska, 1993a),
- D. The task may offer some insight into the history of mathematics and in this way encourage the community to see mathematics as a continuous process of reflection and improvement over time, and provide an opportunity for developing participants' conception of what mathematics is (Arcavi et al., 1982; Fauvel, 1991; Freudenthal, 1981).

The three first criteria relate to both the development of our community of inquiry and the development of algebraic thinking. The last criterion, concerning getting insights into the history of mathematics, arises from my own interest in this area and my wish to share this with the teachers. In addition I consider that this perspective opens for inquiring into one's

own conception of mathematics as a subject matter, and thereby, offers an opportunity to engage in ‘critical alignment’.

3.4.2 Key criteria for the *a posteriori* analysis

In the previous section, I referred to the idea of *a posteriori* analyses which were conducted after each workshop. These *a posteriori* analyses throughout the research took various forms: both formal and informal. Informal analysis took place immediately after each workshop and consisted of my own notes where I recorded my impressions and reflections. These feedbacks constituted the background for a new “thought-experiment” in relation to the next workshop.

A more formal analysis of my data was conducted after the process of collecting my data was over. It was conducted in a way close to the “Grounded Theory” approach introduced by Barney Glaser and Anselm Strauss (1967). I develop this issue later (see Section 3. 5. 3). The findings, which were emerging from the analysis, are presented in Chapter 4.

The informal a posteriori analysis

Concerning the informal *a posteriori* analysis, it consisted of evaluating the mathematical tasks relying on the following criteria:

- E. Did the task motivate and engage all the participants, and in this way address issues concerning the “belonging” to the community?
- F. Did the participants resolve the task using algebraic notation?
- G. Did the analysis of the workshop show any evidence for enhancement of teachers’ algebraic thinking in terms of participating and engaging in the social processes of learning?

Considering the three first criteria from the *a priori* analysis (from A to C) and the three criteria from the informal *a posteriori* analysis (from E to G), I see a correspondence between these as criterion E offers an evaluation of the issue addressed by criterion A, and the same for B and F, and C and G. Especially I want to emphasise the role played by the last criterion G, since it is related to how our group addressed the didactical aim I chose in each workshop, and thereby, to the development of didactical aims. I develop this issue further in Chapter 4, where I offer a formal *a posteriori* analysis of each workshop.

3.4.3 Overview of the nine workshops

The following table gives an overview of the different workshops, when they took place, their main characteristics, and the criteria for choice:

Table 3: Overview of the nine workshops

	Subject matter (didactical aim)	Tasks as a pedagogical means	Criteria for choice	Purpose of the tasks
Workshop I (16.06.04)	The choice and use of algebraic symbols	Cuisenaire-rods	Mainly A (becoming)	To explore the link between the use of manipulatives and symbols
Workshop II (07.09.04)	The choice and use of algebraic symbols	About even and odd numbers	A and C	To explore the generalization of patterns and its expression using symbols
Workshop III (05.10.04)	The power of algebraic notation	Two tasks: One task from Babylonian time and the Calandri's fish problem	C and D	To explore the power of symbolic notations
Workshop IV (10.11.04)	The power of algebraic notation	Viviani's theorem	A , B , and C	To explore the connection between geometry and algebra
Workshop V (30.11.04)	The power of symbolic notation	Four digits palindromes	C	To explore the generalization of patterns and its expression using symbols
Workshop VI (11.01.05)	The meaning of symbols	Two tasks: The Student-Professor task and a task from Diophantus	A and C	To explore the transition from natural language to symbols
Workshop VII (09.03.05)	The meaning of symbols	Two tasks: One concerning area and circumference of a rectangle and a task related to Ole's siblings	A and C	To explore the transition from natural language to symbols

Workshop VIII (10.05.05)	Teachers in charge	Teachers in charge	Mainly A (belonging)	
Workshop IX (14.06.05)	Evaluation of all the workshops	Evaluation of all the workshops	Mainly A (belonging)	

Concerning the observations steps, I was able to follow John four times in his class (09.11.04; 12.11.04; 10.01.05; 20.01.05). Furthermore, I observed and interviewed Mary twice (18.10.04; 27.01.05), and once for Paul (27.01.05).

3.5 Methods used to collect and analyse data

In the previous sections, I addressed “The Why” and gave an in-depth justification for the choice of the methodological approach to my research project. In this section, I address “The How” and explain both the way data have been collected and how these have been analysed. Since I approached the data using a “Grounded Theory” approach, I also present briefly this theoretical approach. In the next section, I explain how I collected and analysed the data following Grounded Theory in a loose way. In the last section, the emphasis is placed on inquiry moves, as an analytical tool.

3.5.1 The process of collecting data

All our workshops and classroom observations were audio-taped. In order to put emphasis on the collaborative part of my role, acting more like a participant and not so much as a researcher, I took the decision not to video-tape our workshops. For the same reason, I experienced as difficult to take field notes during our meetings with the teachers, it was like constantly going in and out of our group in order to be able to reflect on the spot and a way of avoiding this schizophrenic activity (Eisenhart, 1988) was to decide not to take field notes and to solely rely on the recorder during the workshops. In that situation, I felt that my purpose as a researcher was contradicting my purpose as a didactician. These choices imply that my role, at least during the workshops, might be describes as *participant-as-observer* (Gold, 1958) and I can see now that, especially during the first workshops, I was acting as a native, to “go native”(Gold, 1958). On the other hand, during the classroom observations, I was sitting in the back of the class, observing the activity around me, taking field notes, without any intervention. In that case, my role corresponded to the almost *complete observer* (Gold, 1958), removing myself from social interaction. However, I consider that by being there, I am still a participant to some degree (Wagner, 1997).

3.5.2 The process of analysing data

The process of data analysis was conducted after the fieldwork had been completed, and it consisted of listening and re-listening to audio-tapes, reading and re-reading transcriptions of the dialogues between the teachers and myself. It also consisted of reflecting on the data collected, seeking for and coding patterns, elaborating the perceived aspects and making conjectures concerning the relations between these different aspects. Initially, I tried to identify the purpose of the different negotiations of meaning, as these appeared through the workshops. Thereby, I developed several categories and experienced soon the need for regrouping these various categories in some few. This process resulted in six purposes of inquiry: inquiry in a mathematical task, inquiry in community building, inquiry in the other participants' understanding of a mathematical task or their own practice of teaching, and inquiry in Claire's didactical and pedagogical aims (see Section 2. 2. 6). I started by mentioning the categories in the margin of the transcripts, but soon I developed a system consisting of different colours, each corresponding to one category, and thereby the patterns were easier to identify. When excerpts¹¹ seemed to fit into the same category, the different excerpts were compared and sometime a refining of the category was necessary. The emerging categories were constantly refined, compared with each other, and the possibility to relate these categories to those developed within my theoretical framework was critically examined. The results of this analytical process are presented in Chapter 4.

One of the criteria I decided to follow during conducting data analysis was to adopt a holistic perspective on my data. I did not want to split my data in different parts and to analyse these different parts according to different theoretical perspectives. I looked at my data as a whole and tried to comprehend it holistically. I did all transcriptions by myself, and Workshops I, II, IV, VII, IX were transcribed *in extenso*, while Workshops V, VI were partly transcribed. Concerning Workshops III and VIII, I did data reduction. By 'data reduction', I mean listening to the audio tape several times first, and then using a table indicating who is speaking, what the utterance is about, and the number on my audio-recorder. In this way I was able to get a quite detailed overview of the data. I also followed the same procedure with detailed data reduction for the data from classroom observations. As mentioned earlier (see Section 3. 2. 2), I do not present an analysis from classroom observations since

¹¹ I am aware of Wells' (1999) distinction between *episode* and *sequence* as categories for the analysis of the sequential organization of discourse. However, I prefer to use the term *excerpt*, since I found it difficult to differentiate between episode and sequence in my research. Thereby, by referring to *excerpt* I mean a single utterance or a group of utterances.

this was not the aim of my study. The classroom observations constituted the background for the interviews with teachers after a teaching period.

All transcriptions were in Norwegian, and relevant excerpts, in relation to my research questions, were selected and translated into English. The analysis and coding process were conducted in English, while my own thinking was in French.

One of the challenges I had to face while reporting on the findings of my study in Chapter 4 was to choose how to describe my own role during the school year (2004 – 2005) our collaboration lasted. As explained earlier, during that period, I was acting both as a didactician and as a researcher. Today, while I am engaging in the process of presenting my research, I only act as a researcher. In order to differentiate these roles, I chose to talk about myself as ‘Claire’ when I refer to me during the year 2004 – 2005, and therefore, I use the third person ‘she’ when presenting the findings of the analysis of data in Chapter 4.

The aim of this distinction is to help me, as a researcher, to take some distances with myself and the way I was acting during that year, and, thereby to make an attempt to reduce subjectivity.

3.5.3 Introducing Grounded Theory

Before presenting the main features constitutive of a grounded theory approach to educational research, I offer a justification for my choice for using this strategy in my work. As explained in Chapter 2, my theoretical frame derives from Wenger’s theory related to the study of communities of practice, and this theoretical perspective is complex with many different notions which are interrelated and interwoven with each other. However, as I also underlined it, my perspective is slightly different from Wenger’s as I am looking at the *creation* and the *development* of a community of inquiry and therefore some aspects of Wenger’s theory are not relevant for my study as, for example, the notion of legitimate peripheral participation. This means that on one hand I have Wenger’s notions available, and on the other hand I have many pages of transcription of our dialogues, which became data through the research questions, related to the year our collaboration lasts. Nevertheless, I want to argue that it was not possible, in advance, to determine which aspects and notions of Wenger’s theory would be relevant for the analysis of my data. It is through the very process of listening and re-listening, reading, and re-reading again, several times, that some features and patterns emerged and these aspects were then identified to some notions proposed by Wenger. Thereby I did not come to the analytical process without any theory in mind, I believe this is not possible (May, 2001; Wellington, 2000), but I was not able to decide in advance on which aspects of Wenger’s theory I wanted to focus. I would describe the process of immersing into the data and letting some features and patterns emerge and

crystallize as following loosely a grounded theory approach to the analysis of data.

The main aspects of grounded theory: a brief overview

Glaser and Strauss (1967) define an approach to data analysis called *Grounded Theory* as “how the discovery of theory from data – systematically obtained and analysed in social research – can be furthered” (p.1). According to these authors, the aim of this strategy is to avoid an opportunistic use of theories, as illustrated in “exemplifying” where the researcher proposes examples that were selectively chosen for their confirming power of the adopted theoretical perspective (referred as *logico-deductive theorizing*, Glaser and Strauss, 1967, p.5). Very briefly, the main steps of a ‘Grounded Theory’ approach consist of theoretical sampling (collecting data for generating theory), and moving from substantive to formal theory, where substantive theory is described as a strategic link in the formation and generation of grounded formal theory. The general approach to the analysis of data uses the *Constant Comparative Method* whose purpose is to lead to the development and testing of a Grounded Theory.

3.5.4 Inquiry and didactical moves

As explained earlier (see Section 2. 2), I did *not* set up, from the beginning, to create and establish a community of inquiry. Both the dimension of inquiry and my own growth of awareness of it emerged gradually from the analytical process and it appears to be a fundamental characteristic of our community.

In this section, concerning the analysis of data, I would like to present an in-depth description of how the idea of inquiry emerged from the processes related to seeking for and coding patterns.

During the process of reading and re-reading the transcriptions of the dialogues between the teachers and myself, I became gradually interested in some moves, which I called in the beginning “exploratory moves”. The nature of these moves was mainly questions, but not exclusively. The most important features that captured my attention were their purposes, and who among the participants was expressing these exploratory moves. I consider that these moves constituted the “key” to enter into the search for patterns and by identifying and characterizing these moves, I could start to structure my analysis. I used the term “exploratory”, as a metaphor, to convey a sense of “questioning something” or “searching for something”. I consider now that the term “inquiry move” is more accurate and in that sense, I follow Wells’ (1999) definition of inquiry as “a stance toward experiences and ideas – a willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them” (p.121). In my view, this quotation captures some fundamental aspects related to both the devel-

opment of algebraic thinking, and of the development of our community of inquiry. In that sense, these *inquiry moves* are deeply related to the five components of learning (see Section 2. 2. 5), as elaborated from Wenger's (1998) four components of a social theory of learning (meaning, practice, community, and identity) and Graven's (2004) fifth component (confidence). In the next chapter, Chapter 4, I identify these inquiry moves and show how these relate to the different modes of participation within our learning community.

In contrast to inquiry moves, I argued in Section 2. 2. 6, for considering what I called *didactical moves*. This recognition came from seeking to grasp the purpose of some utterances, often a question. During the process of analysing my data, I became gradually aware of the different nature of these utterances, especially my own utterances during the first workshops. Contrary to inquiry moves which are rooted in genuine wondering and/or questioning, I could see that when asking those questions I *knew* the answer. In other words, my didactical aim was to inspire and stimulate the teachers to consider and to move toward the didactical aim which I had chosen for that particularly workshop. I elaborate further on this issue in Chapter 4.

In this chapter, I explained and offered some justifications for the choice of the adopted methodology. In addition, I presented the kind of data which was collected in order to address my research questions and explained how these were analysed. In the next chapter, I present the results of the analysis of my data.

4 Analysis and results

The aim of this chapter is to present the results of the analysis of the data coming both from the workshops and from the interviews and observations of the teachers in their respective classes. One main concern for me, in analysing the data, was to comprehend these in a holistic way.

The research questions, as presented in Section 1. 5, are:

- a) *In what ways is the development of algebraic thinking related to the development of our community of inquiry?*
- b) *What relationships can be discerned between teachers developing algebraic thinking during the workshops and their thinking in relation to their practice in the classroom?*

Through the first question, I seek to follow and describe the processes related to the creation and the development of a community of inquiry focusing on algebraic thinking. From the workshops, interviews and classroom observations I studied, I will, during Chapter 4, highlight emerging themes and issues, and attempt to extract characteristics which seem pervasive to the way the teachers and I collaborated during the school year. These emerging themes and issues arose from the Grounded Theory approach to the data, which I followed (see Section 3. 5. 3).

I propose to address the first research question by zooming *in* on what is happening during each workshop, at a fine grain level, and to follow how algebraic thinking is mediated through a *process of negotiation of meaning*. Furthermore, using Lerman's (1998b) metaphor of "the zoom of a lens", I propose to zoom *out* and to take a larger perspective while following how algebraic thinking is mediated, during the year of our collaboration, through Karpov et al.'s (1998) ideas of metacognitive and cognitive mediation. The second research question addresses a potential link between the teachers' development of algebraic thinking, as it emerges from our community of inquiry, and their own thinking in relation to their respective practice.

The chapter is organised as the following: Section 4. 1 presents evidence of key points as these emerged from the analyses of each workshop. Section 4. 2 deepens the role played by the mathematical tasks, while in Section 4. 3, I exemplify how the notions of *metacognitive* and *cognitive* mediation are addressed in my research.

Another point has to be clarified before presenting the results of the data analysis. It concerns the use of names in this study, for the teachers and for myself, and the way I, as a researcher, address my own participation within our community of inquiry. For ethical reasons, the names of the three teachers have been changed, and they chose themselves the following pseudonyms: Mary, Paul, and John. Furthermore, there is no

point for me to change my name, this is why my utterances are referred as Claire's turns. Nevertheless, it is important to distinguish between Claire, as the researcher, engaging in the process of data analysis and looking at Claire, participating into the workshops, and acting *both* as a didactician and a researcher. This is why I decided to write as referring to myself as "Claire", when I consider my own role during the period our collaboration lasted, while I use "I" when referring to myself, acting today as a researcher, looking at myself acting with the teachers. I argue for using this distinction as it helps me to distinguish between my different roles. I also addressed this issue in Section 3. 2. 2.

4.1 The nine mathematical workshops

In this section I present the results of the analysis of the data coming from the nine workshops which were organised during the period from June 2004 to June 2005 (see Section 3. 4. 3).

In addition, the analysis reveals how the two different roles, Claire acting as a didactician with developmental aims and Claire acting as a researcher with research aims, can in some circumstances be compatible and in other circumstances be in contradiction.

All the excerpts presented in this chapter have been translated from Norwegian. In Appendix 4, I present an example of the data from Workshop I, in Norwegian.

4.1.1 Workshop I: The first steps into our community of inquiry

A priori analysis of the mathematical task

As mentioned before (see Section 3. 4. 1), Claire did an *a priori* analysis of the mathematical tasks according to key criteria related to the development of algebraic thinking, and the establishment and development of our community of inquiry. In the following I present the *a priori* analysis of the mathematical task which Claire presented to the teachers during our first workshop, where Claire's didactical aim and pedagogical means are made explicit.

The data presented in this section have been selected from the first workshop within which the three teachers and Claire participated. The rationale for choosing the mathematical task was twofold: on one hand, Claire's didactical aim was to provoke a discussion concerning the choice and the use of algebraic notation. On the other hand, she had a particular concern about issues related to community building, and in that sense, she considered it important that the task presented during the first workshop enabled the participants to engage collaboratively in a mathematical task, and offered opportunities for discussing issues related to algebraic notation. In addition, Claire had in mind a story from Mary about her bad experience in relation to a previous course for in-service teachers' development where she did not understand the task and thereby

could not engage with it. Therefore, these three aspects were important, for Claire, in choosing an appropriate mathematical task to begin with. As such, these considerations were in accord with the key criteria for the *a priori* analysis, as presented in Section 3.4.1.

Claire decided to present a task related to the Cuisenaire rods as it seemed that such a task had the potential to address the aspects presented above: the Cuisenaire rods are attractive and tangible, and they offer opportunity to address algebraic concepts through hands-on activity. In addition, Claire had experienced through the Mathias-project that teachers easily engaged in tasks related to Cuisenaire rods. Concerning the design of the task, it was inspired by Jaworski (1988).

The main characteristics of Workshop I

In the beginning of this workshop, each participant introduced him/herself in terms of education and professional activities. These aspects have been presented in Section 1.4.

The data, as presented below through several excerpts, illuminate the way our community of inquiry engaged in a task related to Cuisenaire rods. I consider, as a researcher, that the analysis of these different excerpts is important as it reveals how our community of inquiry started, on which common ground and what elements were addressed during this first meeting.

The main characteristics of this first workshop are the following:

- Teachers' reaction to the task
- A discussion concerning the scientific concept of fraction
- The mismatch between Claire, pursuing her didactical aim and one of the teachers' questioning about the "rules" of our community
- Claire's lack of experience, as a didactician
- The dyadic structure of the communication
- The participants discussing students' difficulties with the use of symbols

I will now illustrate each of these characteristics through several excerpts, as presented below.

Workshop I: presentation of excerpts

The workshop began with Claire presenting to the teachers three different formations of Cuisenaire rods (see Figure 7), and inviting them to offer a description of the formations in the following way: first *using words*, and second *in a more mathematical way*.

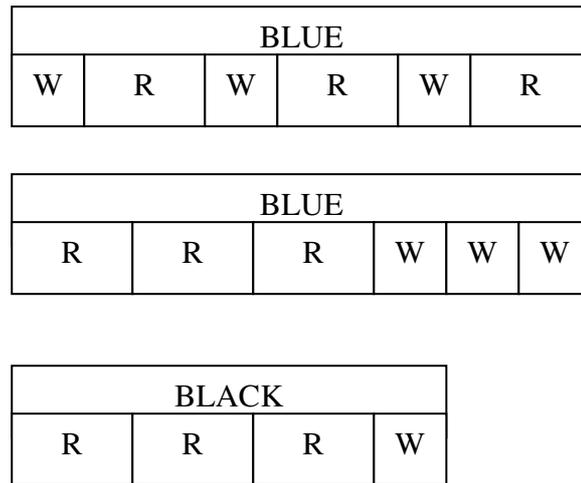


Figure 7: The different formations of Cuisenaire rods

The teachers mentioned that they had never seen such rods before and therefore had never had the opportunity to work with these previously. However, they did express their interest while Claire was arranging the rods on the table. The first excerpt presents both how the teachers react to the task and the discussion among the participants about fractions:

Teachers' reaction to the task

Excerpt 1

40. Claire: Now there are three different arrangements with Cuisenaire-rods on the table. First, I would like you to give a description of what you see, in words ...

41. John: ha, ha! (Mary is laughing)

42. John: I can see three groups, two of the groups look like ... or have the same number, the same colour, that is, it is the same in two of the groups. The third group has a different colour and not as many [rods]. So if this is going to have some value or something like that, I would say that two of these have the same value while the other one, ..., maybe, ..., four units or something like this, less in value, I do not really know because I do not know what these [rods] symbolize, they are organized quite nicely, almost parallel, with a little space between them

43. Claire: ok, Paul?

44. Paul: well, there is not so much to add, only that I thought about fractions when you put these on the table

45. Claire: ok, ..., fractions, why?

46. Paul: yes, because you have the long blue, or black, which represent a whole, then the other represent smaller parts of the whole, and I thought it was possible to think about fractions

47. Mary: yes, I thought about that too, .., first I looked at the colours, and then what was equal, and then I thought about fractions at once, as Paul did, there was some whole [rods] and then some half [rods] beside. So I connected at once with mathematics since this is what we are talking about, with fractions

48. Claire: yes ...

49. John: I would like to get an explanation concerning how you thought about fractions, where could you see fractions here?

50. Mary: These are whole, the blue and the black ones, and then you have smaller parts which might represent half ones and fourth ones and, yes!

51. John: alright! If I would have thought about fractions here, then we had to decide which of these were whole, eventually two wholes, and which of these that should be the whole (unclear) . . . This is the way I see it, I can't see fractions here!

(pause)

The teachers' reaction to the task is visible as responding to Claire's *didactical move* (40), they reacted with surprise (41), with John's exclamation and Mary's laugh. My interpretation of their utterances is that the teachers were not expecting this kind of task, both in terms of using manipulatives and giving a description using *words*, and this was the reason why they were expressing their surprise by laughing (compare with Workshop II where the teachers question Claire's choice of task). I consider the fact that the teachers did *not* formulate more explicitly their surprise as revealing of the stage within which our community was: this was the very first time we met and the first discussion we had all together. Thereby, there is a possibility that the teachers did not have enough confidence in our community of inquiry in order to question the kind of task that Claire proposed for this workshop. In using the term "confidence" I seek to refer to the fact that the teachers were probably unsure about *how* to act during the workshop and what kind of question to ask and, therefore they did not formulate any comments or questions concerning the task, as Claire proposed it.

The discussion concerning the idea of "fraction" started when John, after offering a description with words, wanted to address the mathematics which he supposed was behind these manipulative (42). He started by saying "*if* this is going to have some value or something like that" and formulated his hypothesis, emphasizing the fact that he was unsure both regarding to the value of the rods and to the choice of units. I understand his utterance as offering evidence of John's awareness of the difficulties in connection with the use of manipulatives. Claire was satisfied with John's description and she turned to Paul, inviting him to share his thinking with the other participants (43). It was Paul (44) who introduced first the idea of "fraction". Claire was curious (45) as to what Paul meant. Her repetition of the word "fraction" and question "why?" indicates an *information-seeking inquiry move*. She showed a willingness to seek clarifications and explanations for Paul's suggestion and to make these available to the other participants. In the following utterances (46 to 51), Mary seemed to say that, since the context of the activity was mathematics, she found it reasonable to assume that the proposed task had some connection with mathematics. I consider her recognition of the importance of the context within which the activity is situated as an illustration of the relevance of the social setting within which psychological tools

are addressed and discussed through a process of negotiation of meaning (see Section 2. 5. 3).

John was challenging Paul and Mary and asking “where could you see fractions here?” (49). By asking “where” he seemed to emphasize the difficulty, from his perspective, of establishing a connection between the different formations of Cuisenaire rods, as presented during this workshop, and the scientific concept of fraction. I consider his utterance as an *information-seeking inquiry move*, as he wanted to question Mary and Paul and thereby he showed a willingness to explore further their claims. I understand John’s utterance (51) as a *wondering inquiry move*, since he engaged in the process of exploring the idea of fraction and considered how it might be used in this specific situation in relation to the Cuisenaire rods. However, he clearly argued for the necessity to define, without ambiguity, which rod has to be considered as a whole, i. e. which rod might be considered as a unit. Otherwise, he claims, the scientific concept of fraction cannot be applied in this task. I consider that John’s argument offers an illustration of Vygotsky’s (1986) and Karpov’s (2003) focus on the importance of providing precise verbal definitions when addressing scientific concepts (see Section 2. 5. 1.). In the second excerpt, which followed right after the first one, I offer evidence of a mismatch of focus between Claire, in addressing her didactical aim, and John, in questioning the “rules” of our community.

Mismatch between John and Claire’s focus

Excerpt 2

52. Claire: the next point I was thinking about was, if it was possible to have the same description, but in a mathematical way, because now we used words. Now, we could do it in a mathematical way ...

53. John: is it possible to move these [rods], or do they need to stay like this?

54. Claire: what are you thinking about?

55. John: just do this [John moves the rods], I am not sure I can judge by my eyes. Like this yes, just look at how big these are in relation to each other

56. Mary: hmm, hmm

57. John: that was it, ..., it works ...

58. Mary: two of the small ones are equal to one red

Here Claire’s didactical purpose was to explore the possibility to introduce symbols in order to offer a more mathematical description of the Cuisenaire formations. As a way towards this goal, she referred to “the same description, but *in a mathematical way*” (52), leaving the teachers to interpret what “in a mathematical way” could mean. However, the mismatch between John and Claire’s focus became visible as John asked her (53) whether it was possible to move some of the Cuisenaire rods, while Claire, focusing on her didactical aim, was not prepared for his question and reacted by being surprised and asking him to explain his

thinking (54). John indicated that moving the rods allowed him to check their relative sizes (55). Right after, he put back the rods into their original pattern, while Mary explained her perception of his actions (58). I consider that this excerpt offers evidence of how making the mismatch visible enables the participants to inquire into our mutual engagement within our community of inquiry. John wanted to engage and explore the second part of the task. However, *before* doing this *he needed to ask Claire about moving the rods*, that is, he was asking about the kind of “rules” that Claire wanted to establish for the community. In that sense, it seems that, *from John’s perspective*, Claire would have decided, in advance, the way the community might function. I consider, as a researcher, that this excerpt illustrates how our mutual engagement was negotiated during this first workshop and how through *information-seeking inquiry moves*, John, Mary, and Claire were seeking to establish a common platform before engaging with the task. At the same time, this excerpt shows Claire’s lack of confidence in relation to how the community of inquiry might function, in the sense that she was not prepared for John’s question and, therefore, was unsure about how to respond to it.

In Excerpt 3, I present evidence for Claire’s lack of experience, as a didactician, as she experienced difficulties to articulate what she meant by “in a mathematical way”. In addition, this excerpt illustrates the dyadic structure of the communication, as it occurred during the first workshop. I present Paul’s answer since it was more articulated than those of Mary’s and John’s. The third excerpt starts with Claire inviting the teachers to share with the other participants their thoughts concerning how to formulate a description of the Cuisenaire formations “in a more mathematical way”. Paul’s answer is presented in details in the third excerpt.

Negotiating the meaning of “in a mathematical way”

Excerpt 3

62. Claire: Does anybody want to start?

⋮

82. Claire: ok, Paul?

83. Paul: the respective position is different for all three

84. Claire: yes?

85. Paul: the one has [unclear], and there is a pattern, with each second

86. Claire: ok, and if we want to give a little more accurate description ...

87. Paul: yes, I can take the first one. Here they all are, ..., all, or, all the three red ones are close to each other and all the three white ones are close to each other

88. Claire: ok, and how would you have written it?

89. Paul: now, I don’t understand what you mean?

90. Claire: what you just said

91. Paul: yes ...

92. Claire: all the three

93. Paul: the three red ones stand beside each other, after each other, and the three white ones stand also after each other
94. Claire: yes, what you just said with words now, how would you write it down?
95. Paul: ..., the blue, how the red and the white [rods] stand along the blue, placed in two groups ...
96. Claire: ok, yes, and what about this one?
97. Paul: over there, they form a pattern ...
98. Claire: yes, and how could you write it?
99. Paul: I would say that, ..., the red and the white [rods] stand along the blue, and are placed red, white, red, white, red, white
100. Claire (writing on a flip chart): is this what you mean?
(Claire writes **Blue = R + W + R + W + R + W**)
101. Paul: yes
102. Claire: is that correct?
103. Paul: yes, it is that way I see it
104. Claire: ok, hmm
105. Paul: yes, this is what I mean

After some discussion with Mary (63-81, not included), the negotiation of meaning of “in a more mathematical way” started as Claire turned to Paul and asked him to present his thoughts (82). Paul started by reformulating a description of the Cuisenaire arrangements, provoking Claire’s *didactical move* (86) in which she repeated that the focus now was on offering “a mathematical description”. She used “if *we* want to give a little more accurate description”, putting emphasise on the mutual engagement of the whole group and using the pronoun “we” in an inclusive way. Through Paul’s utterances (87 – 99) it is possible to follow his attempt to make sense of Claire’s didactical move (86), both questioning the meaning of her question (89), and offering a precise description of the Cuisenaire rods (87, 93, 95, 97, 99). Thereby, it seems that, through Paul’s utterances, it is possible to perceive his lack of understanding concerning Claire’s didactical aim, as the coordinator of the workshop. Likewise, Claire’s lack of experience, as a didactician, about offering an explanation of “in a mathematical way” in a more articulated way, is visible through the repetition of her *didactical moves* (86, 88, 94, 98). Claire’s goal was to stimulate Paul to introduce symbols, but she was not able to help him by formulating her question using different terms. Therefore, the way the meaning of the terms “in a mathematical way” has been negotiated can be traced as Claire tried to start from Paul’s explanation (90, 92), while Paul repeated his description, trying to give as much detail as possible (93, 95, 97, 99).

The process of negotiating the meaning of “in a mathematical way” moved forward as Claire took the initiative to write on a flip chart, asking Paul, at the same time, to confirm her interpretation (100). Looking at the notation Claire used, it is a form of shorthand of Paul’s description, taking the first letter of the words “red” and “white”. Paul agreed

(101) with Claire's written formulation and confirmed (103, 105) when Claire asked explicitly if these notation accorded accurately with his description (102). There followed a discussion concerning the possibility to write Paul's description ($\text{Blue} = \mathbf{R} + \mathbf{W} + \mathbf{R} + \mathbf{W} + \mathbf{R} + \mathbf{W}$) in a shorter way, using $3\mathbf{R} + 3\mathbf{W}$. The advantages (shorter notation) and disadvantages (no indication of order of the rods) were acknowledged.

I consider that several aspects emerging from Excerpt 3 are important to emphasise. The first aspect concerns the purpose of the negotiation between Paul and Claire which was to address the meaning of the terms "a description in a mathematical way". During this excerpt we, as researchers, can follow how Paul tried to make sense of what Claire was asking, and, perhaps building on his confidence in Claire as a previous colleague, claiming that he could not engage in the task because he did not understand what Claire, as the coordinator of the workshop, was aiming at. On the other hand, Claire was unsure about how to help Paul to move from a description using words to a mathematical description using symbols. Her lack of confidence, as a didactician, is visible since she was not able to provide explanations or indications to the teachers when they were struggling in order to understand what she meant by "in a mathematical way". In that sense, she was unsure about *how* to move forward to her didactical aim and *how* to promote the use of algebraic notation. In addition, she avoided using the term "symbols" as her aim was to see how the teachers would interpret the idea of "in a mathematical way". As such, this excerpt offers evidence of both Paul's lack of confidence concerning Claire's didactical aim, and Claire's lack of confidence, as a didactician. In that sense, Paul and Claire engaged mutually in the process of negotiating what "a description in a mathematical way" could mean and, by addressing this issue they were able to negotiate how their joint enterprise could look.

The second aspect concerns the choice and the use of symbolic notation. It was Claire who decided to introduce the symbols \mathbf{R} and \mathbf{W} as shorthand for Paul's description. The choice of symbols was *not* discussed, and in that sense, the discussion between Paul and Claire what not about how to choose symbols which would represent Paul's description, rather the discussion concerned the teachers' understanding of Claire's didactical aim.

The third aspect concerns the structure of the discussion during this workshop. In Excerpt 3, I presented the discussion between Paul and Claire. During this discussion neither Mary nor John participated. The same pattern is visible when Claire asked Mary or John about their thinking. Thereby, I consider that one of the characteristics of Workshop I is the strong dyadic nature of the interaction between Claire and the teachers since, as Claire asked the teachers about their thinking in rela-

tion to using words, and using symbols to describe some Cuisenaire-rods arrangements, each teacher answered one after the other, without being interrupted by the other.

After the three excerpts above, when each of the teachers had explained his/her way of writing the Cuisenaire formations, there followed a discussion on how the students use or fail to use symbols. During this discussion the teachers made visible their concern about the difficulties pupils experience when they engage in a task related to algebra. I consider that the issues emerging from the discussion are important since they reveal some of the reasons why the teachers agreed to engage collaboratively with Claire and to work with her during a school year. In that sense, the teachers shared their concern, the problems they experienced in their practice, and their wish to deepen their knowledge and expertise in algebra and algebraic thinking by interacting on the basis of our regular meetings. Furthermore, I consider that the discussion, as presented in Excerpt 4 and in the following utterances, played a central role in terms of community building, since it is through making available to the other participants ones' own teaching experience that the teachers were able to develop an awareness of common teaching experience. I consider that these characteristics bring evidence in terms of establishing a common ground for our community of inquiry (see Section 2. 2. 3. concerning mutual engagement, joint enterprise, and shared repertoire) and conceptualizing our group as a community (Wenger et al., 2002) within which inquiry into own practice plays a central role.

In order to develop further on this issue I want to recall Lindfors' (1999) two types of inquiry acts: *information-seeking* and *wondering*. In addition I proposed to add a third kind of inquiry act: *experience-sharing* (see Section 2. 2. 6). I consider that Excerpt 4 and the utterances, as presented below, offer evidence for this kind of inquiry, where the teachers, by sharing with the other participants their own experience, showed a willingness to develop further their understanding of own teaching practice.

In the following I present an excerpt from our discussion which emerged after working on the mathematical task. I also offer two utterances from Paul and John. I consider these data as important in terms of making visible the participants' concern about students' difficulties with algebra and the use of symbols.

Experience-sharing inquiry acts

Excerpt 4

147. Mary: it is the letters that come in, they make it [to introduce algebra] so difficult at once, I think

148. Paul: they [the letters] mean nothing

149. Mary: yes, yes ...

150. Paul: it is just a letter

151. Mary: yes, they [the pupils] think they [the letters] just are there, they don't understand that they stand as symbols for something

The difficulties Mary and Paul referred to are well known from research literature on algebra: students usually do not comprehend the use of letters as generalized numbers or as variables (Booth, 1984; Küchemann, 1981). According to Mary (147), the introduction of letters, algebraic symbols, is a source of difficulties for students, since, as Paul argued (148), these letters do not endorse any meaning, these are “just” letters (150), and therefore the students do not really know how to operate on these letters (151).

Excerpt 4 illustrates the fact that both Mary and Paul share the same experience concerning the introduction of symbols, and have the same concern about understanding students' difficulties. In addition, Mary and Paul's description is also in accord with Claire's own experience, as she had taught both in lower and higher secondary school, and as she could follow students' struggle with symbols. In that sense, the identification and recognition of students' difficulties with algebra became part of a common understanding between the three teachers and Claire, and as such I consider that it became a basis for our mutual engagement. Thereby, through addressing and making visible this common ground, it seems that the participants were experiencing a sense of “community”.

In the next utterance, Paul explained in more detail how he understood the students' difficulties, from his perspective.

Paul's articulation of students' difficulties

Paul's utterance

155. Paul: I am not sure about this, but I am thinking about those who can make it in mathematics, then it doesn't seem to be so difficult with letters, but for them who have problems, [they] think that mathematics is numbers, and “for God's sake, keep these letters away, I want to work with numbers”, this is what I feel, they are not in that stage where they can work in an abstract way, the fact that letters can symbolize something, they [the pupils] are not there, maybe, ...

I understand Paul's utterance as an attempt to build on previous discussion (see Excerpt 4). It seems that in order to offer a deeper articulation of the students' difficulties, Paul needed to establish a distinction between low and high achieving pupils, the latest he referred to as “those who can make it in mathematics”. Paul seemed to say that, from his per-

spective, low achieving pupils' understanding of mathematics is guided by numbers, as they consider that "mathematics is numbers" which means that a mathematical task is about doing some operations with numbers, without introducing any symbols: "for God's sake, keep these letters away, I want to work with numbers". Paul's explanation for this understanding of mathematics is phrased in cognitive terms, as Paul referred to pupils not being in a "stage" from which they could work in an abstract way. Researchers frequently attribute students' poor performance to some cognitive developmental constraints. For example, Collis (1974) and Küchemann (1981) interpreted students' responses on algebra problems from a neo-Piagetian perspective, while Filloy and Rojano (1989) claimed the existence of an historical and individual "cut-point" separating arithmetic from algebraic thought.

John's articulation of students' difficulties

John's utterance

162. John: I don't know if we talk about the same kind of pupils, if there are several who have problems with mathematics, those who get low grades, when they get letters or an expression with letters, so they are, I think, dependent on learning an algorithm, not necessarily understanding what they are doing, so they rescue themselves by the easiest way, because they learned: this is the way I can do, I have the feeling this is the way [the pupils work], I think

The *experience-sharing* inquiry act which was initiated in Excerpt 4 was followed by John as, building on Paul's utterance, he considered the role played by algorithms in pupils' learning. John referred to pupils who obtained low grades, and claimed that these are dependent on having an algorithm available in order to be able to engage in a task which includes letters or expressions with letters. Using John's terms: "they rescue themselves by the easiest way". The way pupils perceive algorithm is addressed by Arthur Baroody and Herbert Ginsburg (1986), claiming that "For most children, school mathematics involves the mechanical learning and the mechanical use of facts". John contrasted the learning of algorithms with understanding the subject matter, and his distinction might be related to Skemp's (1976) epistemological considerations concerning instrumental and relational understanding.

The informal a posteriori analysis

As explained earlier (see Section 3.4.2), Claire made an informal *a posteriori* analysis right after each workshop where she noticed, in French, her own impressions and reflections. Following on Gravemeijer's (1994a, 1994b) cycle of developmental research, these intuitive analyses constituted the background from which Claire planned a new "thought-experiment" for the next workshop.

After our first workshop Claire wrote:

I have a good feeling after this first workshop: it seems that the teachers engaged well in the task, and that they were interested in working with the Cuisenaire rods. We also had an interesting discussion about students' difficulties with algebra. (from my Diary, 16.06.04, translated from French)

It was from this background that Claire developed an *a priori* analysis or “thought-experiment” for Workshop II. Especially, the positive reaction to the Cuisenaire rods, as experienced during Workshop I, encouraged Claire to envisage the possibility to introduce another kind of manipulatives in relation to the task for Workshop II.

The formal a posteriori analysis

The formal *a posteriori* analysis was conducted *after* finishing collecting data, and therefore cannot be considered as influencing on the *a priori* analysis of the next workshop. When presenting the mathematical task to the teachers, Claire had a well defined didactical and pedagogical strategy. Her didactical aim was twofold: it concerned the choice and the use of algebraic notation and issues related to community building. She chose, as a pedagogical means, to engage in a task related to Cuisenaire rods. The presentation of the different excerpts above has allowed me, as a researcher, to deepen the main characteristics of this workshop, as presented in the beginning of this section. In the following, I offer a synthesis of these aspects.

In the first excerpt the teachers' reaction to the task and their discussion concerning the scientific concept of fraction were presented. Their reaction (just laughing without asking any questions) shows that they were surprised and unsure about how to react to the task and did not know how to act within our community. My interpretation of the teachers' reaction is that these aspects, as explained above, offer evidence of the teachers' lack of confidence in our community of inquiry. Furthermore, I consider the fact that the idea of fraction emerged from a description of the different Cuisenaire formations as evidence for the relevance of the social setting within which the activity is situated (*mathematical* workshop). The second excerpt offered an example of a mismatch between Claire, pursuing her didactical aim and one of the teachers questioning the “rules” of our community. Both their reaction to the task and the questioning about the “rules” bring evidence for a lack of confidence in our community of inquiry in terms of being unsure about *how* to act. The mismatch is still visible in Excerpt 3, as Paul was struggling to understand what Claire meant by “in a mathematical way”. At the same time, Claire was unsure about *how* to explain to Paul what she was aiming for, and thereby the excerpt presented an example of her lack of didactical experience. It was Claire who introduced the symbols **R** and **W**, without discussing with the teachers how to choose these symbols.

Both in Excerpt 4 and in Paul and John's utterances, the purpose of the discussion enabled the participants to make visible and to establish a common ground for our further work, as *all* participants have been engaged in mathematics teaching at lower secondary school and thereby have developed awareness concerning students' difficulties with algebraic notation. Finally, a last characteristic of Workshop I consists of a strong dyadic structure of the communication, as illustrated in Excerpt 3.

I consider that, by highlighting issues concerning the teachers' reaction to the task and the mismatch between Claire and one of the teacher's aim, the formal *a posteriori* analysis offers Claire the opportunity to enhance her didactical knowledge. In addition, the excerpts offer evidence of Claire's growth of understanding as to the didactician role and its associated issues.

4.1.2 Workshop II: Difference/tension between Claire's and the teachers' views

A priori analysis of the mathematical task

The data presented in this section have been selected to provide evidence of key points from Workshop II. The mathematical task presented during this workshop concerned *even and odd numbers* where the design of the task was inspired by Burton (1984). The rationale for choosing this task was almost the same as in Workshop I: on one hand, Claire's didactical aim was to provoke a discussion concerning the choice and use of algebraic notation, and especially to address the standard notation for even/odd number ($2n$ and $2n+1$). On the other hand, she wanted to address further issues related to community building in terms of offering the participants an accessible task within which they could engage collaboratively.

In choosing a task related to even and odd numbers, Claire thought that these scientific concepts were well known for the teachers, as part of their teaching practice and, therefore, all participants would engage easily in the task. Thereby, Claire's goal or target knowledge, as a didactician, was to explore and to give an example of the power of algebraic notation. Furthermore, her pedagogical strategy, as a means to work forward to symbolic notation, was to introduce a mathematical task exploring addition with even and odd numbers. The reason for addressing algebraic notation is that in order to address and enhance algebraic thinking, it is crucial to recognize the importance of the process of being able to search, recognise and identify patterns and underlying structure (Sierpiska, 1993a) and then to seek to express this generality by using symbolic notation (Sfard, 1995), see Section 2. 5. 4. Therefore a central issue was to capture this urge to enable the teachers to get some sort of perception of algebraic essence and this was Claire's motivation for choosing of the task.

In addition, following Gravemeijer (1994a, 1994b), Claire considered that her informal *a posteriori* analysis of Workshop I provided feedback from the practical experience with this first workshop from which she could envision a new teaching experiment during Workshop II (see Section 3.3.3). Therefore, by building on the experience and reflections from the informal *a posteriori* analysis concerning the use of Cuisenaire rods during Workshop I, Claire decided to include the use of manipulatives (small plastic squares) whose purpose was to illustrate the geometrical properties of even and odd numbers. However, the analysis of the data reveals that some tensions emerged in connection with the introduction of these manipulatives.

The main characteristics of Workshop II

The main characteristics of the second workshop are the following:

- The teachers' reaction to the task: questioning its relevance for lower secondary school
- The mismatch between Claire, pursuing her didactical aim, and the teachers' unwillingness to engage further in the exploration of the task by exploring the manipulatives
- Claire's lack of experience as a didactician
- Less emphasis on a dyadic structure of the communication
- The participants' reaction to the scientific concept of proof

I will now illustrate each of these characteristics through several excerpts, as presented below.

Workshop II: presentation of excerpts

The mathematical task started by Claire asking Mary, Paul and John: *What happens when we add even and odd numbers?*

However, *before* getting into the task, the teachers discussed Claire's choice to address even and odd numbers:

Questioning the adequacy of the presented task

Excerpt 1

54. John: isn't it very [relevant] for grade 8?

55. Paul: yes

56. Mary: at least it is there [in grade 8] that they [the pupils] work with even and odd numbers and begin to repeat again

57. John and Paul: yes, yes

58. Mary: and prime [numbers]

My interpretation of John, Paul, and Mary's discussion is that the teachers were questioning the usefulness and the relevance of the task related to even and odd numbers as a means to work at lower secondary level. Through his *information-seeking inquiry move* (54), John engaged with the two other teachers in an evaluation of the task, addressing explicitly the issue concerning the relevance and choice of pedagogical means, as

presented by Claire, and offering details concerning the curriculum for Grade 8 (56, 58). I consider that this excerpt is important in terms of offering evidence of the development of the teachers' confidence in our community, since they felt confident enough to discuss and evaluate the relevance of the task, as proposed by Claire. Contrasting with the teachers' reaction, as presented in Excerpt 1, Workshop I, the teachers were *now* in a position to articulate their questions and to discuss with each other the central issue of choosing a task.

After a silent pause during which they were writing individually in their notepads, trying different numerical examples, the teachers shared their thinking with the other participants.

In Excerpt 2, I offer evidence of how the negotiation of number relationships began. I consider that this second excerpt offers an example of an inquiry act into another participant's understanding of a mathematical task, as Claire was inquiring into Paul's understanding of the even and odd numbers' task.

Claire inquiring into Paul's understanding of the task

Excerpt 2

81. Paul: it depends on how many you take, if you have two or three

82. Claire: two or three what?

83. Paul: yes, either even numbers or odd numbers, whatever it is, then the result will change

84. Claire: can you go a little deeper?

85. Paul: yes, therefore if you just put together even numbers, so it will be, you will never see odd numbers, but if you put together odd numbers then it depends on how many numbers you take, if you take even number of odd numbers (laugh) to put it that way

86. Claire: an even number of odd numbers?

87. Paul: yes (Mary and John are laughing) and for an odd number of odd numbers, the result will then be influenced!

88. Claire: ok, then you get this [result], do you agree, disagree?

89. John: at once, it seems that this is the pattern that ...

Claire's inquiry into Paul's understanding of the even/odd numbers task is visible through her *information-seeking inquiry moves* (82, 84) where she tried to get a deeper understanding of Paul's claim (81). I consider that the nature of Claire inquiry moves was twofold as it addresses inquiry in Paul's understanding of the task, and at the same time it offered Claire the possibility to address issues related to community building, in the sense of sharing and making accessible Paul's understanding to the other participants. Thereby, Paul got the opportunity to express, develop, and make visible his thoughts to the other participants by explaining his findings in a gradually more articulated way (81, 83, 85, 87), distinguishing between adding even numbers together and adding odd numbers together. By repeating the last part of Paul's sentence (86), Claire offered a

synthesis of Paul's result and she invited the others to share the meaning of "an even number of odd numbers". Through her *didactical move* (86), Claire was trying to move the teachers towards her didactical goal (the use of algebraic symbolism). I also consider the repetition of the last part of Paul's sentence as a step into inviting the teachers to share the meaning of "an even number of odd numbers", and in that sense as an important step in community building.

I want to emphasize the fact that Paul (85), and Mary and John (87), were laughing when Paul and Claire were talking about "an even number of odd numbers". I consider, as a researcher, laughing as evidence of community building in the sense of developing confidence in our community of inquiry since the teachers felt free to react spontaneously. In her next utterance (88), Claire actively involved Mary and John in negotiating Paul's meaning with the task. I consider that inviting other participants to discuss and negotiate is another move towards community building. The negotiation of the meaning of the task continued as John seemed to agree with Paul's explanation (89) and showed a willingness to engage further in the exploration of the mathematical task, claiming that "it seems that this is the pattern here".

As explained earlier, Claire's didactical aim, with this task, was to explore and give an example of the power of algebraic symbolism. In Excerpt 3, I present evidence for a mismatch between Claire's, pursuing her didactical aim, and the teachers, as they tried to understand what Claire meant by "in a more mathematical way".

Mismatch between Paul and Claire's focus

Excerpt 3

90. Claire: can we write it [the result], not in words, but in a more mathematical way?

91. Paul: but I have, I have just done it this way, I don't know if it was what you had in mind?

92. Claire: yes, now it is [written] with specific numbers, but what you said, you were talking about a generalization [referring to Paul's utterance 85]

93. Paul: hmm

94. Claire: how would you write it?

95. Paul: oh, yes, now I understand what you ask, so, (laugh), then you have to write even numbers plus even numbers is like even numbers, isn't it, is it what you ...?

The mismatch between Paul and Claire's focus became visible as Claire, recognizing Paul's result, invited the teachers to express it "in a more mathematical way"(90), not only expressing it verbally. Claire's utterance (90) is a *didactical move*, since Claire *knows* very well that it is possible to write the result in a more mathematical way by using standard algebraic symbolism for even and odd numbers ($2n$, $2n + 1$).

Thereby, her didactical challenge consisted of encouraging the teachers to introduce algebraic notation without expressing herself the idea of “symbols”. This particular move is articulated as a question aiming to invite the other participants to work towards Claire’s didactical goal. As a researcher, I also want to emphasize the use of the pronoun “we” in “can *we* write it ...”. Rowland (2000) presents a particular use of this pronoun, as the exclusive “We”, where the speaker is appealing to an unnamed “expert” community in order to add authority to a certain kind of classroom practice. I consider that the way the pronoun “we” is used in Claire’s *didactical move* (90) is, on the contrary, an *inclusive “we”*, where Claire associated herself with the three teachers in the next step of the mathematical task. Therefore, using the pronoun “we” in an inclusive sense addresses issues related to community building, as Claire did not differentiate herself from the teachers.

The mismatch between Paul and Claire’s focus is visible as it seems that Paul’s focus was on what Claire “had in mind” and that he tried to understand what she was aiming for (91 and 95), while Claire’s focus was on moving toward the introduction of algebraic symbolism. Similarly to Workshop I (see Excerpt 3), she did not mention the idea of algebraic symbols, waiting to see how the teachers negotiated the meaning of “in a more mathematical way”. Paul’s reaction (91) showed that he referred to his search for patterns with numerical examples, as written in his notepad, “I have just done it this way”, and he did not understand the meaning of Claire’s question. By asking “I don’t know if it was what you had in mind?”, there is a possibility that Paul showed enough confidence in Claire, as a colleague, by asking if what he did (numerical examples in the notepad) was what Claire, as the responsible for the workshop, had in mind (“what *you* had in mind?”). In that sense, Paul’s question reveals a desire to engage further in our community, and I consider his question (91) as an *information-seeking inquiry move*, which is related to both Claire’s didactical aim and to the community building. Paul’s inquiry move could also suggest a mode of dependence on the didactician, in terms of Paul trying to think about *what Claire wanted him to do*, rather than Paul taking his own initiative in engaging with the task, and perhaps it is not clear to him what “taking initiative” might look like. I will come back to this aspect in Section 4.1.4.

Claire pursued her didactical aim by first agreeing with Paul (92): the result is written with numerical examples (on his notepad), but also by referring to Paul’s previous utterances (85, 87), where his words suggested a generalisation of even and odd numbers. By repeating her *didactical move* (94), Claire emphasized the challenge: how to write “in a more mathematical way” Paul’s generalisation. I consider that the repetition of her didactical moves offers an example of the iterative nature of

didactical acts (see Section 2. 2. 6). The mismatch is still visible as Paul, laughing, said that he understood (95), and he proposed to write addition of even numbers, asking for Claire’s approbation (isn’t it, is it what *you...*?). Again, his question revealed a step toward seeking clarity in the task through searching an understanding of Claire’s purpose. On the other hand, Claire was struggling with *how* to express what she wanted without telling him what to write. In that sense, this excerpt offers evidence of how the negotiation of the meaning of the task, and especially the meaning of “in a more mathematical way” was gradually clarified through addressing the mismatch between Paul and Claire’s focus.

Thereby, I consider that Excerpt 3 offers an example of how the meaning of Claire’s didactical move (90) has been negotiated. Through the repetition of *didactical moves* (90, 92, 94) it is possible to follow Claire, acting as a didactician, as she tried to achieve the transition from expressing patterns which were observed through generic examples, to the introduction of symbols. By referring to Claire, as a didactician, I want to emphasize the fact that from the teachers’ (and particularly Paul in this excerpt) perspective, it seemed that Claire was considered as the leader, the expert, the one who knows. On the other hand, from Claire’s perspective, acting as a didactician meant working and building on the teachers’ ideas and trying to get them to move towards her didactical goal which was the introduction of algebraic symbolism.

In Excerpt 4, I present how the teachers engaged further in the task and addressed Claire’s didactical move: “write the result in a more mathematical way”. Especially, Excerpt 4 presents Paul’s idiosyncratic notation, and how Paul, John, and Mary were engaging collaboratively in the process of expressing Paul’s results.

Working toward algebraic symbolism: Paul’s idiosyncratic notation

Excerpt 4

(Claire invited Paul to write his results on a white board.)

Paul writes: $e. n. + e. n. = e. n.$)

99. Paul: then if you got, hmm ...

100. John: odd numbers

101. Paul: odd numbers, yes, then it is dependent of the number (of odd numbers)

(Paul writes: $o. n. + o. n. = e. n.$)

102. Mary: yes, but the way you wrote it down now, it is yeah (unclear) ...

103. Claire: yes, and I think you said something about having an odd number of odd numbers

104. Paul: yes, then I must look at ... (pause)

105. Claire: can you write, as an example, ...

106. Paul: yes, then if you take, ..., if you write, yes, ... (pause)

107. Claire: if it is difficult to write it generally, you can take three as an example

(Paul writes on the white board: $o. n. + o. n. + o. n. = o. n.$)

I consider that Excerpt 4 brings evidence of how the teachers worked forward to an expression of Paul's results using symbols. This process engaged all three teachers, as John was helping Paul (100) to articulate his thoughts, and Mary (102) was looking critically into what Paul just wrote. Thereby, the meaning of Claire's *didactical move* (90) "in a more mathematical way" was negotiated by all teachers, and I argue that this way of discussing, helping each other, arguing, negotiating, and inquiring into the mathematics is a crucial step in the participants' development of confidence within our community of inquiry. The teachers explored and negotiated collaboratively the meaning of Claire's didactical move, and through inquiry moves into the mathematics (99 to 102), they were getting opportunity to develop confidence both into the mathematics (engaging in what "in a more mathematical way" might mean) and into our community of inquiry (engaging collaboratively in the negotiation of the meaning of these words).

Claire, following the teachers' attempt to express Paul's result in a mathematical way, proposed that they could consider an odd number of odd numbers (103). In that sense, she was acting as a didactician drawing attention to one of the things Paul said (85) concerning adding an odd number of odd numbers in order to help Paul formulating his thoughts. Claire's moves (103, 107) have a didactical purpose, as these aim to help the teachers to direct their awareness to the addition of an odd number of odd numbers. As such, by introducing his idiosyncratic notation "o. n. + o. n. + o. n. = o. n.", Paul showed how he interpreted Claire's suggestion (107).

An important feature emerging from Excerpt 4 concerns the nature of Paul's notation. His notation clearly shows idiosyncratic characteristics (Menzel, 2001) and has a strong syncopated nature (see Section 2. 6. concerning the history of symbolism and algebra). Here, the symbols consisting of the first letter of the words **o**dd and **n**umber, are used as a shorthand for an object (Küchemann, 1981). As such, these notation do *not* correspond to what Claire was expecting from the mathematical task. Still, Paul's answer is relevant, for Claire as a researcher, as it indicates the different steps in Paul's development of algebraic thinking. However, it is *not* satisfactory for Claire, as a didactician, whose aim was to address and use algebraic symbolism for odd and even numbers. There is a possibility that Paul was influenced by Claire's use of symbols during Worksoop I, as she wrote **Blue = R + W + R + W + R + W** to express Paul's thinking. Still, I consider that there is some differences between Paul and Claire's notation: the Cuisenaire rods are not scientific concepts per se, rather they might be considered as support to introduce various scientific concepts. On the other hand, even and odd numbers are examples of scientific concepts which are represented by the standard notation $2n$ and

$2n+1$. In addition Paul's notation is idiosyncratic of nature, and even though there is a sense of generality, this notation is not functional since one can not operate on and with it.

Claire's aim was to guide the teachers to the standard notation used in algebra to denote even and odd numbers ($2n$ and $2n+1$). Considering Paul's answer, Claire's didactical aim was *not* achieved and her struggle here consisted of finding a way to move from Paul's notation to the standard notation for odd and even numbers. This stage is important for Claire the researcher, as it reveals Paul's way of generalizing the observed pattern, as well as for Claire the didactician, as she wanted to go further. As explained in the *a priori* analysis of Workshop II, Claire had a positive experience with the introduction of manipulatives during Workshop I and she wanted to build further on it and to present to the teachers another kind of manipulatives. She had thought of doing this by using small squares in coloured plastic which were chosen and brought in order to illustrate the geometrical representation of the standard notation. The aim with using these manipulatives, considered as a means in Claire's pedagogical strategy, was to illustrate the geometrical properties of even and odd numbers (see Figure 8).

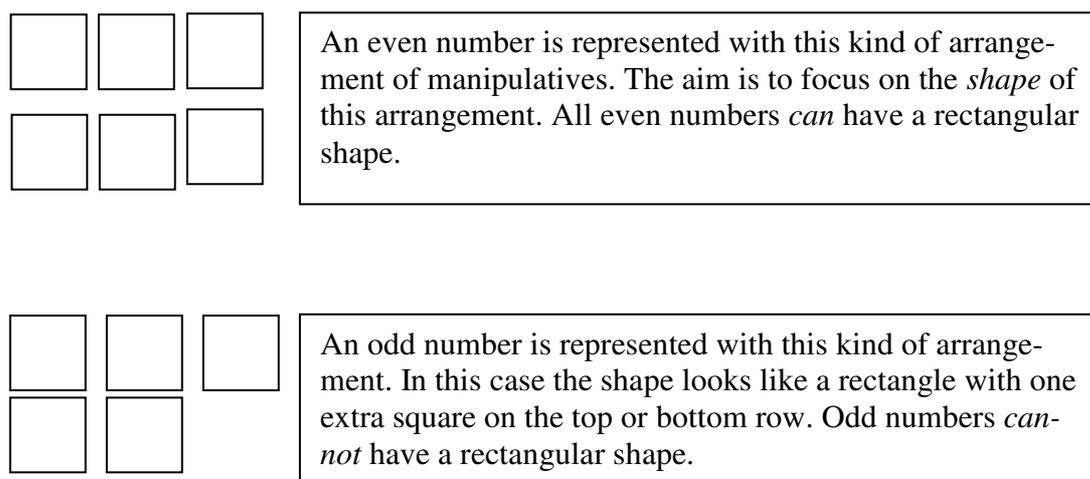


Figure 8: Even and odd numbers' geometrical representation

Here the manipulatives were intended to illustrate how geometric figures can be used to deal with some problems involving even and odd numbers and to offer possibilities to discover properties of these numbers under addition.

Through Excerpt 5, I bring evidence for both how Claire, pursuing her didactical aim, wanted to introduce the manipulatives to the teachers, and how they reacted to these. This excerpt also offers an example of Claire's lack of experience, as a didactician, since she was not able to achieve her didactical aim by using the manipulatives. In addition, the

teachers' attitude towards the scientific concept of proof is presented in the last utterances.

Introducing the manipulatives: the teachers' reaction

Excerpt 5

117. Claire: I took these (the manipulatives) with me as you see, is it possible to use these?

118. Paul (with a loud voice): I would have worked on numbers, I wouldn't have thought about manipulatives

119. Claire: ok?

120. Paul (with a loud voice): automatically

121. Claire: no, ok, you would have worked on numbers, this means that you would have ...

122. Mary: put in numbers here

123. Claire: that is, for example, $8 + 9$ (Claire is referring to what Paul wrote on his notepad) as an example for even number plus odd number?

124. Paul: yes, yes

125. Claire: and you got 17, even number plus odd number (Claire is pointing to what Paul wrote earlier on the white board), it is this one, isn't it?

126. Paul: hmm ...

127. Claire: and then you took $4 + 5$ (Claire is reading from Paul's notepad), this is also even number plus odd number give odd number, and then you took $4 + 6$, even number plus even number give even number, and then $3 + 5$, odd number plus odd number is even number, ok, but now, here we have four examples with numbers, and how can I be sure that this (result) is always valid? here we have four examples and the claim is that this is always valid, can you convince me that the transition (from examples to a generalisation) is valid?

128. Paul: then you touch what is called a mathematical proof and mathematical (unclear) and here I must admit that I am not good at all, to make it clear, I can't deduce general results from things that seems to be like this

129. Mary: I think this was terribly difficult

130. Paul: yes ...

The teachers' reaction came immediately after Claire's *didactical move* where, pursuing her aim to introduce algebraic symbolism, she proposed to use some manipulatives (117). I refer to her question (117) as a didactical move since she knows the answer to that question: yes, it is possible to use the manipulatives, as a geometrical representation of what "even and odd numbers" mean. Mary and Paul reacted immediately (118 to 122), and claimed that they would prefer working on numbers and thereby not consider the introduction of manipulatives. Further support for my interpretation of the teachers' reaction is offered by Paul's utterance, later on during the discussion about the introduction of even and odd numbers, when Claire proposed and showed how to use the manipulatives as a means to illustrate the geometrical properties of even and odd numbers:

Paul's utterance

218. Paul: some [pupils] will probably understand this, I mean, that odd numbers plus odd numbers give even numbers, but here (pointing to the manipulatives) you are going to ordinary [specific] numbers, so they [the pupils] do not manage to see the connection between these manipulatives and numbers. ... To see that this [geometric pattern of the manipulatives] is also valid for numbers. Because these [the manipulatives] are not numbers for them [the pupils], these are just pieces of plastic.

I consider that Paul's utterance (218) offered a clear articulation of the teachers' unwillingness to consider the introduction of manipulatives in the context of working with odd and even numbers, as it seems that Paul was anticipating his pupils' reaction to these manipulatives, considering these as just pieces of plastic and not as representing numbers.

Coming back to Excerpt 5, I consider as important, in terms of Claire's development as a didactician, to follow her response to the teachers: first Claire was surprised (119), as she was not expecting this kind of reaction from the teachers, especially since they had worked so readily with the Cuisenaire rods during Workshop I. Since Paul insisted (120), claiming that he would *automatically* have worked on numbers, it seems that his reaction could be understood as Paul reacting in terms of thinking himself in a teaching situation and introducing a task related to even and odd numbers (218). In that sense, his claims (118, 120) seemed to mean that I, Paul, as a teacher, would have worked with numbers, and would not have thought about introducing manipulatives. After her surprise, Claire felt that she could not pursue her idea concerning the introduction of manipulatives, as a pedagogical means and, at that moment, she felt that she had no choice and had to follow the teachers' suggestion (121).

I consider now, as a researcher, that her reaction shows a lack of didactical experience, as she could have chosen to illustrate the geometrical properties of $4 + 5$ and $4 + 6$ using the manipulatives (compare to Workshop V, where Claire adjusted *momentarily* her pedagogical strategy). The last part of her utterance (121) shows how she tried to adjust her pedagogical strategy and to figure out the consequences of Paul's claims in terms of her didactical aim, as she said "this (Paul's claims) means that you would have ...".

The way Claire adapted her pedagogical strategy is visible as she decided to start from Paul's notepad (123) and to address her didactical aim through the numerical examples which Paul had been working on during the pause. Thereby, Claire asked if $8 + 9$ could be interpreted as an example for addition between even and odd numbers, and she emphasised the connection between Paul's numerical example in his notepad, and what he wrote on the white board (125). Paul's utterances (124, 126)

seemed to indicate that he agreed with Claire's new pedagogical strategy.

Following the same strategy, Claire recalled first several of Paul's numerical examples from his notepad, establishing in each case the connection between the numerical examples and what Paul wrote on the white board (127). In the second part of her utterance, starting from "how can I be sure ...", she addressed again her didactical aim through a didactical move "can you convince me that the transition (from examples to a generalisation) is valid?". The aim of this *didactical move* was to encourage the teachers to introduce and work with algebraic symbolism. Here the idea of proof was addressed and it seemed that Paul and Mary recognized the challenge since they claimed that in order to answer her question it would be necessary to elaborate a proof which they felt rather difficult (128 to 130) (compare with the elaboration of a proof during Workshop IV).

I consider that Paul and Mary's utterances give me, as a researcher, evidence of both their confidence in our community of inquiry and their hesitation in the subject-matter, since they were able to share with the other participants their hesitations in relation to a possible elaboration of a mathematical proof as generalisation of observed patterns. However, I recognise, now, that it could have been possible to use the manipulatives to give an informal proof, and that, for Claire as a didactician, can be seen as a missed opportunity.

The informal a posteriori analysis

Right after the second workshop, Claire wrote her own impressions and reflections:

We had a good meeting today, with an interesting discussion concerning the even/odd numbers task. The challenge consisted of how to move from numerical examples to a generalisation. I was surprised about the way the teachers reacted when I proposed to use the manipulatives, such a difference from Workshop I. (From my diary, 07.09.04, translated from French)

These reflections constituted the background from which Claire elaborated an *a priori* analysis or "thought-experiment" for Workshop III. Especially, she considered as important the fact that the teachers, and especially Paul, used idiosyncratic notation as he tried to generalise his results (see Excerpt 4). The mismatch between Claire's goal, to use the standard algebraic notation $2n$ and $2n+1$, and Paul's own notation inspired Claire to focus on the power of algebraic notation as a topic for the next workshop.

The formal a posteriori analysis

When presenting the mathematical task to the teachers, Claire had a well defined didactical and pedagogical strategy. Her didactical aim was to explore and to give an example of the power of algebraic notation. Her

chosen pedagogical means was to engage in a task related to the addition of even and odd numbers and to explore it through the standard notation $2n$ and $2n+1$ for even and odd numbers.

The presentation of the different excerpts allowed me to deepen my account of the main characteristics of this workshop, as presented in the beginning of this section. In the following, I offer a synthesis of these aspects and establish a link with the main characteristics from Workshop I as a means to trace the development of both our community of inquiry and algebraic thinking.

In the first excerpt the teachers' reaction to the mathematical task was presented. Compared to their reaction during Workshop I (laughing, without any comments), this time the teachers discussed the relevance of the task, asking when in the curriculum pupils work with even and odd numbers. I consider this aspect as a step forward developing confidence in our community of inquiry in terms of being able to address and question Claire's pedagogical decisions. The second excerpt offered an example of how Paul and Claire engaged in negotiating number relationships. I consider the dialogue, as presented in this excerpt, as a parallel to the discussion in Workshop I concerning the description of the Cuisenaire rods *using words* (Workshop I, Excerpt 1). In both cases the teachers were able to formulate, verbally, their thinking concerning the task. The challenge appeared when Claire asked to elaborate the verbal description "in a more mathematical way". The mismatch between Claire, pursuing her didactical aim, and the teachers, questioning about the functioning of our community of inquiry, was illustrated through Excerpt 3, where Paul had difficulties to understand what Claire meant by "write it (Paul's result) in a more mathematical way". Her challenge consisted of encouraging the teachers to introduce algebraic notation for even and odd numbers *without* mentioning the idea of symbols. However, because of her didactical inexperience, she was not able to reformulate her question in a different manner, and therefore her discussion with Paul offers an example of how the mismatch was negotiated: Paul was hesitating, as he was not sure about how to engage in the mathematical inquiry, as proposed by Claire.

Paul introduced his notation "e. n." and "o. n.", as presented in Excerpt 4. Claire did not expect the way Paul generalised his result, using idiosyncratic notation of syncopated nature. Still, his notation has a sense of generality, however it does *not* correspond to the standard algebraic notation for even and odd numbers which Claire was aiming for. On the other hand, Paul's notation could be seen in relation to the notation Claire introduced in Workshop I with the consequence that the teachers might have been influenced by the previous workshop in their choice of notation. In the Excerpt 5, Claire considered that the introduction of the

manipulatives might be a means to work forward to the standard notation used in algebra to denote even and odd numbers ($2n$ and $2n+1$). The teachers' reaction showed clearly some differences and tensions between Claire's and their pedagogical views, as the teachers claimed that they would have preferred working with numbers. The teachers' perspective, and especially Paul's perspective, was not expected by Claire and, as a consequence, she adjusted her pedagogical strategy and worked with numerical examples. Thereby, the possibility to explore the geometrical properties of even/odd numbers was excluded. She changed her pedagogical strategy and, taking numerical examples from Paul's notepad, she emphasised the connection between selected numerical examples and Paul's notation on the white board. In response to her *didactical move* (at the end of utterance 127), the teachers gave evidence of their lack of confidence in mathematics in terms of elaborating a proof for that particular task. I consider that Claire's reaction reflects her inexperience, as a didactician, since a more experienced person might have related the teachers' numerical examples to the manipulatives, asking about the patterns emerging, and thereby having more chance to lead to the standard notation for even and odd numbers. I am, as a researcher, able to recognise this issue, by comparing with later workshops where Claire's experience developed and allowed a wider range of pedagogical possibilities.

Thereby, the tension between the teachers' and Claire's pedagogical view was not solved, in the sense that the standard, historical cultural well established notation $2n$, $2n+1$, which was Claire's didactical aim, was *not* addressed. However, and I want to emphasise this point, from this particular workshop it is *not* possible to deduce information related to the fact that these three teachers know or do not know the standard algebraic notation $2n$, $2n+1$ (this notation is part of the curriculum at Grade 9). All I can say is that, *while engaging with this particular mathematical task*, the teachers did *not* introduce the standard notation. Using Wertsch's (1991) formulation, "it is meaningless to assert that individuals "have" a sign, or have mastered it, without addressing *the ways in which they do or do not use it to mediate their own actions or those of others*" (p.29, my emphasis) (see Section 2. 5. 3).

In addition, they expressed the difficulty they experienced in elaborating a mathematical proof for this task. Therefore, my hypothesis is that *the nature of the mathematical task* might play a crucial role in the way the participants negotiate the mathematical task (compare with Workshop IV). I develop further on this issue later (see Section 4. 2).

Finally, a last characteristic of Workshop II consists of less emphasis on a dyadic structure of communication, since the teachers participated more spontaneously in the discussion, as compared to Workshop I.

What could have been done?

As I see it, there are two possibilities here in order to achieve the didactical aim which was to explore and to give an example of the power of algebraic notation: One is to face and address explicitly the difference or tension between Claire's and Paul's views. This implies insisting, and taking advantage of Claire's position as the course coordinator and using the manipulatives.

Another possibility was to follow Paul's arguments further than Claire did during the workshop, and while using numerical examples, illustrating the same principle as the one exemplified by the geometrical properties of even and odd numbers, as for example in:

$$\begin{array}{r} 15 + 13 \\ (14 + \mathbf{1}) + (12 + \mathbf{1}) \\ 14 + 12 + (\mathbf{1} + \mathbf{1}) \end{array}$$

where the two odd ones from the odd numbers add together to form an even number. Thereby, Claire could have offered this example as a means to a proof for the sum of two odd numbers.

I see these considerations as referring to Claire's learning, as a didactician, and it is through the analysis of this workshop, done afterwards by Claire the researcher, that these possibilities are articulated and made visible. In other words, Claire, as a researcher, is helping Claire to develop her didactical knowledge further.

4.1.3 Workshop III: Offering some insights into the historical development of algebra

A priori analysis of the mathematical task

The analysis of the two previous workshops showed that algebraic symbolism was not addressed in the way Claire wanted it: during Workshop I, there was no discussion concerning the choice of symbols and during Workshop II the standard notation for even and odd numbers was not introduced (see informal *a posteriori* analyses of Workshops I and II). Therefore, Claire's aim for Workshop III was to put emphasis on algebraic symbolism by focussing on the historical development of algebra (see Section 2. 6), and thereby to illustrate and highlight the power of algebraic symbolism. As such this workshop offered the opportunity to address the historical and cultural conventions from which symbolic tools derive their meaning (see Section 2. 5. 3). In order to address her didactical aim, Claire decided to change the organisation Workshop III and thereby the structure of this workshop was different from the one used in Workshops I and II. It consisted for a large part of an overview, given by Claire, of historical perspectives on algebra and, in addition, of

two tasks taken from the history of mathematics, one from Babylon time and one from Renaissance time.

The main characteristics of Workshop III

- Teachers showing interest in the historical perspective on algebra
- Teachers taking initiative in the organisation of the workshop
- Teachers not using algebraic symbolism when exploring the tasks
- Teachers' thinking concerning the possibility to introduce one of the tasks in their teaching practice

In the following I offer a brief description of both Claire's presentation and the two tasks which were presented to the teachers in order to introduce the main aspects, as presented above. Then I emphasise the main outcomes from this workshop.

A short introduction into the historical development of algebra

Claire presented shortly the three stages through which algebra developed (rhetorical, syncopated, and symbolic stage, see Section 2.6) and gave the main characteristics of each of these stages. Claire referred especially to Diophantus, in relation to the introduction of symbols (abbreviations) to represent unknown quantities, and to Descartes, in relation to his distinction between parameters and unknown quantities. She also gave the teachers a copy of the main steps within the development of algebraic notation (see Appendix 1) while focusing on Diophantus and Descartes' notations. Her aim was to emphasise *the link between the choice of symbols and the meaning which these symbols endorse*. As such, Diophantus' notation is difficult to understand, because of the kind of symbols he used, while Descartes' notation is much closer to the standard notation used today, except for the equal sign.

During Claire's presentation the teachers did not ask questions and, from Claire's perspective, they were listening carefully to her talk and showing interest for the different stages of the development of algebraic notation. In order to pursue her didactical aim, which was to illustrate the power of modern algebraic symbolism, Claire proposed two tasks taken from the history of mathematics. Her goal was to illustrate the power of modern algebraic notation by presenting how these tasks were solved at that time, and to compare and contrast these methods with the way we would have solve the tasks, using modern notation.

Two mathematical tasks

The first task was taken from Babylonian time where the solution is found by following the instructions provided by the scribe, formed as a recipe (Friedelmeyer, 1993). The task was as following: "*Find the length and the width of a rectangle when the area is 96 and the semi-perimeter is 20*".

Just after Claire had introduced the task, the teachers started to discuss together: John wondering how to start, and Paul responding that the task might be solved in different ways. Further, John asked if they could start to talk about the task, while Mary asked for some more time as she said she was not ready to present her results. Finally, after a short break, it was Mary who took the initiative: now we can start to discuss the task. The teachers presented and explained how they engaged in the task, and Claire could then present the Babylonian solution and compare it to the modern way of solving the tasks.

I consider that the discussion, as summarized above, brings evidence of how the teachers were developing confidence in our community of inquiry in terms of sharing with the other their hesitations and thinking, taking initiative to ask for more time, and deciding when to start the discussion. Concerning the way the teachers engaged in this task, I want to emphasise the fact that while Paul did not use unknowns, Mary and John tried to introduce these without success: Paul used a “try and fail” method, John introduced the notation l and b for the length and the breadth of the rectangle and questioned why Claire used the notation a and b , while Mary tried to introduce unknowns, but did not succeed.

Thereby, even though the focus of this workshop was on algebraic notation and its development the teachers either did not use unknowns or did not manage to solve the task using unknowns.

After Claire’s presentation of the Babylonian solution and its comparison with modern way of solving the task, the participants discussed the fact that some pupils are keen about having “recipes” which they can directly apply. This issue was recognised by all participants and I consider that this common teaching experience contributed further to community building, and as such it expanded further the common understanding which the participants had experienced during Workshop I in relation to students’ difficulties with the introduction of symbols.

The second task, referred to as the Calandri’s fish problem, offered an illustration of the Rule of the False Position, where one assumes a certain value for the solution, performs the operations as described in the problem, and depending on the error found in the answer, adjusts the initial value using proportions. Claire considered this approach as useful since, according to Radford (2001), there are important structural similarities between false position reasoning and early algebraic thinking (see Section 2. 6).

The task was as following: “*The head of a fish weights $1/3$ of the whole fish, the tail weights $1/4$ and the body weights 300 grams. How much does the whole fish weigh?*”.

Concerning the second task, both John and Paul solved it without using unknowns, while Mary tried to introduce some notation but she was un-

sure about how to proceed. As observed concerning the first task, even though the focus of this workshop was on algebraic notation and its development, the teachers either did not use unknowns or did not manage to solve the task using unknowns. However, John claimed that he could see how the Calandri's fish problem might be relevant for his teaching, since he was teaching about fraction and considered that this task could be presented in that context.

The informal a posteriori analysis

Right after the third workshop, Claire wrote her own impressions and reflections:

I had big expectations about this workshop as I wanted to present some issues from the history of Algebra and to contrast the way a task was solved in Babylonian time comparing to the modern way of solving it. My point was to emphasise the power of modern algebraic notation. I was surprised that the teachers did not use symbols. At the same time, I feel I was acting very much as a "teacher" during my presentation, having a kind of presentation to "my students". Otherwise we had an interesting discussion about students' difficulties with algebra and how they prefer to get some kind of "recipes" from teachers. My challenge for the next workshop will be: how to encourage the teachers to introduce and use algebraic notation? (From my diary, 05.10.04, translated from French)

These reflections constituted the background from which Claire elaborated an *a priori* analysis or "thought-experiment" for Workshop IV. Especially, Claire considered as important the fact that the teachers did not use algebraic notation, even though the focus of this workshop was on the history and power of these. In addition, Claire reported in her reflection the fact that she felt she was acting as a "teacher" giving a lecture on algebraic symbolism. I comment further on this issue below. As Claire wrote in her diary, the challenge for the next workshop consisted of finding and developing an appropriate task in order to stimulate the teachers to use algebraic notation.

The formal a posteriori analysis

In order to summarize the main aspects from Workshop III, I want to emphasise the fact that even though Claire's didactical aim was to put emphasis on algebraic symbolism by focussing on the historical development of algebra, and thereby to illustrate and highlight the power of algebraic symbolism, the teachers explored the tasks either without using unknowns or by experiencing difficulties in introducing these. Therefore, in terms of development of algebraic thinking, this workshop did not bring what Claire was hoping for. However, our community of inquiry did develop further, and I consider that the way the teachers organised their interaction, after Claire introduced the Babylonian task, brings evidence of the participants' development of confidence in the community: on one hand, as the teachers were taking initiative to ask for more time, and deciding when to start the discussion, on the other hand as Claire did

not intervene. Thereby I argue that through the analysis of Workshop III, it is possible to observe that both developments (community of inquiry and algebraic thinking) did not necessarily evolve simultaneously. In Workshop III, it is possible to follow how the community of inquiry was developing further while the participants' development of algebraic thinking was not as Claire expected. However, it seemed, from Claire's perspective, that the teachers were interested by her presentation, and that they were able to recognise similarities between the tasks and their teaching practices: the pupils' preference for "recipe" solutions, and the Calandri's fish problem as relevant in the context of teaching fraction.

As mentioned in the informal *a posteriori* analysis, Claire reported on her feeling concerning her engagement during Workshop III. She talked about "acting very much as a teacher", and I am in a position today where I recognise that this impression influenced my further work as a researcher: I took the decision to approach this workshop through data reduction, (see Section 3. 5.2), and thereby not to transcribe *in extenso* the dialogues from Workshop III.

I recognise that Claire was unsure, at that time, about how to deal with these feelings she noticed right after the workshop, and that she choose to solve the tension by approaching the data through data analysis. I argue, today, for the importance to acknowledge the impact of the feelings a researcher/didactician might have after a working session as these could have further implication on research. An alternative, for Claire, would have been to address this issue in an open manner, by recognising the problem and looking critically at how to organise her further work, as a researcher. I recognise that addressing and making visible this issue helps me, as a researcher, to develop my awareness and competence.

I understand these considerations as referring to Claire's development, both as a didactician and as a researcher, since the tension, as presented above, resulted from a particular mathematical environment where it seemed that Claire's roles as didactician and researcher became in conflict with each other. As such, I consider that I am today, while acting as a researcher, in a position to help Claire in developing further her knowledge.

4.1.4 Workshop IV: Achieving convergence/harmony between the teachers' and Claire's views

A priori analysis of the mathematical task

Until now the different workshops have put emphasis on the introduction of algebraic symbols (Workshop I), generalisation of numerical patterns (Workshop II), and historical development of algebraic symbolism (Workshop III). As mentioned in previous informal *a posteriori* analyses of Workshops II, and III, Claire recognised that the teachers did engage

into the mathematical tasks using idiosyncratic notation (Workshop II) or without using algebraic notation (Workshop III). As part of the *a priori* analysis of Workshop IV, Claire recognised that the challenge consisted of choosing a task which would encourage the teachers to use and explore the power of algebraic notation.

Looking back to the tasks chosen for the previous workshops, their purposes were to express various Cuisenaire formations using symbols (Workshop I), to express the generality of numerical patterns (Workshop II), and to experience the power of algebraic notation by comparing Babylonian and modern solutions of two tasks (Workshop III). The inspiration for choosing the task presented during Workshop IV was two-fold: first Claire experienced, during the Mathias-project, that the teachers consider geometry and algebra as two separate entities and, assuming that the three teachers in our community of inquiry shared the same view, Claire wanted to challenge this understanding of the subject-matter through presenting a task which could address and bridge this separation. Second, one of Claire's supervisors indicated that the Viviani's theorem could be an interesting task to propose since he had a positive experience with this task from a problem solving course.

Therefore, the rationale for Workshop IV was to present a task which offered the opportunity to develop an algebraic proof within a geometrical approach in order to encourage further the teachers to introduce algebraic notation. Furthermore, Claire chose to present Viviani's theorem as a pedagogical means for achieving her didactical aim which consisted of developing an algebraic proof of Viviani's theorem and, thereby, to underline the connection between geometry and algebra. In addition, this theorem was seen in relation to the historical focus, as presented during Workshop III.

The main characteristics of Workshop IV

The main characteristics of the fourth workshop are the following:

- A mix of different kinds of inquiry as the teachers engaged in the mathematical task
- The teachers (John and Mary) taking the initiative to organize the mathematical activity
- Claire working towards her didactical aim
- The teachers' elaboration of an algebraic proof for Viviani's theorem
- One teacher (Paul) taking new initiatives and developing further the mathematical task
- Evidence of the participants developing confidence both in the subject-matter and in our community of inquiry

I will now illustrate each of these characteristics through several excerpts, as presented below.

Workshop IV: presentation of excerpts

This workshop started with Mary offering some reflections about how she experienced having Claire in her class and observing her teaching. It was the first time Claire had visited Mary, and Claire had encouraged her to present her reflections during the next workshop, that is during Workshop IV. I consider Mary's reflections as important in terms of developing an awareness of the possibility to look critically into one's own teaching. I present some extracts of her reflections in Section 4. 1. 9.

Before introducing Viviani's theorem to the teachers, Claire recalled briefly the main aspects from her presentation of the history of algebra while the teachers emphasised both how difficult it was to understand Diophantus' notation and pupils' preference to have solutions presented as recipes. By the end of the discussion, Claire introduced Viviani's theorem, as a task in relation to the focus on the historical perspective. Viviani's theorem states that, in an equilateral triangle, the sum of the distances from a point within the triangle to the sides is equal to the height of the triangle (Figure 9).

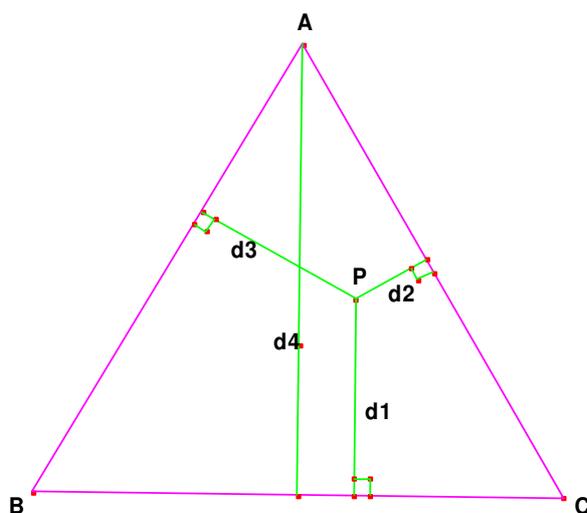


Figure 9: According to Viviani's theorem is
 $d_1 + d_2 + d_3 = d_4$

In Excerpt 1, I present evidence for the teachers' engagement in the task and how they mixed different kinds of inquiry.

Mixing different levels of inquiry

Excerpt 1

60. Claire: What I thought we could work on, it is, hmm, a theorem that was found by an Italian mathematician who lived from 1622 to 1703 and it is like this: if we have an equilateral triangle and a point inside the triangle, then the sum of the distances from the point to the sides of the triangle equals the length of the height in the triangle. (pause) So I thought we could explore this, what is this? and when we have done some work, when we see what this is about, then we could look at if this could be used in the class? Part of it, perhaps not whole of it, just part [referring to construction with compass and ruler], I mean, to explore a little this task. Is this ok for you?

...

64. Mary: yes, I understood, but, hmm, just how to start ...

65. Claire: yes, how do we start with such a task?

66. Mary: hmm, the first thing I will do is to draw a triangle, a equilateral triangle, and then draw the point, yes, ... (unclear)

67. John: yes, so here indeed it is about to look for and maybe to play with the properties of an equilateral triangle and then it could be one of the things that comes, will orient it [the activity?] in that direction if it is what is wanted. I can not tell it is desired yet, isn't it, but let us suppose we want there, that we want to see how it is to construct the perpendicular, take the middle perpendicular for all the sides and all that stuff, and then can we see what we end up with and then can we try to direct them [the pupils] to find it, and so on, maybe, it will be like to try and fail, most like a game. Because, I mean, I have never seen this [task] before, don't think so, so I feel it is outside what they are supposed to teach [to the pupils]

68. Claire: hmm, hmm

69. John: now you are going to learn how to construct a perpendicular in a way, but is it something we can look at here? Some [pupils] like it, some like to ...

70. Mary: construct

71. John: yes, some like to construct, some like to make patterns and like this, and maybe they can find something here

I consider that Excerpt 1 brings evidence of two different levels of inquiry which were already addressed in Claire's utterance (60). On one hand, Claire was inviting the teachers to engage in the mathematical task, referring explicitly to the whole group (inclusive "we"). On the other hand, she proposed to direct the inquiry toward exploring the possibilities for using either the whole task or part of it in class. Her plan was to address inquiry in the task first (mathematical inquiry), and then to move the inquiry into how the task might be implemented in class (didactical inquiry). By finishing her utterance with the question: "is this ok for you?", her aim was to invite the teachers to follow her and to engage with the proposed task. As such, she was addressing issues related to community building in terms of putting emphasis on the participants' joint engagement.

As Mary responded to the mathematical inquiry, she seemed to indicate her hesitation concerning how to engage in the task (64). I want to emphasise the fact this is the first time Mary took the initiative to open

up the discussion in this way. Compared to Workshop III and the Babylonian task, it was John who opened the discussion by recognizing that he was not sure about how to engage in the task. There is a possibility that Mary, building on her experience of working collaboratively during Workshop III, was *now* able to open up the mathematical discussion and to share her hesitation with the other participants. I see these aspects as offering evidence of the participants' development of confidence within our community of inquiry.

Through her response to Mary's hesitation, Claire tried to engage *all* participants (65) "how do *we* start such a task?", using an inclusive "we". Thereby, through her question she wanted to emphasise the collaborative nature of Mary's search as it concerned the whole group in terms of joint enterprise and building further our community of inquiry.

Further evidence for a mix of different levels of inquiry is visible as, while Mary had started to inquiry into the mathematics (66), it seems that John engaged in the task (66-71) by offering pedagogical considerations (67) related to the possibility to adapt the task to his teaching. Furthermore, additional evidence of the mix of different kinds of inquiry is offered as I understand John's use of the pronoun "we" (67, 69) as a means to refer to the community of teachers, contrary to Claire's use (60) where she was referring to our community of inquiry.

Thereby, I consider that Excerpt 1 offers evidence of two different levels of inquiry within which Mary's suggestion (to draw an equilateral triangle) is taken over and elaborated into a pedagogical level, as John considered the different possibilities to implement this activity in his own teaching. This mix of inquiry into mathematics with pedagogical considerations for the classroom was already observed, in a less pronounced way, during Workshop III, as the teachers were able to link the Babylonian task with the pupils' preference for solutions as recipes.

I understand John's long utterance (67, 71) as a *wondering inquiry move* where he considered a hypothetical teaching situation. However, he concluded by expressing his doubts "I feel it [the task] is outside what we are supposed to teach [to the pupils]". In addition, as mentioned earlier, I understand the use of the pronoun "we" as different in these two levels: in Claire's utterance (65), the pronoun is used in the inclusive sense, as including all participants in our community of inquiry, while my interpretation of John's use of the pronoun "we" is that it referred to "we" as mathematics teachers and as such referring to a different community of people.

I want to argue that these considerations illustrate the complexity of mathematical discourse while working collaboratively with teachers, since while Claire presented mathematical and didactical inquiry as two separated entities, the teachers engaged in the mathematical task by of-

fering didactical considerations, and as such, mixing these two levels of inquiry with the possibility of creating ambiguity about terminology, e.g. the use of the pronoun “we”.

Right after these considerations, there was a pause during which our group engaged in the task, following and developing Mary’s suggestion (66): choosing a point P inside the triangle, and comparing the length of the height with the sum of the distances from the point P to each side of the triangle (see Figure 9).

In Excerpt 2, I present evidence for both Claire recognising and valuing the diversity within our community of inquiry, and John and Mary taking initiative to organise the mathematical inquiry.

Claire valuing the diversity of our community of inquiry and John and Mary organising the mathematical inquiry

Excerpt 2

120. Claire: ok, you were faster than me [concerning testing Viviani’s theorem], I am not quite finished yet [to compare the three distances with the height of the triangle], but, let us have a look, so the sum, so it says that the sum, it [Viviani’s theorem] says that it has to be like this all the time...

121. John: then we have to try several [points]

122. Claire: are we convinced or?

123. John: no

124. Claire: is it only by coincidence that it [Viviani’s theorem] is correct?

125. John: think I will have ...

126. Mary: must take all ... (unclear)

127. John: in order to accept so I would have directed a little bit where the points should be, isn’t it, and not as similar as Paul and I have done, isn’t?

128. Paul: hmm, yes ...

129. John: we have already done it [the calculation in order to check the theorem], but I think we should have some more points ...

130. Claire: yes?

131. John: so if you had ten [pupils] in the class that had got it [the calculation in order to check the theorem] correct, then it is ok in a way, shall we take one ...

132. Mary: in the middle

133. John: in the middle too, is it correct?

134. Mary: yes

I consider that the following two key points are illustrated in Excerpt 2: first how Claire recognised and valued the diversity within our community of inquiry, and second how Mary and John took over gradually the responsibility to organise the exploration of the task. I develop these two aspects below.

The first aspect is visible as Claire, by mentioning the fact that she was not finished with her exploration of Viviani’s theorem (120), recognised and made visible for the teachers an important difference between the participants: working with Euclidean geometry, including compass and ruler, constitutes an important part of the teachers’ practice and

thereby it seems that they are experienced in this domain. On the other hand, it was a long time ago Claire had used compass and ruler and she needed some more time to construct the distances from the point P to each sides of the triangle. Thereby, I consider that, by making the difference visible “you were faster than me” which could mean “you are more experienced in that domain than me”, *Claire recognised and valued the teachers’ experience and thereby the diversity within our community of inquiry*. This aspect has been addressed in Section 2. 2. 2, where the dimension of diversity is understood in terms of making mutual engagement possible and productive. As such I consider this recognition as valuable in relation to community building.

In the second part of her utterance, Claire tried to put the focus of the discussion on her didactical aim by emphasising, through a *didactical move* that “it [Viviani’s theorem] says that it has to be like this all the time”. I consider that the second part of her utterance could be seen as reinforcing the aspect of valuing the diversity among the participants, as a characteristic of our community of inquiry: on one hand the teachers are more experienced in constructing with compass and ruler in the context of Euclidean geometry. On the other hand, Claire is more experienced, as a didactican, and she wants to focus back to her aim. Through Claire’s utterance (120) both experiences are made visible and valued, and, I argue for considering *the teachers’ and Claire’s experience as complementary within the particular social setting of our community of inquiry*.

I understand Mary and John’s utterances (125-134) as an attempt to address Claire’s *didactical move* (120). It seems that John and Mary engaged in a discussion about how to organize further inquiry into the task with *John taking the role as the leader of the group* (127, 129) concerning how to organise the activity and where to choose the next point P. As John was taking the role as the leader, it seems that he was also using the inclusive “we”, as Claire did (see Excerpt 1), indicating his willingness to include all participants. In addition, the mix of different levels of inquiry was also visible in Excerpt 2, as John referred to a hypothetical teaching situation (131) where several pupils could have engaged in exploring Viviani’s theorem.

There followed a pause during which the teachers and Claire, after choosing a new point P, constructed the perpendiculars from the point to each side and after taking the sum of the distances, compared to the height of the triangle. During this activity, the teachers commented on the bad quality of the compasses and the difficulty to use these in a teaching situation and, thereby, offering further evidence of mixing inquiry. In Excerpt 3, I present further evidence of how John and Mary continued to organise the mathematical inquiry.

Mary and John taking further initiative in organising mathematical inquiry

Excerpt 3

141. Claire: let us try, shall we take another point?

142. John: yes, you have taken [one point] down in the corner, is there other places we have not taken?

143. Mary and John: (laughing together, unclear)

144. John: do you want so close to the corner, right there! (laugh together)

145. Mary: microscopic

146. John: and you [to Paul] have taken in the middle [of the triangle] and in the middle of the side, take one a little bit closer to one of these, so we can do something

Further evidence of John and Mary taking the role as the leader is offered in Excerpt 3 as, building on previous experience (see Excerpt 2), they continued organising the activity by evaluating Mary's suggestion (144), and proposing a new position for a point P to Paul (146). At the same time, they were laughing together, and my interpretation is that they enjoyed this way of acting and organising the mathematical inquiry. By claiming "so *we* can do something" (146) John seemed to emphasise his role as the leader for organising the mathematical inquiry. In addition, I consider the fact that Claire was *not* participating actively in the negotiation of how to organise the activity within the group as important, since she showed enough confidence in our community of inquiry *not* to intervene in Mary and John's organisation. It seems that through their suggestions (142-146), Mary and John were aiming to answer Claire's *didactical move* (124) where her goal was to encourage the teachers to recognise the necessity to introduce algebraic symbolism, and thereby to elaborate an algebraic proof of Viviani's theorem.

I argue for considering these ways of acting as evidence of John and Mary's confidence in both our community of inquiry and in the mathematics: I consider that aspects concerning confidence in our community can be traced back to Workshops II and III, where the teachers started questioning the proposed task, opening the discussion, sharing with each other hesitations and thinking, asking for more time. In addition, by taking a task from Euclidean geometry, Claire opened for the teachers the possibility to rely on their mathematical knowledge in a domain in which they seemed to feel comfortable. Thereby, I consider that the combination of these two aspects, confidence in our community, as it has developed through the workshops, *and* the mathematical context within which the task was situated (Euclidean geometry), enabled the teachers to take over the leadership of the workshop (John, Mary, and Paul, later in the workshop) and, for Claire, *not* to intervene and to remain silent. I deepen this issue further in the formal *a posteriori* analysis of this workshop.

All participants engaged in constructing and measuring distances. After a short silent pause, Claire addressed the idea of elaborating a proof as presented in Excerpt 4.

Addressing the idea of proof

Excerpt 4

162. Claire: is it [the theorem] correct?

163. John: oh, yes, he was right! (John is laughing)

164. Claire: he was right! are you convinced?

165. John: yes!

166. Claire: is it really correct? (pause)

167. Mary: yes!

168. Claire: it is indeed, ok, can we move a little deeper in this task in order to see what is really happening, why is it [the theorem] correct?

As explained earlier, the participants had developed gradually an understanding of Viviani's theorem, by selecting several positions for a point P and measuring and comparing the distances. Through the repetition of *didactical moves* (162, 164, 166), Claire was seeking for establishing a common understanding and awareness of Viviani's theorem. As such, her goal was to move from several examples to a generalisation. The idea of proof was directly addressed in the last part of her didactical move (168) as Claire asked "why is it [the theorem] correct?".

I consider that, by establishing a common agreement about the fact that Viviani's theorem is correct, Claire put emphasis on the group's joint enterprise (Wenger, 1998) as she invited all participants to engage collaboratively in the elaboration of a proof for Viviani's theorem.

Thereby, she showed an understanding of "proof" as emerging from working on different examples, recognising the generality of a particular example (generic example), being convinced that the result did not depend on the choice of the point P (verification of Viviani's statement), and getting deeper into the task by introducing symbols (providing insight into *why* it is true). As observed before (see Workshops I and II), Claire did *not* mention the idea of symbol, letting the teachers to interpret what "move a little deeper in this task" might mean.

As a means to move forward to her didactical aim, Claire proposed both to consider the equilateral triangle (ABC) as consisting of three smaller triangles (APB; BPC; CPA) (see Figure 10) and to choose a label for the length of the sides. Excerpt 5 presents how symbols were introduced.

Introducing symbols

Excerpt 5

191. Claire: ... now let us see, we have, when we consider the distance from the point, let us call it point P, when we take the distance from the point P to the sides here, I am looking at the small triangle now, one of the three, so this is an equilateral triangle, how can we call the sides? the length of the side, how can we call it?

192. John: call it "a"

193. Claire: "a", so we get "a", "a", "a" [the three sides of the equilateral triangle]

194. John: are we going to look to area or something like this?

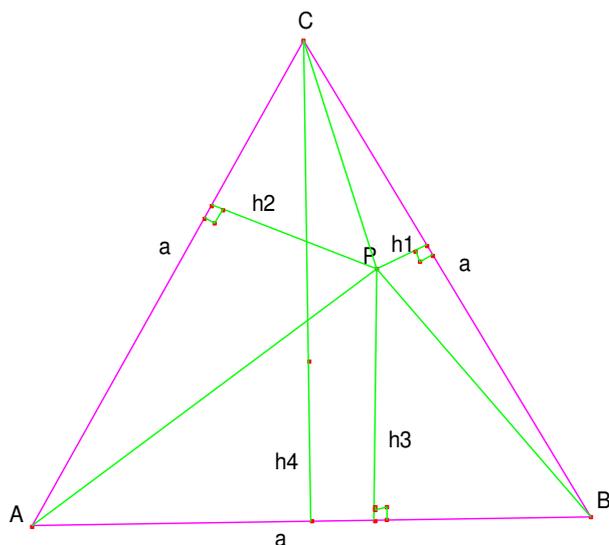


Figure 10: The introduction of symbols

I consider that Claire's didactical aim was clearly visible by the end of her utterance (191), since she asked directly "how do we call the sides?". Her question could be seen as the result of a long didactical act running through the whole workshop and being articulated in different ways: as such, these different *didactical moves* (120, 122, 124, 162, 166, and 168) offer an illustration of the iterative nature of a didactical act (see Section 2. 2. 6). Claire's reference to the possibility to introduce symbols seemed to be understood since John proposed the symbol "a" to denote the length of the side of the triangle. Compared to his choice of symbols during Workshop III in relation to the Babylonian task where John argued for using "l" and "b" (length and breadth) instead of Claire's symbols "a" and "b", it seems that John was able, during Workshop IV, to use symbols without any direct connection to a mathematical object and, as such moving away from symbols used as shorthand (Küchemann, 1981). I consider John's utterance (194) as a *wondering inquiry move*, since he seemed to share with the other participants his idea to explore the area of the triangle, and how it might be used in elaborating an algebraic proof

for Viviani's theorem. I argue for considering this particular step, the introduction of symbols, as significant for our community of inquiry, since it allowed the participants to move forward and to envisage the elaboration of an algebraic proof of Viviani's theorem. In addition, I want to highlight the fact that, contrary to Workshop II where Paul's idiosyncratic notation for even and odd numbers was not functional, the symbol "a" introduced by John was functional and thereby the teachers were now in a position to move forward.

Before introducing the next excerpt, I want to draw attention to the fact that Paul's voice has almost disappeared, except for a very short exclamation (Excerpt 2, utterance 128). It seems that he was not participating actively in the discussion concerning a possible implementation of Viviani's theorem in class (Excerpt 1), and did not participate in how to organise further the mathematical inquiry into the task (Excerpts 2, 3). Claire had noticed Paul's lack of engagement and, both as a didactician and as a researcher, had some concern about him.

After a brief discussion about how to continue with John's idea, there was a pause during which the teachers and Claire elaborated a proof of Viviani's theorem. Excerpt 6 presents evidence of the elaboration of an algebraic proof and it starts as Paul claimed suddenly:

Elaborating an algebraic proof of Viviani's theorem

Excerpt 6

205. Paul (with a loud voice): like this, it [the proof] was smart!

206. Claire: did you arrive at something?

207. Paul (with enthusiasm): yes!

208. John (with enthusiasm): same for me

209. Mary (with enthusiasm): yes, for me

210. John (with a loud voice): then we agree that the area of the three small triangles is as big as the area of the whole [triangle] and we put the following, yes, can we call it an equation or a claim, yes, I call it "a" [the side of the triangle], "a" multiplied by h_3 divided by two, plus "a" multiplied by h_2 divided by two, it is the small height, plus "a" multiplied by h_1 divided by two is equal "a" multiplied by h_4 , which is the big height, divided by two

$$[\text{my interpretation of John's utterance: } \frac{ah_1}{2} + \frac{ah_2}{2} + \frac{ah_3}{2} = \frac{ah_4}{2}]$$

211. Claire: hmm, hmm

212. John: common factors outside, then we get a divided by two, I have h_4 there and a parentheses with the other three heights in there, and then they [h_1 plus h_2 plus h_3] have to be equal to it [h_4] in order to be correct [my interpretation of John's utterance: $h_1 + h_2 + h_3 = h_4$]

The elaboration of an algebraic proof of Viviani's theorem was discussed among the participants first through a kind of *tacit agreement*, before it was made explicit. I mentioned earlier how Paul seemed not to participate actively in the inquiry of Viviani's theorem. Therefore, I want

to highlight the fact that it was Paul who took the initiative to claim spontaneously, with a loud voice, that he got a result, without offering any details, while John and Mary (208, 209) joined him, with an enthusiastic voice. Thereby, it seemed that Paul decided to act more actively within the group, and this interpretation is supported further by the way Paul acted later during the workshop (see Excerpts 7, 8, and 9).

Claire's willingness to question Paul's claim is visible through her *information-seeking inquiry move*, as she sought (206) to share the teachers' enthusiasm. As mentioned earlier, I interpret the teachers' utterances (207, 208, 209) as a kind of *tacit agreement* with Paul's claim (205), although none of them have made explicit what the agreement was about. Therefore, Claire's *information-seeking inquiry move* (206) was formulated in this perspective, aiming to address explicitly and to make accessible to all participants Paul's claim (205). I consider the fact that, it was John who took the initiative to explain his own thinking (210, 212) as if he was talking for the whole group, as a way of following up his way of acting as the leader of the group (see Excerpts 2, 3). As such, this way of acting brings evidence of John's thorough confidence both in the mathematics and in our community of inquiry. Further evidence for the teachers' confidence both in the mathematics and in our community of inquiry is presented in Excerpts 7, 8, and 9.

Exploring and developing further Viviani's theorem: the elaboration of a new task

Before presenting the excerpts, I summarise briefly the participants' discussion in order to offer the context from which these excerpts were taken. The teachers first expressed their satisfaction for elaborating an algebraic proof for Viviani's theorem and then they engaged in a discussion concerning the possibility to use this task in their own teaching practice. Here they differentiated between lower and higher achieving pupils, and compared the different possibilities to adapt the task to the pupils. One of the teachers' suggestions was to present to the pupils a sheet of paper with a drawing of an equilateral triangle. The task would consist of choosing a point P inside the triangle and verifying Viviani's triangle. As a way to develop the task further, the teachers discussed the possibility to take a different starting point and, for example, given the point P to inquire how to construct an equilateral triangle. Claire had seen a similar problem before and she proposed to specify the problem by taking the three distances as 3, 5, and 7 cm from the point P to the sides of the triangle. The teachers discussed further on the advantages (connecting algebra to geometry) and disadvantages (where to begin; why do we, as pupils, need to work on this task) of introducing a similar task. The teachers were engaged in didactical discussion, wondering how

to introduce this task in their own teaching when the focus of the discussion came back to the question of how to construct an equilateral triangle when the distances, from a point P to the sides of the triangle, are 3, 5, and 7 cm.

I consider the fact that the teachers engaged in developing further Viviani's task, *on their own initiative*, as further evidence of their confidence both in mathematics and in our community of inquiry. In order to deepen what I mean, I recall the way I defined a *mathematical environment* (see Section 3.4.1) as being characterised by *both* the mathematical task *and* the social setting within which the task is addressed. Here the ideas of *didactical aim* and *pedagogical means* play a central role. The didactical aim refers to the target knowledge, while the pedagogical means refers to the tools (task, manipulative) used in order to reach the chosen didactical aim. By referring to the teachers' confidence both in mathematics and in our community of inquiry, I seek to capture the teachers' own initiative to propose a different task to the participants, both in terms of didactical aim and pedagogical means.

The following excerpts offer evidence of these aspects, as presented above, since the teachers, and particularly Paul and John, started to explore the new task (how to construct an equilateral triangle when the distances, from a point P to the sides of the triangle, are 3, 5, and 7 cm) by taking each time different starting points. I argue for presenting Paul and John's all three approaches (Excerpts 7 to 9) to the new task, as I consider that these three different ways of engaging and solving the task bring evidence of the richness of their mathematical knowledge. I comment further on this issue after presenting all three excerpts.

All figures, as presented below in this section, have been elaborated during the formal *a posteriori* analysis and represent my own interpretation of the teachers' utterances.

Excerpt 7 started when John offered the result of his thinking. This happened after that all participants were engaged with this new task, which consisted of exploring how to construct an equilateral triangle with the distances 3, 5, and 7 cm from a point P. This new task was *also* new for Claire.

Exploring and developing further Viviani's theorem – approach 1

Excerpt 7

321. John: think at least that it [the construction] should be right, do you think the same as me (to Paul)?

322. Paul: hmm, hmm

323. John: yes, I thought the same as you, to move parallel and then 60°?

324. Paul: yes, yes

325. Claire: are you thinking about angles?

326. Paul: 7 cm, perpendicular, 60°, and then move parallel

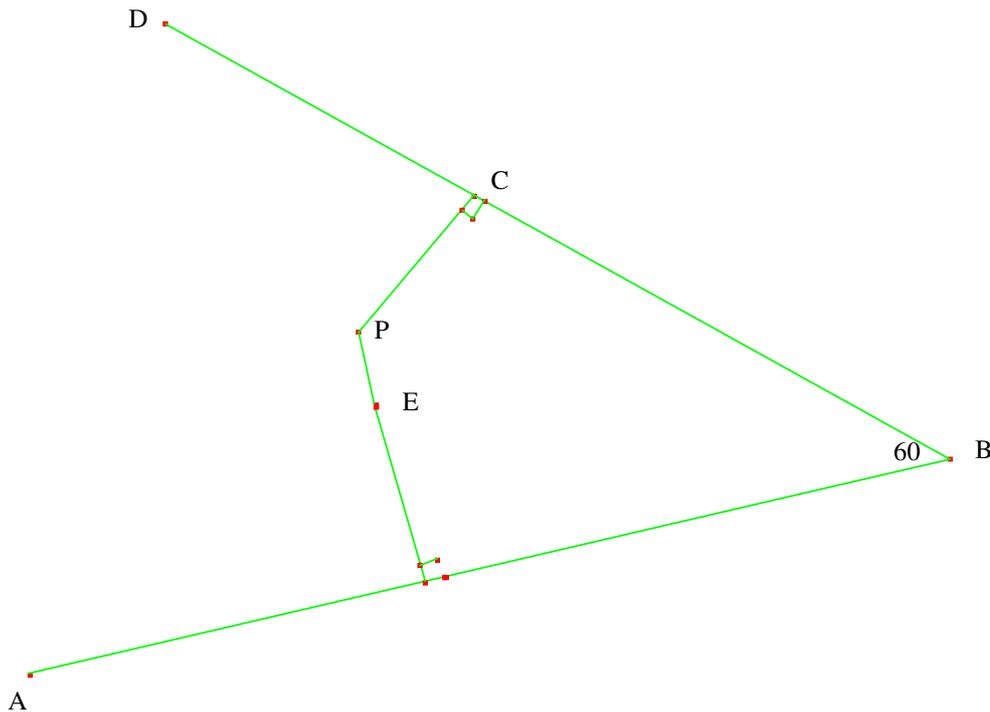


Figure 11: New task, approach 1

Since Paul was expressing himself with short statements, only using words without elaborating his thoughts in sentences, I offer here my own understanding of his claims.

In order to construct an equilateral triangle, Paul and John proposed to take a distance of 7 cm from the point P (segment PC), to construct the perpendicular to PC in C in order to get DB, to construct a 60° angle in B in order to get AB, and finally to move AB parallel until the point E where PE is either 3 or 5 cm, Paul offered no precision about which case he was considering. Paul did not offer further explanations about how to construct the third side of the triangle, but there is a possibility that he would repeat the same procedure in A by constructing a 60° angle and move the segment parallel until it is 3 or 5 cm from P (see Figure 11).

I consider as central the fact that a new task was emerging from the participants' didactical considerations and therefore the teachers took over *both* the organisation of the workshop and the conceptualisation of new tasks. John and Paul engaged in a dialogue where their thinking became explicit only by the end of the dialogue (326). Thereby, it seems that there was a kind of *tacit agreement*, similar to the one observed in Excerpt 6, between John and Paul and that they did not need to make explicit their thinking to each other (321-324). It was only after Claire's question that Paul gave some details (326). As such, Claire's question

was a genuine question (325), an *information-seeking inquiry move*, as she tried to follow Paul and John's reasoning.

The discussion went back to didactical and pedagogical considerations where the teachers tried to relate Paul's solution to their own teaching experience. Suddenly, Paul claimed that he had discovered another way of constructing the equilateral triangle:

Exploring and developing further Viviani's theorem – approach 2

Excerpt 8

372. Paul (with a loud voice): but there is indeed another solution too! when you are here [3 cm from the point P], then you can just take the perpendicular on it, on it, on it, on all three places, and then you get a triangle! but of course you have to suppose that

373. John: equilateral

374. Paul: yes, equilateral, equilateral of course, understand?

375. John: could you take it once more?

376. Paul: yes, if you construct three circles, 3cm, 5cm, 7cm [radius], and then you construct a perpendicular

377. John: a radius

378. Paul: a radius and then you construct a tangent on each of the circles under the consideration that you have to divide by 120° , then it will be an equilateral triangle

379. Mary: hmm, hmm

380. John: that was nice!

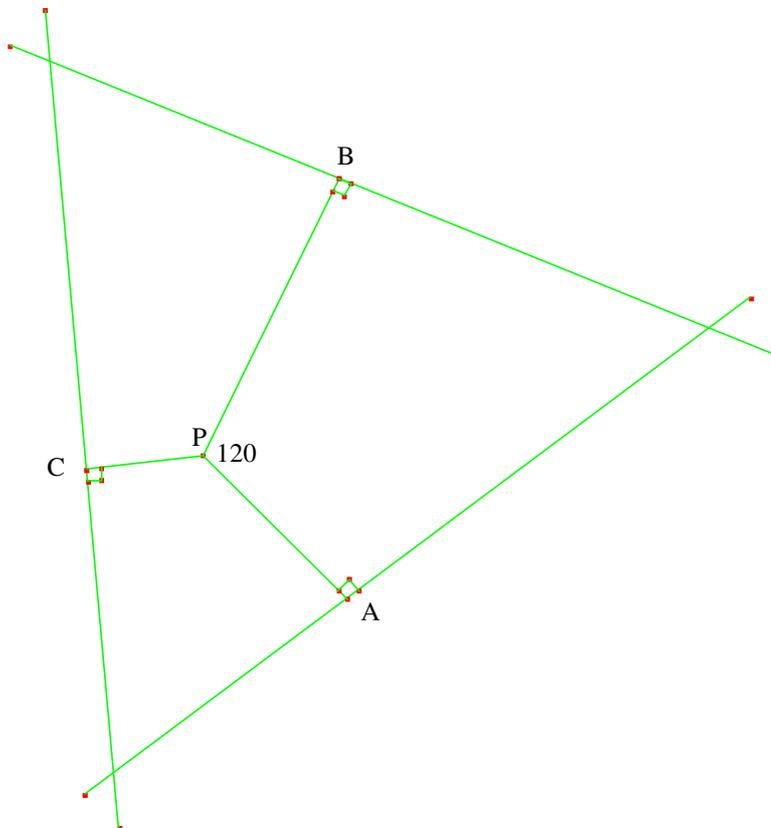


Figure 12: New task, approach 2

Again, I offer here my own understanding of Paul's claims. This time, it seems that Paul proposed to start from the point P and the three distances, $PC = 3$ cm, $PA = 5$ cm, and $PB = 7$ cm, he mentioned to construct circles (376). The next step consists of constructing the perpendicular (tangent on each circle) on each point A, B, C (see Figure 12). According to Paul (378), this procedure is valid when the angles in P are 120° . The discussion went back to didactical and pedagogical considerations when Paul interrupted it by claiming that he had found a third way of solving the problem.

Exploring and developing further Viviani's theorem – approach 3

Excerpt 9

410. Paul (with a loud voice): there is another possibility too! perhaps this is for the (unclear), I don't know, but you can start with a 60° angle, then you construct a parallel 3 cm, a parallel 5 cm, and where these meet, it will be the middle

411. Claire: the middle for what?

412. Paul: yes, yes, for the point P you want to get, and then you can use your compass with 7 cm, and then you will find the last line

413. Claire: yes, yes

414. Paul: but it is perhaps quite the same as before, only that we start with a 60° angle

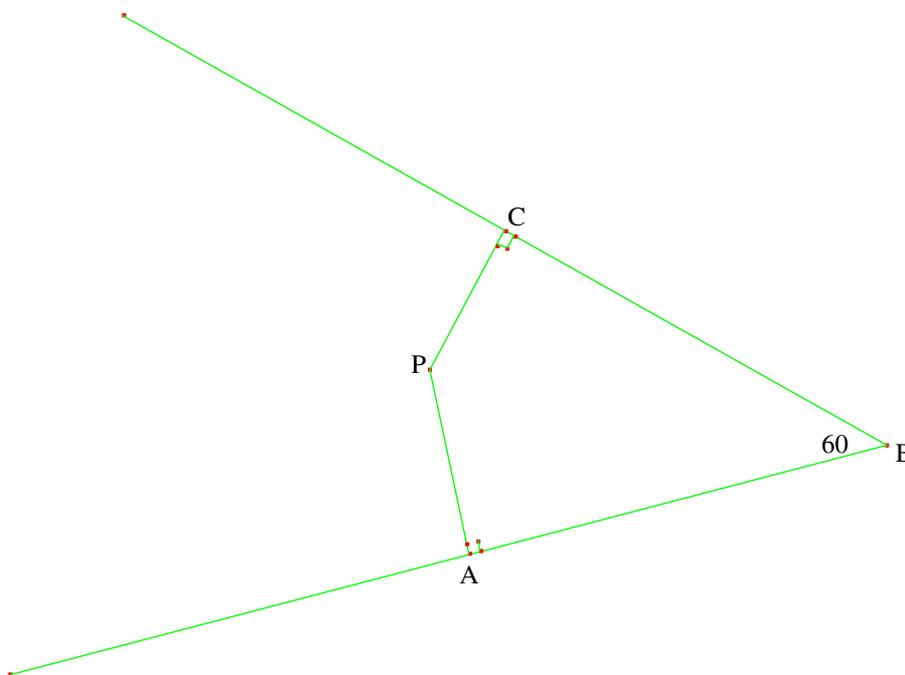


Figure 13: New task, approach 3

My interpretation of Paul's last proposition is illustrated in Figure 13. He seemed to propose to start from a 60° angle in B, and to construct a segment parallel to BC, and 3 cm from C, and another segment parallel to BA, and 5 cm from A. These segments intersect in P (Paul talks about the middle). Then he proposed to construct a circle from P with radius 7 cm, and to construct a segment tangent to the circle with a 60° angle with AB.

Considering the three last excerpts as a whole, I want to highlight the following key aspects: First of all, Paul and John's mathematical discussion occurred in the middle of another discussion which addressed the possibility to use both Viviani's theorem and the new task in the teachers' own teaching practice. This particular feature, consisting of mixing different kinds of inquiry, has been observed before in this workshop (see Excerpt 1), and I argue for interpreting this aspect as further evidence of the teachers' confidence in our community of inquiry as they feel free to weave mathematical inquiry within didactical considerations.

The second aspect I want to address relates to the mathematical inquiry as presented through Excerpts 7 to 9. I argued before, see above, for presenting all three excerpts, as these witness Paul and John's rich mathematical knowledge. Looking closely at Paul and John's argumentation, they were in a position to offer different approaches to the same task, starting respectively from one of the distance from the point P to one side of the triangle and constructing a 60° angle (Excerpt 7), from all three distances (Excerpt 8), or from a 60° angle (Excerpt 9). Using constructivist terms, I would say that Paul and John show relational understanding (Skemp, 1976), as they are able to draw on and relate different mathematical concepts in order to engage within the task. Using the theoretical terms which I introduced in Chapter 2, I would say that the *mathematical environment* which was elaborated during this particular workshop enabled the teachers to move further and expand the proposed task (Viviani's theorem) and thereby offering evidence of their mathematical confidence in our community of inquiry. I argue for considering this aspect as an illustration of Lave and Wenger's (1991) idea of generality of any form of knowledge which they define as "the power to renegotiate the meaning of the past and future in constructing the meaning of present circumstances" (p.34). As such, the teachers were able, while elaborating the meaning of the present task, to draw on and to renegotiate the meaning of previous geometrical tasks they had experienced. At the same time, they were able to engage in didactical discussion and negotiate the possibility to implement this new task on their own teaching. In other words, the generality of their knowledge enabled them to negotiate both past and future actions, while negotiating present mathematical tasks.

The last aspect which I want to highlight relates to the way Paul and John were acting during the three last excerpts. As mentioned earlier (see Excerpts 2 and 3), Mary and especially John were acting as the leaders of the group, organising the mathematical inquiry into the task. There is a possibility that Paul, drawing on the way John and Mary were acting previously, was encouraged to act as a leader and to organise the mathematical inquiry into the new task. In addition, Paul and John were able to engage in mathematical inquiry without making their thinking explicit for the other participants. This aspect, which I called *tacit agreement*, was especially visible in Excerpt 7, where their mathematical inquiry was only interrupted by John and Claire's *information-seeking inquiry moves* (375, 411).

At the same time, Claire chose *not* to intervene, letting the "leadership" pass to Paul and John, and just asking genuine questions through *information-seeking inquiry moves* (325, 411). I argue for considering these aspects, as presented above, as a way of negotiating and developing further our joint enterprise and as evidence of the participants' thorough confidence in our community of inquiry.

The informal a posteriori analysis

After the workshop, Claire wrote her own impressions and reflections:

We had really a nice workshop today, the teachers were so active by the end of it. It was so interesting to see how Paul and John engaged in the new task. It seems that something was happening but I am not sure about what it was. I hope we will have similar experiences in the next workshop. (From my diary, 10.11.04, translated from French).

These reflections constitute the background from which Claire elaborated an *a priori* analysis for Workshop V.

The formal a posteriori analysis

When presenting the mathematical task to the teachers, Claire had a well defined didactical and pedagogical strategy. Her didactical aim was to introduce algebraic notation in the context of Euclidean geometry. She chose, as a pedagogical means, to present Viviani's theorem and to encourage the teachers to elaborate an algebraic proof of the theorem.

The presentation of the different excerpts allowed me to develop the main characteristics of Workshop IV, as presented in the beginning of this section. In the following, I present a synthesis of these aspects.

In Excerpt 1, John and Mary's reaction to the task were presented. The fact that Mary opened the discussion and that John mixed both mathematical considerations with pedagogical issues characterises this first excerpt. I see the mix of inquiry (inquiry in the mathematics and inquiry in the teaching practice) as rooted in Workshop III, where the teachers also considered the relation between the task and their own teaching. In the second excerpt the difference between the participants

was made visible and valued, as Claire recognised the teachers' experience within construction with compass and ruler while Claire's didactical experience was also visible. Thereby, the dimension of diversity was emphasised in the first part of this excerpt and its value for our mutual engagement and joint enterprise was highlighted. Through Excerpt 2, it was possible to follow how Mary and John, particularly, took over as the leaders of the activity, organising the inquiry into Viviani's theorem. This way of acting developed further during Excerpt 3, where John directed where Mary and Paul could choose the next point P. The fact that they were laughing seemed to indicate that they were comfortable with taking over the organisation of the workshop and that they enjoyed this way of acting. In Excerpt 4, Claire's didactical aim was visible as the issue of "being convinced" is discussed. It is through Excerpts 5 and 6 that the participants elaborated an algebraic proof of Viviani's theorem. I want to compare this result with the teachers' claim from Workshop II (excerpt 5) where they recognised how difficult it seemed to "deduce general results from things that seems to be like this" (Paul, 128, Workshop II). By comparing how the teachers approached the idea of proof in Workshop II and in Workshop IV, it is possible to see their development of confidence in the mathematics, based on their confidence with geometry, in terms of being able to elaborate an algebraic proof of Viviani's theorem and expressing their satisfaction afterwards. I mentioned before the hypothesis concerning the importance of the *nature of the mathematical task* (see the formal *a posteriori* analysis of Workshop II). I argue for emphasising the fact that the teachers were able to elaborate an algebraic proof in relation to a task situated in a geometrical approach.

In addition, I argue that one of the characteristics of Workshop IV consists of the recognition of the importance of the combination of confidence in our community *and* the mathematical context within which the task is situated (Euclidean geometry) which enable the teachers to "take over" the organisation of the workshop, and to develop further the task and explore it, as presented in Excerpts 7, 8, and 9. In addition Claire did *not* intervene, and thereby she felt confident enough to let the teachers take over the workshop. As such, her confidence, as a didactician, had improved. I argue for considering the different aspects, as presented in the formal *a posteriori* analysis, as central to Claire's development of knowledge.

4.1.5 Workshop V: Working towards a symbolic formulation

A priori analysis of the mathematical task

As mentioned in the informal *a posteriori* analysis of Workshop IV, Claire experienced the previous Workshop as a successful one, since all participants did engage collaboratively into Viviani's theorem and were able to elaborate an algebraic proof of Viviani's theorem. In addition, the way Paul and John acted during the last part of Workshop IV brings evidence of deep confidence *both* in our community of inquiry *and* in mathematics. As mentioned in the formal *a posteriori* analysis of Workshop IV, it seems that this deep confidence in our community of inquiry could be traced back to Workshops II and III, and that the *mathematical environment* and especially the *nature of the task*, as offered during Workshop IV (task based on Euclidean geometry), played a central role enabling the participants to develop further their confidence in mathematics. As explained in the formal *a posteriori* analysis of Workshop IV, by building on the generality of their mathematical knowledge concerning Euclidean geometry, the teachers were able to elaborate an algebraic proof and to investigate and develop the task further.

Building on the positive experience from Workshop IV, Claire's didactical idea, with Workshop V, was to come back to the idea of generalisation of numerical patterns, as presented during Workshop II, and to see how the participants would respond to this kind of task, taking into consideration the development of confidence as observed in Workshop IV. Therefore, Claire's rationale for the mathematical task offered during Workshop V was based on evidence from prior workshops which led to an envisaging of development of algebraic thinking in relation to generalisation of numerical patterns.

Concerning the choice and design of a particular task, as a pedagogical means to work on generalisation of numerical patterns, Claire decided to present a task related to four digits palindromes (Mason et al., 1982), since she believed that all participants could easily engage in it. This task presented rich numerical patterns and it was possible to elaborate an algebraic proof for it. The task was presented as: "*A friend of mine claims that all palindromes with four digits are exactly divisible by eleven. Are they?*". Thereby, Claire's didactical aim, during Workshop V, was to elaborate an algebraic proof for the divisibility of four digits palindromes by 11.

The main characteristics of Workshop V

The main characteristics of the fifth workshop are the following:

- The participants inquiring into each other's understanding of the mathematical task
- The vagueness of mathematical discourse

- The teachers getting fascinated by the emerging numerical patterns
- Claire adjusting *momentarily* her pedagogical approach
- The participants elaborating an algebraic proof

I will now illustrate each of these characteristics through several excerpts, as presented below.

Workshop V: presentation of excerpts

Before introducing Excerpt 1, which illustrates John's inquiry in Paul's thinking, I want to present briefly the discussion we had before engaging with the mathematical task, since I consider that important issues in terms of *joint enterprise* and *critical alignment* emerged from it.

The discussion concerned John's reflections about how he experienced having Claire in his class and observing his teaching. I present some extracts of his reflections in Section 4.1.9. In addition, the discussion addressed the goal of the interviews and classroom observations, where both Mary and John emphasised that they did not feel that Claire was evaluating or judging their work as teachers. As a researcher, I consider the fact that the teachers introduced this issue, concerning the goal of the interviews and classroom observations, in the discussion as central since it addressed the basis of our joint enterprise and mutual engagement as being supportive and not normative. By being able to bring this issue to the discussion, the teachers showed evidence of their thorough confidence in our community of inquiry.

After the discussion, Claire proposed to the teachers to work on a task concerning palindromes and, on Paul's demand, she had to explain and give examples of what a "palindrome" is. Claire introduced the task as: "*A friend of mine claims that all palindromes with four digits are exactly divisible by eleven. Are they?*", and after some clarifications concerning the purpose of the task, there was a pause during which all participants engaged in the task by exploring different numerical examples on their notepad. It was Mary who first shared her results with the other participants. She explained that by taking different palindromes she could see that all were divisible by 11. John continued by claiming, while laughing, that he did hope there was a system here because he had not the courage to divide any more. Paul agreed with John. The way Paul shared his results and how Mary, John, and Claire inquired into his understanding is presented in Excerpt 1.

Vagueness and ambiguity of the mathematical discourse: example 1

Excerpt 1

36. Paul: there is a system out there at least, there when, hmm, when the digits are consecutive
37. Claire: once more?
38. Paul: when the digits are consecutive like 3, 4; 4, 5; 5, 6; 6, 7, then it [the second digit in the quotient] will always be 1 (laughing)
39. John (with a loud voice): what did you say now? it was four, four digits, wasn't it?
40. Paul: yes, and then you divide by 11, and then you get the answer (unclear)
41. Claire: which is?
42. Paul: 1 in the middle, of the three numbers
43. Mary: when it is (unclear, John is speaking at the same time)
44. John (with a loud voice): I do not understand what you mean!
45. Paul (with a loud voice): if you take 6556
46. John: yes
47. Paul (with the same voice): or 4554, 2332, I mean consecutive with each other and up, then it will always be 1 in the middle
48. Claire: in the answer, when you divide by 11?
49. Paul: yes!
50. Claire: yes, ok
51. John: yes, like this, yes!

The vagueness of mathematical discourse is illustrated as John and Claire engaged in questioning the meaning of Paul's statement (36), searching for details which would provide further explanations. By inquiring into Paul's thinking (37, 39, 41, 44, 48) it seems that both John and Claire were trying to make sense of Paul's claims (36, 38). Until now the analyses of previous workshops have shown that it was Claire, alone, who took the initiative to inquire into the other participants' thinking. In Excerpt 1, it is possible to follow how John, asking with a loud voice, was taking the responsibility, together with Claire, to inquire into Paul's claim (36, 38) and even interrupting Mary (43, 44). My interpretation of John's way of acting, is that, building on the confidence he had acquired through previous workshops (see Workshops III and IV), John felt that he was now in a position where he could interact directly with Mary and Paul.

I consider that Paul's utterances (36, 38, 40, 42), as presented in Excerpt 1, could be seen in relation to his contribution, by the end of Workshop IV (Excerpt 7, utterance 326), where he just mentioned some ideas without adding any explanations. Thereby, it seemed that Paul did not feel any need for articulating his thinking more precisely and this might be the reason why the other participants found his statements vague and ambiguous. There is a possibility that, building on the confidence developed while engaging in the task related to Euclidean geometry (Viviani's theorem from Workshop IV), Paul engaged with the same mathematical

confidence in a task related to generalisation of numerical patterns. The fact that Paul was laughing while explaining his result (38) supports this hypothesis, as it seems that Paul was enjoying the discovery of numerical patterns.

Looking more closely to the way the negotiation of Paul's meaning was achieved, it seems that the purpose of the discussion concerned the clarification of Paul's formulations "when the digits are consecutive" (36) and "it will always be 1 in the middle" (47). In order to address the vagueness of Paul's utterances, Claire, John, and Mary were asking genuine questions, thereby indicating *information seeking inquiry moves* (37, 39, 41, 44, 48). A possibility is that, by offering an example (45), Paul tried to remove the difficulty to understand his result, since both Mary and John seemed to get confused (43-44). It seems that it was after clarifying *what* "will always be one in the middle", that is, the second digit in the quotient after the division, that Claire and John were satisfied (50, 51). See copy of Paul's notepad (Figure 14).

Before moving to Excerpt 2, I want to emphasize two aspects, as these emerged from the analysis of Excerpt 1: the first one relates to the way the inquiry into Paul's understanding was conducted, the second one relates to the reason why it was difficult to grasp Paul's explanation. Concerning the first aspect, the analysis of previous workshops showed that, until now, it was Claire who asked and inquired into the other participants' understanding, as, for example, "can you go a little deeper?" (utterance 84 from Excerpt 2, Workshop II).

Excerpt 1 from Workshop V offers a different picture since *all* other participants (Mary, John, and Claire) did engage in the inquiry, asking genuine questions, claiming their difficulties to understand, seeking for clarifications. All engaged collaboratively and actively (with loud voice), while Paul was arguing and explaining as much as possible, also using a loud voice. These considerations beg the following question: why was it so difficult to understand what Paul meant? Answering this question allows me to address my second aspect. A possibility is that the observed difficulties which Mary, John, and Claire had to grasp Paul's explanation were related to the vagueness of Paul's mathematical discourse and thereby to the ambiguity of the referents for the occurrences of the pronoun "it". This issue is commented on by David Pimm (1987) observing that "like much informal talk, spontaneous discourse about mathematics is full of half-finished and vague utterances" (p.22).

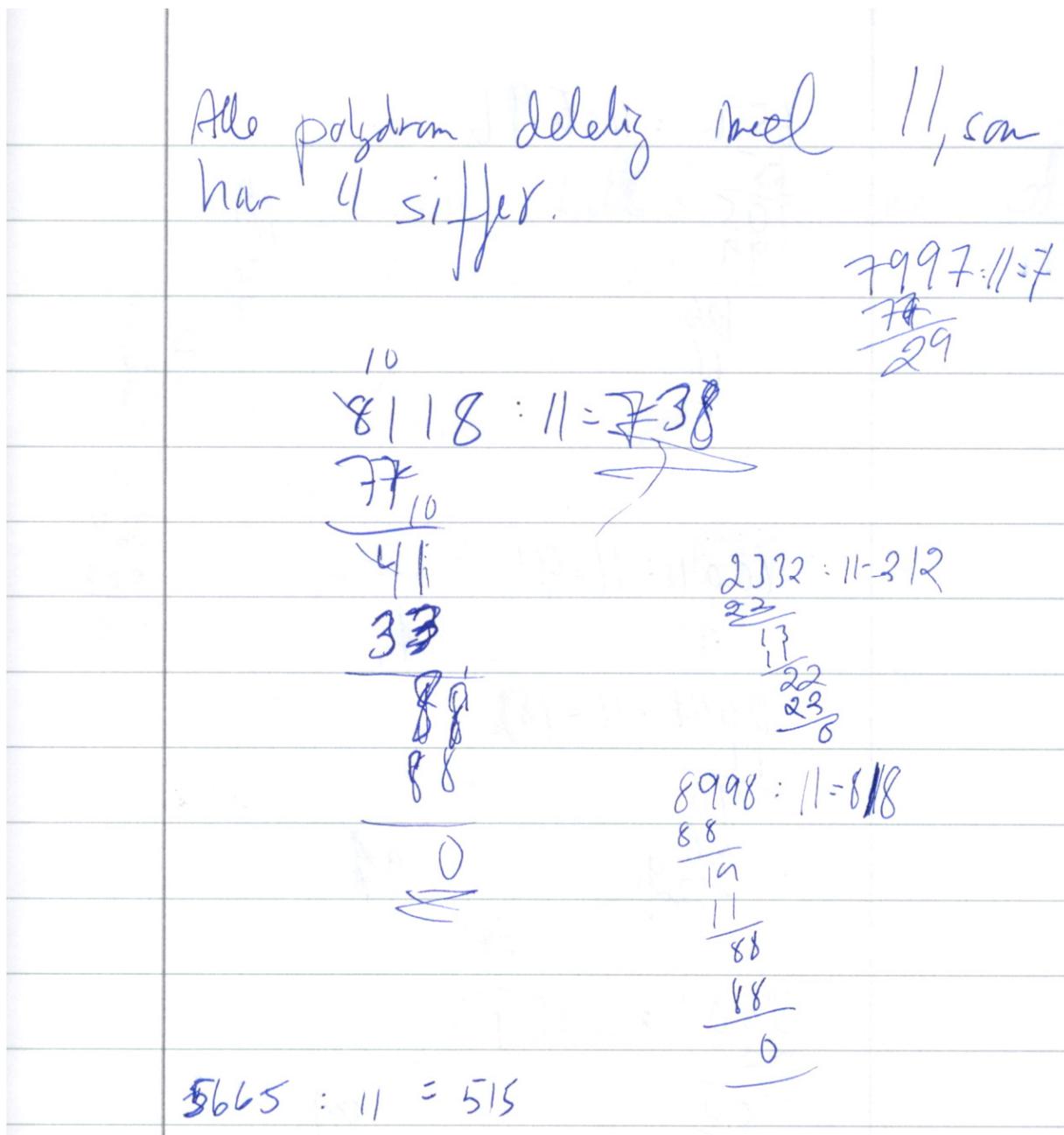


Figure 14: Copy of Paul's notepad concerning the palindrome task

Building on Pimm's observation, Rowland (2000) suggested that the use of the pronoun "it" is an important and distinctive feature of mathematical discourse, acting as a linguistic pointer. I consider these observations as significant in terms of tracing the development of the participants' awareness of the vagueness and ambiguity of mathematical discourse. In addition, it was John and Mary who initiated the inquiry into Paul's thinking.

After the discussion concerning Paul's results, as presented in Excerpt 1, Paul briefly repeated that this was the result he had observed. Then John proposed his way of understanding the task: he adopted a different strategy from Paul, as he started from the smallest four digit palin-

drome, 1001, and looked for what kind of pattern emerges when one gets to the next one. While John and Claire were talking, Paul and Mary were busy with writing in their notepad. Excerpt 2 shows how John explained his result to Claire:

Vagueness and ambiguity of the mathematical discourse: example 2

Excerpt 2

54. John: well, I have taken, hmm, first I saw that they [the palindromes] just increase by 11, because if you want to have, what did you call it? (laughing)

55. Claire: a palindrome

56. John: a palindrome with four digits, then you have to increase by 11 to get it, I mean if you start from the bottom and then go to the next one ...

57. Claire: yes, ok, yes, 1001 and then the next one will be ...

58. John: yes, I mean, not increasing by 11, but the digits that are in the middle are increasing by 11, I mean hundreds increase with 1 and tens increase with 1, and each time in order to get a palindrome, if you go in a systematic way

By claiming (54) that the palindromes were increasing by 11, it seems that John wanted to share and open the discussion with the other participants. I understand his utterance (54) as a *wondering inquiry move*, in the sense that John, after having explored the mathematical task and making some notes of his notepad, wanted to share his result. His focus was on how to get to the next four digits palindrome, when starting from the lowest one (1001). As for Paul in Excerpt 1, there is a possibility that from John's perspective, his utterances (54) and (56) are clear, since John's focus is on the pattern appearing when going from one palindrome to the next (see Figure 15). However, the vagueness of John's explanation was visible as he started claiming that "they just increase by 11" without offering any details (54). It is interesting to see that John offered *spontaneously* some clarifications in (56). There is a possibility that the previous discussion concerning Paul's understanding, as presented in Excerpt 1, made John aware of the need to offer a more detailed and accurate explanation of his observation. However, his use of the terms "to increase by 11 to get it" (56) was still vague. A tentative clarification was given when John added "start from the bottom and then go to the next one". In an attempt to clarify John's claims, Claire showed a willingness to stop and seek an understanding of John's explanations through a *wondering inquiry move* (57). The purpose of this inquiry move was to question the term "the bottom" and to clarify the use of the pronoun "it" in "to increase by 11 to get it". Claire interpreted "the bottom" as the smallest four digit palindrome, which is 1001, and followed John's explanation in order to see how the next palindrome (1111) was generated. Claire's utterance (57) was a *wondering inquiry move* into John's thinking, as she tried to engage in John's observation by capturing key words as "bottom" and "the next one". Claire's search for deeper

explanations gave John (58) the opportunity to refine what he meant by “increase by 11”, and, he was now able to explain, in a more articulated way, that, in order to go systematically from one palindrome to the next one, for example from 1001 to 1111, one has to add 110 to the previous one, in John’s terms “hundreds increase with 1 and tens increase with 1”. Thereby he was able to explain in a more articulated way how it was possible to generate the different palindromes (58), and the ambiguity of his claims was removed.

Through Excerpt 2 it is possible to observe the difficulties related to how meaning was negotiated between the participants. As Paul in the previous excerpt, it seems that John developed a very clear understanding of the pattern which appears when one moves from one palindrome to the next one, starting from the smallest four digits palindrome (1001) and moving to the next one (1111). Here again, the vagueness and ambiguity of mathematical discourse was emphasized. However, it seems that building on the experience of negotiation of meaning, as presented in Excerpt 1, John was able to develop an awareness of the need for offering a more accurate and detailed presentation of his ideas. I consider that the development of awareness concerning the necessity of clear and precise explanation has been emphasized by Vygotsky (1986) as he acknowledged that “the difficulty with scientific concepts lies in their verbalism” (p.148-149) which implied the need for “precise verbal definitions” (Karpov, 2003, p.66).

91	1001	91-11
		+110
101	1111	
		+11
111	1221	
121	1331	
131	1441	
141	1551	
151	1661	

Figure 15: Copy of John’s notepad concerning the palindrome task

Excerpt 3 illustrates how Claire, building on John's example, tried again to address her didactical aim:

Claire adjusting momentarily her pedagogical strategy

Excerpt 3

87. Claire: yes, but for example, if we take 1111, what does this mean? one thousand one hundred and eleven?

88. pause

89. John: just a moment, ..., what did you say?

90. Claire: well, when you say ...

91. John (talking at the same time as Claire, with a louder voice): one thousand one hundred and eleven?

92. Claire: what does it mean? how can you write it?

93. pause

94. John (laughing a little): well, hmm

95. Paul (very low voice): one thousand one hundred and eleven

96. pause (Paul is writing in his notepad)

97. Claire: now I am keen to know what you are writing Paul!

98. Paul (laughing): hmm, hmm

99. Claire: what is it you have found?

100. Paul: yes, I am just finishing to calculate (John is laughing), a little calculation here!

101. pause

102. Paul (laughing a little): if you take 1001, 2002, 3003, 4004, then the difference between the answers [the quotients] is 91! (laughing) this is quite incredible!

103. Claire: beautiful!

104. Paul: (unclear)

105. Mary (clear/loud voice): yes, what is it you have done?

106. Paul: when you take the numbers 1001, 2002, 3003, 4004, then the difference between the answers [the quotients] is 91, for all of them

By encouraging the mathematical inquiry into the meaning of the four digit palindrome 1111, Claire tried to address her didactical aim.

Through her question (87), formulated as a *didactical move*, she wanted to encourage the teachers to move further with the task and to consider the structure behind the numerical patterns. In her utterance, she chose a numerical example and, building on the discussion with John, she asked “what does this mean”, that is, she was aiming to explore the *structure* of this particular number, as a generic example (Rowland, 2000) for four digits palindromes.

Claire's concern was to address her didactical aim and to indicate and encourage the teachers to consider a shift from considering different (and fascinating) numerical patterns to exploring the deeper structure of this task. By asking “what does it mean?”, she sought to indicate that 1111 might be written as $1 \cdot 10^3 + 1 \cdot 10^2 + 1 \cdot 10^1 + 1 \cdot 10^0$. Her didactical suggestion only met silence (88), as the other participants were occupied with writing in their notepads. John (89) asked for some more time, and asked

Claire to repeat her question. I consider the fact that John was asking for some more time before considering Claire's question as evidence for his confidence in our community of inquiry. In that sense, he is now able to decide *when* and *how* he might engage in the proposed inquiry.

Claire's didactical aim was again visible as she made an attempt to repeat her question (90), when John interrupted her (91) and talked at the same time with a loud voice. I consider that the fact that John did not pay so much attention to what Claire said (89), and that he interrupted her and talked at the same time (91) are characteristics of clear engagement and confidence in our community of inquiry. At the same time, these elements show that John was completely absorbed by the discovery of numerical patterns emerging from the task. Paul's fascination for the numerical patterns was also visible (100, 102). Claire's determination in addressing her didactical aim was made explicit as she repeated her *didactical move* (92), adding "how can you write it?" with clear indication to the structure of the task. In that sense, she repeated her first question, as in her previous didactical move (87), and gave some more precise indications of what she was looking for, that is the introduction of algebraic symbolism.

Taking Claire's perspective on the discussion, it seemed that, both John and Paul were just repeating her question without engaging in it. A possibility is that the teachers were completely absorbed by their respective calculations, and thereby Claire only met silence (93, 96) as a response to her didactical moves.

Through Claire's repeated didactical moves (87, 92) it is possible to follow her, as she recognized the teachers' fascination and engagement in different explorations of numerical patterns, and at the same time, as she tried to address her didactical aim by encouraging them to explore further the mathematical task and to introduce algebraic notation in order to articulate the structure of the task. Until now the teachers have presented to the other participants the result of their thinking, as presented in the two first excerpts, and in Excerpt 3, Claire's didactical aim was clearly visible. Since her didactical act, consisting of several didactical moves (87, 92), did not succeed, she decided to *change her pedagogical strategy* and to follow the teachers' discovery of patterns. Through her *information-seeking inquiry move* (97), she showed a willingness to seek to understand what Paul was engaged in. Therefore, I consider that Claire's utterance (97) represented a *shift* from seeking to achieve a didactical aim to inquiring into Paul's thinking. Paul's fascination for the emerging numerical patterns was clearly visible as he asked for more time (100) before addressing Claire's inquiry moves (97, 99). The fact that John was laughing at the same time, seemed to indicate that he recognized the fascination caused by discovering new numerical patterns.

The vagueness and ambiguity of mathematical discourse was again visible as Paul, possibly satisfied with his new numerical discovery, claimed (102) that “if you take 1001, 2002, 3003, 4004, then the difference between the answers is 91!” (see Figure 16). Here again Paul’s utterance offers evidence of the vagueness of mathematical talk, as Paul only mentioned “the difference between the answers” without giving additional comments. I believe he was referring to the fact that when dividing 1001 by 11 one gets 91, 2002 gives 182, and 3003 gives 273. Looking at the differences between the quotients, it is equal to 91 (102). He was quite enthusiastic, as he claimed, while laughing at the same time, that this result was quite incredible. By sharing his enthusiasm (103), Claire showed that she had changed her pedagogical approach *momentarily* and that she was following Paul’s fascination. Again, the vagueness of his mathematical explanation was highlighted by Mary, as speaking with a loud voice, she wanted to inquire into Paul’s result (105).

I consider that Excerpt 3 gives evidence of thorough confidence in our community of inquiry, and in the subject-matter, for all participants, as the teachers were confident enough *not* to follow Claire’s pedagogical approach, as presented at the beginning of the excerpt, and to inquire into the numerical patterns which they had discovered. Especially, Mary was showing her confidence in the community, through her *information-seeking inquiry move* (105) when asking, with a loud voice, for some explanations from Paul. At the same time, Claire showed enough confidence in the community *not* to continue on her didactical aim but, to change her strategy *momentarily*, and follow the teachers’ search for numerical patterns.

The teachers were still fascinated by the numerical patterns they observed and were not paying so much attention to what Claire tried to propose. In Excerpt 4, Claire’s didactical aim was in focus again, as she continued to encourage and support the teachers for introducing symbolic notation in order to elaborate a proof for the divisibility of the four digit palindromes by 11.

$$\begin{array}{r}
 1001 : 11 = 91 \\
 \underline{99} \\
 11
 \end{array}
 \quad \left. \vphantom{\begin{array}{r} 1001 \\ 99 \\ 11 \end{array}} \right\} 91$$

$$\begin{array}{r}
 2002 : 11 = 182 \\
 \underline{11} \\
 90 \\
 \underline{88} \\
 22
 \end{array}
 \quad \left. \vphantom{\begin{array}{r} 2002 \\ 11 \\ 90 \\ 88 \\ 22 \end{array}} \right\} 182$$

$$\begin{array}{r}
 3003 : 11 = 273 \\
 \underline{22} \\
 80 \\
 \underline{77} \\
 33
 \end{array}
 \quad \left. \vphantom{\begin{array}{r} 3003 \\ 22 \\ 80 \\ 77 \\ 33 \end{array}} \right\} 273$$

$$\begin{array}{r}
 4004 : 11 = 364 \\
 \underline{33} \\
 60 \\
 \underline{66} \\
 44
 \end{array}$$

Figure 16: Copy of Paul's notepad concerning the palindrome task

Introducing symbols

Excerpt 4

113. Claire: then how can you write them [the palindromes]? in order to write any palindromes, how many digits do we really need? in order to write a four digits palindromes?

114. John: you mean different?

115. Claire: yes, how many different digits do we need?

116. John: two

117. Claire: two, yes, this means, if we call these two digits "a" and "b", how can we write

[the palindrome] in a general way, a four digits palindrome?

118. pause

119. Claire (John is talking at the same time, low voice): it will be "abba"

120. John: yes, yes, or "baab"

In Excerpt 3, it was possible to follow how Claire did change her pedagogical approach and chose to follow Paul's last result. As presented through Excerpt 4, Claire felt that it was now time to bring the focus

back to her didactical aim and through her *didactical move* (113) she wanted to encourage the teachers to introduce algebraic symbolism. I consider that her didactical move is in the continuation of her previous didactical move (87) and these moves illustrate the iterative nature of didactical acts. John followed her suggestion, but asked for some more detail about Claire's question (114). I consider John's question as evidence for developing a more accurate mathematical language, as emphasised by Pimm (1987), Rowland (2000), Vygotsky (1986) and Karpov (2003). Furthermore, the analysis of previous excerpts (Excerpts 1 and 2) has shown that this issue can be traced back to the beginning of this workshop. By repeating and adding John's remark (115, 117) into her didactical move, Claire put emphasis on the necessity to use a precise and accurate mathematical language, and as such, she valued John's question. The next step toward the elaboration of an algebraic proof of the divisibility of four digit palindromes by 11 consisted of introducing symbols when Claire and then John proposed to write a four digit palindrome as "abba" (119) or "baab" (120).

In the following I offer a summary of the discussion taking place by the end of the workshop. In addition to her aim of developing an algebraic proof for the task, Claire was aware of time issue since the teachers had spent a lot of time in exploring numerical patterns, and she wanted to end this workshop in reasonable time. Thereby, the recognition of this issue resulted in Claire pushing the teachers toward an elaboration of a proof in order to achieve her didactical aim. Claire considered that the next step in the discussion with John, was to ask about the meaning of "abba", and what these symbols stand for. John proposed that the symbols refer to thousands, hundreds, tens, and units. Furthermore, answering Claire's *didactical move*: is it possible to go a little further? John and Mary proposed to group the "a" and "b" together. Mary suggested that $a \cdot 10^3 + b \cdot 10^2 + b \cdot 10^1 + a \cdot 10^0$ could be written as $a \cdot (10^3 + 10^0) + b \cdot (10^2 + 10^1)$. John finished the calculation, pointing to the fact that $10^3 + 10^0$ is equal to 1001 which is divisible by 11. The same reasoning applies for $10^2 + 10^1$ which is equal to 110 and which is also divisible by 11. As Claire asked about the possibility to develop this particular task further, Mary proposed to explore five or six digits palindromes, and to look for patterns.

The informal a posteriori analysis

After the workshop, Claire wrote her impressions and reflections:

I think it was an interesting workshop, but I was surprised by the way the teachers got fascinated by the numerical patterns they found and that it was difficult for me to introduce symbols, since my aim to be elaborate a proof. I think our group was functioning well and the teachers were really engaged this time too. (From my diary, 30.11.04, translated from French)

Based on these reflections, Claire elaborated an *a priori* analysis for Workshop VI.

The formal a posteriori analysis

Through this workshop it is possible to follow how the meaning, which each teacher had elaborated from the given mathematical task, was negotiated between the participants. The analysis reveals that it was necessary to negotiate several aspects of the activity before Claire could address her didactical aim. These aspects concerned the teachers' fascination for numerical patterns, the difficulty the teachers experienced in order to communicate to the other participants the result of their numerical explorations, the ambiguity and vagueness of mathematical discussion, and finally, Claire's struggle to encourage the teachers to move to the use of symbols. In contrast with Workshop II (see Section 4. 1. 2), this way of working toward a symbolic formulation was possible because of the *increase of confidence for all the participants*: for the teachers as they asked, challenged and argued with each other, and for Claire as she was in a position where she could change her pedagogical strategy *momentarily*, recognising and valuing the teachers' fascination for numerical patterns. However, contrary to Workshop II where Claire did not come back to her didactical aim, she was now able to continue to move toward a symbolic expression for the four digit palindromes, and the elaboration of an algebraic proof. I consider that during the processes related to how meaning was negotiated, the participants developed awareness of the vagueness of mathematical language, as showed in Excerpt 2 where John offered some clarifications of his claims *spontaneously*, and thereby of the necessity to focus on precise verbal definitions (Karpov, 2003) in order to pinpoint verbalism, which is one of the difficulties with scientific concepts (Vygotsky, 1986).

Thereby, this workshop provides evidence of the participants' confidence in both our community of inquiry, in the subject-matter, in the sense that the teachers were fascinated by the emerging structure of numerical patterns and were *not* willing to follow Claire's didactical moves and to engage further in the task and introduce algebraic notation. I consider the activity of developing awareness of underlying structure in numerical patterns, as it emerged from this workshop, is relevant for the development of algebraic thinking.

It was after that Claire insisted, through her repeated didactical moves, that the participants moved toward an algebraic formulation of the palindromes. Therefore, I argue that, the tone of the teachers' voice (speaking with loud voice), the fact that they chose *not* to follow Claire's didactical moves in the beginning of this workshop, the fact that they asked for more time and interrupted each other during the discussion, the fact that they engage deeply in exploring the structure of the task through

numerical examples, all these elements bring strong evidence both in terms of the development of our community of inquiry *and* in terms of the development of algebraic thinking.

In addition I want to emphasize Claire's role in this workshop, acting both as a didactician and as a researcher: it could have been interesting, from a researcher perspective, to let the teachers explore the numerical examples as much as they were willing to do, and observe whether or not they would have moved to the stage of using algebraic notation in order to answer the task (proof for the divisibility by 11). However, Claire was very conscious of her role as a didactician whose aim was to elaborate a proof using algebraic symbols. In addition, since the workshop was limited in time, our meeting lasted usually two hours, she decided to push further the teachers through repeated didactical moves and to move to the use of symbols and the elaboration of a proof. Therefore, I claim that in this workshop it is possible to find evidence of the occurrence of conflicting roles between Claire, acting as a didactician, and Claire acting as a researcher. I argue for recognising these issues as central to the development of Claire's knowledge concerning when and how it is appropriate to intervene.

4.1.6 Workshop VI: No need for Claire as a didactician

A priori analysis of the mathematical task

The rationale for the mathematical tasks, which were presented during Workshop VI, is different from the previous workshops: until now Claire had chosen the tasks as a means to provoke discussion concerning the choice and use of algebraic notation and issues related to community building (Workshops I, II), to focus on the historical development of algebraic notation and the power of modern symbolism (Workshop III), to introduce algebraic symbolism in the context of Euclidean geometry (Workshop IV), and to come back to generalisation of numerical patterns (Workshop V). The rationale for the mathematical tasks, as presented during Workshop VI, emerged from Claire's interviews with teachers and classroom observations, as she became sensitive to issues related to translation from natural language into equations: in Mary's class, the pupils experienced difficulties in expressing the relation between path, velocity and time, and Mary commented on these issues during the interviews both before and after her teaching. In John's class, the pupils worked on exercises, written in natural language, where different job situations, including an expression of salary, were proposed. The tasks consisted of translating these situations, described in natural language, into equations and choosing the most beneficial one. From classroom observation, Claire noticed that John used time and explained very carefully the transition from natural language into equations. In addition he

referred to these problems during the interviews both right before and after his teaching.

Therefore, taking inspiration from these experiences with Mary and John, Claire decided to deepen the issues addressed and observed during both interviews and classroom observations, and to choose the transition from natural language to equations as a didactical aim for Workshop VI.

The difficulties related to translation problems have been addressed in research literature before (see for example Herscovics, 1989), mainly from a cognitive perspective with emphasis on the identification of possible cognitive obstacles. Since my study fundamentally is rooted in a sociocultural approach to learning, as I consider learning as social participation, choosing a task related to the translation from natural language into equations might bring quite different issues to the fore.

During Workshop VI, Claire's goal or target knowledge, as a didactician, was to explore and emphasise the transition between natural language and equations with the role played by symbols, and more specifically to emphasize the difference between a *syntactic* translation (the sequence of words maps into a corresponding sequence of symbols), and a *semantic* translation (the meaning of the problem is related to the equation). Furthermore, her pedagogical strategy, as a means to work forward her didactical aim, was to propose two different tasks to the teachers, the first one from Diophantus (Duval, 2000), and the second one known as the student-professor problem (Küchemann, 1981; Herscovics, 1989).

The main characteristics of Workshop VI

The main characteristics of the sixth workshop are the following:

- Claire offering a justification for the choice of the mathematical tasks
- John taking, gradually, the didactical responsibility for one of the tasks
- John and Claire searching for a strategy
- Claire presenting the result of the Diophantus task
- Mary and Paul's inquiry into the Diophantus task

I will now go into more depth on each of these characteristics through several excerpts, as presented below.

Workshop VI: presentation of excerpts

The workshop began with a discussion concerning Claire's observation in John's class the day before this workshop. John explained how he introduced his pupils to equations with two unknowns, starting by solving equations graphically. He explained to the other participants that he chose to change the presentation of the task, as proposed in the text book, in order to emphasise the usefulness of the graphical representa-

tion. The discussion in our group continued by addressing the role of text books in the teachers' preparation, and John shared with the other participants his reflections concerning the order of the chapters as proposed in the textbook he used in his class. I discuss this issue further in Section 4. 1. 9.

Finally, by referring to the way text books are organised and especially by looking at the content table, the participants discussed which understanding of mathematics pupils might have. Especially, the teachers referred to mathematics as a subject-matter divided in different "boxes" with few connections between these, and they mentioned Viviani's theorem as an example which offered the opportunity to address such connections. The fact that the teachers recalled Viviani's theorem brings evidence for this task being part of the shared repertoire of our community of inquiry (Wenger, 1998), as it seemed meaningful for all participants to refer to this particular task.

During the discussion different views on how to present mathematics to pupils were articulated and Claire summarised by emphasising that the aim of the discussion was not to promote one particularly view, but rather to articulate and make visible each other's own understanding of mathematics as a means to look critically at it.

I consider, as a researcher, that by bringing this critical dimension into the discussion, Claire opened for the teachers the possibility to inquire into their own teaching practice and thereby to address issues related to *critical alignment* (Jaworski, 2006). I develop this issue further in Section 4. 1. 9. As such, this discussion between Claire and the three teachers reveals how the roles of Claire, the didactician and Claire the researcher, were complementary. After the discussion, Claire introduced the mathematical tasks to the teachers.

Offering a justification for the choice of the mathematical tasks

Excerpt 1

1. Claire: What I would like to do today, because I have been with you [observation in John's class] and we talked about equations, then I have a task for you: It is two sentences, two different sentences, with words, and I would like to translate these into mathematical symbols. Then, we will see what is happening. Then I would like to relate this [activity] to the exercise you [John] had yesterday on the blackboard in order to make the connection.

2. John: about what?

3. Claire: about salary, as you had yesterday

4. John: oh, yes, yes, yes!

5. Claire: then, let us see, the first sentence, if you can write down, it is a little task from Diophantus, do you remember Diophantus [from Workshop III]?

6. John and Paul are laughing (unclear talk)

7. Claire: and it is like this: (Claire reads, John, Paul, and Mary are writing) "Divide a given number into two numbers with a given difference". That is one has to divide

a given number into two different numbers with a given difference. That was the first one. And the second sentence is as follows: “Six times as many students as professors”. For example: at the university there is six times as many students as professors. Then we can try to translate these sentences and use symbols, and ...

Before introducing the two mathematical tasks to the teachers, it seemed that Claire, through establishing a link between the interviews and classroom observations with John (1-4), wanted to offer some kind of justification for the choice of the tasks to the teachers. This aspect is new, since Claire, until now, had chosen tasks from her own evaluation of which aspects were relevant to address in order to develop algebraic thinking, as explained in the rationale, or *a priori* analysis, of the previous workshops. Now, after visiting both Mary and John, Claire became sensitive to issues mentioned through interviews and classroom observations, and these became sources of inspiration. By making visible to the teachers the reason for choosing these tasks (1, 3), Claire wanted to emphasise the *complementarity of the different elements of the practice of our community* (workshops, interviews, and classroom observations), and, as such, how issues taken from the teachers’ own teaching practice might be relevant for the workshops. This recognition is also related to how our joint enterprise has evolved during the year, as we all are engaged in a collective process of negotiation (Wenger, 1998) concerning algebraic thinking.

Another aspect emerging from Excerpt 1 which I want to point to is that, while introducing the two mathematical tasks, Claire made visible her didactical aim “to translate these [sentences] into mathematical symbols” (1, 7), using the term “symbols” and not only “in a mathematical way” (see Workshops I and II). I consider that the way Claire articulated her didactical aim shows that the community has developed a common understanding of the didactical aims of the workshops and that, as addressed during the previous workshops, the focus is on algebra and on the use of algebraic symbolism. Thereby it seemed that Claire was now in a position to mention and to point *directly* to the use of algebraic symbols and this aspect offers evidence of her confidence and knowledge both in the community of inquiry and in herself, as a didactician.

The last aspect I want to focus on is the way Paul and John’s reacted to Claire’s question “do you remember Diophantus?” (5). I understand the fact that John and Paul were talking and laughing at the same time (6) as evidence of the existence of a shared repertoire: the mathematician’s name was now part of our shared repertoire (Wenger, 1998), and I consider that these two utterances (5, 6) give evidence of the existence of our community of inquiry, within which common experiences develop and reify into shared repertoire and become part of the history of our mutual engagement (Wenger, 1998).

After some clarifications about the purpose of the tasks, there was a pause during which each participant engaged with the tasks, writing on their notepad. In the following excerpts (Excerpts 2 to 7), I present the way our group engaged in the student-professor task. These excerpts follow directly after each other, with no interruption in the discussion.

Excerpt 2, starting right after a pause, shows the negotiation of which task should be discussed first, and how the participants started the negotiation of meaning of the student-professor task:

John's thorough confidence in our community

Excerpt 2

24. Claire: can we take the second one first? have you finished, Paul?

25. Paul: yes, yes, I am finished with the second one, but not with the first one

26. Claire: no, ok, but then let us take the second one first, what, what did you [to Paul] get?

27. Mary (low voice): yes, let us see ... [interrupted by John]

28. John (loud voice): P is equal to S divided by 6, that is what I wrote! [see Figure 17, 18]

29. Claire (repeats slowly): P is equal to S divided by 6

30. John: or one can also turn it around

31. Claire: that means 6 P is equal to S

32. John: yes!

The discussion concerning which task to address first started when Claire, after asking Paul about his results, proposed they engage with the second task and invited Paul to share his thinking with the other participants. Through her *information-seeking inquiry moves* (24, 26), Claire showed a concern about knowing and following how the teachers, and here especially Paul, engaged in the two different tasks. However, it was John who, by claiming his result (28, 30), started to explain his thinking. I understand his utterances as offering evidence of his thorough confidence both in our community of inquiry and in the mathematics as, while interrupting Mary (27), he was offering his results, explaining it in two different ways (28, 30). I suggest that John's confidence can be traced back to previous workshops (III, IV, and V), and especially to Workshop IV where he offered the proof of Viviani's theorem, as speaking for the whole group (see Section 4. 1. 4, Excerpt 6). However, from the copies of John's notepad (Figures 17, 18), it is possible to see how he wrote the sentence Claire had read out in the beginning of the workshop: in fact, John wrote first $P = 6 \cdot S$ which is a syntactic translation of the sentence, but then changed it into a semantic translation $P = \frac{S}{6}$. There is no indication in his note pad about his reasoning behind the change from syntactic to semantic translation, but a possibility might be that, building on the experiences from previous workshops (Workshops II about even and odd numbers, and Workshop V about palindromes), John considered a nu-

merical example of the student-professor sentence and therefore changed into a semantic translation. I will come back to the importance of numerical examples later (see Excerpts 5 and 6).

2) 6 x so many st. som
prof

Figure 17: Copy of John's notepad (6 x so many st. as prof.)

2) ~~P = 6S~~ P = $\frac{S}{6}$

Figure 18: Copy of John's notepad.

Excerpt 3, which comes right after Excerpt 2, illustrates how Mary and Paul presented their own results to the other participants.

Mary and Paul's presentation of own results

Excerpt 3

- 33. Claire: are there others suggestions?
- 34. Mary (with unsure voice): I believe I read a little wrong (laughing), because I took that P is equal to 6 S, then ... (see Figure 19)
- 35. Paul: yes, yes
- 36. Mary: isn't it ... (unclear)
- 37. Paul: 6 times
- 38. Mary (slowly): 6 times as many students ?
- 39. John: yes!
- 40. Mary: so, yes, (pause), so for each, do I think wrong now perhaps?
- 41. John: yes
- 42. Paul: I only thought six students were equal to one professor
- 43. Claire (repeats): six students are equal to one professor
- 44. Mary and Paul: yes, yes (pause)
- 45. Claire: so each time you have six students ...

46. Paul: then you have one professor
47. Claire: yes, then you have one professor (pause)
48. Claire: so you wrote, what was it you wrote Paul?
49. Paul: yes, 6S is equal to prof., one P (pause)

By opening for the other participants to offer their own results, Claire followed the same approach as in utterance (26), which was using an *information-seeking inquiry move* (33). The repetition of these information-seeking inquiry moves emphasises the iterative nature of inquiry and offers an illustration of the nature of an inquiry act, as consisting of a succession of (one or more) inquiry moves (see Section 2. 2. 6). In utterances (34-46) it is possible to follow how Mary and Paul negotiated their own results *on the background of John's claim* (28-30). It seems that Mary was unsure, as she felt the need for formulating a kind of excuse before formulating her result (34), but through getting support both from Paul (35, 37) and John (39), she was able to explain and articulate her reflections (36, 38), even though she was aware of presenting a different answer from John's (34). I understand her unsure voice and laughing as evidence of her doubt in relation to her mathematical result. However, evidence of the existence of the community of inquiry was offered as Paul and John joined her and engaged collaboratively with her in order to articulate and compare her result with John's (28, 30).

In the last part of the excerpt (38-49) it is possible to follow the different roles the participants were taking: John, building on his claim (28, 30) was acting as a didactician, offering an evaluation (41) of Mary's result (40), while Paul and Mary, by explaining their results (38, 46, 49) were trying to make sense of the differences between John's and their own understandings. Through the repetition of "six students are equal to one professor" (43, 45, 47), Claire's aim was to address the *meaning* of the sentence, and to indicate that the difficulties were lying in the way Paul and Mary understood and used the symbols "S" and "P", which is as short hand for students and professors, and not as representing the *number* of students and professors. From a copy of Mary's notepad (Figure 19), it is possible to see how she wrote the sentence that Claire was reading in the beginning of the workshop, and how she translated (*syntactic translation*: the sequence of words maps into a corresponding sequence of symbols) it into symbolic notation.

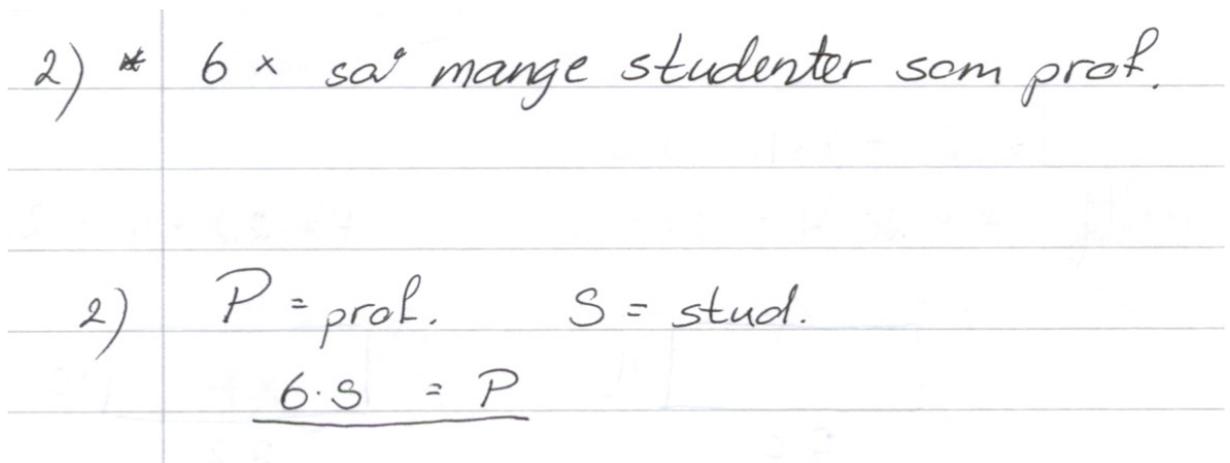


Figure 19: Copy of Mary's notepad (6 x so many students as prof.)

The fact that Claire made a pause after repeating Paul's claim (47) and asking how Paul wrote his result (48) indicate that she was unsure about how to negotiate the meaning of Paul and Mary's results further. Furthermore, through her *information-seeking inquiry move* (48), she showed a willingness really to grasp Paul's understanding in order to consider ways of moving the negotiation further.

By the end of Excerpt 3, there seemed to be a moment of hesitation within our group concerning how to engage further in the process of negotiating the meaning of the task. This hypothesis is supported by evidence as presented through Excerpt 4, where both John and Claire were looking for examples in order to emphasise an important theoretical perspective concerning the differences between a syntactic and a semantic translation.

Searching for a strategy

Excerpt 4

50. John: then, it means, ...(pause)
51. Claire: 6 times as many (at the same time as Paul)
52. Paul: 6 times students is equal to one prof,
53. Claire: yes, 6 times as many students as one prof.
54. Mary and Paul: yes, yes
55. Claire: then, hmm, what does it mean, we have ...
Everybody speaks at the same time

I consider that this short excerpt brings evidence of our mutual engagement in the process of developing a common and shared understanding of the meaning of the student-professor task: On one hand, John and Claire were both hesitating and repeating the task (50, 51, 53, 55), while Paul and Mary, on the other hand, were confirming their results (52, 54). I understand this negotiation as bringing evidence of Claire and John's

engagement in seeking for possibilities to engage further in the negotiation and to address Claire's didactical aim.

Especially the emphasis on the *meaning* of the mathematical task is visible in both John's (50) and Claire's utterances (55). As such the differences between John and, Mary and Paul's results give a clear illustration of the differences between a syntactic and a semantic translation, and thereby the way the discussion evolved offers a *relevant didactical and pedagogical context for elaborating on these issues*. In addition, there is a possibility that John also valued this situation, as he was seeking for finding ways to help Mary and Paul to move from a syntactic to a semantic translation. As such this didactical situation might be useful in relation to the development of his own teaching practice, and I consider this issue as highlighting a key didactical idea. The next excerpt presents how Mary, by offering a numerical example spontaneously, contributed to the development of the negotiation of the student-professor task.

John acting as a didactician: his discussion with Mary

Excerpt 5

56. Mary: if you count 1, 2, 3, 4, 5, 6, then there will be one professor there

57. John: yes, wait, if you have 12000 students, how many professors do you have? so we divide it by 6 then we get the number of professors

58. Mary: yes

59. John: yes, (pause) you cannot take 12000 students multiplied by 6 to get the number of professors. That is really what you are saying

Building on Mary's numerical example (56), John was now in a position to elaborate and develop a strategy for exploring the meaning of Mary's claim, that is the meaning of her syntactic translation. There is a possibility that, building on experiences from previous workshops (Workshops II and V), both Mary and John were now in a position to appreciate and value numerical examples. By engaging in a discussion with her, John showed that he was able to build further on Mary's example and, taking a slightly different numerical example, 12000 students, he formulated in the same utterance (57) both a question "how many professors do you have?" and a response "we divide it [the number of students] by 6 then we get the number of professors". I consider his utterance as a *didactical move*, as John *knows* the answer to his question (57). Furthermore, John developed his argument (59) by following the same numerical example (12000 students) and by emphasising the consequences of Mary's claim "that is really what you are saying". I understand his utterance as an attempt to highlight the difference between a syntactic and a semantic translation.

I consider that this way of negotiating was possible as a result of the thorough confidence the participants have in our community of inquiry which allows them to articulate, support, explore, and challenge each

other and thereby to engage deeply in the negotiation of the student-professor task.

There is a possibility that this experience contributed to the development of Mary and John's teaching knowledge, as they were in a situation where they could see the usefulness of numerical examples, and how these might be used during the negotiation of the meaning of a mathematical task. In Excerpt 6, John's negotiation of Paul's meaning of the task is presented.

John acting as a didactician: his discussion with Paul

Excerpt 6

60. Paul: set up an equation (pause)

61. John: yes, that is?

62. Paul: 6 times students is equal to one prof.

63. John: 6S is equal to one P?

64. Paul: yes!

65. John: but that is not right!

66. Paul: no, well

67. John: yes, but then I understand, but, but, but if we begin to calculate, that is to put in a real number, that is 12000

68. Paul: hmm

69. John: how many professors will you have then? (pause) 6 times 12000 ? then you get 72000, don't you? professors. So one professor must only be one sixth (1/6) student or something like that, isn't it?

70. Paul: yes, yes, yes

71. John: are you with me?

72. Paul: yes, yes, yes

Everybody speaks at the same time
(pause)

I consider that the negotiation of meaning between John and Paul, as presented in Excerpt 6, can be divided into two parts: first, in utterances 60-66, John seemed to follow on Paul's suggestion to "set up an equation" (60), and second in utterances 67-72, where John chose another strategy, offering an numerical example.

During the first part of Excerpt 6, the negotiation of meaning concerned the possibility to write an equation, as proposed by Paul (60). It seemed that John felt that it was his responsibility, as acting as a didactician, to offer an evaluation of Paul's understanding, since, clearly, he rejected (65) Paul's result (62). I consider John's utterance (61) as an *information-seeking inquiry move*, as he was inquiring into what Paul meant by setting an equation. John's didactical approach, during the first part of this excerpt, might be described as getting information (61), and confirming it (63) through a didactical move, since John *knows* that a semantic translation of the proposed mathematical task does not correspond to Paul's claim, and finally giving a response (65).

In the second part of Excerpt 6, John had the responsibility, as a didactician, to offer a new strategy to Paul in order to highlight the differences between a syntactic (Paul's result) and a semantic translation. There is a possibility that building on the experience of the negotiation with Mary (see Excerpt 5), John chose to consider a numerical example in order to achieve his didactical aim. Thereby, John's strategy consisted of choosing a number of students, translating syntactically "6 times 12000", challenging Paul by making visible the consequence of this kind of translation, and asking for confirmation about the differences between these two kind of translation. I consider that John's utterances (69, 71) offered both these different steps, and, in addition, an explanation of a semantic translation. In other words, John's didactical strategy was captured within this utterance as he was able to address the challenge of contrasting and highlighting the differences between a syntactic and a semantic translation (Claire did not use these terms with the teachers).

Focusing on how mathematical symbols are used

Excerpt 7

76. Paul: but then 6 students is equal to one prof.

77. John: corresponds to, no, no, that will be a wrong use of mathematical symbols

78. Mary: yes, but

79. John: in daily speak we can, but for each sixth student we get one professor

80. Paul: yes, that, with that I agree

81. John: yes, (pause)

The last aspect of John's didactical strategy is illustrated in Excerpt 7 in which John emphasised the way mathematical symbols are used. By contrasting the terms "equal" with "correspond" (76, 77), there is a possibility that John's aim was to highlight the fact that the symbol S represents the *number* of students and that the symbol P , the *number* of professors. Thereby John's reference to the loose use of mathematical symbols in everyday language (79) is an illustration of the idea of "vagueness of the mathematical discourse", as emphasized by Pimm (1987) and Rowland (2000).

I argue that, as a researcher, I consider that the negotiation of meaning which is presented through Excerpts 5 and 6 is of great didactical value for *all* the participants in the community of inquiry, including Claire, as Mary and Paul were questioning and challenging John, and how John, acting as a didactician took the responsibility to build on and develop their arguments further. These experiences might be highly relevant for their respective teaching practice. In addition, I want to emphasise the fact that both negotiations were conducted *without* the intervention of Claire, and therefore I argue that John, Paul and Mary were confident enough in the community to develop, explore, challenge, argue, and continue the negotiation together, while Claire, both as a didactician

and as a researcher, was confident enough in the community *not* to interfere in their negotiation of the student-professor task.

As a way to summarise the discussion of the student-professor task, Claire emphasised the importance of the structure and the meaning of the sentence which is translated from natural language into symbols. She illustrated this point by comparing the student-professor task with the first task, the Diophantus task. In addition, she referred to the fact that time for this workshop was close to the end and she spontaneously proposed the result for the Diophantus task, introducing the symbol N for the given number, x for the first number and D for the given difference, while she explained that a translation of the Diophantus task could be written as “N is equal to x plus $x + D$ ”. By presenting the result before the other participants could share their results, Claire showed that her main concern, in this part of the workshop, was to finish as proposed in the schedule.

From Claire’s perspective, her didactical aim was achieved since all participants engaged collaboratively in the negotiation of the student-professor task, and discussed in depth the differences between a syntactic and a semantic translation. Now her priority was on proposing quickly a solution for the Diophantus task in order to conclude the workshop. However, Mary and Paul took the initiative to question Claire about the way the Diophantus task was translated from natural language to an equation. The discussion, as presented in Excerpts 8 and 9, reveals some misunderstanding concerning the formulation of the Diophantus task and especially concerning the term “divide”. Furthermore, the analysis reveals the fact that Claire was not aware of this difficulty. Excerpt 8 starts when Claire had just explained the solution of the Diophantus task and Mary interrupted her.

Inquiring the meaning of “divide”: Mary’s interpretation

Excerpt 8

107. Mary: but, I am not quite sure that I understand the result you just explained ...

108. Claire: yes, ok, then, a certain difference, which means the difference between these two numbers, that is D, isn’t it ?

109. Mary: because I thought, I think that, hmm, you divided that number into two numbers, two different numbers [from Mary’s note pad: $x = x_1 \cdot x_2$]

110. Claire: yes

111. Mary: and it is this one and that one [referring to x_1 and x_2]

112. Claire: yes, no, ok, this one and then that one [referring to x and $x + D$]

(short pause)

113. Claire: then you can say, you can write that N is equal to x_1 and x_2 , and then you know that the difference between x_1 and x_2 is D, this means that x_2 is equal to $x_1 + D$

114. Mary: hmm, hmm

115. Claire: then you can write that N is equal to x_1 plus $x_1 + D$, isn’t it?

116. Mary: ok, ..., now I follow, yes, and then I can go on

117. Claire: yes

118. Mary: ok

By questioning Claire's explanation of the Diophantus task (107), Mary took the initiative to engage in the negotiation of the task and to share with the other participants her difficulties to follow and understand Claire's result. From Mary's notepad, it is possible to see that she understood "to divide a number into two numbers" as referring to a *division*, since she wrote in her note pad x as the product of x_1 and x_2 (Mary's notation). However, it seems to be a mismatch in the discussion since Claire did not directly address the confusion about the term "divide", but explained how the numbers x_1 and x_2 could be written as x and $x + D$ (in Claire's notation). However, it is when Claire explicitly referred to the *sum* of these two numbers (113, 115) that Mary realised her misunderstanding (114, 116). I consider that Claire, during Excerpt 8, interpreted Mary's difficulties as related to how the two numbers were related to each other. As she explained: " x_2 is equal to $x_1 + D$ ". It was only when Paul contributed to the discussion, making explicit the confusion due to the term "divide" that Claire realised Mary's misunderstanding.

Inquiring the meaning of "divide": Paul's interpretation

Excerpt 9

119. Paul: but, I have also interpreted that differently

120. Claire: yes?

121. Paul: I have interpreted that the two numbers should not be plus, but that the

difference should be between the two numbers [from Paul's note pad: $D = \frac{x}{y}$]

122. Claire: oh, oh!, ..., yes, I see ...

123. Paul: yes, yes

124. Claire: ok, yes, hmm

125. Mary: yes, this is where I stopped

126. Paul: ok, now I see ...

By participating in the discussion and claiming his way of understanding the Diophantus task (119), Paul offered support to Mary and made explicit (121) that he did not interpret the sentence as a sum, "the two numbers should not be plus", but as a division between these two numbers x and y , in Paul's notation. Thereby, the ambiguity concerning the term "divide" was made explicit, resulting in Claire's surprise (122, 124). I consider that Claire's reaction (122, 124) brings evidence of the diversity within our community of inquiry, as the terminology used in the Diophantus task belongs to the community Claire is used to be part of, taking a course in history of mathematics and used to Euclid, Diophantus and Descartes' terms, while the teachers belong to their own school community and do not use these terms with the same meaning. However, it is through the recognition and the articulation of these in-

sights that I am, as a researcher, in a position to help Claire, as a didactician, to deepen her awareness of the importance of issues related to terminology, and as such to contribute to the development of her didactical knowledge. I believe that Excerpts 8 and 9 illustrate the way our community of inquiry negotiated the ambiguity resulting from the use of the term “divide” and how the premises for the elaboration of a common repertoire were made visible. It is through the recognition of these differences due to the fact that the participants belong to different communities that the specificity of our community of inquiry became gradually articulate.

Furthermore, it is now, during the process of formulating the results of my analysis that these issues are brought to the fore. Thereby, I am in a position, now, as a more experienced didactician, to recognise that a possibility for avoiding the ambiguity emerging from the use of the term “divide” might be to reformulate the task as: “express a given number as a sum of two numbers with a given difference”. I comment further on the way a particular task is formulated in Section 4. 2. In addition, I recognise the problem of rushing to the didactician’s sophisticated mathematical formulation, due to time issues, and expecting teachers to follow. Nevertheless, the fact that Mary did ask questions (see Excerpt 8) seems like evidence of the growth of community and trust.

The informal a posteriori analysis

After the workshop, Claire wrote her own impressions and reflections:

We had really a nice workshop, and it was so interesting to see how John discussed with Mary and Paul! This student-professor task gave us a nice opportunity to address symbols and what they represent. I think it was a pity that time went so fast, I felt I had to finish rapidly by the end of the workshop. (From my diary, 11.01.05, translated from French)

It was from these reflections that Claire engaged in the elaboration of an *a priori* analysis for the next workshop.

The formal a posteriori analysis

When presenting the two mathematical tasks to the teachers, Claire had a well defined didactical and pedagogical strategy. Her didactical aim was to address the transition from natural language to equations with the role played by symbols, and more specifically to emphasize the difference between a *syntactic* translation (the sequence of words maps into a corresponding sequence of symbols), and a *semantic* translation (the meaning of the problem is related to the equation). Furthermore, her chosen pedagogical strategy, as a means to pursue her didactical aim, was to propose two different tasks to the teachers: a first task from Diophantus, and the student-professor task.

The presentations of the different excerpts allowed me to deepen my account of the main characteristics of this workshop, as presented in the

beginning of this section. In the following, I offer a synthesis of these aspects and highlight the links with previous workshops as a means to trace the development of algebraic thinking within our community of inquiry.

In the first excerpt the way Claire introduced the mathematical tasks was presented. The analysis revealed that, by offering a justification for the choice of the tasks, the complementarity of the different elements of the practice of our community was highlighted since Claire was inspired by the issues mentioned during Mary and John's interviews. This aspect is new, since Claire did not offer such a justification of the choice of the tasks in previous workshops. Evidence for the existence of a shared repertoire, as reported by the teachers' reaction to the name of the mathematician Diophantus, is also presented.

I consider that the analysis, as presented after Excerpts 2 to 7 brings clear evidence of *all* participants', and especially John's, thorough confidence within our community of inquiry. I argue that the development of confidence within our community of inquiry can be traced back to previous workshops (Workshops III, IV, and V) where different aspects of confidence were gradually nurtured.

Through Excerpts 2 to 7 from this section, it is possible to follow John, as he gradually took over the role as the didactician, and engaged with Mary and Paul in the negotiation of the student-professor task: in Excerpt 2, from claiming his result of the student-professor task, to arguing with Mary and Paul as they tried to articulate their findings, while supporting each other (Excerpt 3). Furthermore, the challenge for John and Claire consisted of finding a way to develop further the negotiation of the task where the differences between syntactic and semantic translation were highlighted (Excerpt 4). Finally, building on Mary's numerical example, John took over the responsibility of the student-professor task, *acting as a didactician* with both Mary (Excerpt 5) and Paul (Excerpt 6), and offering them a valuable strategy aiming to elaborate a semantic translation of the task and, thereby highlighting how mathematical symbols were used (Excerpt 7). In addition, the fact that Claire's voice disappeared during these excerpts, as if she was *acting as a silent-participant*, brings evidence of her confidence both in the other participants and in herself, as a didactician, since she decided *not* to intervene.

I recognise now, as I am engaged in the process of presenting the results of my analysis, the value of the different steps, as these emerged during the negotiation of the student-professor task, and I argue that this discussion enabled both Claire, as a didactician, to develop her knowledge further, and the teachers in relation to their teaching knowledge.

In addition, Excerpts 8 and 9 bring insights concerning the importance of the terminology used in the mathematical tasks. As shown in the

analysis of these excerpts, Claire was *not* aware of the difficulty caused by the term “divide” as she presented the solution of the Diophantus task: her main concern was to end the workshop quickly because of time issue. However, through Mary and Paul’s questions, she became gradually aware of the mismatch between the teachers’ understanding of the task, and the way she presented the solution. I argue that, through the recognition of the different communities to which the participants belong, and the difference in the used terminology in each community, important insights are articulated and made visible and, thereby the importance of the *nature and the formulation of the task* is emphasised. The centrality of the role played by the mathematical tasks has been mentioned in previous analyses, as for example in Workshops II, IV, and V. This recognition is also central in the development of Claire’s didactical knowledge. I develop further issues related to the nature and the formulation of the tasks in Section 4. 2.

In addition, I am now, as a researcher engaged in the process of articulating the key elements, as they emerged from the analysing of workshop VI, and thereby I am in the position to recognise, appreciate and value these insights. As such, Claire the researcher is helping, Claire the didactician, to develop her awareness of the value of such didactical situations. I develop these issues further in Chapter 5.

4.1.7 Workshop VII: Developing insights into what the task is about

A priori analysis of the mathematical task

The rationale for the mathematical tasks, as presented during Workshop VII, is based on Claire’s previous experience with Workshop VI. Through that workshop, the importance of issues related to the difference between a syntactic and a semantic translation was highlighted. Thereby, since this issue had emerged from classroom observation (see a *priori analysis* in Section 4. 1. 6), Claire thought that it might be interesting to explore the difference between a syntactic and a semantic translation further and to offer the teachers some tasks related to the same issue. By doing this, it would also offer Claire a possibility to compare how the teachers would engage in new tasks, comparing to what they did during Workshop VI.

In addition, Claire was following at the same time (spring 2005) a doctoral course in Copenhagen concerning French Didactics focusing on the work of Guy Brousseau, Raymond Duval, Yves Chevallard, and Gérard Vergnaud. Especially, Duval’s research was addressing the difference between a syntactic and a semantic translation and, in one of his articles (Duval, 2000), he illustrated this issue through several examples.

Thereby, building on both previous experience through Workshop VI and the doctoral course in Copenhagen, Claire decided to continue ad-

dressing the same didactical aim as in Workshop VI, which was to highlight the difference between a syntactic and a semantic translation. Furthermore, her pedagogical strategy, as a means to work toward her didactical aim, was to propose two different tasks, the first one inspired by Duval's (2000) article, while the second one was elaborated by Claire herself. The first task was as follows: *The perimeter of a rectangle with length "a" and width "b" is 62, the length of the rectangle is increased by 2 meters, and the width is decreased by 1 meter. The area is constant. Write the area of the rectangle before change, write the area after change.* The second task which Claire offered the teachers was: *Ole has three siblings. Ole is ten years older than Per and three years younger than Kari. Ole is five years younger than Jens and together they are 58 years old. How old are Ole, Per, Kari and Jens?*

The main characteristics of Workshop VII

- Clarifying the task: John focusing on what Claire wanted
- Clarifying the task: Paul focusing on the mathematics
- Ambiguity of mathematical discourse: the choice of unknown
- Paul and John organising the negotiation of meaning of the second task

I will now illustrate each of these characteristics through several excerpts, as presented below.

Workshop VII: presentation of excerpts

My aim with presenting data from the two different tasks I offered the teachers is to compare how the meaning of these two tasks has been negotiated among the participants. It seems that during the negotiation of the first task, one of the teachers focused on what Claire wanted to achieve with this task, while during the negotiation of the second task, the teachers' focus was on the mathematics. I illustrate these aspects through the presentation of excerpts, as below.

Clarifying what the task is about – Focusing on what Claire wanted

Excerpt 1

37. John: I wonder, I wonder if I misunderstand the text a little, it says "The perimeter of a rectangle with length a and width b is 62, the length of the rectangle is increased by 2 meters and the width is decreased by 1 meter. The area is constant. Write the area of the rectangle before change, write the area after change" that means that you are looking for an algebraic expression here?
38. Claire: is this the way you understand the question?
39. John: yes, for otherwise it [the area] is constant!
40. Mary: the area is constant
41. John: yes, it is indeed constant, is this what you are asking for?
42. Claire: ok, let me see ...
43. John: yes, yes, yes, no, I mean (John and Mary are laughing)

44. John: yes, yes, alright, I see that you want to direct us toward something like an algebraic solution! But the way it [the task] is phrased, because it says that it [the area] is the same, so it depends on what you are asking for, so here I would have, without doing any calculation, I would have written an expression, because this answer, the area before change, is like that one [area after change], the area is constant!
45. Claire: yes, ok, but here it [the task] says “write the area”, not calculate the area
46. John: no, ok, this was different! ... an algebraic expression, let me see ...

The process of clarifying the purpose of the proposed task started as John recognised and shared with the other participants his hesitation concerning how to understand the task (37). It seems that he chose to go back to the formulation of the task, as offered by Claire, reading it aloud, as for including the other participants in his hesitation. However, by the end of his utterance, he chose to focus on what he perceived Claire was looking for. John (44) seemed to indicate that, since the question was not related to the calculation of the area of the rectangle, another possibility for understanding the task was to investigate Claire’s aim with the task “I see that *you* want to direct us toward something like an algebraic solution”.

I consider this way of engaging in the task as surprising since, during previous workshops, John seemed to have developed great confidence both in the mathematics and in our community of inquiry. Now, I am not questioning his confidence in our community, the fact that he could ask directly what Claire was looking for seems to show that he felt free to address this issue. On the other hand, it seemed that since John was hesitant and unsure concerning the purpose and meaning of the task, his strategy was to ask Claire if she was looking for an algebraic expression. I consider that this way of acting, focusing on what Claire wanted and not on the mathematical task, has been observed during earlier Workshops (see Workshops I and II). The identification of this issue during Workshop VII is central in terms of recognising the importance of the influence of the nature of the mathematical task on the process of negotiating the meaning of the task. I comment further on this issue in the formal a posteriori analysis.

The purpose of Claire’s *information-seeking inquiry move* (38) was both to keep open the possibility for further investigation into the purpose of the task, and to move the focus *away* from her and back to the mathematical task. At the same time, she tried to include Mary and Paul in the inquiry. I understand utterances (39 – 46) as illustrating how the participants negotiated the meaning of the question, trying to clarify what the task was about. By focusing back to the text of the task (45) and contrasting the formulations “*write* the area” with “*calculate* the area”, Claire’s aim was to direct the focus of the negotiation of the meaning back to the task and away from what her purpose was, and as such indi-

cating that the formulation used in the task should be at the core of the discussion.

My interpretation of the discussion is that the teachers' difficulty with the task, as it was proposed, was related to the fact that the rectangle's area *is* constant, as emphasised by the repetition in John's utterances (39, 41, 44) and Mary (40). In other words, the question was not about *calculating* the area after changing the length and width. Thereby, I consider that the negotiation of meaning, as presented in Excerpt 1, concerned the clarification of what the task was about. It seems that the tension between, on one hand John's focus on Claire's purpose and on the other hand Claire's aim to discuss the task was resolved as John recognised and valued the different formulations (46).

Choosing not to engage in the negotiation of meaning

Excerpt 2

51. John: yes, to write the area [of the rectangle] both before and after change. Then I wrote length times width is like area little "f" [my interpretation of John's utterance: A_f] before change, I am not quite sure, I haven't done much more than this but I did this and I called it area before change

52. Claire: yes, area little "f" [my notation: A_f] for area before change, yes and ...

53. John: and after change it has to be length plus 2 times width minus 1, and this is area after which is the same as area before, that means it [the area] will be l plus 2 times b minus 1, which is the same as area before

54. Claire: hmm, hmm

55. John: here are the two expressions I got, so ... (pause)

56. Claire: have you done the same, Paul?

57. Paul: hmm, I don't know, I am not sure ...

58. John: this is the way I thought about area, I have done the same for the circumference

59. Paul: I was not paying attention because I was not quite finished

My purpose, with presenting Excerpt 2, is to draw attention to how Paul chose *not* to engage in the negotiation of the meaning of the task, as initiated by John (51, 53). While John and Claire were engaged in elaborating an expression for the area of the rectangle before change (my interpretation of John's utterance: $A_f = l \cdot b$, where little "f" relates to the Norwegian word "før" meaning "before") and putting emphasis on the area before and after change ($A = (l + 2)(b - 1)$, my interpretation of John's utterance), it seems that Paul was unsure about participating in the discussion (57). There is a possibility that he felt he had to offer a justification for his evasive answer since John tried to summarise his own thoughts (58). According to Paul, the reason why he chose not to pay attention was that he wanted to continue with his own calculations before engaging in the discussion. I referred before to the fact that Paul did not participate actively in the discussion (see Workshop IV, Excerpts 2 and 3). I consider that, as exemplified in Excerpt 2 (see above), it is

possible to follow how Paul chose not to participate in the discussion and how he felt free to justify his position to the other participants explaining that, since he was not finished with his calculations, he chose not to pay attention to the discussion (see also Workshop V, Excerpt 3). I argue for considering this way of acting as evidence for thorough confidence in our community of inquiry.

Shortly after the above exchanges, Paul engaged in the discussion, presenting the result of his thinking. Excerpt 3 presents how Paul engaged in the process of clarifying the purpose of the task. My aim with presenting Paul's way of clarifying the task is to contrast it with John's way of engaging in the task, as presented in Excerpt 1.

Clarifying what the task is about – Focusing on the mathematics

Excerpt 3

71. Paul: I wrote that the area is a times b

72. Claire: ok, and which one was it?

73. Paul: it was before [change], and after [change] I wrote [a] plus 2 and [b] minus 1

74. Claire: yes, yes

75. Paul: and then they [area before and after change] are supposed to be equal

76. Claire: hmm, hmm

77. Paul: then I got that a is equal to $2b$ minus 2 and ...

78. Claire: but what was the task about, what was the question?

79. Paul: write two equations

80. John: oh, yes!

81. Paul: so, this is one of the two equations

82. John: yes, I see [unclear]

83. Paul: and when you

84. Claire: hmm, and then you will ...

85. Paul: well, I don't know, what was it we were supposed to do?

86. Claire: so what equations would you use?

87. Paul: what?

88. Claire: no, I mean, what equations would you use?

89. Paul: no, I was just looking at, I am not there yet, I just wrote, what was it we were supposed to do?

While John was focusing on what he perceived Claire wanted him to do (see Excerpt 1), Paul engaged in the negotiation of the meaning of the task by focusing on the mathematics, and more specifically on writing two expressions, Paul refers to two equations (79, 81). It seems that Paul considered the first equation as resulting from the equality of the area of the rectangle both before and after change (73, 75, 77, 81). Furthermore, there is a possibility that Paul was considering the equation related to the perimeter of the rectangle as the second equation (79, 81). The need for clarifying what the task was about is visible as, while recognising that area is equal (75), Paul seemed to engage in a process of solving an equation (77). As such, Claire's *didactical move* (78) was an attempt to renegotiate what was the meaning of the task and what was the *nature of*

the question in the task (“*write* the area of the rectangle before change, *write* the area after change”). My interpretation is that, although Paul was aware of the nature of the question (79), he still wanted “to do something more” with the two expressions he wrote (85, 89). By using the expression “doing something more”, I want to recall Collis’ (1974) idea of “lack of closure” referring to pupils’ inability to hold unevaluated operations in suspension. I consider that through Paul’s argumentation, as presented in Excerpt 3, it is possible to observe a similar process to “lack of closure”, as Paul seemed to have difficulties in considering the expressions for area of the rectangles before and after change as unevaluated expressions *without* necessarily engaging in solving them (85, 89). Thereby, it seems that Paul was thinking in terms of equations. I consider that this observation is important in terms of following the participants’ development of algebraic thinking, as it witnesses the impact of procedure oriented practices in school on the teachers’ algebraic thinking. The discussion went on with Mary, John and Paul considering possibilities to implement this particular task in their own teaching. After these didactical considerations, the participants engaged with the second task. In Excerpts 4 and 5, I present evidence of the vagueness and ambiguity of mathematical discourse as it emerged from the discussion between the participants.

Ambiguity concerning the choice of unknown: Ole or Per’s age?

Excerpt 4

211. Paul: so this [the task] is very easy!

212. John: yes, yes, so far as I can see it is [very easy]

213. Paul: I think

214. John: yes, as far as I can see it is easy, but I think it can be difficult because they [the pupils] have to, they have to change [the text of the task] a little in order to write an equation, at least as far as I can see. I wrote an equation, they are 58 year old all together, and you have to go to the age of the other [persons in the task], that is you have to change the description

215. Mary: this is where I made the mistake, I can see it now, and then it all getting wrong, I can see it when I start to look at it, I put it quite wrong, I could see it when I started to calculate

216. Claire: what do you mean by quite wrong?

217. Mary: yes, because everything is given in relation to Ole, and when I look at him, then I can see that I wrote a wrong equation

218. Claire: can you give us an example?

219. Mary: yes, because I started with Ole, or I saw that Per, he had to be the unknown, then I wrote, Ole is 10, now we know that Ole is 10 years older than Per, I made a mistake here, yes, I wrote minus ...

220. Paul: but, this is correct!

221. Mary: yes, but this correct!

222. John and Paul: because you took Ole as x

223. Mary: yes, yes, yes!

I consider that Excerpt 4 illustrates the difficulty of engaging in negotiating the meaning of the task since the participants did not explicitly address the choice of unknown in relation to this second task. While presenting some didactical considerations, it seemed that John addressed the ambiguity concerning the choice of unknown as he explained that the description of the task had to be changed (214), but still without giving more detail about what he meant (see Figure 20). I understand John's claim concerning the need for changing the text of the task as referring to the translation from written language to a mathematical expression. It is possible to see, from John's notepad, how he first wrote the task by reformulating "Ole is ten years older than Per" as $Ole: P + 10$; "three years younger than Kari" as $Ole: K - 3$; and "Ole is five years younger than Jens" as $Ole: J - 5$. There is a possibility that, after doing this, John decided to introduce the unknown " x " as referring to Ole's age and to rewrite the ages as $x - 10$ for Per, $x + 3$ for Kari, and $x + 5$ for Jens. The shift between the signs is visible in his notepad (Figure 20), from $P + 10$, $K - 3$, and $J - 5$ to $x - 10$, $x + 3$, and $x + 5$, when x refers to Ole's age. I understand John's claim "they [the pupils] have to change [the text of the task] a little in order to write an equation" as referring to this shift, and as evidence of mixing mathematical inquiry with didactical considerations.

The ambiguity was addressed further as Mary, offering a judgment on her own result, seemed to claim that she wrote a "wrong" equation (215, 217). Through *information-seeking inquiry moves* (216, 218), Claire tried to make explicit what Mary was referring to, and thereby, her aim was to address directly the choice of unknown. I consider that Mary engaged in addressing the issue concerning the choice of unknown as she explained that, since she considered Per's age as the unknown ("I saw that Per, he had to be the unknown"), her mistake, according to her, consisted of writing an equation using a minus sign (219). Thereby, it seems that the ambiguity concerned both the choice of unknown, since Mary chose Per's age while John and Paul claimed that she chose "Ole as x " (222), and to which equation the minus sign was referring.

2	15	Ole	$x + 10$	$K - 3$	$J - 5$	} 58
	5	Per	$x - 10$			
	18	Kari	$x + 3$			
	20	Jens	$x + 5$			

$$x + x - 10 + x + 3 + x + 5 = 58$$

$$4x - 2 = 58$$

$$x = 15$$

Figure 20: From John's notepad

From a copy of her notepad (Figure 21), it is possible to see that Mary did choose Per's age as the unknown, as she wrote x beside the name Per, and from that she could deduce the age of Ole as $(x + 10)$, Kari as $(x + 10) + 3$, and Jens as $(x + 10) + 5$. In addition, I want to emphasise the fact that Mary wrote both Kari and Jens' age as Ole's age in parenthesis $(x + 10)$ before adding the difference between Ole and Kari (+ 3) and Ole and Jens (+ 5). Thus, it seems that Mary's aim was to follow the presentation of the task, where all ages were presented from Ole. However, it is possible to see from her notepad that, when writing the equation, Mary changed the plus sign into a minus sign. It is only by the end of Excerpt 4 that the ambiguity was addressed as both John and Paul claimed that, if Mary took the age of Ole as the unknown (222), it was correct to write a minus sign (220, 221, 223).

I have referred earlier to the ambiguity of mathematical discourse (see Workshop V) and I consider that Excerpt 4 offers further evidence of the need to give clear exposition of mathematical concepts (Vygotsky, 1986) where the necessity to define the unknown, as the age of a chosen person, is emphasised in order to avoid any ambiguity. Thereby, I consider that the issue was not about writing a "wrong" equation, the issue was about having *consistency* between the choice of unknown, and writing the corresponding equation. The ambiguity concerning the choice of un-

known is visible as John and Paul claimed “because you took Ole as x ” (222), while, from a copy of Mary’s notepad (see Figure 21), it is possible to see that she chose Per’s age as the unknown. The ambiguity concerning the choice of unknown is addressed further and clarified in Excerpt 5.

Ole $x+10$
 Per x
 Kari $(x+10)+3$
 Jens $(x+10)+5$ } 58 år

~~$x-10 + x + (x-10) + 3 + (x-10) + 5 = 58$~~

$x-10 + x + x-10 + 3 + x-10 + 5 = 58$
 $4x = 58 + 30 - 3 - 5$
 $4x = 80$

Figure 21: From Mary’s notepad

Exploring further the ambiguity of the choice of unknown: who’s age did you choose as “ x ”?

Excerpt 5

285. John: yes, but here is the change that I was talking about, isn’t it, because from now, you have to turn it [the description], from the information you got here [the text], when you are going to write an equation, then you have to change some sign

286. Mary: yes, because it was something which went wrong when I wrote the equation ...

287. Paul: no, yes, ok, you should not have done this (Mary and John are laughing), I have done it [writing an equation] in another way

288. Mary and John: yes, tell us, tell us!

289. Claire: yes, yes

290. Paul (with a loud voice): ok, I mean I could see who was oldest and who was youngest and like this, and Per was the youngest one, Ole was in the middle, and then were Kari and Jens older [than Ole]

291. Claire: so you could read this from the text?

292. Paul: yes, yes, I could see that Ole was there, Ole there, ok, here is Ole, and then it was Per, he was younger than Ole, 10 years, this means it will be x minus 10, here Ole is x , and Jens is x plus 5, Kari is x plus 3
293. Everybody: hmm, hmm
294. John: who did you choose as x ?
295. Paul: Ole!
296. John: yes, yes
297. Paul: because everything comes from Ole
298. Everybody: yes, yes
299. John: so the expressions you have are just the same as ... (unclear)
300. Paul: yes, yes, just the same as what I have
301. John: yes
302. Paul: but what I wanted was the order, I wanted to see who was oldest and who was youngest
303. Everybody: hmm, hmm
304. Paul: I like to think in a practical way, yes, a little practical

In Excerpt 5, it is possible to follow how the participants clarified the ambiguity concerning the choice of unknown. It seems that an important turn in the clarification of the ambiguity of the task occurred as Paul, interrupting John and Mary's argumentation (285, 286), proposed to explain his result (287), suggesting that he approached the task from a different perspective. According to Paul, the first step consisted of understanding in which order Per, Ole, Kari, and Jens were in relation to their age (290). Responding to Claire's *information-seeking inquiry move* (291), Paul explained that he was able to extract this information from the text. In other words, by reading the task carefully, Paul was able to understand who was oldest and who was youngest, even though all information was given from Ole, and to separate on one side Per, who was younger than Ole, and Kari and Jens who were older than Ole (see Figure 22). I consider John's *information-seeking inquiry move* (294) as crucial as it enables the participants to address directly and to resolve the ambiguity concerning the choice of unknown. My interpretation of John's move is that, by asking "who did you choose as x ?", and I interpret his inquiry move as referring to *who's age* did you choose as x , he recognised implicitly the possibility to have different choice for the unknown, and therefore it was important to clarify *first* who was taken as the unknown (294) *before* engaging further in a discussion concerning writing equations, as presented by Paul (292). I understand the fact that Paul offered a justification for his choice of unknown (297) and put emphasis on the "order", that is who is oldest and who is youngest (302), as an attempt to share with the other participants the different steps in his thinking, and thereby invite the others in understanding how he negotiated the meaning of the task.

②

Kari
5 år
Ole
Per

Kari Jens Ole Per = 58

5 år | 10 år

3 år

$$(x+2) + (x+5) + x + (x-10) = 58$$

Figure 22: From Paul's notepad

By comparing the different approaches Mary, Paul and John had on this task, it is possible to see that there were several possibilities of engaging within Ole's siblings task. However, I want to argue that, because of the ambiguity of the mathematical discussion, as exemplified through Excerpts 4 and 5, the participants had difficulties in understanding each other and in comparing the different results. The necessity of clarifying and making explicit central elements in mathematical discourse has gradually emerged from the recognition of the ambiguity of mathematical discourse, as exemplified in Excerpts 4 and 5.

As mentioned above, I consider that the issue at stake concerned *consistency* between the choice of unknown (Paul chose Ole's age as x , while Mary took Per and John took Ole after changing the description) and expressing the relationship with algebraic symbols. I consider that the richness of the task, as illustrated through Mary, Paul, and John's answers, and the possibility to engage in it through different perspectives were not explicitly addressed in the discussion (see Excerpts 4 and 5) due to the vagueness and ambiguity of the participants' mathematical discourse. Thereby, the lack of clarity concerning the choice of unknown seems to prevent the participants from exploring the richness of the task

and to recognise the possibility to engage with it through different approaches.

The informal a posteriori analysis

After Workshop VII, I wrote in my diary:

I am surprised by the difficulties the teachers had with the first task, and how they engaged with the task. I am not sure why it went like this. Concerning the second task, I think we had a nice discussion with different approaches to it. I told the teachers that they will be responsible for bringing tasks for the next time. It will be interesting to see what kind of tasks they will choose. (From my diary, 09.03.05, translated from French)

These reflections were not directly part of the a priori analysis of the next workshop, since the teachers were in charge for the next workshop.

The formal a posteriori analysis

Through Workshop VII, it is possible to follow how the meaning of two different tasks was negotiated between the participants. The analysis reveals different aspects concerning the participants' development of algebraic thinking, both in terms of clarifying the purpose of a task and highlighting the difference between a syntactic and a semantic translation from language to symbols. I develop these aspects in the following.

Through Excerpt 1, it was possible to follow how John, while expressing his hesitation concerning how to understand the purpose of the task, seemed to focus on what Claire wanted him to do with this task. It was only after Claire's clarification concerning *the nature of the question* (write the area, not calculate the area) that John seemed to understand the purpose of the task. I argue for considering John's way of acting, as he felt free to ask Claire directly about what she wanted, as evidence for thorough confidence in our community of inquiry.

However, my interpretation of his way of engaging in negotiating the meaning of this first task is that his confidence *in the mathematics* seemed to be not as strong as observed during previous workshops (see Workshops IV, V, and VI). I argue for recognising the importance of this fact and for considering it as further support concerning my hypothesis about the importance of the *nature of the mathematical task*. I develop further this issue in Section 4. 2.

While John seemed to focus on what he perceived was Claire's aim with the task, Paul engaged in it by focusing on the mathematics. I consider that, by following the way Paul explained his thinking (Excerpt 3), it seems that his aim was to find an expression for " a ", "I got that a is equal to $2b$ minus 2 and ..." (77). Thereby, Paul seemed to engage in solving an equation, and I argue for considering this way of approaching an algebraic expression (having difficulties in considering expressions as unevaluated expressions without necessarily engaging in solving these) as related to Collis' (1974) "lack of closure". I consider this recognition

as central in terms of understanding and following the participants' development of algebraic thinking. At the same time, I want to emphasise the fact that Excerpt 2 brings further evidence of Paul's thorough confidence in our community of inquiry as he chose not to pay attention to the discussion between John and Claire and as he offered a justification for it, showing that he could decide *when* to engage in the discussion.

The way the participants engaged in negotiating the meaning of the second task (Ole's siblings) was presented through Excerpts 4 and 5. Here, I consider that the issue at stake concerns the vagueness and ambiguity of mathematical discourse, since, due to a lack of clarity concerning the choice of unknown, the participants experienced difficulties in understanding each others' results. The issue concerning the choice of unknown was addressed explicitly in Excerpt 5 where John asked directly ("who did you choose as x "), and my interpretation of John's claim is that he was referring to *who's age* did you choose as x . Thereby, the participants did not have the opportunity to explore the richness of the task and to recognise the different approaches with it due to different ways of choosing the unknown. However, Claire's didactical aim with these tasks was to highlight the difference between a syntactic and semantic translation. My interpretation of the way the second task was negotiated is that the difficulties the teachers experienced with it were caused by the way they performed the translation between written language and symbolic notation. Although the terms "syntactic and semantic translation" were not used during Workshop VII, I consider John's claim ("they [the pupils] have to change [the text of the task] a little in order to write an equation") as evidence of mixing inquiry as he was referring to a semantic translation where the *meaning* of the task comes to the fore. Thereby, in terms of the participants' development of algebraic thinking, I argue for considering the recognition of, what I called, the "lack of closure" of algebraic expressions and the implications of the ambiguity of mathematical discourse as central both to Claire's development of didactical knowledge and the development of her awareness, as a researcher. Furthermore, I understand this workshop as bringing further evidence of the importance of the nature of the mathematical task in the process of negotiating the meaning of it. I develop further this issue in Section 4. 2.

4.1.8 Workshop VIII: Teachers in charge

A priori analysis of the mathematical task

Until now Claire had addressed different didactical aims through various workshops. She had put emphasis on the introduction of algebraic symbols (Workshop I), generalisation of numerical patterns (Workshops II and V), historical development of algebraic symbolism (Workshop III), the link between geometry and algebra (Workshop IV), and the tran-

sition from natural language to equations according to syntactic or semantic translation (Workshops VI and VII). During Workshop VIII, which was happening in March, Claire felt that she could now share with the teachers the responsibility of offering mathematical tasks to the group. Therefore, she had told them, by the end of Workshop VII, that they will have the possibility to bring with them a task for the next workshop. I consider that this aim addressed both issues concerning the existence of our community of inquiry and the development of the participants' algebraic thinking. The first aspect was pinpointed as the ideas of joint enterprise and mutual engagement were realised since the teachers were invited to participate in the process of choosing tasks. Concerning the second aspect, the development of algebraic thinking, it was interesting for Claire, both as a didactician and as a researcher, to follow what kind of task the teachers would select to bring to the workshop. Through the various workshops, the teachers had developed an understanding of the aim of our collaborative work and as they selected a particular task, it would have been possible, for Claire, to observe which aspects they would choose to put emphasis on. This process was interesting for Claire, both as a researcher and as a didactician. As such, the teachers' choice of mathematical tasks could/might give Claire an indication of how they perceived the purpose of our collaborative work.

The main characteristics of Workshop VIII

- Mary choosing a task related to geometrical patterns
- Paul choosing to inquire into one of his pupils' thinking
- John choosing to talk about spreadsheet
- Mary taking initiative to interrupt the task
- John commenting and comparing his choice with Claire's choice of tasks

In the following I offer a brief description of Mary, Paul and John's choice and presentation of tasks in order to illuminate the main aspects, as presented above. Then I put emphasis on the main outcomes from this workshop.

Mary's choice of task

In the following I present a synthesis of the discussion concerning Mary's task. Before presenting her task, Mary explained to the participants that she wanted to offer a task related to algebra, and that she has been looking through her notes from some in-service courses for teachers she had been following. Furthermore, she explained that she found a task which might be suitable for the purpose of this workshop. Mary had made copies of the task, and she distributed to the other participants a sheet of paper with a drawing of a bridge where the task consisted of

looking at the structure of cables used under the construction of the bridge. These cables, which were of “size 4”, consisted of 37 wires which were compressed in a hexagonal shape. The task consisted of finding how many wires were needed in a cable of “size 5”, “size 6”, “size 10”, and “size n ”.

All participants engaged in the task by writing on the sheet of paper. After a long pause, John and Paul shared with the other their thinking: they found the task rather difficult and were hesitating about engaging further in it. Mary recognised that she also found this task difficult, and Mary, Paul and John discussed the possibility to stop the mathematical inquiry and to move to another task. Since Claire agreed with their suggestion, it was decided to stop working with this particular task.

Before presenting Paul and John’s choice of task, I want to comment on the following aspects: first on the type of task Mary chose, and second how the participants engaged in the task. In order to justify her choice of task, Mary explained that she wanted to offer a task related to algebra, and that she had found this particular task among her notes from a course for in-service teachers. Looking at the task from a didactical and pedagogical perspective, it seems that a possible didactical aim was the generalisation of geometrical patterns which was contextualised as inquiring the structure of cables of different sizes. I have no indication about if Mary was thinking in those terms, however, the task she proposed was related to algebra and algebraic thinking in the sense that the different steps consisting of exploring, generalising, and expressing a structure were addressed. Thereby, I consider that she responded to Claire’s demand when presenting that particular task. What seemed to happen during Workshop VIII was that the teachers experienced problems in envisioning how the structure of the first cable, consisting of 37 wires packed in a hexagonal shape, could be expanded to larger cables, and eventually to a cable of “size n “. The teachers did not explain in detail what kind of difficulties they met, they shared only the fact that they found the task difficult and wanted to consider the possibility to move to another task. Claire did not participate in the discussion, she preferred to leave the responsibility to Mary concerning what to decide since it was her task. I comment further on what could have been done with the task later (see the formal *a posteriori* analysis).

Paul’s choice of task

It was now Paul’s turn to offer to the participants a mathematical task. Paul explained that his intention was to bring a copy of the notebook of one of his pupils where Paul found it difficult to understand the answer the pupil offered. Further, Paul explained that the task was about percentage and he had problems to follow the pupil’s reasoning. Unfortu-

nately, Paul could not find the pupil's notebook, and thereby he apologised for not having a task to offer to the participants.

Before presenting John's choice of task, I want to comment on the following aspects: While Mary was looking for a task related to algebra, Paul's plan was to share with the other participants his hesitation, as a teacher, concerning how to understand a pupil's answer. I consider that Paul's way of acting brings evidence of his thorough confidence in our community of inquiry, as he chose to present an episode from his own teaching, recognising his difficulties to deal with the pupil's answer. It is not possible to decide if this "task" was related to algebra, but I understand Paul's search as deeply related to the idea of *inquiry* as he was inviting the other participants to inquire into a pupil's answer. As such, I argue for considering Paul's approach as expanding the idea of joint enterprise and mutual engagement in our community of inquiry, as he put emphasis on an inquiry approach since his aim was to invite the other participants to inquire into a specific aspect of his own teaching practice. It was now John's turn to offer a task to the participants.

John's choice of task

Before presenting his task, John proposed to go to the computer lab as he wanted to talk about algebra in relation to the use of spreadsheet. All participants moved to the computer lab (at Mary and Paul's school) and John started by explaining how he would introduce algebra to his pupils if he had the possibility to do this. However, because of limited access to the computer lab at his school, he recognised that this way of working with pupils was not an alternative for him, as a teacher. According to John, one of the advantages of a spreadsheet is the possibility for a teacher to push pupils from considering a cell as representing a specific number to considering a cell as representing a general number. It seems that for John, this transition, from the particular to the general, was central in the introduction of algebra. John did not offer a particular task to the other participants to engage with, but he developed his argument through the presentation of several examples. During his presentation, John also mentioned the fact that Claire did not present any tasks which were related to the use of ICT. Claire recognised this aspect and explained to the teachers that, due to constraints for limiting the scope of her research, she had decided, in advance, not to address the use of ICT, and therefore, not to include tasks related to ICT.

Even though, I decided, as a researcher, not to address the use of ICT, I want to comment on John's argumentation related to the use of ICT as an introduction tool for algebra. I am aware of the existence of a lot of research literature related to the use ICT, and, although I do not

want to engage in deep considerations concerning spreadsheets, I will comment on John's argument.

The advantages in using spreadsheet as a tool to introduce algebra are commented on by Rosamund Sutherland (1995). According to her, one of the characteristics of expressing mathematical generality is the ability to think about general relationships between objects. Referring to Küchemann's (1981) study and the difficulties pupils have to interpret algebraic symbols as representing general objects, she argues that "when working in a Logo or a spreadsheet environment pupils learn to view a symbol as representing a general number and this may be the most important aspect of work with these computers environments." (p.277). Thereby, from her perspective, the use of spreadsheet encourages and stimulates pupils to move forward and to consider a cell in a spreadsheet as representing a general number. Furthermore, she considers that the next step in the development of algebraic thinking, that is moving from, for example, $(3 * A5 + 7)$ to $(3x + 7)$, should not be too difficult since the pupils have developed an understanding of the spreadsheet symbols and the algebra symbol as representing any number.

I understand John's argumentation as following the same perspective as from Sutherland's (1995) research. However, another approach to the use of spreadsheet in relation to the introduction of algebra is presented in Guiliana Dettori, Rossella Gaturi, Enrica Lemut, and Ljuba Netchitailova's research (1995). According to these authors, the adequacy of a spreadsheet as a tool to teach algebra has to be questioned, since the expressions used in the spreadsheet do not have algebraic character. For example, they contrast the difference in use of the operator "=" in a spreadsheet (assignment) and in algebra (relation), and put emphasis on the fact that it is not possible to handle algebraic variables and relations directly in a spreadsheet environment, only assignments are made. Thereby, they argue that even if the spreadsheet environment might be useful for introducing some aspects of algebra (introduction of the idea of generalisation and distinction between variables and parameters), the results obtained in that environment are inadequate and might be misleading for developing a deeper understanding of the fundamental aspects of algebra. While Sutherland considered the transition between formulae computed by spreadsheet, for example $(3 * A5 + 7)$, to algebraic expressions, in that case $(3x + 7)$, as not so difficult, Dettori et al. claim that a fundamental component of algebra is missing in a spreadsheet environment: the unknown x . They consider that the formulae computed by spreadsheet are not relations but functions and thereby the involved cell names (A5) play the role of functional variables rather than algebraic unknowns. However, despite divergent perspectives on the use of spreadsheet in the introduction of algebra, the importance of the role

played by the teacher is emphasised both by Sutherland and Dettori et al.'s research.

I recognise that John did not address the different aspects of the use of spreadsheet, as exposed above, however I consider his contribution as valuable since the perspective he brought by addressing the use of spreadsheet enabled the participants to expand the approach which Claire has been following since the beginning of the collaboration.

The informal a posteriori analysis

Right after the workshop, Claire wrote her own impressions and reflections:

I think we had a really interesting workshop with quite different approaches. It is a pity that Paul could not find the pupil's notebook. It would have been interesting to follow the discussion concerning the pupil's answer. I was surprised by John's suggestion to discuss spreadsheet since I did not address it this year. I mentioned to the teachers that we will have an evaluation of our work together during our next workshop. (From my diary, 10.05.05, translated from French).

Since the nature of the next workshop was different from the previous ones, Claire's reflections were not part of an *a priori* analysis, or "thought-experiment". However, I consider her reflections as valuable.

The formal a posteriori analysis

In order to summarise the main aspects of Workshop VIII, I want to highlight the fact that the teachers were "in charge" during this workshop, that is the teachers had the responsibility to find and present some mathematical tasks to the other participants. Thereby, Claire was not included in the preparation of this workshop, and she did not know what kind of task the teachers were going to propose.

As explained above, the teachers seemed to follow various approaches concerning the preparation of the workshop. After experiencing working collaboratively for several months, the teachers were aware of Claire's focus on algebra and algebraic thinking, and I understand Mary and John's choice of tasks as an attempt to respond to this aim. It seems that Mary, by offering a task related to the generalisation of geometrical patterns, wanted to address the link between exploring geometrical patterns and expressing the observed structure using algebraic notation. However, the meaning of the task, concerning the arrangement of wires in cables, was not really negotiated since, after engaging with the task individually, the teachers decided not to continue with the task. My interpretation is that the difficulty experienced by the teachers was related to getting an understanding of the structure of the cable "size 4?" and seeing how this structure might be expanded. One possibility might have been to consider a smaller cable with fewer wires and to explore how to expand it. However, Claire felt that this task was not within her respon-

sibility and she agreed when the teachers decided to stop the mathematical inquiry and to move to the next task.

Paul's choice of task seems to be different from Mary's. My interpretation of his choice of task relates to the discussion the participants had during the workshops, where *experience-sharing inquiry* was visible. I addressed this perspective earlier (see Workshop I). It seems that Paul wanted to build on this kind of experience and his aim was to extend this kind of inquiry to examine a pupil's answer. As such, he developed further the idea of sharing with each other teaching experiences, he wanted to propose a specific example from his own teaching practice and to invite the other participants to join him. Thereby, the idea of inquiry is expanded further to include, not only mathematical inquiry, but also inquiring into a specific example from own teaching. Furthermore, I consider that Paul's attempt brings evidence of his thorough confidence in our community of inquiry as he was willing to recognise his difficulty to understand the pupil's answer and wanted to invite the other participants to join him in his search for making sense of the pupil's answer. Unfortunately, Paul was not able to find the pupil's notebook and thereby it was not possible to engage with Paul's task.

I consider that John's choice of task witnessed his concern with how to introduce algebra to pupils. This issue has been addressed during previous workshops (see Workshop I). By including the use of spreadsheet, John took the initiative to introduce a new perspective into the joint enterprise of our community of inquiry and I argue for considering this aspect as evidence of his thorough confidence in our community. In addition, by commenting on the fact that Claire did not address this possibility during previous workshops, it seems that John was looking critically into what kind of task Claire had prepared earlier.

I refer earlier (see Section 3. 3. 2) to the idea of critical alignment as a means to achieve emancipation. I argue for considering these aspects, as explained above, are part of developing "critical alignment", since by offering the teachers the possibility to choose and propose mathematical tasks, they became aware of their situation in our community of inquiry and of the possibilities to improve the focus of our joint enterprise and mutual engagement. As such, these opportunities might offer the teachers the possibility to look critically into their own teaching practices and to align critically with these. In other words, the experience of critical alignment within our community of inquiry might induce a similar attitude in relation to their own practice in school. I develop further this issue in Section 4. 1. 9. I consider that by offering to the teachers the possibility to choose tasks and by reflecting on the outcomes of this workshop, especially the importance of experience-sharing inquiry, I, as a researcher, am helping Claire to enhance her didactical knowledge.

4.1.9 Workshop IX: Evaluation of the workshops and reconsidering the past in order to imagine the future

Before presenting the main characteristics of Workshop IX and some excerpts from it, I offer some utterances taken from previous workshops, as these illuminate central features of the teachers' reflections concerning different aspects of our collaboration.

As mentioned earlier (see Workshop IV), I reported on Mary offering some reflections about how she experienced having Claire in her class and observing her teaching. Since it was the first time Claire was visiting her, Claire had encouraged Mary to present her reflections during the next workshop, that is during Workshop IV. I present her utterance since I consider it as evidence of Mary's development of awareness concerning her own teaching.

From Workshop IV:

6. Mary: yes, what I went through after [during the interview], it was what I thought and felt during this class and which pupils I have been by [helping] and who had trouble and [I] thought through this afterwards and I don't usually do this, but it is something I am aware of, but I don't reflect so much about it.

I consider Mary's utterance as complex and referring to different layers of reflections. In order to make visible and address each of these layers, I suggest dividing her utterance and examining each part carefully. In reporting on her experience having Claire in her class, Mary seemed to refer first to *what* was discussed during the interview after classroom observation concerning her own thinking and feelings ("what I went through after [during the interview], it was what I thought and felt during this class and which pupils I have been by [helping] and who had trouble"). I interpret the next part of her utterance, which is "and [I] thought through this afterwards" as a reflection on what has been discussed during the interview. In addition, Mary seemed to indicate that these reflections came *afterwards*, this means that there is a possibility that it was after the interview was over that Mary started to reflect on what has been discussed with Claire during the interview. I consider that, in the last part of her utterance, which is "and I don't usually do this, but it is something I am aware of, but I don't reflect so much about it.", Mary moved to another level, as she was reflecting on her reflections, and as she could recognised that she did not engage in such reflections usually and, in addition, that she was aware of this fact.

To summarise, my interpretation of Mary's utterance is that she was addressing three layers of reflection: on the first level, Mary reported on *what* was said during the interview, on the second level, Mary *reflected afterwards on what had been said*, on the third level, Mary *reflected on the fact that she was reflecting* on the discussion during the interview. I

argue for considering Mary's utterance as evidence of her development of awareness concerning own teaching and as offering an opportunity to engage in looking critically at own teaching practice. This kind of reflection might be considered as a first step in engaging in critical alignment with one's own practice.

As mentioned earlier, see Workshop V, I reported on John offering some reflections about how he experienced having Claire in his class and observing his teaching. Claire had encouraged John to present his reflections during the next workshop, that is during Workshop V. I present his utterance since I consider it as evidence of John's development of awareness concerning his own teaching.

From Workshop V:

10. John: When you were sitting in my classroom then, (pause), I was concentrated in a totally different way on what I was supposed to do, I mean I was much more listening to the pupils, (pause), I was on a higher activity level, (pause), it is just like being on a quite different level of consciousness then, that I noticed ...

Looking carefully at John's utterance, it seems that there is some kind of similarity between his utterance and Mary's one, as presented above. My interpretation of John's utterance is that it refers also to different layers of reflection. During Workshop V, John was referring to what had been said during the interview with Claire after observing his teaching. Here John was focusing on the impact on him of having Claire sitting in his class during a teaching period. In the first part of his utterance "When you were sitting in my classroom then, (pause), I was concentrated in a totally different way on what I was supposed to do", John seemed to indicate that, as a consequence of having Claire sitting in his class, his level of concentration was different from what it usually was. It seems that John wanted to offer further explanation as he continued "I mean I was much more listening to the pupils", and specified that he was referring to a different level of concentration in relation to his attitude, as a teacher, to the pupils. In the next part of his utterance "I was on a higher activity level, (pause), it is just like being on a quite different level of consciousness then", John seemed to offer a characterisation of what he called previously "to be concentrated in a totally different way", as he used the terms *higher activity level*, and *different level of consciousness*. Finally, I understand the last part of his utterance "that I noticed" as referring to John reflecting on his own reflections concerning his level of consciousness.

To summarise, my interpretation of John's utterance is that he seemed to address three layers of reflection: on the first level, John reported on *the consequence of having Claire in his class observing his teaching* in terms of being in a different level of concentration, on the

second level, John, reflecting on *how* he was acting in class, offered his characterisation of being in a *higher* activity level, and on the third level, John *reflected on the fact that he was reflecting* on his way of acting during that particular teaching period.

Therefore, I argue for considering John's utterance as evidence of his development of awareness concerning his own teaching and how he engaged with it. As such I understand these reflections as offering an opportunity to engage in looking critically at own teaching practice, and similarly to Mary, as a first step in engaging in critical alignment with one's own teaching practice .

In Section 3. 3. 2, I drew on Goodchild (2008) and argued for considering our collaboration as an opportunity, for the teachers, to reconsider the way they align with their respective practice in school and, thereby, to explore possibilities for changing and improving their teaching. Therefore, I consider that the kind of reflection, as exemplified through Mary and John's utterances, might be considered as a first step in engaging in *critical alignment* (Jaworski, 2006) with one's own practice, as they started engaging in a process of looking critically at own teaching. In addition, I consider it important to emphasise the fact that Mary and John's reflections were made visible and available through the interviews with Claire.

The next excerpt I present is taken from Workshop VI, and it concerns the way one of the teachers, John, works with the textbook. The textbook John used is divided into two volumes and the last chapter of the first volume is about functions, which was the theme of John's teaching when Claire was visiting him. During the interview right after John's teaching, Claire asked him what he planned to present to the pupils in the next teaching period. The rationale for Claire's question was that the first chapter in the second volume is about statistics while the second chapter is about functions. My aim was to see how John wanted to organise further his teaching in relation to functions, as a theme.

John developing a critical approach to his own teaching practice

From Workshop VI (data reduction):

132. John: I am very dependant of the textbook, I can notice that, I mean I go from page 1 to page 2, like this. Sometimes we meet, maths. teachers, and we recommend to each other what we can avoid because it will come again next year. We have done that, but we have not looked at the succession of chapters, but when you [Claire] asked me today, isn't it, what will come next, because I will start in a new book [the next volume] and it starts with statistics and goes back to function, different types of functions. Then I thought that it isn't necessary [to follow the order of the new volume] can't I just follow the progression we had in class and continue on what we did with equations with two unknowns, I can jump there, and when I thought about it, this is what I did last year too

133. Claire: so you did the same last year too?

134. John: yes, because I also had grade 10. last year too, but it was simply from practical reason I did that, I can recall that, because I wanted to use the computer lab, but we had problems with it so I had to wait with it [statistics]. But I feel I do not feel secure enough in the subject-matter, even after 20 years, so that I can take the initiative to change bigger things [change the order of chapters] I mean.

135. Claire: so why do you want to change this year?

136. John: because you asked me, yes, this is clear, but this time the reason is completely different from last year, I want to have some continuity in my teaching, so this is completely different from last year, yes.

I consider that through this excerpt it is possible to follow how John engaged in some reflections concerning the organisation of his teaching during and after the interview with Claire. During classroom observation, Claire had been following how John was teaching about graphical representation of functions and since this subject was the last chapter in the first volume of the textbook, Claire, during the interview after classroom observation, asked John what subject-matter will come next. During Workshop VI, John reported on his reflections caused by Claire's question and it seems that John wanted to present first the general context considering himself, as a teacher, in relation to the textbook, recognising his dependence on it. Furthermore, John seemed to contrast the discussion he usually had with his colleagues, discussing about what parts of the curriculum might be taken out in order to avoid repetition, with the interview with Claire. Especially, he mentioned that fact that the order of the chapters was *not* addressed during the discussions with colleagues, but this particular issue came at the fore during the interview. I consider as central the fact that, as John engaged in reflecting on "what comes next", he was able to recall that he changed the order of chapters last year too, but as he compared the *reason for this change*, he could see the difference between some practical issue (problems with the computer lab.) he experienced last year, and his reflection about how to organise his own teaching this year. Now, during the interview he was able to consider the *continuity in his teaching* and, as a consequence of this, he chose *not* to follow the order of the chapters in the second volume of the textbook. Therefore, it seems that, as John engaged in looking critically at his own teaching practice, the issue concerning *continuity in his own teaching* emerged as central for him.

I consider as crucial to emphasise the fact that Mary and John's reflections were made visible and available through the interviews with Claire. Therefore, I understand these *interviews as a means to engage in looking critically at one's teaching practice* since it was during and after the interview situation that the teachers were able to articulate and make visible their thinking. Thereby, I consider that the recognition of the importance of the interviews, as a means to inquire into own practice, and the importance of the workshops, as an arena for sharing and discussing

these reflections, highlights the complementarity of the different components of the practice of our community of inquiry.

In the following I present four excerpts taken from the discussion during Workshop IX. As mentioned earlier, this workshop was different from the previous ones, as it consisted of an evaluation of all workshops which have been organised during the school year. Before the workshop, Claire had prepared a list of all workshops with the dates and the different tasks as proposed during each workshop (see Appendix 2).

From Workshop IX:

A priori analysis of Workshop IX

This workshop was different from the other workshops, as its aim was to offer to the teachers the opportunity to summarize the year our collaboration lasted. Therefore, no mathematical task was proposed, but during this meeting, Claire invited the teachers to look back to all the workshops and to discuss and evaluate these. In order to engage the discussion Claire had decided to distribute to the teachers, at the beginning of Workshop IX, a sheet of paper with the information of all workshops and some questions concerning their impressions about working collaboratively during this year. The aim was to facilitate the discussion as the participants could remember more easily what has been done during the year of our collaboration.

As evidenced through these excerpts, the teachers looked back to their progression within our learning community during the school year, and offered an evaluation of their own work during the workshops. They also related to the thoughts that emerged in relation to their own teaching in class. I consider that the excerpts, as presented below, address both issues related to the building and development of our community and the development of algebraic thinking.

Addressing central features of our community of inquiry and envisaging possibilities for teaching practice

Excerpt 1

80. John: yes, I think it [our collaboration] has been very interesting and to have the opportunity to meet, hmm, to have time, to talk about mathematics, we often referred to mathematics in the classroom, but look, we are persons sitting here and we are concerned with mathematics, isn't it, and it is seldom one has that opportunity, and we talk together, and I feel that what we are talking about is rooted in everyday experiences, in a sense

81. Claire: hmm, hmm

82. John: I also think that it has been interesting and exciting to meet other teachers from another school, even if I know who they are (Mary is laughing), and then it is obvious that you don't need to move far away to see that we have different cultures and I think this aspect has been visible when we discussed, that there are different cultures, and in a sense, this is part of the whole

83. Mary: hmm, hmm

84. John: I mean it, yes

85. Mary: Yes, ..., I think it has been very exciting to have the opportunity to discuss, and to get different opinions, I have been thinking more about things, I do, and then I got ideas, perhaps can we try that way, hmm, think in a different way, then, yes, I felt that I have been thinking more in depth concerning tasks, like is it possible to become more independent of the textbook, to get new impulses, yes, as John says, to do things in a different way than how I do, I think I learnt a lot

86. Claire: hmm

87. Paul: yes, I have the same feeling too. It is quite obvious that to be able to have this kind of discussion here, I mean related to mathematics, this is quite different, and to have time, this is really, really all right. I have got a lot to think about, so now the question is if I manage to put this in application for the autumn, I mean to create some expectations, motivations, perhaps different ways to introduce the subject-matter

88. Claire: hmm

89. Paul: I want to try that! I think it has been really, really exciting ...

I consider that Excerpt 1 brings evidence both of which features of our community the teachers put emphasise on, and of how they might envisage possibility to change their own teaching practice. I develop these aspects as follows: Mary, Paul and John seemed to emphasise particularly *the possibility to address the subject-matter, mathematics, on its own, during our meeting*. By using the terms “opportunity to meet”, “have time to talk about mathematics”, “we are concerned with mathematics”, and “talk together” (John in utterance 80), it seems that John valued the design of the workshops where mathematics came to the fore, as the participants engaged with different tasks. I understand Mary and Paul’s claims as supporting John’s evaluation, since they referred to “it has been very exciting to have the opportunity to discuss” (Mary in utterance 85), and “to be able to have this kind of discussion here, .., related to mathematics, . . , to have time, .., this is really, really all right” (Paul in utterance 87). Thereby, I consider that by organising the workshops as working sessions where the participants could meet on a regular basis and engage with mathematical tasks, Claire opened for the teachers the opportunity to *have time to both engage in mathematics and to talk about mathematics*. In addition, John emphasised the link between the discussions during the workshops and his teaching practice, as he referred to these as “rooted in everyday experiences”. I understand his claim as related to what I called “experience-sharing” inquiry as the teachers often brought elements from their own teaching practice into the discussion during workshops, and thereby they were able to establish a link between our community of inquiry and their own teaching practice.

Another aspect that the teachers seemed to value was the possibility to meet teachers from other schools. Both John and Mary referred to this issue as they talked about “different cultures” (John in 82) and “to get

different opinions” (Mary in 85). In Section 2. 2. 3, I addressed the notion of joint enterprise and I explained how the dimension of *diversity* might be seen as a productive part of it. I understand Mary and John’s claims as *valuing the dimension of diversity within our community of inquiry* as teachers coming from different schools might have developed different cultures and thereby different perspectives and understandings of the subject-matter. It was by engaging in discussion with teachers from different cultures that the teachers got the opportunity to compare, evaluate, and reconsider their own teaching practice. Thereby, I consider that the dimension of diversity is understood as a potential source for looking critically at one’s own practice.

The next aspect I want to emphasise concerns envisaging the consequence of our collaboration for future teaching practice. Both Mary and Paul referred to what they considered as a challenge, which consisted of *establishing a connection between the activities as addressed during the workshops and their own teaching*. Mary reported on having ideas that she would like to try (85), while Paul, recognising that he got a lot of ideas, claimed that “the question is if I manage to put this [got a lot to think about] into practice in the autumn” (87). In other words, both Mary and Paul recognised that during the workshops, they got new ideas and perspectives which they would like to implement in their own teaching practice. The challenge consisted of envisaging *how* to adapt and implement these new ideas. In addition, Paul offered some details concerning what kind of ideas he was referring to. He talked about “to create expectations, motivations, perhaps different ways to introduce the subject-matter”. There is a possibility that he found these elements in the way our group engaged in mathematical tasks during the workshops, and I argue for considering Paul’s characterisation as related to *inquiry*. At the same time, he seemed, strongly, to emphasise his willingness to engage in trying these new ideas in his teaching practice.

Finally, it seems that it is possible, in Mary’s utterance, to identify *elements of potential emancipation* from textbooks and from one’s usual way of teaching. As Mary recognised that she has been through a process where she thought “more in depth concerning tasks”, she was able to consider the alternative of becoming “more independent of the textbook” (85). I develop further several of the aspects, as mentioned above, in the analyses of Excerpts 2, 3, and 4. The discussion continued with John referring to previous in-service courses.

Comparing with previous experiences

Excerpt 2

90. John: ... as a mathematics teacher, I have been through different in-service courses during my career, but I think I manage to focus more on mathematics now, I pay much more attention now to what I do, I pay much more attention to what is in

the textbook, so it is just like if you have increased my level of consciousness somewhat, in relation to what I am doing

91. Claire: do you think this is a consequence of what we have been doing through the year?

92. John: yes, because one has suddenly to sit down, and, hmm, we also discussed mathematics in the teachers' room too, but it is not that kind of discussion, we do not take time to sit down, if something happens, so we have a discussion about methods and problems with motivation and like this, but if we discuss mathematics it is often during the break and then we move on

93. Claire: hmm

94. John: and there is nobody from outside who can conduct the discussion, we don't have someone among our colleagues who can conduct a mathematical discussion like this. Now, I feel that when you came and you proposed some stuff that we discussed together, that is, I mean this has an influence on the other things that I am doing. This is what I really and honestly mean, this is not because you ask for evaluation, this increased my level of consciousness and it feels good to have this after 20 years!

95. Claire: hmm

During his discussion with Claire, John seemed to compare his previous experiences with in-service courses and discussions with colleagues. Thereby, I understand his utterances as developing further one of the aspects which was addressed in Excerpt 1, which concerned *the possibility to address the subject-matter, mathematics, on its own, during our meeting*. John, recalling mathematical discussions with colleagues and comparing with our discussions, seemed to say that there was a difference in both the *quality* ("but it is not that kind of discussion") and in the *purpose* ("discussion about methods and problems with motivation and like this") of these discussions (92). Therefore, according to John, it is important to have someone "from outside who can conduct the discussion". I consider this aspect as central, since from John's perspective, it seems to be crucial for engaging in a *mathematical* discussion to develop a cooperation with someone coming from "outside". Now, John did not offer further detail concerning what "outside" was supposed to mean. Would it be possible for Mary or Paul to be the person coming from "outside" since, as John reported in Excerpt 1 (utterance 82), he was able to notice the different cultures from which the teachers came, or was John referring to a person coming from University?

In addition, John seemed to offer some insights into the consequences, for him as a teacher, of our collaboration. He mentioned the fact that he managed "to focus more on mathematics now", to "pay more attention now to what I do", and also to "pay much more attention to what is in the textbook". I consider that, by putting emphasis on these aspects, John was addressing his development of awareness concerning own teaching practice. This is supported by John's claims "it is just like if you have increased my level of consciousness", and I consider that his claim refers back to what he addressed in the excerpt from Workshop V

(see above) concerning his reaction to Claire sitting in his class. While the two first excerpts addressed general issues concerning our collaboration, I understand Excerpts 3 and 4 as focusing more directly on algebra.

Focus on algebra: valuing different perspectives and implication for teaching practice

Excerpt 3

96. Mary: yes, yes, and about algebra on which you focused so much, because this is a difficult subject-matter in school too, the pupils don't understand what they can use it for, and to find good examples, those from the textbook are always the same, you try to think, how is it we can use it [algebra], perhaps to show some practical examples, I thought a little more about this now and we have been discussing a lot here

97. Paul: hmm, hmm

98. Mary: yes, this is very exciting, hmm, hope we can manage better after a while

99. Paul: yes, because it is, ..., when I remember the way we talked about algebra so we had very different positions

100. Mary: hmm, hmm

101. Paul: and just that, oh, this is the way he thinks about it!

102. Mary: yes, yes, hmm

103. Paul: isn't it, I mean this gives inputs, hmm, because you often believe that the way you do [teach] is ok, but here you get quite other thoughts, and I think this is quite exciting, I think so!

104. Mary: yes, one is quite trapped in the way one teaches, because it is the way we learnt and then we teach the same way, so it is good to get some inputs and help, yes, perhaps I could do it that way

105. Paul: hmm, hmm

106. Mary: yes, because it is difficult

107. Paul: yes, yes, because it is about both the mathematics and what we can use in class, it is clear that when we agreed to be part of this I thought about what was useful for me, what could I apply, I must be that honest, I didn't think about you and your doctoral thesis (laugh)

108. Claire: no, no

109. Paul: and I must say that I am satisfied with this part [relation to usefulness], I have to say that

110. Claire: so what has been useful for you, it was from our discussions?

111. Paul: yes, and what is going to be the big challenge now, this is clear, is to what extent can I get some of the pupils, not all, but some of the pupils to reflect, to wonder, to get some of the pupils to think a little that, hmm, perhaps, and so on, that not everything is obvious. You can say that mathematics is a subject-matter where things are right or wrong, you can go from here to there, ..., but to wonder ...

112. Claire: hmm, yes

In Excerpt 3 Mary and Paul seemed to emphasise the fact that different perspectives on algebra did emerge from the discussion between participants (99, 101). Thereby, according to Paul, through making these different positions visible and bringing them to the fore, one got the possibility to widen own understanding (103), and, as Mary emphasised, to reconsider own teaching practice (104). I consider this recognition as central as it witnesses the link between our practice which developed

within our community of inquiry (see Section 3. 2. 2) and the teachers' own practice within their respective school. It was during the process of engaging with tasks that the participants got the opportunity to address both mathematics and the possibility to implement some ideas in their teaching. Thereby, I understand the tasks and discussions related to these as a means to allow different perspectives to emerge and to be critically addressed. Furthermore, once these different perspective were addressed explicitly, the teachers had the opportunity to look critically into their own teaching practice and to recognise that "you often believe that the way you teach is ok" (Paul, 103), and, "one is quite trapped in the way one teaches" (Mary, 104). In addition, Mary and Paul seemed to report on their own reflections (96, 103) concerning how they organised their teaching like trying to find "good" examples. I argue for considering these elements as first steps into a process of critical alignment with one's own practice.

Another aspect from Excerpt 3 which I want to highlight concerns a possible implication for teaching practice. Paul addressed this aspect, referring to "the big challenge now" (111), as he seemed to wondering *how* to implement some aspects of our practice within his own teaching practice. I consider that by referring to getting some pupils "to reflect, to wonder, to get some of the pupils to think a little" (111) Paul was pointing to what the idea of inquiry might mean in his teaching, that is "a willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them" (Wells, 1999, p.121), see Section 2. 2. 6. Paul already addressed this issue in Excerpt 1 (87), and I interpret this repetition as an indication of the importance, from Paul's perspective, of a willingness to establish the link between the activities, as Paul experienced these within our community of inquiry, and his own teaching practice. In Excerpt 4, John and Mary deepen their views on algebra.

Focus on algebra: possibility to develop a new view on algebra

Excerpt 4

113. John: ... at least I was very sceptical to algebra in lower secondary school, but I believe in what has happened here [in our community of inquiry] and I am much more conscious about the usefulness of algebra when we run into it in mathematics, as pupils see that here, here we have algebra with us. I am still not so very happy about those quadratic formula and that sort of algebra, but it seems to me, hmm, to be amongst us a greater understanding that it is a part of what we have to bring with us in order that they [the pupils] shall understand some other things, and I think that I have seen it better, more clearly, I have been more conscious to see it clearly this year with the group [the pupils] I have had this year than I have done before, I mean, this [algebra] is what we need in other connections, something we have to go through in order to use latter, and I think it paid at the oral assessment, a part of what we have done here, I think that, yes, I think it gave me a lot back, and my view on algebra has

not completely changed, but, hmm, but I see it has a potential for my group also, a bigger value than I was aware of myself

114. Mary: yes, and that was what we talked about once, that mathematics is divided in all these subject-matter, and these are so very separated, and then we say to the pupils that algebra is used to multiply here, and then we get something, and then we go to the next chapter, so they can't see that connection, oh, yes, I used algebra to solve it [the exercise] in a simple way and I think it is useful, as you say, hmm, you can use it [algebra] in other connections

115. John: yes, I mean, what you are pointing to, that I have become more conscious about ...

116. Mary: hmm, hmm

117. John: it was not present

118. Mary: yes, yes

119. John: but I have become more conscious, I mean

120. Mary: yes, but you have seen it, but you have not thought so much about that before now, yes

The possibility to develop a new perspective on algebra and the consequences for teaching were addressed through John's utterance (113). By recalling his own attitude to algebra, I consider that John tried to compare and trace the evolution of his own thinking. Here, it seems that the issue related to the *usefulness of algebra* was central in John's argument, and that this recognition emerged gradually through our collaboration ("I believe in what has happened here"). I understand John's reference to quadratic formula as an example related to his scepticism to algebra in lower secondary school. There is a possibility that John was referring to pupils' learning of algebraic formulae without developing an understanding of the need of using those formulae. According to John, it was through the development of a common understanding of the *need for algebra and algebraic symbolism* within our community of inquiry that he became aware of how to introduce algebra in his teaching, and I understand John's claim as related to my own definition of algebraic thinking¹². By the end of his utterance, John seemed to address himself to his class, explaining that algebra is useful because "this is what we need on other connections, something we have to go through in order to use later".

This shift in the kind of discourse was already observed before (see Workshop IV), and I consider this issue as a characteristic of how teachers inquire into their own teaching practice. My understanding of John's utterance is that he offered to the other participants a summary of his development, seen from his perspective, as he could trace the development

¹² In Section 2. 5. 4, I defined algebraic thinking in the following sense: By addressing and developing algebraic thinking I mean to focus on the need, the choice, the introduction, the use and the meaning attributed to algebraic symbolism and on the way these various components of algebraic thinking are addressed and negotiated within our community of inquiry through inquiry acts.

of his own thinking and the implication for this own teaching practice. The emphasis on the possibility to establish connections between different areas of mathematics is developed further as Mary (114) referred to the need to highlight the continuity between the different chapters, as those are presented in textbooks. There is a possibility that Mary was building further on her experience from Workshop IV, in which the aim was to explore the connection between geometry and algebra, and that she was referring to this issue in a critical way, recalling the separation between different subject-matter like geometry, algebra, functions. In addition, I understand Mary's comments as related to John's recognition of the need to change the order of the chapters in his textbook, as presented above. It seems that in the last utterances (115 to 120), both Mary and John moved to another level of reflection, as they were reflecting on their reflections, as presented through utterances (113 and 114). While John was referring to his recognition of his own level of awareness ("I have become more conscious"), Mary offered her interpretation of his claim, as she seemed to highlight that this level of awareness was gradually emerging during the year of our collaboration.

The formal a posteriori analysis

Looking through my diary I could not find any notes concerning my reflections after this last workshop. However, I can remember that my feelings were mixed, as I was happy with the discussions we had during that last workshop and with our collaboration generally, but at the same time I felt a little sad as it was the last time we had the opportunity to meet together.

Looking through both the different utterances which were taken from Workshops IV, V, and VI, as presented in the beginning of the section, and the different excerpts from Workshop IX, it seems that the analysis revealed several important features of our collaboration during the year.

The first aspect which I consider emerged strongly from the different utterances and excerpts refers to the emergence of different layers in the teachers' reflections. Mary and John's utterances offer examples of this aspect as they were addressing a particular issue, reflecting on that issue, and reflecting on the fact that they were reflecting on it, and as Mary emphasised this process was new for her. Similarly, I understand John's reflection about the continuity in his own teaching (from Workshop VI) as evidence of different layers of reflections as he was able to reflect on his own teaching practice. But the recognition of the emergence of these different layers of reflections begs the following question: why did this process emerge? or what did enable it to happen? and what are the consequences of engaging in this process? As an attempt to answer the two first questions, I want to recall the design of the practice of our community of inquiry. As explained in Section 3. 2. 2, the practice of our com-

munity of inquiry, as the actual realisation of the six-step framework, consisted of nine workshops, but only few classroom observations and interviews due to practical reasons. However, even though I was only twice in Mary's class and four times in John's class, the crucial role played by these observation steps is highlighted in Mary and John's utterances. This is the reason why I argue for considering these observations steps (interviews both before and after classroom observation) as a means to engage in looking critically at one's teaching practice, as these provoked reflections at different layers and thereby stimulated the teachers in engaging in developing an awareness about one's own teaching. Concerning the consequences of engaging in looking critically at one's teaching practice, I argue for considering this process as a first step in questioning and reconsidering one's own practice, as evidenced from Mary and John's utterances, which might develop further in engaging in critical alignment with one's own practice.

However, the observation steps were not the only opportunities to engage in this process. As presented through Excerpts 1 to 4 from Workshop IX, the teachers emphasised several aspects from the mathematical workshops which were central for them. From my interpretation of their utterances (Excerpts 1 and 2), it seems that having the time and the opportunity to engage in mathematical tasks and discuss mathematics is strongly valued by the teachers. Even if this aspect might seem obvious, it was not possible to achieve it through in-service courses (John, 90) or by discussing with colleagues (John, 94). In addition, John seems to point to the necessity to have someone from "outside" who could organise and conduct the discussion. From the teachers' perspective, it seems that these three characteristics, having time, focusing on the mathematics, and some to lead the discussion, created and nurtured a mathematical environment (see Section 3.4.1) which enabled the participants to engage in sharing and comparing each others' perspective on mathematics. Furthermore, by reflecting on these discussions the teachers were able to reconsider and to look critically into their own understanding of mathematics, and more specifically algebra and, into their own teaching practice (Excerpts 3 and 4). This is the reason why I argue for considering the different elements of our practice (mathematical workshops and observation steps) as interdependent and reinforcing each other since the reflections emerging from one component, for example interviews, might be refined in an other component, for example during the workshops.

Evidence of the link between the activities and reflections from our community of inquiry and the teachers' practice is offered differently by each teacher. By John, as he already decided to change the organisation of his teaching (utterance from Workshop VI) and as he was able to no-

tice the development of own awareness, by Paul, as he expressed a strong willingness in trying new ideas in his teaching (Excerpts 1 and 3), and by Mary, as she shown a clear articulation of different layers of reflections on her own teaching practice (utterance from Workshop IV). In addition, both John and Mary addressed more specifically the teaching of algebra as they referred to being sceptical in introducing formulas without relating them to why these are useful (John, 113), and the necessity to establish clearer connections within the curriculum (Mary, 114).

Looking back to the three characteristics which I identified in the teachers' reflections, I consider that during previous sections (see Sections 4. 1. 1. to 4. 1. 8) I offered an elaboration on the dimension "time" since I referred to the participants' development of confidence both in our community of inquiry and in the mathematics which is dependent, among other factors, on time. Concerning the characteristic related to have someone to lead the discussion, I argue for looking back to previous workshops, for example Workshops IV or VI, in order to see that this role can be taken by different persons in the group. Concerning the last characteristic "focusing on the mathematics" I already addressed what kind of mathematical tasks I offered the teachers, as explained in both the *a priori* and the informal and formal *a posteriori* analyses. However, I argue for further investigation into the *nature of the mathematical tasks*, as this aspect strongly emerged from the formal *a posteriori* analyses of the different workshops. This issue is addressed in the next section.

4.2 The role played by the mathematical tasks

The rationale for this section is that, as mentioned in previous sections, it seems that the nature of the mathematical tasks played a crucial role with respect to the way the participants engaged with the tasks. This recognition emerged from an in-depth analysis of my data, as presented in the formal *a posteriori* analyses of the different workshops.

In this section, I introduce and deepen first the different roles played by the mathematical tasks. Then I develop further what I mean by the nature of the mathematical tasks by introducing a coding of the different mathematical tasks which enables me to differentiate and articulate between different characteristics of the tasks. In addition, as presented in the *a priori* analysis of each task, I recall the didactical aim from which each task has been chosen. This coding enables me to compare the way Claire designed the task with how the task functioned during the workshop. In other words, the issue I am trying to capture relates to the following question: to what extent did the task fulfil its purpose and, what kinds of factors were relevant to consider?

4.2.1 The different roles played by the mathematical tasks

Insights within the complex role played by the mathematical tasks, which Claire offered the teachers through the year our collaboration lasted, emerged gradually during my research. I am able now to distinguish between five different roles, as I consider the mathematical tasks as:

- A means to develop our community of inquiry
- To develop the participants' algebraic thinking
- To link different communities
- To reflect on one's own teaching practice
- To reflect on my own development.

I develop each of these aspects further in the following.

The first aspect concerns the mathematical tasks as *a means to develop our community of inquiry*. It was through the participants' collective engagement with the mathematical tasks that our community of inquiry could develop. Therefore, each new task nurtured the development of our community of inquiry further as it enabled the participants to develop their confidence further both in the mathematics and in our community of inquiry.

This leads me to the second aspect which is about considering the mathematical tasks as *a means to develop the participants' algebraic thinking*. As mentioned through the *a priori* analyses of the different workshops, each task has been designed with a particular didactical aim. Therefore, the participants were offered the opportunity to address different aspects of algebraic thinking such as the power of algebraic notation, generalisation of numerical patterns, addressing the link between geometry and algebra, or addressing the translation from written text to algebraic symbols. By focusing on these different aspects, the mathematical tasks enabled the participants to explore and enhance their understandings of algebra and algebraic thinking further.

The third aspect, which emerged strongly from the formal *a posteriori* analyses of the workshops, relates to considering the mathematical tasks as *a means to link our community of inquiry to the teachers' communities in their respective schools*. In Section 2.2.7, I presented Wenger's (1998) idea of *boundary objects* as referring to forms of reification (artefacts, documents, terms) around which communities of practice can organise their interconnections. I understand the mathematical tasks as boundary objects since the teachers, while engaging with these tasks, were able to establish a link with their respective teaching practice, as evidenced by what I called *mixing inquiry* and *experience-sharing inquiry*. Mixing inquiry refers to how the teachers could mix mathematical inquiry into a particular task with pedagogical inquiry related to how

they could implement the task in their teaching practice (see for example Workshop IV). In addition, during our discussions, the teachers were often willing to share with each other aspects of their own teaching practice. As explained in Section 2. 2. 6, I understand *experience-sharing inquiry* as a kind of inquiry where a teacher shares her own experience with other participants resulting in developing her understanding of her own teaching practice further. In addition, I see *experience-sharing inquiry* as related to Wenger's ideas of mutual engagement, joint enterprise, and shared repertoire, since while sharing with each other teaching experiences, the participants elaborated a common basis for mutual engagement and joint enterprise while developing a shared repertoire (see Workshop II with the discussion concerning students' difficulties with symbols). Thereby, it was during the process of engaging with and reflecting on the tasks that the teachers were able to recall previous teaching experiences and by bringing and making these visible, through *experience-sharing inquiry*, they were in a position of establishing links to their communities in their respective school.

As a consequence, and this relates to the fourth aspect of the mathematical task, the teachers were in a position where they had the *opportunity to reflect on their own practice*, looking critically at their own understanding of mathematics and ways to organise teaching. Evidence of this aspect is given through the different excerpts from Workshop IX, where the teachers, by reconsidering their previous teaching experiences in the light of our collaboration, took a critical position which enabled them to envisage possible and different teaching experiences. For example, I understand Paul's utterance (Excerpts 1 and 3) about engaging his pupils in wondering, and reflecting, as his recognition of a possibility to envisage further development of his teaching practice. I argue that, by engaging in inquiry into both the mathematical tasks *and* one's own teaching experience, the teachers and myself were in a position to move to a form of *critical alignment* (Jaworski, 2006) where, "through the exercise of imagination during engagement, alignment can be a critical process in which the individual questions the purposes and implications of alignment with norms of practice" (p.190). This is what happened within our community of inquiry, as evidenced through the teachers' utterances as presented in the different excerpts in Workshop IX. In addition, it seems that one important feature in this process, from the teachers' perspective, was the opportunity they had to discuss with other teachers coming from different schools, or "cultures" as John suggested (Workshop IX, Excerpt 1).

This move to adopting a critical stance to one's own practice leads me to consider the last aspect of the role played by the mathematical tasks which is the tasks as *a means to reflect on my own development*,

both as a researcher and as a didactician. I consider that evidence of my own development is offered as, by comparing the informal *a posteriori* analyses to the formal *a posteriori* analyses of the different workshops, it is possible to trace my own development both as a didactician and as a researcher. Through engaging in a process of planning (*a priori* analyses) and reflecting on (informal *a posteriori* analyses) the different workshops, I engaged in developmental research. In addition, analysing the different workshops enabled me to elaborate formal *a posteriori* analyses which offered me the opportunity to investigate and deepen central issues emerging from these workshops. I consider that this process is related to Freudenthal's recognition of the need to legitimise new knowledge since, as replication is problematic in educational development, "new knowledge will have to be legitimized by the process by which this new knowledge was gained" (1991, p.452). I elaborate further on this issue in Chapter 5. In order to deepen what I mean by the nature of the mathematical tasks, I propose to investigate further the different elements which constitute a task.

4.2.2 The nature of the mathematical tasks: what do I mean?

In order to explain what I mean by *the nature of the mathematics tasks*, I propose to differentiate and to articulate the different dimensions of the tasks. Inspired by a categorisation of tasks proposed by Sandra Crespo (2003) and Anna Sierpiska (2003), I identified tasks used during the different workshops according to the following dimensions: involving which didactical aim Claire had when choosing the task, as addressing *the choice and use of algebraic symbols, the power of algebraic notation, or the meaning of symbols*. In addition, I consider using *known versus unknown mathematical objects* (from my own teaching experience and my interpretation of the teachers' perspective), and *contextualised versus non-contextualised* which addresses if the task is contextualised in a particular setting or not. The next dimension which I introduced is *exploratory versus non-exploratory*. By introducing the dimension *exploratory* I seek to capture if the task allows for engaging with it using different approaches. The exploratory dimension is deepened further in the next coding referring to *allowing for numerical patterns approach versus non-numerical patterns, and/or allowing for geometrical approach versus non-geometrical approach*. In the next dimension, *routine versus non routine* (from my own teaching experience and my interpretation of the teachers' perspective), I seek to put emphasise on *the nature of the question* asked in the task, in other words, my aim is to characterise the kind of question the task is offering. Finally I introduce a coding referring to the *use of manipulatives versus non-use of manipulatives*.

Thereby, this characterisation enables me to code each task which was proposed during the workshops with a string of letters, each pre-

ceded by 0 or 1, and representing the above dimensions. I summarise these aspects in the following table:

Table 4: Coding the mathematical tasks

Dimensions of the task	Coding
Didactical aim as Choice and use of algebraic symbols or Power of algebraic notation or Meaning of symbols	CAS PAN MS
Known versus unknown mathematical objects	MO
Contextualised versus non-contextualised	C
Exploratory versus non-exploratory	E
Allowing for numerical patterns versus non-numerical patterns	NP
Allowing for geometrical approach versus non-geometrical approach	GA
Routine versus non-routine	R
Introducing the use of manipulatives versus non-use of manipulatives	M

For example, the string PAN-1MO-0C- 1E-1NP-0GA-1R-0M represents a task addressing the power of algebraic notation, which involved known mathematical objects, which is not contextualised and which is exploratory. Furthermore, the task allows for numerical patterns approach, but does not allow for a geometrical approach. The nature of the question is routine and the task does not offer the use of manipulatives.

Looking back to the seven mathematical workshops and considering the different tasks which Claire offered the teachers, I can present a coding of each task which enables me to deepen what I mean by the crucial role played by the mathematical tasks. In other words, I understand this coding as enabling me to develop a better understanding of the influence of the mathematical task on the *mathematical environment* (see Section 3. 4. 1) by comparing the coding of a particular task with the formal *a posteriori* analysis of the corresponding workshop. I develop this issue further by the end of this section. However, during the process of coding tasks, I recognise a part of my own judgment, for example regarding if the task is exploratory, routine or not, but I consider that such element of subjectivity is difficult to avoid.

4.2.3 Coding the mathematical tasks

Workshop I – Cuisenaire Rod Formations

The task which was offered the teachers during Workshop I was related to different Cuisenaire rod formations. Two questions were offered the teachers: the first one addressed a description using words, the second question addressed a description “a more mathematical way”. I propose the following coding for this task:

Table 5: Coding the task from Workshop I

Workshop I	
Task	Coding
<i>Can you describe the following Cuisenaire rods formations first using words, and then in a more mathematical way?</i>	CAS-1MO-1E-1C-0R-1NP-1GA-1M

By coding this task in the following way, CAS-1MO-1E-1C-0R-1NP-1GA-1M, I mean that the task offered during Workshop I addressed the choice and use of algebraic symbols as a didactical aim, did involve known mathematical objects (symbols used to represent the different Cuisenaire rods), it was exploratory (allowing for different approaches), contextualised, non-routine (open question), allowing for a numerical patterns approach and for a geometrical approach, and using manipulatives.

In order to consider to which extent a mathematical task fulfilled its purpose, I recall the following criteria for evaluating the mathematical task (Section 3. 4. 2):

- Did the task motivate and engage all the participants, and in this way address issues concerning the “becoming” or “belonging” to the community?
- Did the participants resolve the task using algebraic notation?
- Did the analysis of the workshop show any evidence for enhancement of teachers’ algebraic thinking in terms of participating and engaging in the social process of learning?

From the formal *a posteriori* analysis of Workshop I, I can see that the task did motivate all participants as they engaged collaboratively with it. Therefore my interpretation is that the task did address issues concerning the “becoming” participants in our community of inquiry successfully. However, it was Claire who introduced the symbols **R** and **W**, without discussing with the teachers how to choose the symbols. Still, the participants did address the use of algebraic symbols and particularly the teachers referred to the pupils’ difficulties with this issue. Therefore, I consider that Claire’s didactical aim concerning the choice and use of algebraic symbols was partially achieved. This could be due to Claire’s

inexperience, as a didactician, as she ran into introducing algebraic symbols without allowing the participants to discuss the choice of symbols first, but also to the early stage of our community of inquiry since this was our first workshop. This last aspect is visible in the mismatch between Claire, trying to pursue her didactical aim and one of the teachers questioning about the “rules” of our community, or struggling to understand what Claire meant by “in a more mathematical way”. I understand these characteristics as bringing evidence of both the teachers and Claire as being unsure about *how* to act. Therefore, it seems that issues related to the creation and establishment of our community of inquiry prevented the participants from engaging successfully in the mathematical inquiry of the task.

Workshop II – Odd and even numbers

The task which was offered the teachers during Workshop II was related to even and odd numbers. The task was as following: What happens when we add even and odd numbers? Claire’s didactical aim was to provoke a discussion concerning the choice and use of algebraic notation, similarly to Workshop I, and especially to address the standard notation for even and odd numbers ($2n$ and $2n + 1$). I propose the following coding for this task:

Table 6: Coding the task from Workshop II

Workshop II	
Task	Coding
<i>What happens when we add even and odd numbers?</i>	CAS-1MO-1E-0C-0R-1NP-1GA-1M

By coding this task in the following way, CAS-1MO-1E-0C-0R-1NP-1GA (not addressed during Workshop II)-1M (not addressed during Workshop II), I mean that the task offered during Workshop II addressed the choice and use of algebraic symbols as a didactical aim, did involve known mathematical objects (even and odd numbers), it was exploratory (allowing for using both a numerical patterns approach and a geometrical approach), non-contextualised, non-routine (open question), allowing for numerical patterns and geometrical approaches, and allowing for using manipulatives (small plastic squares).

It is possible to see, from the formal *a posteriori* analysis of Workshop II, that the task did motivate all participants as they engaged collaboratively with it. Therefore, I interpret the task as successful in relation to the aspect of “becoming” participants in our community of inquiry. However, Claire’s didactical aim concerning the choice and use of algebraic notation was *not* achieved since the teachers did not introduce the standard algebraic notation for even and odd numbers and did not

show a willingness to consider the use of the manipulatives as a means to explore the geometrical properties of even and odd numbers. I consider, both as a researcher and as a didactician, that this task was potentially a good task due the number of aspects which could have been addressed but, still, it did *not* fulfil its purpose.

A possible reason is that this task was presented in an early stage of the development of our community of inquiry where issues related to the functioning of the community were still evident, for example when Paul was struggling to understand what Claire meant by “write it (Paul’s result) in a more mathematical way”, seeking to understand what Claire wanted him to do. In addition, I consider that Claire’s inexperience, as a didactician, is central as she was not able to articulate her didactical move in a different way. Her inexperience was also visible as she had to adjust her pedagogical strategy and to work with numerical examples when the teachers were not willing to consider the manipulatives. Even though, it seems that the teachers were starting developing confidence in our community of inquiry, as they were questioning the adequacy of the task and relevance of the task. However, an in-depth mathematical inquiry into the task was prevented by the teachers still struggling to understand Claire’s didactical purpose and by Claire’s didactical inexperience as she was not able to pursue her didactical aim. Thereby, the potentiality of the task, which was addressed in the coding 1NP-1GA, was *not* addressed since a geometrical approach to even and odd numbers was not considered during that workshop, and, as such, the purpose of the task, in terms of algebraic symbolisation, was not fulfilled.

Workshop III – Historical perspectives on algebra

As presented in the *a priori* analysis of Workshop III (see Section 4. 1. 3), Claire’s didactical aim was to put emphasis on algebraic symbolism by focusing on the historical development of algebra (see Section 2. 6). This was done by dividing the workshop into two parts: first as Claire offered an overview of historical perspectives on algebra, and, in the second part by offering the teachers two tasks. The first task was taken from Babylonian time while the second task was the Calandri’s fish problem. Claire’s goal was to put emphasis on the power of modern algebraic notation by contrasting how these tasks were solved at that time with how it is possible to solve them when using modern notation. I propose the following coding for the first task:

Table 7: Coding the first task from Workshop III

Workshop III	
1. Task	Coding
<i>Find the length and the width of a rectangle when the area is 96 and the semi-perimeter is 20</i>	PAN-1MO-0E-1C-1R-0NP-1GA-0M

By coding this task in the following way, PAN-1MO-0E-1C-1R-0NP-1GA-0M, I mean that the first task offered during Workshop III addressed the power of algebraic notation as a didactical aim, did involve known mathematical objects (length and width of a rectangle), it was non-exploratory, contextualised, routine, not allowing for numerical patterns, but allowing for a geometrical approach with no use of manipulatives.

From the formal *a posteriori* analysis of the first task from Workshop III, I can see that all participants engaged collaboratively with this task. Further evidence of “becoming” participants in our community of inquiry is offered as the teachers started organising their interaction. However, the teachers engaged with the task without using unknowns or by experiencing difficulties in introducing these. Therefore, I consider that Claire’s didactical aim was partially achieved as *she* could point to and demonstrate the contrast between a Babylonian and a modern way of solving the task, but her aim was that *the teachers* could experience the contrast by themselves as they could engage in solving the task with modern algebraic notation. Concerning the second task, I propose the following coding:

Table 8: Coding the second task from Workshop III

Workshop III	
2. Task	Coding
Calandri’s fish problem: <i>The head of a fish weights $\frac{1}{3}$ of the whole fish, the tail weights $\frac{1}{4}$ and the body weights 300 grams. How much does the whole fish weight?</i>	PAN-1MO-1E-1C-1R-0NP-0GA-0M

By coding this task in the following way, PAN-1MO-1E-1C-1R-0NP-0GA-0M, I mean that the second task offered during Workshop III addressed the power of algebraic notation as a didactical aim, did involve known mathematical objects (fractions and relation between numbers), it was exploratory, contextualised, routine, not allowing for numerical patterns or geometrical approach, and with no use of manipulatives.

From the formal *a posteriori* analysis of Workshop III of this second task, I can see that all participants did engage collaboratively with this second task. Further evidence of “becoming” participants in our commu-

nity of inquiry is offered as the teachers started organising their interaction by taking initiative to ask for more time or deciding when to start to discuss the task. However, similarly to the first task, the teachers engaged with the task without using unknowns or by experiencing difficulties in introducing these. Therefore, I consider that Claire’s didactical aim was partially achieved as *she* could point to and demonstrate the contrast between the Rule of the False Position and modern way of solving the task, but her aim was that *the teachers* could experience the contrast by themselves by solving the task with modern algebraic notation.

As a summary it seems that Claire’s didactical aim, which was to put emphasise on the power of modern algebraic notation, was *partially* achieved with both tasks, since this issue was actually brought to the fore and emphasised for the participants, but it was *Claire* who addressed and discussed the contrast between these different ways of solving the tasks, *not* the teachers. At the same time, the analysis brings evidence of the development of our community of inquiry in terms of “belonging” since the teachers were able to organise their interaction, while Claire chose *not* to intervene. Thereby, even though the formal *a posteriori* analysis did not bring so much evidence in terms of development of algebraic thinking, the participants continued to develop confidence in our community of inquiry.

Workshop IV – Viviani’s theorem

As presented in the *a priori* analysis of Workshop IV (see Section 4. 1. 4), Claire’s didactical challenge was to choose a task which would encourage the teachers to use and explore the power of algebraic notation. In addition, she wanted to address a task which could bring a geometrical and an algebraic approach. As a pedagogical means she decided to offer the teachers Viviani’s theorem. This theorem states that, in an equilateral triangle, the sum of the distances from a point within the triangle to the sides of the triangle is equal to the height of the triangle (see Figure 9, Section 4. 1. 4). I propose the following coding for the Viviani task:

Table 9: Coding the task from Workshop IV

Workshop IV	
Task	Coding
<p>Viviani’s theorem: <i>In an equilateral triangle, the sum of the distances from a point within the triangle to the sides of the triangle is equal to the height of the triangle</i></p>	<p>PAN-1MO-0E-1C-1R-0NP-1GA-0M</p>

By coding this task in the following way, PAN-1MO-0E-1C-1R-0NP-1GA-0M, I mean that the task offered during Workshop IV addressed the power of algebraic notation as a didactical aim, did involve known

mathematical objects (triangle, distances, height), it was non-exploratory, contextualised, routine (verification of a theorem), did not allow for numerical patterns approach, but allowing for a geometrical approach with no use of manipulatives.

Looking back to the formal *a posteriori* analysis of Workshop IV, it is possible to find strong evidence of the motivation of the participants as all engaged with the task, as they were able to elaborate an algebraic proof for the Viviani's theorem and, in addition, to develop further the proposed task into a new one (how to construct an equilateral triangle when the distances from a point P are known). This recognition begs the following question: what is particular about this task in order to enable the teachers to achieve so well? By using the expression "so well" I want to refer to the fact that they did introduce symbols, used them as a means to elaborate an algebraic proof. In addition, the teachers were able to develop the mathematical task further.

In order to address this question, I want to recall what I labelled *mathematical environment* (see Section 3.4.1). I defined a mathematical environment as follows: "by presenting a particular task within a specific social environment, a didactician creates a mathematical environment whose characteristics depends *both* on the mathematical task and on the social setting". A possibility is to reformulate my question as: what is particular with this *mathematical environment* in order to enable the teachers to achieve so well? By doing this change I seek to expand my perspective and to take into consideration additional issues to those related exclusively to the nature of the mathematical task. Starting with the coding of the Viviani task, PAN-1MO-0E-1C-1R-0NP-1GA-0M, I want to put emphasise on the fact that the task allows for a geometrical approach, or more precisely, the task *starts from a geometrical approach* (drawing an equilateral triangle), in other words, the didactical aim was presented in the mathematical context of Euclidean geometry. It seems that this aspect is important for the teachers as they started engaging in the mathematical inquiry and, *at the same time*, considering possibilities to implement this activity in their own teaching (see Excerpt 1 in Workshop IV). There is a possibility that since the task is situated in a Euclidean context, a content familiar to them, this aspect enabled the teachers to engage *both* in mathematical exploration and in didactical explorations, and thereby to inquire deeply into the mathematics and to elaborate an algebraic proof of the Viviani's theorem. Thereby, it seems that this particular mathematical context, a Euclidean context, was supporting for the teachers. My interpretation is that this context is familiar to the teachers and that it enabled them to establish a link between our practice within our community of inquiry and their own teaching practice. This

suggestion is supported further by observing a mix of different levels of inquiry during the whole workshop.

In addition, by asking “what is particular with this *mathematical environment* in order to enable the teachers to achieve so well?” I want to point to the social setting and more precisely to the development of our community in terms of “belonging” to the community. As evidenced in the formal *a posteriori* analysis, the teachers took the initiative to organise the mathematical inquiry, to present and explain an algebraic proof, and to develop Viviani’s theorem further on their own initiative. Thereby, they were able to modify a routine task, as the coding indicated, into a non-routine one.

I argue for considering all these aspects as bringing strong evidence of the “belonging” (Wenger, 1998) to our community of inquiry, in terms of engagement and imagination with the task, in addition to the teachers’ fluency with algebraic notation. Thereby, *within this specific mathematical environment*, Claire’s didactical aim was actually fulfilled.

Workshop V - Palindromes

As presented in the *a priori* analysis of Workshop V (see Section 4. 1. 5), Claire’s didactical aim was to continue on the power of algebraic notation and to come back to the idea of generalisation of numerical patterns, as it was addressed in Workshop II. Her goal was to see how the participants would respond to that kind of task, given the mathematical fluency and confidence in our community of inquiry as observed during Workshop IV. A task related to four digit palindromes was chosen as a pedagogical means. I propose the following coding for this task:

Table 10: Coding the task from Workshop V

Workshop V	
Task	Coding
<p>The four digit palindromes task: <i>A friend of mine claims that all palindromes with four digits are exactly divisible by eleven. Are they?</i></p>	PAN-0MO-1E-0C-1R-1NP-0GA-0M

By coding this task in the following way, PAN-0MO-1E-0C-1R-1NP-0GA-0M, I mean that the task offered during Workshop V addressed the power of algebraic notation as a didactical aim, did involve unknown mathematical objects to the teachers (four digit palindromes), it was exploratory, non-contextualised, routine (proof of divisibility by 11), allowing for a numerical pattern approach, but not allowing for a geometrical approach, and without use of manipulatives.

Looking back to the formal *a posteriori* analysis of Workshop V, it seems that it was necessary to negotiate several aspects of the activity *before* Claire could address her didactical aim. These aspects related to

the teachers' fascination for emerging numerical patterns, and the difficulty the teachers experienced when communicating to the other participants the result of their numerical explorations, due to the ambiguity and vagueness of mathematical discourse. It seemed that the teachers did not feel a need for introducing symbols since they engaged in exploring further other numerical patterns. It was only after Claire's emphasis on moving forward to introducing algebraic symbols, and after she changed *momentarily* her pedagogical strategy, that the teachers were able to elaborate an algebraic proof for the divisibility of four digit palindromes by 11. From Claire's perspective there is, on one hand, a resistance to the introduction of symbols, and, on the other hand, a recognition of the importance of the discovery, exploration and investigation of numerical patterns, as a means to grasp and express some structure (see Chapter 5).

As mentioned above, Claire's goal, during Workshop V, was to revisit the generalisation of numerical patterns as a didactical aim, taking into consideration the participants' mathematical fluency and confidence in our community of inquiry. Therefore, I propose to compare with Workshop II, where the generalisation of numerical patterns also was addressed, and the way I coded the offered mathematical tasks:

- Workshop II was coded as: CAS-1MO-1E-0C-0R-1NP-1GA-1M
- Workshop V was coded as: PAN-0MO-1E-0C-1R-1NP-0GA-0M.

From comparing the coding of the two tasks, it is possible to see that the didactical aim was slightly different, even if I consider that these are closely related since the choice and use of algebraic symbols might be seen as a preliminary step before exploring the power of algebraic notation. I argue that the unknown nature of mathematical objects (palindromes) did not seem to interfere, since, after giving to the teachers the definition of a four digit palindrome they were able to engage with the task. I coded both tasks as exploratory and non-contextualised, the difference in the coding seems to be on the *routine* versus *non-routine* dimension. The non-routine code refers to the nature of the question (What happens when we add even and odd numbers?), while I consider the question related to four digit palindromes as routine (are all four digit palindromes divisible by 11?).

In addition, considering the *mathematical environment*, that is the environment whose characteristics depends *both* on the mathematical task and on the social setting, it seems that during Workshop II the mathematical environment included *both* a non-routine task (open question) and a community which was in an early process of "becoming" established (all participants were unsure how to act). Thereby, there is a possibility that the combination of these two factors had an important

impact on the development of Workshop II and might explain the reason why Claire’s didactical aim was not achieved.

On the other hand, the mathematical environment of Workshop V was different since, according to my coding, the mathematical task related to four digit palindromes was a routine task (non-open question), and all participants had developed more confidence in our community of inquiry (the teachers asking, challenging, and arguing with each other while Claire was adjusting momentarily her didactical strategy). I want to argue for considering the achievement of Claire’s didactical aim, the elaboration of a proof for the divisibility of four digit palindromes by 11, as closely related to *both* the dimension of routine in the coding of the task and the development of our community of inquiry.

This recognition begs the following question: what would have happened if Claire had reversed the order of the tasks, that is if she had presented the palindromes task during the second workshop and the even/odd numbers task during the fifth workshop. I consider that the recognition of the importance of these factors helps me to envisage potential further development of my research.

Workshop VI – Syntactic versus semantic translation

As presented in the *a priori* analysis of Workshop VI (see Section 4. 1. 6), Claire’s didactical aim was to address the translation from natural language to symbols, and more specifically, the differences between syntactic and semantic translation. In order to address her didactical aim, she offered the teachers two tasks. The first task was from Diophantus, the second one was the student-professor task. I propose the following coding for the first task:

Table 11: Coding the first task from Workshop VI

Workshop VI	
1. Task	Coding
<p>The Diophantus task: <i>Divide a given number into two numbers with a given difference</i></p>	MS-1MO-1E-0C-0R-0NP-0GA-0M

By coding this task in the following way, MS-1MO-1E-0C-0R-0NP-0GA-0M, I mean that the first task offered during Workshop VI addressed the meaning of symbols, since the focus was on the difference between syntactic and semantic translation, it did involve known mathematical objects, it was exploratory, non-contextualised, non-routine, did not allow for numerical patterns or geometrical approach and with no use of manipulatives.

From the *a posteriori* analysis of Workshop VI, it is possible to see the difficulty this task represented for the teachers, since the term “divide” was interpreted differently by Claire and by the teachers. Again, as in the

even/odd numbers task, the task is of non-routine nature with a formulation which is unusual for modern mathematics. However, this aspect was addressed *directly* by the teachers as they questioned Claire’s interpretation of the task. I argue that it was possible to make visible and address this tension due to the stage of development of our community of inquiry. In other words, the tension was addressed directly by the teachers after Claire’s presentation of the semantic translation of the task. Thereby, the task fulfilled its purpose and the meaning of symbols was addressed since all participants had developed enough confidence in the community. Thus, within this particular *mathematical environment*, the level of development of our community contributed to the achievement of the didactical goal. Concerning the second task, I propose the following coding:

Table 12: Coding the second task from Workshop VI

Workshop VI	
2. Task	Coding
The Student-professor task: <i>Six times as many students as professors</i>	MS-1MO-1E-1C-1R-0NP-0GA-0M

By coding this task in the following way, MS-1MO-1E-1C-1R-0NP-0GA-0M, I mean that the second task offered during Workshop VI addressed the meaning of symbols, since the focus was also on the difference between syntactic and semantic translation, it did involve known mathematical objects, it was exploratory, contextualised, routine, did not allow for numerical patterns or geometrical approach and with no use of manipulatives.

As presented in the *a posteriori* analysis of this second task, this task also fulfilled its purpose since the participants were able to elaborate a semantic translation of the task formulated in natural language. However, I want to argue for recognising the importance of how the didactical aim (meaning of the symbols) has been gradually achieved since, from the beginning it was John, only, who had translated semantically, while both Mary and Paul had translated syntactically. It was only after John’s numerical example, that the teachers developed an in-depth understanding of the meaning of the symbols, contrasting a symbol as used as an object with a symbol as used as a variable. Thereby, as for the first task, the second task fulfilled its purpose and the meaning of symbols was addressed since all participants had developed enough confidence in the community. However, I want to emphasise the fact that it was *the teachers* who engaged in inquiring into the task and making explicit the difficulties related to translation problems. Thereby, within this particular *mathematical environment*, the level of development of our community contributed to the achievement of the didactical goal.

Workshop VII – Syntactic versus semantic translation

As presented in the *a priori* analysis of Workshop VII (see Section 4. 1. 7), Claire’s didactical aim was to explore the difference between a syntactic and semantic translation further. In order to address her didactical aim she offered the teachers two tasks: The first task was from Duval, while the second one was elaborated by Claire herself. I propose the following coding for the first task:

Table 13: Coding the first task from Workshop VII

Workshop VII	
1. Task	Coding
<p style="text-align: center;">The task from Duval:</p> <p><i>The perimeter of a rectangle with length “a” and width “b” is 62, the length of the rectangle is increased by 2 meters, and the width is decreased by 1 meter. The area is constant. Write the area of the rectangle before change, write the area after change.</i></p>	<p>MS-1MO-1E-1C-0R-0NP-1GA-0M</p>

By coding this task in the following way, MS-1MO-1E-1C-0R-0NP-1GA-0M, I mean that the first task offered during Workshop VII addressed the meaning of symbols, it did involve known mathematical objects, it was exploratory, contextualised, non-routine, did not allow for numerical patterns approach, but did allow for geometrical approach and with no use of manipulatives.

From the *a posteriori* analysis of Workshop VII, it is possible to see that the purpose of this task was difficult to achieve since the teachers either asked Claire directly and focused on what she wanted them to do or engaged in solving an equation. Therefore, it seems that the fact that the task was *non-routine*, as evidenced by the nature of the question, provoked difficulties for the teachers. In addition, I want to emphasise the fact that these difficulties were observed even though the community was established and all participants had developed confidence in our community of inquiry. Therefore, within this particular *mathematical environment*, the challenge offered by the mathematical task, and more particularly by the non-routine character of the task, offered an important challenge to the teachers and it was only after Claire’s focus on the nature of the question (write the area, not calculate the area) that the teachers were able to elaborate a semantic translation of the task. A possible explanation of the difficulties the teachers met with this task, is that this kind of question is not usual in their teaching practice and therefore, they tried to relate to some aspects of their practice which were familiar. I propose the following coding for the second task:

Table 14: Coding the second task from Workshop VII

Workshop VII	
2. Task	Coding
<i>Ole has three siblings. Ole is ten years older than Per and three years younger than Kari. Ole is five years younger than Jens and together they are 58 years old. How old are Ole, Per, Kari and Jens?</i>	MS-1MO-1E-1C-1R-0NP-0GA-0M

By coding this task in the following way, MS-1MO-1E-1C-1R-0NP-0GA-0M, I mean that the second task offered during Workshop VII addressed the meaning of symbols, since the focus was on the difference between syntactic and semantic translation, it did involve known mathematical objects, it was exploratory, contextualised, routine, did not allow for numerical patterns or geometrical approach, and with no use of manipulatives.

As explained in the formal *a posteriori* analysis of Workshop VII, this task fulfilled its purpose as the teachers were able to differentiate a syntactic translation from a semantic one and to address both the ambiguity of mathematical discourse and more particularly the central question concerning the choice of the unknown. Therefore, I argue that this particular *mathematical environment* enabled the teachers to ask critical questions concerning the meaning of symbols and, based on their confidence in our community of inquiry, to challenge and argue with each other in order to fulfil the purpose of the task.

4.2.4 The importance of the dimension routine versus non-routine

Comparing these two tasks offered during Workshop VII, it seems that the dimension related to *routine* versus *non-routine* is central in understanding the differences in how the purpose of the tasks was fulfilled. The importance of that particular dimension was emphasised earlier, in comparing the coding of the task offered in Workshop II with the task offered in Workshop V. Therefore, I argue for focusing on the dimension *routine* versus *non routine*, as I consider that this dimension captures the kind of question (s) asked in the task, and to classify the tasks according to this dimension:

Table 15: Classification of tasks according to the dimension *routine* versus *non-routine*

Routine	Non-routine
Tasks from Workshops III, IV, V. Workshop VI, Student-professor task Workshop VII, Ole's siblings	Tasks from Workshops I, II. Workshop VI, Diophantus-task Workshop VII, a rectangle with length ...

In order to explore further the consequences of classifying the tasks according to the dimension of *routine* versus *non-routine*, I want to recall what I labelled *mathematical environment* (see Section 3. 4. 1). I defined

a mathematical environment as, by presenting a particular task within a specific social environment, a didactician creates a mathematical environment whose characteristics depends *both* on the mathematical task and on the social setting. Thereby, as a consequence of introducing a classification of the tasks according to the dimension of *routine* or *non-routine* tasks, it is possible to follow how Claire, acting both as a researcher and as a didactician, created different mathematical environments. Looking back to the formal *a posteriori* analyses of the different workshops, it is actually possible to distinguish that difference. I develop this issue further in the following.

4.2.5 Looking back to the formal *a posteriori* analyses of the workshops

Considering first the *a posteriori* analyses of the workshops whose tasks were classified as *routine*, it seems that the teachers engaged with the different mathematical tasks with an increasing confidence in mathematics. By considering the *a posteriori* analyses in chronological order, it is possible to trace how the teachers gradually started to take initiative to ask for more time for investigating the task, and decided when to start the discussion (Workshop III). Further evidence of the participants' development of confidence in our community of inquiry shows how they took the leadership concerning the organisation of mathematical inquiry, developing the proposed task further (Workshop IV), getting fascinated by numerical patterns and even not paying attention to Claire's didactical moves (Workshop V). During Workshop VI (the student-professor task), the teachers were taking the role of the didactician by proposing and elaborating on a numerical example, and finally as they were able to recognise the issue concerning the choice of unknown (Workshop VII, Ole's siblings task). I argue for considering these different aspects as characteristics of the teachers' development of confidence in mathematics and also in our community of inquiry, since these two components seem to be strongly interdependent and mutually constituent.

On the other hand, considering the *a posteriori* analyses of the workshops whose tasks were classified as *non-routine*, it seems that the teachers engaged with the tasks differently, as they were focusing on what Claire meant by "in a (more) mathematical way" (Workshops I and II), as they addressed the mismatch between Claire's presentation of the Diophantus task and their own understandings due to the difficulty caused by the term "divide" (Workshop VI, the Diophantus task), and as they focused on Claire's purpose with the task as they seemed disturbed by the fact that the area of the rectangle was unchanged after transformation of the rectangle (Workshop VII, first task). I argue for considering these different aspects as evidence of the challenges the teachers met when engaging with these tasks. However, I recognise that issues con-

cerning the creation of our community of inquiry were interfering with mathematical inquiry during the first workshops, but this argument cannot be used as regarding to Workshop VII. By that time the teachers had developed thorough confidence in our community of inquiry, as evidenced in the *a posteriori* analyses of previous workshops, and therefore I consider that a possible explanation of the teachers' difficulties relates to the *non-routine* dimension of this particular task.

My aim, with presenting this distinction between the different tasks, was to deepen and make visible what I meant by the nature of the mathematical tasks. By considering the different challenges which each task offered the teachers, I argue for valuing *both* tasks showing *routine and non-routine* dimension, as these enabled all participants, that is including Claire, to develop an awareness of the crucial role of this particular dimension. Thereby, I consider that, even if the tasks contributed differently to the participants' development of algebraic thinking, routine and non-routine tasks acted as complementary and I argue for the need to address and engage with both types of mathematical tasks.

During the elaboration of this section, I tried to address Sierpinska's (2003) concern about task analysis. While reporting on research reports, she asked:

In particular, were mathematical tasks used as tools in research? How were they presented? Was their choice justified and discussed? ... I consider the design, analysis and empirical testing of mathematical tasks, whether for the purpose of research or teaching, as one of the most important responsibilities of mathematics education. (p.12)

Therefore, I argue for considering this section and both the *a priori* and *a posteriori* analyses of the different mathematical tasks as an attempt to address Sierpinska's questions.

In the next section, Section 4. 3, I propose to use Lerman's (1998b) metaphor of the "zoom of a lens" and by zooming *out*, I am in a position to adopt a larger perspective which enables me to follow how algebraic thinking was mediated, during the year of our collaboration, using Karpov et al.'s (1998) notions of *metacognitive* and *cognitive* mediation.

4.3 Zooming out: addressing participation through cognitive and metacognitive mediation

Until now, I have been offering a fine grain analysis of each workshop (see Sections 4. 1. 1 to 4. 1. 9) and I have elaborated on what I called *the nature of the mathematical tasks* (see Section 4. 2). My aim, with this section, is to adopt a wider perspective on our collaborative work during the year, and to zoom *out* from the close analysis of each workshop, to a more holistic perspective on the functioning of our community of inquiry during the year our collaboration lasted. Adopting this perspective enables me to address the link between *participation within our community*

of inquiry and cognitive and metacognitive mediation (Karpov et al., 1998) (see Section 2. 5. 3). The former derives from Wenger (1998) who defined participation in a community of practice as:

... the social experience of living in the world in terms of membership in social communities and active involvement in social enterprises. Participation in this sense is both personal and social. It is a complex process that combines doing, talking, thinking, feeling, and belonging. It involves our whole person, including our bodies, minds, emotions, and social relations. (Wenger, 1998, p.55-56)

As explained in Section 2. 2. 6, the dimension of *inquiry* emerged from the analysis of my data and I conceptualised this characteristic of our collaboration by referring to the work of Elliott (2005), Jaworski (2005a, 2006), and Lindfors (1999). Thereby, I understand the idea of *participation within our community of inquiry* as going beyond and expanding Wenger's definition of participation in a community of practice.

The idea of participation is developed further by defining learning as involving "transformation of participation in collaborative endeavour" (Rogoff et al., 1996, p.388). The notion of *transformation of participation* is based on the idea that "learning and development occur as people participate in the sociocultural activities of their community, transforming their understanding, roles, and responsibilities as they participate" (p.390). This was the case in my study since, in our community of inquiry, we learned and developed as we participated in the sociocultural activities of the different workshops and the observation steps consisting of interviews and classroom observations. Therefore, as a researcher, I am able to trace the participants' transformation of understanding, roles, and responsibilities as we all participated in the joint enterprise and mutual engagement of the practice of our community of inquiry. During all workshops, we addressed algebra and algebraic thinking through engaging both in inquiry into the mathematical tasks and by sharing with each other previous teaching experiences (see *a posteriori* analyses of the different workshops). During the negotiation of meaning of each mathematical task, our understanding became visible and accessible to the other participants and therefore, I am in a position to trace the *transformation of participation* as it emerged within our community of inquiry. I shall come back to this aspect later.

Wenger, in his quotation as presented above, refers to participation in a community of practice as a complex process. In Section 2. 2. 1, I argued for considering that the "complex process" of participation is exemplified in my research through *negotiation of meaning* since, as all participants engaged with the mathematical tasks, the meaning of the different mathematical tasks was negotiated between the participants and thereby we all engaged mutually in the social enterprise of our community of inquiry. As explained in Section 2. 5, I argued for taking into account the specificity of the social setting within my research (addressing

mathematical learning) and therefore, I proposed to integrate Vygotsky's ideas of *mediation* and *scientific concepts* (see Section 2. 5) and more specifically the *metacognitive* and *cognitive mediation* of Karpov et al. (1998) (see Section 2. 5. 1). Thereby, I consider that the elaborated theoretical framework is both *relevant* to the social setting of my research (addressing *mathematical* learning) and *coherent* since both Vygotsky's and Wenger's work are rooted in a sociocultural approach to learning.

Furthermore, I understand Wenger's *negotiation of meaning* as closely related to Kozulin's (2003) *mediation of meaning* since "mediation of meaning is an essential moment in the acquisition of psychological tools, because symbolic tools derive their meaning *only* from the cultural conventions that engendered them." (p.26, my emphasis). In this quotation Kozulin (2003) emphasizes the cultural conventions which engendered the symbolic tools. In the case of algebraic symbolism, the cultural conventions refer to the historical development of algebraic notation and the way the community of mathematicians uses it. However, I argued before (see Section 2. 5. 3) for adding another aspect which, I believe, plays a central role in the process of mediation of meaning: the *social setting* within which psychological tools are addressed, discussed, and explored. Thereby I proposed to take into consideration *both* the importance of the cultural conventions from which the meaning of psychological tools emerged *and* the social setting, and more particularly the *mathematical environment* (see Section 3. 4. 1), within which psychological tools are addressed. Thereby, I propose to rephrase the quotation from Kozulin as "mediation of meaning is an essential moment in the acquisition of psychological tools, because symbolic tools derive their meaning *both* from the cultural conventions that engendered them *and* from the mathematical environment within which these are presented". This acknowledgement implies for my study that the practice *and* the nature of the mathematical tasks are components of the social setting and, as such, play a central role. This recognition is pragmatic as it emerged from the analysis of my data (see Section 4. 2).

My interpretation of the link between these two concepts (negotiation of meaning and mediation of meaning) is that, as the participants engaged in negotiating the meaning of a particular mathematical task, meaning is mediated between the participants and algebraic thinking. In other words, I consider that engaging in the process of negotiating the meaning of a mathematical task is a necessary step in the process of mediation of meaning between the participants and algebraic thinking. Furthermore, a central dimension of my study concerns *the nature of our engagement* during the process of negotiating the meaning of a mathematical task. This recognition begs the following question: *how* did we engage in the process of negotiating the meaning of mathematical tasks?

I argue for considering our engagement as rooted in *inquiry* and characterised by the six different kinds of inquiry¹³ which I identified (see Section 2. 2. 6). Thereby, I understand *inquiry as a mediating tool between the participants engaging in the activities of our practice and algebraic thinking*.

In order to clarify the difference between psychological tools and symbolic tools, as addressed in the quotation from Kozulin's (2003), I follow Kozulin (1998)¹⁴ explaining that:

Unlike material tools, which serve as conductors of human activity aimed at external objects, psychological tools are internally oriented, transforming the inner, natural psychological processes into higher mental functions. In their external form psychological tools are symbolic artifacts such as signs, symbols, languages, formulae, and graphic devices. (p.14)

According to Kozulin (2003), the way meaning is mediated plays a crucial role within the process of acquisition (or appropriation) of psychological tools, where psychological tools are defined as “symbolic artefacts – signs, symbols, texts, formulae, graphic-symbolic devices – that help individuals master their own “natural” psychological functions of perception, memory, attention, and so on” (Kozulin, 1998, p.1). I argue for considering my research as offering a conceptualisation of what *mediation of meaning* could mean in the specific social setting of my study.

Concerning the appropriation of psychological tools, Karpov and Haywood (1998) relate *cognitive mediation* to the acquisition of scientific concepts representing the core of some subject-matter. Furthermore, I argued earlier (see above) for considering mediation of meaning as an essential moment in the acquisition of psychological tools, since symbolic tools derive their meaning *both* from the cultural conventions that engendered them *and* from the mathematical environment within which these are presented. However, I find it inappropriate to talk about the participants' acquisition or appropriation of such and such scientific concepts in the context of my research. Therefore, I rather address cognitive mediation in terms of *developing one's own awareness and enhancing one's own understanding of scientific concepts in relation to algebraic thinking* (see Figure 23, later in this section). Thereby, by adopting this perspective, I am in a position to address cognitive mediation (the development and enhancement of one's own understanding of scientific concepts) through the idea of negotiation and mediation of meaning, as these

¹³ I proposed to differentiate between inquiry in a mathematical task, inquiry in community building, inquiry in the other participants' understanding of a mathematical task or their own teaching practice, and inquiry in Claire's didactical and pedagogical aims.

¹⁴ Kozulin uses both the term “symbolic artifacts” (1998) and the term “symbolic tools” (2003) as referring to “letters, codes, mathematical signs” (p. 26). In the following I choose to use symbolic tools as referring to “signs, symbols, languages, formulae, and graphic devices” as external form of psychological tools.

occurred in the specific social setting of my study. For example, I consider that the mathematical environment observed during Workshop II did not allow the participants to develop their awareness and enhance their understanding of scientific concepts of even and odd numbers in relation to algebraic standard notation (see the formal *a posteriori* analysis of Workshop II in Section 4. 1. 2).

In addition to cognitive mediation, Karpov and Haywood (1998) define *metacognitive mediation* as the acquisition of semiotic tools of *self-regulation*, such as *self-planning*, *self-monitoring*, *self-checking*, and *self-evaluating* (see Section 2. 5. 1). I referred earlier in this section to transformation of participation and I consider that the dimension of *inquiry* and its strong emergence in my research enables me to add the aspect of *self-reflecting* in relation to one's own teaching practice: as the participants engaged in sharing with each other their own teaching experience and opened up a possibility for looking critically upon it, they engaged in critical alignment "in which it is possible for participants to align with aspects of practice while critically questioning roles and purposes as a part of their participation for ongoing regeneration of the practice" (Jaworski, 2006, p.190). This was the case in my research as the participants engaged, each in different ways, in asking critical questions about their teaching practice (see Section 4. 1. 9). Thereby, I argue for considering my research as enabling me to address a potential link between Karpov and Haywood's (1998) *metacognitive mediation* and Jaworski's (2006) *critical alignment* through recognising the crucial role played by *inquiry*.

As a consequence of these considerations, I can refer in my study, on one hand, to how algebraic thinking is addressed, during the year our collaboration lasted, through metacognitive and cognitive mediation. On the other hand, I can refer to what is happening during each workshop and consider how algebraic thinking is mediated, at a fine grain level, by the way the meaning of each mathematical task is negotiated between the participants. In other words, using Lerman's (1998b) metaphor of "the zoom of a lens", by zooming *in* on what is happening between the participants during each workshop as they inquire into the mathematical tasks, Wenger's ideas of *negotiation of meaning*, *practice*, *community*, and *identity* act as theoretical tools describing the particular social setting within which the activity is situated. By zooming *out* and looking at what is happening between the participants during the year, the Vygotskian ideas of *mediation of meaning*, *metacognitive* and *cognitive mediation* and Wenger's notion of *participation* act as theoretical tools describing the general cultural conventions and social settings within which the activity is situated. As a result of these considerations I understand Wenger's "complex process of participation" as exemplified in the fol-

lowing way in my research. I consider *participation within our community* of inquiry as consisting of, on one hand, engaging in the negotiation of meaning of the mathematical tasks through addressing inquiry and didactical acts, and, on the other hand, developing awareness and understanding of scientific concepts (cognitive mediation) and following the acquisition of semiotic tools of self-regulation (metacognitive mediation) (see Figure 23). Thereby, to trace the transformation of participation, as it emerged within our community of inquiry, means to follow the development of these aspects, as presented above, during the year our collaboration lasted (see Chapter 4).

Until now I have described how the meaning of mathematical tasks has been negotiated, by using inquiry as mediating between the participants and algebraic thinking, and resulting in the acquisition (or appropriation) of psychological tools such as mathematical signs or algebraic symbols. However, as evidenced through the formal *a posteriori* analyses of the different workshops (see Sections 4. 1. 1 to 4. 1, 9), challenges, mismatches and difficulties appeared during the negotiation of meaning of some of the mathematical tasks. As a consequence, there is a possibility that the meaning of the mathematical tasks has been mediated differently from workshop to workshop, according to how the negotiation has evolved during that particular workshop. In terms of acquisition (or appropriation) of psychological tools such as mathematical signs or algebraic symbols, the difficulties observed during the workshops might prevent the appropriation of psychological tools which were addressed. For example, the didactical aim of Workshop II was to address the choice and use of algebraic symbols and, more particularly, to address the use of standard notation of even and odd numbers, which is $2n$ and $2n+1$. However, as explained in the formal *a posteriori* analysis of that workshop, the participants did not address and use this notation and, thereby, this particularly task did *not* fulfil its purpose.

In Section 4. 2, I introduced a coding for each mathematical task in order to develop further what I meant by *the nature of the mathematical tasks*. In addition, I argued for taking into account the *mathematical environment*, that is a specific social environment whose characteristics depends *both* on the mathematical task and on the social setting. Therefore, even if the mathematical task offered the teachers during Workshop II was potentially a good task, there is a possibility that the difficulties observed during that workshop were caused by the non-routine character of the task (see the coding of that task in Section 4. 2) *and* by the early stage of our community of inquiry. The last aspect was evidenced by Claire's didactical inexperience and by the teachers' questions concerning the functioning of our community. I consider that these aspects caused difficulties in the negotiation of meaning of the task and thereby,

in the mediation of meaning. In an attempt to summarise how these different notions relate to each other, I propose the following figure. Here scientific concepts are understood as examples of psychological tools:

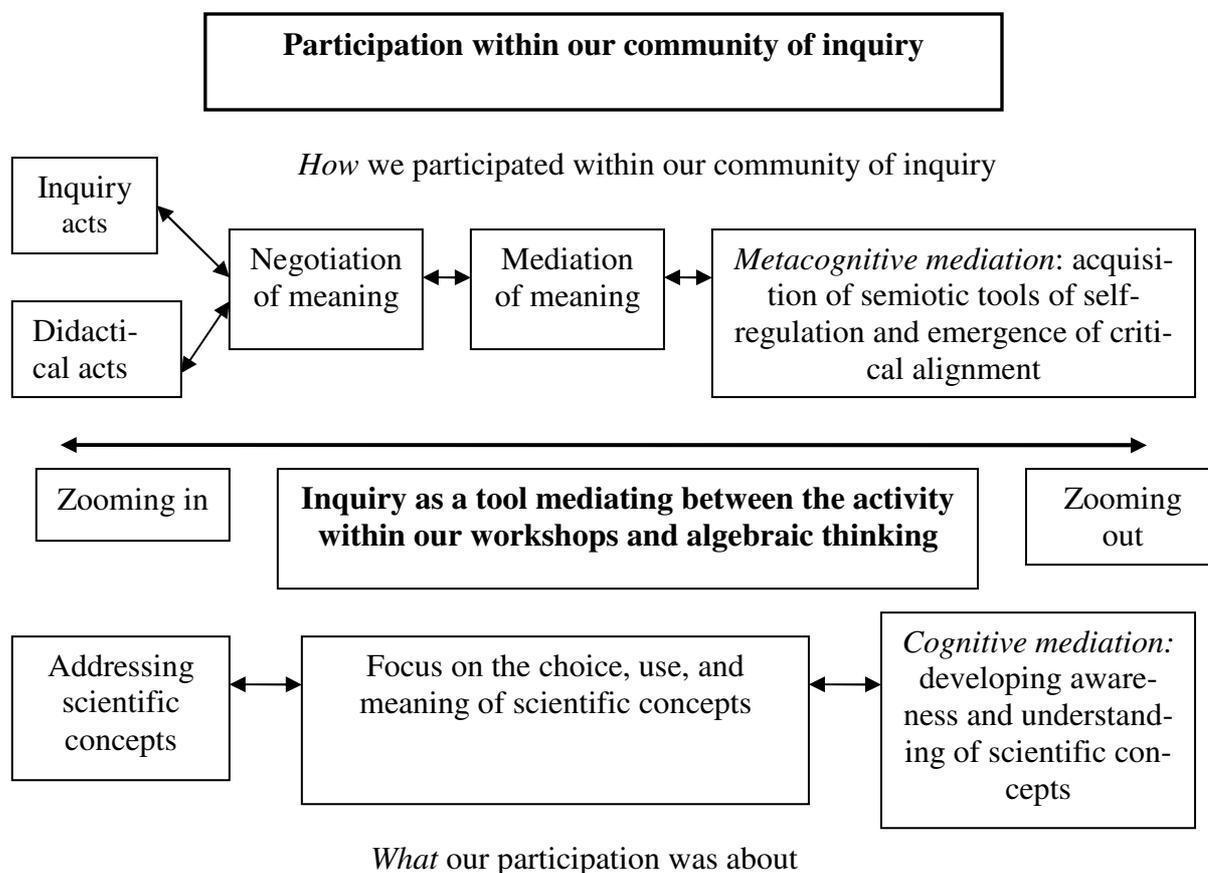


Figure 23: Participation within our community of inquiry: From inquiry/didactical acts and addressing scientific concepts to metacognitive and cognitive mediation

In this figure, I offer a conceptualisation of the idea of participation within our community of inquiry where the links between the central ideas which form my theoretical framework (see Chapter 2) are made explicit. Using Gravemeijer’s (1994a) words, “global basic theory is elaborated and refined in local theories” (p.452). This is what happened in my research as the global theories from Wenger (1998), Vygotsky (1978, 1986), and Karpov et al. (1998) were elaborated and refined further in a local theory which I presented in my Chapter 2. Furthermore, according to Gravemeijer’s (1994a), “Vice versa, the more general theory can be reconstructed by analysing local theories” (p.451). Thereby, Gravemeijer’s quotation begs the following question: can the more general theory concerning mathematical learning be reconstructed by analysing the local theory which I elaborated in my research? I believe that, within the specific social context of my research with a focus on alge-

braic thinking, I am in a position where I can reconstruct the more general theory of learning (Wenger, 1998) by expanding it to include *mathematical* learning (Vygotsky, 1978, 1986; Karpov et al., 1998) rooted in the idea of *inquiry* (Elliott, 2005; Jaworski, 2005a, 2006; Lindfors, 1999; Wells, 1999).

4.3.1 Addressing metacognitive mediation: the emergence of different modes of participation

In order to trace the transformations in participation within our community of inquiry, I followed the chronological order of the workshops and tried to capture the main characteristics of each one while trying to identify what mode of participation was emerging from the formal *a posteriori* analysis. As a result, I propose the following modes of participation:

- The ‘novice-expert like’ mode
- The ‘questioning’ mode
- The ‘reflective’ mode
- The ‘taking-over’ mode
- The ‘didactical’ mode
- The ‘silent-participant’ mode

In the following paragraphs, I develop each of these modes of participation. In order to be able to trace the development of our community of inquiry in terms of metacognitive mediation, I considered it important to identify the characteristics of the mode of participation which emerged during the first workshop. Recalling the formal *a posteriori* analysis of Workshop I, the central features were the teachers’ reaction to the mathematical task, asking no questions, and the mismatch between Claire’s focus on her didactical aim and the teachers being unsure about the ‘rules’ of our community of inquiry. This mismatch was also visible as Claire, introducing the expression ‘in a more mathematical way’, was unable to provide explanations to the teachers, while the teachers focused on what Claire wanted to achieve. In addition, the analysis of this first workshop showed a strong dyadic structure of the communication within the group. Based on these features, I propose to characterise this mode of participation as *novice-expert like* mode. In other words, this mode of participation enables me to capture and make visible the first steps of the creation of our community of inquiry and, from this recognition, I am able to compare the mode of participation as it emerged from Workshop I, with the different modes of participation in the following workshops in order to trace their evolution. However, I want to put emphasis on the fact that the expression *novice-expert like* mode of participation does not refer to Claire as the expert and the teachers as the novices. From my perspective as a researcher, I understand *all* participants as novices within our community of inquiry. By using this expres-

sion I seek to convey what I perceive from the *teachers' perspective*, since it seems that they consider Claire as the expert and themselves as the novices.

The *questioning* mode of participation emerged from the fine grain analysis of Workshop II as it seems that the teachers started to question the choice of the mathematical task, asking about the relevance of the task for lower secondary school. In addition, the teachers questioned the adequacy of introducing manipulatives as a means to illustrate the geometrical properties of even and odd numbers. I consider the teachers' questions as crucial, as these questions enabled the participants to define what our joint engagement and mutual enterprise might look like. It was through addressing central questions concerning the functioning of our group that all participants could, jointly, develop further their confidence in our community of inquiry. In addition, as mentioned in the formal *a posteriori* analysis of Workshop II, Claire was not prepared for this kind of question and she was surprised by the teachers' questions. Thereby, it seems that, in terms of development of our community of inquiry, it was important to address these questions concerning the functioning of our group *before* engaging with the mathematical tasks. However, I am in a position, today, where I recognise the importance of these issues and, at the same time, I acknowledge that the nature of our practice did not allow Claire to address these issues at that time. This might be due to a lack of awareness of a set of interwoven evolving forms as the development related to algebraic thinking, the development of our community and the development of expertise.

The *reflective* mode of participation refers to the fact that the teachers showed a willingness to share with each other their own teaching experience concerning the pupils' difficulties with algebra and more particularly in relation to the use of symbols (Workshop II). For example, based on Claire's emphasis on the differences between modern notation and Babylonian notation, the teachers mentioned the pupils' preference for having recipes when solving problems. This characteristic of bringing aspects of one's own teaching practice into the discussion was present during several workshops (see Workshop IV where I mentioned mixed inquiry) and I argue for recognising this mode of participation as central to the development of our community of inquiry as it enabled the participants to establish a common base for joint enterprise and mutual engagement and as such developing a sense of belonging to our community of inquiry. In addition, as mentioned above, this mode of participation enabled the participants to look critically into their own teaching practice and thereby to engage in *critical alignment* by inquiring into their roles and purposes in relation to their own teaching practice.

The *taking-over* mode of participation emerged from the formal *a posteriori* analysis of Workshop IV, as the teachers engaged in developing further the Viviani task, and explored a new task while mixing the mathematical exploration with didactical considerations. I chose to label this mode of participation as ‘taking-over’ since I seek to capture the teachers’ own initiative to propose and engage with a different task, both in terms of didactical and pedagogical means. In other words, the teachers were able to create a *different* mathematical environment (see Section 3.4.1) by introducing a different task from the one Claire proposed at the beginning of Workshop IV. As mentioned in the formal *a posteriori* analysis, I argued for considering the mode of participation as strong evidence of the participants’ thorough confidence both in the mathematics and in our community of inquiry, as they were able to organise the mathematical inquiry, to elaborate an algebraic proof of Viviani’s theorem, and to develop the task further.

Concerning the *didactical* mode of participation, I argue that this mode developed strongly during Workshop VI. During that workshop, John, building on Mary’s numerical example, was able to value and develop it further as a means to illustrate the difference between a syntactic and a semantic translation. Furthermore, he was able to emphasise the crucial role played by symbols by contrasting symbols used as an object with symbols used as a variable. In other words, John was acting as a didactician, adopting Claire’s didactical aim and thereby showing thorough confidence both in the mathematics and in our community of inquiry.

The last mode of participation which I identified was the *silent-participant* mode. In the analysis of Workshop IV, I mentioned the fact that Paul’s voice was missing during the exploration of Viviani’s theorem as he seemed not to participate in how to organise further the mathematical inquiry into the task. As I wrote: Claire had noticed Paul’s lack of engagement and had concern about him, both as a didactician and as a researcher. However, later during that workshop, Paul did participate very actively as he proposed several ways of exploring and developing further Viviani’s theorem. Therefore, I argue for considering Paul’s mode of participation during the first part of Workshop IV as ‘silent-participant’ since he *was* participating in the joint enterprise, but without expressing any utterances. Once this particular mode of participation is identified, I am in a position of recognising it during several other workshops. For example, Claire moved into this mode of participation as she chose *not* to intervene when John and Mary organised the mathematical inquiry (Workshop IV), and when John negotiated the meaning of the task with Paul and Mary (Workshop VI). I argue for recognising the importance of this mode of participation as deciding *not* to participate

might witness strong confidence both in the mathematics and in our community of inquiry. As explained in the beginning of this section, there is a possibility that the meaning of the mathematical tasks has been mediated differently from workshop to workshop, according to how the negotiation has evolved during that particular workshop. By identifying several modes of participation I am in a position where I can make explicit how the meaning of each task has been negotiated and therefore I can address mediation of meaning by considering to which extent a mathematical task fulfilled its purpose or not. I understand cognitive mediation in terms of developing one's awareness and enhancing one's understanding of scientific concepts in relation to algebraic thinking as closely related to the mathematical fulfilling its purpose or not.

In the previous paragraph, I addressed metacognitive mediation where the idea of participation stands central. According to Karpov and Haywood (1998), this kind of mediation relates to the acquisition of semiotic tools of self-regulation, self-planning, self-monitoring. In addition, I proposed to add the aspect of *self-reflecting* in relation to looking critically into one's own teaching practice. Thereby, I consider the six different modes of participation, as presented above, as a possible conceptualisation of what metacognitive mediation means in the particular social setting of my research.

4.3.2 Addressing cognitive mediation

As mentioned earlier, Karpov and Haywood (1998) relate cognitive mediation to the acquisition of scientific concepts representing the core of some subject-matter, as they define "mediation of meaning is an essential moment in the acquisition of psychological tools, because psychological tools derive their meaning only from the cultural conventions that engendered them" (Kozulin, 2003, p.26). However, I rather address cognitive mediation in terms of *developing one's own awareness and enhancing one's own understanding of scientific concepts in relation to algebraic thinking* (see Figure 23). Thereby, by adopting this perspective, I am in a position to address cognitive mediation through mediation of meaning and through the different modes of participation. In addition, I argued for taking into consideration the nature of the mathematical tasks. From the formal *a posteriori* analyses of the different workshops, it is possible to follow how the teachers, through engaging with the different mathematical tasks, gradually developed their awareness and enhanced their understanding of the importance of defining a unit when addressing the scientific concept of fraction (Workshop II), and the power of modern algebraic notation (Workshop III). Furthermore, the link between Euclidean geometry and developing an algebraic proof was addressed in Workshop IV, while the difficulties related to the vagueness of mathematical language and the importance of developing an aware-

ness of underlying structure in numerical patterns were emphasised during Workshop V. In addition, I consider that the teachers developed further their awareness concerning the difference between syntactic and semantic translation and were able to emphasise and contrast the use of symbols as an object with symbols used as variable (Workshop VI). Finally, Claire offered the teachers further opportunities to enhance algebraic thinking through addressing importance of the choice of unknown and of the formulation of a task (Workshop VII).

As a summary, I argue for considering both these two aspects, metacognitive and cognitive mediation, as deeply interwoven and mutually constituent within the particular social setting of my research, since as explained earlier (see Section 4. 2), I argued for recognising the crucial role played by the nature of the mathematical tasks in relation to both metacognitive (modes of participation) and cognitive (developing further one's own understanding of scientific concepts) mediation. By using the expression "crucial role" I seek to address the issue of a task fulfilling its purpose since, as explained earlier, I consider that to what extent a mathematical task fulfilled its purpose depends both on the nature of the mathematical task and the social setting, which in my study refers to the stage of development of our community of inquiry. However, it is possible to observe, as during Workshop VII, some tensions in the negotiation of a mathematical task (the first task where the issue was about to *write* the area of a rectangle before and after change), even if the participants had shown strong confidence in our community. Referring to the coding of the this particular task, its non-routine question, due to the way the question was formulated, provoked some difficulties in the negotiation of the meaning of that task (see Section 4. 1. 7). In addition, I want to emphasise the importance of recognising the crucial role played by the *questioning* mode of participation, since, it seems that the teachers needed to ask questions about the functioning of our community of inquiry and Claire's intentions *before* they could engage in inquiry with the mathematical tasks. Therefore, my hypothesis is that this mode of participation is a *necessary* step within the creation and the development of our community of inquiry. In relation to cognitive mediation, I refer to the development of one's own awareness and enhancement of understanding of scientific concepts in relation to algebraic thinking. In addition, during the formal *a posteriori* analyses of the different workshops, I emphasised aspects related to my own learning. In the next chapter, Chapter 5, I propose to deepen these issues and to make visible different elements of my own learning related both to Claire, as a didactician, and as a researcher.

5 My Own Development both as a Researcher and as a Didactician

The purpose of this chapter is to present to the reader some aspects concerning my own development both as a researcher and as a didactician. The rationale for offering this perspective is that this reflection helps me in understanding and articulating my own development both as a researcher and as a didactician. In addition, I understand this chapter as enhancing my thesis since it explains the development of my theoretical perspectives and adds weight to my interpretations and analyses.

I consider it important to develop a deep understanding of my role, acting both as a researcher and as a didactician, in creating a community of inquiry through which both the teachers and myself can learn algebra and develop a better understanding of algebraic thinking. This understanding will then be central for the teachers in becoming better able to create an effective learning environment in their classroom, and for me in becoming better able to create an effective learning environment for the teachers with whom I am working. I agree with Konrad Krainer (2008) arguing for considering teacher educators' reflection on their own practice as a source of motivation for teachers to engage in reflecting on their own teaching practice. Krainer refers to 'an ethical facet' where "we do not only demand activities of those for whose growth we are co-responsible, but we demand it also of ourselves" (p.177). This implies that we, as teacher educators or didacticians, need also to face and grapple with the challenges we propose to the teachers, or using Jaworski' (2008) words 'walking the talk'. Thereby, I see this chapter as a response to Jaworski's (2008) suggestion to reflect on how one's own findings from research had influenced one's own thinking and impacted one's own practice. In my case, the findings emerging from my research have an impact both on my work as a didactician collaborating with teachers, and on my work as a researcher engaging in the practice of researching.

In Section 3.3.3, I presented the main aspects of developmental research and I argued for considering my research as following this methodological approach. Therefore, I understand this chapter as closely related to Freudenthal's (1991) assertion:

... developmental research means: experiencing the cyclic process of development and research so consciously, and reporting on it so candidly that it justifies itself, and that this experience can be transmitted to others to become like their own experience. (p.161)

This is the case in my study since, by offering the reader a reflection over my own learning process during these years as a doctoral student, I am in a position to report on my experience "so candidly" that it might

be explained and shared¹⁵ with others. In addition, this perspective enabled me to meet Freudenthal's (1991) demand for "an attitude of self-examination on the part of the developmental researcher: a state of permanent reflection" (p.161).

The issues I seek to address in this chapter are related to the following question: how can I make and justify my decision to adopt one particular theoretical perspective rather than another, and how did my own understanding of algebraic thinking evolve during my research? In addition I trace my development as it emerged from Chapter 4. However, I want to emphasise that I do not consider it appropriate to address my own development in terms of this aspect having an impact on me *exclusively* as a researcher or that aspect having an impact on me *exclusively* as a didactician. I rather address the different issues, as presented in this chapter, as having respectively a more pronounced impact on me as a researcher or other issues as having a more pronounced impact on me as a didactician. Thereby, I do not want to impose a clear differentiation between my role as a researcher or a didactician, and instead, I argue for considering the deep complementarities and interrelations of both roles.

In the first section of this chapter I recall the rationale for my research. My goal in doing so is to emphasise that my interest in working collaboratively with teachers emerged from insights I developed during the Mathias-project which I initiated, developed and implemented in a primary school before engaging in research. In the second section of this chapter, I explain what kind of data I used in order to address my goal and to trace my own development.

The following sections of this chapter address elements of my own development according to the following dimensions: the elaboration of my theoretical framework (Section 5. 3), the evolution of my own understanding of algebraic thinking (Section 5. 4), and the insights emerging from the analyses of the different workshops (Section 5. 5). Through these three dimensions, I am in a position to point to elements which influenced my own development, both as a researcher and as a didactician, to address central issues as presented earlier in this chapter, and to increase the depth in which research observations and interpretations are made and justified.

¹⁵ I rather use *explained and shared with others* than *transmitted to others* since I want to avoid using a transmission metaphor.

5.1 Recalling the rationale for the study

As presented in Sections 1. 2. and 3. 2. 1, the rationale for this study was a desire to get a better understanding of what ‘working *with* teachers’ might mean in relation to algebra and algebraic thinking. Through my engagement in the Mathias-project, I had the opportunity to experience working collaboratively with teachers, however, as underlined in Section 1. 2, I was not engaged, at that stage, in research since my work was of a developmental nature and, thereby, I was not guided by research questions. Therefore, I consider that this research study has allowed me to look critically at and deepen the idea of “working with teachers” through addressing my research questions, as presented in Section 1. 5. In using the expression “working *with* teachers”, and putting emphasis on the preposition “with”, my focus was on *the interaction between the teachers and me*, and therefore, my goal was to elaborate a theoretical framework which would enable me to conceptualise what “working with teachers” might mean and, at the same time, to pinpoint the specificity of our collaboration which is addressing a *mathematical* setting with a focus on *algebraic thinking*.

I am in a position today where I can articulate and make visible these issues. However, I recognise that my decision concerning the choice of theoretical perspective, when I started to engage in research, was more intuitive in nature and not as well articulated as it is today. I develop further on this issue in Section 5.3.

5.2 Looking back to my own development: What kind of data?

In order to trace my own development I considered three kinds of data. The first one consists of a diary I wrote during the year I collaborated with the teachers, the second relates to the résumés I wrote after each supervision meeting we had, between my supervisors and myself, and finally the last kind of data consists of comparing the informal and formal *a posteriori* analyses of the different workshops (see Chapter 4) as a means to trace my own development.

While working collaboratively with the three teachers during one school year, I wrote some notes in a diary, mainly after each workshop. I already presented excerpts from this in the informal *a posteriori* analyses of each workshop.

Until today (November 2008), I have 15 résumés from supervision meetings. Each meeting was audio-taped and I made some notes from listening to the audio-tapes. These notes, which I called résumés, were presented to my supervisors during the next meeting. Working for the elaboration of this chapter consisted of reading through all these résumés and through the notes from my diary, synthesizing them and identifying

the main features which were emerging in relation to the three main directions, as mentioned above.

5.3 Addressing the way the theoretical framework has been established

As mentioned above, the rationale for my study emerged from my previous experience during the Mathias-project. Thereby, my approach was pragmatic since I was interested in investigating further a particular social setting (my collaboration with teachers) with a particular focus (mathematical learning and more specifically algebraic thinking).

Thereby, I moved from developmental work to engaging in research (see Section 1. 3). As a consequence of this recognition, it seemed that adopting a sociocultural research paradigm was a suitable approach in order to address the collaboration between the three teachers and myself. Furthermore, Wenger's theoretical frame, where the ideas of meaning (Section 2. 2. 1), practice (Section 2. 2. 2), community (Section 2. 2. 3) and identity (see Section 2. 2. 4) are brought to the fore, seemed to be an appropriate theoretical frame within which I could conceptualise elements of our collaboration. However, when engaging in the elaboration of the theoretical framework for my thesis, I decided in advance to follow two criteria in relation to the elaboration of my theoretical framework: the first criterion concerned the *relevance* of my theoretical frame while the second criterion addressed the *coherence* of it.

As explained in Section 2. 5, I used the term *relevance* in relation to my search to elaborate a framework which enabled me to address mathematical learning and more specifically algebraic thinking. However, I recognised quite soon that Wenger's theory offered a conceptualisation of learning in rather general terms and, therefore, I sought to expand his theory further in order to pinpoint the specificity of my research. Now, to expand Wenger's theory means to add another theoretical perspective to Wenger's one. This implies the combination of different theoretical perspectives, and thereby, as I understand it, the necessity to ask questions about the compatibility of epistemological positions present within these different theoretical perspectives.

By introducing the criterion of *coherence*, my goal was to pinpoint the compatibility of epistemological issues I seek to achieve since I considered it crucial to address and make visible this issue. The issue related to the criterion of *relevance* implied that the chosen frame should enable me to address *learning* considered in a social setting with a particular focus on *mathematical* learning, or learning in a *mathematical* context. On the other hand, the demand for *coherence* forced me to realise the affordances and constraints of choosing a situated perspective on learning.

The decision of conceptualising *learning* according to a sociocultural approach was taken from the beginning of my engagement in research since, as explained in Section 1. 2., I was interested in deepening the idea of “working with teachers”, and in bringing out central features of this collaboration. In order to present an overview over the different tracks which I followed while developing my theoretical frame, I consider it important to mention that right from the beginning of the conception of my research study, I introduced the idea of *reflection*. I used this term in an attempt to capture a kind of meta-level activity as resulting from engaging within the mathematical tasks offered the participants during the workshops. This term, *reflection*, played an important role in the conceptualisation of my research, as it influenced the elaboration of my theoretical framework and my research questions. I recognise, today, that the way I used the term reflection did not allow me to address and articulate the complexity which progressively was emerging from my research.

The use of the term *reflection* implied that my theoretical framework had to link an approach to learning from a sociocultural viewpoint, with a focus on learning in a mathematical context and particularly on algebraic thinking, and, at the same time, to articulate how the idea of “reflection” might be understood in this frame. In the following part of this section, I trace the different stages through which I went in order to clarify my position. Then, by the end of this section, I give my interpretation of the different attempts I made in elaborating my theoretical framework. In addition, I develop further the reason why I saw the elaboration of a theoretical frame with the combination of Wenger’s theory with another theory rooted in a cognitive approach to learning as problematic.

Before presenting some extracts from the résumés of supervision meetings, I want to comment on the following aspect: In Sections 1. 5 and 2. 2, I emphasised that the dimension of *inquiry* emerged strongly during the analytical process and became a fundamental aspect of our community. However, before I was able to identify and make visible this dimension, I characterised our community as a *learning community*. This is the reason why the term learning community appeared in several of my earlier résumés.

In the résumé from 26.11.04, the articulation of the research questions is questioned, and I mentioned the need for having emphasis on the crucial relation between algebraic thinking and our community:

In order to make the deep connection between algebra and communities more visible, I have to refine my research questions.

My analysis has to be related to my research questions:

- To which extent is it possible to provoke teachers’ reflections concerning algebra through the creation of a learning community?
- What are the central features of this learning community?
- What are the central features of teachers’ reflections concerning algebra?

I consider that the challenge is to be able to draw general characteristics concerning learning communities from the observations of our community.

(Résumé of supervision meeting, 26.11.04)

From these notes, it is possible to see that my focus, at that time, was on *teachers'* learning and more specifically on *teachers'* reflections. This focus has gradually evolved over time and I am in a position, today, where I recognise the importance of addressing *both* the teachers' and my own learning. Looking at my résumé from supervision meeting in November 2004, it seems that the idea of "reflection" played an important role, since I addressed the collaboration between the teachers and myself in terms of "to provoke teachers' reflections" and to look for "central features of teachers' reflections".

A critique of the formulation of the research questions is addressed in the résumé of our next supervision meeting as I wrote:

By looking at the way these [research questions] are formulated (to which extent ...), the answer should give a kind of measure. This kind of question is not researchable and might be replaced by a "what is the nature" question. In that sense, a "what" question allows me to look at teachers' reflections and I can say something about it. I can ask: what is the nature of these reflections because I can observe them. So, I am asking about something observable.

(Résumé of supervision meeting, 28.01.05)

I consider the recognition of what is a researchable question as an important step in my development, as a researcher. The formulation of research questions is so determinant for the research project, as the study aims to answer these, that their formulation needs to undergo several phases of refinement. At the same time, the formulation of suitable research questions is deeply related to the choice or elaboration of the theoretical framework. I develop this issue further by the end of this section. In order to help me during the process of reformulation of the research question, I tried to articulate the purpose of my study in a more visible way:

A way of formulating research questions is to make clear and explore the aim of the research. The aim of the research is to look at the deep connection between algebra and communities, or more precisely to look at the way in which developing a community of this sort can lead towards a deeper understanding of algebraic thinking. The first research question might be refined as: what is the nature of teachers' reflections, and how do these relate to the creation of a learning community?

(Résumé of supervision meeting, 28.01.05)

Thereby I had three ideas emerging from my research questions which had to be linked through the elaboration of my theoretical frame. As I mentioned before, I had already decided to adopt Wenger's theory concerning *Communities of Practice*, so the challenge consisted of adding a theoretical perspective which would allow me to focus on learning in a mathematical context and to account for how the idea of "teachers' reflections" fits in that frame.

In the following, I present the impact three doctoral courses (doctoral course in Copenhagen during Spring 2005; a summer school in Norway in June 2006; and a seminar in London focusing on Tall's three worlds in December 2006) had on helping me to develop further my thinking. I chose to present these three since I consider they had a major influence on the development of my research in terms of helping to refine my thinking concerning the elaboration of my theoretical framework.

I proposed a tentative answer to the challenge offered by the elaboration of my theoretical frame after following a doctoral course in Copenhagen concerning French Didactics. Through this course the work of Brousseau, Chevallard, Duval, and Vergnaud was presented, and as an assessment, we had to write an essay in which one of the four French theoretical frames was supposed to be linked to our own research. At that time, it seemed to me that Duval's frame might be relevant in addressing the *mathematical* aspect of learning. Especially, during the doctoral course in Copenhagen, I presented Duval's (2000) article "L'apprentissage de l'algèbre et le problème cognitif de la désignation des objets". It was clear from the way Duval presented his work, arguing for the need for students' developing new cognitive practices when introduced to algebra, that Duval's theoretical framework was articulated within a cognitive approach to learning. As explained above, adopting his approach might create problems in relation to the criterion of *coherence* when combining different theoretical perspectives. An evaluation of the essay is reported in the résumé of our next supervision meeting as I wrote:

The purpose of the essay was to use at least one of the French theoretical frameworks in our study. I tried Duval's semiotic approach as a way to address algebra in my study. A central question is: why is a cognitive approach relevant here, and how does the theory of Duval help me in my analysis of Paul's notations? (Résumé of supervision meeting, 04.03.05)

My last question was referring to the notations introduced by Paul (e. n. and o. n.) during Workshop II about even and odd numbers, as I was searching in research literature for a possible conceptualisation of Paul's notation. By questioning the relevance of a cognitive approach, I addressed epistemological issues concerning the commensurability of a sociocultural approach to learning and a cognitive approach to *mathematical* learning. I wrote further in the résumé:

I have to reflect on the issue of ways of linking the two parts of my research: the idea of community, community of learners, in which Lave and Wenger (1991) and Wenger (1998) are related, and I have the development of algebraic thinking and concepts in which I relate Duval. But is the work of Duval helping me in the understanding of the community? (Résumé of supervision meeting, 04.03.05)

In addition to the difficulties related to the introduction of Duval's cognitive approach into my theoretical frame, I also struggled with the idea of

“reflection”. This challenge is addressed in the résumé of our next supervision meeting, where I started questioning “reflection” and its origin:

Concerning the notion of “reflection”:

Where does this notion come from?, and how is it related to the notion of community? An answer could be that it is part of the expectations of the community that people will reflect (this was introduced by me right in the beginning when I asked the teachers if they wanted to collaborate with me).

(Résumé of supervision meeting, 08.04.05)

It seems, from this excerpt, that the idea of reflection has been part of the design of the study since the beginning, and my notes in my diary confirm this aspect:

I feel quite nervous today as I am going to meet the three teachers [Mary, Paul, and John] for the first time. I really hope that the activities related to the Cuise-naire-rods will be interesting, that we will have a good discussion and engage in reflections, and that the teachers get a positive impression of our workshop with a willingness to continue during the next semester.

(Notes from my diary, 16.06.04, translated from French)

I suppose that when I wrote these notes in my diary, I used the idea of “reflection” to convey a kind of activity at a meta-level: in the sense of engaging within the tasks presented during the workshops, and then reflecting on these. At that time, deeper considerations concerning the elaboration of a coherent theoretical framework within which “reflection” had to fit, were not part of my agenda.

My next attempt to build a coherent framework was reported in the résumé of the supervision meeting from 09.12.05. This attempt still relates to the analysis of Paul’s notation from Workshop II and was inspired by Radford’s (1996) chapter: *Some reflections on teaching algebra through generalization*. Here, Radford questions the status of knowledge which is obtained by generalisation. In order to address this issue he proposed to consider the epistemological status of generalisation and the nature and complexities of generalisation as these appear in the algebra classroom. My questions, in considering Radford’s model were:

Concerning the analysis of Workshop II:

Radford (1996) tries to theorize the procedure of generalization but it seems that there is a need for more clarification concerning what the different levels mean. The same question arises concerning what a “symbolic system” means. Looking to the notation used by the three teachers (e. n. and o. n.), we could say that, in a primitive way, they have a symbolic system, but it is not *the* symbolic system that I would like ($2n$ and $2n+1$) them to have. Then the question is: How does Radford’s model differentiate between these two kind of symbolic systems? In my analysis I wrote: The difficulties arise when the result has to be written using algebraic notation. But what do I mean by *algebraic notation*, and is there some kind of characteristics I can expect from some kind of notation that I could call *algebraic notation*? As I see it, the notation introduced by Paul is not functional, that means no operation can be performed on it. In order to define what I mean with “algebraic notation” the term functional could be one of the characteristics

algebraic notation has to fulfil. What are the other characteristics? These issues are not addressed in Radford (1996).

(Résumé of supervision meeting, 09.12.05)

I consider that through these extracts it is possible to follow my struggle in addressing the nature and qualities of a notation and, especially Paul's notation. As this extract shows, the issues related to building a coherent and relevant theoretical framework are deeply related to the way I define algebraic thinking and algebraic notation. Considering this struggle today, I see that finding "an answer" to how to understand Paul's notation was not *only* a matter of immersing myself more deeply into the research literature, but also of looking critically into my *own* understanding of algebra and algebraic thinking. These issues are addressed in the next section in this chapter. As such, I am now in a position where I can identify my struggle as evidence of the interdependency of the issues related to the elaboration of my theoretical framework and the articulation of how algebraic thinking is addressed in my research.

As my writings were taking form, I presented to my supervisors a chapter within which I tried to relate theoretical, and methodological considerations with analysis of data. The aim was to address clearly and make visible the link between the different aspects of my study. The résumé of the supervision meeting shows a consideration of using the Teaching Triad (TT), as developed in Potari and Jaworski (2002), in an attempt to conceptualise the relationship between the three teachers and myself. From my diary:

I can see that the degree of complexity in my research is rapidly increasing. Therefore I am concerned about ways of controlling/limiting this development, or at least looking critically into it. I want to stop and to go back and take a critical look at my theoretical and analytical framework:

1. I want to further develop Wenger's theory. Is it necessary, can I give some justifications for this? Yes, I can; my work is fundamentally about observing and describing the *creation* of a learning community and these aspects are not directly addressed by Wenger.
2. I have developed an analytical frame consisting of three different layers (modes of participation, focus on reflections, algebraic thinking). Is it necessary to have these three layers? Can I give some justifications for this? Yes, I can, I want to address *both* the developmental aspect of our learning community and the development of algebraic thinking. I consider that, in mathematics education the *mathematics* has to be visible. Therefore the analytical frame as presented allows me to address both social aspects (development of our learning community) and mathematical aspects (algebraic thinking).
3. In relation to the analysis of Mary, Paul, and John's reflections I consider the possibility to use the TT, as exposed above. Is it necessary to use the TT, and can I justify its use? what are the other alternatives? Here I am not so sure and need to read more about the TT and reflect about how it would fit in my work before I can give any answer.

(Notes from my diary, 01.06.06, translated from French)

These notes, as presented above, were included in the chapter I sent to my supervisors and, in the résumé of the supervision meeting (22.06.06), the issue related to *coherence* was critically addressed:

About the draft of the analysis chapter:

In this draft I had mixed theoretical, methodological considerations, and analysis of data with my own reflections (excerpts from my notes, 01.06.06). In this draft I express my concerns (p.16) about the increasing complexity of the theoretical and analytical framework that I am using or consider to use. It includes:

Wenger's "community of practice", my analytical framework consisting of three layers (modes of participations, teachers' reflections, algebraic thinking), Frege's analysis of meaning (as consisting of two complementary phenomena: denotation and sense) and the possibility to use the Teaching Triad. I question the relevance and the adequacy of these different perspectives. In my view the *coherence* within my work is one of the most important issues to address within a PhD-thesis, and therefore I want to look really critically and justify the need for these theoretical frameworks before using them.

It seems also, emerging from my analysis that I am able to answer to several issues other than those addressed in my research questions. More particularly, I can follow the process of *building our learning community* through the different modes of participation.

(Résumé of supervision meeting, 22.06.06)

From the perspective I have today, I can see myself searching in the research literature and struggling with the elaboration of a suitable frame which might allow me to address algebraic thinking without introducing epistemological tensions. Another aspect which is reported in this résumé concerns the results of data analysis as it seems that the richness of the data enables me to get insights into the processes related to the creation and development of our community. As mentioned earlier, I did not use the term *community of inquiry* since the dimension of inquiry had not yet emerged.

Another possibility, concerning the elaboration of my theoretical framework, is presented during the same meeting. I had just participated in a summer school in Norway (June 2006) and I present one of the suggestions I got there since I consider that it helped me to address fundamental epistemological considerations in my research. The suggestion related to the possibility to include Tall's (2004) three worlds of mathematics in my framework in order to address the development of algebraic thinking. I referred to this possibility in the résumé of the supervision meeting:

Before considering the possibility to use Tall's framework in my work I need to go to some depth epistemologically in order to rationalize Tall's worlds (grounded into the work of Piaget, Dienes, Bruner, Skemp) and a socio-cultural based research.

One of the main challenges for my study will be to be able to combine, in a coherent way, the participational and the algebraic aspect of this research. Maybe Tall's framework is helpful, it is potentially helpful, it is also potentially dangerous, if I try to work with two (incompatible) paradigms at the same time! To go

to some depth epistemologically means to address questions like: where is knowledge? Epistemology is about *the nature of knowledge* and *how* we see knowledge, *where* we see knowledge. Taking Wenger's socio-cultural practice where knowledge is in the practice and comparing with Tall's three worlds theory where knowledge comes through the construction of the individuals, then these offer two completely different ways of seeing and conceptualizing knowledge, these theories have a different measure, they are incommensurable. Coming back to my study (the three teachers and myself and how I see knowledge growing within the group), it is relevant to ask: Do I consider knowledge as being rooted in the activity of our group or in our practice we are engaging in? or do I see knowledge as a construction within the mind of the individuals? The same kind of questions could be asked taking the notion of reflection: what does "reflection" look like if we (researchers) try to characterize this notion in a socio-cultural way? what does "reflection" look like if we (researchers) try to characterize this notion in a constructivist way?

*I consider these questions as essential because I realize that even if I have situated my research within a socio-cultural approach I am still thinking about "knowledge" and "reflections" in constructivist terms. Therefore, being able to articulate and to make visible this *décalage* (mismatch) is crucial for my study.* (Résumé of supervision meeting, 22.06.06)

Through my repeated confrontation of epistemological tensions, I gradually realized that I had to look critically into my *own* understanding of "knowledge", "algebraic thinking" and "reflection", and that these difficulties might reveal the challenges I faced by adopting a theoretical perspective concerning learning, by following Wenger, which was really new and unusual for me. This issue is addressed further during the next supervision meeting:

Concerning the possibility to conduct the analysis in Tall's terms: the issue is not that I can't use Tall's framework, it is possible to include Tall's three worlds of mathematics, but I have to account for it. This means to make visible and to look critically into the sharp edges of the theories and to pull out the kinds of challenges the combination of these two theories (Wenger's and Tall's) address. (Résumé of supervision meeting, 22.08.06)

In addition to the challenge consisting of integrating Tall's theory in my theoretical frame, I had to look through the research literature in order to trace the way the idea of "reflection" had been addressed:

A lot has been written about reflection from a constructivist approach because it is related to cognition. Sociocultural theories do not ignore the cognitive aspect, but it is addressed differently because they have a different epistemological approach. From my data, it seems possible to pull out instances of the teachers reflecting together, where these instances are deeply socially rooted. Here, I have to define what I mean by the term *reflection*.

Another important aspect is to realize that the domain of my study is *the enhancement of teachers' algebraic thinking*, but it seems that I put a lot of emphasis on the notion of *reflection*, and the purpose doesn't say anything about reflection. I have to look critically into the way my research questions, in which reflection is central, are related to the purpose of my study. (Résumé of supervision meeting, 22.08.06)

As underlined in the last sentences from the résumé, it seems to be a mismatch between the purpose of my study (studying the collaboration between the three teachers and myself in a social setting focusing on algebraic thinking) and the way the research questions are formulated. One alternative was to refine my research questions, another is to reconsider the relation between the research questions and the purpose of the study and to “define what I mean by the term *reflection*”. In an attempt to address this last issue, I read and wrote about the idea of “reflection” from the perspectives offered by Dewey, Piaget, Skemp, Freudenthal, Schön, and Carr and Kemmis. My aim was to look for possibilities to link “reflection” to Wenger’s framework. Looking back to the struggle I experienced in elaborating a *coherent* theoretical framework, I recognised that, behind the difficulties related to the term *reflection*, I was addressing the following question: how is the mind addressed from a constructivist perspective compared to a sociocultural approach. Using Cobb’s (1994) terms, I was struggling with the question: where is the mind? According to Cobb, these kinds of claims, made by adherents of constructivist or sociocultural perspectives on learning who consider the mind as either in the head or in the individual-in-social-action, are essentialist assumptions and involve a denial of responsibility. Therefore, he argues for presenting “pragmatic justifications [which] reflect the researcher’s awareness that he or she has adopted a particular position for particular reasons” (Cobb, 1994, p.19). This was the case in my study: as explained earlier in this chapter, I started engaging in research with a desire to develop a deep understanding of what “working *with* teachers” might mean. This was the reason why I find necessary to focus on the interaction and collaboration between us (the three teachers and I) and not on the actively cognising teacher.

The next challenge was related to how to address algebraic thinking. Issues concerning the commensurability of Wenger’s and Tall’s theories are addressed through the résumé of the supervision meeting:

My main concern in writing the second draft of this chapter [theory chapter] was to respond to the comment: How does all this fit into your overall theoretical perspective?

My point is the following: My research is rooted in the work of Lave (1988), Lave and Wenger (1991), and Wenger (1998), and these perspectives offer a general view of learning (situated learning, learning as increased participation). But I am primarily looking at *mathematical* learning and this focus has to be visible and addressed in the theoretical framework. As a consequence of this focus, the issue at stake is how to address and to overcome the differences between the epistemological bases for each of these theories (Wenger in a sociocultural approach to learning, Tall in a constructivist one). This is the reason why I need to talk about Piaget, Bruner, and Skemp in my theory chapter.
(Résumé of supervision meeting, 20.10.06)

In my notes, I also remarked that, by its nature, mathematics education has to face these epistemological tensions:

One could ask here if this is necessary [to address epistemological tensions], or if it is possible just to take Tall's theory and use it. I consider this issue to be really important for mathematics education as an interdisciplinary field and, going further, the scope of my work: many concepts that we (researchers) use are taken from other fields (psychology, mathematics, sociology, philosophy, social anthropology, etc.). Can these notions be taken as such and used/operationalized in mathematics education, or do we need to re-define and examine their epistemological basis before adopting them in our theoretical frames? I think that this question deserves to be addressed in a critical way.

(Notes from my diary, 27.11.06, translated from French)

Furthermore, the struggle consisting of establishing a relation between Wenger and the idea of "reflection" is underlined during the next meeting:

Writing about reflection, I quoted Dewey, Piaget, Skemp, Freudenthal, Schön. These authors relate to a more constructivist way of thinking. Kemmis doesn't fit in this description, Kemmis relates to critical theory, and the position of critical theory takes us behind positivism and interpretivism into taking a critical viewpoint. This position could be related to the critical viewpoint in the development of the thinking within our group. The link between what I wrote on reflection and Wenger's theory needs to be further developed. In my theoretical framework I have Wenger, Kemmis (reflection), and Tall. How do they fit together?

(Résumé of supervision meeting, 05.12.06)

By that time, I felt that it was difficult to move further, since I considered that I had to look critically into epistemological issues and to make clear potential incompatibilities between the different theoretical approaches. I acknowledged the importance of this issue and, at the same time, I had to recognise that my research was not exclusively of a theoretical nature but primarily rooted in the practice I had established with the three teachers.

This was the reason why I decided to consider the possibility to address *mathematical* learning from a sociocultural approach to learning by introducing Vygotsky's theoretical frame in order to address algebraic thinking. In an attempt to refine my thinking I made an overview of two possible theoretical frames: Wenger and Tall versus Wenger and Vygotsky and looked critically into the advantages and disadvantages in each case (see the following two pages). I consider, today, this initiative as particularly useful as it helped me to realise, recognise, and question my assumptions concerning the role played by a theoretical frame.

Theoretical framework with Tall's three worlds (Wenger + Tall)

Advantages:

- Wenger's theory addresses learning in general terms,
 - Tall's theory allows me to specify what learning in a mathematical context means.
 - Tall's notion of "journey" (Tall, 2004, p.285) could be exemplified by
 - the results of my research through the different modes of participation
 - and by the different aspects of what I called "negotiation of didactical/pedagogical strategies"
- ⇒ therefore I can see how Tall's three worlds could be useful in the analysis of my data.

Disadvantages:

Essentially of epistemological nature:

- Wenger's theory is rooted in a socio-cultural view on learning, learning as a participation in social settings
- Tall's theory is rooted in a cognitive view on learning, mainly emerging from Piaget and Bruner
- since I consider the issue of *coherence* as central in my work, I want to present a good justification for the possibility to elaborate a coherence frame in which both Wenger and Tall are included
 - therefore my plan was to look at the possibility to understand Tall's three worlds from a Vygotskian perspective with
 - Tall's first world \approx Vygotsky's spontaneous concepts
 - Tall's second world \approx Vygotsky's psychological tools
 - Tall's third world \approx Vygotsky's scientific concepts

Now the question is: is it possible for me to give an in depth justification for this? I can see a danger here of being too ambitious and to give only a superficial justification. Both Confrey (1995)¹⁶ in her article *How compatible are radical constructivism, sociocultural approaches, and social constructivism*, and Wertsch and Penuel (1996)¹⁷ in their article *The individual-society antinomy revisited: productive tensions in theories of human development, communication, and education* have addressed similar issues and in her conclusion Confrey warns: "although the theories of Piaget and Vygotsky are undeniable useful in analyzing schooling, the frameworks are not easily reconciled at a deep theoretical level" (p.222). I need to take this warning seriously!

Now do I need Tall's theory at all? What aspects are offered by Tall's theory which are not present in Vygotsky's theory?

(Résumé from a seminar in London focusing on Tall's three worlds of mathematics, 18.12.06)

¹⁶ in *Constructivism in Education*, edited by Leslie Steffe and Jerry Gale.

Theoretical framework without Tall's three worlds (Wenger + Vygotsky)

Advantages:

- since Wenger's theory is rooted in a Vygotskian perspective on learning, these two theories belong to the "same family of learning theories"
 - ⇒ the epistemological obstacles mentioned in the previous alternative are avoided
 - Vygotsky's theory allows me to focus on learning in a mathematical context, but of course with a different perspective than the one adopted by Tall.
 - as underlined by Vygotsky (1986, p.146)¹⁸, Schmittau (2003, p.226)¹⁹, Kozulin (2003, p.35)²⁰, and Karpov (2003, p.65-68)²¹ the role played by instruction in the acquisition of scientific concepts is crucial. This aspect of the theory could be exemplified by
 - looking at my own role within our learning community and
 - how the scientific concepts discussed during the different workshops are *consolidated* (not introduced) by putting emphasis on the role played by algebraic symbolism. (This aspect could be related to Lerman's (2001)²² remark concerning Even's et al. (1996)²³ article: "Learning is through cognitive conflict, brought about by the situations the teachers encounter. Thus Piaget's model of learning through adaptation is extended into *adult learning*. Is this a suitable model?" (p.42, my emphasis)).
- ⇒ therefore I also can see how Vygotsky's theory could be useful in the analysis of my data.

Disadvantages:

- in a sense I miss the hierarchical organization as described in Tall with a progression through the three worlds, but why is this aspect important for me and therefore how do I understand mathematical progression?
 - why did I find Tall's three worlds so attractive?
 - ⇒ I need to make visible my own underlining assumptions concerning mathematics

These aspects, as presented in this résumé, will be further elaborated and developed in our next supervision meeting.

(Résumé from a seminar in London focusing on Tall's three worlds of mathematics, 18.12.06)

¹⁷ in *The Handbook of Education and Human Development*, edited by David Olson and Nancy Torrance.

¹⁸ in *Thought and Language*, edited by Alex Kozulin.

¹⁹ in *Vygotsky's educational theory in cultural context*, edited by Alex Kozulin et al.

²⁰ in *Vygotsky's educational theory in cultural context*, edited by Alex Kozulin et al.

²¹ in *Vygotsky's educational theory in cultural context*, edited by Alex Kozulin et al.

²² in *Making sense of mathematics teacher education*, edited by Fou-Lai Lin and Thomas J.Cooney

²³ in PME proceedings, Spain, I, 119-134.

This summary, as presented in the two pages above, turned out to be really helpful. Especially by addressing the disadvantages in Wenger's and Vygotsky's frame, *I realized that I needed to look critically into my own conceptualization of mathematics*. Today I am able to make the conjecture that it was not by coincidence that I tried several frames which were rooted in a cognitive view on learning, these corresponded to my own personal understanding of the idea of learning, and as long as I did not open for looking critically into my own assumptions, I was, in a sense trapped, in these tensions.

Another important challenge was addressed during our first supervision meeting in January 2007. I have been struggling with the idea of "reflection" since the beginning of my study, as my notes from June 2004 show. These notes were written just before meeting the teachers, and I already referred to the idea of "reflection". As mentioned through this section, I found it really difficult to link this concept in a coherent way to my theoretical frame. This central issue was addressed during the January meeting and the question of the use of terminology, and especially in relation to "reflection" was discussed:

There is "something" going on within our community, and until now I called "it" *reflection*. Because I used this notion of "reflection" I thought it was necessary to look back in the literature and to trace the antecedents of "reflection". The problem was to define "reflection" as a community act.

Coming back to the data, is it possible to characterize what is going on in other terms? Is it possible to see "it" in terms of the new unit of analysis (our community)? Is there some notions in Wenger that could be useful? What about using the notion of "inquiry"?

(Résumé of supervision meeting, 26.01.07)

It might appear that the process of establishing a coherent and relevant theoretical framework is linear, finite and straightforward. In practice, at least as I experienced it, it was cyclic and complex, in the sense of trying again several different frames to focus on *mathematical* learning and ways to integrate the idea of "reflection" that would "fit" with my frame. Therefore, by questioning the antecedents of "reflection" and rather referring to the processes which were going on in our community as "something", was really helpful as I realized that I was trapped in a particular terminology.

Before addressing my own development concerning algebraic thinking, I would like to go back to my struggle while elaborating my theoretical framework, as described above. Today, I am in a position where I am able to recognise and to identify what, I perceive, are the reasons behind this struggle. I referred earlier to the possibility that the choice of different frames (Duval; TT; Tall), all rooted in a cognitive view of learning, originated in my personal understanding of the idea of learning. I consider, today, that choosing a particular theoretical framework might

be understood as putting “theory to work”. By using these terms, I seek to refer to the fact that the choice of a particular theoretical frame also implies the choice of particular types of research questions, of a particular kind of phenomena which might be investigated, and of a particular kind of knowledge which emerges from the research (Cobb, 2007). Retrospectively, I recognise that the choice of theoretical frameworks rooted in a cognitive approach to learning was more intuitive in nature than critically addressed. A possibility might be that this cognitive tradition corresponds to my own culture and understanding of learning, which I never expressed so clearly as today, and, as using Sfard’s (1998) formulation, when a metaphor “is so strongly entrenched in our minds that we would probably never become aware of its existence if another, alternative metaphor did not start to develop” (p.6). This was the case for me, as a researcher, and I am in a position to recognise, today, that the process of becoming aware of another metaphor or perspective, in my case the *participation* metaphor, and its consequences for the research process were challenging. However, my goal is *not* to claim that it is impossible to combine different theoretical approaches rooted in different epistemological perspectives on learning, I recognise only that this enterprise would have been beyond the scope of my research. All I can say is that engaging within “the process of comparing and contrasting (theoretical) perspectives provides a means both of deepening our understanding of the research traditions in which we work, and of enabling us to de-center and develop a basis for communication with colleagues whose work is grounded in different research traditions” (Cobb, 2007, p.7). I consider this recognition as a central aspect of my own learning as a researcher.

5.4 Addressing the way algebraic thinking is understood

The role played by the idea of “algebraic thinking” is crucial in the thesis, but what do I mean by “algebraic thinking”? Furthermore, did my understanding of algebraic thinking evolve during this research?

Through this section I propose to consider my own development in terms of understanding what “algebraic thinking” means and, in order to trace my development, I start from the way the different mathematical tasks proposed to the teachers during this year have been chosen. I recall (see Section 3. 3. 3) that I defined my research as following a developmental research methodological approach and one of the central aspects is the idea of “thought experiment”. Cobb (1998), referring to Gravemeijer’s (1994b) writings concerning developmental research, underlines that in instructional development:

... the designer initially conducts an anticipatory thought experiment in order to formulate conjectures about both 1) possible trajectories for students’ learning

and 2) the means that might be used to support and organize that learning. These tentative conjectures are then tested and modified during the teaching experiment on the basis of an ongoing analysis of classroom events. It is here that the second major aspect of developmental research, classroom-based analysis, comes to the fore. (Cobb, 1998, p. 33)

The implications of this quotation for tracing my own development are the following: what kind of “anticipatory thought experiment” did I perform in the *a priori* analysis; that is what kind of “possible trajectories for teachers’ learning” and what kind of “means that might be used to support and organize that learning” did I imagine? As addressed in Section 3.4.3, the mathematical tasks I proposed to the teachers were chosen as pedagogical means to address the following didactical aims:

- The choice and use of symbols
- The power of symbolic notation
- Some historical perspectives on algebra
- Exploration of the relation between geometry and algebra
- Addressing the difference between a syntactic and a semantic translation

It seems that in my “anticipatory thought experiment”, the role played by symbols is emphasized through these different tasks, and was a major concern when I chose the tasks. In the short evaluation I did after each workshop the use or eventually the lack of use of algebraic symbolism was a central criterion for the choice of the next task. In that sense, I anticipated that the possible trajectories for teachers’ learning had to include the use of algebraic symbolism. Therefore, it might be useful to recall an extract from a supervision meeting where I shared my concerns in relation to the research project, as a whole:

I wanted to share with you both my reflections concerning the research project and especially the fact that even if we (the teachers and I) have been through the cycle several times, I wonder what if nothing happens? Perhaps I had a feeling of frustration and disappointment and I tried to understand if it was because of the tasks I proposed or should I separate the three teachers and look at the development for each of them? More generally I had a lot of questions and it was good to share with both of you my worries. Perhaps I expect too much, too soon!! Now what I get does not fit with my expectations, but I can learn something from that too.

(Résumé of supervision meeting, 28.01.05)

I consider that my claims “I wonder what if nothing happens?” and “Perhaps I expect too much, too soon” reveal a lot about my *own* expectations and understanding of the research project, at that time, and these reactions indicate a gap between my “anticipatory thought experiments” and the actual realisation of the workshops. I consider that two aspects are central here: first, how did I understand “teachers’ learning”, and second how did I understand “teachers’ algebraic thinking”?

As explained in the previous section, adopting the theoretical approach on learning as situated was a process which gave me challenges. I recognize, today, that my understanding of learning was close to a cognitive approach and, using Sfard's (1998) words, which might be characterized as:

Since the time of Piaget and Vygotski, the growth of knowledge in the process of learning has been analyzed in terms of concept development. Concepts are to be understood as basic units of knowledge that can be accumulated, gradually refined, and combined to form ever richer cognitive structures. The picture is not much different when we talk about the learner as a person who constructs meaning. (Sfard, 1998, p.5)

Therefore, as explained by the end of Section 5.3, by developing an awareness of the meaning and implications of the "participation metaphor" (Sfard, 1998) forced me to realize that I experienced, what I perceived as, a kind of incompatibility between my research project and my own previous assumptions. In other words, I realised that I was socialised into a constructivist view of learning. Addressing these questions helped me to make these tensions visible and to understand, more deeply, the implications of considering learning from the "participation metaphor". According to Sfard (1998), this transition might be addressed as:

The terms [knowledge, concept] that imply the existence of some permanent entities have been replaced with the noun "knowing", which indicates action. ...

The talk about the states has been replaced with attention to activities. In the image of learning that emerges from this linguistic turn, the permanence of *having* gives way to the constant flux of *doing*. (p.6)

I consider that this quotation expresses clearly the shift in perspective on learning which was necessary for me to go through in order to be coherent with my theoretical frame.

The second aspect addresses "teachers' algebraic thinking". By asking: "Perhaps I expect too much, too soon", my concern was related to the fact that the teachers seemed to engage in the tasks without necessarily using symbolic notation. I presented this issue to the supervision meeting in November 2005:

Looking back to the workshops that I had transcribed and these two workshops (III and V) I had the impression that the use of algebra was very limited in that sense that the teachers solved the tasks *without* using algebraic notation, for example in the fish problem (workshop III) and the task about palindromes (workshop V). To be honest I was a little depressed when I came to our meeting.

These considerations raise the following questions:

- what does this (teachers not using algebraic notation) mean for my study?
- what does this mean in a broader sense?
- why am I concerned? after all these are my data and I can't change them.
- what is the basis of my concern?

Having in mind these questions forced me to recognize the fact that I was expecting the teachers to solve the tasks using algebra. To be more precise my expecta-

tions were that teachers' conception of algebra was close to mine. The next question is: what is my conception of algebra? I guess for me algebraic thinking is: to look at patterns, being able to express generality using variables, being able to recognize unknown quantities and express them in general terms using variables. Perhaps I was unconsciously expecting/hoping that the teachers would have solved the tasks in a similar way as I do? Being now aware of this I can clearly see how important it is to make visible the preconceptions/expectations we have as researchers.

(Résumé of supervision meeting, 11.11.05)

and from the supervision meeting 27.01.06:

The central question is: does algebraic thinking require fluency with the symbols? (cf. Kieran's (1989a) article). The implication of Kieran's claim is that just generalizing is NOT doing algebra! but maybe this claim could be challenged (young children developing their algebraic thinking without going into the formalities of symbolization). For Kieran algebra requires the formal symbolization, but if I (only) learn to manipulate symbols, am I doing algebra?

I consider these questions as crucial, and particularly I understand my own reflection concerning the necessity to address and make visible my own preconceptions and expectations as a step into taking a critical stand in research. In addition, maybe Kieran's (1989a) quotation could be challenged. Recalling her claim:

I suggest that, for a meaningful characterization of algebraic thinking, it is not sufficient to see the general in the particular; one must also be able to express it algebraically. Otherwise we might only be describing the ability to generalize and not the ability to think algebraically. Generalization is neither equivalent to algebraic thinking, nor does it even require algebra. For algebraic thinking to be different from generalization, I propose that a necessary component is the use of algebraic symbolism to reason about and express that generalization. (p.165)

As presented in Section 2. 5. 4, the importance of abstractions and generalisations of certain aspects of numbers is also emphasised by Sierpinska (1993a), arguing for considering that "algebraic thinking develops upon the arithmetic thinking and transcends it through generalization" (p.105-106). She develops her view further in the following quotation:

Algebraic thinking is based on the generalization of one's own arithmetical operations and thoughts and is, therefore, characterized by free acting in and on the arithmetical domain. In algebra, arithmetic expressions can be transformed, combined according to the general laws of arithmetic operations and not just calculated, "executed" as in the frame of arithmetic thinking. Operations are independent from the particular arithmetic expressions they are involved in. For an arithmetically thinking schoolchild $2+3$ is 5, period. For the algebraically thinking adolescent $2+3$ is a particular case of $a+b$, where a, b are any real numbers. For the algebraically thinking adolescent, *arithmetical operations are special cases of the more general algebraic notions*. (p.106, my emphasis)

I understand Sierpinska's quotation as emphasising the contrast between considering $(2+3)$ and $(a+b)$ as *mathematical objects*, and $2+3$ as a *mathematical procedure* meaning adding 2 and 3 with the answer 5, in other words considering 5 as a result of the process of adding 2 and 3.

Thereby, the focus is moved away from a procedural to a structural component of algebraic thinking.

I recognize that my own understanding of algebraic thinking has evolved and developed during the research process and, as a result, I am able to present a conceptualization of algebraic thinking which is in accord with Sierpinska's claims. Therefore, in relation to my first research question (In what ways is the development of algebraic thinking related to the development of our community of inquiry?), I consider that this section presents evidence of how the development of our community of inquiry did influence the development of my *own* understanding of algebraic thinking.

As explained in Section 2.5.4, I use the idea of algebraic thinking in the following sense: By addressing and developing algebraic thinking, I mean to focus on the need, the choice, the introduction, the use and the meaning attributed to algebraic symbolism and on the way these various components of algebraic thinking are addressed and negotiated within our community through inquiry acts. Thereby, I understand my own development as moving from a focus on symbol manipulations as a means to explore particular algebraic structures to developing an awareness of the importance of the discovery, exploration and investigation of patterns, aiming to grasp and express some algebraic structure. Therefore, I am in a position, today, where I can characterise the ability to express the observed structure by using standard algebraic notation as a result of algebraic thinking and not as a condition *sine qua non* for it. In other words, understanding where I came from (Master thesis on Galois Theory) helped me to look critically into the tensions I experienced in my research. This recognition enables me to understand my reaction, as presented through my claims "Perhaps I expect too much, too soon", as it was necessary for me to re-examine critically how I consider algebraic thinking.

5.5 Elements of my own learning emerging from Chapter 4

I consider that aspects of my own learning, mainly in relation to me as a didactician, are visible from Chapter 4. I proposed first to compare the informal with the formal *a posteriori* analyses of the different workshops and second to identify elements of my own learning from the formal *a posteriori* analyses.

As one considers the *a posteriori* analyses of the nine workshops, it becomes clear that there is an important difference between the informal and formal *a posteriori* analyses. As explained in Chapter 4, I wrote the informal *a posteriori* analyses just after the workshops and it seems, today, that I noticed my impressions in an intuitive way, using terms as

“good feelings” or “nice workshop” (Workshops I, IV, and VI), valuing the quality of our discussions concerning pupils’ difficulties with different aspects of algebra (Workshops I, II, III, V). At the same time I was able to recognise some challenges as “how to move from numerical examples to generalisation” (Workshops II and V), “how to encourage the teachers to introduce and use algebraic notation” (Workshop III). It is also possible to see that I recognised some elements of our participation “something was happening but I am not sure about what it was” (Workshop IV). I argue that, even if these informal *a posteriori* analyses were rather short, they offer evidence of the recognition of some central elements within our collaboration. It was after engaging in the process of conducting an in-depth analysis of each workshop that I was able to pull out these elements and to search for possible explanations and relations between these elements. In the following part of this section, I propose to consider the formal *a posteriori* analyses of the different workshops and to identify elements of my own learning.

The first aspect concerns the teachers’ reaction to the proposed mathematical tasks. As explained in the *a priori* analyses of each workshop, I had a well defined didactical and pedagogical aims when preparing the workshops. However, it is possible to see how the teachers reacted to the tasks, by being unsure about how to act (Workshop I), questioning the “rules” of our community of inquiry *before* engaging in the tasks (Workshop II). This recognition is significant as it puts emphasis on the importance of addressing *explicitly* the functioning of our community of inquiry before engaging with mathematical tasks. Otherwise this aspect may interfere or even prevent the participants from participating in the mathematical inquiry. This is the reason why I argue for recognising the critical role played by the “questioning” mode of participation (see Section 4. 3).

The second aspect refers to the observed mismatch between my didactical aim and chosen pedagogical means and the teachers’ interpretation of these. As explained in the formal *a posteriori* analysis of Workshop II, the teachers were not willing to consider the use of manipulatives as a means to illustrate the geometrical properties of even and odd numbers. As a consequence, I had to adjust my pedagogical strategy and thereby, I had not the possibility to introduce standard algebraic notation. A similar situation appeared during Workshop V when the teachers got fascinated by the emergence of several numerical patterns and seemed not to be interested in the introduction of algebraic symbols. However, in that case, I adjusted *momentarily* my pedagogical strategy and, as a result, the participants were able to elaborate an algebraic proof of the divisibility of four digit palindromes by 11. I argue for considering the

evolution of my way of acting as strong evidence of the development of my didactical knowledge.

The third aspect I want to address concerns having someone coming from the “outside” in order to conduct a mathematical discussion (Workshop IX). From the teachers’ utterances, it seems that it is necessary to have a participant coming from “outside” as a leader for conducting the workshops. However, the analyses of the different workshops showed that this role has been assumed not only by me, but also by Paul and John, developing further the mathematical task (Workshop IV), building on numerical example in order to emphasise the difference between syntactic and semantic translation (Workshop VI) or asking clarification concerning the choice of unknown (Workshop VII). Thereby, central aspects of mathematical inquiry were emphasised as the importance of the *mathematical environment* (Section 3. 4. 1) and thereby of *the nature of the mathematical tasks* (Section 4. 2), the importance of the *exploratory step* within algebraic thinking (Section 2. 5. 4), and the importance of *developing an awareness of the vagueness and ambiguity of mathematical discourse* (Rowland, 2000). In Section 4. 3, I tried to capture these ways of acting and to make this recognition visible by using the terms of “taking-over” and “didactical” modes of participation. On the other hand, while some of the participants were moving into these modes of participation, I am in a position, today, where I can characterise my own way of acting as “silent-participant” mode of participation. I argue for recognising the importance of these aspects as they contribute to the development of my didactical knowledge, and for valuing the process of engaging in research as it enabled me to deepen the informal *a posteriori* analyses into the formal *a posteriori* analyses, as presented in Sections 4. 1. 1 to 4. 1. 9.

The last aspect relates to the importance of realising our joint enterprise and mutual engagement through inviting the teachers to share the responsibility of organising a workshop. As explained in the formal *a posteriori* analysis of Workshop VIII, the teachers were in a position to bring *new* elements into our practice, such as inquiry into a pupil’s answer or the role played by spreadsheet in the introduction of algebra, and, thereby, they were looking critically into the way our collaboration has been organised. I argue for considering these aspects as elements of “critical alignment” with our community and as potential elements for engaging in “critical alignment” with their own teaching practice. This recognition is central to the development of my didactical knowledge as it might influence the organisation of my future collaboration with teachers, and I see it as resulting from engaging in research.

Before introducing the final chapter of my thesis, I want to put emphasis on the fact that the recognition of these different aspects influenc-

ing my didactical knowledge, as mentioned above, emerged from *engaging in research* and, this is the reason why I consider that my role as a researcher enables me to develop further my role as a didactician. On the other hand, the aspects presented above might play a central role in research as they could indicate routes for further research. This is the reason why I understand research and didactics as deeply interwoven and mutually constituent.

In order to point to possible routes for further research it is necessary to make clear what aspects of my research are deeply dependent of the context of my research and what aspects might be generalizable. I address these issues in my concluding chapter.

6 Conclusions and Discussion

In Chapter 4, I offered the results of the analyses of the nine workshops we had during the year and, in addition, I elaborated on the idea of *the nature of the mathematical tasks* and of several emerging *modes of participation* in Sections 4. 2 and 4. 3.

Furthermore, in Chapter 5, I presented insights within my own development following three directions: the elaboration of my theoretical framework, my understanding of algebraic thinking and elements of my own learning which emerged from Chapter 4.

In this final chapter, I propose to look back to my research questions, to emphasise the main findings and to present conclusions. In addition, I indicate implications and directions for further research.

6.1 Recalling the aim of my study

The aim of this thesis was to research the way a community of inquiry addresses and develops algebraic thinking and shows evidence of learning through engaging in social participation (see Section 1. 5). In order to address this aim, the following research questions were formulated²⁴:

1. *In what ways is the development of algebraic thinking related to the development of our community of inquiry?*
2. *What relationships can be discerned between teachers developing algebraic thinking during the workshops and their thinking in relation to their practice in the classroom?*

Through the first research question, I was able to research the processes related both to the creation and to the development of our community of inquiry, with focus on the development of algebraic thinking. Furthermore, I sought to elaborate a theoretical frame which would enable me to give an account for the development of algebraic thinking within a community of inquiry consisting of three teachers and a didactician/researcher. This implied developing Wenger's *community of practice* further with the idea of *inquiry*. In addition, in order to elaborate a relevant and coherent framework, I addressed the development of algebraic thinking through *scientific concepts* (Vygotsky, 1978, 1986) and *cognitive and metacognitive mediation* (Karpov et al., 1998).

The second research question addressed the potential link between the teachers' development of algebraic thinking within our community of inquiry and their thinking related to their own teaching practice. In terms of Wenger's theory, these concerns relate to the recognition of the fact that the teachers are both members of the school community to which

²⁴ Originally, the first research question was phrased in terms of *learning community* (see Chapter 5).

they belong and, during the school year of our collaboration, members of our community of inquiry. By questioning the possibility for the existence of relationships between these two communities, I refer to the teachers' experience of *multimembership*, and the possibility to bring some element of one practice into another, also called *brokering* (see Section 2. 2. 7). This issue is related to Lave and Wenger's (1991) idea of generality of any form of knowledge which they define as "the power to renegotiate the meaning of the past and future in constructing the meaning of present circumstances" (p.34).

6.2 What are the main findings?

The theoretical framework, which I elaborated (see Chapter 2) according to the criteria of *relevance* and *coherence*, allowed me to conceptualise learning as development of knowledge while participating in social practice. Using Ryan and Williams' (2007) terms, "learning mathematics is part of learning to *act purposefully* with mathematics, and so is a process of becoming active with mathematics" (p.154, my emphasis). It is in this sense that I view our (the three teachers and myself) development of algebraic thinking: learning to act purposefully with algebra and algebraic thinking, as situated with respect to the development of our community of inquiry within which our collaboration took place. Thereby, my focus, as a didactician and researcher, is on the development of our community of inquiry, which encompasses the different modes of participation and the way the didactical and pedagogical strategies were negotiated, as well as on the specific algebraic issues as these were judged worthy of discussion and engagement.

Before presenting the main findings of my study, I propose to consider Figure 24 as a means to capture the central features of the practice in our community of inquiry. The cyclic and iterative nature of the workshops is central in this conceptualisation as the development of confidence in the mathematics and in our community of inquiry emerged from one workshop and was refined constantly in the following ones. In addition, each new task nurtured further the development of our community of inquiry as it enabled the participants to develop their confidence further both in the mathematics and in our community of inquiry. However, I want to put emphasise on the fact that it seemed, on one hand, that the confidence in our community was growing smoothly from workshop to workshop, while, on the other hand, the confidence in mathematics seemed to be highly dependent on the nature of the mathematical task (see Section 4. 2). Furthermore, the interrelation between *negotiation of meaning* and *modes of participation* needs to be brought to the fore since, while engaging in negotiating the meaning of a task, new modes of participation emerged and, in the next step, these new modes of par-

ticipation allowed the teachers and myself to engage more deeply in the processes of negotiating the meaning of a task. Thereby, there is a cyclic process between these two dimensions within each workshop.

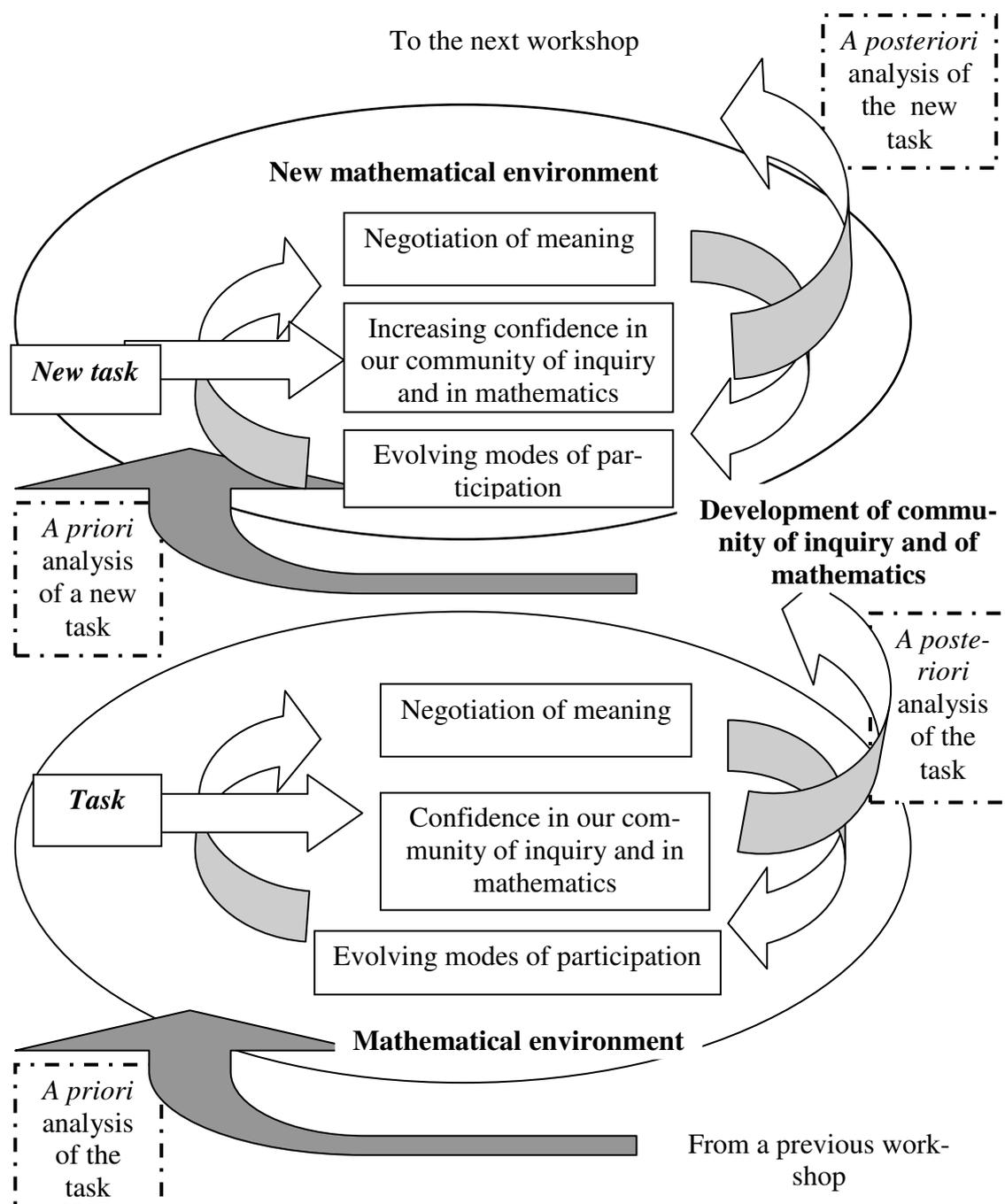


Figure 24: the cyclic process of the mathematical workshops

This cyclic process between negotiation of meaning and evolving modes of participation is central to what I called the *mathematical environment*. As explained in Section 3. 4. 1, by presenting a particular task within a specific social setting, a didactician creates a mathematical environment

whose characteristics depends *both* on the mathematical task and on the social setting. Thereby, I consider that a new mathematical environment is created as a workshop starts since the social setting is in constant evolution, due to the development of confidence within our community of inquiry and in the mathematics, and a new task is presented.

In addition, this figure captures another characteristic of our community of inquiry: while all participants (the three teachers and myself) were engaged in working collaboratively during the workshops, the *a priori* and *a posteriori* analyses were my own responsibility (represented by a dotted line in Figure 24). As explained in the *a priori* analyses of each workshop (see Chapter 4), the rationale for each task changed from workshop to workshop, coming both from my own focus on the choice, use and power of algebraic symbolism but also from issues emerging from classroom observations as, for example, the difficulties in translating from natural language to symbols (syntactic and semantic translation).

6.2.1 The main findings in terms of development of algebraic thinking

As explained in Chapter 2 (see Section 2. 5), the theoretical basis of my study on algebraic thinking has been conceptualised by elaborating further on Wenger's theory by going back to the Vygotskian ideas of *scientific concepts* and *mediation of meaning*. Furthermore, Karpov and Kozulin's *cognitive* and *metacognitive mediation* helped me to elaborate a *relevant* and *coherent* theoretical framework. However, before being able to formulate and making visible the results of my study, I had to re-question and look critically into my own understanding of what it means to work, as a researcher, within a socio-cultural approach to learning and to be faithful to that theoretical position when extending it to include mathematical learning. In addition, I also experienced tensions between the way algebraic thinking was addressed (or the lack of addressing algebraic thinking, as I perceived it at the beginning of our collaboration) between the participants and what I expected would happen during my research (see Chapter 5). These two points of tensions (elaboration of my theoretical framework and understanding of algebraic thinking) forced me to face, address and discuss these issues with my supervisors, and by engaging in this process I was able to recognise and make visible *my own assumptions* concerning what I meant by *learning*, (see Section 5. 3) and what I understood by *algebraic thinking* (see Section 5. 4) until I started engaging in research. Thereby, I became gradually aware of the reason why I experienced a kind of incompatibility between my research project and my own assumptions. In other words, I realised that I was socialised into a constructivist view of learning without addressing it explicitly. Similarly, my understanding of algebraic thinking was largely

influenced by my previous work on Galois theory, where the focus is on exploring various algebraic structures using algebraic notation. It has been both challenging and fascinating to deepen my own understanding of “learning” and addressing “algebraic thinking”, but as a result of dealing with and overcoming these difficulties, I recognise, today, that I developed a deeper understanding of the meaning of learning mathematics as learning to *act purposefully* with mathematics (Ryan and Williams, 2007), and I argue for considering my study as an example of what these terms might mean. In addition, my own understanding of the nature of algebraic thinking has evolved as I am today in a position where I recognise and value the importance of what I called the *exploratory steps* in algebraic thinking (see Section 2. 5. 4).

Therefore, *now*, I am in a position where I am able to describe the participants’ development of algebraic thinking in terms of developing one’s awareness and enhancing one’s understanding of scientific concepts in relation to algebraic thinking (cognitive mediation). In addition, I argued for considering the participants’ development of algebraic thinking in terms of emergence of different modes of participation (metacognitive mediation). Evidence of the participants’ development of awareness and understanding of scientific concepts is offered in the formal *a posteriori* analyses of the different workshops through addressing, for example, fractions, even and odd numbers, equilateral triangles, height and lengths in a triangle, symbols as variables versus symbols used as an object. The same formal *a posteriori* analyses offered also evidence of the emergence of the participants’ different modes of participation as starting from the *novice-expert* mode to the *questioning* mode, the *reflective*, *taking-over*, *didactical*, and *silent-participant* mode.

6.2.2 The main findings in terms of development of our community of inquiry

While I decided from the beginning of my research to focus on algebraic thinking (see Section 1. 2), the dimension of inquiry was *not* present from the start, it emerged gradually from the process of engaging in the analysis of my data. Therefore, during the elaboration of my theoretical framework, I started with Wenger’s theory and the idea of *community of practice*, in order to deepen the meaning of “working *with* teachers”. In other words, my starting point was a problem, or a question, which I wanted to explore and then I had to search for how to elaborate an appropriate theoretical framework (see Section 5. 3) and identify a suitable research methodology (See Section 3. 3. 3) which would enable me to engage in researching the particular issue concerning working collaboratively with teachers.

Through a fine grain analysis of my data, I was able to identify and characterise the participants' joint enterprise and mutual engagement as related to inquiry as defined by Wells (1999). This recognition enabled me to differentiate between different kinds of inquiry acts (see Section 2. 2. 6) and to refine my conceptualisation of our community of practice as a *community of inquiry*. Building on Elliott (2005) and Lindfors (1999) helped me to identify and characterise different kinds of inquiry moves while Jaworski (2005a, 2006) offered me the possibility to recognise inquiry as a dimension which enabled the participants to look critically at their own practice.

I argue for recognising the centrality of inquiry in metacognitive and cognitive mediation since it is through engaging in inquiry into the different mathematical tasks that the participants were able to develop their awareness and understanding of the different scientific concepts. Furthermore, it is through engaging in inquiry into the tasks that the participants started to develop confidence in our community of inquiry. Evidence of that development is offered in the formal *a posteriori* analyses of the different workshops as the participants started asking questions about the relevance of the proposed task, or challenging, explaining, and arguing with each other or deciding to remain silent. In addition, it was through engaging in looking critically at their own practice that the participants were able to move into the first step of critical alignment. Evidence of this dimension was offered in Section 4. 1. 9 and in Chapter 5.

6.2.3 Looking back to my first research question

Addressing my first research question: *in what ways is the development of algebraic thinking related to the development of our community of inquiry?*

As explained before, my first research question addressed the interrelation between the development of our community of inquiry and the development of algebraic thinking. I consider that evidence presented through the formal *a posteriori* analyses of the different workshops shows that these two developments are deeply interwoven and mutually constituent.

By zooming in on the different workshops and looking at a fine grain level, I have described the processes of creation of our community of inquiry as these emerged from the formal *a posteriori* analyses of the two first workshops: from acting according to the *novice-expert like* mode of participation to the *questioning* mode with starting asking questions about the relevance of the offered mathematical task and about the functioning of our community of inquiry. However, I want to put emphasis on the fact that the expression *novice-expert like* mode of participation does not refer to Claire as the expert and the teachers as the novices. From my perspective as a researcher, I understand *all* participants as

novices within our community of inquiry. By using this expression I seek to convey what I perceive from the *teachers' perspective*, since it seems that they considered Claire as the expert and themselves as the novices.

I argue that these two modes of participation were crucial in the development of *both* the participants' confidence in our community of inquiry and in the development of algebraic thinking since the analysis of the two corresponding workshops (Workshops I and II), from where these modes emerged, showed that the participants needed to address and clarify issues concerning the functioning of our community of inquiry *before* engaging with the tasks. In other words, issues related to the functioning of our community of inquiry seemed to constitute an obstacle which prevented the participants from engaging deeply with the negotiation of the mathematical tasks.

Looking through the workshops in a chronological order, I have suggested that the participants developed confidence in our community of inquiry gradually (starting asking questions, taking initiative for organising the mathematical inquiry, explaining and challenging each other, choosing when to participate or not). Therefore, the development of algebraic thinking is nurtured and supported by the development of confidence in our community of inquiry. However, there is strong evidence of the importance of *the nature of the mathematical task* in relation to the development of algebraic thinking. As explained in Section 3.4.1, I defined a *mathematical environment* as, 'by presenting a particular task within a specific social setting, a didactician creates a mathematical environment whose characteristics depends *both* on the mathematical task and on the social setting'. By comparing Workshop IV (where the mathematical task concerning Viviani's theorem was offered) with Workshop VII (where the first task concerned writing the area of a rectangle before and after change), it is possible to capture the crucial role played by the nature of the mathematical task: Workshop IV was organised in November 2004, Workshop VII in March 2005. Between these two workshops the participants developed further confidence in our community of inquiry, as evidenced through the formal *a posteriori* analyses of Workshops V and VI. Thereby, based on these observations, one could conclude that the mathematical environment in Workshop VII would enable the participants to fulfil the didactical aim in a more knowledgeable way than in Workshop IV. However, the formal *a posteriori* analyses of these two workshops show that the *mathematical environments* of these two workshops were rather different: During Workshop IV the teachers engaged well with the task and were able to elaborate an algebraic proof of Viviani's theorem. In addition, they were able to develop further the proposed task. In Workshop VII, the teachers

seemed to be unsure about how to engage with the first task, focusing on what Claire wanted rather than focusing on the mathematical task. Thereby, even though the participant's confidence in our community of inquiry had developed between Workshop IV and VII, establishing a "better" social setting in Workshop VII than in Workshop IV, the didactical aim was fulfilled in a more fluent way during Workshop IV than during Workshop VII. My hypothesis is that *the nature of the mathematical task* played a crucial role here, and it seems that a task situated in a Euclidean context enabled the teachers to engage fluently with the task and influenced positively the mathematical environment, while the nature of the question in the first task of Workshop VII (write the area of a rectangle before and after change) created some tension in the way the meaning of the task was negotiated. The same argumentation is valuable for comparing Workshop IV with Workshops II and V, that is a Euclidean context compared to a numerical patterns context. It seems that the teachers were more fluent with algebra and algebraic thinking within a geometrical context than within a context related to the generalisation of numerical patterns. This is the reason why I introduced a coding for each task (see Section 4. 2), as an attempt to capture the mathematical context within with each problem was situated and the nature of the question. Therefore, this recognition begs the following questions: within which context should a problem²⁵ be situated in order to fulfil a particular didactical aim? and how should the question of the task be formulated in order to foster a specific mode of participation? I come back to these issues in the last section of this chapter concerning the implications of my research.

6.2.4 Looking back to my second research question

Addressing my second research question: *what relationships can be discerned between teachers developing algebraic thinking during the workshops and their thinking in relation to their practice in the classroom?*

As explained earlier, my second research question enabled me to address the potential link between the teachers' development of algebraic thinking within our community of inquiry and their thinking related to their own teaching practice. In particular, using Wenger's terminology, I seek to identify *boundary objects* (artefacts, documents, terms, concepts) around which our community of inquiry and the teachers' community in their respective schools could organise interconnections. I argue for considering *the mathematical tasks* as an example of boundary objects since, while engaging with these, the teachers were able to envisage possible

²⁵ In Section 3. 4. 1, I proposed to define *a mathematical task as a contextualised problem*. In other words, a mathematical task is what people actually do within the context of a specific social setting.

implementations of these tasks in their own teaching. I referred to this process of interweaving mathematical inquiry with didactical inquiry as *mixing-inquiry*. In terms of the participants' development of confidence in our community of inquiry, I have suggested that the teachers engaged with the mathematical task while mixing didactical inquiry in a more fluent way as their confidence in our community was growing.

In addition, I consider that my second research question opens possibility to consider evidence of a potential link between our community of inquiry and the teachers' respective school community in terms of the teachers' reflections on the year of our collaboration. Evidence of their reflections is offered in Section 4. 1. 9, as during the last workshop the participants engaged in an evaluation of all workshops. Thereby, they could address and make visible potential links between the practice of our community of inquiry and their own practice in their respective school. I am referring to Mary's utterance where she addressed her reflections on her own teaching practice. I also refer to John's reflections after having me sitting in his class where he referred to different levels of consciousness. Here I see inquiry in one's own experiences as closely related to metacognitive awareness and *critical alignment* (Jaworski, 2006). Further evidence presented in Section 4. 1. 9 shows how the teachers were able to re-consider their practice. For example, John looked critically at the way the textbook was organised and thereby, he decided to change the order of the chapters in order to achieve a coherent continuity in his teaching. Furthermore, Paul seemed to emphasise his experience during the different workshops as he showed a willingness to engage his pupils in reflecting, wondering, and thinking.

My understanding of the teachers' attempts to link elements from our practice with their own teaching practice is that the teachers showed evidence of metacognitive awareness *not only* in relation to algebraic thinking, but also in relation to *their own teaching practice as a whole*. Thereby, I argue that while engaging with mathematical tasks designed with the aim to address algebraic thinking, the teachers engaged, at the same time, in looking critically into their own teaching practice from a holistic perspective, and considering elements which could be elaborated: for John developing an awareness of different layers of reflections on his teaching and, in addition, reflecting on the organisation of his teaching, which implies not necessarily following the text book; for Mary developing an awareness of different layers of reflection on her teaching; and for Paul trying to implement inquiry in his teaching practice, as he talked about offering to his pupils the possibility to wonder, to think, and to reflect on mathematical tasks.

I claim that these findings indicate that mixing mathematical inquiry with didactical inquiry is a fundamental characteristic of the teachers'

mathematical discourse and that this dimension of mixing inquiry enables the teachers to address potential links between our community of inquiry and their own community in their respective school. Furthermore, it seems that by addressing these potential links the teachers started to look critically into their own teaching practice and developed metacognitive awareness which is closely related to critical alignment. I consider the recognition of the importance of mixing-inquiry, as presented above, as central to teacher educators who aim to contribute to developing the teaching practice of the teachers they collaborate with.

6.3 Strengths and limitations of the thesis

Based on the criteria for the evaluation of the scientific quality of research, as proposed in the literature (Bassey, 1999; Bryman, 2001; Cohen, Manion & Morrison, 2000; Jaworski, 1994; Kilpatrick, 1993; Pring, 2000; Sierpiska, 1993b; Wellington, 2000), I propose to examine the research reported in this thesis against the following criteria: relevance, validity, objectivity, rigour, and trustworthiness.

6.3.1 Relevance

According to Sierpiska (1993b), the issue of relevance might be addressed in relation to both the research question(s) and the research result(s). In addition it is important to ask (Kilpatrick, 1993; Sierpiska, 1993b): relevant for *whom*? A way of engaging in this search is to distinguish between theoretical versus pragmatic relevance (Kilpatrick talks about basic or applied research) and to hope that the intersection between these two domains is not empty. In my research I tried to contribute to knowledge in the field of mathematics education as I examined in detail the development of algebraic thinking, taking a situated perspective on learning. At the same time I hope my work might contribute to the enhancement of practice, in relation to the building of community of inquiry between teachers and didacticians. In that sense my study could be considered as belonging to the intersection between theoretically and pragmatically relevant research. However, following Sierpiska (1993b), and in order to address the issue completely, I want to continue with the question: theoretically and pragmatically relevant for whom? I consider that through my contributing to the enhancement of knowledge within the field of mathematics education, I am addressing my work to the research community in the academic world. Nevertheless, I hope that the results of my study also can be useful for teachers who might express the desire to develop their practice further and engage in collaborative work with didacticians and/or other teachers. I consider that the insights presented in my thesis concerning the processes of development of algebraic thinking as seen in relation to the processes concerning the development of our community of inquiry are valuable both for researchers

and for teacher educators. I see my research as significant for researchers in terms of proposing a relevant and coherent theoretical framework for addressing mathematical learning, with focus on algebraic thinking, within a socio-cultural approach to learning. Furthermore, by offering an in-depth description of the challenges I met when elaborating my framework (see Section 5. 3) and by making visible my underlying assumptions, I invite other researchers to follow a similar development through their research. In other words, this study might offer an opportunity for other researchers to engage with addressing the same kind of questions: Within which theoretical approach is my research situated? and why?, in other words: what are my underlying assumptions? I argue for the necessity to address these fundamental issues clearly as they might have a crucial impact on the way research is conducted.

Furthermore, I see my research as relevant to teacher educators in terms of exploring the meaning of “working collaboratively *with* teachers”. I argue that developing insights related to this issue is central for teacher educators in order to engage in developing better understanding of what it means to establish collaboration with teachers, and in order to develop our own knowledge, as teacher educators. This was the case in my research since I presented an in-depth description of the creation and development of our community of inquiry and of my own learning both as a didactician working with in-service teachers and as a researcher.

I recognise that my research was situated in a sociocultural approach to learning and, thereby, the findings need to be understood within this theoretical paradigm. This implies that, by adopting this particular theoretical approach to learning, I used the available theoretical tools like, for example, *negotiation of meaning* from Wenger (1998) and *mediation of meaning* from Kozulin (2003). In addition, I tried to go beyond these theoretical tools in an attempt to capture their meaning in the specific context of my research. This implied addressing the following questions: *how* is the idea of mediation of meaning addressed within the specific context of my research? and, similarly, *how* is the idea of negotiation of meaning addressed within the specific context of my research? In Section 4. 3, Figure 23, I explained my understanding of the relation between these two theoretical concepts, since I argued for considering mathematical meaning as *mediated* through engaging in a process of negotiating the meaning of the tasks. Similarly, I argued for considering that the meaning of the mathematical tasks was *negotiated* through different kinds of inquiry acts.

Recognising the nature of research as situated within a particular theoretical approach creates the possibility to reframe my research within a different theoretical approach to learning. This implies that I would engage in a similar process of seeking to understand how the theoretical

tools offered by that (new) theoretical paradigm could be understood within the specific context of my research. Especially, I want to draw on Cobb (1994) arguing for considering constructivist and sociocultural perspectives as complementary rather than as a dispute between opposing approaches to learning, and, thereby, “to explore ways of coordinating constructivist and sociocultural perspectives in mathematics education” (p.13). I understand his view as being in accord with both Sfard (1998) and Williams and Linchevski’s (1998) position, pleading for considering “learning theories to incorporate the psychological with the social, and the metaphor of concepts as mental objects to coexist with the metaphors of learning as ‘participation’ in social processes and in communities of practice” (Williams et al., 1998, p.155). I recognise this perspective as promising and challenging even though I acknowledge the difficulties I met during the elaboration of my theoretical framework (see Chapter 5). Furthermore, considering a different theoretical perspective begs the following question: would it be possible that some results could still be valuable within another theoretical approach to learning? for example, I am thinking particularly about the recognition of the central role played by the nature of the mathematical tasks and the recognition of the importance of the different steps within the development of algebraic thinking, as presented in Section 2. 5. 4. I consider that by addressing this question I offer the possibility that results emerging from my research might be more or less relevant in other theoretical frameworks.

6.3.2 Validity

According to Wellington (2000), “validity can be seen as a measure of the confidence in, credibility or plausibility of a piece of research” (p.201). Furthermore, Bryman (2001) distinguishes between internal and external validity. The former refers to issues of causality within the study, while the later addresses issues of generalization of the results of a study beyond the specific research context.

As mentioned in Section 3. 1, I consider this study as an exploratory project and therefore my aim is not to provide definitive answers but to contribute to the development of theoretical knowledge within the area of mathematics education. Addressing research issues through a case study raises the question of external validity or generalization. A way of dealing with the criterion of validity is to use Bassey’s (1999) idea of “fuzzy generalizations”. He writes:

A fuzzy generalization carries an element of uncertainty. It reports that something has happened in one place and that it may also happen elsewhere. There is a possibility but no surety. There is an invitation to ‘try it and see if the same happens to you’. (p.52)

Cooney (1994), referring to the many theoretical frameworks emerging from “collecting interesting stories” (p.627), questions the possibility for

the local theories about teachers to contribute to a broader picture and to elaborate a more general theory about teacher education. I see my study as a local story, addressing the cooperation between three teachers from lower secondary school and a didactician from University, in a Norwegian context. I recognise the implications of working with only three teachers regarding aspects of generalisability from such a small sample. Therefore, I choose to follow Bassey (1999) and to acknowledge an element of uncertainty while addressing issues of generalization of the results of my study beyond the specific research context.

In order to report from this local story, I elaborated a local theory which enabled me to conceptualise my research. However, in Section 3.3.3, I presented the central features of developmental research and more particularly I referred to the research cycle which is a cyclical process between global theories and local theories. Using Gravemeijer's (1994a) terms, "global theory is concretized in local theories. Vice versa, the more general theory can be reconstructed by analysing local theories" (p.451). As explained in Chapter 2, I started from a problem, or a question, which I wanted to explore (working *with* teachers) and then I had to search for how to elaborate an appropriate and suitable theoretical framework. I decided to choose Wenger's work as I considered that this frame enabled me to capture the "with". Furthermore, in order to address the specificity to my research (*mathematical learning*), I decided to go back to Vygotsky's work and to go beyond and expand Wenger's theory. These decisions were taken in accord with the criteria of *relevance* and *coherence* which I decided to follow, in advance, in my research. It was during the process of analysing the data that the dimension of *inquiry* appeared and I extended my theoretical frame in order to include that dimension. Thereby, by zooming *in* on each workshop, I followed and described the different inquiry moves during the negotiation of meaning of each mathematical task. Likewise, by zooming *out*, I followed and described the emergence of the different modes of participation during the year of our collaboration. In other words, my local theory enabled me to tell a local story. As a researcher, I am interested in how the analysis of my local theory might contribute to the reconstruction of the more general theory.

I argue for considering my research as expanding Wenger's work to include processes related to the *creation* and *establishment* of a community, which is of inquiry by nature, in my research. In addition I understand my research as deepening particularly Wenger's idea of *negotiation of meaning* since I am now able to answer the question: *how* was meaning negotiated? In the case of my research, the participants negotiated meaning by engaging in inquiry with the mathematical tasks. I argue for considering these insights as examples of how the more general

theory can be reconstructed by analysing the local theory which I used in my study. Likewise, I understand my research as deepening Kozulin's idea of *mediation of meaning* since I am now able to answer the question: *how* is meaning mediated? In my study, meaning was mediated as all participants engaged in negotiating the meaning of the mathematical tasks.

6.3.3 Objectivity and rigour

Both Kilpatrick (1993) and Sierpiska (1993b) underline the importance of the criterion of objectivity as it should rule out obvious bias from research. The aim is that "researchers should attempt to identify the biases they bring to their work and then provide as much evidence as possible concerning how those biases may have distorted their findings" (Kilpatrick, 1993, p.23). I recognise, today, that I engaged in my study with biases concerning how to understand learning, starting implicitly from a constructivist understanding, and, therefore, the influence of my personal biases became visible in my choices of theoretical approaches as I made an attempt to address *mathematical* learning. In Chapter 5, I presented my own development and the struggles I experienced as I tried to elaborate the theoretical framework for my thesis. I consider that writing this chapter helped me to make visible and articulate my theoretical biases and the challenges I faced adopting a situated perspective on learning. However, it might be seen as an element of subjectivity in my research that I did not discuss with the three teachers the analysis and the results of the research process. I recognize, in retrospect, that my research could have a greater degree of objectivity if the teachers had the opportunity to comment on the initial analysis. However, the formal analysis of my data emerged gradually and it was not clearly articulated until recently. Thereby, I argue that it would have been difficult for the teachers to look at my results and make sense of these without being able to recall the detail of our interaction.

Similarly, I also recognise that I could have brought further evidence supporting my interpretation of the data if I had used video-recording. However, I took the decision to rely on audio-recorder only as I did not want to shift between the roles of participant in our community and of camera operator. Therefore, I decided to engage completely in the collaboration with the three teachers as I saw it as crucial, especially in the process of creation of our community of inquiry.

I consider that I addressed the criterion of rigour, which might be characterized as exactitude, accuracy, and precision, through my whole study since I tried to offer as much justification as possible concerning the way my theoretical frame has been elaborated and the identification of my research methodology. In addition, by making visible my underlying assumptions, I made visible and addressed the biases I brought to

research. This study has been conducted in the spirit of being *relevant* (addressing mathematical learning) and *coherent* (addressing epistemological issues), and at the same time, with a concern for going in depth and exemplifying the different theoretical ideas I used.

6.3.4 Trustworthiness

The issue of trustworthiness could be captured in the question: “How can an inquirer persuade his or her audiences (including self) that the findings of an inquiry are worth paying attention to, worth taking account of?” (Lincoln and Guba, 1985, p.290). I recognize that during the process of analysing the data I did not have a research associate who could help me through challenging my interpretations. However, through the regular meetings with my supervisors and their comments on my writings, I had the opportunity to present and argue for my interpretation of the findings. These meetings were really useful as these allowed me to share with my supervisors the tensions I experienced in my research (see Chapter 5) and to face and go beyond them. In fact I did not share only the concerns and challenges I met in my research, but I shared also the moments where I felt I was able to grasp and articulate the complexity of my research and getting fascinated by it. In addition, I want to argue that through my participation in several conferences and summer schools I had the opportunities to present, discuss, and get comments from other researchers. In that sense, the role played by these “critical friends” (Bassey, 1999) has been of great importance for my work. Thus the degree of trust expressed through these critiques of my work and my subsequent response to them gives me confidence to believe in the trustworthiness of what I present.

6.4 Theoretical contributions of the thesis

I engaged in research with a desire to address a question which emerged from my previous experience with the Mathias-project: developing an understanding of what “working *with* teachers” meant. Taking this question as a starting point, I decided that a sociocultural approach to learning was the most suitable theoretical frame within which to situate my research. Wenger’s work offered me the following theoretical tools: *meaning*, *practice*, *community*, and *identity*. Thereby, I was able to conceptualise our community in terms of *mutual engagement*, *joint enterprise*, and *shared repertoire*. In addition, I referred to the practice (the implemented practice) of our community, and the notion of *meaning* enabled me to conceptualise the mathematical discussion the three teachers and I had during the workshops. I put less emphasis on the idea of *identity* since my aim was not to trace the development of each participant, although I characterised the identity of the emerging community as a *community of inquiry*. However, Wenger’s theory uses community of

practice while I developed in my research, as explained above the idea of community of inquiry. As mentioned earlier, the dimension of inquiry emerged gradually from the process of analysing my data and it became a significant characteristic of our community. I consider that it was important to capture this dimension, as it emerged from the analysis of my data, in order to be able to conceptualise it. Thereby, the theoretical approach developed by Elliott (2005) and Lindfors (1999) enabled me to conduct a fine grain analysis of my data by identifying different inquiry moves, while using Jaworski's (2006) perspective on critical inquiry involving metacognitive awareness helped me to articulate theoretically the teachers' thinking. Against this theoretical background, I am able, as a researcher, to address and analyse learning, as understood within a sociocultural approach, both at a fine grain level and in general terms. However, my study aims to address the development of algebraic thinking and, therefore, I argued for the necessity of elaborating further my theoretical frame in order to capture the specificity of *mathematical learning*. This demand arises from the criterion of *relevance* of my theoretical framework which I decided to adopt in advance. Another criterion which I followed was related to the *coherence* of my theoretical frame, and, this is the reason why I went back to Vygotsky (see Chapter 5), and used *metacognitive* and *cognitive mediation* from Karpov and Kozulin's works. The rigour of my analysis depends of the coherence of the elaborated framework, and I argue that, in the case of my study, I was able to develop both a relevant and a coherent theoretical framework. However, as explained above, this was not possible until I was able to face and recognise which assumptions I had brought to my research and why these created tensions while I tried to conceptualise learning and algebraic thinking.

One of the main arguments I put forward in my thesis consists of the recognition of the importance played by the nature of the mathematical task in relation to the development of algebraic thinking. I conceptualised each workshop as a mathematical environment depending *both* on the mathematical task and on the social setting. Thereby, since the social setting was in constant improvement due to the participants' development of confidence in our community of inquiry, the extent to which a task fulfilled, or not, its didactical purpose, depended on the mathematical context within which the task was contextualised. Emerging from my research, it seemed that, for the teachers in our community, an Euclidean context was more suitable than a context involving the generalisation of numerical patterns. Again, recognising an element of uncertainty (Bassey, 1999), there is a possibility that this observation might be true for teachers more generally. In addition, analysis suggested that the di-

mension of *non-routine* versus *routine* (open question or not) had to be taken into account during the process of designing mathematical tasks.

During the process of engaging collaboratively with the teachers and, later, analysing my data I developed my own understanding of the nature of algebraic thinking. As explained in Chapter 5, engaging in research enabled me to recognise where I came from: from working with a Master thesis on Galois Theory, with focus on symbol manipulation as a means to explore particular algebraic structures, to valuing what I called the *exploratory step* (see Section 2.5.4) and considering *the use of standard algebraic notation as a result of algebraic thinking* and not as a condition *sine qua non* for it.

In the next section of this chapter, I draw on the insights that developed in this research, from a theoretical perspective and from the empirical findings, in order to indicate potential outcomes in terms of educational practice and directions for further research.

6.5 The way ahead: further issues to educational practice and research

I believe that this research has brought light to some of the ideas related to the collaborative work between teachers and a didactician focusing on algebraic thinking: community of inquiry, practice, negotiation of meaning, mediation of meaning, cognitive and meta-cognitive mediation, and scientific concepts. In my study, I used these concepts as theoretical tools in order to conceptualise my perception of central elements from our collaboration. In concluding my thesis I discuss potential implications that my study may have towards the collaboration between didacticians and teachers working both with pre-service and in-service teachers. In addition, I indicate the implications of my understanding of algebraic thinking, as presented in Section 2.5.4, for the way algebra is taught in schools.

My findings seem to emphasise the crucial role played by inquiry in the development of collaboration between teachers and didacticians and in the development of awareness of scientific concepts. Furthermore, such developments seem to be closely related to each other since a deep collaboration between the participants will influence the way they inquire into mathematics and, vice versa, as they engage with inquiring into mathematics, the participants will develop their collaboration further. However, as emphasised in my research, it seems that it is central to address and discuss explicitly the ‘rules’ related to the collaboration *before* engaging in mathematical inquiry, as these might constitute an obstacle and prevent the participants from concentrating on the mathematics. I see these insights particularly important in relation to collaboration

with in-service teachers since there is no pre-given structure for the collaboration, as in pre-service courses for student-teachers.

Another important result for my research concerns the practice established between the participants. Since developmental work or research does not happen in vacuum, it is often organised around the presentation of mathematical tasks. As my research showed, it is important to make clear what is the didactical aim which the meeting or workshop is going to address and why this particular task has been chosen. In addition, an analysis of the nature of the proposed mathematical task is central in developing an understanding of its impact on the way the participants engage with it. Likewise, an in-depth *a posteriori* analysis of the collaboration or workshop might contribute to the recognition of central aspects, like the emergence of different modes of participation or the difficulty of communicating results due to the vagueness of mathematical discourse, aspects which are worth making visible and commenting on. This recognition begs the following question: how often are these issues directly addressed in research reports presenting this kind of collaboration?

As shown in my research, in addressing these issues, as presented above, the didactician or teacher educator gets an opportunity to reflect on his/her way of acting and therefore he/she might engage in looking critically into his/her own development. I want to argue for the necessity, for a researcher, to engage in this process of meta-reflection as it enables one to reveal, challenge and re-consider implicit assumptions which might bias the research process. According to Jaworski (2008), authors do not usually incorporate paragraphs reporting on their own learning while conducting their research. However, some researchers, encouraged to reflect on the impact of their findings on their practice, reported on the value of engaging in such reflection. I agree strongly with emphasis on reporting on one's own learning process and its impact on our practice, as researchers, and I consider my research as an example of what this issue might look like.

In this research I developed the following understanding of algebraic thinking: By addressing and developing algebraic thinking, I mean to focus on the need, the choice, the introduction, the use and the meaning attributed to algebraic symbolism and on the way these various components of algebraic thinking are addressed and negotiated within our community through inquiry acts (Section 2. 5. 4). Furthermore, I emphasised the discovery, exploration and investigation of patterns, aiming to grasp and express some *structure*. Thereby, my goal was to move the focus away from symbol manipulation to symbols as a means to express structure. As a result I argued for considering the ability to express the observed structure by using standard notation as a *result* of algebraic thinking and not as a condition *sine qua non* for it.

What are the consequences of this claim? This result offers the possibility for changes in the way algebra is addressed in schools: from moving the focus away from focusing on the rules of symbol manipulation per se to introducing activities and tasks which would enable the pupils to explore and inquire into different algebraic structures as part of which experience with symbol manipulation follows. Thereby, there is a possibility that the question pupils usually ask, according to Mary, Paul and John, “why do we need this?” could be addressed differently: there is a need for introducing symbolic notation as a means to capture the structure which you (the pupils) just explored. I consider that this recognition is also relevant for student-teachers since the understanding of algebraic thinking they develop during their teacher education courses will potentially influence their future practice. Furthermore, as my findings have shown, I argue for recognising the importance of offering tasks aiming to address algebraic thinking situated in a context related to generalisation of numerical patterns. In addition, the dimension of routine versus non-routine seems to be central in relation to whether or not the didactical aim will be fulfilled. These dimensions are captured in an analysis of the nature of the mathematical task, as exemplified in Section 4. 2.

Through this research, I suggested implications for researchers, policy makers, teacher educators, and teachers. This study has been conducted in the spirit of being relevant and coherent, and, in addition, in addressing and making visible my underlying assumptions. My hope is that other researchers, policy makers, teacher educators, and teachers could benefit from this contribution.

7 References

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8 Appendices

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Development of algebraic notation		
250 AD	Diophantus	$\text{K}^\tau \beta \quad \text{ση} \wedge \Delta^\tau \epsilon \quad \text{M} \delta \quad \text{ἐστὶ} \quad \mu \delta;$ $2x^3 + 8x - (5x^2 + 4) = 44.$
680	Brahmagupta	$ya \ ka \ 7 \ bha \ k(a)12 \ ru \ 8 \quad \tau xy + \sqrt{12} - 8$ $ya \ v(a) \ 3 \ ya \ 10 \quad = 3x^2 + 10x.$
1494	Pacioli	Trouame .I.n ^o . che giöto al suo qdrat ^o faccia .12. $x \quad + \quad x^2 = 12.$
1514	Van den Hoecke	4Se. - 51Pri. - 30N dit is ghelijc 45. $4x^2 - 51x - 30 = 45.$
1521	Ghaligai	I □ e 32C ^o - 320 numeri. $x^2 + 32x = 320.$
1525	Rudolff	Sit I ½ aequatus 12 \mathcal{R} - 36. $x^2 = 12x - 36.$
1545	Cardano	cubus \bar{p} 6 rebus aequalis 20. $x^3 + 6x = 20.$
1553	Stifel	2 \mathcal{R} A + 2½. aequata. 4335. $2x^2 + 2x^2 = 4,335.$
1557	Recorde	14. \mathcal{R} . + .15. \mathcal{Q} = 71. \mathcal{Q} . $14x + 15 = 71.$
1559	Buteo	I ◇ P 6ρ P 9 I ◇ P 3ρ P 24. $x^2 + 6x + 9 = x^2 + 3x + 24.$
1572	Bombelli	$\frac{6}{I. p.} \quad \frac{3}{8}$. Eguale à 20. $x^6 + 8x^3 = 20.$
1585	Stevin	3 [⊙] + 4 egales à 2 ⁽¹⁾ + 4. $3x^3 + 4 = 2x + 4.$
1591	Viète	I QC - 15 QQ + 85C - 225Q + 274N aequatur 120. $x^5 - 15x^4 + 85x^3 - 225x^2 + 274x = 120.$
1631	Harriot	aaa - 3 bba = +2 ccc. $x^3 - 3b^2x = 2c^3.$
1637	Descartes	$yy \succ cy - \frac{cx}{b} y + ay - ac.$
1693	Wallis	$x^4 + bx^3 + cxx + dx + e = 0.$

Oppsummering av våre møter (14.06.05)

Workshop I (16.06.04): Cuisenaire staver:

1. a) Kan dere gi en beskrivelse av hva dere ser med ord
1. b) Kan dere gi en beskrivelse av hva dere ser på en matematisk måte

Workshop II (07.09.04): Om partall og oddetall:

1. Hva skjer når vi adderer partall og oddetall?

Workshop III (05.10.04): Historisk perspektiv på algebra

1. Finn lengde og bredde av et rektangel når arealet er 96 og halv omkrets er 20
2. Hodet til en fisk veier $\frac{1}{3}$ av hele fisken, halen veier $\frac{1}{4}$ og kroppen veier 300 gr. Hvor mye veier hele fisken?

Workshop IV (10.11.04): Vivianis teorem

1. I en rettvinkel trekant er summen av avstandene fra et punkt til sidene lik høyden av trekanten.

Workshop V (30.11.04): Palindromer

1. En av mine venner sier at alle firesifrede palindromer er delelige med 11. Er de det?

Workshop VI (11.01.05): Fra tekst til symboler

1. Oppgave fra Diophantus: Del et gitt tall mellom to tall med en gitt differanse
2. Student-professør oppgave: det er seks ganger så mange studenter som professorer

Workshop VII (09.03.05): Fra tekst til symboler

1. Oppgave fra Duval: Omkretsen av et rektangel med lengde "a" og bredde "b" er 62, lengden av rektangelet øker med 2 meter, bredden minker med 1 meter. Arealet er konstant. Skriv arealet før forandring, skriv arealet etter forandring.
2. Ole har tre søsken. Ole er ti år eldre enn Per og tre år yngre enn Kari. Ole er fem år yngre enn Jens og til sammen er de 58 år. Hvor gammel er Ole, Per, Kari og Jens?

Workshop VIII (10.05.05): Mary, John og Paul er ansvarlige for opplegget

Workshop IX (14.06.05): Oppsummering

I tillegg kan vi se på følgende spørsmål:

- Hva har dette opplegg betydd for deg?
 - som en person som er interessert i matematikk
 - som lærer i matematikk

 - Har det vi har gjort sammen hatt noen innvirkning på
 - ditt syn på matematikk, algebra?
 - din måte å undervise matematikk, algebra?

 - Hvilke inntrykk sitter du igjen med?

 - Hvis du tenker tilbake til dine forventninger når vi begynte, har vi innfridd noen?
-

Appendix 3: Transcription Keys

Audio transcripts

,	Comma
.	Full stop
?	Question mark
!	Exclamation mark
...	Pause, not exceeding 3 seconds
(pause)	Pause, exceeding 3 seconds
[]	Explanations
()	Description of non-verbal sounds or actions
⋮	Omitted discourse

Workshop I:

Excerpt 1

40. Claire: nå det står tre forskjellige arrangement med Cuisenaire-staver på bordet. For det første har jeg lyst at dere gir en beskrivelse av hva dere ser, med ord ...

41. John: ha, ha! (Mary ler)

42. John: jeg ser tre grupperinger, der to av grupperingene ser ut til ... eller inneholder samme antall, samme farge, altså, det er det samme i to av gruppene. Den tredje gruppa er av en annen farge og litt færre antall. Så hvis det skal forestille noen verdier eller noe sånn, så vil jeg si at to av disse har samme verdien, mens den andre har, ..., kanskje, ..., fire enheter eller noe sånn mindre i verdi, akkurat det vet jeg ikke for jeg vet ikke hva de symboliserer, de er plassert nokså ordentlig, nesten parallelt, med litt avstand mellom seg

43. Claire: ok, Paul?

44. Paul: vel, det er ikke så mye å føre til, jeg bare tenkte brøk når du la dem på bordet

45. Claire: ok, ..., brøk, hvorfor?

46. Paul: jo, for du har den lange blå eller sorte som representerer en hel, så representerer de andre mindre deler av en enhet, så jeg tenkte at det lar seg gjøre å tenke brøk

47. Mary: ja, jeg tenkte på det også, ..., først så jeg på fargene da, og så hva som var likt, og så tenkte jeg brøk med en gang, tenkte som Paul gjorde, det var noen hele og så kommer de halve ved siden av. Så jeg forbandt med en gang med matematikk siden det er det vi snakker om, med brøk

48. Claire: ja ...

49. John: jeg skulle gjerne ha litt forklaring på hvordan dere tenkte brøk, hvor dere tenker brøk her?

50. Mary: de er hele, de blå og de sorte, så har du de små delene som kan representere noen halve og kvarte og ..., ja!

51. John: ja vel! hvis jeg skulle tenke brøk her, så måtte vi bestemme for hvilken av disse var hel, eventuelt to hele, hvilken av de som skulle være den hele (uklart) ... Det er sånn jeg ser det, jeg ser ikke brøkene her!

(pause)

Excerpt 2:

52. Claire: det neste jeg tenkte på var, om vi kunne ha den samme beskrivelsen, men på en matematisk måte, fordi nå har vi brukt ord. Nå, vi kunne gjøre det på en matematisk måte ...

53. John: er det lov å flytte på de, eller skal de ligge sånn?

54. Claire: hva tenker du på?

55. John: bare gjør sånn, jeg er ikke sikker på om øyemålet mitt er god nok. Sånn ja, bare se hvor store de er i forhold til hverandre

56. Mary: hmm, hmm

57. John: det var det, ..., det går opp det der ...

58. Mary: to av de små er like lange som en rød

Excerpt 3:

62. Claire: er det noen som har lyst å begynne?

:

82. Claire: ok, Paul?

83. Paul: innbyrdes plassering er ulikt for alle tre

84. Claire: ja?

85. Paul: den ene har [uklart], og det er et mønster med annen hver

86. Claire: ok, og hvis vi skal gi en litt mer nøye beskrivelse ...
87. Paul: ja, jeg kan ta den første. Der står alle de, ..., alle, eller, alle tre røde står nær hverandre og alle tre hvite står nær hverandre
88. Claire: ok og hvordan ville du ha skrevet det?
89. Paul: nå, jeg skjønner ikke hva du mener?
90. Claire: det du sa nå nettopp
91. Paul: ja ...
92. Claire: alle tre
93. Paul: de tre røde står nær hverandre, etter hverandre, og de tre hvite står også etter hverandre
94. Claire: ja, det du sa med ord nå, hvordan ville du ha skrevet det?
95. Paul: ..., den blå, hvordan de røde og de hvite ligger langs med den blå, plassert i to grupper ...
96. Claire: ok, ja, og hva med den?
97. Paul: der, de danner et mønster ...
98. Claire: ja, og hvordan kunne du ha skrevet det?
99. Paul: da ville jeg ha sagt at, ..., de røde og de hvite ligger langs med den blå, plassert rød, hvit, rød, hvit, rød, hvit.
100. Claire (skriver på flippover): er det det du mener?
(Claire skriver: $\text{Blue} = R + W + R + W + R + W$)
101. Paul: ja
102. Claire: er det riktig?
103. Paul: ja, sånn som jeg ser det
104. Claire: ok, hmm
105. Paul: ja, det er det jeg mener

Excerpt 4:

147. Mary: det er de bokstavene som kommer inn, de vanskeliggjør det med en gang, tror jeg

148. Paul: de betyr ikke noe

149. Mary: ja, ja, ...

150. Paul: det er bare en bokstav

151. Mary: ja, de tror de skal stå der bare, de skjønner ikke det der at de skal symbolisere noe

Pauls utsagn:

155. Paul: jeg er ikke sikker om det, jeg tenker på de som får det til med matematikken, så pleier det ikke det med bokstaver å være så vanskelig, men for de som sliter, tenker sånn at matematikk det er tall, og ”for Guds skyld, hold de bokstavene vekk, jeg vil jobbe med tall”, det er det jeg føler, de er ikke kommet til den der abstrakte nivå hvor de kan abstrahere, at bokstaver symbolisere noe, de er vel ikke der, kanskje, ...

Johns utsagn:

162. John: jeg vet ikke om vi snakker om samme kategori eller, er det flere som sliter med matematikk, de som ligger nede i den nederste delen av karakterskala, når de får sånne bokstaver eller et eller annet bokstav i et uttrykk, så de er, tror jeg, avhengig at de har lært seg algoritme, ikke nødvendigvis forståelse av det de holder på med, altså de redder seg på det aller enkleste måte, ved at de har lært: sånn skal jeg gjøre det, jeg har den følelsen av at slik er det, det tror jeg

Appendix 5: Copy of the Norwegian Social Science Data Services (NSD) – Status of data.

Statusskjema

for forsknings- og studentprosjekt som medfører meldeplikt eller konsesjonsplikt
(jf. personopplysningsloven og helseregisterloven med forskrifter)

Statusskjema sendes per post,
e-post eller faks, i ett eksemplar, til:

Norsk samfunnsvitenskapelig datatjeneste
Personvernombudet for forskning
Harald Hårfagres gate 29
5007 BERGEN

personvernombudet@nsd.uib.no / Telefaks: 55 58 96 50 / Telefon: 55 58 21 17

DAGLIG ANSVARLIG		VED STUDENTPROSJEKT	
Navn (fornavn - etternavn): BARBARA JAWORSKI HANS ERIK BORGERSEN		Navn (fornavn - etternavn) på student: CLAIRE VAUGELADE BERG	
Telefon:	E-postadresse: Barbara.jaworski@uia.no	E-postadresse: Claire.v.berg@uia.no	Telefon:
PROSJEKTNUMMER OG PROSJEKTITTEL			
Nummer: 11466			
Tittel: Developing Algebraic Thinking in a Community of Inquiry			
Status for prosjektet			
Datamaterialet er anonymisert* <input checked="" type="checkbox"/>			
Datamaterialet er ikke anonymisert <input type="checkbox"/>			
Gi en begrunnelse for videre behandling av personopplysninger, ny dato for prosjektslutt og hva som da vil skje med datamaterialet: Prosjektet er forlenget ut året 2008. Datamaterialet blir lagret ved UiA.			
* For at datamaterialet skal være anonymisert må opplysningene ikke på noe som helst måte kunne identifisere enkeltpersoner, verken direkte gjennom navn eller personnummer, indirekte gjennom bakgrunnsvariabler, eller gjennom navneliste/koplingsnøkkel eller krypteringsformel og kode. Å anonymisere et datamateriale innebærer vanligvis å slette/makulere navnelister, og ev. kategorisere eller slette indirekte personidentifiserbare opplysninger. Dersom det er tatt lyd- eller bildeopptak i forbindelse med prosjektet må disse også anonymiseres/slettes/makuleres eller sladdes dersom datamaterialet skal være anonymt.			
Arkivering ved NSD			
For arkivering av data hos NSD ber vi om at arkiveringsskjema fylles ut og sendes inn sammen med data og nødvendig dokumentasjon. For videre informasjon, se www.nsd.uib.no/personvern/arkivering/pvo_arkiveringsskjema.cfm .			
Dersom data først kan overføres på et senere tidspunkt, angi dato: _____			

Er det spørsmål i forbindelse med utfylling av skjemaet, ta gjerne kontakt med Personvernombudet hos NSD, telefon 55 58 21 17

Appendix 6: Copy of the correspondence with NSD concerning making data anonymous.

Jeg bekrefter herved at datamaterialet er anonymisert, og at koblingsnøkkelen er slettet.

Mvh

Claire V. Berg

> Viser til statusskjema mottatt 25.3.08.
>
> Det framkommer at prosjektslutt er utsatt til 31.12.08, men at
> datamaterialet er anonymisert.
> Anonyme data kan oppbevares fritt. Dersom datamaterialet er anonymisert, det vil si at direkte og indirekte personidentifiserende opplysninger er slettet, og at koblingsnøkkelen er slettet, ønsker ombudet at dette bekreftes per e-post. Vi vil deretter registrere prosjektet som avsluttet.
>
> Svar imøteses.
>
> --
> Vennlig hilsen
>
> Anne-Mette Somby
> Fagkonsulent
>
> Norsk samfunnsvitenskapelig datatjeneste AS
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>
