Masteroppgave i Matematikkdidaktikk

Procedural and conceptual knowledge among upper secondary students:

The case of second degree polynomials

Av

Erik Masson

Masteroppgaven er gjennomført som et ledd i utdanningen ved Universitetet i Agder og er godkjent som sådan. Denne godkjenningen innebærer ikke at universitetet innestår for de metoder som er anvendt og de konklusjoner som er trukket.

Veileder:

Martin Carlsen

Universitetet i Agder, Kristiansand

Dato

Juni 2010

Foreword

Writing this Master's degree thesis has taken longer than I initially expected. I would first like to thank my supervisor Martin Carlsen, for helping me focus my ideas into a concrete project, and for his advice along the way. I would then like to thank Christina and our daughter Frida for bearing with me throughout this long process, as well as Unni for her moral support. Last but not least I am grateful to the teacher and his students who volunteered for this project.

Sammendrag:

Denne oppgaven dreier seg om begrepskunnskap og prosedyre kunnskap av andregradspolynomer.

Jeg forsøker å kartlegge en gruppe elevers kunnskap om dette emnet.

Forskningsspørsmålene jeg stiller er følgende:

- Hvor fortrolige er elevene med algoritmene knyttet til problemløsning der andregradspolynomer inngår?
- Kan en finne indikasjoner på at elevene innehar begrepskunnskap om emnet?

Jeg gir en oversikt over relevant litteratur, med fokus på hvordan disse to kunnskapsformene står i forhold til hverandre, hvordan de påvirker hverandre, hvordan man kan undervise for at elever skal tilegne se dem (eventuelt med digitale hjelpemidler), og hvordan de kan la seg påvise hos elever.

Deltakerne i studien tilhører en førsteklasse på en videregående skole i Norge. Elevene følger programmet matematikk 1T, det mest teori - orienterte matematikk faget på det trinnet i videregående skole. Jeg har samlet inn data i form av elevers svar på en test og intervju av elever utvalgt på bakgrunn av deres besvarelser.

Av den kvalitative analysen av disse dataene framkommer det at elevene har varierende kunnskap om relevante prosedyrer, og at få av dem er i stand til å bruke de så effektivt som mulig. Med effektivt menes for eksempel ved minst mulig bruk av manipulasjoner. Noen av feilene elevene gjør kunne sannsynligvis blitt unngått dersom de hadde hatt større konseptuel kunnskap om emnet. Noen elever utviser en noe begrenset begrepskunnskap.

Summary

This study is concerned with conceptual and procedural knowledge of second degree polynomials.

I try to give an account of a group of students' knowledge of this topic

The research questions I pose are the following:

- How fluent are the students with respect to the algorithms for problem solving where quadratic polynomials are involved?

- Can one find indications that the students possess any conceptual knowledge of the subject?

I provide an overview of the relevant literature, with a focus on how these knowledge types relate to each other, how they influence one another, which instructional strategies are suited to instill them (also with the use of computer software), and how they can be assessed.

The participants in this study belong to the first grade of an upper secondary school in Norway. The students are taking a course called mathematics 1T, which is the most theoretically oriented course in this grade of upper secondary. I have gathered data in the form of students' answers on a test, and the interviews of students selected on the basis of their performances on the test.

The qualitative analysis of the data reveals that the students have a varying grasp of the relevant procedures, and that few of them are able to use procedures effectively. Some of the mistakes made on the test could possibly be avoided if the students had greater conceptual knowledge of the subject matter. Some students show a superficial degree of conceptual knowledge.

Table of content

1. INTRODUCTION

During the spring of 2007 I took part in a course called *Learning and Teaching of Mathematics*, during which some student colleagues and I participated in what was called Mathematics Education Research Group 13 (or MERG 13). The course involved some data gathering at an upper secondary school, where some first year students were learning to solve quadratic inequalities. It was during this process that I was made aware that students who could correctly solve those inequalities could also fail to recognize that the roots of a given polynomial could be inferred directly from that polynomial's factored form (for example: $(x + 2)(x - 3) = 0 \Leftrightarrow x = -2$ or $x = 3$). This would force these students to use an unnecessarily laborious method to solve inequalities if these were presented to them with the polynomial in its factored form. As a prospective mathematics teacher, I found it interesting to investigate how this lack of knowledge could be remedied, and this led me to delve into the theoretical framework provided by the research on conceptual and procedural knowledge of mathematics. Having taught mathematics myself, I have on several occasions encountered students who, while high achieving, seemed to have a lack of knowledge of the rationale the procedures they used was built on. Some have even openly complained about this, saying for example: "I know I am doing this right, I just don't understand what I'm doing". The frustration conveyed in that statement gives one reason why teaching for conceptual knowledge is a goal that may be worth pursuing, if only to motivate those students.

The MERG project being somewhat limited in duration and scope, I wanted to pursue this line of inquiry in the present master degree thesis, and study in closer detail the procedural and conceptual knowledge of quadratic polynomials among upper secondary students. The research questions I pose are therefore the following:

- How fluent are the students with respect to the algorithms for exercise solving where quadratic polynomials are involved?

- Can one find indications that the students possess any conceptual knowledge on the subject?

In this text I will use the term fluent on several occasion, and I use the word as one would use it for describing a language skill. Fluency with respect to the aforementioned algorithms is the ability to use them to acquire the correct results.

Hiebert and Lefevre (1986) define procedural knowledge as "*...made up of two distinct parts. One part is composed the formal language, or symbol representation system of mathematics. The other part consists of the algorithms, or rules, for completing mathematical tasks."* They define conceptual knowledge as "*...knowledge that is rich in relationships"* meaning that it lies not within isolated units of mathematical information but rather in the connections between those units that one subject has established.

I will first elaborate on these definitions by presenting an overview of the existing literature on this topic, which also deals with the relationships between the two kinds of knowledge, how they influence one another, what advantages there are to acquiring them both, and how to instill them in students. Building on that foundation, I will define what is meant by procedural and conceptual knowledge of quadratic polynomials, and thereby try to justify my choice of method for data gathering. The data will then be presented and analyzed, and the results discussed in an effort to answer the research questions.

2. THEORETICAL FRAMEWORK

Here I will present an overview of the literature on the topic of conceptual and procedural knowledge of mathematics. I will first elaborate on the definitions given in the introduction, and address them from a cognitive perspective. Then the instructional perspective will be presented before, and finally some of the recent debate regarding this issue will be presented. It is of some importance to mention that the two forms of knowledge have been addressed by other using a different terminology.

Skemp (1976) writes about *instrumental and relational understanding*, and by the former he means what we so far have defined as procedural knowledge. By the latter he means, in his own words *"…knowing both what to do and why"*, which in our terminology translates into procedural knowledge enriched by conceptual knowledge. We have yet to address the issue of whether conceptual knowledge can exist without procedural knowledge (and vice versa), and we leave that to a later section in this text.

Mellin Olsen (1984), whom Skemp claims made him aware of the two types of mathematical understanding uses the terms *rule perception and structure perception* (my translation). The first is defined as knowledge of how mathematics is used through the application of rules and principles and the second is *"the understanding of how the rule is tied to its structure, meaning why the rule has become what it is"* (Mellin-Olsen, 1984, pp 32, my translation). These terms are more removed from our terminology than Skemp's but are still related to it in that they deal with the knowledge of algorithms and how they are related to mathematical theory.

While those alternative representations are worth mentioning, the bulk of more recent literature pertaining to this subject uses the terms conceptual and procedural knowledge. Moreover, some of the most recent work on this topic has further developed and enriched our theoretical framework, as will be presented towards the end of this section.

2.1 Procedural knowledge and conceptual knowledge

In the text quoted in the introduction, Hiebert and Lefevre elaborate on their definition in several ways, for example by highlighting an important difference between procedural and conceptual knowledge, namely the fact that while the first does involve a measure of awareness of relationships, those relationships are concerned with the order in which operations are to be carried out: "*…the primary relationship in procedural knowledge I "after", which is used to sequence subprocedures and superprocedures linearly".* To illustrate what they mean by subprocedures and superprocedures, one can use the example of a quadratic inequality, the solving of which requires the sequential execution of several subprocedures. First one finds the roots of the corresponding quadratic equation (first subprocedure), then one uses this result to factorize the polynomial (second subprocedure), and setting up a sign diagram (third subprocedure) one investigates the sign of the polynomial for different values of the unknown, etc. The sum of all those procedures composes a superprocedure. To reiterate: the main relationship in procedural knowledge is that which links subprocedures linearly, whereas conceptual knowledge – according to the authors – may consist of relationships of many kinds.

According to the authors there are two ways in which conceptual knowledge can be acquired: first by establishing a connection between already existing pieces of knowledge, second by establishing a connection a unit of knowledge one has previously acquired and one that just has been learned. They illustrate this by presenting the example of a fictional girl cognitively linking her knowledge of decimal place value with that of the addition algorithm, alternatively linking the two as she possesses one and learns the other, and establishing the link after having possessed both of them for some time.

The authors recognize that their definitions are fallible in the sense that there can exist some forms of mathematical knowledge which do not fit ideally into either of their categories, and that they should consequently be treated as the endpoints of a continuum on which mathematical knowledge lies. They should therefore not be treated as distinct, mutually exclusive entities but as two facets of mathematical knowledge in general, which exert an influence upon –and can potentially strengthen – one another. Thus, they state that conceptual knowledge can leads to benefits in procedural knowledge by helping students make sense of the algorithms they use thereby making them easier to recall and apply, and by helping them appraise the outcome of their procedures. In turn, knowledge of procedures can act as a catalyst for the development of concepts (Gelman and Meck, 1986; Baroody and Ginsburg, 1986).

There are however certain circumstances that can act against the development of relationships between pieces of mathematical knowledge, the most obvious one being lack of knowledge itself. Thus Hiebert and Wearne (1986) find that learning about decimal numbers can be hindered by students' insufficient knowledge of common fractions. The other hindrance is the *"tendency to compartmentalize knowledge"* (Hiebert & Lefevre, 1986), by which knowledge acquired can become tied to the situation in which it was learned, thus preventing the establishment of relationships with previously acquired knowledge. In early years of child development, conceptual and procedural knowledge are tightly tied together (Sinclair & Sinclair, 1986), some claim that conceptual knowledge precedes procedural knowledge (Gelman & Meck, 1986), others claim it may be the other way around (Baroody & Ginsburg, 1986). But as students progress through school, the proclivity to acquire contextualized – and therefore somewhat isolated – knowledge increases (Hiebert & Lefevre, 1986).

2.2 Instructional point of view

The fact that procedural knowledge is composed of the ordered application of several steps to solve mathematical problems entails that it can lend itself to rote learning. While an algorithm doesn't necessarily have to be learned by rote, it is important to realize that it can, so that mathematics instruction geared towards rote learning of procedures can provide students with a measure of academic success if they are able to successfully remember the procedures and use them appropriately in tests and exams. However, limiting students' mathematical ability solely to the application of algorithms to solve textbook exercises can have several negative consequences, as the study mentioned below indicates.

Boaler (1998) claims that mathematics lessons using such an approach can be experienced by students as tedious and monotonous. Moreover it can instill students with a sense that learning mathematics consists essentially of remembering rules without necessarily understanding them. In Boaler's study, some pupils who were taught in this way also experienced a conflict with their own attitude towards mathematics, because they believed mathematics also should involve some understanding and sense making. Skemp (1976) tells of an unusually clever boy who also experiences this conflict, which leads him at first to loathe mathematics, then to appreciate and enjoy it once he is taught in another way. Another consequence of emphasis on rote learning is coined *cue based behavior* (Boaler, 1998), meaning a tendency for students to use non mathematical clues to determine what is expected in an exercise, sometimes leading them to give up on a problem because it does not conform with their expectation of it. The same issue is addressed by Schoenfeld (1988) according to whom students instructed in ways that promote only procedural knowledge gain a compartmentalized knowledge of mathematics which can hinder their subsequent learning of the subject.

The studies presented above stress the importance of providing students with instruction that can also promote conceptual knowledge, and the present section will address some possible methods for achieving this. Firstly however, it is important to be aware of some factors that can inhibit the practice of such instruction.

In an exploration of the case of an American student teacher: Ms Daniels, Eisenhart, Borko, Brown, Jones and Agard (1993) reveal that despite a strong commitment to teach for conceptual as well as procedural knowledge (a commitment shared by her mathematics methods course instructor, her cooperating teachers and the administrators of her placement schools), she was hindered in achieving just that by a combination of factors. The first of these was limits in her own knowledge base, making her more comfortable with providing rules of thumb and memory aides than explanations likely to provide her pupils with some conceptual underpinnings for the algorithms they were studying. Then came the need to prepare her students for skill oriented test, which entailed a strong focus on acquiring and practicing procedural skills, this need being emphasized by her placement schools. While her instructor demonstrated various methods of teaching for conceptual knowledge, those methods were themselves interpreted in procedural terms, sometimes leading her to remember them incompletely. To Eisenhart et al. the case of Ms Daniels is emblematic of a mathematics education system which creates tensions and pressures on educators, which result in an overemphasis on teaching for procedural skills.

Peled and Segalis (2005) mention a renewed demand from the Israeli ministry of education for mathematics educators to contribute to students' mastery of procedures and propose that the knowledge of algorithms can be harnessed to promote conceptual knowledge after procedures have been appropriated. In their study, a group of students receive instruction that help them formulate general principles for subtraction based on their previous knowledge of subtraction in different domains (whole numbers, decimal numbers, fractions). The majority of student receiving such instruction (coined "*mapping instruction*") succeeded in generalizing their procedures and subsequently improved both their domain specific performances and their ability to transfer their knowledge to word problems. Their generalizations make them aware that the rationale for regrouping in order to execute subtraction is based on the principle that one always subtracts from each other things that are alike in some way (tens, tenths, hundreds, fractions with same denominator). These connections happen at a higher level of abstraction (Hiebert & Lefevre, 1986) then their domain specific knowledge of each subtraction procedures. The students' improvement in procedural skills after gains in their conceptual knowledge is also illustrated by Perry (1991) and Rittle Johnson, and Alibali (1999) according to whom such gains resulted in generation of correct procedures.

Rittle Johnson, Siegler and Alibali (2001) also find that improvement in conceptual knowledge leads to improvements in procedural knowledge, but also that improvements in the latter result in improvements in the former. These gains are achieved in an iterative process. Rittle Johnson and Koedinger (2002) investigate whether greater benefits are gained by focusing first on conceptual then on procedural instruction or by alternating between the two forms of instruction. They find that the iterative approach leads to greater gains especially with respect to procedural knowledge.

2.2.1 Impact of language

Language being an important tool for thinking and reasoning, it can be interesting to investigate what influence it has on knowledge of mathematics. Some languages may be better adapted to promote mathematical concepts, as demonstrated by Miura and Okamoto (2003). In their study they argue that Asian languages (both written and spoken) derived from ancient Chinese (Chinese, Japanese and Korean) are better suited to promote knowledge of decimal place than European languages. The reason for this is that they convey the place value of each digit within a number with more accuracy. Moreover there is greater consistency between the spoken and written forms of numbers (for example twenty is read two ten(s) in Japanese, and eighty which is read four-twenty in French is simply eight ten(s) in Japanese). They argue further that this idiosyncrasy of the Asian language gives Asian children advantages over European and American children in counting, place value understanding and consequently addition and subtraction performance.

Simpson and Zakaria (2004), investigate 16 Malay students' use of language in solving problems (with mathematical content) from a chemistry course involving among other rates of reaction and half life. After having separated the students into two groups ("conceptual" and "procedural") with respect to their degree of conceptual knowledge of differentiation, they find that while the ability to solve chemistry problem varied significantly in both groups, the students belonging to the "conceptual" group" made use of many linking words (then, because). The "procedural" students, even those who were very successful at solving their chemistry problems, used very few of those words. To the authors, this characteristic of the "conceptual" students suggests that they are able to link their knowledge of mathematics and chemistry in a mutually reinforcing way.

Writing to learn mathematics (WTLM) is a strategy that has been advocated as helping students make sense of the mathematics they learn (Porter & Masingila, 2000). In this study the authors define WTLM as *"writing that involves articulating and explaining mathematical ideas for the purpose of deepening one's understanding"* (Porter & Masingila, 2000). They investigate the claim that WTLM induces deeper conceptual understanding by comparing two groups of student in an introductory calculus course, one group using WTLM and one not. They do not find any significant difference between the two groups with regard to their conceptual and procedural knowledge, but call for further research addressing writing as a learning tool and whether the act of articulating (not necessarily in writing) mathematical ideas and concept can have a positive effect on conceptual knowledge.

This latter issue is addressed to some extent by Berthold and Renkl (2009), who propose to investigate (among other things) whether the use of *"multiple external representations"* (equations, diagrams, tables, graphs) can promote conceptual knowledge of probability among high school students. Their findings suggest that these MER do not foster conceptual knowledge by themselves, but that they do when used in conjunction with *"self explanation prompts"* whereby students are prompted to provide explanations of rationale or principles to justify their use of mathematical content while solving problems. Self explanation is however found to have potentially damaging effect on acquiring procedural knowledge, since it also can foster incorrect explanations.

2.2.2 Use of computers

The use of computers can also result in improvements for conceptual knowledge, as indicated by Lee (2004), whose case study involved three pre-service teachers' use of computers for mathematical problem solving. The students made use of several computer software packages such as excel, Algebra X-presser, Math-view and Geometer's sketchpad. A key factor contributing to the students' improvements in conceptual knowledge of trigonometric function was according to Lee the fact that the computer programs helped them visualize the effect of parameters on the behavior of function, by making it possible to dynamically influence them. Similar results were obtained by O'Callaghan (1998), who compared the effects of a computer intensive algebra course with that of a traditional curriculum on students' conceptual knowledge of functions. O'Callaghan also mentions the opportunity for self exploration that the computer provided as an aid to conceptual knowledge. Other contributing factors were according to him the possibility to investigate functions in several different representation systems.

Heid (1988) also compared to different instructional approaches, one a traditional calculus course in college where fifteen weeks were spent on skill acquisition, the other a computer intensive program twelve weeks long and three subsequent weeks spent on skill development. While the students in the traditional course spent their time practicing their computational skills, the other group let computer programs do most calculations and were left free to explore the behavior of several kinds of functions. This helped them improve their conceptual knowledge of the subject matter by making more mental space available to reflect on general characteristics of the function. The students having spent only the three final weeks on developing computational skills also fared almost as well on the final examination as those who had spent the entirety of the course on traditional skill acquisition, which according to Heid challenges the claim that one cannot effectively grasp concepts without some prior knowledge of procedures.

2.3 Concept maps as a tool for assessment

While it is common to assess conceptual knowledge of mathematics by interviewing subjects (Star, 2007), Grevholm (2000) suggests that the use of concept maps may yield more bountiful results. Concept maps have been defined thus by their inventor:

> *Concept maps are graphical tools for organizing and representing knowledge. They include concepts, usually enclosed in circles or boxes of some type, and relationships between concepts indicated by a connecting line linking two concepts. Words on the line, referred to as linking words or linking phrases, specify the relationship between the two concepts.* (Novak & Cañas, 2008)

Since they consist of concepts of a more or less general nature, and make apparent the many relationships existing between those concepts, they seem indeed a powerful tool for assessing conceptual knowledge.

One use of concept map is described by Williams (1998), in whose study three different sets of concept maps were compared. The first set was drawn by a group of students enrolled in a traditional calculus course in a state university, the second by a group of students following a calculus curriculum with a greater emphasis on modeling and technology. The third set of maps was drawn by eight professors with PhDs in mathematics. Comparing the two student groups’ maps, Williams claims she found subtle differences between the groups among other things with respect to their ability to connect function to practical application. There were also clear differences between the maps drawn by students and those drawn by the professors, indicating foreseeable differences between the subject groups conceptual knowledge of functions.

2.4 Recent debate

In a fairly recent research commentary, Star (2005) proposes that the widely used definitions of conceptual and procedural knowledge contain some underlying assumptions that make them ill suited to characterize certain forms of mathematical knowledge. According to him the definitions proposed by Hiebert and Lefevre (1986) are lacking in the sense that they amalgamate knowledge type and knowledge quality. Thus the definition of procedural knowledge integrates not only what is known but also how procedures are known (Star, 2005, p 408). Conceptual knowledge as it is commonly defined also suffers from the same entanglement, which leads to call for adding a dimension to both the definitions, by taking type and quality into account. The common definition for procedural knowledge corresponds to superficial procedural knowledge, and the one for conceptual knowledge to deep conceptual knowledge. Some of his latest work on flexibility (Star & Seifert, 2006; Rittle – Johnson & Star, 2007) provides an example of what deep procedural knowledge could be. He defines flexibility as the combination of knowledge of multiple procedures and the ability to create new ones (Star & Seifert, 2006, p 282). According to him, flexibility is a factor that has to be taken into account when evaluating procedural knowledge because it not only belongs to the realm of procedural fluency but also gives it a new dimension which is not readily encompassed by Hiebert and Lefevre’s (1986) definition. As mentioned above they state that the primary relationship inherent in procedural knowledge is that which ties subprocedures sequentially. But since there may exist several ways of sequencing algebraic manipulations to solve an equation, if a subject recognizes that fact and makes use of it to consistently to solve equations with the least possible amount of manipulations, that subject demonstrates a procedural knowledge that is deep in quality. This claim is challenged by Baroody, Feil, & Johnson (2007) who claim that the kind of deep procedural knowledge Star describes actually does require a measure of conceptual knowledge.

3. METHOD

In this section I will present my data gathering method, justify my choice of subject matter and present the test while justifying the aim of the exercises.

3.1 Participants

A class of 25 first year students of an upper secondary school in Norway participated in this study. They were following a curriculum meant to prepare them for further studies in college or university, as opposed to other possible curricula in Norway which are more oriented towards the mastery of a craft, or geared toward various jobs in the service sector. The course they were engaged in was called mathematics 1T, and is one of two possible choices in the first year of upper secondary, the other (mathematics 1P) being somewhat less challenging. They were taught by a very experienced teacher, who made use of the recent textbook “Mathematic 1T” designed to meet the requirements made by the latest educational reform in Norway (Kunnskapsløftet).

3.2 Challenges

They were several challenges tied to the process of data gathering, especially due to the challenging nature of the course the students were following. The gap in mathematical content between the last grade in junior high school and this one is significant, and the students are thus facing a serious challenge when attending mathematics 1T. Having taught (albeit briefly) this course myself, I was aware of that fact and also of the resulting time constraint both the students and their teacher were placed under. This time constraint was exacerbated by the fact that my data gathering occurred towards the end of their semester, during a period ripe with evaluations and exams of many kinds. The reason for this tardiness was my need for the students to have been coursed in every aspect of quadratic polynomials that were part of the curriculum. This led me to fear for intruding in their learning environment, and prompted me to design my data gathering protocol in such a way that I was confident would not waste their time, and perhaps benefit their acquisition of knowledge.

Another challenge occurred when gathering the required authorizations to conduct the study. When requesting from the students and their parents that the students participate in a potential interview, I did not explicitly request their permission to access their answers to the test I had designed. This resulted in the official body responsible for the approval of my study (Personvernombudet for forskning) demanding that I refrain from gathering the answer papers of those students who had not volunteered for a possible interview. Out of 25 students, only ten volunteered, and two gave me permission (after the interviews had taken place) to gather their papers after I had formally requested access to them. While I had intended to study and analyze 25 papers and base my choice of four interview subjects on these, I was left with ten from which to choose interview subjects, and at present with a total of twelve papers to analyze.

3.3 Choice of subject matter

While my interest for students' understanding of quadratic polynomials was awakened during the MERG project by something of a coincidence, the choice of continuing with that mathematical subject in the present study was not an idle one. Quadratic polynomials are a recurring theme in the curriculum of mathematics 1T, and thus students are confronted with them – or with other concepts and procedures related to them – throughout the whole school year. They are first taught the quadratic identities, then quadratic equations during the study of functions, and at the end of the school year they are confronted with them again while dealing with the derivative of functions. In the course of the year the students thus deal with functions, the algebra (factorization and expansion) of binomials, quadratic equations and inequalities and finally derivatives, all of which having some possible relevance and relationships with quadratic polynomials. This makes the subject ideal to attempt to determine whether students are aware of the many relationships that can be established, in other words to assess their conceptual knowledge of that particular mathematical subject.

3.4 Defining conceptual and procedural knowledge of quadratic polynomials

Investigating the students' procedural and conceptual knowledge of quadratic polynomials requires a clearly defined picture of what these knowledge types might be.

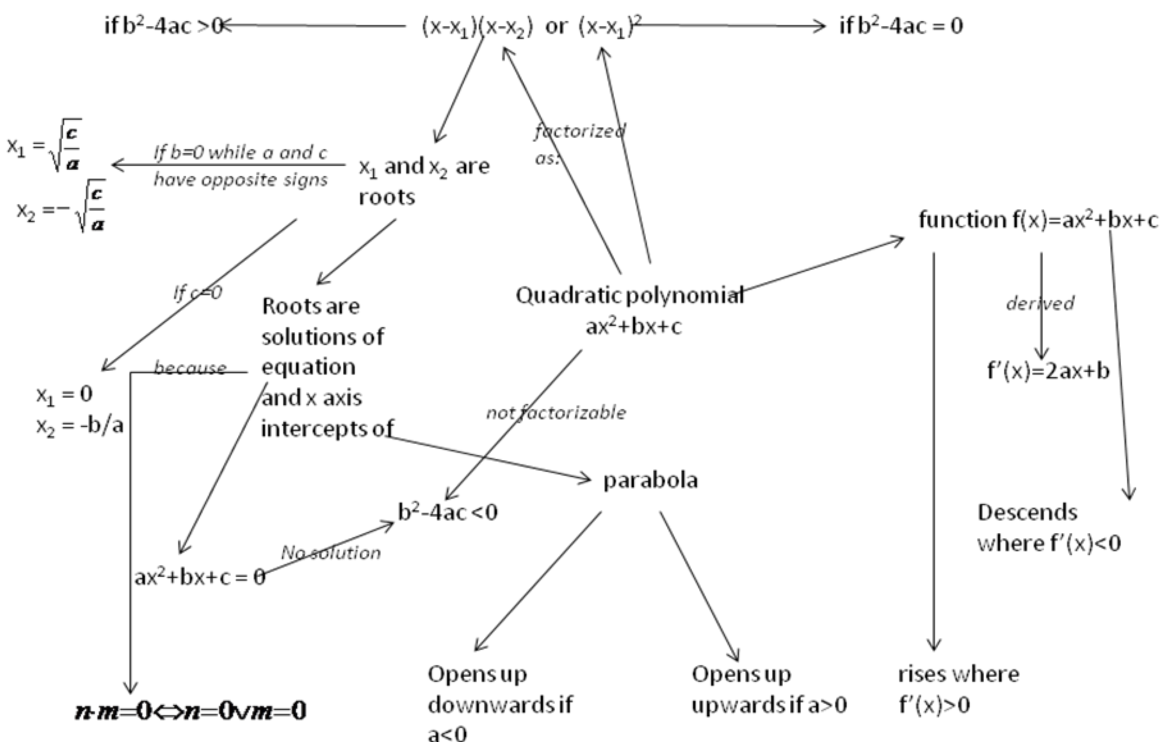
Procedural knowledge of this subject matter is not only defined here as the ability to perform a certain number of algorithms which are described below, but also takes into account the students' flexibility as a measure of the quality of their procedural knowledge. The test described below is intended to give an impression of flexibility by formulating equations and inequalities in slightly different ways to promote the use of different procedures.

As far as conceptual knowledge is concerned, a possible way of defining it here is by the help of the concept map below. If a student is shown to be aware of a number of relationships depicted on the concept map, that student can be said to possess conceptual knowledge.

3.5 Concept map and relevant procedures

By studying the curricular content of the course mathematics 1T, I drew the following concept map for quadratic polynomials.

Concept map for quadratic polynomials



The students have not had any instruction about complex numbers, therefore these are left out of the concept map. Any quadratic equation for which $b^2 - 4ac < 0$, they will regard as having no solution.

I have also left out information about quadratic inequalities, for the purpose of maintaining a certain degree of decipherability. I made some inferences – based on my conversations with the teacher and the content of their textbook – about what the students would know about such inequalities. The procedure they had been taught to solve quadratic inequalities involved first of all finding the polynomials' roots (or rather the corresponding equation's solution, then factorizing the polynomial by using those. The students would then draw a sign diagram with one row for the unknown x, one for each factor and one for the product of these factors. They would then investigate the sign of each factor as x spanned the real numbers, and deduce the sign of the factors products (i.e the polynomial). The students would then give their solution in the form of an interval by looking at the table and finding the relevant values of the unknown with respect to the inequality.

The example below illustrates this procedure for the inequality: $x^2 - x - 6 > 0$

$x^2 - x - 6 = 0 \Leftrightarrow \quad x = -2$ or $x = 3$

Table 1: sign diagram for the inequality $x^2 - x - 6 > 0$

x		-2		3	
(x+2)	----------------------	0	+++++++++++++++++++	++	+++++++++++++
(x – 3)	---------------------	-----	--------------------------------	0	+++++++++++++
(x+2)(x – 3)	+++++++++++++	0	--------------------------------	0	+++++++++++++

$x^2 - x - 6 > 0$ when $x \in \langle\leftarrow, -2\rangle \cup \langle 3, \rightarrow\rangle$

Most of the exercises in the textbook Sinus 1T deal with inequalities having solutions of this form, meaning that their corresponding equations have two real solutions. Since quadratic equations also may have only one or even no real solutions, one interesting avenue of investigation was how students would solve inequalities with such corresponding equations. I therefore included such inequalities in the test I will present in the next section. I also wanted to test whether the students were fluent with the graphical implication of all the possible solution types for quadratic inequalities.

A few other issues have to be explained about the above concept map: the algorithm coined completing the square, upon which a proof for the general formula for the roots of a quadratic polynomial is based has not been included in the map. Although I unfortunately did not make any inquiries to find out whether the students had been taught that procedure, I nonetheless assumed that they had not. Using that algorithm on a particular polynomial, one can find the roots for it, and that process requires some degree of fluency with the following three algebraic identities:

1. $(a + b)^2 = a^2 + 2ab + b^2$

2. $(a - b)^2 = a^2 - 2ab - b^2$

3. $a^2 - b^2 = (a + b) \cdot (a - b)$

These identities I knew the students had been taught at the onset of the school year, and their relevance to quadratic polynomials extends beyond the algorithm for completing the square. The polynomial $x^2 + 4x + 4$ can for example be factorized directly as $(x + 2)^2$ by using the first of these identities.

The items to the far left of the concept map concern the cases where the coefficients b or c equal zero. I did not expect the students to be able to express those cases in their general form. An example of a polynomial satisfying one of these conditions is $x^2 - 9 = 0$, which can be factorized as $(x – 3)(x + 3)$ again by using the third identity mentioned above. This case, and also the ones where the c coefficient is set to zero also provide us with interesting lines of inquiry, which I will outline in the next section.

I have also left out the justification for an equation having no solution if $b^2 – 4ac < 0$, which is the impossibility to extract the square root of a negative real number. I was unsure of what the textbook mentioned of this fact and also of what their teacher had said on the subject, but both the test and the interviews address whether the students are aware of that principle.

Regarding the derivatives of functions involving quadratic polynomials, a common procedure explained in the textbook is the differentiation of a quadratic polynomial and the construction of a sign diagram to investigate the sign of the derivative with respect to the variable x and the deduced properties of the given function. This procedure is illustrated in the example below:

$g(x) = x^2 + 4x + 4$

$g'(x) = 2x + 4$

$g'(x) = 0 \Leftrightarrow 2x + 4 = 0$

$\Leftrightarrow x = - 2$

$g(-2) = (-2)^2 + 4 \cdot (-2) + 4$

$= 4 – 8 + 4$

$= 0$

Table 2: sign diagram for the function g(x)

x		-2	
g'(x)	---------------------------------	0	+++++++++++++++++++++
g(x)	↘	0	↗

It is that procedure I expected the students to be familiar with. Connections can also be made between this procedure, its underlying principles, the graphical representation of the function and other elements relevant to quadratic polynomials. I will mention some of them in connection with the test and interviews.

It should be pointed out that the above concept map is lacking in many respects. It is not hierarchically organized, and some linking words are placed with concepts instead of along the lines connecting them. I became aware of concept maps too late in the course of this study to be able to consider using them more extensively in my evaluation of the students' conceptual knowledge. I could for example have instructed the students in drawing concept maps, and used those in conjunction with the test, but my own difficulties drawing such a map have made me confident that this method would have been very time consuming if it were to yield conclusive results. And as I pointed out above, I had great concerns about intruding too much in the students' learning environment.

3.6 The test

The test took place the last day of April this year and the students had a total of 90 minutes to work on it. Their teacher assured them just before they got started that they would not be graded on it so they should only regard it as training. I asked the students not to use their graphing calculators and also to refrain from deleting anything from their answer paper but instead to underline their mistakes if they felt they had made any.

In the present section I will describe the test in detail and explain my intentions with the many relationships existing between its parts.

<u>1 - Løs likningene:</u>

a) $x^2 - x - 6 = 0$
b) $(x + 2)(x - 3) = 0$
c) $x^2 + 4 = 0$

The first exercise involves three kinds of quadratic equation. The students were familiar with the general quadratic formula and had been taught to use it to solve such equations. Since they were used to utilizing a program on their graphing calculators to provide them with the roots of polynomials, the first thing this exercise tested them for was whether they actually remembered the general formula. Moreover, since the polynomial in b is none other than the factorized form of that in a, the students answers could reveal whether they recognized it as such. It was also of some interest to see whether the students would mention the principle according to which a product equaling zero must contain a factor equaling zero. Finally the third polynomial is a special case where the first degree coefficient is set to zero. It can be solved by applying the quadratic formula, but also by isolating x^2 to the left of the equal sign using an algorithm for solving linear first degree equations. It could also be solved without any algebraic manipulations, by pointing out that the sum of two positive numbers (one strictly positive and the other possibly zero) never can equal zero. The alternative methods for solving b and c require less manipulation of symbols, I therefore regard as more efficient and less time consuming. I also assumed that the students would employ them (if they knew them) for that reason, in order to save both time and energy.

2 - Løs ulikhetene:

a) $x^2 > x + 6$
b) $x^2 < -4$
c) $(x - 2)(x - 1) \leq 0$

The second exercise involved quadratic inequalities, the first two of which were derived from 1.a and 1.c. Those two also required a little manipulation before appearing as a polynomial being greater or smaller than zero. The reason why I had them corresponding to equations the students would already have solved (assuming they solved the exercises in order, which would be apparent from their answers) was to check whether they actually made use of their previously acquired results. To express that in theoretical terms, the students should already have implemented a subprocedure contained within the superprocedure for solving these inequalities. So these exercises would to some extent test whether they were able to regard that subprocedure as isolated from the superprocedure they were familiar with.

An alternative, graphically based method of solving the first of those inequalities was provided to me by one of the students answers, and I will present it along with the results.

Exercise b could also be solved using the following argument: a positive or null number (x^2) can never be less than a negative one. One could also solve it using the same argument as above for 1.c.

The final inequality didn't have any direct relationship with any of the previous equations, but was also given in a factorized form to check if the students would proceed with the sign diagram directly or expand the algebraic expression in order to obtain the coefficients necessary to use the quadratic formula.

3 - Funksjonsdrøfting:

1 - Funksjonen f er gitt ved $f(x) = x^2 - 3x + 2$

a) Finn f'(x)
b) Finn eventuelt topp- eller bunnpunkt
c) I hvilket intervall stiger f ? I hvilket intervall synker f ?

The above exercise dealt with the determination and use of the derivative of a quadratic function. The b question, tasking the students to determine the position of the corresponding parabolas maximum or minimum point was somewhat vague since it didn't explicitly task them to mention whether that point actually was a maximum or a minimum. If I had explicitly required that information before the students had determined (in 3.1.c) where the function ascended and where it descended I could have tested whether the students were aware of the impact the second degree coefficient has on the parabola. Since the function corresponded to the polynomial in 2.c, the students could also have determined the nature of the extreme point by from their sign diagram they likely would have constructed while solving that question.

The construction of a sign diagram is not necessary for answering the above exercises, and I did not task the students with constructing one either.

2 - Funksjonen g er gitt ved $g(x) = x^2 + 4x + 4$

a) Lag en fortegnslinje for den deriverte av g
b) Tegn grafen til g
c) Løs ulikheten: $(x + 2)^2 \leq 0$

In this exercise the students are explicitly asked to use a sign diagram to determine the function's properties, and to draw the graphical representation of it. The last question tasks them to solve an inequality in which the polynomial from the function g is presented in its general form. The less time consuming way of solving that inequality is to consider the expression $(x + 2)^2$ as a squared number which can never be less than zero, making the only possible solution $x = -2$. The most laborious method consists in expanding the polynomial and constructing a sign diagram, and the middle way would be to directly construct a sign diagram with one row for each $(x + 2)$ factor. I was interested in seeing what procedure the students would choose, and also if they would make some use of the graph they had drawn in 3.2.b either to check their result or to propose a solution based on the graph.

3 - Funksjonen h er gitt ved $h(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x + 4$

a) Hvor befinner topp- og bunnpunktene til h seg?
b) I hvilket intervall stiger h?

In this final exercise the students could use their previously constructed (in 2.a) to help them answer task b, since the derivative of h(x) was none other than the polynomial they encountered in 2.a. The construction of a sign diagram was again unnecessary. The question 3.3.a was (as was 3.1.b) intended to check if the students made a common mistake I have encountered in my (by no means extensive) experience teaching functions and their derivatives. The procedure taught to find the Cartesian coordinates of a maximum or minimum consists in finding the x coordinate by finding the value of x for which the derivative equals zero. The y coordinate is then found by inserting the relevant value of x in the function expression. The mistake that students sometimes make is inserting the value x in the derivative instead, which consistently yields a point placed on the x axis of the Cartesian plane.

To summarize, my intentions with this test were the following: the equations and inequalities were given in various forms to check if the students could use several kinds of procedures in solving the exercises. In addition, the answers the students arrived at could be analyses to make assertions as to how many relationships they were able to detect between the parts of various parts of the test. This information could then be analyzed in order to make some informed conjectures regarding the students' procedural and conceptual knowledge of quadratic polynomials, with the help of the concept map from section 3.5.

3.7 Interviews

Four interviews were conducted and video – recorded at the school two weeks after the students had taken the test. I had analyzed the ten answer papers I had been authorized to gather and chosen four candidates I wished to study in further details. I personally contacted all of them to select an appropriate time for the interviews. They were conducted in a semi structured way, and began with a few questions destined to get a sense of how the students viewed their instruction, and mathematics as a subject. The second part of the interviews was task based and destined to enhance my understanding of each individual student's conceptual knowledge. Since I had singled out the students for different reasons, the tasks they were given also varied for each individual student. Before the onset of each interview, the students were assured that the video I recorded would be watched only by myself, and would finally be deleted after a period of one year.

It should be mentioned that one candidate, Maria, had previously been a student of mine while I taught science at her school during the previous semester. The short length of my engagement as a teacher at that school (due to a paternity leave) made me doubt that this state of affairs would significantly influence her responses on the test and during the interview.

I had decided to volunteer no clues during the task based part of the interviews, and to avoid commenting the students' responses and solutions to the tasks. To compensate for my not providing the answers to the tasks while the camera was running I volunteered the answers to each tasks and my reasons for confronting each student with them when the interview was over. I also proposed to answer any question the students should have about the project and my reasons for interviewing them.

The students were chosen on the basis of their performance on the test. Maria was chosen because of her seemingly strong grasp of the procedures involved and her effectiveness in solving the exercises, meaning that she made use of her previously acquired results. Ole was chosen because he did not do so, and was also one of the weakest students procedurally. Christina performed well procedurally, and was also effective in using the results she arrived at, though less consistently then Maria. Finally, Helene performed moderately well on the test, but she demonstrated on a couple of occasions that she used her conceptual knowledge as a means to check the validity of her results. The main purpose of the interviews was to investigate if there was a noticeable relationship between the students procedural knowledge (or lack thereof) and their conceptual knowledge. To that aim, three interviews were transcribed and scrutinized but I did not find that they yielded any more insight into the students' conceptual and procedural knowledge, so I chose to leave them out of the analysis section.

Finally, preceding the interview the teacher had (on his own initiative) made the students work in groups of two to three on the test, in order to discuss the exercises and correct their mistakes. Since I based the interviews on some of the students' answers on the test, this correction session may have influenced the outcome of the interviews, by permitting them to perform better on the interview tasks than they would have otherwise. I didn't see any clear signs that it had, but the possibility cannot be dismissed.

4. ANALYSIS

In this section I will present and analyze the students' answers to the test, focusing on the algorithms they used for each exercise. I will also examine some of the mistakes made, and endeavor to establish whether they can be attributed to computational errors, or to some other reasons. One important area of investigation is whether some mistakes can be said to indicate a lack of conceptual knowledge. I will also try to debate whether the choice of some algorithms over others may be said to indicate greater fluency and/or conceptual knowledge.

4.1 Test results

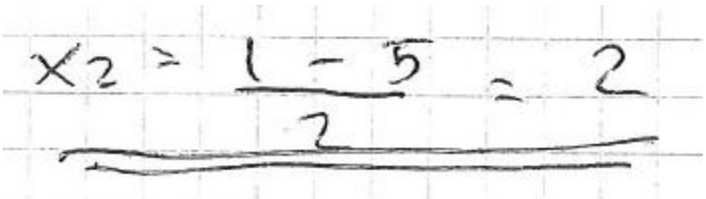
In this section I will present the students answers to the test, in the same order as the exercises were given.

4.1.1 First exercise

a) $x^2 - x - 6 = 0$

To solve the first equation, all twelve students resorted to the quadratic formula (they took the test without access to their calculators, which they commonly use to compute the roots of polynomials). Three students made the same mistake towards the end, while computing the negative solution of the equation. An example can be seen in the figure below:

Figure 4.1: Bjørn's mistake



Failing to add the negative sign before the solution may be considered to be due to inattention on the part of the student. It is noteworthy that one of the students – Michelle – made this mistake consistently throughout the test each time the polynomial $x^2 - x - 6$ appeared either in an equation or an inequality (three times). This forces one to consider the possibility that she has misunderstood relative numbers, however unlikely this may seem. However, since she correctly computes a similar operation towards the end of the test, inattention remains the most likely hypothesis. The answers to this first exercise give the unequivocal indication that all students have mastered the basic procedure for solving second degree equation.

b) $(x + 2)(x - 3) = 0$

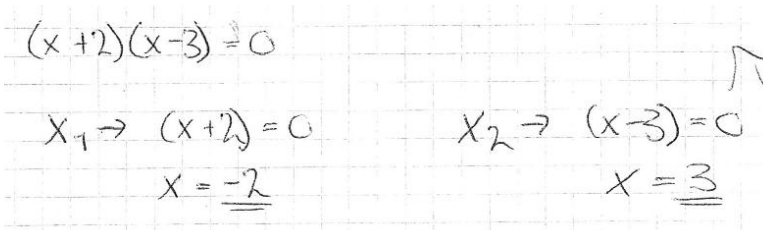
The second equation yielded more varied results from the students. As mentioned in the method section, it was intended to test whether they would solve it using a different procedure than the first one. They had incentive to do so because it would have been less time consuming. Since the assumption that they would use the least possible amount of time seems reasonable, it is very likely that those who did not use the alternate procedure did not know it.

Eight out of twelve students expanded the expression to the left and proceeded to solve the equation using the same procedure as for the first one. The interview subjects Ane, Helene and Ole belong in that category although the latter made a mistake while expanding and apparently moved on to the next exercise. He finally solved the equation towards the end of the test, this time after having expanded correctly.

These students' choice of procedure seems to indicates that they are not familiar with the alternate, quicker method for solving this equation, and consequently with the logical implication of a product equaling zero. It also generates a few questions, not the least of which is: do they immediately recognize a second degree equation even though there is no x squared in the expression to the left? The method they use in exercise 2-a) for solving the corresponding (i.e. based on the same polynomial) inequality suggests that they do, since that method requires them to factorize after having found the roots. Consequently these students can be said to have established one link on the concept map drawn above, namely the one between a second degree polynomial and it's factored form. However this link is of a somewhat tenuous nature since they do not immediately recognize -2 and 3 as the solutions they are looking for, so while they do seem to be aware that a quadratic polynomial can be written in at least two different ways, they are not familiar with what can be inferred from the factored form. Another possibility that cannot be dismissed is that they are used to expanding an algebraic expression when they set out to solve an equation regardless of the equation's degree. There is also one puzzling question which has to be addressed: why do they not simply refer to their previous answer instead of computing the solutions all over again. It seems unlikely that they should not recognize the first equation after having expanded the expression, so it is possible that they believe it is expected of them to perform the computation again.

The remaining four students did not expand the expression. Among these, two explicitly mentioned the fact that one factor of a product equaling zero has to be zero (henceforth dubbed the null product equivalence), and they did so using a sentence. The third (Jo) did not, but the way she wrote her answer suggests that he was aware of it.

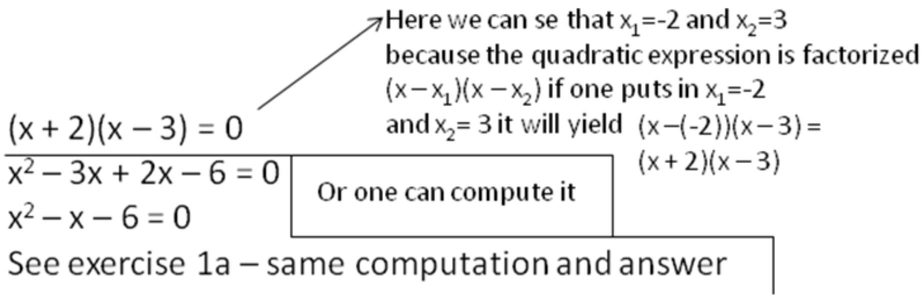
Figure 4.2: Jo's answer



The fact that Jo sets up one first degree equation for each factor indicates that he is aware of the implication of the product of these factors being zero. These three students show that they have established the corresponding connection on the concept map, and that connection is of a more general nature than the one between a polynomial and its factored form. One way to test the robustness of that connection would have been to task them with solving a similarly factorized equation, with three or more factors. If they could do that successfully this would mean that they were able to adapt their previously acquired knowledge to somewhat new problem solving situations, and therefore that their use of the null product equivalence was not contingent on their knowingly dealing with quadratic equations.

The fourth student (Maria, one of the interview subjects) did not expand the expression, but did not either mention the null product equivalence. Instead she pointed out in a sentence that this was a case of a factorized quadratic expression. Since she wrote with a pencil, and her hand in was copied and scanned, her writing is barely legible on the scanned document, though it is somewhat better on the copy I made. I have therefore chosen to reproduce what she wrote as faithfully as I could:

Figure 4.3: Maria's reproduced answer



It looks as though she initially expanded the expression, because the computations are written in order, in the same way she solves the exercises throughout the whole test. Then, realizing that her computations yielded the same equation she had solved in the previous exercise, she added the sentence as a side note. It also possible that she expanded the expression after having written the sentence (perhaps to illustrate her point), however this seems the least likely of the two hypothesis given the way the sentence is framed to the side of the computations with an arrow pointing towards it. One thing that is certain is that she did not feel compelled to reiterate her calculations from the previous exercise, since she did not bother to do it and instead referred to 1a.

In any case, she established a connection between the polynomial and its factored form even though one cannot be sure whether she did during the course of the test or whether she had done so before the test. If the latter is the case one is given to wonder how many other students established the same connection at the same time, and did not write down that they had.

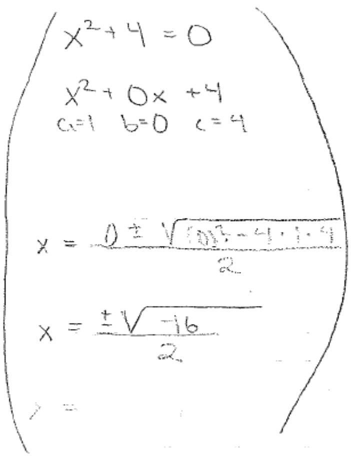
c) $x^2 + 4 = 0$

The range of solutions offered by the students for this exercise was much more varied than on the previous questions. Four students started solving the equation like they would have solved one of the first degree, by deducing that the x squared must equal -4. Then they proceeded to point out that the square root of a negative number could not be extracted (as mentioned above, they had not been taught anything about complex numbers). Among those students were three interview subjects: Ole, Helene and Ane. The use of this method could be said to indicate a beneficial degree of flexibility, since the students were able to use a feature of this equation (the fact that the coefficient b was zero) to their advantage, allowing them to solve the exercise with fewer computations than they could have made had they used the general quadratic formula. Another student – Bjørn – also applied that method, albeit without success because he inferred that the x squared would equal 4. He then found only one solution for the equation, namely two. While the first mistake is most likely due to inattention, the second shows a lack of understanding of square numbers, unless of course it was due to forgetfulness.

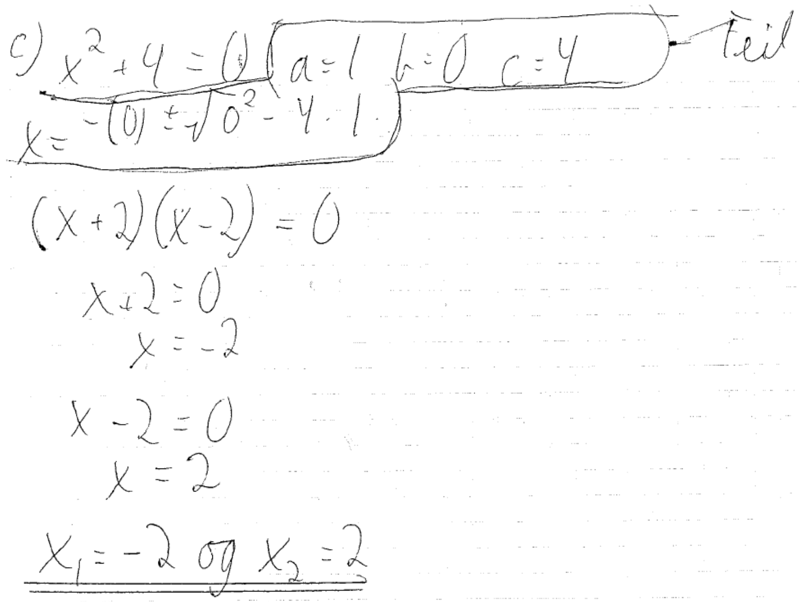
Three students successfully used that formula and concluded by pointing out that the square root of a negative number does not exist. More interestingly, two students started applying the formula but did not follow through, apparently for different reasons. Their answers merit closer examination, and are thus shown below.

Figure 4.4: Michelle and Are's answers

Michelle



Ole

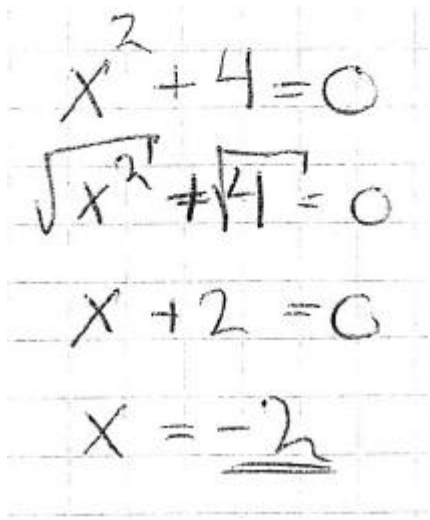


Michelle started applying the quadratic formula, and stopped when she was confronted with the solution for x containing the square root of a negative number. She put her whole calculation within a pair of brackets and stopped working on the exercise. As has been mentioned above, the brackets were one way the students had to signal that they believed they had made a mistake. Though she had accomplished all the necessary calculations, she was unable to conclude anything from them, and considered instead that she had made a mistake somewhere. This belief can be attributed to two things, the first being the fact that the coefficient b equaled zero, the other being the negative number under the root sign. Michelle and her peers had presumably been confronted with these situations less often than they had dealt with equations having one or two solutions and with all coefficients different from zero. Oddly enough, when confronted with an inequality consisting of the same polynomial, she concluded that the inequality had no solutions. So when facing an exercise that she was somewhat less familiar with, she was not consistently able to draw the appropriate conclusion even though she apparently possessed all the knowledge required to successfully solve it.

Are also begun solving the exercise with the quadratic formula, but like Michelle he gave up and explicitly signaled that he had made a mistake. He then proceeded to solve the equation as if the third quadratic identity could be applied. It is not possible to be sure whether he believed that using the quadratic formula really would yield an incorrect result or whether he wrote "wrong" because he found that using the third quadratic identity was a simpler and therefore better way to proceed with this exercise. If the former is true then Are did not realize that two different procedures should yield the same results if they both are applied adequately. Of course in this case the third quadratic identity was not appropriate because the coefficient c was positive. The interview subject Maria also mistakenly used that procedure. Whether the mistake is due to inattention or to the students having misunderstood the quadratic identity is not possible to say.

The last student's answer is hard to qualify, it is shown below.

Figure 4.5: Jo's answer



This mistake shows that Jo lacks some fluency with squared numbers, since what he writes implies that the square root of a sum equals the sum of the roots. His method also gives rise to several questions: would he consistently apply this procedure to such equations? Was this the first time he did so? If it was, what prompted him to do so?

To sum up, seven students showed that they could be opportunistic, and profit from the idiosyncrasy of this equation to use a less time consuming method for solving it. Four of those managed to correctly solve the equation. Two students correctly solved the equation using the general quadratic formula.

4.1.2 Second exercise

a) $x^2 > x + 6$

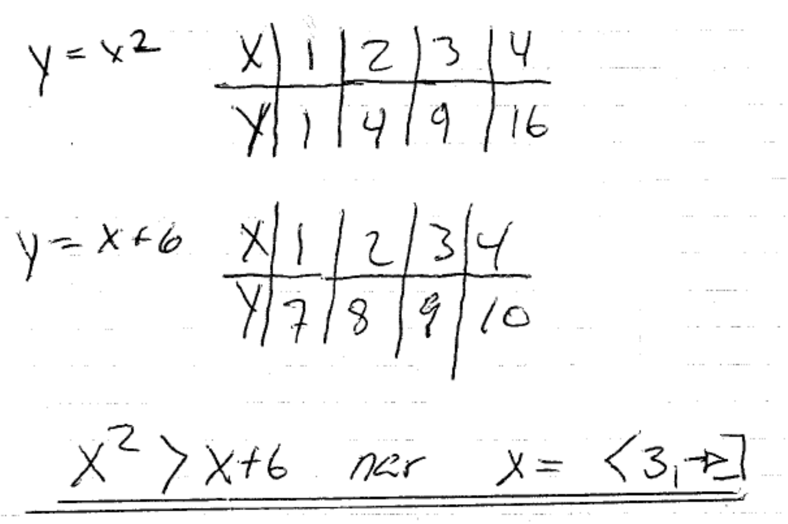
On this exercise, six students used their solution for the corresponding equation from 1) a). Three of those were the interview subjects Maria, Helene and Ane. All five students managed to successfully (somewhat) solve the inequality, though Ane made a mistake when writing down the solution, as did Jo and Helene. The mistake consisted in giving the solution as a union of intervals but by including the values -2 and 3, in other words solving the inequality $x^2 \geq x + 6$. Since this kind of mistake is made with some consistency by the same students, as well as others, later on in the test, I am not inclined to characterize it as being due to inattention. It is more likely due to the fact that the students are not wholly fluent in the ways to write intervals, and more importantly with the meaning of these notations. This begs the question: does their lack of understanding limit itself to the writing of intervals or does it extend to the theoretical meaning for the solutions of quadratic inequalities, as well as the graphical representation of these solutions? It will be easier to answer this question at the end of this analysis.

The remaining students did not use their answer to the corresponding equation to facilitate their work. Again, this could mean that they thought it was expected of them to reproduce all the steps in the superprocedure for solving inequalities, but it is also possible that they had such a shallow understanding of that superprocedure that they could not divide it into several subprocedures that also have meaning independently of each other. This could explain that they failed to recognize the fact that they already had done half the work in exercise 1) a).

Three of those students proceeded to solve the inequality successfully (the interview subject Ole belongs in that category), two included the values -2 and 3 in their intervals and one just solved the equation and left it at that.

The last student (Jarle) used the quadratic formula – incidentally making the same mistake with one of the solutions as he had done in the first exercise and thus finding the solutions 2 and 3 – and did not seem to remember how to set up a sign diagram. However he knew that the solution to the inequality had to be an interval of numbers so he used a table of values to find the interval.

Figure 4.6: Second part of Jarle's answer



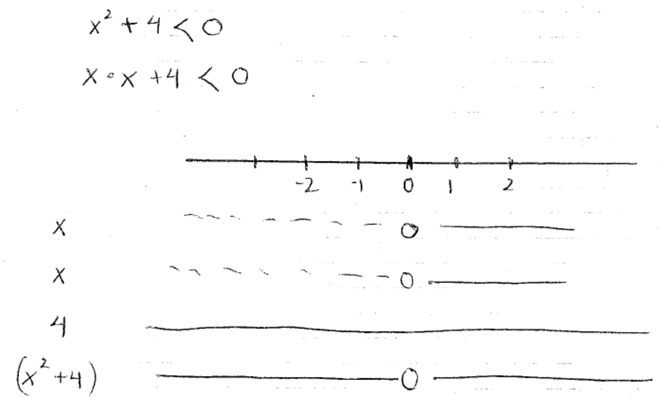
By using these two tables Jarle concluded that the inequality was satisfied for values of x strictly superior to 3. He only got half the answer, and as mathematical arguments go his was not valid because it was inductive, but the answer is less interesting than the method he used. The fact that he first solved the corresponding equation strongly suggests that he intended to set up a sign diagram, since the solutions to that equation have no apparent bearing on the tables of values depicted above. Moreover it is unlikely that he would not use the method he had been taught in class to solve an exercise on a test. So one can be reasonably sure that he used these tables because he did not remember the next steps to take to use a sign diagram but wanted to try and solve the inequality in spite of that. It is not known whether he came up with his method then and there or if he had applied previously on other exercises, but the former seems to be the most likely. If he had used this method before it is quite probable that he would have asked his teacher for feedback on it, and would consequently have either abandoned it or refined it somehow (for example by also investigating for negative values of x), depending on what the teacher would have said. Assuming he "invented" this method during the course of the test, he demonstrated a readiness to experiment that could be used as a pedagogical tool. I will elaborate on this when dealing with the pedagogical implications of these results.

b) $x^2 < -4$

The answers to these inequalities were very diverse. Three students, Maria, Ole and Jakob correctly solved $x^2 < 4$ by applying the third quadratic identity to factorize the polynomial $x^2 - 4$. Maria and Ole had also mistakenly solved the corresponding equation in 1) c). As it was in the previous case it is not possible to infer much from their mistake, but in exercise 1) c) the mistake could have been due to their confusing $x^2 - 4$ with $x^2 + 4$. In this case however the cause could have been that they made a mistake while "moving the 4 to the left side of the < sign".

Four students (three of them the interview subjects: Ole, Helene and Ane) argued with a sentence that a number squared could never be inferior to a negative number. Ole and Helene did so instantly, but Ane began by setting up a sign diagram:

Figure 4.7: Ane's sign diagram



Right underneath this diagram she made the same argument as the other two. What is striking is that the sentence does not seem to be a logical consequence of the diagram. It would have been if she had not had a line for the number 4, and if the last had been for the x^2 alone instead of $x^2 + 4$. In that case the diagram would have shown that the product of x with itself is always positive. There are also some mistakes in the diagram, firstly she deduced the sign of a sum from the sign of the sums elements, and secondly she found that $x^2 + 4$ equaled zero for $x = 0$. The two mistakes are related in that they seem to stem from her applying the proprieties of a product to a sum. They reveal that she has incomplete understanding of the underlying principles of the sign diagram. At the same time she exhibited a stronger conceptual knowledge of squared numbers than the students mentioned below (as do Ole and Helene), and was able to use that to her advantage in that exercise.

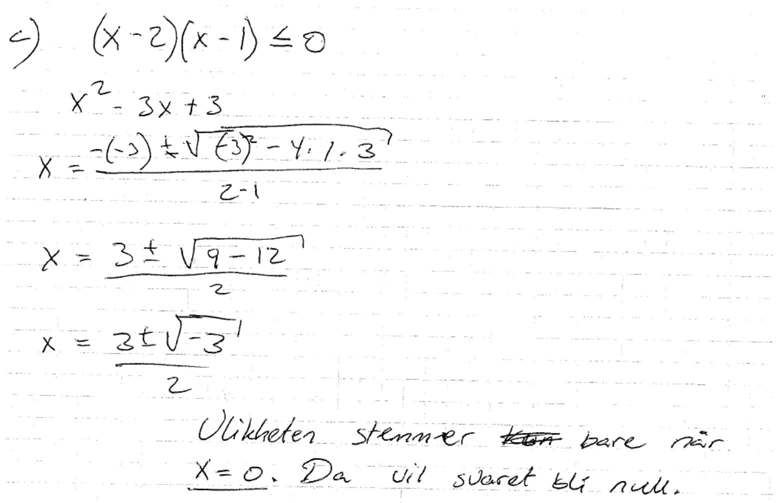
Three students applied the general quadratic formula, one of those simply by referring to what she had done in 1) c), stating "it is the same problem as in 1 c". And finally one student simply wrote that he did not know how to solve the exercise, and the last one (Jo) referred to his answer from 1) c), depicted in figure 4.5 and concluded that x must be less then - 2.

Seeing the way some of the students phrased their answers, especially those who applied the quadratic formula, one gets the sense that they believe they are solving an equation. I regretted afterwards the fact that I had not given them the opposite problem to solve, namely $x^2 > -4$ to see whether they would have contented themselves with solving the equation and concluded that there existed no solutions, even though the inequality is true for all real numbers. As will be seen below, the interviews indicate that this hypothesis cannot be dismissed.

c) $(x - 2) \cdot (x - 1) \leq 0$

Seven students expanded the expression, and four of those made appropriate use of a sign diagram (the interview subjects Ole and Helene belong in that category). Among the three remaining students who expanded the expression, two contented themselves with correctly solving the equation. The last one made a mistake while expanding: instead of multiplying (-1) and (-2) he added 2 and one together, yielding the polynomial $x^2 - 3x + 3$.

Figure 4.8: Jarle's answer



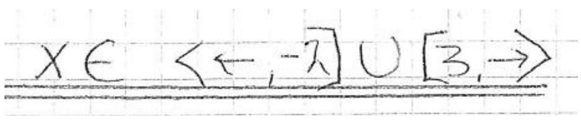
He had made the same mistake while expanding the expression in 1) b), which then resulted in him getting the number 21 under the root sign. He did not compute anything else on that exercise, most likely because he did not have a calculator. In the answer depicted above the expansion mistake led him to compute a solution for x containing the square root of – 3. While he had shown in 1) c) that he was aware that this was impossible, he did not point that fact out in this exercise, and instead concluded that the inequality only had the solution x = 0. Since he did not have a calculator, it is possible that he believed that $3 \pm \sqrt{-3}$ would yield zero. It is not possible to know how he thought in this instance, but what is apparent is that he consistently made no connection between the roots of a polynomial and that polynomial's factored form. Two factors lead to that conclusion: firstly the fact that he expanded a factorized expression both times he was confronted with one, and secondly the fact that he did not realize his mistakes despite them leading to solutions which were not consistent with the factored expressions.

While those students who did not expand the expression arguably showed a greater degree of conceptual knowledge than those who did, there is one mistake which was made by both groups of students which needs to be examined. That mistake has been mentioned earlier, consists in giving the solutions for the opposite inequality. In this case the students solved the inequality $(x – 2) \cdot (x – 1) \geq 0$, (one of them did not include the values of x that yielded zero).

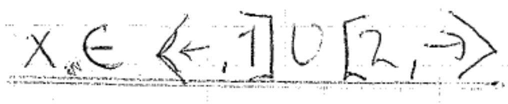
This mistake is interesting because it is not dependent on how successful the students are with setting up a sign diagram, since the diagram gives the solution for an equality regardless of which way the inequality sign points. So it is either the students' interpretation of the diagram that is at fault, or some lack of understanding of intervals (or their notations) on their part. Looking back at the student's answers for exercise 2) a) the last hypothesis finds itself strengthened. Indeed, in both exercises three students wrote the solution as a union of intervals with the roots of the polynomial included.

Figure 4.9: Jo's answers on 2)a) and 2)c)

2) a)



2) c)



In the first case the roots of the polynomials were not supposed to be included, and in the second they were part of the solution, but as the endpoints of the closed interval [1,2]. In this case the students' incorrectly concluded on both exercises by writing the same solution type, but the answers were incorrect for different reasons. This could indicate that they wrote the solution in this manner because they expect solutions to inequalities to be written like this, which in turn would imply that they have limited understanding of intervals or their notations.

4.1.3 Third exercise

3)1)a) $f(x) = x^2 – 3x + 2$

Find f'(x)

All students correctly found f'(x) in this exercise, which they all answered by writing down the expression for f, and the one for f' right below it. This is a quite strong indicator that they possess a good grasp of the differentiation procedure for quadratic functions.

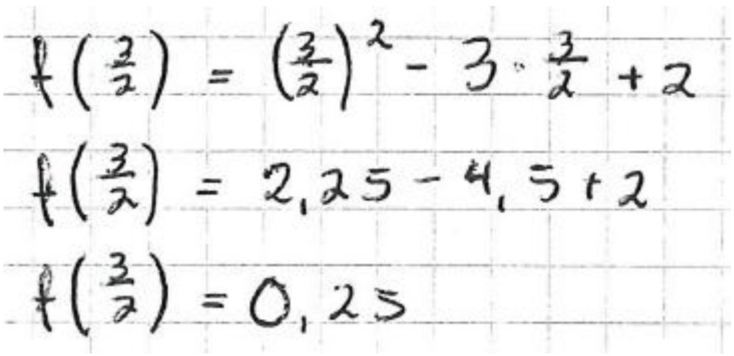
3)1)b) Find the maximum or minimum point of the function

3)1)c) Under what interval does f increase? Under what interval does f decrease?

These two questions are addressed simultaneously because the answers given by the students in c) depended entirely of their answers in b). All students drew a sign diagram in b), which provided them with the means to conclude for both questions. In addition, it is of some interest to examine some discrepancies between the results obtained from the sign diagrams and some students' subsequent conclusions.

Four students succeeded in finding the x coordinate of the minimum point, and they all concluded after having drawn a sign diagram. The sign diagram provided them with more information than just that coordinate, since it also made apparent the fact that the extreme point of the function was a minimum, by showing that the functions decreased for values of x inferior to $\frac{3}{2}$, and increased for subsequent values of x. Consequently the diagram made apparent the answer to the next question. While two of those students contented themselves with providing the x coordinate of the bottom point, Helene was the only one who correctly substituted x with $\frac{3}{2}$ in the expression for f to find the corresponding y coordinate of the bottom point, thereby completely answering the question. Jakob did the same computation, but mistakenly equated $1{,}5^2$ to 2. Here it is of some importance to remind the reader that the students did not have access to their calculator, so the way the Helene and Jakob approached their computation is of the utmost significance. The final part of Helene's answer is shown below.

Figure 4.10: Helene's computation



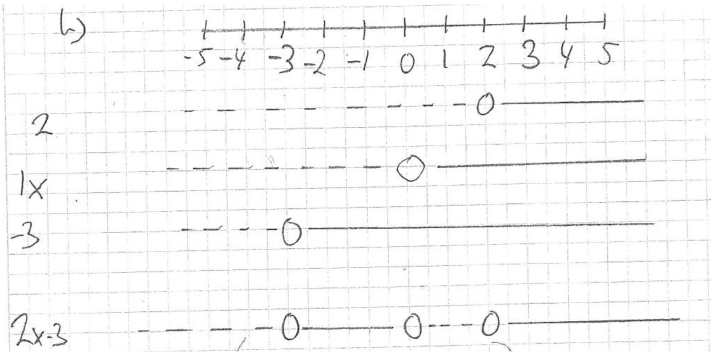
It is reasonable to assume that no students could compute the square of a decimal number without calculator, but the students could have possessed the necessary prerequisites to compute that of a fraction. It is therefore likely that Helene equated $\left(\frac{3}{2}\right)^2$ to $\frac{9}{4}$ and subsequently found the value 2,25.

Apparently it did not strike Jakob to use the same method, so Helene showed a greater degree of fluency with rational numbers in that she was able to shift from one notation from the other as suited her purposes.

Four students made mistakes while drawing a sign diagram, and those mistakes, albeit not always identical, seem to stem from the same misapprehensions.

Sign diagrams are based on the principles governing the sign of a product as related to the sign of the factors. In this case the students seemed to apply the same principles to the sign of sums and differences. In addition two students made gave the constants varying signs as x spanned the set of real numbers. One example is shown below.

Figure 4.11:Ole’s diagram



Ole then concluded that the function had minimums at -3 and 2, and a maximum at 0.

He decomposed the expression into three parts without regard to the operations that connected the numbers. While the two first lines in this diagram are indeed the factors of a product, the third is not. He also consistently wrote that the sign of the two constants varied with respect to x. Michelle drew a similar diagram but she only had the sign of the factor 2 varying while the sign of -3 was always negative. This generated one less change of sign for the expression 2x – 3.

Karl’s diagram was set up identically to Ole’s, but Karl did not give have the constants vary as a function of x, so it yielded a result that was closer to the actual characteristics of the quadratic functions, namely that it first decreased then increased. Kristin did not separate 2 from x, and got the same result as Karl did.

While these four students made a similar mistake in setting up their diagram, they should be divided into two categories, since only Ole and Michelle made the sign of the constant vary. This indicates that they had not understood the rationale underlying sign diagrams since they did not recognize that the number line applied only to the variable of the function, and consequently that any line in the diagram not containing x should be given a constant sign. This could therefore be characterized as a lack of conceptual knowledge in addition to a lack of procedural knowledge (the last because the students did not manage to successfully draw their diagram). Another argument for a lack in conceptual knowledge is the fact that Ole and Michelle apparently did not connect the function f with its graphical representation as a parabola: if they had, they would have had to revise their conclusions, which had the f function changing orientation three and two times respectively. Of course, one cannot exclude the possibility that Ole and Michelle actually did make the connection and realized their conclusion was erroneous but chose not to correct it. Had that been the case, it is still reasonable to assume that they would have signaled their awareness aware of a mistake, as they had done earlier on some occasions. Their other attempts at diagrams will be scrutinized to check whether they consistently made the same mistakes.

The last argument for lack of conceptual knowledge does not necessarily apply to Kristin and Karl, who gave a conclusion consistent with the general properties of a parabola, namely that it changes monotonicity only once. In addition the two students seem to have understood that the number line represents the different values of the variable. Their only mistake resided in decomposing the derivative into the arguments of a sum, setting up a line for each argument. This yielded a zero point for the derivative at x = 0 and therefore only one extreme point at x = 0. If not submitted to a close scrutiny, the result would not conflict with the general properties of a parabola, namely that of changing orientation only once. Kristin and Karl could have made a mental check of this and found that their result was consistent in that respect. To spot a mistake they would have had examine their result more thoroughly, or recognize the function as the expanded form of the polynomial they had worked on in exercise 2)c) and realize that the extreme point of a parabola could not be situated outside the interval containing the roots of the polynomial defining that curve.

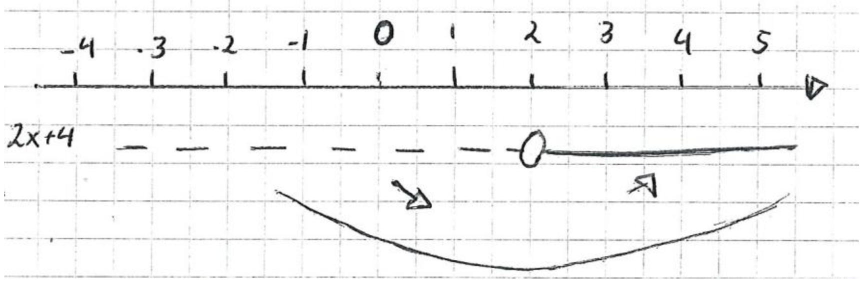
Finally Are did not draw a sign diagram but correctly found the value of x which yielded a zero value for the derivative. He started out by finding the expression for the derivative and wrote: “top or bottom point happens when f’(x) = 0”. After having followed through and found the value 1,5 for x, he signaled his belief that he had made a mistake. He then proceeded to computing the value of the derivative for x = 0, and after having found the value -3 concluded that f increased for x ∈ ⟨←, 0⟩ and decreased for x ∈ ⟨0, →⟩. It is not possible to know how he came to this conclusion.

3)2) $g(x) = x^2 + 2x + 4$

a) Draw a sign diagram for the derivative of g.

All the students successfully computed the derivative of the function g, and seven of them also drew correct sign diagrams for g’, while one did not draw any. The mistakes that were made by the remaining four can roughly be sorted into two categories. The first is very likely due to inattention, and was made by Helene, who had successfully found the point (0,-2) for which g’ equaled zero, but when drawing her diagram right underneath , she made the expression 2x + 4 equal zero for x = 2.

Firgure 4.12: Helene’s diagram



This led to an interesting development, which shall be presented in the analysis of the students’ graphs.

While the three other students did not draw exactly the same diagrams, their mistakes were all a result of their apparent misunderstanding of sign diagrams, and as such were similar to those analyzed above, in section 3)1)c). For example, Ole drew a diagram similar to the one depicted in figure 4.11 (which represents Ole’s own diagram for exercise 3)1)c). He showed consistency in his error, while Michelle – who had made the same mistake as him in the previous exercise – did not and drew a correct diagram. Are also drew one line for each elements of the sum 2x + 4, but did not make the sign of the constant 4 vary with respect to x as Ole had done. His conclusion was therefore that g’(x) was negative for values of x inferior to 0, and positive for values greater than zero. In other words his answer to that question was similar to Kristin’s diagram for exercise 3)1)c).

Bjørn did not make the mistake of decomposing the sum in the same fashion as the two students above, but instead factorized the expression 2x + 4 to 2(x + 4) and drew one line for each factor. However, he made the sign of the constant vary as x spanned the number line. This yielded two zeroes for g'(x), one at x = - 2 and one at x = 0.

As for the previous exercise there were two misconceptions underlining the mistakes made. One was that lines in a diagram not containing the variable still could have a varying value and sign, and the other that the each elements of a sum was to be treated as though they were the factors of a product. These two errors were sometimes compounded and sometimes not, and though they could still yield conclusion which would seem correct if not submitted to close scrutiny these conclusions would still be consistently false. It is worth noting that this particular sort of sign diagram was not drawn by any student in the exercise dealing with quadratic inequalities. This fact will be examined in the section devoted to discussion.

b) Draw the graph for g

Among the seven students who had drawn correct sign diagrams for g', four drew correct graphical representations of g. All of those first computed some corresponding values of x and g(x) which they put in a value table. Apparently they plotted those points in a coordinate system and proceeded to join them with a curve. Using this method made the students able to draw a correct curve without necessarily making any use of the sign diagram they had drawn in the previous question. In fact, apart from Helene there is no clear indication that any student did so, on the other hand some students did overlook some contradictions between their diagram and their graph. Those contradictions stemmed either from an error in the diagram (as in Are's case) or an error in the computation of points in the coordinate system (as is apparently the case of Kristin). While Are, whose diagram was described above drew a correct curve, Kristin had made a correct diagram but still placed the bottom point elsewhere than at (-2,0) on her graph, looked more like two straight lines meeting at a curved angle than a parabola.

Helene's diagram is shown in figure 4.12. As was mentioned above, she had found that the value of x for which g' was zero to be -2 but as can be seen in the figure she drew her diagram with the value 2 for the x coordinate of the bottom point. To answer 3)2)b) she proceeded to find three point that would lie on the curve, the first of which was to be the bottom point. Using the value 2 she found that point to be (2,16). She then found the points (4,36) and (0,4), and argued that g passing through that last one was not consistent with the placement of the bottom point she had found. She concluded that she must have made a mistake somewhere and left it at that. In this case she showed that she was aware of the connection between her diagram and the curve, as well as a readiness to critically examine her result in the light of the properties she knew the curve must have.

Karl and Michelle both drew the graph for g', though Michelle signaled her belief that she was mistaken by drawing a set of parenthesis around her graph. She had filled a table with some x and y values that belong to the function g'. Apparently she made another attempt at drawing a graph for g since she proceeded to fill another table, this time with values belonging to g(x). She was possibly unsure of what to make of that table however, because she not only did not make use of it, and instead also put that table between a set of parenthesis. The fact that she correctly drew the graph for g' based on a table of values, but did not make a drawing based on the second table is not easily explained. One conjecture is that she did not immediately know what to make the values in her table, and left the exercise at that because she was pressed for time. This hypothesis is emitted because there is not any trace of her attempting the last exercise, and she seems to have stopped after 3)2)c).This could also mean that she simply deemed the very last exercise to be two difficult and did not see fit to even attempt it, or did not know how to begin. That last exercise did stand out as being the only one dealing with a third degree polynomial function. Karl's drawing is likely the result of inattention, because he labeled his y axis g(x) and his curve g(x) = 2x + 4. It can be ruled out that he consistently confused a function with its derivative because he successfully solved the last exercise and even drew the shape of h(x)'s graph.

c) Solve the inequality $(x + 2)^2 \leq 0$

Jakob and Are were the only students who managed to solve this exercise successfully, by setting up a sign diagram with one line for each x+2 factor. Next to his diagram, Are also solved the corresponding equation, which he solved by using the null product principle (which he had also used earlier in the test). Because of this it is unclear which procedure he based his conclusion on, and it is possible that he realized that both could be used with success.

The rest of the students expanded the polynomial expression and applied the quadratic formula to solve the corresponding equation. In addition, all but one of them either concluded by giving the solution of the equation or that of the opposite inequality. Maria was the exception here because she signaled that her use of the quadratic formula was unnecessary, and proceeded to set up a diagram similar to that Jakob. She then correctly concluded that the only possible solution to the inequality was x = - 2, but wrote next to her conclusion: "is that possible? This is interesting because in the previous exercise she had drawn a correct graph of the function g, which should have provided her with a confirmation for her result. This either indicates a lack of conceptual knowledge: that she doesn't know the connection between the curve and the roots of the corresponding equation on the concept map, or that she has established a tenuous connection which limits itself to the case where a polynomial has two distinct roots. As Star (2005) distinguishes between knowledge type (conceptual and procedural) and knowledge quality (superficial or deep), Maria's conceptual knowledge could be superficial because it is not adaptable to novel situations. What Maria (as well as Jakob and Are) does demonstrate is a strong grasp of the procedure for solving a quadratic inequality. That the other students use a correct if laborious procedure but conclude erroneously suggests that the application of the procedure itself is not supported by a abstract knowledge of how the solutions for an inequality are interpreted, namely as the values of x making the statement in the inequality true.

3)3) h(x) = $\frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x + 4$

a) Where are the function's top- and bottom points located?

Ten students successfully found the expression for the derivative function of h, and three of those (Maria, Jo and Jakob) referred to their answer from exercise 1)a) instead of repeating the procedure for finding the solutions required to find the x coordinates of the top and bottom points. Jakob was the only one who correctly inserted the x coordinates into the function to calculate the y coordinates of the top and bottom points. He successfully found the y value corresponding to x = 3 but made a slight mistake calculating the other one. The question was arguably too vaguely phrased as the coordinated were not explicitly demanded, so it is possible that he was the only one to do that is due to the other nine students' interpretation of the question. They did not answer the question explicitly and instead drew a sign diagram similar to the one in exercise 2)a). Maria did attempt to find the y coordinates of the points, but she inserted the x values – 2 and 3 into the derivative, which provided her with the growth coefficient at those points. She did not interpret them as such, thus revealing a lack of understanding of how a function and its derivative are connected.

b) In what interval does h increase?

Those students who had already drawn a sign diagram for the derivative of h used it to good effect to conclude on the intervals where the function was increasing. This shows that the students see the connection between the sign of the derivative and the monotonicity of a function, as shown on the concept map. Whether this qualifies as conceptual knowledge is arguable. It does show that the students make a connection between two pieces of knowledge, but at the same time the students may simply conclude use that connection as a part of a procedure for finding out where a function increases or decreases.

5. DISCUSSION

The most striking element that transpires from the analysis is the students' knowledge of the longest procedures (especially solving quadratic inequalities) is characterized by its' ordered nature, as described by Hiebert and Lefevre (1986). As a result the students often seem forced to solve a quadratic inequality by applying an ordered number of steps even though they are in possession of information that should enable them to dispense with some of them. An example of that are students who did not use exercise 1)a) to solve 2)a). This also indicates that those same students have difficulties making sense of the meaning of the subprocedures they apply, and what information can be gained from them. The students who are able to attach meaning to the subprocedures they have previously applied show that they have deeper procedural knowledge, as defined by Star (2005).

Some students demonstrate that they are aware of some of the connections depicted on the concept map: those between the sign of $b^2 - 4ac$ and the existence of roots for a quadratic polynomial and those between the sign of the derivative and the monotonicity properties of a function primarily. Those connections are arguably integrated in the procedure the students use to solve equations and find out whether a function increases or not. So the awareness of those connections is not necessarily a sign of conceptual knowledge. Alternatively this can be considered as evidence of the overlapping nature of procedural and conceptual knowledge.

Helene demonstrates that she establishes connections between a functions' expression and the shape of its curve, and is able to use that knowledge to appraise her result, by checking its consistency with what she knows a curve is supposed to look like. And while most students apparently also make a connection between a quadratic function and a parabola, Maria demonstrates that it can be a tenuous one in exercise 3)2)c).

6. CONCLUSION

The analysis revealed that the students had consistently mastered the shortest procedures related to quadratic polynomials, namely solving equations and finding (and to some extent interpreting) the derivative of a quadratic function.

The longest procedures were the ones that posed problems to some of the students, possibly because they had learned those as ordered steps without being aware of the rationale for each step. Their knowledge of the longest procedures seemed to lack the support of conceptual knowledge.

Some indications of conceptual knowledge were present, in that the students seemed to have made connection between pieces of knowledge on the concept map, but those connections can arguably be qualified as part of their procedural knowledge of the subject.

7. REFERENCES

Baroody, A., J., Feil, Y., & Johnson, A., R. (2007). An alternative reconceptualization of procedural and conceptual knowledge. *Journal for Research in Mathematics Education*, 38(2), pp 115-131

Baroody, A. J., & Ginsburg, H. P. (1986). The relationship between initial meaningfull and mechanical knowledge of mathematics. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp 75-112). Hillsdale, NJ: Erlbaum

Berthold, K., & Renkl, A. (2009). Instructional aids to support a conceptual understanding of multiple representations. *Journal of educational psychology*, 101(1), pp 70-87

Boaler, J. (1998). Open and closed mathematics: Student experiences and understandings. *Journal for Research in Mathematics Education*, *29*, 41-62

Eisenhart, M., Borko, H., Underhill, R., Brown, C., Jones, D., & Agard, P. (1994). Conceptual knowledge falls through the cracks. *Journal for Research in Mathematics Education*, 24 (1), 8-40

Gelman, R., & Meck, E. (1986). The notion of principle: The case of counting. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp 29-58). Hillsdale, NJ: Erlbaum

Grevholm, B. (2000). Research on student teachers' learning in mathematics and mathematics education. In: Proceedings of the Ninth International Congress on Mathematical Education.

Heid, K., M. (1988). Resequencing skills and concepts in applied calculus using the computer as a tool. *Journal for Research in Mathematics Education*, 19(1), pp 3-25

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp 1-27). Hillsdale, NJ: Erlbaum

Hiebert, J., & Wearne, D. (1986). Procedures over concepts: The acquisition of decimal number knowledge. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp 199-223). Hillsdale, NJ: Erlbaum

Lee, K., J. (2004). A case study of procedural and conceptual knowledge construction in the computer environments. *Journal of the Korea Society of Mathematical Education Series D: Research in Mathematical Education*, 8(2), pp 81-93

Mellin-Olsen, S. (1984). *Eleven, matematikken og samfunnet: En undervisninglære*. Bekkestua: NKI-forlaget.

Miura, I., T., & Okamoto, Y. (2003). Language supports for mathematics understanding and performance. In A. J. Baroody, & A. Dowker (Ed.), *The development of arithmetic concepts and skills: Constructing adaptive expertise* (pp 229-242). NJ: Erlbaum

Novak, J., D., & Cañas, A., J. (2008-01/ 2006-01). *The theory underlying concept maps and how to construct and use them* Downloaded 16, June, 2009, from http://cmap.ihmc.us/Publications/ResearchPapers/TheoryCmaps/TheoryUnderlyingConceptMaps.htm

O'Callaghan, B., R. (1998). Computer intensive algebra and students' conceptual knowledge of functions. *Journal for Research in Mathematics Education*, 29(1), pp 21-40

Peled, I., & Segalis, B. (2005). It's not too late to conceptualize: Constructing a generalized subtraction schema by abstracting and connecting procedures. *Mathematical Thinking and Learning*, 7(2), 207-230

Perry, M. (1991). Learning and transfer: Instructional conditions and conceptual change. *Cognitive Development*, 6, 449-468

Porter, M., K., & Masingila, J.,O. (2000). Examining the effects of writing on conceptual and procedural knowledge in calculus. *Educational Studies in Mathematics,* 42, pp 165-177

Rittle-Johnson, B. & Alibali, M.W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology*, 91(1), 1-16

Rittle-Johnson, B., & Koedinger, K., R. (2002). Comparing instructional strategies for integrating conceptual and procedural knowledge. In: Proceedings of the Annual Meeting [of the] North American Chapter of the International Group for the Psychology of Mathematics Education.

Rittle-Johnson, B., Siegler, R.S., & Alibali, M.W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology,* 93(2), 346-362

Rittle-Johnson, B., & Star, J.,R. (2007). Does comparing solution methods facilitate procedural and conceptual knowledge? An experimental study on learning to solve equations. *Journal of Educational Psychology*, 99(3), pp 561-574

Schoenfeld, A. H. (1988). When good teaching leads to bad results: The disasters of "well-taught" mathematics courses. *Educational Psychologist, 23,* 145-166

Simpson, A., & Zakaria, N. (2004). Making the connection: procedural and conceptual students' use of linking words in solving problems. In: Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education. Vol 4 (pp 201-208)

Sinclair, H., & Sinclair, A. (1986). Children's mastery of written numerals and the construction of basic number concepts. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp 59-74). Hillsdale, NJ: Erlbaum

Skemp, R.R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching*, *77*, 20-26

Star, J., R. (2005). Research commentary: Reconceptualizing procedural knowledge. *Journal for Research in Mathematics Education*, 36(5), pp 404-411

Star, J., R. (2007). Research commentary: a rejoinder foregrounding procedural knowledge. *Journal for Research in Mathematics Education*, 38(2), pp 132-135

Star, J., R., & Seifert, C. (2006). The development of flexibility in equation solving. *Contemporary Educational Psychology*, 31, pp 280-300

Williams, C., G. (1998). Using concept maps to assess conceptual knowledge of function. *Journal for Research in Mathematics Education*, 29(4), pp 414-421

8. APPENDIX

Appendix 1: Test

Tirsdag 31 Mars

Oppgavesett til forskningsprosjekt

1 - Løs likningene:

d) $x^2 - x - 6 = 0$
e) $(x + 2)(x - 3) = 0$
f) $x^2 + 4 = 0$

2 - Løs ulikhetene:

d) $x^2 > x + 6$
e) $x^2 < -4$
f) $(x - 2)(x - 1) \leq 0$

3 - Funksjonsdrøfting:

1 - Funksjonen f er gitt ved $f(x) = x^2 – 3x + 2$

d) Finn f'(x)
e) Finn eventuelt topp- eller bunnpunkt
f) I hvilket intervall stiger f ? I hvilket intervall synker f ?

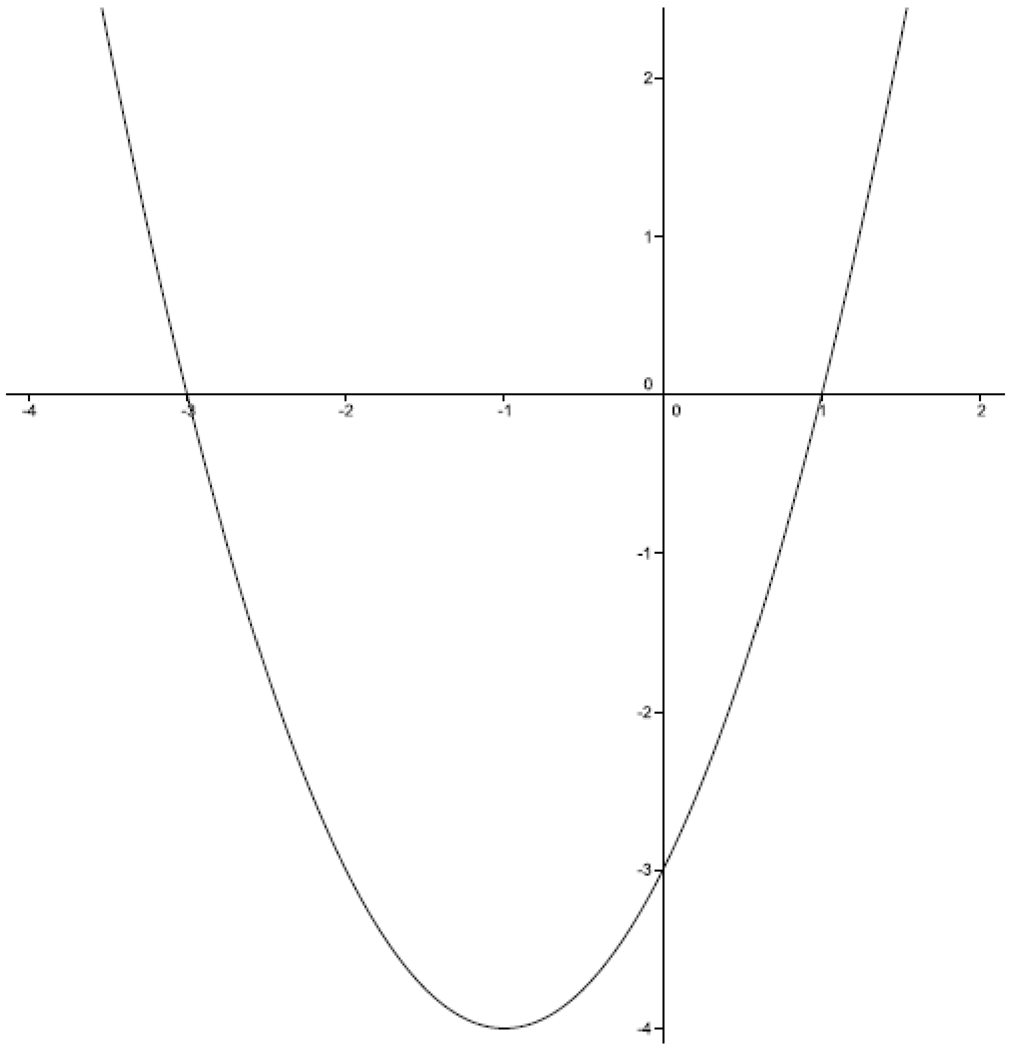
2 - Funksjonen g er gitt ved $g(x) = x^2 + 4x + 4$

d) Lag en fortegnslinje for den deriverte av g
e) Tegn grafen til g
f) Løs ulikheten: $(x + 2)^2 \leq 0$

3 - Funksjonen h er gitt ved $h(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x + 4$

c) Hvor befinner topp- og bunnpunktene til h seg?
d) I hvilket intervall stiger h?

Appendix 2: Interview sheet 1



$f(x) = x^2 + 2x - 3$

Appendix 3: Interview sheet 2

Funksjonen $g(x) = x^2 - 2x + 3$ har et topp- eller bunnpunkt i A(1,3)

Løs ulikheten $g(x) < 0$

Appendix 4: Interview transcript Ole

Interview, Ole

E: Interviewer

O: Ole

(): Pauses

(*Italic*)*:* Action

=: overlapping speech

…: interrupted speech

((Inaudible))

E: Først har jeg noen spørsmål om timene. Kan du si hva en typisk mattetime er?

O: Eh, ja, det kan jeg godt. Eeeh, vi kommer, også kommer læreren, litt seint. Også, begynner han først med å prate, så går han gjennom en del av leksa, og det vi skal… det temaet vi skal ha om i timen, det er nytt tema vær gang, stort sett. Så når det er dobbel time så bruker han gjerne en halvtime på å gå gjennom den delen…

E: En halv time?

O: Ja. En halv klokketime altså på å gå gjennom forskjellige ting om det temaet, liksom hva som er poenget med det, og åssen framgangsmåten er og greier, så får vi ofte regne etterpå. Da regner vi, oppgaver.

E: Da regner dere for dere sjølv eller to og to eller i grupper?

O: To og to.

E: Ja. Pleier du å samarbeide mye med naboen eller jobber du mest sjølv?

O: Vi samarbeider ganske godt. Vi er venner da så, det blir mye samarbeid.

E: Føler du du får noe ut av samarbeidet? Mer en du=

O: = Ja absolutt=

E: = ville gjort hvis du jobba sjølv

O: Det synes jeg.

E: Eh, hva… Kan du si meg hva det er som hjelper med å ha noen å samarbeide med?

O: Ja vi får diskutert hvis det er noe vi lurer på=

E: = Mmhm=

O: = så vi tar mindre tid fra læreren, hvis han skal hjelpe andre, så får vi tenkt oss litt fram til forskjellige problemløsninger, åssen man kan løse ting. Det er ganske produktivt.

E: Okay. Jeg har en påstand som jeg lurer på hvor enig i du er: matematikk er en samling regler og formler som må pugges for å kunne løse oppgaver.

O: Det er jeg i ganske stor grad enig i.

E: Ja?

O: Fordi, hvis man ikke kan framgangsmåten så funker det dårlig=

E: =mmhm

O: =å løse oppgaver.

E: Føler du at du kan klare å tenke deg fram til framgangsmåte sjølv? Uten å ha lært en regel på forhånd?

O: Mmm nei, det er ikke så stor grad.

E: Nei?

O: Jeg må som regel kunne regelen, hvis jeg skal kunne…

E: Mmhm

O: …komme fram til noe særlig.

E: Ja. Okay. Eeeh neste spørsmål er ganske generelt, så kanskje ((uforståelig)) a svare på av den grunn, men vi prøver. Hvordan pleier du å gjøre for å løse en matte oppgave? Bare tenkt deg en hvilken som helst oppgave som blir lagt fram. Kan du si noe om hvilken framgangsmåte som du bruker?

O: Eeh ja. Først så kikker jeg på oppgaven, og ser hva den dreier seg om…

E: Mmhm.

O: … og hvis jeg da kan regelen så bruker jeg de reglene på oppgaven. For eksempel hvis det er () andregradslikning eller () regne ut areal av et eller annet eller noe sånt…

E: Mmhm.

O: … så bare tenker jeg ut den regelen som må brukes i denne oppgaven også løser jeg den.

E: Mmhm. Vet du () hvordan vet du til enhver tid hvilken regel som passer hvor?

O: Da må jeg kjenne igjen oppgaven, og vite hvor () hva slags regel () hva oppgaven går utpå på en måte.

E: Mmhm.

O: En arealsetning eller () om det er sinus eller hva det er.

E: Mmhm.

O: jeg må bare se i oppgaven, og huske de reglene jeg kan.

E: Ja () okay. Ja, da har jeg noen få oppgaver til deg (*rekker fram blankt ark og penn).* Ta den, og den.

O: Okay.

E: Eh, x i andre pluss ni () er større enn null.

O: (*skriver ned ulikhet)*. Ja () skal jeg bare løse den?

E: Ja.

O: Da begynner jeg å sette x i andre på den ene sida. Så setter jeg er lik. Også flytter jeg ni over, så blir det x i andre er lik () minus ni. (*har skrevet ned* $x^2 = -9$) (). Og det kan ikke løses ser jeg ((uforståelig)). Hvis vi tar kvadratrot så blir det x er lik (*har skrevet x=*). Kvadratroten av et tall kan ikke bli () man kan ikke gange to tall og få et negativt tall sp akkurat sånn som det er gjort der, det kan ikke løses.

E: Mmhm. () Jeg ser du begynte med å løse likning.

O: Ja. Det tror jeg ikke ble riktig (*smiler, og begynner å skrive*) jeg må ofte prøve og feile litt…

E: Ja

O: … for å komme fram til noe. (*skriver:* $x^2 + 9 > 0$) Da bare tar jeg å deler () på kvadratrot () på den ene siden av null () altså bare sånn som det står i starten…

E: Mmhm

O: …så blir det x pluss tre større enn null (*skriver* $x + 3 > 0$). Så () flytter jeg da tre over sånn (*skriver* $x > -3$). Så blir det x større enn minus tre.

E: Mmhm. Okay. ()

O: Ja, x større enn minus tre.

E: Eeh. Tror du du kunne kommet fram til det () ved å gå denne veien (*peker på den Oles første forsøk på løsning, der han kom fram til* $x^2 = -9$) på en måte? Jeg så at du begynte med å løse likningen, ikke sant?

O: Mmhm.

E: Det er det du pleier å gjøre når du skal løse sånne ulikheter, du løser først likningen også…

O: Ja

E: … lager du fortegnsskjema.

O: ja fortegnskjema pleier jeg ikke å bruke. Jeg bare regner det ut sånn.

E: Mmhm. Okay. () kan du si meg hvorfor dette ikke går (*peker på likning* $x^2 = -9$) hvorfor går det ikke an å ta kvadratroten av et negativt tall?

O: eh jo, for da må du få et tall som () ganget med seg selv også blir negativt () minus tre for eksempel, og det () et tall ganget med seg selv kan ikke bli negativt.

E: Mmhm. () Ja.

O: Det er bare regelen vi har hatt om.

E: Ja. () Eeh neste er, en ulikhet også: to x pluss fire er større enn null.

O: (*skriver:* $2x + 4 > 0$) Ja. Da deler jeg begge to på to (*deler begge leddene i addisjonen på to, skriver:* $2x/2 + 4/2 > 0$) så får jeg x pluss to er større enn null (*skriver samtidig* $x + 2 > 0$) () x større enn minus 2 (*skriver* $x > -2$). Da flytter jeg over.

E: Ja. Mmhm. Neste er en funksjon som heter f av x er lik x i andre pluss fire minus tre.

O: (*skriver samtidig* $f(x) = x^2 + 4 – 3$).

E: Så vil jeg gjerne vite når … nei unnskyld pluss fire x minus tre.

O: (*legger til en x bak fireren på det han skrev*)

E: Og jeg vil vite når funksjonen stiger.

O: Ja. Det er greit. Da begynner jeg å finne den deriverte av x…

E: Ja.

O: … er lik 2 x pluss fire (*skriver samtidig: $f'(x) = 2x + 4$)* så setter jeg () faktoriserer, setter utenfor likhetstegn, nei parentes mener jeg () så får jeg 2 parentes x pluss to (*legger til på samme ligne $2(x + 2)$* så setter jeg det inni et sånt () fortegnskjema (*begynner å tegne fortegnskjema mens han snakker)* så tar jeg først den som står utenfor parentesen toeren, to () det blir da pluss hele veien (*første rad på skjema for toeren som alltid er positiv*), også setter jeg den andre delen x pluss to () og hvis det skal være null så må da x være minus to (*legger til en rad, der $x + 2$ er negativ fram til -2 og positiv deretter*). Også () putter jeg hele uttrykket på bånn så blir det minus, null, og pluss. (*Legger til en siste rad, der han setter uttrykket $2(x + 2)$ til å være negativt fram til verdien -2, lik null for $x = -2$, og positivt deretter.)* Så bunnpunktet er da på minus to.

E: Ja. Når er det funksjonen stiger da?

O: Da stiger funksjonen når den er over x minus to.

E: Mmhm. Okay.

O: Eh? Det stemte det?

E: Ja. () Neste er da, kanskje en gammel kjenning, x i andre () minus x, minus 6, er () mindre enn null.

O: (*skriver: $x^2 – x – 6 < 0$)* Ja, da begynner jeg å sette opp en sånn likning, da får jeg (*skriver samtidig som han snakker $x^2 – x – 6 = 0$*) x i annen minus x minus seks () er lik null. Så blir det (*Skriver samtidig $a=1$, $b=-1$, $c=-6$*) a er lik en, b er lik minus 1, c er lik minus seks. Sa setter jeg opp en sånn andregradsformel…

E: Mmhm

O: (*skriver samtidig som han snakker, den generelle andregradsformelen med koeffisientene a, b, og c*), x er lik () minus b pluss minus kvadratroten av () b i annen minus fire ganger a () c oppå to ganger a. Også bare setter jeg i de egentlige tallene. (*Skriver samme formelen på nytt, men erstatter koeffisientene med sine respektive verdier, bortsett fra b som han erstatter med seks. Han oppdager feilen*) der skrev jeg litt feil (*og retter det opp ved å skrive -1 i stedet. Fortsetter med beregningene, og kommer fram til de to løsningene 3 og 2 etter å ha oversett at han mangler et minustegn foran toeren.*) Ja, så får jeg da to svar, at x kan være enten tre eller to.

E: Ja. Hva er det du gjør etterpå?

O: (*lang pause*) det har jeg egentlig… det har jeg glemt… åssen jeg skulle gå fram da. (*lang pause*)

E: Du har glemt det?

O: Ja.

E: Okay. () (*rekker fram interview sheet 1*) Denne funksjonen, x i andre pluss to x minus tre.

O: Ja.

E: Med denne grafen her (*peker på parabelen på arket*) Kan du fortelle meg når denne funksjonen er mindre enn null?

O: () ja det kan jeg. Altså denne… (*peker på parabelen*) når grafen er mindre enn null?

E: Ja.

O: Eeeh det er når den er, til venstre for bunnpunktet, (*peker på den delen av parabelen som er til venstre for bunnpunktet*) der. Når den synker () er den mindre enn null. Bunnpunktet er der (*peker omtrent der bunnpunktet til parabelen ligger*), og når den er til høyre for det bunnpunktet så er grafen positiv. Også er den null, når den er akkurat på bunnpunktet.

E: Mmhm. Okay. () Siste (*rekker fram interview sheet to*). Du har en funksjon g av x som er lik x i andre minus to x pluss tre. Også er det sånn at den har enten topp eller bunnpunkt i punktet 1, 3.

O: Mhm.

E: Kan du bruke denne informasjonen for å løse den ulikheten som står her (*peker på ulikheten som står på arket*) g av x er mindre enn null?

O: (*lang pause*) Mmmm. Ja jeg vet at funksjonen er lik null, når x er lik en. (*skriver* $g(x) < 0$. *Skriver så* $x^2 – 2x + 3 < 0$) () så da må x være (*skriver:* $x1< 1$) x må være mindre enn en.

E: Okay. Hvordan kom du fram til det?

O: Da tenkte jeg at det står der (*peker på funksjonen g på arket*) at den der skal være mindre enn null…

E: Mhm.

O: og hvis det skal stemme så må x… så må x være lavere enn en. For hvis x er lik en så blir det null. Jeg bare tenkte logisk fram til det.

E: Mmhm.

O: Det virka logisk i hvert fall.

E: Ja. Okay men da var det alt jeg hadde.

O: Okay.

E: Da avslutter vi (*skrur av kamera*).

Appendix 5: Interview transcript Ane

Interview Ane

E: Interviewer

A: Ane

(): Pauses

(*Italic*)*:* Action

=: overlapping speech

…: interrupted speech

((Inaudible))

E: Først lurer jeg på om du kan fortelle meg hvordan dere pleier å jobbe i timene.

A: Sånn generelt?

E: Ja. Beskrive en typisk mattetime.

A: Vi begynner med… som regel så er det først om den leksa vi hadde til timen, sånn om det var noen problemer med den og sånn og hvis det var det så går han gjennom det…

E: ja…

A: Og så begynner han på nytt stoff, og så går han gjennom, generelt på tavla, også begynner vi å jobbe med oppgaver til det, også kan han gjerne ta noen minutter igjen, etter en stund, så gå gjennom noe nytt, også jobber vi med oppgaver til det, også får vi lekser til stoffet.

E: Ja. Hvor lang tid pleier dere å bruke på de forskjellige fasene? Hvor lang tid bruker han på tavla for eksempel?

A: Han bruker jo fem-ti minutter på å gjennomgå leksa også… han bruker… Vi har jo en og en halv time til sammen som regel så det blir jo sikkert tre kvarter i forhold til tavle.

E: Mmhm.

A: Det varierer litt, for noen timer så sitter vi å jobber veldig mye og andre timer får vi veldig mye på tavla så… får vi bare litt sånn jobb på slutten.

E: Mmhm.

A: Også blir det mye å gjøre hjemme istedenfor.

E: Ja. Når dere jobber med oppgaver, er det da to og to så vidt jeg har skjønt

A: Ja. Vi sitter to og to. Men, jeg jobber to og to, som regel, med han jeg sitter sammen med. Men det er sikkert noen som sitter å jobber sånn, helt for seg selv og.

E: Ja.

A: Men vi sitter ved siden av noen.

E: Mmhm. Hva er det du… Foretrekker du å jobbe to og to? Det er det du pleier å gjøre?

A: Det er det jeg pleier å gjøre, for jeg sitter ved siden av han. Men det gjør ingenting å jobbe aleine, for når jeg jobber aleine så føler jeg at jeg må konsentrere meg mer selv, og tenke mer sjølv, enn når jeg sitter sammen med noen andre.

E: Mmhm.

A: Så det er, det er litt forskjellig egentlig…

E: …Ja…

A: hva jeg foretrekker.

E: Hva synes du kan være fordelene med å ha noen å jobbe med?

A: Det er jo det at hvis jeg lurer på en sånn liten ting… så kan jeg på en måte spørre han også "å ja, okay", og så kan jeg jobbe videre istedenfor å måtte sitte i ti minutter et kvarter å vente på læreren. For han har jo som regel, altfor mange som trenger hjelp.

E: Ja.

A: Så jeg blir ikke stående fast så lenge, for jeg merker hvis det er noe som jeg lurer på, også kan ikke han ved siden av hjelpe meg, så blir det til at jeg må sitte veldig lenge å vente. Det blir så unødvendig bruk av tid føler jeg.

E: Ja. Eh, neste har jeg en påstand som jeg lurer på hvor enig du er i. Det er matematikk er en samling regler og formler som må pugges for å løse oppgaver.

A: Altså du må jo pugge og lese regle for å skjønne noe, men det er jo veldig mye, du må tenke deg til ting. Det er jo logikk i veldig mye og sånn…

E:… Ja…

A: Det er jo ikke bare en samling formler og regler, du må jo tenke, det er jo arbeidsmåter hvordan du gjør det, også er ofte flere måter du kan gjøre ting på også er det da velge riktig måte og…

E: Når du sier det er flere måter å gjøre ting på, mener du det er flere regler som kan fungere, for en ting?

A: Ja… Ja.

E: Mmh. () Eh neste spørsmål er ganske generelt, men jeg lurer på om du kan svare på det likevel, hvordan pleier du å gå fram når du skal løse en matteoppgave?

A: Eh. Først så må jeg lese oppgaven, også må jeg finne ut () hva de spør etter. Hva det egentlig er jeg skal regne ut, også hvis det er figurer så begynner jeg å tegne figurene, også hvis det er figurer blir det til av jeg bare sitter å se… må jeg bare se om noe bare "aha, ok", også hvis jeg egentlig ikke har peiling på hva jeg skal gjøre så begynner jeg bare å skrive noe, bare gjøre noe, også som regel etter hvert så kommer jeg på noe å gjøre.

E: Ja. Ok. Da går vi løs på oppgavene (*rekker fram papir og penn*). Den første er en ulikhet: x i andre pluss en er mindre enn null.

A: Skal jeg gjøre den eller skal jeg bare…

E:… Ja…

A: x i andre?

E: Pluss en, er mindre enn null.

A: Ok. (*begynner å løse den tilsvarende likningen, ved å bruke algoritmen for løsning av førstegradslikninger og kommer fram til at x må være lik kvadratroten av minus 1*). Det blir jo feil da.

E: Jeg ser du begynner å løse likningen.

A: Mmhm. (*Setter så opp likningen på nytt og ser ut som hun sjekker om uttrykket* $(x-1)(x+1)$ *blir lik* x^2+1*dersom det utvides)*. Hvordan blir det da?

E: Prøvde du å faktorisere?

A: Ja. ((uforståelig)). Gjorde jeg den på prøven?

E: Du gjorde noe liknende.

A: Noe liknende ja. () Men jeg må jo nødt til å faktorisere det hvis ikke så… Går det ikke.

E: Hvorfor er du nødt til å faktorisere det?

A: For hvis jeg faktoriserer det kan jeg sette det i et fortegnsskjema, for å se () for å se hvor () nullpunktene er og sånne ting.

E: Mmhm. Det at du nå ikke kommer fram med å løse likningen.

A: Ja.

E: Hva er det forteller deg?

A: At ikke jeg har kommet så veldig langt i den oppgaven (*ler*). At ikke jeg kan gjøre noen ting egentlig.

E: Ja. Men forteller det deg noe om selve oppgaven, om selve likningen, eller selve ulikheten?

A: At ikke jeg kan faktorisere den, at den ikke har noe løsning.

E: Ja. Ok.

A: Det kan jo hende at jeg ikke får det til og da (*ler*) det er jo noe muligheter for det og.

E: Skal vi prøve på neste?

A: Ja ok.

E: x i andre pluss ni er mindre enn null.

A: (*begynner å skrive. Bruker samme metode som oppgaven ovenfor, men stopper opp etter å ha skrevet* $x^2+9=0$). Men det går ikke an å si at det er lik minus ni heller. ()

E: Du får det samme problemet som…

A: For det at det står pluss ((uforståelig)). () Også kan jeg ikke sette noe x utenfor for det er ikke noe x der (*peker på nieren*), også kan jeg ikke sette noe tall utenfor for det er ikke noe tall der (*peker på x-en*). ((uforståelig)) du kan ikke ta kvadratroten av et minustall.

E: Hvorfor kan du ikke det egentlig?

A: For det at hvis du tar minus ganger minus så får du pluss.

E: Ja?

A: Men hvis du tar x minus (*skriver (x – 3)(x + 3)* det blir jo feil det og, da blir det (*skriver mens hun snakker*) x i annen () minus tre x () minus tre x () pluss ni. Da blir det plutselig minus seks x og det skal ikke stå der. Men hvis du tar pluss så får du minus ni.

E: Men hva hvis du da ikke prøver med å begynne å løse likningen?

A: Hva skal jeg da begynne med? Å ikke løse den?

E: Nei, men du begynner med en ulikhet ikke sant?

A: Ja.

E: Også begynner du umiddelbart med å prøve å løse likningen. Men hvis du prøver å bare forholde deg til selve ulikheten?

A: (*tenker litt*). Hvordan kan du gjøre det da? Du må jo uansett finne en x.

E: Ja. Hvis… hvis det ikke hadde stått x i andre med bare x, hvordan hadde du gjort det da?

A: Hvis det bare hadde stått (*skriver mens hun snakker*) x pluss ni større enn null, så kunne du sagt x er mindre enn minus ni.

E: Ja. Hvis du da går tilbake til der det står x i andre pluss ni.

A: Da blir det jo at x i andre er mindre enn minus ni.

E: Ja.

A: Men det kan da heller ikke ha noen løsning da.

E: Nei…

A:…hvis du skal ha et tall som er under minus ni, så må det bli et minus tall…

E: …ja…

A: og x i andre () minus ganger minus det blir jo pluss.

E: Ja.

A: Da har ikke den heller noen løsning.

E: Nei. () hva med den over da?

A: (*skriver mens hun snakker)* x i andre, er mindre enn minus en.

E: Nei forresten () x i andre pluss en er større enn null.

A: (*skriver mens hun snakker*) pluss en er større enn null. X i annen er større enn minus en.

E: Ja.

A: Den kan være null da. Det kan jo være () ja det går jo an det.

E: Ja.

A: Gjør det ikke det? (*ler*)

E: Hva kan x være for at…

A: …x kan være () null

E: Ja.

A: Den kan være større. En to tre fire () for så vidt.

E: Mmhm.

A: Da er det jo større enn minus en.

E: Ja. () føler du at du har løst oppgaven nå?

A: Ikke det der (*peker på* $x^2 + 1 < 0$)

E: Hva mener du med det der?

A: At x kan være null og større. Det er ikke () jeg hadde aldri skrevet på en oppgave x kan være null eller større jeg hadde blitt helt gal hvis jeg ikke hadde funnet noen tall.

E: (*flirer*)

A: (*smiler*)

E: Ok. Ja. Skal vi prøve på neste?

A: Ok.

E: Nok en ulikhet. Det er to x pluss fire er større enn null.

A: (*begynner å skrive, løser oppgaven veldig raskt, ved å bruke algoritmen for løsning av lineære likninger*)

E: Ok. Også en funksjon. F av x er lik x i andre pluss fire x minus tre. Også lurer jeg på når denne funksjonen stiger.

A: Når den stiger?

E: Ja. For hvilke verdier av x den stiger.

A: Det blir litt lengre oppgave. Eller mener du at jeg skal kunne se det med en gang?

E: Nei, ikke direkte.

A: (*begynner å derivere funksjonen*) f derivert av x. (*faktoriserer den deriverte funksjonen* $f'(x) = 2x + 4$ *til* $2(x + 2)$, *og setter opp et fortegnsskjema med en rad for faktoren 2 og en rad for faktoren* $(x + 2)$, *og siste rad for hele produktet av faktorene*) den stiger fra minus uendelig til minus to () også stiger fra minus to til pluss uendelig.

E: Kan du si hva koordinatene til nullpunktet er?

A: Det blir hvert fall at x () lik minus to

E: Ja.

A: Også må jeg sette det inni det der da (*peker et sted mellom funksjonsuttrykket og det deriverte uttrykket*)

E: Inni?

A: Må sette det inni funksjonen for å finne y verdien.

E: Mmhm.

A: Skal jeg finne y verdien eller skal jeg bare?

E: Nei.

A: Ok.

E: Hvis du ser på dette arket her (*rekker fram interview sheet 1*) funksjonen x i andre pluss to x minus tre, også den tilhørende figuren. Så lurer jeg på om du kan fortelle meg når funksjonen er mindre enn null?

A: Når funksjonen er mindre enn null?

E: Ja.

A: Når x er mellom minus tre og en.

E: Ja. En siste en (*rekker fram interview sheet 3*). En funksjon g(x) er lik x i andre minus to x pluss tre, og den blir du fortalt har et topp eller bunnpunkt i punktet en, tre.

A: Mmhm.

E: Så lurer jeg på om du kan løse ulikheten g av x er mindre enn null.

A: Må bare ta en ny side (*blar i arket hun har fått utdelt, og begynner å løse den tilsvarende likningen ved å bruke andregradsformelen, helt til hun finner ut at hun må finne kvadratroten av minus åtte*). Men det går jo ikke an.

E: Hvorfor går det ikke an?

A: To pluss minus kvadratroten av minus åtte (*peker på den siste linjen i beregningene sine*). Det blir jo ikke noe tall det.

E: Det er igjen det problemet med å ta kvadratroten av et minustall.

A: Ja. Jeg har hvert fall lært at man ikke kan gjøre det.

E: Ja.

A: () kan man gjøre det da?

E: Nei.

A: Nei! Det går jo ikke an da.

E: () hva sier du om ulikheten da?

A: Da kan det heller ikke ha noen løsning heller.

E: Nei? Ser du noen som helst måte å bruke den informasjonen på (*peker på at funksjonen har et ekstrempunkt i (1,3)*)

A: At x-en er en, i topp eller bunnpunkt.

E: Ja. Vet du om det der er topp eller bunnpunkt?

A: Jeg vet at det er en av delene (*smiler*)

E: Ja.

A: (*skriver ulikheten* $1^2 - 2\cdot 1 + 3 < 0$*, og regner seg fram til* $2 < 0$). Det sa meg ingenting, egentlig.

E: Nei…

A:…(*flirer*) det var bare mer forvirrende.

E: Ja. () Da tenker jeg at vi gir oss.

A: Ok.

Appendix 6: Interview transcript Maria

Interview, Maria

E: Interviewer

M: Maria

(): Pauses

(*Italic*)*:* Action

=: overlapping speech

…: interrupted speech

((Inaudible))

I: Ja. Først lurer jeg på om du kan fortelle meg hvordan dere pleier å jobbe i timene.

M: Vi begynner=

I: =Beskriv den typiske=

M: = Å ja, ok. Vi begynner ofte med at vi går gjennom noen lekseoppgaver som ikke vi har skjønt. Hvis det er noen problemer med lekser vi har, også pleier vi å fortsette med det vi holder på med. Så tar læreren å underviser litt () i det vi skal videre med lissom, også jobber vi eventuelt med noen oppgaver selv. Med akkurat det han har snakka om.

I: Mmm. Eeh, hvor lenge… Hvor lang tid bruker læreren på å undervise…

M: …eeh det er forskjellig, men det kan være opp til en time og lissom, tre kvarter, halvtime det er veldig forskjellig=

I: =Mmm=

M: = men det pleier å være en stund. Så jobber vi litt med oppgaver, i slutten av timen igjen.

I: Ja. Jobber dere aleine med oppgaver eller i grupper?

M: eeh vi sitter på pulten to og to lissom så vi pleier å samarbeidet litt to og to.

I: Mmm. () Får du noe ut av å samarbeide med hverandre?

M: Ja.

I: Ja? På hvilken måte?

M: Eeh, hvis for eksempel ho jeg sitter ved siden av lurer på noe så kan jeg hjelpe ho og, så kan ho hjelpe meg så, vi hjelper hverandre lissom. Jeg lærer av å lære vekk og også, bli lært.

I: Mmm. Ja. () Eeh, jeg har en påstand som jeg skal komme med, så lurer jeg på hvor enig du er i den, om du er veldig enig, litt enig, litt uenig eller veldig uenig.

M: Mmm.

I: Matematikk er en samling regler og formler som må pugges for å kunne løse oppgaver.

M: Jeg er kanskje ikke helt enig. Vi… Man trenger jo… vi… Mer enn man kanskje tror, å lære det lissom. () Men for mange kan det kanskje virke som det bare er formler og kjedelig men, det er kanskje egentlig ganske nyttig ()

I: Ja. Har du et eksempel=

M: =eeeh=

I: =sånn umiddelbart på hvorfor det kan være nyttig?

M: eeh. (). Læreren viste et eksempel i timen en gang om () hva var det det var (). Eeh Noe som hadde med en bro å gjøre der det var lissom, matte involvert da, som viste at man lissom bruker det da, i jobb og i… Ja.

I: Mmm. Eeh ja. Synes du det er mye pugg i mattematikk overhode eller synes du at du kan () skape noe skjøl på en måte…

M: ()

I: … kan komme på ting skjøl

M: Ja på en måte hvis du kan formlene så er det lettere å, tenke seg til ting skjøl også… Hvis du kan alle reglene og sånn, så kan du på en måte veldig lett bruke de. Og () kombinere de litt for eksempel () ja

I: Ja. Eeeh. Nå et litt generelt spørsmål. Kanskje vanskelig å svare på men vi får se. Kan du fortelle meg hvordan du pleier å gjøre når du har en generell matteoppgave foran deg. Hvordan du pleier å jobbe, hvordan du pleier å løse den?

M: Eeeh det kommer helt an på hvordan den oppgaven er men () eeh for det første så sjekker jeg for å se om jeg har lest den riktig, og eeh, hvis den er vanskelig så bruker jeg tegninger eller andre ting, for å klare å løse den da. Og skjemaer og forskjellig som kan hjelpe meg selv om det ikke står at vi skal lage det i oppgaven=

I: =Mmm=

M:= eh, men så hvis den er lett så bare løser jeg den som man skal løse den.

I: Ja. () Ok. Da har jeg noen få oppgaver igjen (*Gir M penn og papir)*. Den første er x i andre minus ni, er lik null.

M: (*gjør seg klar til å skrive*). Skal jeg regne den ut?

I: Ja.

M: Skal jeg forklare samtidig som jeg gjør det lissom eller skal jeg…

I: … Det kan du gjerne gjøre.

M: Ok det er vel sånn at når det er to tall som kan eeh bruke kvadrat… nei bruke kvadrat… eeh kvadratroten, og et minus, altså det første tallet er positivt det andre er negativt, så kan du bruke tredje kvadratsetning for å finne nullpunktene. Og da kan jeg se at nullpunktene her er minus tre og pluss tre. Også ((uklart)), også lager jeg fortegnsskjema (*lager fortegnsskjema, med en rad for (x – 3), en for (x+3) og en for produktet av disse. Når hun skal til å skrive konklusjonen: at (x -3)(x+3) = 0 når x tilhører… så innser hun at fortegnsskjemaet var unødvendig*). Men jeg trenger ikke å gjøre det her når det er lik null! Da er det bare det som står der da (*peker på nullpunktene hun fant i begynnelsen og ler).*

I: Hva med x i andre pluss ni er lik null?

M: (*gjør seg klar til å skrive)* Ja det går ikke. Eeh eller da må jeg bruke andregradsformelen da for det går bare an å bruke tredje kvadratsetning hvis det er minus (*begynner å skrive opp a=1, b=0, c=9)* Jeg vet ikke om det her går jeg tror egentlig ikke det går men (*setter opp regnestykket for å regne ut andregradsformelen med verdiene for a, b, og c)*

I: Hva er det som ikke går, å bruke denne formelen?

M: Jeg tror ikke det går opp, var det lik null forresten?

I: Ja.

M: (*fortsetter med andregradsformelen)* det går ikke fordi, minus trettiseks, det går ikke an å ta kvadratroten av et negativt tall.

I: Mmm. Vet du hvorfor det ikke går an å ta kvadratroten av et negativt tall?

M: Det er et imaginært tall, det er alt jeg vet (*ler)*

I: Et imaginært tall?

M: Det eeh går ikke an å så, man må jo ha, for å få et minustall må man gange det med en pluss, et positivt tall og et negativt tall, men hvis man skal ta kvadratroten må de to tallene være like og da går det ikke fordi en av de må være minus hvis det skal bli, hvis svaret skal bli minus.

I: Og dette med imaginært tall det er Læreren som har sagt?

M: Ja.

I: Ok. Eeh, hva med x i andre pluss ni er større enn null?

M: (). Da har det vel egentlig ikke noe å si… eller… først må jeg gjøre det om til lik, og da blir jo svaret som der (*peker på forrige oppgave*) det går ikke.()

I: Nei?

M: ((Uklart))

I: Eeh, hva med ulikheten 2 x + 4 er større enn null?

M: (*begynner umiddelbart å skrive, setter opp 2x+4=0, og under dette skriver a=1, b=0, c=0. Setter så opp regnestykke for andregradsformelen med koeffisientene a b og c. Dette gir, under rottegnet:* $0^2 - 4 \cdot 1 \cdot 4$*.)* Det går ikke det her heller.() Jo… (*faktoriserer uttrykket til 2·(x+2), og lager fortegnsskjema med en rad for 2, som alltid er positiv, en rad for x+2, negativ fram til -2 og positiv deretter, og en rad for produktet av faktorene. Setter til slutt opp løsningen slik:* $x \in (-2, \rightarrow)$). Sånn.

I: Ja. Ok. Også før jeg viser deg noen ark så har jeg en til. Det er en funksjon f av x, som er lik x to…

M: … to x lissom?

I: Nei x i andre, pluss fire x, pluss tre. Så lurer jeg på når denne funksjonen synker.

M: (*Løser likningen* $x^2+4x+3=0$ *som hun pleier, ved å sette a=1, b=4, c=3 og sette opp andregradsformelen med disse koeffisientene. Under rottegnet regner hun:* $4^2-4 \cdot 1 \cdot 3= 8-12$*. Hun oppdager slurvefeilen, ler og slår seg i panna)* åtte minus tolv. Hjelp.

I: Hva er det?

M: Ikke noe.(*fortsetter med sine beregninger, men gjør en annen slurvefeil som fører henne til løsningen:* $x_1=-4, x_2=0$*. Nøler)* Jeg tror det blir feil det her.

I: Hva er det du har gjort? Du har brukt andregradsformelen…

M: … Ja, så fant jeg ut at x en lik minus fire men da blir x to lik 0 delt på to og det blir null, men jeg vet ikke ((uklart)). Eeh () (*lett irritert)* å jeg skulle finne ut når det synker, åh jesus. Ja jeg må finne nullpunktene først det er sant. Eeh ()

I: Må du finne nullpunktene for å finne når funksjonen synker?

M: Nei jeg kan ta den deriverte. (*finner den deriverte av funksjonen: 2x+4. Tegner en pil derfra til fortegnsskjemaet hun lagde til forrige oppgave)* Det blir det samme som der. Bare (*skriver svaret slik: f(x) synker når x* $\in \langle \leftarrow, 2\rangle$) Sånn.

I: Ja du bruker det fortegnskjemaet som du hadde lagd=

M: =Ja=

I:= fra før. Mmm. Ja, OK. () Så var det på dette arket her (*rekker fram interview sheet 1*) så har du en funksjon f av x som er gitt med figuren. Lurer på om du kan fortelle meg nå denne funksjonen er mindre enn null.

M: (). Når den er under her (*peker på den delen av parabelen som befinner seg under x aksen på figuren)*

I: Ja () for hvilke verdier av x er det?

M: minus tre og () og en

I: Mmm. Ja. Og den siste (*rekker fram interview sheet 2*). Funksjonen g, x i andre minus 2 x pluss tre har et bunnpunkt () eller toppunkt… () som er da en tre. Så lurer jeg på om du kan løse ulikheten g av x er mindre enn null.

M: Mener du koordinatene er en og tre?

I: Ja

M: her, på arket (*peker på interview sheet 2*), eller der (*peker på arket hun har brukt tidligere*)

I: Helst der ja (*peker på svar arket til Maria*)

M: Ok. (*skriver opp funksjonsuttrykket for g av x, og under: g(x)<0.*)

I: Er det noe å hente fra den første linjen etter din mening?

M: Eeh. Når x er en () så () Hvis det skal være topp eller bunnpunkt så må det vel være ()

I: Så må det være? Tror du det er topp eller bunnpunkt?

M: Det må være topp fordi hvis det er bunn så () har den ikke noe nullpunkt. Men jeg vet ikke om…()

I: Må den ha et nullpunkt?

M: Eeh. Nei, jeg vet ikke. Jeg tror det er toppunkt i hvert fall. Eehm, vent… () Ok men stigningen er i hvert fall null i det punktet så da kan jeg ta () den deriverte kanskje eller noe? Jeg bare prøver, jeg vet ikke om det er riktig.

I: Ja

M: (*Finner den deriverte g'(x) = 2x – 2 og faktoriserer den til 2·(x – 1). Setter så opp fortegnskjema som hun gjorde tidligere, og finner ut at punktet må være bunnpunkt.)*

I: Hva kommer du fram til?

M: Må bare sjekke en ting (*gjør noen beregninger som jeg ikke klarte å tyde i ettertid. Ser ut som hun undrer seg over noe.*) ().

I: Er det noe som ikke stemmer?

M: Eemh. Jeg vet ikke om jeg har gjort riktig men, deg tok først den deriverte av, av den=

I:=Ja=

M: =og, da finner man jo ut ((uklart)), også, da fant jeg at den er null på en men den er ikke. Å jo, da må det ()

I: Hva er det som er null på en?

M: x. Men x er en

I: Ja

M: Så er y null () nei det går jo ikke det.((Uklart))

I: Ja du har satt inn verdien en i den deriverte

M: Ja, ja

I: Da får du null.

M: Ja fordi stigningen er null når x er en. Fordi der er et topp eller bunnpunkt selvfølgelig.

I: Mmm.

M: Eeh, så det vil si at den synker før den blir en, og den stiger etter. Så det vil si at det er ett bunnpunkt.

I: Kan du bruke den informasjonen til å løse denne ulikheten?

M: Å ja, eh når den er mindre enn null da? () Eeh. Da er nullpunktet en. Hvordan går det når det er ett nullpunkt. Når hele den greia der x i annen minus to x pluss tre er lik… nei () jeg skal finne når hele den er størr… nei mindre enn null ()

I: Ja

M: Eehm. (). Ja er det ikke bare… Fordi når den er ((uklart)) Er det sånn? (*Peker på hva hun har skrevet: g(x)<0 når x* $\in \langle \leftarrow, 1 \rangle$). Er det riktig?

I: Nei.

M: For det er den deriverte egentlig det er lissom når den går nedover ikke når den er mindre enn null.

I: Ja. Men jeg… jeg er litt vrien () på disse. Men jeg tror egentlig at vi kan avslutte

M: Ja.

I: Mmm. Takk for at du var med. (*reiser seg for å slå av kamera*)

M: (*Reiser seg*). Kan jeg se hva de riktige svarene var?