A Stochastic Search on the Line-Based Solution to Discretized Estimation

Anis Yazidi[1], Ole-Christoffer Granmo[1], and B. John Oommen[2,⋆]

[1] Dept. of ICT, University of Agder, Grimstad, Norway
[2] School of Computer Science, Carleton University, Ottawa, Canada

Abstract. Recently, Oommen and Rueda [11] presented a strategy by which the parameters of a binomial/multinomial distribution can be estimated when the underlying distribution is nonstationary. The method has been referred to as the Stochastic Learning Weak Estimator (SLWE), and is based on the principles of *continuous* stochastic Learning Automata (LA). In this paper, we consider a new family of stochastic *discretized* weak estimators pertinent to tracking time-varying binomial distributions. As opposed to the SLWE, our proposed estimator is *discretized*, i.e., the estimate can assume only a finite number of values. It is well known in the field of LA that discretized schemes achieve faster convergence speed than their corresponding continuous counterparts. By virtue of discretization, our estimator realizes extremely fast adjustments of the running estimates by jumps, and it is thus able to robustly, and very quickly, track changes in the parameters of the distribution after a switch has occurred in the environment. The design principle of our strategy is based on a solution, pioneered by Oommen [7], for the Stochastic Search on the Line (SSL) problem. The SSL solution proposed in [7], assumes the existence of an Oracle which informs the LA whether to go "right" or "left". In our application domain, in order to achieve efficient estimation, we have to first *infer* (or rather *simulate*) such an Oracle. In order to overcome this difficulty, we rather intelligently construct an "Artificial Oracle" that suggests whether we are to increase the current estimate or to decrease it. The paper briefly reports conclusive experimental results that demonstrate the ability of the proposed estimator to cope with non-stationary environments with a high adaptation rate, and with an accuracy that depends on its resolution. The results which we present are, to the best of our knowledge, the first reported results that resolve the problem of discretized weak estimation using a SSL-based solution.

Keywords: Weak Estimators, Learning Automata, Non-Stationary Environments, Stochastic Point Location.

⋆ *Chancellor's Professor*; *Fellow: IEEE* and *Fellow: IAPR*. The Author also holds an Adjunct Professorship with the Dept. of ICT, University of Agder, Norway. The third author is grateful for the partial support provided by NSERC, the Natural Sciences and Engineering Research Council of Canada.

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1 Introduction

The problem of estimating the parameters of a distribution, when these parameters change with time, is far from trivial. The reason why the problem is intrinsically difficult is because "traditional" estimation methods rely on the long term properties of the estimation process to conclude on the *quality* of the estimate. Consequently, they attempt to obtain convergence properties that are as "strong" as possible.

Estimators generally fall into various categories including the Maximum Likelihood Estimates (MLE) and the Bayesian family of estimates. Estimates from these families are well-known for having good computational and statistical properties. However, the basic premise for establishing the quality of estimates, including these, is based on the assumption that the parameters being estimated do not change with time, i.e, the distribution is assumed to be stationary. Thus, as mentioned above, it is desirable that the estimate converges to the true underlying parameter with probability 1, as the number of samples increases.

As argued in [11], there are numerous real-life problems in which the environment is non-stationary, and the parameter being estimated changes with time. Once can, for example, be dealing with an ensemble of biased dice, which leads to a multinomially distributed random variable where the parameter (the vector of probabilities for choosing the possible sides) is *switched*, perhaps, periodically, to possibly a new random value. Such a scenario demonstrates the behavior of a non-stationary environment. Thus, in this case, the goal of an estimator scheme would be to estimate the parameter, and to be able to adapt to any changes occurring in the environment. In other words, the algorithm must be able to detect the changes and estimate the new parameter after a *switch* has occurred in the environment. If one uses strong estimators (i.e., estimators that converge w.p. 1), it is impossible for the learned parameter to change rapidly from the value to which it has converged, resulting in poor time-varying estimates.

As opposed to the traditional MLE and Bayesian estimators, Oommen and Rueda, in [11], used the principles of *continuous* stochastic Learning Automata (LA) to propose a Stochastic Learning Weak Estimator (SLWE). Their scheme yielded a *weak* estimator which could learn the parameters of a binomial or multinomial distribution when the underlying distribution is nonstationary. As opposed to their scheme, in this paper, we propose a *discretized* algorithm. However, rather than merely utilize the process of discretization by "uniformly slicing" the parameter space, we propose a novel discretized weak estimator in which the principle of inferring how the updates are to be made, is *itself* based on a meta-learning strategy, explained, in detail, below. Our new estimator can be shown to converge to the true value fairly quickly, and to "*unlearn*" what it has learned so far to adapt to the new, "*switched*" environment. Further, as in the case of the SLWE, the convergence of the estimate is weak, i.e., with regard to the first and second moments.

The principle with which we design the estimator, referred to to as the Stochastic Search on the Line Based Discretized Weak Estimator (SSLDE), is based on the solution to the Stochastic Search on the Line (SSL) problem, pioneered by Oommen in [7].

It is pertinent to mention that the SSL solution presented in [7] is not directly relevant to the problem currently studied. The reason for this is the following: The method proposed in [7] explicitly assumes the existence of an Oracle which informs the learning mechanism whether to go “right” or “left”. Such an Oracle is non-existent in the estimation problem. In our application domain, in order to resolve this paradox for estimation, we have to first *infer* (or rather *simulate*) such an Oracle. Thus, to overcome this difficulty, we first intelligently construct an “Artificial Oracle” that suggests whether we are to increase the current estimate or to decrease it.

The analytic results derived and the empirical results obtained demonstrate that the SSLDE estimator is able to cope with non-stationary environments with a high adaptation rate and accuracy, that, as one can expect, is dependent on the choice of the resolution parameter.

With regard to their applicability, apart from the problem being of importance in its own right, weak estimators admit a growing list of applications in various areas such as intrusion detection systems in computer networks [15], spam filtering [19], ubiquitous computing [6], fault tolerant routing [10], adaptive encoding [12], and topic detection and tracking in multilingual online discussions [14]. We ourselves are also considering the application of such estimators in mobile adaptive architectures.

2 State-of-the-Art

Traditionally available methods that cope with non-stationary distributions resort to the so-called *sliding window* approach, which is a limited-time variant of the well-known MLE scheme. The latter model is useful for discounting stale data in the stream of observations. Since the data samples arrive continuously, only the most recent observations are used to compute the current estimates. Thus, the data elements occurring outside the “current window” is forgotten and replaced by the new data. The problem with using sliding windows is the following: If the time window is too small the corresponding estimates tend to be poor. As opposed to this, if time window is too large, the estimates prior to the change of the parameter have too much influence on the new estimates. Moreover, the observations during the entire window width must be maintained and updated during the process of estimation.

Apart from the sliding window approach, many other methods have been proposed, which deal with the problem of detecting change points during estimation. In general, there are two major competitive sequential change-point detection algorithms: Page’s Cumulative Sum (CUSUM) [2] detection procedure and the Shiryaev-Roberts-Pollak detection procedure. In [13], Shiryayev used a Bayesian approach to detect changes in the parameters of the distribution, where the change points were assumed to obey a geometric distribution. CUMSUM is motivated by a maximum likelihood ratio test for the hypotheses that a change occurred. Both approaches utilize the log-likelihood ratio for the hypotheses that the change occurred at the point, and that there is no change.

Inherent limitations of CUMSUM and the Shiryaev-Roberts-Pollak approaches for on-line implementation are the demanding computational and memory requirements. In contrast to the CUMSU and the Shiryaev−Roberts−Pollak approaches, our SSLDE avoids the intensive computations of ratios, and does not invoke hypothesis testing.

A description of the state-of-the-art would not be complete without mentioning the SLWE work of Oommen and Rueda [11] (cited above) which is based on the principles of *continuous* stochastic Learning Automata (LA). As opposed to this, our scheme resorts to *discretizing* the probability space [1,5,9,16], and performing a controlled random walk on this discretized space. It is well known in the field of LA that discretized schemes achieve faster convergence speed than continuous schemes [1,8]. By virtue of discretization, our estimator realizes fast adjustments of the running estimates by jumps, and it is thus able to robustly track changes in the parameters of the distribution after a switch has occurred in the environment.

A brief history of the science of discretization in the field of LA is not out of place. The concept of discretizing the probability space was pioneered by Thathachar and Oommen in their study on Reward-Inaction LA [16], and since then that it has catalyzed a significant research in the design of discretized LA [1,5,9,3,4]. Recently, there has been an upsurge of research interest in solving resource allocation problems based on novel discretized LA [3,4]. In [3,4], the authors proposed a solution to the class of *Stochastic Nonlinear Fractional Knapsack* problems where resources had to be allocated based on incomplete and noisy information. The latter solution was applied to resolve the web-polling problem, and to the problem of determining the optimal size required for estimation.

In a previous research work, the authors of this current work also devised a discretized weak estimator that is able to cope with non-stationary binomial and multinomial distributions. That estimate, referred to as the Stochastic Discretized Weak Estimator, has been analyzed and descibed elsewhere [17], and space does not permit us to submit a comprehensive comparison here. In all brevity, we mention that the latter scheme can be seen to be a more "fidel" counterpart of the continuous SLWE [11] since, in both algorithms (SLWE and SDWE), the mean of the final estimate is independent of the scheme's learning parameter.

We now proceed to present the new estimator, i.e., the Stochastic Search on the Line Based Discretized Weak Estimator (SSLDE).

3 The Estimator for Binomial Distributions

We assume that we are estimating the parameters of a binomial distribution. This distribution is characterized by two parameters, namely the number of trials and the parameter characterizing each Bernoulli trial. We assume that the number of observations is the number of trials. We seek to estimate the Bernoulli parameter for each trial.

Let X be a binomially distributed random variable, which takes on the value either "1" or "2". We choose to use these values instead of the more common used notation "0" or "1" to make the notation consistent when we consider the multinomial case. It is assumed that the distribution of X is characterized by the parameter vector $S = [s_1, s_2]^{\mathrm{T}}$. In other words,

X = "1" with probability s_1

X = "2" with probability s_2, where $s_1 + s_2 = 1$.

Let $x(t)$ be a concrete realization of X at time 't'. We intend to estimate S, i.e, s_i for $i = 1, 2$. We achieve this by maintaining a running estimate of $P(t) = [p_1(t), p_2(t)]^{\mathrm{T}}$ of S where $p_i(t)$ represents the estimate of s_i at time t, for $i = 1, 2$. Our proposed SSLDE works in a discretized manner. In fact, we enforce the condition that $p_i(t)$ takes values from a finite set, i.e, $p_i(t) \in \{0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}, 1\}$, where N is a user-defined integer parameter. N is called the "resolution" parameter and determines the stepsize Δ ($\Delta = \frac{1}{N}$) relevant to the updating of the estimates. A larger value of N will ultimately imply a more accurate convergence to the unknown parameter S. However, a smaller value of N will hasten the convergence rate, but will forfeit some accuracy. Initially, we assign $p_1(0) = p_2(0) = \frac{N}{2}$, where N is assumed to be an even integer. Let $r_1(t)$ be the state index of the estimator, implying that $p_1(t) = \frac{r_1(t)}{N}$. Thereafter, the value of $r_1(n)$, is updated as follows, depending on whether $r_1(t)$ is greater or less than $\frac{N}{2}$:

- **Case 1**: $r_1(t) \geq \frac{N}{2}$
  If $x(t)$ = "1" and $rand() \leq \frac{N}{2r_1(t)}$ and $r_1(t) < N$
$$r_1(t) := r_1(t) + 1 \tag{1}$$
  Else If $r_1(t) > 0$
$$r_1(t) := r_1(t) - 1 \tag{2}$$
  Else
$$r_1(t) := r_1(t) \tag{3}$$
- **Case 2**: $r_1(t) < \frac{N}{2}$
  If $x(t)$ = "2" and $rand() \leq \frac{N}{2(N-r_1(t))}$ and $r_1(t) > 0$
$$r_1(t+1) := r_1(t) - 1 \tag{4}$$
  Else If $r_1(t) < N$
$$r_1(t) := r_1(t) + 1 \tag{5}$$
  Else
$$r_1(t) := r_1(t), \tag{6}$$

where $p_2(t+1) = 1 - p_1(t+1)$ and $rand()$ is a uniform random number generator function. In the interest of simplicity, we omit the time index t, whenever there is no confusion, and thus, P implies $P(t)$.

The main theorem that we present concerns the convergence of the vector P which estimates S as per Equations (1 - 6). We claim that the mean of P converges exactly to S as N goes to infinity. In the interest of brevity, this proof is omitted here.

Theorem 1. *Let X be a binomially distributed random variable, and $P(t)$ be the estimate of S at time t obtained by Equations (1 - 6). Then:*

$$Lim_{N\to\infty}E[P(\infty)] \to S.$$

The proof of this result is quite involved and can be found in [18]. □

Remarks: A few remarks regarding our method for updating the estimates are not out of place. Indeed:

- First of all, it is pertinent to mention that although the rationale for updating is similar to that of the SSL algorithm [7], there are some fundamental differences. Unlike the latter, which explicitly assumes the existence of an "Oracle", in this case, our scheme simulates such an entity.
- Secondly, at this juncture, we emphasize that unlike the work of [7], the probability that the Oracle suggests the move in the correct direction, is not constant over the states of the estimator's state space. This is quite a significant difference, which basically reduces our model to a Markov Chain with state-dependent transition probabilities.
- The crucial issue is that by means of a random number generator function, we, hopefully elegantly, construct an *Artificial* Oracle that is **informative** as per the definition of [7], i.e, the Artificial Oracle's suggestions to either increase or decrease the estimate are correct with a probability always larger than 0.5. In that sense, whenever $p_1(t)$ is less than s_1, the Oracle directs the scheme to increase $p_1(t)$ with a probability greater than 0.5, and vice versa. The converse is true whenever $p_1(t)$ is greater than s_1.
- The main difference between our estimator and the SDWE, presented in [17], is that in the latter, the mean of the final estimate is independent of the scheme's learning coefficient, N. Consequently, we can say that the SDWE is a more fidel counterpart version of the SLWE, where the mean does not depend on *its* parameter, λ, as well. Moreover, in the case of the SDWE, at the internal states (i.e. $0 < r_1 < N$), there is a non zero probability that the estimate remains unchanged at the next time instant. As opposed to this, in our present updating scheme, the machine never stays at the same state at the next time instant, except at the end states. Therefore, it can be better characterized to be true to the essence of the SSL problem where the environment directs the LA to move to the right or to the left, and never directs it to stay at the same position.

4 Experimental Results

In this section, we evaluate the new family of discretzied estimators in non-stationary environments as well as in stationary environments. In the interest of brevity, we merely cite a few specific experimental results that highlight the salient properties of our approach.

4.1 Optimality Property

In this set of experiments, we experimentally verify the asymptotic optimality property of the estimator, as stated in Theorem 1. The results obtained have been recorded in Table 1, which summarizes the performance of the estimator, for a wide range of resolution parameters, N, and for two different values of s_1, namely $s_1 = 0.352$ and $s_1 = 0.827$. The resulting performance is reported in terms of the asymptotic true value of $E[p_1(\infty)]$, where $E[p_1(\infty)]$ is obtained using a single run experiment consisting of 10^7 iterations.

Note that an alternative manner to compute the true value of $E[p_1(\infty)]$ requires using its closed form expression. The reader will observe that by using a significant number of iterations (in our case, 10^7 iterations) the computations render the difference between the theoretical and experimental value of $E[p_1(\infty)]$ unobservable.

The experimental results confirm that the optimality property is valid. Indeed, $E[p_1(\infty)]$ consistently approaches s_1 as we increase the resolution. For example, for a resolution N equal to $1,000$, the final terminal value represents an error less than 0.002%.

Table 1. The true value of $E[p_1(\infty)]$ for various values of the resolution parameter, N, and for two different values of s_1

N	$s_1 = 0.352$	$s_1 = 0.827$
6	0.32	0.764
10	0.3371	0.798
60	0.3424	0.830
100	0.3472	0.8311
200	0.3496	0.8291
400	0.3503	0.8279
1000	0.3511	0.8277

4.2 Rate of Convergence

This set of experiments were conducted so that we could better understand the transient behavior of the chain associated with the estimator, and to thus perceive the rate of convergence of p_1 for different resolution configurations. To accomplish this, we fixed s_1 to be 0.9123, while we increased the memory N from $N = 6$ to $N = 40$. The quantity p_1 was then estimated by averaging it over $1,000$ experiments. In order to understand the effect of the resolution on the rate of convergence, we report the number of iterations required to reach a value that is 95% of the terminal value of $E[p_1(\infty)]$.

From Figure 1(a), we see that it took *only* 5 time instants for the algorithm to reach 95% of $E[p_1(\infty)]$ for a resolution $N = 6$. This, we believe, is remarkable. Further, from Figure 1(b), we see that 95% of $E[p_1(\infty)]$ was attained within 17 iterations when we set N to have the value 20. Similarly, for the results depicted in Figure 1(c), we chose N to be 30, where we see that it took *only* 25 time instants to converge to 95% of $E[p_1(\infty)]$. Finally, in Figure 1(d), when we set N to 40, we record the required number of iterations was as low as 35! We believe that these outstanding results speak for themselves.

Observe that as we increased the memory, the estimator spent more time to converge to the optimal value of $E[p_1(\infty)]$, which, of course, is understandable.

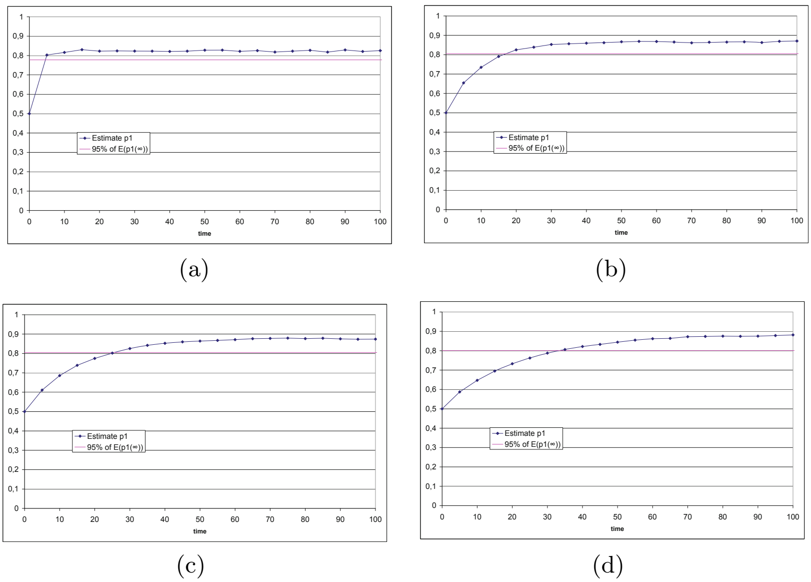
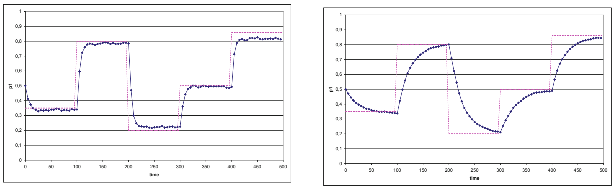


Fig. 1. This figure depicts the transient behavior of the chain as a function of time, where we plot (a) $p_1(t)$ for a memory size $N = 6$, (b) $p_1(t)$ for a memory size $N = 20$, (c) $p_1(t)$ for a memory size $N = 30$, and (d) $p_1(t)$ for a memory size $N = 40$

4.3 Performance in Dynamic Environments

In this experiment, we were interested in understanding the characteristics of the estimator when interacting with a non-stationary environment (i.e., when s_1 changed with time). To do this, we modeled a non-stationary environment by altering the parameter of the binomial distribution, s_1, at every 100^{th} time slot.

In particular, the parameter s_1 that we used in the experiments was drawn sequentially every 100^{th} time instant from the vector $R = [0.35, 0.8, 0.2, 0.5, 0.86]$. To be more specific, between time instants 0 and 100, s_1 was equal to 0.35, between instants 100 and 200, s_1 was equal to 0.8, and so on. In Figure 2(a), we have plotted the average value of p_1 over the set of experiments using a continuous line, and the target parameter s_1 using a discontinuous (dashed) line, when $N = 10$. Similarly, in Figure 2(b), we have reported the results of achieving the same (as in Figure 2(a)), except that, in this case, the resolution parameter was set as $N = 40$. From Figures 2(a) and 2(b), we observe that the instantaneous value of p_1 were well able to track the target distribution (drawn using the



(a) (b)

Fig. 2. Ability of the scheme to track the target distribution with (a) $N = 10$, and with (b) $N = 40$

discontinuous line) in a near-optimal manner, which we believe is quite fascinating! The use of this strategy to achieve time-varying testing, is obvious!

5 Conclusion

In this paper, we have presented a novel estimator, referred to as the Stochastic Search on the Line Based Discretized Weak Estimator (SSLDE), that is suitable for estimation in non-stationary environments. The design and foundations motivating the SSLDE are based on the principles of the pioneering solution of Oommen to the Stochastic Search on the Line problem [7]. By virtue of the SSLDE, we have shown that discretizing the probability space offers a new promising approach for the design of weak estimators. In fact, comprehensive simulation results demonstrate that the new estimator is able to cope with non-stationary environments with both a high adaptation rate and accuracy. To the best of our knowledge, this paper represents the first reported results that resolve *discretized* weak estimation using a SSL-based solution. The generalization of the scheme to handle multinomial distributions is currently being investigated.

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