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Robust Adaptive H_∞ Synchronization of Master-Slave Systems with Discrete and Distributed Time-Varying Delays and Nonlinear Perturbations

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Abstract: This paper establishes an adaptive synchronization problem for the master and slave structure of linear systems with nonlinear perturbations and mixed time-varying delays, where the mixed delays comprise different discrete and distributed time-delays. Using an appropriate Lyapunov-Krasovskii functional, some delay-dependent sufficient conditions and an adaption law which include the master-slave parameters are established for designing a delayed synchronization law in terms of linear matrix inequalities. The controller guarantees the $\boldsymbol{H}_\infty$ synchronization of the two coupled master and slave systems regardless of their initial states. Particularly, it is shown that the synchronization speed can be controlled by adjusting the update gain of the synchronization signal. A numerical example is given to show the effectiveness of the method.

Keywords: Adaptive Synchronization; Master-slave systems; Delay; $\boldsymbol{H}_\infty$ performance; Nonlinear perturbations

1. INTRODUCTION

Synchronization is a basic motion in nature that has been studied for a long time, ever since the discovery of Christian Huygens in 1665 on the synchronization of two pendulum clocks. The results of chaos synchronization are utilized in biology, chemistry, secret communication and cryptography, nonlinear oscillation synchronization and some other nonlinear fields. The first idea of synchronizing two identical chaotic systems with different initial conditions was introduced by Pecora and Carroll (1990), and the method was realized in electronic circuits. The methods for synchronization of the chaotic systems have been widely studied in recent years, and many different methods have been applied theoretically and experimentally to synchronize chaotic systems, such as feedback control (Alvarez and Curiel, 1997); Gao et al., 2006); Karimi and Maass, 2009); Wen et al., 2006); Lu and van Leeuwen, 2006), adaptive control (Cao and Lu, 2006; Liao and Tsai, 2000; Yan et al., 2006; Fradkov et al., 2000), backstepping (Park, 2006) and sliding mode control (García-Valdovinos, 2007). Recently, the theory of incremental input-to-state stability to the problem of synchronization in a complex dynamical network of identical nodes, using chaotic nodes as a typical platform was studied by Cai and Chen (2006).

On the other hand, delay systems represent a class of infinite-dimensional systems largely used to describe propagation and transport phenomena or population dynamics (Hale and Verduyn Lunel, 1993). The presence of a delay in a system may be the result of some essential simplification of the corresponding process model. The delay effects problem on the stability of systems including delays in the state and/or input is a problem of recurring interest since the delay presence may induce complex behaviors (oscillation, instability, bad performances) for the schemes (Wang et al., 2006; Wang et al., 2005; Gao et al., 2006). Some recent views and improved methods pertaining to the problems of determining robust stability criteria and robust control design of uncertain time-delay systems have been reported see for example (Karimi and Gao, 2008). In the past few decades increased attention has been devoted to the problem of robust delay-independent stability or delay-dependent stability and stabilization via different approaches (for example, model transformation techniques (Han, 2002), the improved bounding techniques (Mou et al., 2008), and the properly chosen LKFs (He et al., 2007)) for a number of different neutral systems with delayed state and/or input, parameter uncertainties and nonlinear perturbations (Karimi, 2008; Wang et al., 2006; Lam et al., 2005; Zhang et al., 2008).

On the synchronization problems of systems with time-delays and nonlinear perturbation terms, we see that there have been some research works (Karimi and Gao, 2010; Sun et al. 2007; Wang and Cao, 2009). In (Yan et al., 2006), the adaptive decentralized synchronization of master–slave large-scale time-varying delayed systems with unknown signal propagation delays was investigated based on the Lyapunov stability theorem. In (Cao and Lu, 2006), based on the invariant principle of functional differential equations, an analytical and rigorous adaptive feedback scheme is proposed for the synchronization of almost all kinds of coupled identical neural networks with time-varying delay, which can be chaotic, periodic, etc. In (Wang et al. 2008), the synchronization problem is studied for a class of stochastic complex networks with time delays. By utilizing Lyapunov functional form based on the idea of 'delay fractioning', the stochastic analysis techniques and the properties of Kronecker product are employed to establish delay-dependent synchronization criteria that guarantee the globally asymptotically mean-square synchronization of the addressed delayed networks with stochastic disturbances. So the development of synchronization methods for master-slave systems with time-varying delays using delay-dependent adaptive synchronization is important and has not been fully investigated in the past and remains to be important and challenging. This motivates the present study.

In this paper we contribute to the further development of the adaptive synchronization problem for a class of master-slave systems with nonlinear perturbations and mixed time-delays, where the mixed delays comprise different discrete and distributed time-delays. Some sufficient conditions and an adaption law which include the master-slave parameters are obtained by using the LKFs method and linear matrix inequality (LMI) techniques. Then, the controller is developed based on the available information of the size of the discrete and distributed delays so as to guarantee that the controlled slave system can be synchronized with the master system regardless of their initial states. Particularly, the synchronization speed can be controlled by adjusting the update gain of the synchronization signal. All the developed results are expressed in terms of convex optimization over LMIs and tested on a representative example to demonstrate the feasibility and applicability of the proposed synchronization approach.

This paper is organized as follows. In Section 2, the model of master-slave systems with both time-varying discrete and distributed delays and nonlinear perturbations is described. In Section 3, the discrete-delay-dependent distributed-delay-dependent adaptive synchronization is derived based on LMIs. In Section 4, a numerical example is given to verify our results. Finally, in Section 5, a conclusion is given.

2. PROBLEM DESCRIPTION

Consider a model of master and slave systems with mixed discrete and distributed time-varying delays and nonlinear perturbations as

$$\begin{cases} \dot{x}_m(t) = A_1 x_m(t) + A_2 x_m(t-h(t)) + A_3 \int_{t-\tau(t)}^{t} x_m(s)\, ds \\ \qquad + N_1 f_1(t; x_m(t)) + N_2 f_2(t, x_m(t-h(t))), \\ x_m(t) = \phi(t), \qquad t \in [-\kappa, 0], \\ z_m(t) = C_1 x_m(t) + C_2 x_m(t-h(t)) + C_3 \int_{t-\tau(t)}^{t} x_m(s)\, ds \end{cases} \tag{1}$$

and

$$\begin{cases} \dot{x}_s(t) = A_1 x_s(t) + A_2 x_s(t-h(t)) + A_3 \int_{t-\tau(t)}^{t} x_s(s)\, ds \\ \qquad + N_1 f_1(t; x_s(t)) + N_2 f_2(t, x_s(t-h(t))) + B u(t) + D w(t), \\ x_s(t) = \varphi(t), \qquad t \in [-\kappa, 0], \\ z_s(t) = C_1 x_s(t) + C_2 x_s(t-h(t)) + C_3 \int_{t-\tau(t)}^{t} x_s(s)\, ds \end{cases} \tag{2}$$

where $\kappa := \max\{h_M, \tau_m\}$, $x_m(t), x_s(t)$ are the $n \times 1$ state vector of the master and slave systems, respectively and $u(t)$ is the $r \times 1$ control input. The time-varying vector valued initial functions $\phi(t)$ and $\varphi(t)$ are continuously differentiable functionals and $f_i(.,.)$ are also time-varying vector-valued functions. The time-varying delays are satisfying

$$0 < h(t) \le h_M, \qquad \dot{h}(t) \le h_D. \tag{3a}$$

$$0 < \tau(t) \le \tau_M, \qquad \dot{\tau}(t) \le \tau_D. \tag{3b}$$

Assumption 1. The functions $f_i : \Re^+ \times \Re^n \to \Re^{n'}$ are continuous and satisfy $f_i(t,0) = 0$ and the Lipschitz conditions, i.e.,

$$\|f_i(t, x_0) - f_i(t, y_0)\| \le \|\Gamma_i (x_0 - y_0)\|$$

for all t and for all $x_0, y_0 \in \Re^n$ such that Γ_i are some known matrices.

Remark 1. The model (1)-(2) can describe a large amount of well-known dynamical systems with time-delays, such as the Logistic model, the chaotic models with time-delays and the artificial neural network model with discrete time-delays. In real application, these coupled systems can be regarded as interacting dynamical elements in the entire system, such as physical particles, biological neurons, ecological populations, genic oscillations, and even automatic machines and robots. A feasible coupling design for successful synchronization leads us to fully command the intrinsic mechanism regulating the evolution of real systems, to fabricate emulate systems, and even to remotely control the machines and nodes in networks with large scales (Pecora and Carroll, 1990; Gao et al. 2006; Park, 2006; Gao et al. 2008).

Assumption 2. The full state variables $x_s(t)$ and $x_m(t)$ are available for measurement.

Now, it is required to synchronize the slave system with the master system at the same time. The synchronization error of the master and slave systems (1)-(2) is defined as $e(t) = x_m(t) - x_s(t)$, then the error dynamics between (1)-(2), namely synchronization error system, can be expressed by

$$\begin{aligned} \dot{e}(t) &= A_1 e(t) + A_2 e(t-h(t)) + A_3 \int_{t-\tau(t)}^{t} e(s)\, ds + N_1 \hat{f}_1(t; e(t)) \\ &\quad + N_2 \hat{f}_2(t; e(t-h(t))) - B u(t) - D w(t) \end{aligned} \tag{4}$$

where $\hat{f}_1(t; e(t)) := f_1(t; x_m(t)) - f_1(t; x_m(t) - e(t))$ and $\hat{f}_2(t; e(t-h(t))) := f_2(t; x_m(t-h(t))) - f_2(t; x_m(t-h(t)) - e(t-h(t)))$.

From Assumption 1, the corresponding uncertainty set is denoted by

$$\Xi_i(e(t)) := \{\hat{f}_i(t, e(t)) : \|\hat{f}_i(t, e(t))\| \le \|\Gamma_i e\|\} \tag{5}$$

The problem to be addressed in this paper is formulated as follows: given the master-slave systems (1)-(2) with both discrete and distributed time-delays, find a delay-dependent adaptive synchronization control $u(t)$ for the slave system (2) so that the state of the slave system can follow that of a master model, i.e., $\lim_{t\to\infty} e(t) = 0$.

3. MAIN RESULTS

In this section, we propose sufficient conditions for the solvability of the adaptive synchronization problem of the master-slave systems (1)-(2) using the Lyapunov method.

Define the following Lyapunov-Krasovskii functional

$$V(t) = V_1(t) + V_2(t) + V_3(t) + q\rho(t)^2, \tag{6}$$

with $\rho(0) = 0$ and

$$V_1(t) = e(t)^T P e(t),$$

$$V_2(t) = \int_{t-h(t)}^{t} e(\xi)^T S e(\xi)\, d\xi + \int_{t-h(t)}^{t} \int_{\xi}^{t} e(s)^T R\, e(s)\, ds\, d\xi,$$

$$V_3(t)=\int_{t-\tau(t)}^{t}\Big[\int_{s}^{t}e(\theta)^T\,d\theta\Big]U_1\Big[\int_{s}^{t}e(\theta)\,d\theta\Big]\;ds+\int_{0}^{\tau(t)}\int_{t-s}^{t}(\theta-t+s)e(\theta)^T U_1 e(\theta)\;d\theta\,ds$$

where $\rho(t)\in\Re$ denotes the adaptation errors which will be defined later. Define the H_∞ performance measure

$$J_\infty=\int_0^\infty[(z_m(t)-z_s(t))^T(z_m(t)-z_s(t))-\gamma^2 w^T(t)w(t)]\;dt\,. \qquad (7)$$

Now, to establish the H_∞ performance measure for the system (1)-(2), assume zero initial condition, then we have $V(t)\big|_{t=0}=0$. Consider the index J_∞ in (7), then along the solution of (4) for any nonzero $w(t)$ there holds

$$\begin{aligned}J_\infty &\le \int_0^\infty[(z_m(t)-z_s(t))^T(z_m(t)-z_s(t))-\gamma^2 w^T(t)w(t)]\;dt\\ &\qquad -V(x(t))\big|_{t=0}+V(x(t))\big|_{t=\infty} \qquad (8)\\ &\le \int_0^\infty H[u(t),w(t),\hat f_1(t,e(t)),\hat f_2(t,e(t-h(t)))]\;dt\end{aligned}$$

where the function

$$\begin{aligned}&H[u(t),w(t),\hat f_1(t,e(t)),\hat f_2(t,e(t-h(t)))]\\ &\quad:=(z_m(t)-z_s(t))^T(z_m(t)-z_s(t))-\gamma^2 w^T(t)w(t)+\dot V(t)\end{aligned}$$

is called a Hamiltonian function. It is well known that a sufficient condition for achieving robust disturbance attenuation, i.e. $J_\infty<0$, is that the inequality

$$\begin{aligned}&H[u(t),w(t),\hat f_1(t,e(t)),\hat f_2(t,e(t-h(t)))]<0,\\ &\qquad\forall w\in L^2,\hat f_i(t,e(t))\in\Xi_i(e(t)),\; i=1,2\end{aligned} \qquad (9)$$

results in an $V(t)$ which is strictly radially unbounded.

Theorem 1. Under Assumptions 1-2, the master-slave systems (1)-(2) with the different discrete and distributed time-varying delays can be synchronized if there exist the scalar $\gamma>0$, matrices L_1,L_2,L_3 and positive-definite matrices X, $\tilde S$, $\tilde R$, $\tilde U_1$ such that the following LMI holds

$$\tilde\Pi:=\begin{bmatrix}\tilde\Pi_{11} & A_2X-BL_2 & 0 & A_3X-BL_3 & N_1 & N_2 & -D & XC_1^T & X\Gamma_1^T & 0\\ * & -(1-h_D)\tilde S & 0 & 0 & 0 & 0 & 0 & XC_2^T & 0 & X\Gamma_2^T\\ * & * & -(1-h_D)\tilde R & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ * & * & * & -(1-\tau_D)\tilde U_1 & 0 & 0 & 0 & XC_3^T & 0 & 0\\ * & * & * & * & -I & 0 & 0 & 0 & 0 & 0\\ * & * & * & * & * & -I & 0 & 0 & 0 & 0\\ * & * & * & * & * & * & -\gamma^2 I & 0 & 0 & 0\\ * & * & * & * & * & * & * & -I & 0 & 0\\ * & * & * & * & * & * & * & * & -I & 0\\ * & * & * & * & * & * & * & * & * & -I\end{bmatrix}<0 \qquad (10)$$

where $\tilde\Pi_{11}=A_1X+XA_1^T-BL_1-L_1^TB^T+\tilde S+h_M\tilde R+\tau_M^2\,\tilde U_1$. Then, the adaptive synchronization controller is given by

$$u(t)=K_1e(t)+K_2e(t-h(t))+K_3\int_{t-\tau(t)}^{t}e(s)\,ds-\rho(t)\frac{B^TX^{-1}e(t)}{\|B^TX^{-1}e(t)\|} \qquad (11)$$

for all $\|B^TX^{-1}e(t)\|\neq 0$, otherwise $u(t)=0$, with the adaptation law

$$\dot\rho(t)=q^{-1}\|B^TX^{-1}e(t)\|,\quad \rho(0)=\rho_0 \qquad (12)$$

where $K_i=L_iX^{-1}$, $i=1,2,3$ and the positive constants q and ρ_0 are specified by the designer.

Proof. We will prove the Theorem by showing that the control law (11) will guarantee the inequality of (10).
By using the Jensen's Inequality, Lemma 3 in Appendix, and the properties of the time delays, derivatives of $V_i(t)$, $i=1,\cdots,3$, are given, respectively, by

$$\dot V_1(t)=2\dot e(t)^TPe(t) \qquad (13a)$$

$$\begin{aligned}\dot V_2(t)&=e(t)^T(S+h(t)R)e(t)-(1-\dot h(t))e(t-h(t))^TSe(t-h(t))\\ &\quad-(1-\dot h(t))\int_{t-h(t)}^{t}e(s)^T\,R\,e(s)\;ds\,d\xi\\ &\le e(t)^T(S+h_MR)e(t)-(1-h_D)e(t-h(t))^TSe(t-h(t))\\ &\quad-(1-h_D)\left(\int_{t-h(t)}^{t}e(s)^T\;ds\right)R\left(\int_{t-h(t)}^{t}e(s)\;ds\right)\end{aligned} \qquad (13b)$$

$$\begin{aligned}\dot V_3(t)&=-(1-\dot\tau(t))\Big[\int_{t-\tau(t)}^{t}e(\theta)^T\,d\theta\Big]U_1\Big[\int_{t-\tau(t)}^{t}e(\theta)\;d\theta\Big]\\ &\quad+2\int_{t-\tau(t)}^{t}(\theta-t+\tau(t))e(t)^TU_1e(\theta)\;d\theta+\int_0^{\tau(t)}se(t)^TU_1e(t)\,ds\\ &\quad-\int_0^{\tau(t)}\int_{t-s}^{t}e(\theta)^TU_1e(\theta)\;d\theta\,ds\\ &\le\int_{t-\tau(t)}^{t}(\theta-t+\tau(t))[e(t)^TU_1e(t)+e(\theta)^TU_1e(\theta)]\;d\theta\\ &\quad-(1-\dot\tau(t))\Big[\int_{t-\tau(t)}^{t}e(\theta)^T\,d\theta\Big]U_1\Big[\int_{t-\tau(t)}^{t}e(\theta)\;d\theta\Big]+\int_0^{\tau(t)}se(t)^TU_1e(t)\,ds\\ &\quad-\int_{t-\tau(t)}^{t}(\theta-t+\tau(t))e(\theta)^TU_1e(\theta)\;d\theta\\ &=\tau_M^2e(t)^TU_1e(t)-(1-\tau_D)\Big[\int_{t-\tau(t)}^{t}e(\theta)^T\,d\theta\Big]U_1\Big[\int_{t-\tau(t)}^{t}e(\theta)\;d\theta\Big]\end{aligned} \qquad (13c)$$

From (5), we have

$$-\hat f_1(t,e(t))^T\hat f_1(t,e(t))+e(t)^T\Gamma_1^T\Gamma_1e(t)\ge 0 \qquad (14a)$$

$$-\hat f_2(t;e(t-h(t)))^T\hat f_2(t;e(t-h(t)))+e(t-h(t))^T\Gamma_2^T\Gamma_2e(t-h(t))\ge 0 \qquad (14b)$$

Substituting $u(t)$ by

$$u(t)=K_1e(t)+K_2e(t-h(t))+K_3\int_{t-\tau(t)}^{t}e(s)\,ds-\rho(t)\frac{B^TPe(t)}{\|B^TPe(t)\|} \qquad (15)$$

for all $\|B^TPe(t)\|\neq 0$, and from (13) and adding the left sides of equations (14) into $\dot V(t)$, we get

$$
\begin{aligned}
&H[u(t),w(t),\hat f_1(t,e(t)),\hat f_2(t,e(t-h(t)))]=(C_1e(t)+C_2e(t-h(t))\\
&\quad +C_3\int_{t-\tau(t)}^{t}e(s)\,ds)^T(C_1e(t)+C_2e(t-h(t))+C_3\int_{t-\tau(t)}^{t}e(s)\,ds)\\
&\quad -\gamma^2w(t)^Tw(t)+\dot V_1(t)+\dot V_2(t)+\dot V_3(t)+2q\rho(t)\dot\rho(t)\\
&\le e(t)^T(S+h_MR+\tau_M^2U_1+C_1^TC_1+\Gamma_1^T\Gamma_1)e(t)+2\dot e(t)^TPe(t)\\
&\quad +e(t-h(t))^T(-(1-h_D)S+\Gamma_2^T\Gamma_2+C_2^TC_2)e(t-h(t))+2q\rho(t)\dot\rho(t)\\
&\quad +2e(t)^TC_1^TC_2e(t-h(t))+2e(t)^TC_1^TC_3\int_{t-\tau(t)}^{t}e(s)\,ds\\
&\quad +2e(t-h(t))^TC_2^TC_3\int_{t-\tau(t)}^{t}e(s)\,ds-\hat f_1(t,e(t))^T\hat f_1(t,e(t))\\
&\quad -\hat f_2(t;e(t-h(t)))^T\hat f_2(t;e(t-h(t)))-\gamma^2w(t)^Tw(t)\\
&\quad -(1-h_D)\left(\int_{t-h(t)}^{t}e(s)^T\,ds\right)R\left(\int_{t-h(t)}^{t}e(s)\,ds\right)\\
&\quad +\left(\int_{t-\tau(t)}^{t}e(s)^T\,ds\right)(-(1-\tau_D)U_1+C_3^TC_3)\left(\int_{t-\tau(t)}^{t}e(s)\,ds\right)
\end{aligned}
\tag{16}
$$

By using some matrix norm operations, the above inequality can be rewritten as

$$
\begin{aligned}
&H[u(t),w(t),\hat f_1(t,e(t)),\hat f_2(t,e(t-h(t)))]\le\chi(t)^T\Pi\chi(t)\\
&\qquad +2\rho(t)e(t)^TPB\frac{B^TPe(t)}{\|B^TPe(t)\|}+2q\rho(t)\dot\rho(t)
\end{aligned}
\tag{17}
$$

where $\chi(t)=[e(t),\ e(t-h(t)),\ \int_{t-h(t)}^{t}e(\tau)\,d\tau,\ \int_{t-\tau(t)}^{t}e(\tau)\,d\tau,$ f1t,et, f2t,et−h(t), $w(t)]$ and

$$
\Pi=\begin{bmatrix}
\Pi_{11} & P(A_2-BK_2)+C_1^TC_2 & 0 & P(A_3-BK_3)+C_1^TC_3 & PN_1 & PN_2 & -PD\\
* & -(1-h_D)S+\Gamma_2^T\Gamma_2+C_2^TC_2 & 0 & 0 & 0 & 0 & 0\\
* & * & -(1-h_D)R & 0 & 0 & 0 & 0\\
* & * & * & -(1-\tau_D)U_1+C_3^TC_3 & 0 & 0 & 0\\
* & * & * & * & -I & 0 & 0\\
* & * & * & * & * & -I & 0\\
* & * & * & * & * & * & -\gamma^2I
\end{bmatrix}
\tag{18}
$$

where $\Pi_{11}=P(A_1-BK_1)+(A_1-BK_1)^TP+S+h_MR+\tau_M^2U_1+C_1^TC_1+\Gamma_1^T\Gamma_1$. Substituting the relation (12) in (17) results in

$$H[u(t),w(t),\hat f_1(t,e(t)),\hat f_2(t,e(t-h(t)))]\le\chi(t)^T\Pi\chi(t)\tag{19}$$

Let $\zeta=\mathrm{diag}\{X,X,X,X,I,I,I\}$ with $X:=P^{-1}$. Premultiplying ζ and postmultiplying ζ^T to the matrix inequality $\Pi<0$ and considering $\tilde S=XSX$, $\tilde R=XRX$, $\tilde U_1=XU_1X$ and $L_i=K_iX$, $i=1,2,3$, result in the LMI (10) by applying Schur complement lemma. Moreover, the condition $\Pi<0$ implies either $J_\infty<0$ or

$$\dot V(t)\le -e(t)^TC_1^TC_1e(t)\le 0\tag{20}$$

for $w(t)\equiv 0$. Then, we have $V(t)<V(0)$. It can be easily seen that the inequality (20) holds for all $\|B^TPe(t)\|=0$ (with $u(t)=0$ and $\rho(t)=\rho_0$), as well. Integrating the inequality (20) from 0 to t, it yields

$$V(0)\ge V(t)+\int_0^t e(s)^TC_1^TC_1e(s)\,ds\ge\int_0^t e(s)^TC_1^TC_1e(s)\,ds\tag{21}$$

Since the term $V(0)$ is positive and finite, the following limit exists and is finite

$$\lim_{t\to\infty}\int_0^t e(s)^TC_1^TC_1e(s)\,ds=\lim_{t\to\infty}\int_0^t\left|e(s)^TC_1^TC_1e(s)\right|ds\,.\tag{22}$$

Thus according to Barbalat lemma (in Appendix), we obtain

$$\lim_{t\to\infty}e(t)^TC_1^TC_1e(t)=0\,.\tag{23}$$

Then, the synchronization of master-slave systems with mixed time-delays and nonlinear perturbations is achieved under the neutral-delay-dependent adaptive synchronization law (11). Moreover, it is clear that the Lyapunov function (6) results in

$$V(t)\ge\lambda_{\min}(P)|e(t)|^2+\lambda_{\min}(S)\int_{t-h(t)}^{t}|e(\xi)|^T\,d\xi\tag{24}$$

and one gets

$$V(0)\le\Delta\|\zeta\|^2\tag{25}$$

where $\Delta=\lambda_{\max}(P)+h_M\lambda_{\max}(S)+\frac{1}{2}h_M^2\lambda_{\max}(R)+\frac{1}{3}\tau_M^3\lambda_{\max}(U_1)+\frac{1}{6}\tau_M^3\lambda_{\max}(U_1)$. Therefore, we have

$$|e(t)|^2+\varpi\int_0^t|e(t)|^2\,ds\le\frac{1}{\lambda_{\min}(P)}[\Delta\|\zeta\|^2+q\rho_0^2]\,.\tag{26}$$

where $\varpi:=\lambda_{\min}(C_1^TC_1)/\lambda_{\min}(P)$. The substitution

$$Z(t):=\int_0^t|e(t)|^2\,ds\tag{27}$$

reduces the inequality (26) to the following first order differential inequality

$$\dot Z(t)+\varpi Z(t)\le\frac{1}{\lambda_{\min}(P)}[\Delta\|\zeta\|^2+q\rho_0^2]\tag{28}$$

where $Z(0)=0$. Solving the inequality (28) gives

$$Z(t)\le\frac{1-e^{-\varpi t}}{\varpi\lambda_{\min}(P)}[\Delta\|\zeta\|^2+q\rho_0^2]\,,\tag{29}$$

so transforming back to the variable $|e(t)|^2$ in (24), we find that

$$|e(t)|^2\le|e(0)|^2+\frac{1}{\lambda_{\min}(P)}e^{-\varpi t}[\Delta\|\zeta\|^2+q\rho_0^2]\tag{30}$$

or

$$|e(t)|\le|e(0)|+e^{-\varpi t/2}[\sqrt{\frac{\Delta}{\lambda_{\min}(P)}}\|\zeta\|+\sqrt{\frac{q}{\lambda_{\min}(P)}}|\rho_0|]\tag{31}$$

which shows that the difference operator of the synchronization error system $e(t)$ is globally exponentially bounded with an exponential decay rate $\varpi/2$, which depends on the matrices C_1 and P. Therefore, synchronization speed can be controlled by adjusting positive constants q and ρ_0. Furthermore, from inequality (21), it is observed that $V(t)$ in

(6) is bounded since $V(0)$ is finite. This implies that $e(t)$ and $\rho(t)$ are bounded for all $t>0$. Moreover, the state $x_m(t)$ of the master model is always bounded, then it is concluded that the state $x_s(t)$ is also bounded. This completes the proof. ■

Corollary 1. Consider the following two master and slave systems without time delays:

$$\begin{cases} \dot{x}_m(t) = A_1 x_m(t) + N_1 f_1(t; x_m(t)), \\ z_m(t) = C_1 x_m(t) \end{cases} \tag{32}$$

and

$$\begin{cases} \dot{x}_s(t) = A_1 x_s(t) + N_1 f_1(t; x_s(t)) + B u(t) + D w(t), \\ z_s(t) = C_1 x_s(t) \end{cases} \tag{33}$$

Under Assumptions 1-2 and for a given scalar $\gamma > 0$, the master-slave systems (32)-(33) can be synchronized when the adaptive synchronization controller is given by

$$u(t) = K_1 e(t) - \rho(t) \frac{B^T X^{-1} e(t)}{\left\| B^T X^{-1} e(t) \right\|} \tag{34}$$

for all $\left\| B^T X^{-1} e(t) \right\| \neq 0$, otherwise $u(t)=0$, with the adaptation law

$$\dot{\rho}(t) = q^{-1} \left\| B^T X^{-1} e(t) \right\|, \quad \rho(0) = \rho_0 \tag{35}$$

where $K_1 = L_1 X^{-1}$ and the positive constants q and ρ_0 are specified by the designer and the matrix L_1 and the positive-definite matrix X are solutions of the following LMI

$$\begin{bmatrix} \vec{\Pi}_{11} & N_1 & -D & XC_1^T & X\Gamma_1^T \\ * & -I & 0 & 0 & 0 \\ * & * & -\gamma^2 I & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -I \end{bmatrix} < 0 \tag{36}$$

with $\vec{\Pi}_{11} = A_1 X + X A_1^T - B L_1 - L_1^T B^T$.

Remark 2. The results presented in Theorem 1 are depended on the upper bounds of the time-varying discrete and the distributed delays and the upper bounds of their derivative, as well. These give a less conservative design than the available delay-independent results in Yan et al. (2006). Therefore, the treatment in the present paper is more general.

4. SIMULATION RESULTS

In this section, we will verify the proposed methodology by giving an illustrative example. We solved LMI (10) by using Matlab LMI Control Toolbox, which implements state-of-the-art interior-point algorithms and is significantly faster than classical convex optimization algorithms. The example is given below.

Consider the master-slave systems (1)-(2) with the following state-space matrices for an aircraft model

$$A_1 = \begin{bmatrix} -0.2 & 0.01 \\ -0.1 & -0.5 \end{bmatrix};\ A_2 = \begin{bmatrix} -0.1 & 0.2 \\ 0.01 & -0.1 \end{bmatrix};\ A_3 = \begin{bmatrix} -0.3 & -0.1 \\ 0.1 & -0.15 \end{bmatrix};$$

$$B = D = \begin{bmatrix} 1 \\ 1 \end{bmatrix};\ N_1 = N_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};\ C_1 = [1 \quad 1];\ C_2 = C_3 = [1 \quad 1];$$

$$f_1(t, x(t)) = f_2(t, x(t)) = \begin{bmatrix} 0.5(|x_1(t)+1| - |x_1(t)-1|) \\ 0.5(|x_2(t)+1| - |x_2(t)-1|) \end{bmatrix}.$$

The delays $h(t) = \tau(t) = (1-e^{-t})/(1+e^{-t})$ satisfy $0 \le h(t) = \tau(t) \le 1$ and $\dot{h}(t) = \dot{\tau}(t) \le 0.5$, where $x_m(t) = [x_{1m}(t), x_{2m}(t)]^T$, $x_s(t) = [x_{1s}(t), x_{2s}(t)]^T$.

It is required to design the synchronization signal (11) with the adaptive law (12) such that the trajectories of the slave subsystem and master subsystem (1)-(2) can be synchronized. To this end, in light of Theorem 1, we solved the LMI (10) for $\gamma = 0.8$ and obtained

$$X = \begin{bmatrix} 0.0042 & 0.0076 \\ 0.0076 & 0.0364 \end{bmatrix}.$$

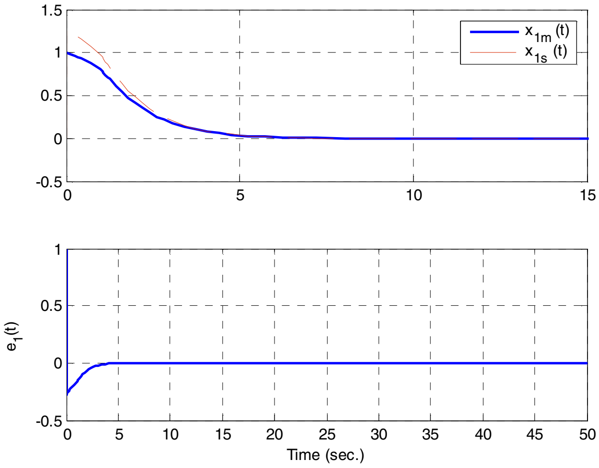

Fig. 1. Time responses of the first state of the master-slave systems and the related synchronization error.

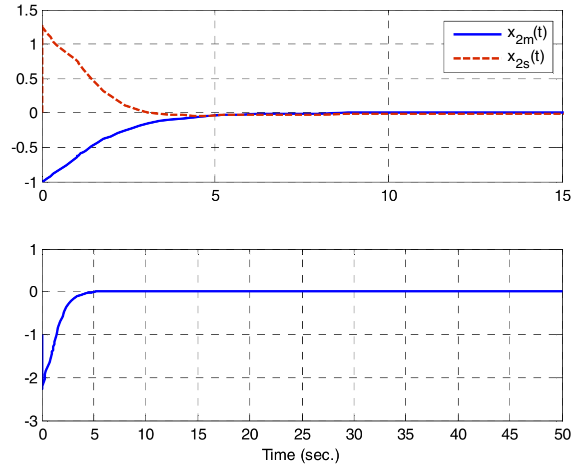

Fig. 2. Time responses of the second state of the master-slave systems and the related synchronization error.

For simulation purposes, we set values of the designed parameters as $q=10$, $\rho_0 = 1$ with the following initial conditions

$$x_m(t) = [1, -1]^T, \qquad t \in [-1, 0],$$
$$x_s(t) = 0, \qquad t \in [-1, 0].$$

and an exogenous disturbance input is set as

$$w(t) = \frac{1}{1+\sqrt{t}}, \quad t \ge 0.$$

Now, by applying the synchronization signal (11) with the adaptive law (12) and the parameters above, the temporal evolution of each variable of the master-slave systems $x_{1m}(t), x_{2m}(t)$, $x_{1s}(t)$, $x_{2s}(t)$ with the related synchronization errors, i.e., $e(t) = x_s(t) - x_m(t)$, are shown in Figures 1-2. It is seen that the synchronization errors $e_1(t) = x_{s1}(t) - x_{m1}(t)$ and $e_2(t) = x_{s2}(t) - x_{m2}(t)$ converge to zero. Moreover, the adaptation parameter $\rho(t)$ is depicted in Figure 3.

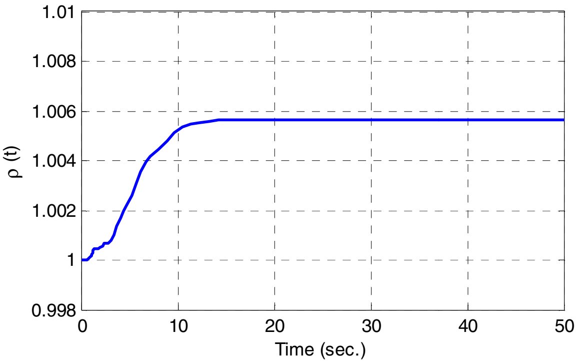

Fig. 3. Time response of the adaptation parameter.

5. COCLUSION

In this paper an adaptive H_∞ synchronization problem was proposed for the master and slave structure of linear systems with nonlinear perturbations and mixed time-varying delays, where the mixed delays comprise different discrete and distributed time-delays. Using an appropriate Lyapunov-Krasovskii functional, some delay-dependent sufficient conditions and an adaption law which include the master-slave parameters were established for designing a delayed synchronization law in terms of linear matrix inequalities. The controller guarantees the H_∞ synchronization of the two coupled master and slave systems regardless of their initial states. Particularly, it was shown that the synchronization speed can be controlled by adjusting the update gain of the synchronization signal.

APPENDIX

Lemma 1. (Park, 1999) (*Jensen's Inequality*) Given a positive-definite matrix $P \in \Re^{n\times n}$ and two scalars $b > a \geq 0$ for any vector $x(t) \in \Re^n$, we have

$$\int_{t-b}^{t-a} x^T(\omega) P x(\omega)\, d\omega \geq \frac{1}{b-a} \left(\int_{t-b}^{t-a} x(\omega)\, d\omega \right)^T P \left(\int_{t-b}^{t-a} x(\omega)\, d\omega \right).$$

Lemma 2. (*Barbalat lemma* (Popov, 1973)) If $w : \Re \to \Re$ is a uniformly continuous function for $t \geq 0$ and if the limit of the integral

$$\lim_{t\to\infty} \int_0^t |w(\lambda)|\, d\lambda$$

exists and is finite, then $\lim_{t\to\infty} w(t) = 0$.

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