

An LMI Approach to Quantized H_∞ Control of Uncertain Linear Systems with Network-Induced Delays

Hamid Reza Karimi, Ningsu Luo, Mauricio Zapateiro

Abstract—This paper deals with a convex optimization approach to the problem of robust network-based H_∞ control for linear systems connected over a common digital communication network with norm-bounded parameter uncertainties. Firstly, we investigate the effect of both the output quantization levels and the network conditions under static quantizers. Secondly, by introducing a descriptor technique, using Lyapunov-Krasovskii functional and a suitable change of variables, new required sufficient conditions are established in terms of delay-range-dependent linear matrix inequalities for the existence of the desired network-based quantized controllers with simultaneous consideration of network induced delays and measurement quantization. The explicit expression of the controllers is derived to satisfy both asymptotic stability and a prescribed level of disturbance attenuation for all admissible norm bounded uncertainties. One example is utilized to illustrate the design procedure proposed in this paper.

I. INTRODUCTION

Networked control systems (NCS) in which control and communication issues are combined together, and all the delays and limitations of the communication channels between sensors, actuators, and controllers are taken into account has become an enabling technology for many military, commercial and industrial applications. The study of NCSs is an interdisciplinary research area, combining both network and control theory. That is, in order to guarantee the stability and performance of an NCS, analysis and design tools based on both network and control parameters are needed. Modeling, analysis, and design of NCSs have received increasing attention in recent years (see [1], [2], [8] and [21]).

However, due to network bandwidth restriction, the insertion of communication network in the feedback control loop inevitably leads to communication delays and makes the analysis and design of NCSs complex. Communication delays can deteriorate the performance of NCSs and even can destabilize the systems when they are not considered in the design of NCSs. So far, a variety of efforts have been devoted

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Hamid Reza Karimi is with the Faculty of Technology and Science, University of Agder, N-4898 Grimstad, Norway. hamid.r.karimi@uia.no

Ningsu Luo is with the Institute of Informatics and Applications, University of Girona, Girona, Spain. ningsu.luo@udg.edu

Mauricio Zapateiro is with the Department of Applied Mathematics III, Universitat Politècnica de Catalunya, Barcelona, Spain. mauricio.zapateiro@upc.edu

to analyzing NCSs with communication delays (see, e.g., [3], [4],[15]-[19] and the references therein). Specifically, [1] and [20] analyzed the stability of NCSs and obtained stability regions using a hybrid systems technique. [10] presented linear matrix inequality (LMI) conditions for obtaining maximum allowable delay bounds, which guarantee the stability of NCSs. Based on Lyapunov-Razumikhin function method, [19] presented conditions on the admissible bounds of data packet loss and delays for NCSs in terms of LMIs. Based on stochastic control theory, optimal controller design of NCSs with stochastic network delays was investigated in ([7], [12]). For other control schemes, we refer readers to the survey ([14]). Recently, the problem of output feedback control for networked control systems (NCSs) with limited communication capacity was studied by Tian et al. in [13].

In this paper, we contribute to the further development of a convex optimization approach to the problem of robust network-based H_∞ control for uncertain linear systems connected over a common digital communication network. Here, We consider the case where quantizers are static and the parameter uncertainties are norm bounded. Firstly, we propose a new model to investigate the effect of both the output quantization levels and the network conditions. Secondly, by introducing a descriptor technique, using Lyapunov-Krasovskii functional and a suitable change of variables, new required sufficient conditions are established in terms of delay-dependent linear matrix inequalities (LMIs) for the existence of the desired network-based quantized controllers with simultaneous consideration of network induced delays and measurement quantization. The explicit expression of the controllers is derived to satisfy both asymptotic stability and a prescribed level of disturbance attenuation for all admissible norm bounded uncertainties. A numerical example is provided to illustrate the effectiveness of the approach presented in this paper.

The notations used throughout the paper are fairly standard. I_n and 0_n represent, respectively, n by n identity matrix and n by n zero matrix; the superscript T stands for matrix transposition; \mathfrak{R}^n denotes the n -dimensional Euclidean space; $\mathfrak{R}^{n \times m}$ is the set of all real m by n matrices. The matrices \hat{I} and \tilde{I} are defined, respectively, as $\hat{I} := [I, 0]$ and $\tilde{I} := [0, I]$. $\|\cdot\|$ refers to the Euclidean vector norm or the induced matrix 2-norm and $diag\{\dots\}$ represents a block diagonal matrix. $\lambda_{min}(A)$ and $\lambda_{max}(A)$ denote, respectively, the smallest and largest eigenvalue of the square matrix A . The operator $sym\{A\}$ denotes $A + A^T$ and $\lceil \cdot \rceil$ is the operation of round. The notation $P > 0$ means that P is real symmetric and positive definite and the symbol $*$ denotes the elements below the

main diagonal of a symmetric block matrix.

II. SYSTEM DESCRIPTION

Consider the following continuous-time system with time-varying structured uncertainties:

$$\dot{x}(t) = (A + \Delta A(t))x(t) + Du(t) + (B + \Delta B(t))w(t), \quad (1)$$

$$y(t) = Cx(t), \quad (2)$$

$$z(t) = Gx(t), \quad (3)$$

where $x(t) \in \mathfrak{R}^n$ is the state vector, $y(t) \in \mathfrak{R}^m$ is the measured output, considered as the control input; $w(t) \in \mathfrak{R}^l$ and $z(t) \in \mathfrak{R}^r$ are the disturbance and the signal to be estimated, respectively. The coefficient matrices A, B, C, G are real matrices with appropriate dimensions. The time-varying structured uncertainties $\Delta A(t)$ and $\Delta B(t)$ are said to be admissible if the following form holds

$$[\Delta A(t) \quad \Delta B(t)] = M_1 F(t) [L_a \quad L_b], \quad (4)$$

where L_a, L_b are constant matrices with appropriate dimensions; and $F(t)$ is an unknown, real, and possibly time-varying matrix with Lebesgue measurable elements, and its Euclidean norm satisfies

$$\|F(t)\| \leq 1, \quad \forall t. \quad (5)$$

We are interested in investigating the stability property of systems when the observer undergoes quantization and delays. This kind of problem arises in scenarios in which a finite bandwidth channel lies in the feedback loop and introduces a delay.

In this paper, a quantizer means a piecewise constant function $q: \mathfrak{R}^p \rightarrow \mathcal{Q}$, where \mathcal{Q} is a finite subset of \mathfrak{R}^l . We will use quantized measurements of the form

$$q_\mu(z) := \mu q\left(\frac{z}{\mu}\right) = \begin{cases} \mu M \Delta, & \frac{z}{\mu} > (M + 0.5)\Delta \\ -\mu M \Delta, & \frac{z}{\mu} < -(M + 0.5)\Delta \\ \mu \Delta \left\lceil \frac{z}{\mu} \right\rceil, & \left| \frac{z}{\mu} \right| \leq (M + 0.5)\Delta \end{cases} \quad (6)$$

where $\mu > 0$ and the range of this quantizer is μM and the quantization error is $\mu \Delta$ ([11]).

The problem considered here is to design the signal $u(t)$ by a network-based quantized controller of a general structure described by

$$\dot{x}_f(t) = A_f x_f(t) + B_f \mu_{1k} q_1\left(\frac{y(i_k h)}{\mu_{1k}}\right) \quad (7)$$

$$u(t) = C_f x_f(t), \quad t \in [i_k h + \eta_k^{sc}, i_{k+1} h + \eta_{k+1}^{sc}) \quad (8)$$

where $x_f(t)$ is the controller state vector, $\mu_{1k} q_1\left(\frac{y(i_k h)}{\mu_{1k}}\right)$ is the quantized plant output with $i_k h$ as the sampling instant of the sensor and h as the sampling period, $u(t)$ is the control signal and A_f, B_f, C_f are appropriately dimensioned matrices to be designed. η_k^{sc} denotes the transmission delay from sensor to the controller. When considering the network conditions from the controller to the plant output, the quantized output signal can be expressed as

$$\mu_{2k} q_2\left(\frac{u(j_k h)}{\mu_{2k}}\right). \quad (9)$$

Define $\eta_1(t) = t - i_k h - \eta_{1m}$ for $t \in [i_k h + \eta_k^{sc}, i_{k+1} h + \eta_{k+1}^{sc})$ and $\eta_2(t) = t - j_k h - \eta_{2m}$ for $t \in [j_k h + \eta_k^{ca}, j_{k+1} h + \eta_{k+1}^{ca})$ with a natural assumption on the network induced delays as follows

$$\eta_{1m} \leq \eta_k^{sc} \leq \eta_{1M} \quad (10)$$

$$\eta_{2m} \leq \eta_k^{ca} \leq \eta_{2M} \quad (11)$$

where constants η_{im} and η_{iM} , $i = 1, 2$, denote the minimum and maximum delays, respectively. η_k^{ca} denotes the transmission delay from the controller to the actuator. Then, from (10)-(11) we have

$$0 \leq \eta_i(t) \leq \bar{\eta}_i \quad (12)$$

where $\bar{\eta}_i := \eta_{iM} - \eta_{im}$. We assume that the values in both sets $\{i_1, i_2, i_3, \dots\}$ and $\{j_1, j_2, j_3, \dots\}$ are ordered as follows $i_{k+1} > i_k$ and $j_{k+1} > j_k$, which means that there is no wrong packet sequences in the network, and satisfy the following conditions, respectively,

$$(i_{k+1} - i_k)h + \eta_k^{sc} < \eta_{1M} \quad (13)$$

$$(j_{k+1} - j_k)h + \eta_k^{ca} < \eta_{2M} \quad (14)$$

Furthermore, it is noting that there are $n - 1$ continuous packets dropped or lost if $i_{k+1} - i_k = n (n \geq 2)$ ([18]).

Replacing $i_k h$ and $j_k h$ in the quantized plant and controller outputs with $t - \eta_{1m} - \eta_1(t)$ and $t - \eta_{2m} - \eta_2(t)$, respectively, in (7) and (9), we obtain

$$\begin{aligned} \dot{x}_f(t) &= A_f x_f(t) + B_f \mu_{1k} q_1\left(\frac{Cx(t - \eta_{1m} - \eta_1(t))}{\mu_{1k}}\right) \\ &= A_f x_f(t) + B_f Cx(t - \eta_{1m} - \eta_1(t)) + B_f \delta_1(t) \end{aligned} \quad (15)$$

and, for $t \in [j_k h + \eta_k^{ca}, j_{k+1} h + \eta_{k+1}^{ca})$,

$$\mu_{2k} q_2\left(\frac{u(j_k h)}{\mu_{2k}}\right) = C_f x_f(t - \eta_{2m} - \eta_2(t)) + \delta_2(t) \quad (16)$$

where

$$\begin{aligned} \delta_1(t) &= \mu_{1k} q_1\left(\frac{Cx(t - \eta_{1m} - \eta_1(t))}{\mu_{1k}}\right) \\ &\quad - Cx(t - \eta_{1m} - \eta_1(t)) \end{aligned} \quad (17)$$

and

$$\begin{aligned} \delta_2(t) &= \mu_{2k} q_2\left(\frac{C_f x_f(t - \eta_{2m} - \eta_2(t))}{\mu_{2k}}\right) \\ &\quad - C_f x_f(t - \eta_{2m} - \eta_2(t)) \end{aligned} \quad (18)$$

By connecting the plant (1)-(3) and the controller (7)-(8) and from the Leibniz-Newton formula, i.e.

$$\begin{aligned} X(t - \eta_{1m} - \eta_1(t)) &= X(t - \eta_{1m}) \\ &\quad - \int_{t - \eta_{1m} - \eta_1(t)}^{t - \eta_{1m}} \dot{X}(s) ds \end{aligned} \quad (19)$$

we obtain the following closed-loop system as

$$\dot{X}(t) = (\bar{A} + \Delta \bar{A}(t))X(t) + \bar{D}_1 X(t - \eta_{2m})$$

$$\begin{aligned}
& -\bar{D}_1 \int_{t-\eta_{2m}-\eta_2(t)}^{t-\eta_{2m}} \dot{X}(s) ds + \bar{B}_1 X(t - \eta_{1m}) \\
& -\bar{B}_1 \int_{t-\eta_{1m}-\eta_1(t)}^{t-\eta_{1m}} \dot{X}(s) ds + \bar{B}_2 \delta_1(t) + \bar{D}_2 \delta_2(t) \\
& + (\bar{B}_3 + \Delta \bar{B}_3(t)) w(t)
\end{aligned} \tag{20}$$

and

$$z(t) = \bar{C}_1 X(t) \tag{21}$$

where $X(t) = [x(t)^T, x_f(t)^T]^T$ and

$$\begin{aligned}
\bar{A} &= \begin{bmatrix} A & 0 \\ 0 & A_f \end{bmatrix}, \Delta \bar{A}(t) = \begin{bmatrix} \Delta A(t) & 0 \\ 0 & 0 \end{bmatrix}, \bar{B}_1 = \begin{bmatrix} 0 & 0 \\ B_f C & 0 \end{bmatrix}, \\
\bar{D}_1 &= \begin{bmatrix} 0 & DC_f \\ 0 & 0 \end{bmatrix}, \bar{B}_2 = \begin{bmatrix} 0 \\ B_f \end{bmatrix}, \bar{D}_2 = \begin{bmatrix} D \\ 0 \end{bmatrix}, \bar{B}_3 = \begin{bmatrix} B \\ 0 \end{bmatrix}, \\
\Delta \bar{B}_3 &= \begin{bmatrix} \Delta B(t) \\ 0 \end{bmatrix}, \bar{C}_1 = [G \ 0]
\end{aligned}$$

Finally, the problem of robust network-based H_∞ control for uncertain linear systems with both the output quantization levels and the network conditions can be expressed as below.

Problem: Given system (1)-(3), design the controller (7)-(8) such that the augmented system (20)-(21) from $w(t)$ to $z(t)$ is asymptotically stable with a prescribed H_∞ performance γ , that is $\|z(t)\|_2^2 < \gamma^2 \|w(t)\|_2^2$ under zero initial conditions for all admissible uncertain parameters.

III. H_∞ PERFORMANCE ANALYSIS

In this section, we investigate the problem of H_∞ performance analysis for nominal system (1)-(3) with no uncertainties and exactly known controller matrices. Specifically, we will be concerned with the conditions under which the closed-loop system with finite delay components is asymptotically stable from $w(t)$ to $z(t)$ with an H_∞ performance γ .

Theorem 1. Given the positive constants γ, Δ_i and the matrices A_f, B_f, C_f , if there exist positive-definite matrices $P_1, R_1, R_2, S_1, S_2, Q_1, Q_2, Z_1, Z_2, T_1, T_2$ and matrices $P_2, P_3, H_1, H_2, U_1, U_2, N_{i,j} (i = 1, 2, \dots, 4; j = 1, 2, \dots, 10)$ of appropriate dimensions such that the following LMIs hold

$$\begin{bmatrix} \Pi & \eta_{1m}\chi_1 & \eta_{2m}\chi_2 & \eta_{1m}\chi_3 & \eta_{2m}\chi_4 \\ * & -\eta_{1m}T_1 & 0 & 0 & 0 \\ * & * & -\eta_{2m}T_2 & 0 & 0 \\ * & * & * & -\eta_{1m}Q_1 & 0 \\ * & * & * & * & -\eta_{2m}Q_2 \end{bmatrix} < 0 \tag{22}$$

$$\begin{bmatrix} H_i & U_i \\ * & Z_i \end{bmatrix} \geq 0 \quad (i = 1, 2) \tag{23}$$

with $\chi_i = [N_{i,1}^T, N_{i,2}^T, \dots, N_{i,10}^T, 0]^T$ ($i = 1, 2, \dots, 4$), $\Pi = \Pi^T = [\Pi_{i,j}]_{i,j=1,2,\dots,11}$, $\bar{N}_i = N_{1,i} + N_{2,i} + N_{3,i} + N_{4,i}$, $P = \begin{bmatrix} P_1 & 0 \\ P_3 & P_2 \end{bmatrix}$ and $\Pi_{1,1} = \text{sym} \left\{ P^T \begin{bmatrix} 0 & I \\ \bar{A} & -I \end{bmatrix} \right\} + \sum_{i=1}^2 \bar{\eta}_i H_i + \text{diag} \{ R_1 + S_1, \sum_{i=1}^2 \eta_{im} Q_i + 2\bar{\eta}_i Z_i + \eta_{im} T_i \} + \text{sym} \{ \bar{N}_1 \hat{I} \}$, $\Pi_{1,2} = P^T \begin{bmatrix} 0 \\ \bar{B}_1 \end{bmatrix} - N_{1,1} + \hat{I} \bar{N}_2^T$, $\Pi_{1,3} = U_1 - P^T \begin{bmatrix} 0 \\ \bar{B}_1 \end{bmatrix} - N_{1,1} +$

$$\begin{aligned}
& \hat{I} \bar{N}_3^T, \Pi_{1,4} = -N_{3,1} + \hat{I} \bar{N}_4^T, \Pi_{1,5} = P^T \begin{bmatrix} 0 \\ \bar{D}_1 \end{bmatrix} - N_{2,1} + \hat{I} \bar{N}_5^T, \\
& \Pi_{1,6} = -N_{4,1} + \hat{I} \bar{N}_6^T, \Pi_{1,7} = U_2 - P^T \begin{bmatrix} 0 \\ \bar{D}_1 \end{bmatrix} + \hat{I} \bar{N}_7^T, \\
& \Pi_{1,8} = P^T \begin{bmatrix} 0 \\ \bar{B}_2 \end{bmatrix} + \hat{I} \bar{N}_8^T, \Pi_{1,9} = P^T \begin{bmatrix} 0 \\ \bar{D}_2 \end{bmatrix} + \hat{I} \bar{N}_9^T, \\
& \Pi_{1,10} = P^T \begin{bmatrix} 0 \\ \bar{B}_3 \end{bmatrix} + \hat{I} \bar{N}_{10}^T, \Pi_{1,11} = [\bar{C}_1 \ 0]^T, \Pi_{2,2} = \\
& -R_1 - R_2 - \text{sym} \{ N_{1,2} \}, \Pi_{2,3} = -\frac{\Delta_1^2}{M_1^2 \mu_{1k}^2} \hat{I}^T C^T C \hat{I} - N_{1,3}^T, \\
& \Pi_{2,4} = -N_{1,2} - N_{1,4}^T, \Pi_{2,6} = -N_{4,2} - N_{1,6}^T, \Pi_{2,7} = -N_{1,7}^T, \\
& \Pi_{2,8} = -N_{1,8}^T, \Pi_{2,9} = -N_{1,9}^T, \Pi_{2,10} = -N_{1,10}^T, \\
& \Pi_{3,3} = -\bar{\eta}_1^{-1} Z_1 + \frac{\Delta_1^2}{M_1^2 \mu_{1k}^2} \hat{I}^T C^T C \hat{I}, \Pi_{3,4} = -N_{3,3}, \\
& \Pi_{3,5} = -N_{2,3}, \Pi_{3,6} = -N_{4,3}, \Pi_{4,4} = -R_2 - \text{sym} \{ N_{3,4} \}, \\
& \Pi_{4,5} = -N_{2,4} - N_{3,5}^T, \Pi_{4,6} = -N_{4,4} - N_{3,6}^T, \Pi_{4,7} = -N_{3,7}^T, \\
& \Pi_{4,8} = -N_{3,8}^T, \Pi_{4,9} = -N_{3,9}^T, \Pi_{4,10} = -N_{3,10}^T, \Pi_{5,5} = \\
& S_2 - S_1 - \text{sym} \{ N_{2,5} \} + \frac{\Delta_2^2}{M_2^2 \mu_{2k}^2} \hat{I}^T C_f^T C_f \hat{I}, \Pi_{5,6} = -N_{4,5} - N_{2,6}^T, \\
& \Pi_{5,7} = -N_{2,7}^T - \frac{\Delta_2^2}{M_2^2 \mu_{2k}^2} \hat{I}^T C_f^T C_f \hat{I}, \Pi_{5,8} = -N_{2,8}^T, \Pi_{5,9} = -N_{2,9}^T, \\
& \Pi_{5,10} = -N_{2,10}^T, \Pi_{6,6} = -S_2 - \text{sym} \{ N_{4,6} \}, \Pi_{6,7} = -N_{4,7}^T, \\
& \Pi_{6,8} = -N_{4,8}^T, \Pi_{6,9} = -N_{4,9}^T, \Pi_{6,10} = -N_{4,10}^T, \\
& \Pi_{7,7} = -\bar{\eta}_2^{-1} Z_2 + \frac{\Delta_2^2}{M_2^2 \mu_{2k}^2} \hat{I}^T C_f^T C_f \hat{I}, \Pi_{8,8} = \Pi_{9,9} = -I, \\
& \Pi_{10,10} = -\gamma^2 I, \Pi_{11,11} = -I \text{ and other elements } \Pi_{i,j} \text{ for } j \geq i \\
& \text{are equal to zero. Then, system (22)-(23) is asymptotically} \\
& \text{stable with the } H_\infty \text{ performance level } \gamma > 0.
\end{aligned}$$

Proof. Firstly, we represent (20) in an equivalent descriptor model form as

$$\begin{cases} \dot{X}(t) = \xi(t), \\ 0 = -\xi(t) + \bar{A}X(t) + \bar{B}_1 X(t - \eta_{1m}) + \bar{D}_1 X(t - \eta_{2m}) \\ - \bar{B}_1 \int_{t-\eta_{1m}-\eta_1(t)}^{t-\eta_{1m}} \dot{X}(s) ds - \bar{D}_1 \int_{t-\eta_{2m}-\eta_2(t)}^{t-\eta_{2m}} \dot{X}(s) ds \\ + \bar{B}_2 \delta_1(t) + \bar{D}_2 \delta_2(t) + \bar{B}_3 w(t) \end{cases} \tag{24}$$

Define the Lyapunov-Krasovskii functional [9]

$$V(t) = \sum_{i=1}^5 V_i(t) \tag{25}$$

where

$$V_1(t) = X(t)^T P_1 X(t) := [X(t)^T \quad \xi(t)^T]^T T P \begin{bmatrix} X(t) \\ \xi(t) \end{bmatrix},$$

$$\begin{aligned}
V_2(t) &= \int_{t-\eta_{1m}}^t X(s)^T R_1 X(s) ds \\
&+ \int_{t-\eta_{1m}}^{t-\eta_{1m}} X(s)^T R_2 X(s) ds
\end{aligned}$$

$$\begin{aligned}
V_3(t) &= \int_{t-\eta_{2m}}^t X(s)^T S_1 X(s) ds \\
&+ \int_{t-\eta_{2m}}^{t-\eta_{2m}} X(s)^T S_2 X(s) ds
\end{aligned}$$

$$\begin{aligned}
V_{i+3}(t) &= \int_{-\eta_{iM}}^t \int_{t+\theta}^t \xi(s)^T Q_i \xi(s) ds d\theta \\
&+ 2 \int_{-\eta_{iM}}^{-\eta_{im}} \int_{t+\theta}^t \xi(s)^T Z_i \xi(s) ds d\theta \\
&+ \int_{-\eta_{iM}}^{-\eta_{im}} \int_{t+\theta}^t \xi(s)^T T_i \xi(s) ds d\theta
\end{aligned}$$

with $i = 1, 2$ and $T = \text{diag}\{I, 0\}$. Differentiating $V_1(t)$ in t we obtain

$$\begin{aligned}
\dot{V}_1(t) &= 2X(t)^T P_1 \dot{X}(t) = 2 \begin{bmatrix} X(t)^T & \xi(t)^T \end{bmatrix} P^T \begin{bmatrix} \dot{X}(t) \\ 0 \end{bmatrix} \\
&= 2 \begin{bmatrix} X(t)^T & \xi(t)^T \end{bmatrix} P^T \left\{ \begin{bmatrix} 0 & I \\ \bar{A} & -I \end{bmatrix} \begin{bmatrix} X(t) \\ \xi(t) \end{bmatrix} \right. \\
&\quad + \begin{bmatrix} 0 \\ \bar{B}_1 \end{bmatrix} X(t - \eta_{1m}) + \begin{bmatrix} 0 \\ \bar{D}_1 \end{bmatrix} X(t - \eta_{2m}) \\
&\quad - \begin{bmatrix} 0 \\ \bar{B}_1 \end{bmatrix} \int_{t-\eta_{1m}-\eta_1(t)}^{t-\eta_{1m}} \xi(s) ds - \begin{bmatrix} 0 \\ \bar{D}_1 \end{bmatrix} \int_{t-\eta_{2m}-\eta_2(t)}^{t-\eta_{2m}} \xi(s) ds \\
&\quad \left. + \begin{bmatrix} 0 \\ \bar{B}_2 \end{bmatrix} \delta_1(t) + \begin{bmatrix} 0 \\ \bar{D}_2 \end{bmatrix} \delta_2(t) + \begin{bmatrix} 0 \\ \bar{B}_3 \end{bmatrix} w(t) \right\} \quad (26)
\end{aligned}$$

By Lemma 1 (in Appendix), it is clear that

$$\begin{aligned}
&-2 \begin{bmatrix} X(t)^T & \xi(t)^T \end{bmatrix} P^T \begin{bmatrix} 0 \\ \bar{B}_1 \end{bmatrix} \int_{t-\eta_{1m}-\eta_1(t)}^{t-\eta_{1m}} \xi(s) ds \\
&\leq \int_{t-\eta_{1M}}^{t-\eta_{1m}} \xi(s)^T Z_1 \xi(s) ds + \bar{\eta}_1 \begin{bmatrix} X(t) \\ \xi(t) \end{bmatrix}^T H_1 \begin{bmatrix} X(t) \\ \xi(t) \end{bmatrix} \\
&+ 2 \begin{bmatrix} X(t) \\ \xi(t) \end{bmatrix}^T (U_1 - P^T \begin{bmatrix} 0 \\ \bar{B}_1 \end{bmatrix}) \int_{t-\eta_{1m}-\eta_1(t)}^{t-\eta_{1m}} \xi(s) ds \quad (27)
\end{aligned}$$

and, similarly,

$$\begin{aligned}
&-2 \begin{bmatrix} X(t)^T & \xi(t)^T \end{bmatrix} P^T \begin{bmatrix} 0 \\ \bar{D}_1 \end{bmatrix} \int_{t-\eta_{2m}-\eta_2(t)}^{t-\eta_{2m}} \xi(s) ds \\
&\leq \int_{t-\eta_{2M}}^{t-\eta_{2m}} \xi(s)^T Z_2 \xi(s) ds + \bar{\eta}_2 \begin{bmatrix} X(t) \\ \xi(t) \end{bmatrix}^T H_2 \begin{bmatrix} X(t) \\ \xi(t) \end{bmatrix} \\
&+ 2 \begin{bmatrix} X(t) \\ \xi(t) \end{bmatrix}^T (U_2 - P^T \begin{bmatrix} 0 \\ \bar{D}_1 \end{bmatrix}) \int_{t-\eta_{2m}-\eta_2(t)}^{t-\eta_{2m}} \xi(s) ds \quad (28)
\end{aligned}$$

Differentiating other Lyapunov terms in (25) give

$$\begin{aligned}
\dot{V}_2(t) + \dot{V}_3(t) &= X(t)^T (R_1 + S_1) X(t) - X(t - \eta_{1m})^T \\
&\times (R_1 - R_2) X(t - \eta_{1m}) - X(t - \eta_{2m})^T S_2 X(t - \eta_{2m}) \\
&\quad - X(t - \eta_{1M})^T R_2 X(t - \eta_{1M}) \\
&\quad - X(t - \eta_{2m})^T (S_1 - S_2) X(t - \eta_{2m}) \quad (29)
\end{aligned}$$

and, using Jensen's Inequality in Lemma 2 (in Appendix), one gets, for $i = 1, 2$,

$$\dot{V}_{i+3}(t) \leq \xi(t)^T (\eta_{iM} Q_i + 2\bar{\eta}_i Z_i + \eta_{im} T_i) \xi(t)$$

$$\begin{aligned}
&- \int_{t-\eta_{iM}}^t \xi(s)^T Q_i \xi(s) ds - \int_{t-\eta_{iM}}^{t-\eta_{im}} \xi(s)^T Z_i \xi(s) ds \\
&- \bar{\eta}_i^{-1} \int_{t-\eta_{im}-\eta_i(t)}^{t-\eta_{im}} \xi(s)^T ds Z_i \int_{t-\eta_{im}-\eta_i(t)}^{t-\eta_{im}} \xi(s) ds \\
&- \int_{t-\eta_{im}}^t \xi(s)^T T_i \xi(s) ds. \quad (30)
\end{aligned}$$

Moreover, from the Leibniz-Newton formula, the following equations hold for any matrices $\{N_i\}_{i=1}^{10}$ with appropriate dimensions, for $i = 1, 2$:

$$2v(t)^T T_i (X(t) - X(t - \eta_{im})) - \int_{t-\eta_{im}}^t \xi(s) ds = 0 \quad (31)$$

$$2v(t)^T T_{i+2} (X(t) - X(t - \eta_{iM})) - \int_{t-\eta_{iM}}^t \xi(s) ds = 0 \quad (32)$$

where $v(t) := \text{col}\{X(t), \xi(t), X(t - \eta_{1m}), \int_{t-\eta_{1m}-\eta_1(t)}^{t-\eta_{1m}} \xi(s) ds, X(t - \eta_{1M}), X(t - \eta_{2m}), X(t - \eta_{2M}), \int_{t-\eta_{2m}-\eta_2(t)}^{t-\eta_{2m}} \xi(s) ds, \delta_1(t), \delta_2(t), w(t)\}$ is an augmented state vector. According to the property of the quantizers $q_i(\cdot)$ and using the Leibniz-Newton formula, we readily obtain

$$0 \leq -\delta_i(t)^T \delta_i(t) + \frac{\Delta_i^2}{M_i^2 \mu_{ik}^2} (x(t - \eta_{im})$$

$$- \int_{t-\eta_{im}-\eta_i(t)}^{t-\eta_{im}} \dot{x}(s) ds)^T C^T C (x(t - \eta_{im})$$

$$- \int_{t-\eta_{im}-\eta_i(t)}^{t-\eta_{im}} \dot{x}(s) ds). \quad (33)$$

Now, to establish the H_∞ performance measure for the system (1)-(3), assume zero initial condition, then we have $V(t)|_{t=0} = 0$. Consider the index J_∞ in the form $J_\infty = \int_0^\infty [z(t)^T z(t) - \gamma^2 w(t)^T w(t)] dt$, then along the solution of (1) for any nonzero $w(t)$ there holds

$$J_\infty \leq \int_0^\infty [e(t)^T e(t) - \gamma^2 w(t)^T w(t) + \dot{V}(t)] dt \quad (34)$$

From (26)-(30), (34) and adding the left and right sides of equations (31)-(32) and (33), respectively, into $\dot{V}(t)$, we get

$$J_\infty \leq \int_0^\infty v(t)^T \Sigma v(t) dt \quad (35)$$

where $\Sigma := \Pi + \eta_{1M} \chi_1 T_1^{-1} \chi_1^T + \eta_{2M} \chi_2 T_2^{-1} \chi_2^T + \eta_{1M} \chi_3 Q_1^{-1} \chi_3^T + \eta_{2M} \chi_4 Q_2^{-1} \chi_4^T$. Now, if $\Sigma < 0$, then $J_\infty < 0$ which means that the L_2 -gain from the disturbance $w(t)$ to the controlled output $z(t)$ is less than γ . By applying Schur complements, we find that $\Sigma < 0$ is equivalent to (22). \triangleleft

IV. ROBUST H_∞ CONTROL DESIGN

In this section we investigate the robust network-based H_∞ control design problem for system (1)-(3) with the norm bounded uncertainty parameters defined in (4).

Theorem 2. Consider system (1)-(3) with the quantizer given in (6). Given positive constants ε, γ and Δ_i , there exist a network-based quantized controller in the form of (7)-(8) such that the closed-loop system (20)-(21) is asymptotically stable with an H_∞ disturbance attenuation level γ if there exist the scalar $\rho > 0$, positive-definite matrices $P_1, R_1, R_2, S_1, S_2, Q_1, Q_2, Z_1, Z_2, T_1, T_2$ and matrices $\hat{P}_1, \hat{P}_2, P_{22}, \bar{P}_{11}, W_1, W_2, W_3, H_1, H_2, U_1, U_2, N_{i,j}$ ($i = 1, 2, \dots, 4; j = 1, 2, \dots, 10$) of appropriate dimensions and satisfying (23) and the LMI

$$\begin{bmatrix} \tilde{\Pi} & \Gamma_d^T & \rho \Gamma_e^T \\ * & -\rho I & 0 \\ * & * & -\rho I \end{bmatrix} < 0 \quad (36)$$

with

$$\tilde{\Pi} := \begin{bmatrix} \hat{\Pi} & \eta_{1m}\chi_1 & \eta_{2m}\chi_2 & \eta_{1M}\chi_3 & \eta_{2M}\chi_4 \\ * & -\eta_{1m}T_1 & 0 & 0 & 0 \\ * & * & -\eta_{2m}T_2 & 0 & 0 \\ * & * & * & -\eta_{1M}Q_1 & 0 \\ * & * & * & * & -\eta_{2M}Q_2 \end{bmatrix}$$

$$\Gamma_d = \begin{bmatrix} \varepsilon M_1^T P_2 & M_1^T P_2 & \dots & 0 & 0 \\ L_a & 0 & \dots & 0 & L_b \end{bmatrix} \text{ with } \hat{\Pi} = \hat{\Gamma}^T = [\hat{\Pi}_{i,j}]_{i,j=1,2,\dots,11},$$

and $P_2 = \begin{bmatrix} P_{11} & P_{11} \\ P_{22} & P_{22} \end{bmatrix}$ and

$$\hat{\Pi}_{1,1} := \text{sym} \left\{ \begin{bmatrix} \varepsilon \begin{bmatrix} P_{11}^T A & W_1 \\ P_{11}^T A & W_1 \end{bmatrix} & P_1 - \varepsilon P_2^T \\ \begin{bmatrix} P_{11}^T A & W_1 \\ P_{11}^T A & W_1 \end{bmatrix} & -P_2^T \end{bmatrix} \right\} \\ + \sum_{i=1}^2 \bar{\eta}_i H_i + \text{diag} \left\{ R_1 + S_1, \sum_{i=1}^2 \eta_{iM} Q_i + 2\bar{\eta}_i Z_i + \eta_{im} T_i \right\} \\ + \text{sym} \{ \tilde{N}_1 \hat{I} \},$$

$$\hat{\Pi}_{1,2} = (\varepsilon \hat{I}^T + \tilde{I}^T) \begin{bmatrix} W_2 C & 0 \\ W_2 C & 0 \end{bmatrix} - N_{1,1} + \tilde{I}^T \tilde{N}_2^T, \\ \hat{\Pi}_{1,3} = U - (\varepsilon \hat{I}^T + \tilde{I}^T) \begin{bmatrix} W_2 C & 0 \\ W_2 C & 0 \end{bmatrix} + \tilde{I}^T \tilde{N}_3^T, \quad \hat{\Pi}_{1,5} = \\ (\varepsilon \hat{I}^T + \tilde{I}^T) \begin{bmatrix} 0 & DW_3 \\ 0 & DW_3 \end{bmatrix} - N_{2,1} + \hat{I} \tilde{N}_5^T, \quad \hat{\Pi}_{1,7} = U_2 - (\varepsilon \hat{I}^T + \\ \tilde{I}^T) \begin{bmatrix} 0 & DW_3 \\ 0 & DW_3 \end{bmatrix} + \hat{I} \tilde{N}_7^T, \quad \hat{\Pi}_{1,8} = (\varepsilon \hat{I}^T + \tilde{I}^T) \begin{bmatrix} W_2 \\ W_2 \end{bmatrix} + \tilde{I}^T \tilde{N}_8^T \text{ and} \\ \text{other elements } \hat{\Pi}_{i,j} \text{ are equal to their counterpart elements in the matrix } \Pi. \text{ Moreover, if the above conditions are} \\ \text{feasible, desired controller gain matrices are given by } C_f = (\bar{P}_{11}^T)^{-1} W_3 \text{ and}$$

$$[A_f \quad B_f] = (P_{22}^T)^{-1} [W_1 \quad W_2]$$

where $D^T = U [\hat{D} \quad 0] V^T$, $P_{11} = V \begin{bmatrix} \hat{P}_1 & 0 \\ 0 & \hat{P}_2 \end{bmatrix} V^T$, $\bar{P}_{11} = U \hat{D} \hat{P}_1 \hat{D}^{-1} U^T$ with the unitary matrices U, V and a diagonal matrix \hat{D} with positive diagonal elements in decreasing order.

Proof. The state-space matrices \bar{A} and \bar{B}_3 in (22) are replaced with $\bar{A} + M_1 F(t) L_a$ and $\bar{B}_3 + M_1 F(t) L_b$, respectively. By considering $P_3 = \varepsilon P_2$ with $D^T P_{11} = \bar{P}_{11} D^T$ and introducing change of variables

$$[W_1 \quad W_2] = P_{22}^T [A_f \quad B_f] \quad (37)$$

$$W_3 = \bar{P}_{11}^T C_f \quad (38)$$

then the inequality (22) is equivalent to the following condition:

$$\tilde{\Pi} + \text{sym} \{ \Gamma_d^T F(t) \Gamma_e \} < 0 \quad (39)$$

By Lemma 3 (in Appendix), a necessary and sufficient condition for (39) is that there exists a scalar ρ such that

$$\tilde{\Pi} + \rho^{-1} \Gamma_d^T \Gamma_d + \rho \Gamma_e^T \Gamma_e < 0 \quad (40)$$

then, applying Schur complements, we find that (40) is equivalent to (36). \triangleleft

Remark 2. In Theorem 2, the results are expressed within the framework of LMIs, which can be easily computed by the interior-point method.

V. NUMERICAL RESULTS

Consider an uncertain linear system with the system matrices given by

$$A = \begin{bmatrix} -1 & 0.4 \\ 0.2 & 0.1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, G = [1 \quad 1],$$

$$L_a = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0.05 \end{bmatrix}, L_b = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, M_1 = I, F(t) = c \sin(t)$$

where $|c| \leq 1$. It is assumed that the network-induced delay bounds are given by $\eta_{1m} = \eta_{2m} = 20ms$, $\eta_{1M} = \eta_{2M} = 200ms$. In addition, the quantizer parameters in (5) are assumed to be $\Delta_1 = \Delta_2 = 0.1$, $M_1 = M_2 = 5$ and the sampling period $h = 20ms$. By using the convex problem in Theorem 2 with a constant $\varepsilon = 0.1$, we obtain the minimum guaranteed performance in terms of the feasibility of (23) and (36) as $\gamma = 0.45$.

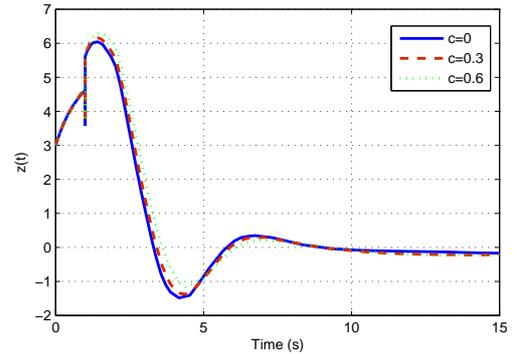


Fig. 1. Controlled output signals for the plant.

With the initial conditions $x(0) = [1, 1]^T$ and $x_f(0) = [0, 0]^T$, and an exogenous disturbance input as a unit step function within $[0, 1]$, then the controlled output signal $z(t)$

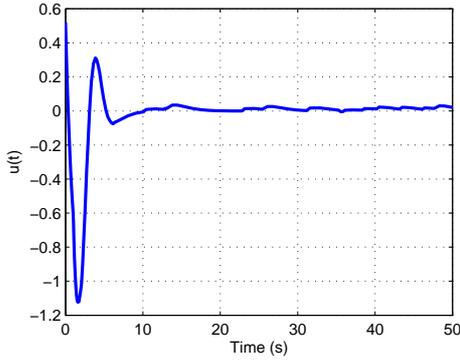


Fig. 2. Quantized H_∞ Control signal.

is plotted in Figure 1 for different values of the parameter c , i.e., $\{0, 0.3, 0.6\}$. The quantized H_∞ output feedback Control signal is also shown in Figure 2 in the case $c = 0.3$.

VI. CONCLUSION

we have investigated the problem of robust quantized H_∞ output feedback control for linear systems with norm-bounded uncertainties connected over a common digital communication network. Firstly, we studied the effect of both the output quantization levels and the network conditions. Secondly, by introducing a descriptor technique, using Lyapunov-Krasovskii functional and a suitable change of variables, new required sufficient conditions were established in terms of delay-range-dependent linear matrix inequalities for the existence of the desired network-based quantized controllers with simultaneous consideration of network induced delays and measurement quantization. The explicit expression of the controllers was derived to satisfy both asymptotic stability and a prescribed level of disturbance attenuation for all admissible norm bounded uncertainties.

APPENDIX

Lemma 1: ([6]) For any arbitrary column vectors $a(t), b(t)$, matrices $\Phi(t), H, U$ and W the following inequality holds:

$$-2 \int_{t-r}^t a(s)^T \Phi(s) b(s) ds \leq \int_{t-r}^t \begin{bmatrix} a(s) \\ b(s) \end{bmatrix}^T \begin{bmatrix} H & U - \Phi(s) \\ * & W \end{bmatrix} \begin{bmatrix} a(s) \\ b(s) \end{bmatrix} ds$$

where $\begin{bmatrix} H & U \\ * & W \end{bmatrix} \geq 0$.

Lemma 2: ([5]) (*Jensen's Inequality*) Given a positive-definite matrix $P \in \mathcal{R}^{n \times n}$ and two scalars $b > a \geq 0$ for any vector $x(t) \in \mathcal{R}^n$, we have

$$\int_{t-b}^{t-a} x(w)^T P x(w) dw \geq \frac{1}{b-a} \left(\int_{t-b}^{t-a} x(w) dw \right)^T P \left(\int_{t-b}^{t-a} x(w) dw \right)$$

Lemma 3: Given matrices $Y = Y^T, D, E$ and F of appropriate dimensions with $F^T F \leq I$, then the following matrix inequality

$$Y + \text{sym}\{DFE\} < 0$$

holds for all F if and only if there exists a scalar $\varepsilon > 0$ such that

$$Y + \varepsilon DD^T + \varepsilon^{-1} E^T E < 0.$$

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