Robust Delay-Dependent H_∞ Control of Uncertain Time-Delay Systems with Mixed Neutral, Discrete and Distributed Time-Delays and Markovian Switching Parameters

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Abstract— **The problem of robust mode-dependent delayed state feedback H_∞ control is investigated for a class of uncertain time-delay systems with Markovian switching parameters and mixed discrete, neutral and distributed delays. Based on the Lyapunov-Krasovskii functional theory, new required sufficient conditions are established in terms of delay-dependent linear matrix inequalities for the stochastic stability and stabilization of the considered system using some free matrices. The desired control is derived based on a convex optimization method such that the resulting closed-loop system is stochastically stable and satisfies a prescribed level of H_∞ performance, simultaneously. Finally, two numerical examples are given to illustrate the effectiveness of our approach.**

I. Introduction

In recent years, more attention has been devoted to the study of stochastic hybrid systems, where the so-called Markov jump systems. These systems represent an important class of stochastic systems that is popular in modeling practical systems like manufacturing systems, power systems, aerospace systems and networked control systems that may experience random abrupt changes in their structures and parameters [1]-[10]. Random parameter changes may result from random component failures, repairs or shut down, or abrupt changes of the operating point. Many such events can be modeled using a continuous time finite-state Markov chain, which leads to the hybrid description of system dynamics known as a Markov jump parameter system [11]-[13]; such a description will be utilized in the paper. The state of a Markov jump parameter system is described by continuous range variables and also a random discrete event variable representing the regime of system operation. A great number of results on robust stability, stabilization, H_∞ control and filtering problems related to such systems have been reported in the literatures ([1], [3], [4]-[6], [8]-[9], [14]). For example, H_∞ control has been discussed in [3], [4], [9], filtering problem has been discussed in [5], [8] and [15], stability and stabilization problems have been considered in [1], [6] and [16], respectively. More recently, the fault detection problem for a class of discrete-time Markov jump linear system with partially known transition probabilities was investigated in [17]. The proposed systems are more general, which relax the traditional assumption in Markov jump systems that all the transition probabilities must be completely known.

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On another research front line, time delays for many dynamic systems have been much investigated; see for example [18]. Time-delayed systems represent a class of infinite-dimensional systems largely used to describe propagation and transport phenomena or population dynamics. Delay differential systems are assuming an increasingly important role in many disciplines like economic, mathematics, science, and engineering. For instance, in economic systems, delays appear in a natural way since decisions and effects are separated by some time interval. The delay effects on the stability of systems including delays in the state and/or input is a problem of recurring interest since the delay presence may induce complex behaviors for the schemes [18]-[24].

On the other hand, stability of neutral delay systems proves to be a more complex issue because the system involves the derivative of the delayed state. Especially, in the past few decades increased attention has been devoted to the problem of robust delay-independent stability or delay-dependent stability and stabilization via different approaches for linear neutral systems with delayed state and/or input and parameter uncertainties (see [20]-[28]). Among the past results on neutral delay systems, the LMI approach is an efficient method to solve many control problems such as stability analysis and stabilization [29]-[31] and H_∞ control problems [32]-[35]. It is also worth citing that some appreciable works have been performed to design a guaranteed-cost (observer-based) control for the neutral system performance representation [36]-[38]. Furthermore, from the published results, it appears that general results pertaining to robust mode-dependent delayed state feedback H_∞ control for uncertain Markovian jump systems with mixed discrete, neutral and distributed delays are few and restricted ([39]-[45]), despite its practical importance, mainly due to the mathematical difficulties in dealing with such mixed delays. Hence, it is our intention in this paper to tackle such an important yet challenging problem.

In this paper, we are concerned to develop an efficient approach for robust H_∞ control problem of uncertain time-delay systems with Markovian switching parameters and mixed discrete, neutral and distributed delays. The main merit of the proposed method is the fact that it provides a convex problem such the delay-dependent control gains can be found from the LMI formulations. New required sufficient conditions are established in terms of delay-range-dependent LMIs combined with the Lyapunov-Krasovskii method for the existence of the desired control such that the resulting closed-loop system is stochastically stable and satisfies a prescribed level of H_∞ performance, simultaneously. Numerical examples are given to illustrate the use of our results. The main contribution of the paper is three folds: 1) for the addressed problem, the assumption that the random jumping process appears in the system with neutral, discrete and distributed delays is different from the existing results. Based on the proposed design method, controllers can be obtained by solving a set of LMIs; 2) delay-dependent and delay-discretization techniques are applied successfully into the analysis and synthesis results; 3) a Lyapunov-Krasovskii functional-based method is provided to derive a new form of the bounded real lemma (BRL) for the system under consideration.

The remainder of this paper is organized as follows. The problem of robust control design for uncertain time-delay systems with Markovian switching parameters and mixed time-delays and norm-bounded time-varying uncertainties and some preliminaries are provided in Section II. Section III presents the results on performance analysis and H_∞ control synthesis. Section IV gives two illustrative examples. At last we conclude the paper in Section V.

Notation: The notations used throughout the paper are fairly standard. I and 0 represent identity matrix and zero matrix; the superscript $'T'$ stands for matrix transposition. $\|.\|$ refers to the Euclidean vector norm or the induced matrix 2-norm. $diag\{\cdots\}$ represents a block diagonal matrix and the operator $sym(A)$ represents $A + A^T$. Let $\Re^+ = [0,\ \infty)$ and $\mathcal{E}\{.\}$ denotes the expectation operator with respect to some probability measure $\mathcal{P}$. If $x(t)$ is a continuous $\Re^n$-valued stochastic process on $t \in [-\kappa, \infty)$, we let $x_t = \{x(t+\theta): -\kappa \le \theta \le 0\}$ for $t \ge 0$ which is regarded as a $C([-\kappa, 0]; \Re^n)$-valued stochastic process. The notation $P > 0$ means that P is real symmetric and positive definite; the symbol $*$ denotes the elements below the main diagonal of a symmetric block matrix.

II. Problem Description

Consider a class of uncertain time-delay systems with Markovian switching parameters and mixed neutral, discrete and distributed delays and norm-bounded time-varying uncertainties represented by

$$\begin{aligned}\dot{x}(t) - A_4(r(t))\dot{x}(t-d) &= (A_1(r(t)) + \Delta A_1(t,(r(t))))x(t) \\ &+ (A_2(r(t)) + \Delta A_2(t,(r(t))))x(t-h) + (A_3(r(t)) \\ &+ \Delta A_3(t,(r(t)))) \int_{t-\tau}^{t} x(s)\,ds + (A_5(r(t)) + \Delta A_4(t,(r(t))))x(t-d) \\ &+ (B_1(r(t)) + \Delta B_1(t,(r(t))))u(t) + (B_2(r(t)) + \Delta B_2(t,(r(t))))w(t),\end{aligned} \tag{1a}$$

$$x(t) = \phi(t), \qquad t \in [-\kappa,\ 0] \tag{1b}$$

$$r(t) = r_0, \qquad t \in [-\kappa,\ 0] \tag{1c}$$

$$\begin{aligned}z(t) &= C(r(t))x(t) + C_d(r(t))x(t-d) + C_h(r(t))x(t-h) \\ &+ C_\tau(r(t)) \int_{t-\tau}^{t} x(s)\,ds + D(r(t))u(t),\end{aligned} \tag{1d}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $w(t) \in L_2^s[0, \infty)$ and $z(t) \in \Re^z$ are state, input, disturbance and controlled output, respectively. $A_i(r(t)), B_i(r(t)), C(r(t)),\ C_d(r(t)), C_h(r(t)), C_\tau(r(t))$ and $D(r(t))$ are matrix functions of the random jumping process $\{r(t)\}$. $\{r(t), t \ge 0\}$ is a right-continuous Markov process on the probability space which takes values in a finite space $S = \{1,2,\ldots,s\}$ with generator $\Pi = [\pi_{ij}]\,(i, j \in S)$ given by

$$P\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \pi_{ij}\Delta + o(\Delta), & \text{if } i \ne j \\ 1 + \pi_{ii}\Delta + o(\Delta), & \text{if } i = j \end{cases} \tag{2}$$

where $\Delta > 0$, $\lim_{\Delta\to 0} o(\Delta)/\Delta = 0$ and $\pi_{ij} \ge 0$, for $i \ne j$, is the transition rate from mode i at time t to mode j at time $t+\Delta$ and $\pi_{ii} = -\sum_{j=1, j\ne i}^{j=s} \pi_{ij}$. The time-varying function $\phi(t)$ is continuous vector valued initial function and h, d and τ are constant time delays with $\kappa := max\{h, d, \tau\}$. Moreover, the norm-bounded uncertainties are defined as follows:

$$\Delta A_i(t, r(t)) = H_1(r(t))\Delta(t, r(t))E_i(r(t)), \quad i = 1,2,\cdots,4 \tag{3a}$$

$$\Delta B_j(t, r(t)) = H_1(r(t))\Delta(t, r(t))E_{4+j}(r(t)), \quad j = 1,2 \tag{3b}$$

where $\Delta(t, r(t))$ is the uncertain time-varying matrix function of the random jumping process, which satisfies $\Delta^T(t, r(t))\Delta(t, r(t)) \le I$ for $\forall t \ge 0; r(t) = i \in S$ and $E_i(r(t))$ and $H_1(r(t))$ are known real constant matrices of the random jumping process with appropriate dimensions.

Remark 1. The model (1) can describe a large amount of well-known dynamical systems with time-delays, such as the delayed Logistic model, the chaotic models, the artificial neural network models, and the predator-prey model with time delays.

Remark 2. It is shown in [44] that the $C([-\kappa, 0]; \Re^n) \times S$ −valued process $(x_t, r(t))$ is a time homogenous strong Markov process. Furthermore, the stability in distribution of solutions to stochastic neutral differential delay equations with Markovian switching parameters can be proved by Lyapunov function type methods [45].

Definition 1. Uncertain time-delay system (1) with Markovian switching parameter in (2) is said to be stochastically mean square stable if, when $u(t) = 0$, for any finite $\phi(t) \in \Re^n$ defined on $[-\kappa, 0]$, and $r_0 \in S$ the following condition is satisfied

$$\mathcal{E}\{\|x(t)\|^2\} \le c \sup_{-\kappa \le s \le 0} \|x(s)\|^2, \ \ t > 0$$

where $x(t)$ is the trajectory of the system state from initial system state $\phi(0)$ and initial mode r_0, and c is a positive constant.

Definition 2. The H_∞ performance measure of the system (1) is defined as $J_\infty = \mathcal{E}\left(\int_0^\infty [z^T(t)z(t) - \gamma^2 w^T(t)w(t)]\ dt\right)$, where the positive scalar γ is given.

The weak infinitesimal operator $\mathcal{L}V(.)$ of the stochastic process $\{(x_t, r(t)), t \geq 0\}$, acting on $V \in C(\Re^n \times \Re^n \times \Re^+ \times S)$ at the point $\{t, x(t), r(t) = i\}$, is given by (see Lemma 3.1, [43])

$$\begin{aligned}\mathcal{L}V(x, x_t, t, i) &= \lim_{\Delta \to 0^+} \frac{1}{\Delta}\{\mathcal{E}[V(x(t+\Delta), x_{t+\Delta}, t \\ &\quad + \Delta, r(t+\Delta) | x(t), x_t, t, r(t) = i)] \\ &\quad - V(x, x_t, t, i)\} \\ &= V_t(x(t) - A_4(i)x(t-d), t, i) + (\dot{x}(t) - A_4(i)\dot{x}(t-d))^T V_x(x \\ &\quad - A_4(i)x(t-d), t, i) \\ &\quad + \sum_{j=1}^{s} \pi_{ij} V(x(t) - A_4(i)x(t-d), t, j)\end{aligned}$$

where

$$V_t(x, x_t, t, i) = \frac{\partial V(x, x_t, t, i)}{\partial t},$$

$$V_x(x, x_t, t, i) = \left(\frac{\partial V(x, x_t, t, i)}{\partial x_1}, \cdots, \frac{\partial V(x, x_t, t, i)}{\partial x_n}\right)^T.$$

Then the generalized Itô formula reads for $V(x, x_t, t, i)$ as follows:

$$\begin{aligned}&\mathcal{E}\left\{V\left(x(t) - A_4(r(t))x(t-d), t, r(t)\right)\right\} = \\ &\mathcal{E}\{V\left(x(0) - A_4(r(0))\phi(-d), 0, r(0)\right)\} \\ &\quad + \mathcal{E}\{\int_0^t \mathcal{L}V\left(x(x), x_s, s, r(s)\right) ds\}\end{aligned}$$

Remark 3. Let us consider

$$\begin{aligned}&\dot{x}(t) - A_4(r(t))\dot{x}(t-d) = A_1(r(t))x(t) + A_2(r(t))x(t-d), \\ &x(t) = \phi(t), \qquad t \in [-d, 0], \\ &r(t) = r_0, \qquad t \in [-d, 0],\end{aligned}$$

with $V(x, x_t, t, i) = x^T P_{1i} x$. Then the operator $\mathcal{L}V(.)$ associated with the system above has the form

$$\begin{aligned}&\mathcal{L}V(x, x_t, i) = \\ &\quad 2(x(t) - A_4(i)x(t-d))^T P_{1i} (\dot{x}(t) - A_4(i)\dot{x}(t-d)) + \\ &\sum_{j=1}^{s} \pi_{ij}(x(t) - A_4(i)x(t-d))^T P_{1i} (x(t) - A_4(i)x(t-d))\end{aligned}$$

It can be shown that

$$\mathcal{L}V(x, x_t, i) = \begin{bmatrix} x(t) \\ x(t-d) \end{bmatrix}^T \Theta_i \begin{bmatrix} x(t) \\ x(t-d) \end{bmatrix}$$

where

$$\begin{aligned}\Theta_i &= \begin{bmatrix} sym(P_{1i}A_1(i)) & -A_1(i)^T P_{1i} A_4(i)^T + P_{1i}A_2(i) \\ \star & -sym(A_4(i)^T P_{1i} A_2(i)) \end{bmatrix} \\ &\quad + \begin{bmatrix} I_n \\ -A_4(i)^T \end{bmatrix} \left(\sum_{j=1}^{N} \pi_{ij} P_{1j}\right) \begin{bmatrix} I_n \\ -A_4(i)^T \end{bmatrix}^T\end{aligned}$$

Assumption 1. The full state variable $x(t)$ is available for measurement.

In this paper, the author's attention will be focused on the design of the following robust mode-dependent delayed state feedback H_∞ control law,

$$u_i = K_i\, x(t) + K_{di}\, x(t-d) + K_{hi}\, x(t-h) + K_{\tau i} \int_{t-\tau}^{t} x(s)\, ds \tag{4}$$

where the matrices $K_i, K_{di}, K_{hi}, K_{\tau i}$ of the appropriate dimension is to be determined such that for any $r(t) = i \in S$ the resulting closed-loop system is stochastically stable and satisfies an H_∞ norm bound γ, i.e. $J_\infty < 0$.

Remark 4. Note that the state feedback control in (4) is the general form. When $K_{di} = K_{hi} = K_{\tau i} = 0$ it is just the instantaneous state feedback. Moreover, if the closed-loop system can be stabilized with $K_{di} = K_{hi} = K_{\tau i} = 0$, in order to make the controller simpler and more practical, we should use the control law as $u_i = K_i\, x(t)$.

Lemma 1. [46] (*Jensen's Inequality*) Given a positive-definite matrix $P \in \Re^{n \times n}$ and two scalars $b > a \geq 0$ for any vector $x(t) \in \Re^n$, we have

$$\int_{t-b}^{t-a} x^T(\omega) P x(\omega)\, d\omega \geq \frac{1}{b-a}\left(\int_{t-b}^{t-a} x(\omega)\, d\omega\right)^T P \left(\int_{t-b}^{t-a} x(\omega)\, d\omega\right).$$

Lemma 2. Given matrices $Y = Y^T$, D, E and F of appropriate dimensions with $F^T F \leq I$, then the matrix inequality

$$Y + sym(D F E) < 0$$

holds for all F if and only if there exists a scalar $\varepsilon > 0$ such that

$$Y + \varepsilon D D^T + \varepsilon^{-1} E^T E < 0.$$

III. Main Results

In this section, we first investigate both the stochastic stability and H_∞ performance of the system (1) with norm-bounded uncertainty parameters. A new delay-dependent stochastic stability condition by a discretization technique is proposed in Theorem 1. Then, we will show the procedure to design the controller gains K_i, K_{di}, K_{hi}, $K_{\tau i}$, which guarantee the resulting closed-loop system is stochastically stable and satisfies an H_∞ norm bound γ.

Theorem 1. Let $h_1 = \frac{h}{N}$, $h_2 = \frac{d}{N}$ be given for any positive integer N. The time-delay system (1) with Markovian switching parameters in (2) and without the norm-bounded uncertainties in (3) is stochastically mean square stable with an H_∞ performance level $\gamma > 0$, if there exist some matrices $P_{2i}, P_{3i}, H_j, Q_{l,r}, R_{j,r} = R_{j,r}^T$, $T_{j,r} = T_{j,r}^T$, and positive definite matrices P_{1i}, U_1, U_2, S_j $(j, r = 0, 1, \cdots, N; l = 1, 2, \cdots, s)$ satisfying the following LMIs

$$\begin{bmatrix} P_{1i} & Q_i \\ * & R + S \end{bmatrix} > 0, \tag{5a}$$

$$\begin{bmatrix} U_1 & -U_1 \\ * & S_d \end{bmatrix} > 0\,, \tag{5b}$$

$$\begin{bmatrix} U_2 & -U_2 \\ * & S_d \end{bmatrix} > 0\,, \tag{5c}$$

$$\Pi_i := \begin{bmatrix} \Xi_{ei} & D^s & O^s & D^a & O^a \\ * & -S_d - R_{ds} & 0 & 0 & 0 \\ * & * & -H_d - T_{ds} & 0 & 0 \\ * & * & * & -3U_1 & 0 \\ * & * & * & * & -3U_2 \end{bmatrix} < 0 \tag{5d}$$

where $R_{ds} = h_1[R_{r-1,j-1} - R_{r,j}]$, $T_{ds} = h_2[T_{r-1,j-1} - T_{r,j}]$, $R = [R_{r-1,j-1}]$, $R_{ds} = h_1[R_{r-1,j-1} - R_{r,j}]$ $(r, j = 1,2,\cdots,N)$ and

$$\Xi_{ei} = \begin{bmatrix} \Sigma_{1i} + \mathrm{diag}\{C_i^T C_i, 0\} & \begin{bmatrix} P_{2i}^T A_{2i} - Q_{iN} + C_i^T C_{hi} \\ P_{3i}^T A_{2i} \end{bmatrix} & \begin{bmatrix} \Sigma_{2i} \\ P_{3i}^T A_{5i} \end{bmatrix} & \begin{bmatrix} (P_{2i}^T - P_{1i})A_{4i} \\ P_{3i}^T A_{4i} \end{bmatrix} & \begin{bmatrix} P_{2i}^T A_{3i} + C_i^T C_{\tau i} \\ P_{3i}^T A_{3i} \end{bmatrix} & \begin{bmatrix} P_{2i}^T B_{2i} \\ P_{3i}^T B_{2i} \end{bmatrix} \\ * & -S_N + C_{hi}^T C_{hi} & 0 & 0 & 0 & 0 \\ * & * & \Sigma_{3i} & A_{4i}^T P_{1i} A_{4i} & 0 & 0 \\ * & * & * & -U_1 & 0 & 0 \\ * & * & * & * & C_{\tau i}^T C_{\tau i} - U_2 & 0 \\ * & * & * & * & * & -\gamma^2 I \end{bmatrix}$$

$$D^s = h_1 \begin{bmatrix} \begin{pmatrix} -2Q_{i1}^a + R_{0,1}^s \\ +\sum_{j=1}^{s} \pi_{ij} Q_{j1}^s \end{pmatrix} & \begin{pmatrix} -2Q_{i2}^a + R_{0,2}^s \\ +\sum_{j=1}^{s} \pi_{ij} Q_{j2}^s \end{pmatrix} & \cdots & \begin{pmatrix} -2Q_{iN}^a + R_{0,N}^s \\ +\sum_{j=1}^{s} \pi_{ij} Q_{jN}^s \end{pmatrix} \\ Q_{i1}^s & Q_{i2}^s & \cdots & Q_{iN}^s \\ -R_{N,1}^s & -R_{N,2}^s & \cdots & -R_{N,N}^s \\ \begin{pmatrix} 2A_{4i}^T Q_{i1}^a \\ -\sum_{j=1}^{s} \pi_{ij} A_{4i}^T Q_{j1}^s \end{pmatrix} & \begin{pmatrix} 2A_{4i}^T Q_{i2}^a \\ -\sum_{j=1}^{s} \pi_{ij} A_{4i}^T Q_{j2}^s \end{pmatrix} & \cdots & \begin{pmatrix} 2A_{4i}^T Q_{iN}^a \\ -\sum_{j=1}^{s} \pi_{ij} A_{4i}^T Q_{jN}^s \end{pmatrix} \\ -2A_{4i}^T Q_{i1}^a & -2A_{4i}^T Q_{i2}^a & \cdots & -2A_{4i}^T Q_{iN}^a \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$D^a = h_1 \begin{bmatrix} R_{0,1}^a + \sum_{j=1}^{s} \pi_{ij} Q_{j1}^a & R_{0,2}^a + \sum_{j=1}^{s} \pi_{ij} Q_{j2}^a & \cdots & R_{0,N}^a + \sum_{j=1}^{s} \pi_{ij} Q_{jN}^a \\ Q_{i1}^a & Q_{i2}^a & \cdots & Q_{iN}^a \\ -R_{N,0}^a & -R_{N,1}^a & \cdots & -R_{N,N-1}^a \\ -\sum_{j=1}^{s} \pi_{ij} A_{4i}^T Q_{j1}^a & -\sum_{j=1}^{s} \pi_{ij} A_{4i}^T Q_{j2}^a & \cdots & -\sum_{j=1}^{s} \pi_{ij} A_{4i}^T Q_{jN}^a \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$O^s = h_2 \begin{bmatrix} T_{0,1}^s & T_{0,2}^s & \cdots & T_{0,N}^s \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ T_{N,1}^s & T_{N,2}^s & \cdots & T_{N,N}^s \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$O^a = h_2 \begin{bmatrix} T_{0,1}^a & T_{0,2}^a & \cdots & T_{0,N}^a \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ T_{N,0}^a & T_{N,1}^a & \cdots & T_{N,N-1}^a \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$S_d = \mathrm{diag}\{S_0 - S_1, S_1 - S_2, \cdots, S_{N-1} - S_N\}\,,$$

$$H_d = \mathrm{diag}\{H_0 - H_1, H_1 - H_2, \cdots, H_{N-1} - H_N\}\,,$$

$$Q_i = [Q_{i0}, Q_{i1}, \cdots, Q_{iN}]\,,$$

$$S = 1/h_1\ \mathrm{diag}\{S_0, S_1, \cdots, S_N\}\,,$$

with $P_i = \begin{bmatrix} P_{1i} & 0 \\ P_{2i} & P_{3i} \end{bmatrix}$ and

$$\Sigma_{1i} = sym\left(P_i^T \begin{bmatrix} 0 & I \\ A_{1i} & -I \end{bmatrix}\right) + diag\{\textstyle\sum_{j=1}^{s} \pi_{ij} P_{1j} + sym(Q_{i0}) + S_0 + H_0 + \tau^2 U_2, U_1\},$$

$$\Sigma_{2i} = P_{2i}^T A_{5i} + C_i^T C_{di} + \left(\sum_{j=1}^{s} \pi_{ij} P_{1j} - Q_{i0}\right) A_{4i}\,,$$

$$\Sigma_{3i} = C_{di}^T C_{di} - H_N + \sum_{j=1}^{s} A_{4i}^T \pi_{ij} P_{1j} A_{4i}\,,$$

$$Q_{ip}^s = (Q_{ip} + Q_{i(p-1)})/2\,,$$

$$Q_{ip}^a = (Q_{ip} - Q_{i(p-1)})/2\,,$$

$$R_{p,q}^s = (R_{p,q} + R_{p,q-1})/2\,,$$

$$R_{p,q}^a = (R_{p,q} - R_{p,q-1})/2\,,$$

$$T_{p,q}^s = (T_{p,q} + T_{p,q-1})/2\,,$$

$$T_{p,q}^a = (T_{p,q} - T_{p,q-1})/2\,.$$

Proof. The time-delay system (1) with Markovian switching parameters in (2) and the mixed neutral, discrete and distributed delays and without norm-bounded uncertainties is of the following form,

$$\dot{x}(t) - A_{4i}\,\dot{x}(t-d) = A_{1i}\,x(t) + A_{5i}x(t-d) + A_{2i}\,x(t-h) + A_{3i}\int_{t-\tau}^{t} x(s)\,ds + B_{2i}\,w(t), \tag{6a}$$

$$z(t) = C_i\,x(t) + C_{di}x(t-d) + C_{hi}x(t-h) + C_{\tau i}\int_{t-\tau}^{t} x(s)\,ds, \tag{6b}$$

The notations $A_{ji}, B_{ji}, C_i, C_{di}, C_{hi}, C_{\tau i}$ and D_i stand for $A_j(i)$, $B_j(i), C(i), C_d(i), C_h(i), C_\tau(i)$ and $D(i)$, respectively. It is noting that the Markov process $\{r(t)\}$ takes values in the finite space S.

According to Remark 1, it is clear that the process $\{(x_t, r(t)), t \geq 0\}$ is a Markov process with initial state $(\phi(.), r_0)$. Now, we choose a stochastic Lyapunov-Krasovskii functional candidate $V(.,.,.,.): \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \times S \to \mathbb{R}^+$ for the system (6) as

$$V(x, x_t, t, i) = \sum_{i=1}^{4} V_i(x, x_t, t, i) \tag{7}$$

where

$$V_1(x, x_t, t, i) = x(t)^T P_{1i}\, x(t) + 2x(t)^T \int_{-h}^{0} Q(\xi, i)\, x(t+\xi)\ d\xi,$$

$$V_2(x, x_t, t, i) = \int_{-h}^{0} x(t+\xi)^T S(\xi) x(t+\xi)\ d\xi + \int_{-h}^{0}\int_{-h}^{0} x(t+s)^T R(s,\xi)\ x(t+\xi)\ ds\, d\xi,$$

$$V_3(x, x_t, t, i) = \int_{t-d}^{t} \dot{x}(s)^T U_1\, \dot{x}(s)\ ds + \int_{-d}^{0} x(t+\xi)^T H(\xi) x(t+\xi)\ d\xi + \int_{-d}^{0}\int_{-d}^{0} x(t+s)^T T(s,\xi)\ x(t+\xi)\ ds\, d\xi,$$

$$V_4(x, x_t, t, i) = \int_{t-\tau}^{t} [\int_{s}^{t} x(\theta)^T\ d\theta] U_2 [\int_{s}^{t} x(\theta)\, d\theta]\ ds + \int_{0}^{\tau}\int_{t-s}^{t} (\theta - t + s) x(\theta)^T\ U_2\ x(\theta)\ d\theta\, ds$$

where $R(s,\xi) = R(s,\xi)^T, S(\xi) = S(\xi)^T, T(s,\xi) = T(s,\xi)^T$ and $H(\xi) = H(\xi)^T$ are continuous matrix functions and $Q(\xi, i)$ ($i \in S$) is a mode-dependent matrix function. Differentiating $V_1(x, x_t, i)$ in *t* we obtain

$$\begin{aligned}
&LV_1(x, x_t, t, i) \\
&= 2(\dot{x}(t) - A_{4i}\, \dot{x}(t-d))^T \{P_{1i}\,(x(t) - A_{4i}x(t-d)) + \int_{-h}^{0} Q(\xi, i)\ x(t+\xi)\ d\xi\} \\
&+ \sum_{j=1}^{s}((x(t) - A_{4i}x(t-d)))^T \pi_{ij} P_{1j}\,((x(t) - A_{4i}x(t-d))) \\
&+ 2(x(t) - A_{4i}x(t-d))^T (\int_{-h}^{0} Q(\xi, i)\ \dot{x}(t+\xi)\ d\xi + \sum_{j=1}^{s}\int_{-h}^{0} \pi_{ij} Q(\xi, j)\ x(t+\xi)\ d\xi)
\end{aligned} \tag{8}$$

Differentiating other Lyapunov terms in (7) give

$$LV_2(x, x_t, t, i) = 2\int_{-h}^{0} \dot{x}(t+\xi)^T S(\xi) x(t+\xi)\ d\xi + 2\int_{-h}^{0}\int_{-h}^{0} \dot{x}(t+s)^T R(s,\xi)\ x(t+\xi)\ ds\, d\xi, \tag{9}$$

$$\begin{aligned}
LV_3(x, x_t, t, i) &= \dot{x}(t)^T U_1\, \dot{x}(t) - \dot{x}(t-d)^T U_1\, \dot{x}(t-d) \\
&+ 2\int_{-d}^{0} \dot{x}(t+\xi)^T H(\xi) x(t+\xi)\ d\xi \\
&+ 2\int_{-d}^{0}\int_{-d}^{0} \dot{x}(t+s)^T T(s,\xi)\ x(t+\xi)\ ds\, d\xi,
\end{aligned} \tag{10}$$

$$\begin{aligned}
LV_4(x, x_t, t, i) &= -[\int_{t-\tau}^{t} x(\theta)^T\ d\theta] U_2 [\int_{t-\tau}^{t} x(\theta)\ d\theta] + 2\int_{t-\tau}^{t} x(\theta)^T\, U_2 [\int_{s}^{t} x(\theta)\, d\theta]\ ds \\
&+ \int_{0}^{\tau} s\, x(t)^T U_2\, x(t)\, ds - \int_{0}^{\tau}\int_{t-s}^{t} x(\theta)^T\, U_2\, x(\theta)\ d\theta\, ds \\
&\leq \int_{t-\tau}^{t} (\theta - t + \tau)[x(t)^T\, U_2 x(t) + x(\theta)^T R_5 x(\theta)]\ d\theta + \int_{0}^{\tau} s\, x(t)^T U_2\, x(t)\, ds \\
&- [\int_{t-\tau}^{t} x(\theta)^T\ d\theta] U_2 [\int_{t-\tau}^{t} x(\theta)\ d\theta] - \int_{t-\tau}^{t} (\theta - t + r_1) x(\theta)^T\, U_2\, x(\theta)\ d\theta \\
&\leq \tau^2\, x(t)^T U_2\, x(t) - [\int_{t-\tau}^{t} x(\theta)^T\ d\theta] U_2 [\int_{t-\tau}^{t} x(\theta)\ d\theta]
\end{aligned} \tag{11}$$

Moreover, from (6a), the following equation holds for any matrices P_{2i}, P_{3i} $(i \in S)$ with appropriate dimensions:

$$\begin{aligned}
&2(x(t)^T P_{2i}^T + \dot{x}(t)^T P_{3i}^T)(-\dot{x}(t) + A_{4i}\, \dot{x}(t-d) + A_{1i}\, x(t) + A_{5i}x(t-d) \\
&+ A_{2i}\, x(t-h) + A_{3i} \int_{t-\tau}^{t} x(s)\, ds + B_{2i}\ w(t)) = 0
\end{aligned} \tag{12}$$

From the obtained derivative terms in (8)-(11) and adding the left-hand side of the equation (12) into $\mathcal{L}V(x, x_t, t, r(t))$, we obtain the following result for $\mathcal{L}V(x, x_t, t, r(t))$,

$$\begin{aligned}
&LV(x, x_t, t, i) \leq \chi^T(t)\Xi_i\chi(t) + 2(\dot{x}(t) - A_{4i}\, \dot{x}(t-d))^T \int_{-h}^{0} Q_i(\xi)\ x(t+\xi)\ d\xi \\
&+ 2\ (x(t) - A_{4i}x(t-d))^T \int_{-h}^{0} \sum_{j=1}^{s} \pi_{ij} Q_j(\xi)\ x(t+\xi)\ d\xi \\
&- 2\dot{x}(t)^T P_{3i}^T A_{3i} \int_{-\tau}^{0} x(t+\xi)\ d\xi - 2x(t-h)^T \int_{-h}^{0} R(-h,\xi)\ x(t+\xi)\ d\xi \\
&- \dot{x}(t-d)^T U_1\, \dot{x}(t-d) - 2x(t-d)^T \int_{-d}^{0} T(-d,\xi)\ x(t+\xi)\ d\xi \\
&- \int_{-h}^{0}\int_{-h}^{0} x(t+s)^T (\frac{\partial}{\partial s} R(s,\xi) + \frac{\partial}{\partial \xi} R(s,\xi))\ x(t+\xi)\ ds\, d\xi \\
&- \int_{-h}^{0} x(t+\xi)^T \dot{S}(\xi) x(t+\xi)\ d\xi + 2x(t)^T \int_{-h}^{0} (R(0,\xi) - Q(\xi, i))\ x(t+\xi)\ d\xi \\
&+ 2x(t-d)^T A_{4i}^T \int_{-h}^{0} Q(\xi, i)\ x(t+\xi)\ d\xi - 2x(t)^T P_{2i}^T A_{3i} \int_{-\tau}^{0} x(t+\xi)\ d\xi \\
&- \int_{-d}^{0} x(t+\xi)^T \dot{H}(\xi) x(t+\xi)\ d\xi - 2(x(t)^T + x(t-d)^T A_{4i}^T) P_{1i} A_{4i} \dot{x}(t-d) \\
&- \int_{-d}^{0}\int_{-d}^{0} x(t+s)^T (\frac{\partial}{\partial s} T(s,\xi) + \frac{\partial}{\partial \xi} T(s,\xi))\ x(t+\xi)\ ds\, d\xi \\
&+ 2(x(t)^T P_{2i}^T + \dot{x}(t)^T P_{3i}^T)(A_{4i}\, \dot{x}(t-d) + B_{2i}\ w(t)) \\
&- [\int_{t-\tau}^{t} x(\theta)^T\ d\theta] U_2 [\int_{t-\tau}^{t} x(\theta)\ d\theta]
\end{aligned} \tag{13}$$

where $\chi(t) \coloneqq col\{x(t), \dot{x}(t), x(t-h), x(t-d)\}$ and

$$\Xi_i = \begin{bmatrix} \Sigma_{1i} & \begin{bmatrix} P_{2i}^T A_{2i} - Q_i(-h) \\ P_{3i}^T A_{2i} \end{bmatrix} & \begin{bmatrix} P_{2i}^T A_{5i} - (Q_i(0) + \sum_{j=1}^N \pi_{ij} P_{1j}) A_{4i} \\ P_{3i}^T A_{5i} \end{bmatrix} \\ * & -S(-h) & 0 \\ * & * & -H(-d) + A_{4i}^T \sum_{j=1}^N \pi_{ij} P_{1j} A_{4i} \end{bmatrix}.$$

According to the discretization technique in [19], the delay intervals $[-h, 0]$ and $[-d, 0]$ are, respectively, divided into N segments $[\theta_p, \theta_{p-1}]$ and $[\hat{\theta}_p, \hat{\theta}_{p-1}], p = 1, \cdots, N$, of equal length (or uniform mesh case), where $\theta_p = -p\, h_1$ and $\hat{\theta}_p = -p\, h_2$. For instance, this scheme divides the square $[-h, 0] \times [-d, 0]$ into $N \times N$ small squares $[\theta_p, \theta_{p-1}] \times [\theta_q, \theta_{q-1}]$ and each small square is further divided into two triangles. It is easily seen using [19] and [35] that although the LKF candidate for the nonuniform mesh case is no more complicated than the uniform mesh case, it is not the case for the LKF derivative condition. Also, a uniform mesh is not possible for the incommensurate delay case and is not practical in the case of commensurate delays with small common factor. In the sequel, $Q_i(.), S(.), H(.), R(.,.)$ and $T(.,.)$ are chosen to be piecewise linear, i.e. $Q_i(\theta_p + \kappa\, h_1) = (1-\kappa) Q_{ip} + \kappa\, Q_{i(p-1)}$, $S(\theta_p + \kappa\, h_1) = (1-\kappa) S_p + \kappa\, S_{p-1}$, $H(\hat{\theta}_p + \kappa\, h_2) = (1-\kappa) H_p + \kappa\, H_{p-1}$, where

$$R(\theta_p + \kappa h_1, \theta_q + \beta h_1) = \begin{cases} (1-\kappa) R_{pq} + \beta R_{p-1,q-1} + (\kappa - \beta) R_{p-1,q}, & \kappa \ge \beta \\ (1-\beta) R_{pq} + \kappa R_{p-1,q-1} + (\beta - \kappa) R_{p,q-1}, & \kappa < \beta \end{cases}$$

and

$$T(\hat{\theta}_p + \kappa h_2, \hat{\theta}_q + \beta h_2) = \begin{cases} (1-\kappa) T_{pq} + \beta T_{p-1,q-1} + (\kappa - \beta) T_{p-1,q}, & \kappa \ge \beta \\ (1-\beta) T_{pq} + \kappa T_{p-1,q-1} + (\beta - \kappa) T_{p,q-1}, & \kappa < \beta \end{cases}$$

with $\dot{S}(\xi) = h_1^{-1}(S_{p-1} - S_p)$, $LQ(\xi, i) = h_1^{-1}(Q_{i(p-1)} - Q_{ip})$, $\dot{H}(\xi) = h_2^{-1}(H_{p-1} - H_p)$, $\frac{\partial}{\partial \xi} R(\xi, \theta) + \frac{\partial}{\partial \theta} R(\xi, \theta) = h_1^{-1}(R_{p-1,q-1} - R_{p,q})$ and $\frac{\partial}{\partial \xi} T(\xi, \theta) + \frac{\partial}{\partial \theta} T(\xi, \theta) = h_2^{-1}(T_{p-1,q-1} - T_{p,q})$. Thus, one obtains

$$2\dot{x}(t)^T \int_{-h}^{0} Q(\xi, i)\, x(t+\xi)\, d\xi = 2 h_1 \sum_{p=1}^{N} \dot{x}(t)^T \int_0^1 [Q_{ip}^s + (1-2\kappa) Q_{ip}^a]\, x(t + \theta_p + \kappa h_1)\, d\kappa \tag{14a}$$

$$2\sum_{j=1}^{s} \pi_{ij} (x(t) - A_{4i} x(t-d))^T \int_{-h}^{0} Q(\xi, j)\, x(t+\xi)\, d\xi = 2 h_1 \sum_{j=1}^{s} \sum_{p=1}^{N} \pi_{ij} (x(t) - A_{4i} x(t-d))^T \int_0^1 [Q_{jp}^s + (1-2\kappa) Q_{jp}^a]\, x(t + \theta_p + \kappa h_1)\, d\kappa \tag{14b}$$

$$2x(t-h)^T \int_{-h}^{0} R(-\tau_1, \xi)\, x(t+\xi)\, d\xi = 2 h_1 \sum_{p=1}^{N} x(t-h)^T \int_0^1 [R_{N,p}^s + (1-2\kappa) R_{N,p-1}^a]\, x(t + \theta_p + \kappa h_1)\, d\kappa \tag{14c}$$

$$2x(t-d)^T \int_{-d}^{0} T(-d_2, \xi)\, x(t+\xi)\, d\xi = 2 h_2 \sum_{p=1}^{N} x(t-d)^T \int_0^1 [T_{N,p}^s + (1-2\kappa) T_{N,p-1}^a]\, x(t + \hat{\theta}_p + \kappa h_2)\, d\kappa \tag{14d}$$

$$\int_{-h}^{0} x(t+\xi)^T \dot{S}(\xi)\, x(t+\xi)\, d\xi = \sum_{p=1}^{N} \int_0^1 x(t + \theta_p + \kappa h_1)^T (S_{p-1} - S_p)\, x(t + \theta_p + \kappa h_1)\, d\kappa \tag{14e}$$

$$\int_{-d}^{0} x(t+\xi)^T \dot{H}(\xi)\, x(t+\xi)\, d\xi = \sum_{p=1}^{N} \int_0^1 x(t + \hat{\theta}_p + \kappa h_2)^T (H_{p-1} - H_p)\, x(t + \hat{\theta}_p + \kappa h_2)\, d\kappa \tag{14f}$$

$$\int_{-h}^{0}\int_{-h}^{0} x(t+s)^T \left(\frac{\partial}{\partial s} R(s,\xi) + \frac{\partial}{\partial \xi} R(s,\xi)\right) x(t+\xi)\, ds\, d\xi = h_1 \sum_{q=1}^{N} \sum_{p=1}^{N} \int_0^1 x(t + \theta_p + \beta h_1)^T (R_{p-1,q-1} - R_{p,q})\, x(t + \theta_p + \kappa h_1)\, d\kappa\, d\beta \tag{14g}$$

$$\int_{-d}^{0}\int_{-d}^{0} x(t+s)^T \left(\frac{\partial}{\partial s} T(s,\xi) + \frac{\partial}{\partial \xi} T(s,\xi)\right) x(t+\xi)\, ds\, d\xi = h_2 \sum_{q=1}^{N} \sum_{p=1}^{N} \int_0^1 x(t + \hat{\theta}_p + \beta h_2)^T (T_{p-1,q-1} - T_{p,q})\, x(t + \hat{\theta}_p + \kappa h_2)\, d\kappa\, d\beta \tag{14h}$$

$$2x(t)^T \int_{-h}^{0} (-Q(\xi, i) + R(0, \xi))\, x(t+\xi)\, d\xi = 2 h_1 \sum_{p=1}^{N} x(t)^T \int_0^1 (-2Q_{ip}^a + R_{0,p}^s + (1-2\kappa) R_{0,p}^a)\, x(t + \theta_p + \kappa h_1)\, d\kappa \tag{14i}$$

$$2x(t-d)^T A_{4i}^T \int_{-h}^{0} Q(\xi, i)\, x(t+\xi)\, d\xi = 2 h_1 \sum_{p=1}^{N} x(t-d)^T A_{4i}^T \int_0^1 2Q_{ip}^a\, x(t + \theta_p + \kappa h_1)\, d\kappa \tag{14j}$$

and

$$2x(t)^T \int_{-d}^{0} T(0, \xi)\, x(t+\xi)\, d\xi = 2 h_2 \sum_{p=1}^{N} x(t)^T \int_0^1 (T_{0,p}^s + (1-2\kappa) T_{0,p}^a)\, x(t + \hat{\theta}_p + \kappa h_2)\, d\kappa \tag{14k}$$

Now, from (13)-(14), one has

$$\begin{aligned} & z^T(t) z(t) - \gamma^2 w^T(t) w(t) + LV(x, x_t, i) \\ & \le \chi_e^T(t) \Xi_{ei} \chi_e(t) - \int_0^1 \phi_e(\kappa;\alpha)^T S_d\, \phi_e(\kappa;\alpha)\, d\kappa \\ & - \int_0^1 \hat{\phi}_e(\kappa;\alpha)^T H_d\, \hat{\phi}_e(\kappa;\alpha)\, d\kappa + 2\chi_e^T(t) \int_0^1 (D^s + (1-2\kappa) D^a)\, \phi_e(\kappa;\alpha)\, d\kappa \\ & - \int_0^1 \phi_e(\kappa;\alpha)^T d\kappa R_{ds} \int_0^1 \phi_e(\kappa;\alpha)\, d\kappa + 2\chi_e^T(t) \int_0^1 (O^s + (1-2\kappa) O^a)\, \hat{\phi}_e(\kappa;\alpha)\, d\kappa \\ & - \int_0^1 \hat{\phi}_e(\kappa;\alpha)^T d\kappa T_{ds} \int_0^1 \hat{\phi}_e(\kappa;\alpha)\, d\kappa \end{aligned}$$

$$=\chi_e^T(t)(\Xi_{ei}+D^s\widetilde{U}_1 D^{s^T}+O^s\widetilde{U}_2 O^{s^T}+\frac{1}{3}(D^a\widetilde{U}_1 D^{a^T}+O^a\widetilde{U}_2 O^{a^T}))\chi_e(t)$$
$$-\int_0^1\int_0^1 \phi_e(\kappa;\alpha)^T R_{ds}\phi_e(s;\alpha)\, d\kappa\, ds-\int_0^1\int_0^1 \hat{\phi}_e(\kappa;\alpha)^T T_{ds}\hat{\phi}_e(s;\alpha)\, d\kappa\, ds$$
$$-\int_0^1 \phi_D(\kappa;\alpha)^T\Theta_1\,\phi_D(\kappa;\alpha)\, d\kappa-\int_0^1 \phi_O(\kappa;\alpha)^T\Theta_2\,\phi_O(\kappa;\alpha)\, d\kappa \tag{15}$$

with

$$\chi_e(t)=\mathrm{col}\{\chi(t),\dot{x}(t-d),\int_{t-\tau}^{t} x(\theta)\, d\theta, w(t))\},$$
$$\chi_D(t):=(D^s+(1-2\kappa)D^a)^T\chi_e(t),$$
$$\chi_O(t):=(O^s+(1-2\kappa)O^a)^T\chi_e(t),$$
$$\phi_D(\kappa;\alpha):=[\chi_D(t)^T,\phi_e(\kappa;\alpha)^T]^T,$$
$$\phi_o(\kappa;\alpha):=[\chi_o(t)^T,\ \hat{\phi}_e(\kappa;\alpha)^T]^T,$$
$$\phi_e(\kappa;\alpha)=\mathrm{col}\{x(t+\theta_1+\kappa h_1),x(t+\theta_2+\kappa h_1),\cdots,x(t+\theta_N+\kappa h_1)\},$$
$$\hat{\phi}_e(\kappa;\alpha)=\mathrm{col}\{x(t+\hat{\theta}_1+\kappa h_2),x(t+\hat{\theta}_2+\kappa h_2),\ \cdots,x(t+\hat{\theta}_N+\kappa h_2)\},$$
$$\Theta_1:=\begin{bmatrix}\widetilde{U}_1 & -I\\ * & S_d\end{bmatrix}>0,$$
$$\Theta_2:=\begin{bmatrix}\widetilde{U}_2 & -I\\ * & H_d\end{bmatrix}>0,$$

where $\widetilde{U}_1=U_1^{-1}$ and $\widetilde{U}_2=U_2^{-1}$. Assume $\Theta_i>0, i=1,2$. Then use the Jensen Inequality (Lemma 1) to the fourth and fifth terms in (15), we have

$$\int_0^1 \phi_D(\kappa;\alpha)^T\Theta_1\,\phi_D(\kappa;\alpha)\, d\kappa$$
$$\geq\begin{bmatrix}\chi_e^T(t) & \phi_e(\kappa;\alpha)^T\end{bmatrix}\begin{bmatrix}D^s\widetilde{U}_1 D^{s^T} & -D^s\\ * & S_d\end{bmatrix}\begin{bmatrix}\chi_e(t)\\ \phi_e(\kappa;\alpha)\end{bmatrix}$$
$$\int_0^1 \phi_O(\kappa;\alpha)^T\Theta_2\,\phi_O(\kappa;\alpha)\, d\kappa$$
$$\geq\begin{bmatrix}\chi_e^T(t) & \hat{\phi}_e(\kappa;\alpha)^T\end{bmatrix}\begin{bmatrix}O^s\widetilde{U}_2 O^{s^T} & -O^s\\ * & H_d\end{bmatrix}\begin{bmatrix}\chi_e(t)\\ \hat{\phi}_e(\kappa;\alpha)\end{bmatrix}$$

Using the above inequalities in (15) we conclude that

$$z^T(t)z(t)-\gamma^2w^T(t)w(t)+LV(x,x_t,i)\leq\ \tilde{\chi}_e(t)^T\tilde{\Xi}_{ei}\tilde{\chi}_e(t) \tag{16}$$

where $\tilde{\chi}_e(t)=\left[\chi_e(t)^T,\int_0^1\phi_e(\kappa;\alpha)^T\,d\kappa,\ \int_0^1\hat{\phi}_e(\kappa;\alpha)^T\,d\kappa\right]^T$ and

$$\tilde{\Xi}_{ei}=\begin{bmatrix}\Xi_{ei}+\frac{1}{3}(D^a\widetilde{U}_1 D^{a^T}+O^a\widetilde{U}_2 O^{a^T}) & -D^s & -O^s\\ * & -S_d-R_{ds} & 0\\ * & * & -H_d-T_{ds}\end{bmatrix} \tag{17}$$

On the other hand, for a prescribed $\gamma>0$ and under zero initial conditions, J_∞ can be rewritten as

$$J_\infty\leq\mathcal{E}\left(\int_0^\infty[z^T(t)z(t)-\gamma^2w^T(t)w(t)]\ dt+V(x,x_t,t,i)|_{t\to\infty}-V(x,x_t,t,i)|_{t=0}\right)$$
$$=\mathcal{E}\left(\int_0^\infty[z^T(t)z(t)-\gamma^2w^T(t)w(t)+LV(x,x_t,t,i)]\ dt\right) \tag{18}$$

and the condition $\tilde{\Xi}_{ei}<0$ means that the condition $J_\infty<0$ is satisfied, and by applying Schur complement on the forst element of the matrix $\tilde{\Xi}_{ei}$, one gets LMI (5d). On the other hand, let $\zeta_i=diag\{U_i,I\}$. Premultiplying ζ_i and postmultiplying ζ_i^T to the inequalities $\Theta_i>0, i=1,2$, one obtains LMIs (5b)-(5c). Moreover, the condition $J_\infty<0$ for $w(t)=0$ implies $\mathcal{E}\{\mathcal{L}V(x,x_t,t,i)\}<0$. Then, we have

$$\mathcal{E}\{\mathcal{L}V(x,x_t,i)\}\leq-\sigma_1\mathcal{E}\{x(t)^Tx(t)\} \tag{19}$$

where $\sigma_1=\min\{\lambda_{min}(-\tilde{\Xi}_{ei}), i\in S\}$, then $\sigma_1>0$. By Dynkin's formula, we have

$$\mathcal{E}\{V(x,x_t,i)\}-\mathcal{E}\{V(\phi(0),r_0,0)\}\leq-\sigma_1\mathcal{E}\left\{\int_0^t x(s)^Tx(s)\,ds\right\} \tag{20}$$

or

$$\mathcal{E}\left\{\int_0^t x(s)^Tx(s)\,ds\right\}\leq\sigma_1^{-1}V(\phi(0),r_0) \tag{21}$$

Moreover, if the LMI condition (5a) (see [19]) is satisfied, the following LKF condition holds

$$\mathcal{E}\{V(x,x_t,t,i)\}\geq\sigma_2\mathcal{E}\{x(t)^Tx(t)\} \tag{22}$$

where $\sigma_2=\min\{\lambda_{min}(P_i), i\in S\}$. In a manner similar to [9], from (21) and (22), we obtain

$$\mathcal{E}\{x(t)^Tx(t)\}\leq-\sigma_1\sigma_2^{-1}\mathcal{E}\left\{\int_0^t x(s)^Tx(s)\,ds\right\}+\sigma_2^{-1}V(\phi(0),r_0).$$

hence

$$\mathcal{E}\left\{\int_0^t x(s)^Tx(s)\,ds\right\}\ \leq\ \sigma_1^{-1}\left[1-e^{-\sigma_1\sigma_2^{-1}t}\right]V(\phi(0),r_0),$$

or

$$\lim_{t\to\infty}\mathcal{E}\left\{\int_0^t x(s)^Tx(s)\,ds\right\}\ \leq\ \sigma_1^{-1}V(\phi(0),r_0)$$

which indicates that, from Definition 1, the system in (6) with Markovian switching parameters in (2) is stochastically mean square stable. This completes the proof. ■

Remark 5. Note that the matrix $P_i=\begin{bmatrix}P_{1i} & 0\\ P_{2i} & P_{3i}\end{bmatrix}$ (or, equivalently, the matrix P_{3i}) is non-singular due to the fact that the only matrix which can be negative definite in the first block on the diagonal of LMI (5d) is $\Sigma_{1i}<0$.

Remark 6. If the switching modes are not considered, i.e. $S=\{1\}$, the jump linear system is simplified into a general linear system with nonlinearities and time delays. Then it is easy to conclude a criterion from Theorem 1, which can be used to determine the stability of such a system.

In the following, we present a condition for the stability of the time-delay system (1) with Markovian switching parameters in (2) and norm-bounded uncertainties in (3).

Corollary 1. Let $h_1 = \frac{h}{N}$, $h_2 = \frac{d}{N}$ be given for any positive integer N. The time-delay system (1) with Markovian switching parameters in (2) is stochastically mean square stable with an H_∞ performance level $\gamma > 0$, if there exist a scalar μ_i, matrices $P_{2i}, P_{3i}, H_j, Q_{l,r}, R_{j,r} = R_{j,r}^T$, $T_{j,r} = T_{j,r}^T$, and positive definite matrices P_{1i}, U_1, U_2, S_j $(j, r = 0,1,\cdots,N; l = 1,2,\cdots,s)$ satisfying the LMIs (5a)-(5c) and

$$\begin{bmatrix} \Pi_i & \Gamma_{di} & \mu \Gamma_{ei} \\ * & -\mu_i I & 0 \\ * & * & -\mu_i I \end{bmatrix} < 0 \tag{23}$$

where $\Gamma_{di} = \left[H_{1i}^T, 0, \cdots, 0\right]^T$, $\Gamma_{ei} = \left[E_{1i}, 0, E_{2i}, E_{4i}, 0, E_{3i}, 0, \cdots, 0\right]$.

Proof. If the matrices $A_{1i}, A_{2i}, A_{3i}, A_{5i}$ in (6a) are replaced with $A_{1i} + \Delta A_{1i}(t)$, $A_{2i} + \Delta A_{2i}(t)$, $A_{3i} + \Delta A_{3i}(t)$ and $A_{5i} + \Delta A_{4i}(t)$, respectively, then (5d) with the admissible uncertainties (3) is equivalent to the following condition:

$$\Pi_i + \mathrm{sym}\left(\Gamma_{di}^T \Delta_i(t) \Gamma_{ei}\right) < 0. \tag{24}$$

It is noting that the notations $\Delta A_{ji}(t)$ and $\Delta B_{ji}(t)$ stand for $\Delta A_j(t,i)$ and $\Delta B_j(t,i)$, respectively. By Lemma 2, a necessary and sufficient condition for (24) is that there exists a scalar $\mu_i > 0$ such that

$$\Pi_i + \mu_i^{-1} \Gamma_{di}^T \Gamma_{di} + \mu_i \;\; \Gamma_{ei}^T \Gamma_{ei} < 0 \tag{25}$$

then, applying Schur complements, we find that (25) is equivalent to LMI (23). ■

Note that the delay-dependent stability condition in Theorem 1 covers a special case $N = 1$ (without discretization technique). Then we have the following corollary.

Corollary 2. The time-delay system (6) with Markovian switching parameters in (2) is stochastically mean square stable with an H_∞ performance level $\gamma > 0$, if there exist some matrices $P_{2i}, P_{3i}, H_j, Q_{l,r}, R_{j,r} = R_{j,r}^T$, $T_{j,r} = T_{j,r}^T$, and positive definite matrices P_{1i}, U_1, U_2, S_j $(j, r = 0,1;\ l = 1,2,\cdots,s)$ satisfying the LMIs (5) with $R_{ds} = h\left[R_{0,0} - R_{1,1}\right]$, $T_{ds} = d\left[T_{0,0} - T_{1,1}\right]$, $R = \left[R_{0,0}\right]$ and

$$\Xi_{ei} = \begin{bmatrix} \Sigma_{1i} + \mathrm{diag}\{C_i^T C_i, 0\} & \begin{bmatrix} P_{2i}^T A_{2i} - Q_{i1} + C_i^T C_{hi} \\ P_{3i}^T A_{2i} \end{bmatrix} & \begin{bmatrix} \Sigma_{2i} \\ P_{3i}^T A_{5i} \end{bmatrix} & \begin{bmatrix} (P_{2i}^T - P_{1i}) A_{4i} \\ P_{3i}^T A_{4i} \end{bmatrix} & \begin{bmatrix} P_{2i}^T A_{3i} + C_i^T C_{\tau i} \\ P_{3i}^T A_{3i} \end{bmatrix} & \begin{bmatrix} P_{2i}^T B_{2i} \\ P_{3i}^T B_{2i} \end{bmatrix} \\ * & -S_1 + C_{hi}^T C_{hi} & 0 & 0 & 0 & 0 \\ * & * & \Sigma_{3i} & A_{4i}^T P_{1i} A_{4i} & 0 & 0 \\ * & * & * & -U_1 & 0 & 0 \\ * & * & * & * & C_{\tau i}^T C_{\tau i} - U_2 & 0 \\ * & * & * & * & * & -\gamma^2 I \end{bmatrix}$$

$$D^s = h \begin{bmatrix} -2Q_{i1}^a + R_{0,1}^s + \sum_{j=1}^{s} \pi_{ij} Q_{j1}^s \\ Q_{i1}^s \\ -R_{1,1}^s \\ 2A_{4i}^T Q_{i1}^a - \sum_{j=1}^{s} \pi_{ij} A_{4i}^T Q_{j1}^s \\ -2A_{4i}^T Q_{i1}^a \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$D^a = h \begin{bmatrix} R_{0,1}^a + \sum_{j=1}^{s} \pi_{ij} Q_{j1}^a \\ Q_{i1}^a \\ -R_{1,0}^a \\ -\sum_{j=1}^{s} \pi_{ij} A_{4i}^T Q_{j1}^a \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$O^s = d \begin{bmatrix} T_{0,1}^s \\ 0 \\ 0 \\ T_{1,1}^s \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad O^a = d \begin{bmatrix} T_{0,1}^a \\ 0 \\ 0 \\ T_{1,1}^a \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

with $S_d = S_0 - S_1$, $H_d = H_0 - H_1$, $Q_i = [Q_{i0}, Q_{i1}]$ and $S = 1/h_1 \ \mathrm{diag}\{S_0, S_1\}$.

Now we are in the position to solve the stabilization problem of the system (1). Based on Theorem 1, we can obtain a mode-dependent delayed state feedback H_∞ control law in the form of (4) in the following theorem.

Theorem 2. Let $h_1 = \frac{h}{N}$, $h_2 = \frac{d}{N}$ be given for any positive integer N. Under Assumption 1, a state feedback controller given in the form (4) exists such that the time-delay system (1) with Markovian switching parameters in (2) is stochastically stable with an H_∞ performance level $\gamma > 0$, if there exist some scalars δ_i, μ_i, matrices $\bar{P}_{2i}, \widetilde{H}_j, \widetilde{Q}_{l,i}, \tilde{R}_{j,r} = \tilde{R}_{j,r}^T, \tilde{T}_{j,r} = \tilde{T}_{j,r}^T, L_i, L_{di}, L_{hi}, L_{\tau i}$, and positive definite matrices $\tilde{P}_{1i}, \widetilde{U}_1, \widetilde{U}_2, \tilde{S}_j$ $(j, r = 0,1,\cdots,N; l = 1,2,\cdots,s)$, satisfying the following LMIs

$$\begin{bmatrix} \widetilde{P}_{1i} & \hat{Q}_i \\ * & \hat{R} + \hat{S} \end{bmatrix} > 0, \tag{26a}$$

$$\begin{bmatrix} \widetilde{U}_1 & -\widetilde{U}_1 \\ * & \widetilde{S}_d \end{bmatrix} > 0, \tag{26b}$$

$$\begin{bmatrix} \widetilde{U}_2 & -\widetilde{U}_2 \\ * & \widetilde{S}_d \end{bmatrix} > 0, \tag{26c}$$

$$\begin{bmatrix} \hat{\Pi}_i & \hat{\Gamma}_{di} & \mu \hat{\Gamma}_{ei} \\ * & -\mu_i I & 0 \\ * & * & -\mu_i I \end{bmatrix} < 0 \tag{26d}$$

where $\hat{R} = [\tilde{R}_{r-1,j-1}]$, $\hat{R}_{ds} = h_1[\tilde{R}_{r-1,j-1} - \tilde{R}_{r,j}]$, $\hat{T}_{ds} = h_2[\tilde{T}_{r-1,j-1} - \tilde{T}_{r,j}]$ $(r, j = 1,2,\cdots,N)$, where $\hat{\Gamma}_{di} = [H_{1i}^T,\ \delta_i H_{1i}^T,\ 0,\cdots,0]^T$, $\hat{\Gamma}_{ei} = [E_{1i} + E_{5i}L_i,\ 0,\ E_{2i} + E_{5i}L_{hi},\ E_{4i} + E_{5i}L_{di},\ 0, E_{3i} + E_{5i}L_{\tau i},\ E_{6i}\bar{P}_{2i},\ 0,\cdots,0]$ and

$$\hat{\Pi}_i := \begin{bmatrix} \hat{\Xi}_{ei} & -\widetilde{D}^s & -\widetilde{O}^s & \widetilde{D}^a & \widetilde{O}^a \\ * & -\widetilde{S}_d - \hat{R}_{ds} & 0 & 0 & 0 \\ * & * & -\widetilde{H}_d - \hat{T}_{ds} & 0 & 0 \\ * & * & * & -3\widetilde{U}_1 & 0 \\ * & * & * & * & -3\widetilde{U}_2 \end{bmatrix},$$

$$\hat{\Xi}_{ei} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & \Lambda_{14} & \Lambda_{15} & \Lambda_{16} & \Lambda_{17} \\ \star & -\tilde{S}_N & 0 & 0 & 0 & 0 & \Lambda_{27} \\ \star & \star & \Lambda_{33} & \varpi_i^2 \tilde{P}_{1i} & 0 & 0 & \Lambda_{37} \\ \star & \star & \star & -\widetilde{U}_1 & 0 & 0 & 0 \\ \star & \star & \star & \star & -\widetilde{U}_2 & 0 & \Lambda_{57} \\ \star & \star & \star & \star & \star & -\gamma^2 I & 0 \\ \star & \star & \star & \star & \star & \star & -I \end{bmatrix},$$

where

$$\Lambda_{11} = sym\left(\begin{bmatrix} A_{1i}\bar{P}_{2i} + B_{1i}L_i & \tilde{P}_{1i} - \bar{P}_{2i} \\ \delta_i\left(A_{1i}\bar{P}_{2i} + B_{1i}L_i\right) & -\delta_i\bar{P}_{2i} \end{bmatrix}\right) + diag\{\sum_{j=1}^{s}\pi_{ij}\tilde{P}_{1j} + sym(\tilde{Q}_{i0}) + \tilde{S}_0 + \tilde{H}_0 + \tau^2\widetilde{U}_2, \widetilde{U}_1\}\},$$

$$\Lambda_{12} = \begin{bmatrix} A_{2i}\bar{P}_{2i} + B_{1i}L_{hi} - \tilde{Q}_{iN} \\ -\delta_i(A_{2i}\bar{P}_{2i} + B_{1i}L_{hi}) \end{bmatrix},$$

$$\Lambda_{13} = \begin{bmatrix} A_{5i}\bar{P}_{2i} + B_{1i}L_{di} + \varpi_i(\sum_{j=1}^{s}\pi_{ij}\tilde{P}_{1j} - \tilde{Q}_{i0}) \\ -\delta_i A_{5i}\bar{P}_{2i} + B_{1i}L_{di} + \varpi_i(\sum_{j=1}^{s}\pi_{ij}\tilde{P}_{1j} - \tilde{Q}_{i0})) \end{bmatrix},$$

$$\Lambda_{14} = \begin{bmatrix} A_{4i}\bar{P}_{2i} - \varpi_i\tilde{P}_{1i} \\ \delta_i A_{4i}\bar{P}_{2i} \end{bmatrix},$$

$$\Lambda_{15} = \begin{bmatrix} A_{3i}\bar{P}_{2i} + B_{1i}L_{\tau i} \\ -\delta_i(A_{3i}\bar{P}_{2i} + B_{1i}L_{\tau i}) \end{bmatrix},$$

$$\Lambda_{16} = \begin{bmatrix} B_{2i}\bar{P}_{2i} \\ \delta_i B_{2i}\bar{P}_{2i} \end{bmatrix},$$

$$\Lambda_{17} = \begin{bmatrix} C_i^T\bar{P}_{2i} + D_i^T L_i \\ 0 \end{bmatrix},$$

and $\Lambda_{33} = -\widetilde{H}_N + \lambda_{\max}^2(A_{4i})\sum_{j=1}^{s}\pi_{ij}\tilde{P}_{1j}$, $\Lambda_{27} = D_i^T L_{hi}$, $\Lambda_{37} = D_i^T L_{di}$, $\Lambda_{57} = D_i^T L_{\tau i}$ with $\varpi_i := \lambda_{\max}(A_{4i})$

$$\widetilde{D}^s = h_1\begin{bmatrix} \begin{pmatrix} -2\widetilde{Q}_{i1}^a + \widetilde{R}_{0,1}^s \\ + \sum_{j=1}^{s}\pi_{ij}\widetilde{Q}_{j1}^s \end{pmatrix} & \begin{pmatrix} -2\widetilde{Q}_{i2}^a + \widetilde{R}_{0,2}^s \\ + \sum_{j=1}^{s}\pi_{ij}\widetilde{Q}_{j2}^s \end{pmatrix} & \cdots & \begin{pmatrix} -2\widetilde{Q}_{iN}^a + \widetilde{R}_{0,N}^s \\ + \sum_{j=1}^{s}\pi_{ij}\widetilde{Q}_{jN}^s \end{pmatrix} \\ \widetilde{Q}_{i1}^s & \widetilde{Q}_{i2}^s & \cdots & \widetilde{Q}_{iN}^s \\ -\widetilde{R}_{N,1}^s & -\widetilde{R}_{N,2}^s & \cdots & -\widetilde{R}_{N,N}^s \\ \begin{pmatrix} \varpi_i(2\widetilde{Q}_{i1}^a \\ -\sum_{j=1}^{s}\pi_{ij}\widetilde{Q}_{j1}^s) \end{pmatrix} & \begin{pmatrix} \varpi_i(2\widetilde{Q}_{i2}^a \\ -\sum_{j=1}^{s}\pi_{ij}\widetilde{Q}_{j2}^s) \end{pmatrix} & \cdots & \begin{pmatrix} \varpi_i(2\widetilde{Q}_{iN}^a \\ -\sum_{j=1}^{s}\pi_{ij}\widetilde{Q}_{jN}^s) \end{pmatrix} \\ -\varpi_i 2\widetilde{Q}_{i1}^a & -\varpi_i 2\widetilde{Q}_{i2}^a & \cdots & -\varpi_i 2\widetilde{Q}_{iN}^a \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$\widetilde{D}^a = h_1\begin{bmatrix} \widetilde{R}_{0,1}^a + \sum_{j=1}^{s}\pi_{ij}\widetilde{Q}_{j1}^a & \widetilde{R}_{0,2}^a + \sum_{j=1}^{s}\pi_{ij}\widetilde{Q}_{j2}^a & \cdots & \widetilde{R}_{0,N}^a + \sum_{j=1}^{s}\pi_{ij}\widetilde{Q}_{jN}^a \\ \widetilde{Q}_{i1}^a & \widetilde{Q}_{i2}^a & \cdots & \widetilde{Q}_{iN}^a \\ -\widetilde{R}_{N,0}^a & -\widetilde{R}_{N,1}^a & \cdots & -\widetilde{R}_{N,N-1}^a \\ \varpi_i\sum_{j=1}^{s}\pi_{ij}\widetilde{Q}_{j1}^a & \varpi_i\sum_{j=1}^{s}\pi_{ij}\widetilde{Q}_{j2}^a & \cdots & \varpi_i\sum_{j=1}^{s}\pi_{ij}\widetilde{Q}_{jN}^a \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$\widetilde{O}^s = h_2 \begin{bmatrix} \widetilde{T}^s_{0,1} & \widetilde{T}^s_{0,2} & \cdots & \widetilde{T}^s_{0,N} \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \widetilde{T}^s_{N,1} & \widetilde{T}^s_{N,2} & \cdots & \widetilde{T}^s_{N,N} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$\widetilde{O}^a = h_2 \begin{bmatrix} \widetilde{T}^a_{0,1} & \widetilde{T}^a_{0,2} & \cdots & \widetilde{T}^a_{0,N} \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \widetilde{T}^a_{N,0} & \widetilde{T}^a_{N,1} & \cdots & \widetilde{T}^a_{N,N-1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$\widetilde{S}_d = \mathrm{diag}\{\widetilde{S}_0 - \widetilde{S}_1, \widetilde{S}_1 - \widetilde{S}_2, \cdots, \widetilde{S}_{N-1} - \widetilde{S}_N\},$$

$$\widetilde{H}_d = \mathrm{diag}\{\widetilde{H}_0 - \widetilde{H}_1, \widetilde{H}_1 - \widetilde{H}_2, \cdots, \widetilde{H}_{N-1} - \widetilde{H}_N\},$$

$$\hat{Q}_i = [\widetilde{Q}_{i0}, \widetilde{Q}_{i1}, \cdots, \widetilde{Q}_{iN}],$$

$$\hat{S} = 1/h_1 \ \mathrm{diag}\{\widetilde{S}_0, \widetilde{S}_1, \cdots, \widetilde{S}_N\},$$

with

$$\begin{aligned} \widetilde{Q}^s_{ip} &= (\widetilde{Q}_{ip} + \widetilde{Q}_{i(p-1)})/2, \\ \widetilde{Q}^a_{ip} &= (\widetilde{Q}_{ip} - \widetilde{Q}_{i(p-1)})/2, \\ \widetilde{R}^s_{p,q} &= (\widetilde{R}_{p,q} + \widetilde{R}_{p,q-1})/2, \\ \widetilde{R}^a_{p,q} &= (\widetilde{R}_{p,q} - \widetilde{R}_{p,q-1})/2, \\ \widetilde{T}^s_{p,q} &= (\widetilde{T}_{p,q} + \widetilde{T}_{p,q-1})/2, \\ \widetilde{T}^a_{p,q} &= (\widetilde{T}_{p,q} - \widetilde{T}_{p,q-1})/2. \end{aligned}$$

Moreover, the controller gains in (4) can be designed as $K_i = L_i \bar{P}_{2i}^{-1}, K_{di} = L_{di}\bar{P}_{2i}^{-1}, K_{hi} = L_{hi}\bar{P}_{2i}^{-1}, K_{\tau i} = L_{\tau i}\bar{P}_{2i}^{-1}$.

Proof. It can be easily seen that the resulting closed-loop system (1) with (4) is of the following form,

$$\begin{aligned} \dot{x}(t) - A_{4i}\,\dot{x}(t-d) = N_{xi}\,x(t) + N_{di}x(t-d) + N_{hi}\,x(t-h) \\ + N_{\tau i}\int_{t-\tau}^{t} x(s)\,ds + (B_{2i} + \Delta B_{2i}(t))\,w(t), \end{aligned} \tag{27a}$$

$$\begin{aligned} z(t) = (C_i + D_iK_i)\,x(t) + (C_{di} + D_iK_{di})x(t-d) \\ + (C_{hi} + D_iK_{hi})x(t-h) + (C_{\tau i} + D_iK_{\tau i})\int_{t-\tau}^{t} x(s)\,ds, \end{aligned} \tag{27b}$$

where $N_{xi} := A_{1i} + \Delta A_{1i}(t) + (B_{1i} + \Delta B_{1i}(t))K_i$, $N_{hi} := A_{2i} + \Delta A_{2i}(t) + (B_{1i} + \Delta B_{1i}(t))K_{hi}$, $N_{di} := A_{5i} + \Delta A_{4i}(t) + (B_{1i} + \Delta B_{1i}(t))K_{di}$, $N_{\tau i} := A_{3i} + \Delta A_{3i}(t) + (B_{1i} + \Delta B_{1i}(t))K_{\tau i}$. It is noting that the notations $\Delta A_{ji}(t)$ and $\Delta B_{ji}(t)$ stand for $\Delta A_j(t,i)$ and $\Delta B_j(t,i)$, respectively. Then, we choose $P_{3i} = \delta_i P_{2i}, \delta_i \in \mathbb{R}$, where δ_i is a tuning scalar parameter (which may be restrictive). From Remark 5, by performing a congruence transformation $diag\{\bar{P}_{2i}^T, \ldots, \bar{P}_{2i}^T, \mathrm{I}, \bar{P}_{2i}^T, \ldots, \bar{P}_{2i}^T\}$, where $\bar{P}_{2i} := P_{2i}^{-1}$, to both sides of (24), applying Schur complements and considering $L_i = K_i\bar{P}_{2i}$, $L_{di} = K_{di}\bar{P}_{2i}$, $L_{hi} = K_{hi}\bar{P}_{2i}$, $L_{\tau i} = K_{\tau i}\bar{P}_{2i}$ result in

$$\hat{\Pi}_i + \mathrm{sym}(\hat{\Gamma}_{di}^T\,\Delta_i(t)\hat{\Gamma}_{ei}) < 0. \tag{28}$$

By Lemma 2, a necessary and sufficient condition for (28) is that there exists a scalar $\mu_i > 0$ such that

$$\hat{\Pi}_i + \mu_i^{-1}\,\hat{\Gamma}_{di}^T\,\hat{\Gamma}_{di} + \mu_i\ \hat{\Gamma}_{ei}^T\,\hat{\Gamma}_{ei} < 0 \tag{29}$$

then, applying Schur complements, we find that (29) is equivalent to LMI (26d). It is noting that the symbol $\tilde{g}$ stands for $\bar{P}_{2i}^T g \bar{P}_{2i}$ for any matrices g, for instance $\tilde{P}_{1i} = \bar{P}_{2i}^T P_{1i}\bar{P}_{2i}$. On the other hand, let $\zeta_i = diag\{\bar{P}_{2i}^T, \bar{P}_{2i}^T\}$. Premultiplying ζ_i and postmultiplying ζ_i^T to the LMIs (5b)-(5c), one obtains LMIs (26b)-(26c). This completes the proof. ∎

Remark 7. By setting $\eta = \gamma^2$ and minimizing η subject to LMIs (26), we can obtain the optimal H_∞ performance level $\gamma^\star$ (by $\gamma^\star = \sqrt{\gamma}$) and the corresponding control gains as well.

Remark 8. The reduced conservatism of Theorems 1-2 benefit from the construction of the Lyapunov-Krasovskii functional in (7), introducing some free weighting matrices to express the relationship among the system matrices and neither the model transformation approach nor any bounding technique are needed to estimate the inner product of the involved crossing terms. It can be easily seen that results of this paper is quite different from existing results in the literature in the following perspective. The Markovian jump structures at most of references, for instance [9], [14] and [39] consider a retarded time-delay systems and in compare to our case do not center on mixed time-delays, i.e., the results in the references above cannot be directly applied to the Markovian jump systems with different neutral, discrete and distributed time delays and nonlinear perturbations.

Remark 9. Note that the corresponding condition developed using the discretized LKF method will allow to overcome the conservatism of the bounds proposed using other time-domain approach. However, we approach the *optimal* bound, in the sense 'necessary and sufficient', if the grid size tends to zero, which is *expensive* in terms of computational effort [18]-[19]. On the other hand, the discretization technique of LKFs developed in this paper is based on LMIs. It is clear that the standard LMI has a polynomial-time complexity. Therefore, the size of the corresponding LMIs is an important problem to be considered if we are interested in further refinements. In this sense, the LMI simplification proposed by Gu in [47] can be used to simplify the conditions above.

IV. Simulation Results

In this section, with the aid of MATLAB LMI Toolbox [48], we use two numerical examples to illustrate the effectiveness and advantage of our design methods.

Example 1. We give an example for the application of the theoretical results to a realistic neutral delay differential equations problem. Here the delay elements are used for modeling transmission lines, and partial element equivalent circuits (PEEC) model. One of the PEEC models used in the literature (see, e.g. [18] and [49]) is given by

$$C_0\,\dot{y}(t) + C_1\,\dot{y}(t-\tau) + G_0\,y(t) + G_1\,y(t-\tau) = B\,u(t, t-\tau)$$

where C_0 is diagonal, and τ is the delay (retarded mutual coupling between partial inductances and current sources). The associated neutral system is

$$\dot{y}(t) - N\,\dot{y}(t-\tau) = L\,y(t) + M\,y(t-\tau)$$

with L, M, N appropriately defined. The matrices for our example are

$$\frac{L}{100} = \begin{bmatrix} -7 & 1 & 2 \\ 3 & -9 & 0 \\ 1 & 2 & -6 \end{bmatrix},$$

$$\frac{M}{100} = \begin{bmatrix} 1 & 0 & -3 \\ -0.5 & -0.5 & -1 \\ -0.5 & -1.5 & 0 \end{bmatrix},$$

$$N = \frac{1}{72}\begin{bmatrix} -1 & 5 & 2 \\ 4 & 0 & 3 \\ -2 & 4 & 1 \end{bmatrix}.$$

If the switching modes are not considered, i.e. $S = \{1\}$, the stability criterion of Theorem 1 for different values of the parameter N, i.e. $N = \{1,\ 2, 3\}$, is compared with those of [25], [27] and [28] for the above system in Table 1. Hence, for this example, the stability criterion we derived for linear time-delay systems is less conservative than those reported in [25], [27] and [28]. Note that the result of [49] is a delay-independent stability analysis which guarantees a feasible solution for an upper bound of the delay τ=1.

Table 1
The upper bound of the time delay for stability analysis.

Delay bound	[25]	[28]	[27]	Th.1 N = 1	Th.1 N = 2	Th.1 N = 3
τ	0.43	1.1413	1.5022	1.6405	1.6537	1.6851

Example 2. Consider a continuous-time uncertain system (1) with two Markovian switching modes and the following state-space matrices

$$A_1(1) = \begin{bmatrix} -1 & 0 \\ 0.2 & -1.2 \end{bmatrix};\ A_1(2) = \begin{bmatrix} -0.1 & 0 \\ 0.2 & -0.1 \end{bmatrix};$$

$$A_2(1) = \begin{bmatrix} 0.01 & -0.04 \\ 0.02 & 0.01 \end{bmatrix};\ A_2(2) = \begin{bmatrix} 0.1 & -0.04 \\ 0.02 & 0.1 \end{bmatrix};$$

$$A_3(1) = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix};\ A_3(2) = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.2 \end{bmatrix};$$

$$A_4(1) = \begin{bmatrix} 0 & 0.1 \\ 0 & 0.1 \end{bmatrix};\ A_4(2) = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.5 \end{bmatrix};$$

$$B_1(1) = \begin{bmatrix} 2 \\ 1 \end{bmatrix};\ B_1(2) = \begin{bmatrix} 1.5 \\ 0.1 \end{bmatrix};$$

$$B_2(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix};\ B_2(2) = \begin{bmatrix} 0.5 \\ 0.1 \end{bmatrix};$$

$$C(1) = C(2) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix};$$

$$D(1) = D(2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix};$$

$$S_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};\ S_2 = 1;$$

$$H_1(1) = H_1(2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix};$$

$$E_1(1) = E_1(2) = \begin{bmatrix} 1 & 0.1 \end{bmatrix};$$

$$E_2(1) = E_2(2) = \begin{bmatrix} 0.5 & 0.2 \end{bmatrix};$$

$$E_3(1) = E_3(2) = \begin{bmatrix} 0.2 & 0.3 \end{bmatrix};$$

$$E_4(1) = E_4(2) = 0.3;$$

$$E_5(1) = E_5(2) = 0,$$

$$h = 0.8,\ d = 0.3,\ \tau = 0.5.$$

The following transition matrix is considered.

$$\pi = \begin{bmatrix} -0.5 & 0.5 \\ 0.3 & -0.3 \end{bmatrix}$$

A realization of the jumping mode is plotted in Fig. 1, where the initial mode is assumed to be $r_0 = 1$.

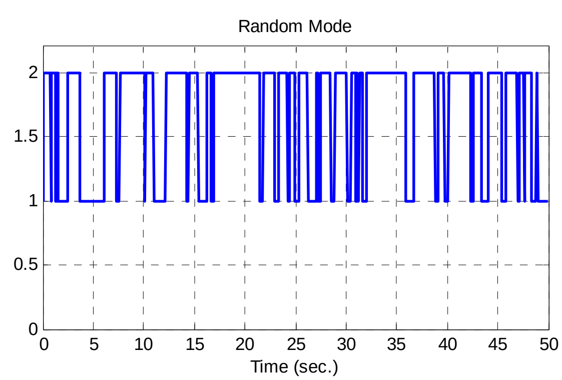

Fig 1. Random jumping mode.

Table 2
γ_{optimal} comparision w.r.t. N .

	N = 1	N = 2	N = 3
γ_{optimal}	0.4015	0.3750	0.3625

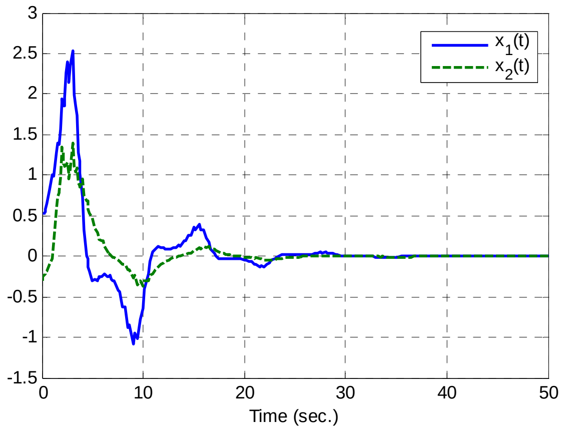

Fig 2. State responses.

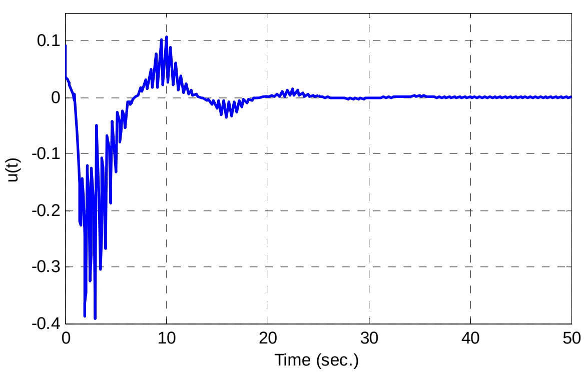

Fig 3. Control signal.

Using Matlab LMI Control Toolbox, LMIs (6) are solved for different values of the parameter N , i.e. $N \in \{1,2,3\}$, and corresponding values of the parameter γ in optimal H_∞ performance measure, γ_{optimal} , are obtained and shown in Table 2. It is easily seen that the parameter γ_{optimal} is decreased as the parameter N is increased.

For simulation purpose, we simply choose a unit step in the time interval $[1, 2]$ as the disturbance, $\Delta(t) = \sin(t)$ as the norm-bounded uncertainty. The simulation results are shown in Fig. 2 and Fig. 3. Responses of two states of the closed-loop system is depicted in Figure 1 under the initial condition $x(0) = [0.5 \quad -0.3]^T$. It is seen from Figure 1 that the closed-loop system is asymptotically stable. The corresponding control signal (37) is shown in Figure 2.

V. Conclusion

The problem of robust mode-dependent delayed state feedback H_∞ control was proposed for a class of uncertain systems with Markovian switching parameters and mixed discrete, neutral and distributed delays. New required sufficient conditions were derived in terms of delay-dependent linear matrix inequalities for the stochastic stability and stabilization of the considered system using some free matrices and the Lyapunov-Krasovskii functional theory. The desired control is derived based on a convex optimization method such that the resulting closed-loop system is stochastically stable and satisfies a prescribed level of H_∞ performance, simultaneously. Future work will investigate fault detection and mode-dependent mixed time delays for Markovian jump systems with partially known transition probabilities (see more details in [15]-[17]).

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