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Robust Regulation with an H_∞ Constrain for Linear Two-Time Scale Systems

Hamid Reza Karimi, *Senior Member, IEEE*

***Abstract*— In this paper, the problem of robust multi-objective control design with an H_∞ constrain is studied for a class of linear two-time scale systems. The design is based on a new modelling approach under the assumption of norm-boundedness of the fast dynamics. In this method, a portion of the fast dynamics is treated as a norm-bounded perturbation in the design by its maximum possible gain. In this view, the problem of robust multi-objective control design is performed only for the certain dynamics of the two-time scale system, whose order is less than that of the original system. One illustrative example is used to demonstrate the validity of the proposed approach.**

I. Introduction

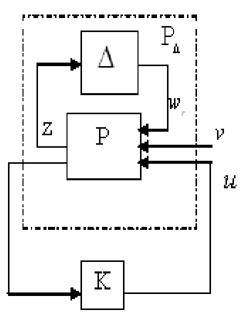
Control of two-time scale systems has been intensively studied for the past three decades and a popular approach adopted to handle these systems is based on the so-called reduced technique; see, e.g. [1]-[2]. The composite design based on separate designs for slow and fast subsystems has been systematically reviewed in [3]-[4]. The stability problem (ε-bound problem) in two-time scale systems differs from conventional linear systems, which can be designed as: characterizing an upper bound ε_0 of the positive perturbing scalar ε such that the stability of a reduced-order system would guarantee the stability of the original full-order system for all $\varepsilon \in (0, \varepsilon_0)$. Researchers have tried various ways to find either the stability bound ε_0 or a less conservative lower bound for ε_0, see for instance the references [5]-[8]. Although numerous ways have been presented to compute the bound ε^*, unfortunately, only some of the conservative bounds of ε were achieved. Recently, the authors in [5] proposed the robust stability analysis and stability bound improvement of ε in the two-time scale systems by using linear fractional transformations and structured singular values (μ) approach. In [9], a computational method based on Haar wavelets to the problem of optimal control of linear singularly perturbed systems is studied.

The research on two-time Scale systems in the H_∞ sense is of great practical importance, particularly in the last few years; see, e.g. [10]-[12]. The H_∞ optimal control of two-time scale linear systems, under either perfect state measurements or imperfect state measurements, for both finite and infinite horizons has been investigated in [13] and [14] via a differential game theoretic approach. Shi and Dragan in [11] also studied the design of a composite linear controller based on the slow and fast dynamics, such that both stability and a prescribed H_∞ performance for the full-order system are achieved and in this line, they could solve the problem of robust control for the above system with time-varying norm-bounded parameter uncertainty. The authors in [15] proposed how to perform order-reduction of a balanced system using theory of singular perturbations that can produce very good accuracy at high frequencies particularly for systems that have lightly damped highly oscillatory modes. Recently, the robust stability and disturbance attenuation for a class of uncertain two-time scale systems has been investigated in [6]. In [5], the problem of H_∞ control for linear two-time scale systems is investigated. The authors' attention is focused on the robust regulation of the system based on a new modelling approach under the assumption of norm-boundedness of the fast dynamics. In the proposed approach, the fast dynamics are treated as a norm-bounded disturbance (dynamic uncertainty). Also, the proposed strategy is applied to a single-link flexible arm in [16]. The authors in [17] proposed the problem of designing a robust H_∞ output feedback controller using a linear matrix inequality (LMI) approach for a class of singularly perturbed systems described by a Takagi–Sugeno fuzzy model. More recently, the problem of H_∞ control of discrete-time singularly perturbed systems was studied in [18]. A new sufficient condition, which ensures the existence of state feedback controllers such that the resulting closed-loop system is asymptotically stable while satisfying a prescribed H_∞ norm bound, is obtained. This condition is in terms of an LMI, which is independent of the singular perturbation parameter. Moreover, robustness and reliability of decentralized stochastic singularly-perturbed computer controlled systems with multiple time-varying delays was studied in [19]. A robust passive stability criteria was derived in [20] for uncertain singularly markov jump systems with time delays.

The contribution of this paper is three-fold: first, this paper extends the previous work [21] on multi-objective control synthesis for robust regulation with an H_∞ constrain (RR H_∞C) for linear two-time scale systems; second, based on a

H.R. Karimi is with the Department of Engineering, Faculty of Engineering and Science, University of Agder, N-4898 Grimstad, Norway (e-mail: hamid.r.karimi@uia.no).

new modelling approach, a dynamical model for the system under consideration is presented such that a portion of the dynamics may be treated as a norm-bounded dynamic uncertainty; third, a robust multi-objective H_∞ control is designed only for the certain dynamics of the two-time scale system, whose order is less than that of the original system. Clearly, it means that the proposed approach deals with only those two-time scale systems where the fast subsystem is norm-bounded. Although, this might be considered as a restriction on systems under consideration, it covers many control systems, for instance mechanical systems having two types, i.e., slow and fast, behaviours. In this view, the synthesis is performed only for certain dynamics of the system [5]. In this view, the problem of robust multi-objective H_∞ control design is performed only for the certain dynamics of the two-time scale system, whose order is less than that of the original system. It should be noted that this scheme is significantly different from the conventional approaches of order reduction for linear two-time scale systems. The controller synthesis problem addressed in this paper is to design (if possible) an admissible controller that solves the problem of RRH_∞C based on the internal model principle. Two examples are provided to illustrate the efficiency of the proposed approach.



$$P \sim \left[\begin{array}{c|ccc} A & B_1 & B_r & B_2 \\ \hline C_1 & 0 & D_{1r} & D_{12} \\ C_2 & D_{21} & D_{2r} & 0 \end{array}\right]$$

Fig. 1. Setup for the RRGBU.

II. MULTI-OBJECTIVE H_∞ CONTROL

In this section, the results of [21] concerning multi-objective H_∞ suboptimal control with the controller constrained to achieve robust closed-loop regulation is reviewed. Throughout this section, the finite-dimensional linear time-invariant system

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) + D\,u(t) \end{aligned} \tag{1a-b}$$

is represented by

$$T_{yu} \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{2}$$

where T_{yu} is the transfer function mapping input u to output y.

Let Δ denote the map from z to v and P_Δ be as shown in Figure 1. Assume that the L_2 norm of Δ is bounded by some positive number. The problem of Robust Regulation in the presence of Gain-Bounded Uncertainty (RRGBU) may be stated as follows: Given a real number $\gamma_v > 0$, design a controller such that for all gain-bounded Δ with $\|\Delta\|_\infty < \gamma_v^{-1}$,

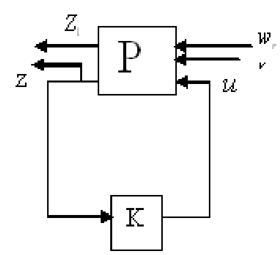
i. The controller internally stabilizes P_Δ,
ii. The regulated output $e(t)$ converges to zero as $t \to \infty$,
iii. The convergence property holds for all plants in some neighbourhood of P_Δ in the sense of the graph topology.

It is shown in [21] that the RRGBU problem is equivalent to a certain multi-objective problem that will be discussed in the context of Figure 2. The multi-objective problem is to design a single controller that solves both the robust regulation problem (from $w_r \equiv 0$ to Z_1 with $v \equiv 0$) and the H_∞ suboptimal control problem (from v to Z with $w_r \equiv 0$).

Definition 1. The multi-objective problem is to design a controller K for a given real number $\gamma > 0$ such that

1) K internally stabilizes P.
2) $T_{Z_1 w_r}(j\omega_k) = 0$ for $k = 1,\ldots,N$.
3) property 2) holds for all plants in some neighbourhood of P in the graph topology, and
4) $\|T_{Zv}\|_\infty < \gamma$.

Objectives 1) through 3) in Definition 1 constitute the standard problem of robust regulation as defined previously. Objective 4) is the usual H_∞ norm requirement. This multi-objective problem will be called the problem of robust regulation with an H_∞ constraint (RRH_∞C).



$$P \sim \left[\begin{array}{c|ccc} A & B_1 & B_r & B_2 \\ \hline C_1 & 0 & D_{1r} & D_{12} \\ C_2 & D_{21} & D_{2r} & 0 \end{array}\right]$$

Fig. 2. Setup for the RRH_∞C.

Note that our proposed design technique is based on state feedback. By the Internal Model Principle [22], any controller that solves the output regulation problem, internally incorporates a model of the dynamical system generates the reference trajectories.

Theorem 1. [21] Let K be any finite-dimensional linear time-invariant controller. Then K solves the RRGBU problem for P if and only if K solves the RRH_∞C problem for P.

Definition 2 (*Internal model matrices*). $\tilde{A}$ and $\tilde{B}$ are internal model matrices associated with the robust regulation problem determined by $\omega_1,\ldots,\omega_N$ if these matrices satisfy

a) $spec(\widetilde{A}) = \{\pm j\,\omega_1, \ldots, \pm j\,\omega_N\}$,

b) every eigenvalue of $\widetilde{A}$ has multiplicity l.

c) $\widetilde{A}$ is diagonalizable, and

d) $(\widetilde{A}, \widetilde{B})$ is controllable.

Remark 1. Matrices $\widetilde{A}$ and $\widetilde{B}$ satisfying a)-d) of Definition 2 can be realized in many different ways. According to [21], the following realization is given here.

With every frequency ω_k to be regulated against, associate system matrices $\widetilde{A}_k$ and $\widetilde{B}_k$ are as follows. If $\omega_k = 0$, choose integrator dynamics

$$\widetilde{A}_k = 0 \in R^{l \times l}, \quad \widetilde{B}_k = I \in R^{l \times l}. \tag{3}$$

If $\omega_k \neq 0$, choose harmonic oscillator dynamics

$$\widetilde{A}_k = \begin{bmatrix} 0 & \omega_k I \\ -\omega_k I & 0 \end{bmatrix} \in R^{2l \times 2l}, \qquad \widetilde{B}_k = \begin{bmatrix} 0 \\ I \end{bmatrix} \in R^{2l \times l}. \tag{4}$$

Now set

$$\widetilde{A} = diag\,\{\widetilde{A}_1, \ldots, \widetilde{A}_N\}, \qquad \widetilde{B} = [\widetilde{B}_1^T \quad \ldots \quad \widetilde{B}_N^T]^T. \tag{5}$$

It is straightforward to verify that $\widetilde{A}$ and $\widetilde{B}$ as constructed do, in fact, satisfy requirements a)-d) of Definition 2.

We make the following standard assumptions on P.

A1) $C_2 = I$ and $D_{21} = 0$.

A2) (A, B_2) is stabilizable.

A3) (C_1, A, B_1) has no uncontrollable/unobservable modes on the imaginary axis

A4) $D_{12}^T C_1 = 0$ and $D_{12}^T D_{12} = I$.

Theorem 2. [21] Let the plant P of Fig. 3 satisfy the standard assumptions A1)-A4). Then the following are equivalent.

i) There exists an admissible controller for P that solves the robust regulation problem from w_r to Z_1 at the frequencies $\omega_1, \ldots, \omega_N$ while also making $\|\mathrm{T}_{\mathrm{Z}v}\|_\infty < \gamma$ (the RRH_∞C problem).

ii) There exists an admissible controller for P that renders $\|\mathrm{T}_{\mathrm{Z}v}\|_\infty < \gamma$, and

$$\gamma^2\,\mathrm{T}_k\,B_2\,B_2^T\,\mathrm{T}_k^* > \mathrm{T}_k\,B_1\,B_1^T\,\mathrm{T}_k^*, \qquad k = 1, \ldots, N \tag{6}$$

where $\mathrm{T}_k \triangleq [I_l \quad 0]\,(j\omega_k I - \overline{A})^{-1}$ and $\overline{A} \triangleq A + (\gamma^{-2}\,B_1\,B_1^T - B_2\,B_2^T)X$, and X is the positive semi-definite, stabilizing solution to

$$A^T\,X + X\,A + X(\gamma^{-2}\,B_1\,B_1^T - B_2\,B_2^T)\,X + C_1^T\,C_1 = 0. \tag{7}$$

Moreover, if either (hence both) of these conditions hold, then a controller K that solves the RRH_∞C problem for P is given by

$$\mathrm{K} \sim \left[\begin{array}{c|c} \widetilde{A} & \widetilde{B}[I_l \quad 0] \\ \hline B_2^T\,L^T\,W^{-1} & -B_2^T\,X - B_2^T\,L^T\,W^{-1}\,L \end{array}\right]. \tag{8}$$

Here L is the unique solution to $L\overline{A} - \widetilde{A}L = \widetilde{B}[I_l \quad 0]$. the intemal model matrices $\widetilde{A}$ and $\widetilde{B}$ satisfy a)-d) of Definition 2, and W is a positive-definite matrix that satisfies the Lyapunov inequality

$$\widetilde{A}\,W + W\,\widetilde{A}^T + L(\gamma^{-2}B_1\,B_1^T - B_2\,B_2^T)\,L^T < 0. \tag{9}$$

III. MAIN RESULTS

The system under consideration, with slow and fast dynamics is described in the standard two-time Scale form by

$$\dot{x}_s = a_{11}\,x_s + a_{12}\,x_f + b_1\,u + b_{1r}\,w_r, \qquad x_s(0) = \overline{\eta} \tag{10a}$$

$$\varepsilon\,\dot{x}_f = a_{21}\,x_s + a_{22}\,x_f + b_2\,u + b_{2r}\,w_r, \qquad x_f(0) = \overline{\xi} \tag{10b}$$

$$y = C\,x_s + F\,x_f \tag{10c}$$

where $a_{11} \in R^{n\times n}$, $a_{12} \in R^{n\times m}$, $a_{21} \in R^{m\times n}$, $a_{22} \in R^{m\times m}$, $b_1 \in R^{n\times k}$, $b_2 \in R^{m\times k}$, $C \in R^{r\times n}$, $F \in R^{r\times m}$, $b_{1r} \in R^{n\times s}$ and $b_{2r} \in R^{m\times s}$ are the certain matrixes and $x_s = [x_{s1}, x_{s2}, \ldots, x_{sn}]^T \in R^n$, $x_f = [x_{f1}, x_{f2}, \ldots, x_{fm}]^T \in R^m$, $y(t) \in R^r$ and $u(t) \in R^k$ represent the state vectors of the slow and fast dynamics and measured output and control input, respectively, and $w_r(t) \in R^s$ is an exogenous input. Also, $\overline{\eta}$ and $\overline{\xi}$ are, respectively, the initial states of $x_s(t)$ and $x_f(t)$. The singularly perturbed parameter ε is nonnegative and always represents the response time of the fast dynamics.

In this paper, our objective is to view a portion of the fast dynamics as norm-bounded uncertainty. Therefore, we call them as unmodeled dynamics. Although the unmodeled term refers to a subsystem whose dynamics are not known, it is used here to emphasize that the complete characteristics of this subsystem will not be utilized in the synthesis. If this is feasible, then the synthesis has to satisfy the design specifications only for the "known dynamics", hereafter referred to as the plant nominal dynamics. The unmodeled dynamics, on the other hand, may be considered as a subsystem that is connected to the plant nominal dynamics. In this section, our objective is to apply the above concept to a linear two-time Scale system. The extension to a nonlinear two-time scale system should in principle be feasible and is a topic for future work.

3.1 A New Modelling Approach

Suppose the fast dynamics are stable. We will show that a portion of the fast dynamics may be treated as a norm-bounded uncertainty and the remaining part can be augmented to the slow dynamics. In this view, (10) will read:

Nominal system:

$$E\,\dot{X} = A_X\,X + A_{Xv}\,v + B_X\,u + B_r\,w_r \tag{11a}$$

$$y = C_2\,X + D_{21}\,v \tag{11b}$$

Uncertain system:

$$\begin{aligned} \varepsilon\,\dot{v} &= A_{vX}\;X + A_v\,v + B_v\,u + B_w\,w_r \\ &\triangleq A_v\,v + \begin{bmatrix} A_{vX} & B_w & B_v \end{bmatrix} \mathrm{Z} \end{aligned} \tag{11c}$$

$$y_v = v \tag{11d}$$

$$\mathrm{Z} = C_1\;X + D_{12}\,u + D_{1r}\,w_r \triangleq \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} u + \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} w_r \tag{11e}$$

such that

$$E \triangleq \begin{cases} diag(I_n, \varepsilon I_{i-1}) & i>1 \\ I_n & i=1 \end{cases}, \; A_X = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

$$A_{Xv} = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}, \; B_X = \begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix}, \; B_r = \begin{bmatrix} b_{1r} \\ b_{Xr} \end{bmatrix},$$

$$C_1 = \begin{bmatrix} C_{11} & C_{12} \end{bmatrix}, \; A_{vX} = \begin{bmatrix} A_{v1} & A_{v2} \end{bmatrix}, \; C_2 = \begin{bmatrix} C_{21} & C_{22} \end{bmatrix}.$$

From comparison between (10) and (11), we find

$$a_{11} = A_{11}, a_{12} = \begin{bmatrix} A_{12} & B_{12} \end{bmatrix}, b_1 = B_{12}, a_{21} = \begin{bmatrix} A_{21}^T & A_{v1}^T \end{bmatrix}^T,$$

$$a_{22} = \begin{bmatrix} A_{22} & B_{21} \\ A_{v2} & A_v \end{bmatrix}, b_2 = \begin{bmatrix} B_{22}^T & B_v^T \end{bmatrix}^T, b_{2r} = \begin{bmatrix} b_{Xr}^T & B_w^T \end{bmatrix}^T.$$

In which $v = (v_i, v_{i+1}, ..., v_m)^T$ is the vector of fast dynamics, which is to be treated as a norm-bounded uncertainty and $X = (x_s^T, x_v^T)^T$ (which $x_v = (v_1, v_2, ..., v_{i-1})^T$ for $i > 1$) is the vector of *certain dynamics*, where i is the index of the first state of the *uncertain dynamics*. The value of this index will be determined in Remark 2 and the vectors η and ξ are the initial states. Therefore, the slow and fast vectors with dimension $n+m$ are partitioned into two parts. One part represents the plant nominal dynamics with dimension $n+i-1$, while the remaining part represents the uncertain dynamics with dimension $m-i+1$.

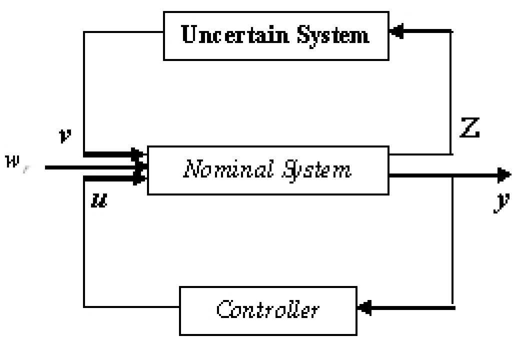


Fig. 3. General Block Diagram of Overall Closed Loop System

The above representation is shown schematically in Figure 3. The nominal plant has two inputs $(v \quad u)$ and two outputs $(Z \quad y)$. The first input represents the disturbances to be rejected (exogenous input). The second input is the control input that is used for feedback design. The controlled output Z represents a penalty variable as well as a cost of the control input needed to achieve the prescribed goal. The second output is the measurement output that is made on the system. This is used to generate the control input, which in turn is the tool we have to minimize the effect of the exogenous input on the controlled output. A constraint that is imposed is that the mapping from the measurement to the control input should be such that the closed loop system is internally stable. The effect of the exogenous input on the controlled output after closing the loop is measured in terms of their energies and the worst-case disturbance of the closed-loop H_∞ norm which is simply the L_2 induced norm.

Suppose the objective is to only stabilize the system, i.e., the system has no exogenous input. By virtue of the *small gain theorem*, if the nominal plant is stable, the overall system would remain stable if the product of the L_2 gains of the nominal plant and unmodeled dynamics is less than one. It is clear that in the case of an unstable nominal plant, one has to first stabilize the system by designing the control law based on the measured output and then apply the small gain theorem to ensure stability. Consequently, only the dominant part of the states in the model (11a) will be considered for the propose of synthesis. In the following, we make the following standard assumptions on the nominal system (11a)-(11b).

A5) Regarding to the *nominal system*, we assume that $(A_X \quad B_X \quad C_2)$ and $(A_X \quad A_{Xv} \quad C_1)$ are stabilizable-detectable and stabilizable-detectable, respectively, and rank of matrix D_{12} is k and rank of matrix D_{21} is r.

A6) The structured dynamic uncertainty $\Delta(s)$ is assumed to be internally asymptotically stable whose H_∞ norm is less than or equal to γ_1, i.e., $\|\Delta(s)\|_\infty \le \gamma_1$. In the frequency-domain one has:

$$\Delta(s) = (\varepsilon s I - A_v)^{-1} \begin{bmatrix} A_{vX} & B_v \end{bmatrix} \qquad (12)$$

where $\Delta(s)$ denotes the open-loop transfer function from $Z(t)$ to $v(t)$ in (6).

In this paper according to [5], the structure of the H_∞ controller is determined for the *nominal system* (11a)-(11b) such that the sufficient condition of *small gain theorem* is satisfied, i.e.,

$$\|T_{Zv}\|_\infty \cdot \|\Delta\|_\infty < 1 \qquad (13)$$

where $\|T_{Zv}\|_\infty = \sup_{v \in L_2} \|Z\|_2 / \|v\|_2$.

Remark 2. The following procedure illustrates the steps to be taken for finding the minimum value for *i*, i.e. the index of state after which the next states are to be considered as uncertainty. The procedure is as follows:

Step 1. Set i =1.

Step 2. Construct Equations (11a)-(11e).

Step 3. Find the upper bound of γ_1 and γ_2, i.e., $\|\Delta(s)\|_\infty \le \gamma_1$ and $\|T_{Zv}\|_\infty \le \gamma_2$.

Step 4. Check if $\gamma_1 . \gamma_2 < 1$

- * If yes, go to step 5.
- * If no, set $i = i+1$ and go to step 2.

Step 5. Construct the controller from the information obtained above in the next subsection.

Since the above procedure begins with i =1, it will always result in the minimum value for *i.*

IV. CONTROL DESIGN

In this section, the RRH_∞C problem presented in the previous section is investigated such that an admissible controller is derived for the nominal and uncertain systems (11). According to Theorem 2, an admissible controller can be designed for nominal system so that the four conditions (1)-(4) in Definition 1 are satisfied. Since the output of the nominal system includes all nominal system states and disturbance states (uncertain system), the control synthesis is done with full information (FI). Therefore, the desired controller is given as follows:

$$\mathrm{K}\sim\left[\begin{array}{c|c}\widetilde{A} & [\widetilde{B}[I_l \quad 0] \quad 0] \\ \hline B_2^T E^{-1} L^T W^{-1} & [-B_2^T E^{-1} X - B_2^T E^{-1} L^T W^{-1} L \quad 0]\end{array}\right] \qquad (14)$$

It is worth noting that the control structure (14) is possibly depending on the perturbation parameter ε. Therefore this dependency will be disappeared by using the theory of composite feedback controls for two-time scale systems. The following Theorem can proceed to resolve this problem.

Theorem 3. [7] Consider the two-time scale system (10) with $u(t)\equiv 0$. If the matrices a_{22} and $\overline{a}=a_{11}-a_{12}\,a_{22}^{-1}\,a_{21}$ are stable, the following relation holds:

$$\left\| \mathrm{T}_{yw_r} \right\|_\infty = \max\left\{ \left\| \mathrm{T}_{y_s w_{rs}} \right\|_\infty , \left\| \mathrm{T}_{y_f w_{rf}} \right\|_\infty \right\} \qquad (15)$$

with $\mathrm{T}_{y_f w_{rf}}(s) := F(sI-a_{22})^{-1}b_2$, $\mathrm{T}_{y_s w_{rs}}(s) := \overline{C}(sI-\overline{a})^{-1}\overline{b}+\overline{F}$, $\overline{C}=C-F\,a_{22}^{-1}\,a_{21}$, $\overline{b}=b_{1r}-a_{22}^{-1}\,b_{2r}$, $\overline{F}=-F\,a_{22}^{-1}\,b_{2r}$, where the indexes s and f indicate the slow and fast parts of the corresponding variables, respectively.

The above Theorem states that if a composite controller is designed for the reduced-order slow subsystem with a disturbance attenuation level γ, $\left\| \mathrm{T}_{yw_r} \right\|_\infty$ will not be smaller than γ unless $\left\| \mathrm{T}_{y_f w_{rf}} \right\|_\infty$ (for the fast subsystem) is smaller than γ. According to Theorem 3, it can be concluded that the conditions 2 and 4 (in Definition 1) in the design of the admissible controller for the nominal system met the same conditions for slow and fast subsystems and main controller is obtained based on the theory of composite feedback controls for slow and fast subsystems for all $\varepsilon\in[0,\ \varepsilon^*)$. It is desirable to further increase the upper limit of the disturbed parameter, i.e. ε^*.

V. SIMULATION RESULTS

Consider a single–link flexible manipulator with six modes of deflection as the fast dynamics, taken from [16]. The state space model in this case is

$$E\dot{x}=Ax+Bu \qquad (16a)$$

$$y=Cx \qquad (16b)$$

with $x=(y,\dot{\theta},\overline{x}_1^T,\ldots,\overline{x}_m^T)^T$, $\overline{x}_i=(\delta_{1i}\quad \delta_{2i})^T$, $\widetilde{w}_i=\varepsilon\,w_i$ and

$$A=\begin{bmatrix}\overline{A}_0 & \vec{A}\\ 0 & \tilde{A}\end{bmatrix},\ \overline{A}_0=\begin{bmatrix}0 & L\\ 0 & -\alpha\end{bmatrix},\ \tilde{A}=diag(\overline{A}_1,\ldots,\overline{A}_m),$$

$$\vec{A}=\begin{bmatrix}r_1 & \cdots & r_m\\ 0 & \cdots & 0\end{bmatrix},\ B=[B_0^T,\overline{B}_1^T,\ldots,\overline{B}_m^T]^T,\quad C=\begin{bmatrix}1 & 0_{2\times m+1}\end{bmatrix},$$

$$r_i=\begin{bmatrix}-\dfrac{\varepsilon\,\phi_i(t)\,\xi_i}{\widetilde{w}_i\sqrt{1-\xi_i^2}} & 0\end{bmatrix},\ \overline{A}_i=\begin{bmatrix}-\xi_i\,\widetilde{w}_i & \widetilde{w}_i\sqrt{1-\xi_i^2}\\ -\widetilde{w}_i\sqrt{1-\xi_i^2} & -\xi_i\,\widetilde{w}_i\end{bmatrix},$$

$$B_0=\tfrac{1}{I_T}\begin{pmatrix}0\\1\end{pmatrix},\ \overline{B}_i=\frac{\widetilde{w}_i\,\dot{\phi}_i(0)}{I_T}\begin{bmatrix}-\sqrt{1-\xi_i^2}\\ \xi_i\end{bmatrix}.$$

And y is the tip position, m is the arbitrarily large number of flexible modes, α denotes the corresponding pole of the rigid dynamics, (ω_i,ξ_i) are the frequency and damping ratio of the $\boldsymbol{i}$ th deflection mode, L is the length of the link, I_T is the total inertia about the armature and ϕ_i represents **i** th mode shape.

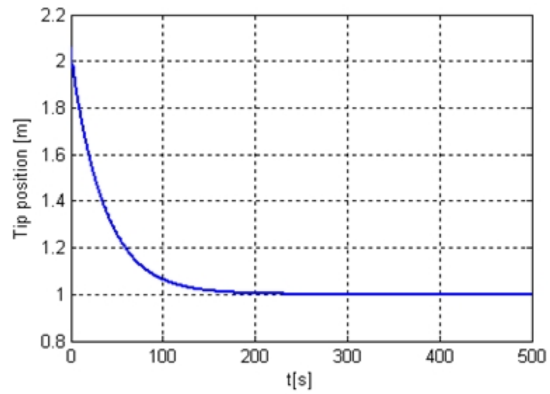


Fig. 4. Response of the tip position.

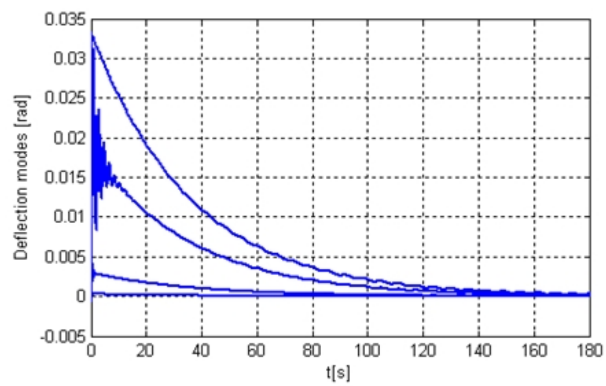


Fig. 5. Response of the deflection modes of the nominal system.

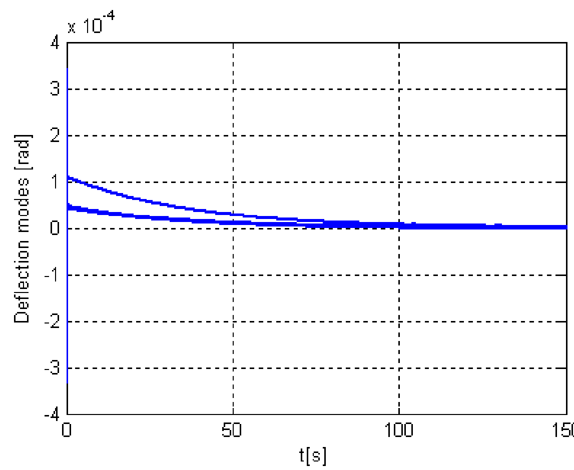


Fig. 6. Response of the deflection modes of the uncertain system.

The link parameters as well as the natural modes and the corresponding damping ratios used for design and simulation are given in Tables A.1 and A.2 in [5]. Table A.3 in [5] gives the pole-zero locations of 14^{th} –order model of the single-link flexible manipulator considered in [23]. The objective is to design a controller so that the tip position $x_{s1}(t)$ tracks a step input. According to the modelling approach presented in Section 3.1, three deflection modes are eligible to be considered as uncertainty [16]. Consider $\varepsilon = 0.011$, $\gamma = 1$ and $x_{s1}(0) = 2$ and the initial value of other dynamics are zero. We apply the state feedback controller (14) to nominal system. Figures 4 and 5 depict the regulation of tip position ($x_{s1}(t)$) and other states (deflection modes) of the nominal system, also Figure 6 depicts the regulation of uncertainty dynamics (Δ). Moreover, the controller has been depicted in Figure 7.

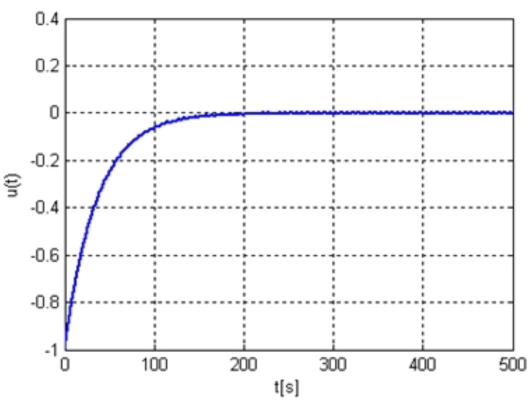


Fig. 7. H_∞ **controller.**

VI. CONCLUSIONS

The problem of robust multi-objective H_∞ control design for linear two-time scale systems based on a new modeling approach under the assumption of norm-boundedness of the fast dynamics was studied in this paper. In the proposed approach, the fast dynamics are treated as a norm-bounded uncertainty and the portion that is treated as a perturbation is incorporated in the design by its maximum possible gain. In this view, the problem of robust multi-objective H_∞ control design was performed only for the certain dynamics of the two-time scale system, whose order is less than that of the original system. The controller synthesis problem addressed in this paper is to design (if possible) an admissible controller that solves the problem of robust regulation with an H_∞ constrain (RRH_∞C) based on the internal model principle. The effectiveness of the approach was presented in the simulation results.

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