New Delay-Dependent Exponential H_∞ Synchronization for Uncertain Neural Networks With Mixed Time Delays

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***Abstract*—This paper establishes an exponential H_∞ synchronization method for a class of uncertain master and slave neural networks (MSNNs) with mixed time delays, where the mixed delays comprise different neutral, discrete, and distributed time delays. The polytopic and the norm-bounded uncertainties are separately taken into consideration. An appropriate discretized Lyapunov–Krasovskii functional and some free-weighting matrices are utilized to establish some delay-dependent sufficient conditions for designing delayed state-feedback control as a synchronization law in terms of linear matrix inequalities under less restrictive conditions. The controller guarantees the exponential H_∞ synchronization of the two coupled MSNNs regardless of their initial states. Detailed comparisons with existing results are made, and numerical simulations are carried out to demonstrate the effectiveness of the established synchronization laws.**

***Index Terms*—H_∞ performance, linear matrix inequality (LMI), neural networks (NNs), synchronization, time delay, uncertainties.**

I. Introduction

IN the last few years, synchronization in neural networks (NNs) has received a great deal of interest among scientists from various fields [1]–[5]. To better understand the dynamical behaviors of different kinds of complex networks, an important and interesting phenomenon to investigate is the synchrony of all dynamical nodes. In fact, synchronization is a basic motion in nature that has been studied for a long time, ever since the discovery of Christian Huygens in 1665 on the synchronization of two pendulum clocks. The results of chaos synchronization are utilized in biology, chemistry, secret communication and cryptography, nonlinear oscillation synchronization, and some other nonlinear fields. The first idea of synchronizing two identical chaotic systems with different initial conditions was introduced by Pecora and Carroll [6], and the method was realized in electronic circuits. The methods for synchronization of the chaotic systems have widely been studied in recent years, and many different methods have theoretically and experimentally been applied to synchronize chaotic systems, such as feedback control [7]–[12], adaptive control [13]–[17], backstepping [18], and sliding-mode control [19], [20]. Recently, the theory of incremental input-to-state stability to the problem of synchronization in a complex dynamical network of identical nodes, using chaotic nodes as a typical platform, has been studied in [21].

On the other hand, in practice, due to the finite switching speed of amplifiers or the finite speed of information processing, time delays, including delays in the state or in the derivative of the state, are often encountered in hardware implementation [22]–[26], which may be a source of oscillation, divergence, and instability in NNs. Another type of time delays, namely, distributed time delays, have begun to receive research attention [27], [28]. The main reason is that, since an NN usually has a spatial nature due to the presence of an amount of parallel pathways of a variety of axon sizes and lengths, continuously distributed delays should be introduced in modeling of the NNs over a certain duration of time such that the distant past has less influence compared to the recent behavior of the state [27]. Therefore, the stability problems of NNs with mixed time delays have gained great research interest [29]–[35]. Recently, both delay-independent and delay-dependent sufficient conditions have been proposed to verify the asymptotical or exponential stability of delayed NNs (see, for instance, [36]–[46] and references therein). Furthermore, many results have been reported on the stability analysis issue for various NNs with distributed time delays, such as recurrent NNs [47], [48], bidirectional associative memory networks [49], Hopfield NNs [50], and cellular NNs [51]. It is noted that both discrete and distributed time delays have recently been considered in [27], [28], [33], and [38]. It can be realized that, in [4], [9], and [52]–[56], several sufficient conditions in terms of linear matrix inequalities (LMIs) were presented to solve the synchronization and estimation problems of NNs with time delays. Huang *et al.* [56] studied the exponential synchronization problem for a class of chaotic Lur'e systems by using delayed feedback control by employing an integral inequality and introducing several slack variables to reduce the conservatism of the developed synchronization criterion. In [54], the problem of synchronization for stochastic discrete-time drive–response networks with time-varying delay was investigated by employing the Lyapunov functional method combined with the stochastic analysis, as well as the feedback control technique. The advantage of this approach was that a less conservative condition, which depends on the lower and upper bounds of the time-varying delay, was obtained. Furthermore, from the published results, it appears

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that general results pertaining to exponential synchronization of master–slave systems with mixed neutral, discrete, and distributed delays and an H_∞ performance criteria are few and restricted, despite its practical importance, mainly due to the mathematical difficulties in dealing with such mixed delays. Hence, it is our intention in this paper to tackle such an important yet challenging problem.

In this paper, we contribute to the further development of an exponential H_∞ synchronization method for a class of uncertain master and slave neural networks (MSNNs) with mixed time delays, where the mixed delays comprise different neutral, discrete, and distributed time delays. Both the polytopic and the norm-bounded uncertainties are separately taken into consideration. An appropriate discretized Lyapunov–Krasovskii functional (DLKF) is constructed to establish some delay-dependent sufficient conditions for designing delayed state-feedback control as a synchronization law in terms of LMIs under less restrictive conditions by introducing some free-weighting matrices. Then, the controller is developed based on the available information of the size of the discrete and distributed delays to guarantee that the controlled slave system can exponentially synchronize with the master system regardless of their initial states. It is shown that the decay coefficient can easily be calculated by solving the derived delay-dependent conditions. All the developed results are expressed in terms of convex optimization over LMIs and tested on a representative example to demonstrate the feasibility and applicability of the proposed approach.

The notations used throughout this paper are fairly standard. I_n and 0_n represent the $n \times n$ identity matrix and the $n \times n$ zero matrix, respectively; the superscript "T" stands for matrix transposition; $\Re^n$ denotes the n-dimensional Euclidean space; and $\Re^{n\times m}$ is the set of all real $m \times n$ matrices. The vector v_i denotes the unit column vector having a "1" element on its ith row and zeros elsewhere. $\|\cdot\|$ refers to the Euclidean vector norm or the induced matrix 2-norm, and $\mathrm{diag}\{\cdots\}$ represents a block diagonal matrix. $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the smallest and the largest eigenvalue of the square matrix A, respectively. The operator $\mathrm{sym}\{A\}$ denotes $A + A^T$. The notation $P > 0$ means that P is real symmetric and positive definite, and the symbol $*$ denotes the elements below the main diagonal of a symmetric block matrix. In addition, $L_2[0,\infty)$ is the space of square-integrable vector functions over $[0,\infty)$. Matrices, if the dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

II. Problem Description

In this paper, the problem of characterizing the delay-dependent coupling technique for the synchronization of a class of MSNNs with mixed time delays is considered. More specifically, consider the master neural network, which is described as follows:

$$\begin{cases} \dot{x}(t) = -Ax(t)+W_1 f\,(x(t))+W_2 g\,(x(t-\tau_1)) \\ \qquad\quad + W_3\dot{x}(t-\tau_2)+W_4 \int\limits_{t-\tau_3}^{t} h\,(x(s))\,ds+o \\ x(t) = \phi(t), \qquad t\in[-\overline{\tau},0] \\ z_x(t) = C_1x(t)+C_2x(t-\tau_1)+C_3 \int\limits_{t-\tau_3}^{t} h(x(s))ds \end{cases} \tag{1a-c}$$

with $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^n \in \Re^n$, where $x_i(t)$ are the master system's state vector associated with the ith neuron, and $z_x(t) \in \Re^s$ is the controlled output of the master network. $f(x(t)) = [f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t))]^T$, $g(x(t-\tau_1)) = [g_1(x_1(t-\tau_1)), g_2(x_2(t-\tau_1)), \ldots, g_n(x_n(t-\tau_1))]^T$, and $h(x(t)) = [h_1(x_1(t)), h_2(x_2(t)), \ldots, h_n(x_n(t))]^T$ denote the activation functions, $A = \mathrm{diag}\{a_i\} > 0$, the vector $o = [o_1, o_2, \ldots, o_n]^T$ is the constant external input, and the constant scalars $\tau_i \geq 0$, for $i = 1, 2, 3$, denote the known neutral, discrete, and distributed time delays, respectively, with $\overline{\tau} := \max\{\tau_1, \tau_2, \tau_3\}$. If all $\tau_i = 0$, the network (1) has no time delay. The time-varying vector-valued initial function $\phi(t)$ is a continuously differentiable functional.

Now, given the master signal $x(t) = x(t, \phi(t))$, we are to design a feasible coupling technique to realize the synchronization between two identical neural networks with different initial conditions. The slave neural network is described as follows:

$$\begin{cases} \dot{y}(t) = -Ay(t)+W_1 f(y(t))+W_2 g\,(y(t-\tau_1)) \\ \qquad\quad + W_3\dot{y}(t-\tau_2)+W_4 \int\limits_{t-\tau_3}^{t} h(y(s))\,ds+Ew(t)+u(t)+o \\ y(t) = \varphi(t), \qquad t\in[-\overline{\tau},0] \\ z_y(t) = C_1y(t)+C_2y(t-\tau_1)+C_3 \int\limits_{t-\tau_3}^{t} h(y(s))\,ds \end{cases} \tag{2a-c}$$

with $y(t) = [y_1(t), y_2(t), \ldots, y_n(t)]^n \in \Re^n$, where $y_i(t)$ are the slave system's state vector associated with the ith neuron; $u(t) \in \Re^n$ is a coupled term, which is considered as the control input; $w(t) \in \Re^q$ is the disturbance; $z_y(t) \in \Re^s$ is the controlled output of the slave network; and $\varphi(t)$ is a continuously differentiable functional.

Remark 1: The models (1) and (2) can describe a large amount of well-known dynamical systems with time delays, such as the delayed logistic model, the chaotic models with time delays, the artificial neural network model with discrete and distributed time delays, and the predator–prey model with distributed delays. In real application, these coupled systems can be regarded as interacting dynamical elements in the entire system, such as physical particles, biological neurons, ecological populations, and even automatic machines and robots. A feasible coupling design for successful synchronization leads us to fully command the intrinsic mechanism regulating the evolution of real systems, to fabricate emulate systems, and even to remotely control the machines and nodes in networks with large scales [6], [17], [20], [24], [53].

One can define a difference operator $\nabla : C([-\overline{\tau}, 0], \Re^n) \to \Re^n$ such that

$$\nabla x_t = x(t) - W_3\, x(t-\tau_2). \tag{3}$$

Definition 1 [22]: The difference operator ∇ is said to be stable if the zero solution of the homogeneous difference equation $\nabla x_t = 0$, $t \geq 0$, $x_0 = \Psi \in \{\Phi \in C([-\overline{\tau}, 0]) : \nabla\Phi = 0\}$, is uniformly asymptotically stable.

The stability of the difference operator ∇ is necessary for the stability of the MSNNs (1) and (2).

Assumption 1: It follows from [22] that a delay-independent sufficient condition for the asymptotic stability of the MSNNs (1) and (2) is that all the eigenvalues of the matrix W_3 are inside the unit circle, i.e., $\lambda_{\max}(W_3) < 1$.

Definition 2 [56]: The MSNNs (1) and (2) are globally exponentially synchronized if there exist scalars $\alpha > 0$ and $M \geq 1$ such that

$$|e(t)| \leq M e^{-\alpha t}\left[\|\zeta\| + \|\dot{\zeta}\|\right]$$

where $\zeta(t) \in C([-\overline{\tau}, 0]; \Re^n)$ is an initial condition, and $e(t) = x(t) - y(t)$ is the synchronization error such that α and M are called the exponential decay rate and the decay coefficient, respectively.

Let $\hat{e}(t) = e^{\alpha t} e(t)$. The error dynamics between the MSNNs (1) and (2), namely, the synchronization error network, can be expressed by

$$\begin{cases} \dot{\hat{e}}(t) = -(A - \alpha I)\hat{e}(t) + W_1 \hat{\psi}_1(\hat{e}(t)) \\ \qquad + W_2 e^{\alpha\tau_1}\hat{\psi}_2(\hat{e}(t-\tau_1)) - \alpha e^{\alpha\tau_2} W_3 \hat{e}(t-\tau_2) \\ \qquad + e^{\alpha\tau_2} W_3 \dot{\hat{e}}(t-\tau_2) + W_4 \int\limits_{t-\tau_3}^{t} e^{\alpha(t-s)}\hat{\psi}_3(\hat{e}(s))\, ds \\ \qquad - E\hat{w}(t) - \hat{u}(t) \\ \hat{e}(t) = \phi_e(t;\alpha), \qquad t \in [-\overline{\tau}, 0] \\ \hat{z}_e(t) = C_1 \hat{e}(t) + C_2 e^{\alpha\tau_1}\hat{e}(t-\tau_1) + C_3 \int\limits_{t-\tau_3}^{t} e^{\alpha(t-s)}\hat{\psi}_3(\hat{e}(s))\, ds \end{cases} \tag{4a-c}$$

where $\hat{u}(t) = e^{\alpha t}u(t)$, $\hat{w}(t) = e^{\alpha t}w(t)$, $\hat{z}_e(t) = \hat{z}_x(t) - \hat{z}_y(t) = e^{\alpha t}(z_x(t) - z_y(t))$, $\hat{\psi}_1(\hat{e}(t)) = [\hat{\psi}_{11}(\hat{e}_1(t)), \hat{\psi}_{12}(\hat{e}_2(t)), \ldots, \hat{\psi}_{1n}(\hat{e}_n(t))]^T$, $\hat{\psi}_2(\hat{e}(t-\tau_1)) = [\hat{\psi}_{21}(\hat{e}_1(t-\tau_1)), \hat{\psi}_{22}(\hat{e}_2(t-\tau_1)), \ldots, \hat{\psi}_{2n}(\hat{e}_n(t-\tau_1))]^T$, and $\hat{\psi}_3(\hat{e}(t)) = [\hat{\psi}_{31}(\hat{e}_1(t)), \hat{\psi}_{32}(\hat{e}_2(t)), \ldots, \hat{\psi}_{3n}(\hat{e}_n(t))]^T$ with $\hat{\psi}_{1i}(\hat{e}_i(t)) = e^{\alpha t}(f_i(x_i(t)) - f_i(y_i(t)))$, $\hat{\psi}_{2i}(\hat{e}_i(t)) = e^{\alpha t}(g_i(x_i(t)) - g_i(y_i(t)))$, $\hat{\psi}_{3i}(\hat{e}_i(t)) = e^{\alpha t}(h_i(x_i(t)) - h_i(y_i(t)))$, and $\phi_e(t;\alpha) = e^{\alpha t}(\phi(t) - \varphi(t))$.

In this paper, we make the following assumption for the neuron activation functions in (1) and (2), which is more general than the descriptions on the conventional sigmoid activation functions, as well as the recently popular Lipschitz-type activation functions.

Assumption 2 [27]: The nonlinear functions $f_i(s)$, $g_i(s)$, and $h_i(s)$, respectively, for any $i = 1, \ldots, n$, satisfy

$$f_i^- \leq \frac{f_i(s_1) - f_i(s_2)}{s_1 - s_2} \leq f_i^+$$
$$g_i^- \leq \frac{g_i(s_1) - g_i(s_2)}{s_1 - s_2} \leq g_i^+$$
$$h_i^- \leq \frac{h_i(s_1) - h_i(s_2)}{s_1 - s_2} \leq h_i^+$$

where f_i^-, f_i^+, g_i^-, g_i^+, h_i^-, and h_i^+ are some constants.

Remark 2: According to Assumption 2, one can easily check that, for any $i = 1, \ldots, n$, the functions $\hat{\psi}_{1i}(\hat{e}_i(t))$, $\hat{\psi}_{2i}(\hat{e}_i(t))$, and $\hat{\psi}_{3i}(\hat{e}_i(t))$, respectively, satisfy

$$f_i^- \leq \frac{\hat{\psi}_{1i}(\hat{e}_i(t))}{\hat{e}_i(t)} \leq f_i^+$$
$$g_i^- \leq \frac{\hat{\psi}_{2i}(\hat{e}_i(t))}{\hat{e}_i(t)} \leq g_i^+$$
$$h_i^- \leq \frac{\hat{\psi}_{3i}(\hat{e}_i(t))}{\hat{e}_i(t)} \leq h_i^+.$$

The problem to be addressed in this paper is formulated as follows: Given the delayed MSNNs (1) and (2) with a prescribed level of disturbance attenuation $\gamma > 0$, find a driving signal $\hat{u}(t)$ of the form

$$\hat{u}(t) = K_1\hat{e}(t) + K_2\hat{e}(t-\tau_1) + K_3 \int\limits_{t-\tau_3}^{t} \hat{e}(s)ds \tag{5}$$

where the matrices $\{K_i\}_{i=1}^3$ are the control gains to be determined such that

1) the synchronization error network (4) is globally exponentially stable;
2) under zero initial conditions and for all nonzero $w(t) \in L_2[0,\infty]$, the H_∞ performance measure, i.e., $J_\infty = \int_0^\infty [\hat{z}_e^T(t)\hat{z}_e(t) - \gamma^2\hat{w}^T(t)\hat{w}(t)]dt$, satisfies $J_\infty < 0$ (or the induced L_2-norm of the operator form $\hat{w}(t)$ to the controlled outputs $\hat{z}_e(t)$ is less than γ).

In this case, the MSNNs (1) and (2) are said to be asymptotically stable with an H_∞ performance measure.

Remark 3: The delay-dependent coupling (5) utilizes the available information of the size of the discrete and distributed delays. However, in many real applications, if the information of the size of the delays is not available for feedback, a memoryless coupling, i.e., $\hat{u}(t) = K_1\hat{e}(t)$, will be designed to synchronize the master and slave systems. Recently, in [52], global synchronization has been given for an array of coupled delayed NNs with a linear diffusive hybrid coupling, containing constant discrete and distributed delay coupling. In comparison, our model extends the model structure in [52] to MSNNs with a hybrid coupling, containing constant, neutral, discrete, and distributed delay coupling.

III. Main Results

In this section, we present our new sufficient conditions for the solvability of the problem of the delayed state-feedback control design using the Lyapunov method and an LMI approach.

Theorem 1: Let $h_i = \tau_i/N$, $i = 1, 2$, be given for any positive integer N. Under Assumptions 1 and 2, a state feedback controller given in (5) exists such that the controlled slave system (2) exponentially synchronizes with the master system (1) with the H_∞ performance level $\gamma > 0$ and an exponential decay rate $\alpha > 0$, if there exist some scalars δ, σ_i, ρ_i, and λ_i $(i = 1, 2, \ldots, N)$, matrices P_2, L_1, L_2, L_3, Q_i, S_i, H_i, $R_{i,j} = R_{i,j}^T$, and $T_{i,j} = T_{i,j}^T$ $(i, j = 0, 1, \ldots, N)$, and positive-definite matrices P_1, Z_1, Z_2, U_1, U_2, $\overline{U}_1$, and $\overline{U}_2$ satisfying the following LMIs:

$$\Sigma := \begin{bmatrix} P_1 & \tilde{Q} \\ * & \tilde{R} + \tilde{S} \end{bmatrix} > 0 \tag{6a}$$

$$\begin{bmatrix} \overline{U}_1 & -\overline{U}_1 \\ * & S_d \end{bmatrix} > 0 \tag{6b}$$

$$\begin{bmatrix} \overline{U}_2 & -\overline{U}_2 \\ * & H_d \end{bmatrix} > 0 \tag{6c}$$

$$\Pi := \begin{bmatrix} \hat{\Xi}_e & D^s & O^s & D^a & O^a \\ * & -S_d - R_{ds} & 0 & 0 & 0 \\ * & * & -H_d - T_{ds} & 0 & 0 \\ * & * & * & -3\overline{U}_1 & 0 \\ * & * & * & * & -3\overline{U}_2 \end{bmatrix} < 0 \tag{6d}$$

where $\tilde{R} = [R_{i-1,j-1}]_{i,j=1,2,\ldots,N+1}$, $R_{ds} = h_1[R_{i-1,j-1} - R_{i,j}]_{i,j=1,2,\ldots,N}$, and $T_{ds} = h_2[T_{i-1,j-1} - T_{i,j}]_{i,j=1,2,\ldots,N}$. $\hat{\Xi}_e$, Σ_{11}, Σ_{15}, and Σ_{55} are expressed in the equations shown at the bottom of the page, and

$$D^s = h_1 \begin{bmatrix} 2Q_1^a + R_{0,1}^s & 2Q_2^a + R_{0,2}^s & \cdots & 2Q_N^a + R_{0,N}^s \\ Q_1^s & Q_2^s & \cdots & Q_N^s \\ -R_{N,1}^s & -R_{N,2}^s & \cdots & -R_{N,N}^s \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$D^a = h_1 \begin{bmatrix} R_{0,1}^a & R_{0,2}^a & \cdots & R_{0,N}^a \\ Q_1^a & Q_2^a & \cdots & Q_N^a \\ -R_{N,0}^a & -R_{N,1}^a & \cdots & -R_{N,N-1}^a \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$O^s = h_2 \begin{bmatrix} T_{0,1}^s & T_{0,2}^s & \cdots & T_{0,N}^s \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ T_{N,1}^s & T_{N,2}^s & \cdots & T_{N,N}^s \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$O^a = h_2 \begin{bmatrix} T_{0,1}^a & T_{0,2}^a & \cdots & T_{0,N}^a \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ T_{N,0}^a & T_{N,1}^a & \cdots & T_{N,N-1}^a \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

with $S_d = \mathrm{diag}\{S_0 - S_1, S_1 - S_2, \ldots, S_{N-1} - S_N\}$, $H_d = \mathrm{diag}\{H_0 - H_1, H_1 - H_2, \ldots, H_{N-1} - H_N\}$, $\tilde{Q} = [Q_0, Q_1, \ldots, Q_N]$, $\tilde{S} = 1/h_1\, \mathrm{diag}\{S_0, S_1, \ldots, S_N\}$, $Q_p^s = (Q_p + Q_{p-1})/2$, $Q_p^a = (Q_p - Q_{p-1})/2$, $R_{p,q}^s = (R_{p,q} + R_{p,q-1})/2$, $R_{p,q}^a = (R_{p,q} - R_{p,q-1})/2$, $T_{p,q}^s = (T_{p,q} + T_{p,q-1})/2$, $T_{p,q}^a = (T_{p,q} - T_{p,q-1})/2$, $F^+ = \mathrm{diag}\{f_1^+, f_2^+, \ldots, f_N^+\}$, $G^+ = \mathrm{diag}\{g_1^+, g_2^+, \ldots, g_N^+\}$, $H^+ = \mathrm{diag}\{h_1^+, h_2^+, \ldots, h_N^+\}$, $F^- = \mathrm{diag}\{f_1^-, f_2^-, \ldots, f_N^-\}$, $G^- = \mathrm{diag}\{g_1^-, g_2^-, \ldots, g_N^-\}$, $H^- = \mathrm{diag}\{h_1^-, h_2^-, \ldots, h_N^-\}$, $\Lambda_1 = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_N\}$, $\Lambda_2 = \mathrm{diag}\{\rho_1, \rho_2, \ldots, \rho_N\}$, and $\Lambda_3 = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$. The decay coefficient can be calculated by

$$M = \sqrt{\frac{\max\{\Delta_1, \Delta_2\}}{\lambda_{\min}(\Sigma)}}$$

with $\Delta_1 = \lambda_{\max}(P_1) + \tau_1 \lambda_{\max}(Q_p)_{p=0}^N + \tau_1 \lambda_{\max}(G^{+T} Z_1 G^+) + \tau_1\, \lambda_{\max}(S_p)_{p=0}^N + \tau_1^2 \lambda_{\max}(R_{p,p})_{p=0}^N + \tau_2\, \lambda_{\max}\,(H_p)_{p=0}^N + \tau_2^2 \lambda_{\max}(T_{p,p})_{p=0}^N + (1/2)\tau_3^3(\lambda_{\max}(H^{+T} Z_2 H^+) + \lambda_{\max}(U_2))$, and $\Delta_2 = \tau_2 \lambda_{\max}(U_1)$. Moreover, the controller gains in (5) can be designed as $K_i = (P_2^T)^{-1} L_i$ $(i = 1, 2, 3)$.

Proof: To prove the theorem, choose a Lyapunov–Krasovskii functional (LKF) candidate as

$$V(t) = \sum_{i=1}^{6} V_i(t) \tag{7}$$

where

$$V_1(t) = \hat{e}(t)^T P_1 \hat{e}(t) + 2\hat{e}(t)^T \int_{-\tau_1}^{0} Q(\xi)\hat{e}(t+\xi)\, d\xi \tag{8a}$$

$$V_2(t) = \int_{t-\tau_1}^{t} \hat{\psi}_2(\hat{e}(s))^T Z_1 \hat{\psi}_2(\hat{e}(s))\, ds + \int_{t-\tau_2}^{t} \dot{\hat{e}}(s)^T U_1 \dot{\hat{e}}(s) ds \tag{8b}$$

$$V_3(t) = \int_{-\tau_1}^{0} \hat{e}(t+\xi)^T S(\xi) \hat{e}(t+\xi)\, d\xi + \int_{-\tau_1}^{0}\int_{-\tau_1}^{0} \hat{e}(t+s)^T R(s,\xi) \hat{e}(t+\xi)\, ds\, d\xi \tag{8c}$$

$$V_4(t) = \int_{-\tau_2}^{0} \hat{e}(t+\xi)^T H(\xi) \hat{e}(t+\xi)\, d\xi + \int_{-\tau_2}^{0}\int_{-\tau_2}^{0} \hat{e}(t+s)^T T(s,\xi) \hat{e}(t+\xi)\, ds\, d\xi \tag{8d}$$

$$\hat{\Xi}_e = \begin{bmatrix} \Sigma_{11} & \begin{bmatrix} -L_2 - Q_N + e^{\alpha\tau_1} C_1^T C_2 \\ -\delta L_2 \end{bmatrix} & \begin{bmatrix} -\alpha P_2^T W_3 e^{\alpha\tau_2} \\ -\alpha\delta P_2^T W_3 e^{\alpha\tau_2} \end{bmatrix} & \begin{bmatrix} P_2^T W_3 e^{\alpha\tau_2} \\ \delta P_2^T W_3 e^{\alpha\tau_2} \end{bmatrix} & \begin{bmatrix} -L_3 \\ -\delta L_3 \end{bmatrix} & \Sigma_{15} \\ * & -S_N + e^{2\alpha\tau_1} C_2^T C_2 & 0 & 0 & 0 & 0 \\ * & * & -H_N & 0 & 0 & 0 \\ * & * & * & -U_1 & 0 & 0 \\ * & * & * & * & -U_2 & 0 \\ * & * & * & * & * & \Sigma_{55} \end{bmatrix}$$

$$\Sigma_{11} = \mathrm{sym}\left(\begin{bmatrix} -P_2^T(A-\alpha I) - L_1 & P_1 - P_2^T \\ -\delta P_2^T(A-\alpha I) - \delta L_1 & -\delta P_2^T \end{bmatrix}\right) + \mathrm{diag}\left\{\mathrm{sym}(Q_0) + S_0 + H_0 + \tau_3^2 U_2 - F^+\Lambda_1 F^- - G^+\Lambda_2 G^- - H^+\Lambda_3 H^-, U_1\right\}$$

$$\Sigma_{15} = \begin{bmatrix} \begin{bmatrix} P_2^T W_1 + \frac{1}{2}(F^+ + F^-)\Lambda_1 \\ \delta P_2^T W_1 \end{bmatrix} & \begin{bmatrix} \frac{1}{2}(G^+ + G^-)\Lambda_2 \\ 0 \end{bmatrix} & \begin{bmatrix} \frac{1}{2}(H^+ + H^-)\Lambda_3 \\ 0 \end{bmatrix} & \begin{bmatrix} P_2^T W_4 + C_1^T C_3 \\ \delta P_2^T W_4 \end{bmatrix} & \begin{bmatrix} P_2^T W_2 e^{\alpha\tau_1} \\ \delta P_2^T W_2 e^{\alpha\tau_1} \end{bmatrix} & \begin{bmatrix} -P_2^T E \\ -\delta P_2^T E \end{bmatrix} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Sigma_{55} = \mathrm{diag}\left\{-\Lambda_1, Z_1 - \Lambda_2, \tau_3^2 Z_2 - \Lambda_3, -e^{-2\alpha\tau_3} Z_2 + C_3^T C_3, -Z_1, -\gamma^2 I\right\}$$

$$V_5(t)=\int_{t-\tau_3}^{t}\left[\int_{s}^{t}\hat{\psi}_3\left(\hat{e}(\theta)\right)^T d\theta\right] Z_2\left[\int_{s}^{t}\hat{\psi}_3\left(\hat{e}(\theta)\right)d\theta\right] ds + \int_{0}^{\tau_3}\int_{t-s}^{t}(\theta-t+s)\hat{\psi}_3(\hat{e}(\theta))^T Z_2\hat{\psi}_3(\hat{e}(\theta))\,d\theta\,ds \tag{8e}$$

$$V_6(t)=\int_{t-\tau_3}^{t}\left[\int_{s}^{t}\hat{e}(\theta)^T d\theta\right] U_2\left[\int_{s}^{t}\hat{e}(\theta)\,d\theta\right] ds + \int_{0}^{\tau_3}\int_{t-s}^{t}(\theta-t+s)\hat{e}(\theta)^T U_2\hat{e}(\theta)\,d\theta\,ds \tag{8f}$$

where $Q(\xi)$, $R(s,\xi)=R(s,\xi)^T$, $S(\xi)=S(\xi)^T$, $T(s,\xi)=T(s,\xi)^T$, and $H(\xi)=H(\xi)^T$ are continuous matrix functions.

Derivatives of $V_i(t)$, $i=1,\ldots,5$, are given, respectively, by

$$\begin{aligned}\dot{V}_1(t)&=2\dot{\hat{e}}(t)^T\left[P_1\hat{e}(t)+\int_{-\tau_1}^{0}Q(\xi)\hat{e}(t+\xi)\,d\xi\right] + 2\hat{e}(t)^T\int_{-\tau_1}^{0}Q(\xi)\dot{\hat{e}}(t+\xi)\,d\xi\\ &=2\dot{\hat{e}}(t)^T\left[P_1\hat{e}(t)+\int_{-\tau_1}^{0}Q(\xi)\hat{e}(t+\xi)\,d\xi\right] + 2\hat{e}(t)^T\int_{-\tau_1}^{0}Q(\xi)\dot{\hat{e}}(t+\xi)\,d\xi\end{aligned} \tag{9a}$$

$$\dot{V}_2(t)=\hat{\psi}_2(\hat{e}(t))^T Z_1\hat{\psi}_2\left(\hat{e}(t)\right)+\dot{\hat{e}}(t)^T U_1\dot{\hat{e}}(t)-\hat{\psi}_2(\hat{e}(t-\tau_1))^T \times Z_1\hat{\psi}_2\left(\hat{e}(t-\tau_1)\right)-\dot{\hat{e}}(t-\tau_2)^T U_1\dot{\hat{e}}(t-\tau_2) \tag{9b}$$

$$\dot{V}_3(t)=2\int_{-\tau_1}^{0}\dot{\hat{e}}(t+\xi)^T S(\xi)\hat{e}(t+\xi)\,d\xi + 2\int_{-\tau_1}^{0}\int_{-\tau_1}^{0}\dot{\hat{e}}(t+s)^T R(s,\xi)\hat{e}(t+\xi)\,ds\,d\xi \tag{9c}$$

$$\dot{V}_4(t)=2\int_{-\tau_2}^{0}\dot{\hat{e}}(t+\xi)^T H(\xi)\hat{e}(t+\xi)\,d\xi + 2\int_{-\tau_2}^{0}\int_{-\tau_2}^{0}\dot{\hat{e}}(t+s)^T T(s,\xi)\hat{e}(t+\xi)\,ds\,d\xi \tag{9d}$$

$$\begin{aligned}\dot{V}_5(t)=&-\left[\int_{t-\tau_3}^{t}\hat{\psi}_3\left(\hat{e}(\theta)\right)^T d\theta\right] Z_2\left[\int_{t-\tau_3}^{t}\hat{\psi}_3\left(\hat{e}(\theta)\right) d\theta\right]\\ &+2\int_{t-\tau_3}^{t}(\theta-t+\tau_3)\hat{\psi}_3\left(\hat{e}(t)\right)^T Z_2\hat{\psi}_3\left(\hat{e}(\theta)\right)d\theta\\ &+\int_{0}^{\tau_3}s\hat{\psi}_3\left(\hat{e}(t)\right)^T Z_2\hat{\psi}_3\left(\hat{e}(t)\right)ds\\ &-\int_{t-\tau_3}^{t}\int_{t-\theta}^{\tau_3}\hat{\psi}_3\left(\hat{e}(s)\right)^T Z_2\hat{\psi}_3\left(\hat{e}(s)\right)ds d\theta\\ \le&\int_{t-\tau_3}^{t}(\theta-t+\tau_3)\left[\hat{\psi}_3\left(\hat{e}(t)\right)^T Z_2\hat{\psi}_3\left(\hat{e}(t)\right) + \hat{\psi}_3\left(\hat{e}(\theta)\right)^T Z_2\hat{\psi}_3\left(\hat{e}(\theta)\right)\right]d\theta\\ &-\left[\int_{t-\tau_3}^{t}\hat{\psi}_3\left(\hat{e}(\theta)\right)^T d\theta\right] Z_2\left[\int_{t-\tau_3}^{t}\hat{\psi}_3\left(\hat{e}(\theta)\right) d\theta\right]\\ &+\int_{0}^{\tau_3}s\hat{\psi}_3\left(\hat{e}(t)\right)^T Z_2\hat{\psi}_3\left(\hat{e}(t)\right)ds\\ &-\int_{t-\tau_3}^{t}(\theta-t+\tau_3)\hat{\psi}_3\left(\hat{e}(\theta)\right)^T Z_2\hat{\psi}_3\left(\hat{e}(\theta)\right)d\theta\\ =&\,\tau_3^2\hat{\psi}_3\left(\hat{e}(t)\right)^T Z_2\hat{\psi}_3\left(\hat{e}(t)\right)\\ &-\left[\int_{t-\tau_3}^{t}\hat{\psi}_3\left(\hat{e}(\theta)\right)^T d\theta\right] Z_2\left[\int_{t-\tau_3}^{t}\hat{\psi}_3\left(\hat{e}(\theta)\right) d\theta\right]\\ \le&\,\tau_3^2\hat{\psi}_3\left(\hat{e}(t)\right)^T Z_2\hat{\psi}_3\left(\hat{e}(t)\right)\\ &-e^{-2\alpha\tau_3}\left[\int_{t-\tau_3}^{t}e^{\alpha(t-\theta)}\hat{\psi}_3\left(\hat{e}(\theta)\right)^T d\theta\right] Z_2\\ &\times\left[\int_{t-\tau_3}^{t}e^{\alpha(t-\theta)}\hat{\psi}_3\left(\hat{e}(\theta)\right) d\theta\right]\end{aligned} \tag{9e}$$

and similarly

$$\dot{V}_6(t)\le\tau_3^2\hat{e}(t)^T U_2\hat{e}(t)-\left[\int_{t-\tau_3}^{t}\hat{e}(\theta)^T d\theta\right] U_2\left[\int_{t-\tau_3}^{t}\hat{e}(\theta)\,d\theta\right]. \tag{9f}$$

According to Remark 2, we have

$$-\left(\hat{\psi}_{1i}\left(\hat{e}_i(t)\right)-f_i^+\hat{e}_i(t)\right)^T\left(\hat{\psi}_{1i}\left(\hat{e}_i(t)\right)-f_i^-\hat{e}_i(t)\right)\ge 0 \tag{10a}$$

$$-\left(\hat{\psi}_{2i}\left(\hat{e}_i(t-\tau_1)\right)-g_i^+\hat{e}_i(t-\tau_1)\right)^T \times\left(\hat{\psi}_{2i}\left(\hat{e}_i(t-\tau_1)\right)-g_i^-\hat{e}_i(t-\tau_1)\right)\ge 0 \tag{10b}$$

$$-\left(\hat{\psi}_{3i}\left(\hat{e}_i(t)\right)-h_i^+\hat{e}_i(t)\right)^T\left(\hat{\psi}_{3i}\left(\hat{e}_i(t)\right)-h_i^-\hat{e}_i(t)\right)\ge 0 \tag{10c}$$

which are, respectively, equivalent to

$$\overleftrightarrow{\psi}_1(t)^T\Delta_{f_i}\overleftrightarrow{\psi}_1(t)\ge 0 \tag{11a}$$

$$\overleftrightarrow{\psi}_2(t-\tau_1)^T\Delta_{g_i}\overleftrightarrow{\psi}_2(t-\tau_1)\ge 0 \tag{11b}$$

$$\overleftrightarrow{\psi}_3(t)^T\Delta_{h_i}\overleftrightarrow{\psi}_3(t)\ge 0 \tag{11c}$$

where $\overleftrightarrow{\psi}_i(t):=[\hat{e}(t)^T,\hat{\psi}_i(\hat{e}(t))^T]^T$, and

$$\Delta_{f_i}:=\begin{bmatrix}-f_i^+f_i^-v_iv_i^T & \frac{f_i^++f_i^-}{2}v_iv_i^T\\ * & -v_iv_i^T\end{bmatrix}$$

$$\Delta_{g_i}:=\begin{bmatrix}-g_i^+g_i^-v_iv_i^T & \frac{g_i^++g_i^-}{2}v_iv_i^T\\ * & -v_iv_i^T\end{bmatrix}$$

$$\Delta_{h_i}:=\begin{bmatrix}-h_i^+h_i^-v_iv_i^T & \frac{h_i^++h_i^-}{2}v_iv_i^T\\ * & -v_iv_i^T\end{bmatrix}.$$

Moreover, from (4) and (5), the following equation holds for any matrices P_2 and P_3 with appropriate dimensions:

$$
\begin{aligned}
&2\left(\hat{e}(t)^T P_2^T + \dot{\hat{e}}(t)^T P_3^T\right)\\
&\quad\times\Bigg(-\dot{\hat{e}}(t) - (A+K_1-\alpha I)\hat{e}(t) - K_2\hat{e}(t-\tau_1) + W_1\hat{\psi}_1\left(\hat{e}(t)\right)\\
&\qquad + W_2 e^{\alpha\tau_1}\hat{\psi}_2\left(\hat{e}(t-\tau_1)\right) - \alpha W_3 e^{\alpha\tau_2}\hat{e}(t-\tau_2)\\
&\qquad + W_3 e^{\alpha\tau_2}\dot{\hat{e}}(t-\tau_2) - K_3\int_{t-\tau_3}^{t}\hat{e}(s)ds\\
&\qquad + W_4\int_{t-\tau_3}^{t} e^{\alpha(t-s)}\hat{\psi}_3\left(\hat{e}(s)\right)ds - E\hat{w}(t)\Bigg) = 0. \qquad (12)
\end{aligned}
$$

Using the obtained derivative terms in (9) and adding the left-hand sides of (11) and (12) into, we obtain the following result for $\dot{V}(t)$:

$$
\begin{aligned}
\dot{V}(t) \le\ & \chi^T(t)\Xi\chi(t) + 2\dot{\hat{e}}(t)^T\int_{-\tau_1}^{0} Q(\xi)\hat{e}(t+\xi)d\xi\\
& - 2\dot{\hat{e}}(t)^T\int_{-\tau_3}^{0} P_3^T K_3\hat{e}(t+\xi)d\xi\\
& - 2\hat{e}(t-\tau_1)^T\int_{-\tau_1}^{0} R(-\tau_1,\xi)\hat{e}(t+\xi)d\xi\\
& - 2\hat{e}(t-\tau_2)^T\int_{-\tau_2}^{0} T(-\tau_2,\xi)\hat{e}(t+\xi)d\xi\\
& - \dot{\hat{e}}(t-\tau_2)^T U_1\dot{\hat{e}}(t-\tau_2) - \int_{-\tau_1}^{0}\int_{-\tau_1}^{0}\hat{e}(t+s)^T\\
& \times\left(\frac{\partial}{\partial s}R(s,\xi) + \frac{\partial}{\partial\xi}R(s,\xi)\right)\hat{e}(t+\xi)dsd\xi\\
& - \int_{-\tau_1}^{0}\hat{e}(t+\xi)^T\dot{S}(\xi)\hat{e}(t+\xi)d\xi\\
& + 2\hat{e}(t)^T\int_{-\tau_1}^{0}\left(R(0,\xi) - \dot{Q}(\xi)\right)\hat{e}(t+\xi)d\xi\\
& - 2\hat{e}(t)^T\int_{-\tau_3}^{0} P_2^T K_3\hat{e}(t+\xi)d\xi\\
& + 2\hat{e}(t)^T\int_{-\tau_2}^{0} T(0,\xi)\hat{e}(t+\xi)d\xi\\
& - \int_{-\tau_2}^{0}\hat{e}(t+\xi)^T\dot{H}(\xi)\hat{e}(t+\xi)d\xi\\
& - \left[\int_{t-\tau_3}^{t}\hat{e}(\theta)^T d\theta\right] U_2\left[\int_{t-\tau_3}^{t}\hat{e}(\theta)\,d\theta\right]\\
& - \int_{-\tau_2}^{0}\int_{-\tau_2}^{0}\hat{e}(t+s)^T\left(\frac{\partial}{\partial s}T(s,\xi) + \frac{\partial}{\partial\xi}T(s,\xi)\right)\\
& \times\hat{e}(t+\xi)dsd\xi + \hat{\psi}_2\left(\hat{e}(t)\right)^T Z_1\hat{\psi}_2\left(\hat{e}(t)\right)\\
& - \hat{\psi}_2\left(\hat{e}(t-\tau_1)\right)^T Z_1\hat{\psi}_2\left(\hat{e}(t-\tau_1)\right)\\
& + \tau_3^2\hat{\psi}_3(\hat{e}(t))^T Z_2\hat{\psi}_3\left(\hat{e}(t)\right) + 2\left(\hat{e}(t)^T P_2^T + \dot{\hat{e}}(t)^T P_3^T\right)\\
& \times\Bigg(W_1\hat{\psi}_1(\hat{e}(t)) + W_2 e^{\alpha\tau_1}\hat{\psi}_2(\hat{e}(t-\tau_1)) + W_3 e^{\alpha\tau_2}\dot{\hat{e}}(t-\tau_2)\\
& \qquad + W_4\int_{t-\tau_3}^{t} e^{\alpha(t-s)}\hat{\psi}_3\left(\hat{e}(s)\right)ds - E\hat{w}(t)\Bigg)\\
& - e^{-2\alpha\tau_3}\left[\int_{t-\tau_3}^{t} e^{\alpha(t-\theta)}\hat{\psi}_3\left(\hat{e}(\theta)\right)^T d\theta\right] Z_2\\
& \times\left[\int_{t-\tau_3}^{t} e^{\alpha(t-\theta)}\hat{\psi}_3\left(\hat{e}(\theta)\right)d\theta\right] + \sum_{i=1}^{n}\sigma_i\overleftrightarrow{\psi}_1(t)^T\Delta_{f_i}\overleftrightarrow{\psi}_1(t)\\
& + \sum_{i=1}^{n}\rho_i\overleftrightarrow{\psi}_2(t)^T\Delta_{g_i}\overleftrightarrow{\psi}_2(t) + \sum_{i=1}^{n}\lambda_i\overleftrightarrow{\psi}_3(t)^T\Delta_{h_i}\overleftrightarrow{\psi}_3(t)
\end{aligned}
\qquad (13)
$$

where $\chi(t) := \mathrm{col}\{\hat{e}(t), \dot{\hat{e}}(t), \hat{e}(t-\tau_1), \hat{e}(t-\tau_2)\}$, and

$$
\Xi = \begin{bmatrix}
\hat{\Sigma}_{11} & \begin{bmatrix} -P_2^T K_2 - Q(-\tau_1) \\ -P_3^T K_2 \end{bmatrix} & \begin{bmatrix} -\alpha e^{\alpha\tau_2}P_2^T W_3 \\ -\alpha e^{\alpha\tau_2}P_3^T W_3 \end{bmatrix}\\
* & -S(-\tau_1) & 0\\
* & * & -H(-\tau_2)
\end{bmatrix} \qquad (14)
$$

with

$$
\begin{aligned}
\hat{\Sigma}_{11} = \mathrm{sym}&\left(P^T\begin{bmatrix} 0 & I \\ -(A+K_1-\alpha I) & -I \end{bmatrix}\right)\\
&+ \mathrm{diag}\left\{\mathrm{sym}\left(Q(0)\right) + S(0) + H(0) + \tau_3^2 U_2, U_1\right\}
\end{aligned}
$$

where $P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}$.

According to the discretization technique in [23] and [24], the delay intervals $[-\tau_1, 0]$ and $[-\tau_2, 0]$, respectively, are divided into N segments $[\theta_p, \theta_{p-1}]$ and $[\hat{\theta}_p, \hat{\theta}_{p-1}]$, $p = 1, \ldots, N$, of equal length (or uniform mesh case), i.e., $h_i = \tau_i/N$, $i = 1, 2$, where $\theta_p = -ph_1$ and $\hat{\theta}_p = -ph_2$. For instance, this scheme divides the square $[-\tau_1, 0]\times[-\tau_1, 0]$ into $N\times N$ small squares $[\theta_p, \theta_{p-1}]\times[\theta_q, \theta_{q-1}]$, and each small square is further divided into two triangles. It is easily seen using [24, Lemma 7.7] that, although the LKF candidate for the nonuniform mesh case is no more complicated than the uniform mesh case, it is not the case for the LKF derivative condition. In addition, a uniform mesh is not possible for the incommensurate delay case and is not practical in the case of commensurate delays with a small common factor. In the sequel, $Q(\cdot)$, $S(\cdot)$, $H(\cdot)$, $R(\cdot,\cdot)$, and $T(\cdot,\cdot)$ are chosen to be piecewise linear. i.e., $Q(\theta_p + \kappa h_1) = (1-\kappa)Q_p + \kappa Q_{p-1}$, $S(\theta_p + \kappa h_1) = (1-\kappa)S_p + \kappa S_{p-1}$, and $H(\hat{\theta}_p + \kappa h_2) = (1-\kappa)H_p + \kappa H_{p-1}$, where

$$
\begin{aligned}
&R(\theta_p + \kappa h_1, \theta_q + \beta h_1)\\
&\quad= \begin{cases} (1-\kappa)R_{pq} + \beta R_{p-1,q-1} + (\kappa-\beta)R_{p-1,q}, & \kappa \ge \beta\\ (1-\beta)R_{pq} + \kappa R_{p-1,q-1} + (\beta-\kappa)R_{p,q-1}, & \kappa < \beta \end{cases}\\
&T(\hat{\theta}_p + \kappa h_2, \hat{\theta}_q + \beta h_2)\\
&\quad= \begin{cases} (1-\kappa)T_{pq} + \beta T_{p-1,q-1} + (\kappa-\beta)T_{p-1,q}, & \kappa \ge \beta\\ (1-\beta)T_{pq} + \kappa T_{p-1,q-1} + (\beta-\kappa)T_{p,q-1}, & \kappa < \beta \end{cases}
\end{aligned}
$$

with $\dot{S}(\xi)=h_1^{-1}(S_{p-1}-S_p)$, $\dot{Q}(\xi)=h_1^{-1}(Q_{p-1}-Q_p)$, $\dot{H}(\xi)=h_2^{-1}(H_{p-1}-H_p)$, $(\partial/\partial\xi)R(\xi,\theta)+(\partial/\partial\theta)R(\xi,\theta)=h_1^{-1}(R_{p-1,q-1}-R_{p,q})$, and $(\partial/\partial\xi)T(\xi,\theta)+(\partial/\partial\theta)T(\xi,\theta)=h_2^{-1}(T_{p-1,q-1}-T_{p,q})$. Thus, one obtains

$$2\dot{\hat{e}}(t)^T\int_{-\tau_1}^{0}Q(\xi)\hat{e}(t+\xi)d\xi = 2\dot{\hat{e}}(t)^T\sum_{p=1}^{N}h_1\int_0^1\left[Q_p^s+(1-2\kappa)\ Q_p^a)\right]\hat{e}(t+\theta_p+\kappa h_1)d\kappa \tag{15a}$$

$$2\hat{e}(t-\tau_1)^T\int_{-\tau_1}^{0}R(-\tau_1,\xi)\hat{e}(t+\xi)d\xi = 2\hat{e}(t-\tau_1)^T\sum_{p=1}^{N}h_1\int_0^1\left[R_{N,p}^s+(1-2\kappa)R_{N,p-1}^a\right]\times\hat{e}(t+\theta_p+\kappa h_1)d\kappa \tag{15b}$$

$$2\hat{e}(t-\tau_2)^T\int_{-\tau_2}^{0}T(-\tau_2,\xi)\hat{e}(t+\xi)d\xi = 2\hat{e}(t-\tau_2)^T\sum_{p=1}^{N}h_2\int_0^1\left[T_{N,p}^s+(1-2\kappa)T_{N,p-1}^a\right]\times\hat{e}(t+\hat{\theta}_p+\kappa h_2)d\kappa \tag{15c}$$

$$\int_{-\tau_1}^{0}\hat{e}(t+\xi)^T\dot{S}(\xi)\hat{e}(t+\xi)d\xi = \sum_{p=1}^{N}\int_0^1\hat{e}(t+\theta_p+\kappa h_1)^T(S_{p-1}-S_p)\hat{e}(t+\theta_p+\kappa h_1)d\kappa \tag{15d}$$

$$\int_{-\tau_2}^{0}\hat{e}(t+\xi)^T\dot{H}(\xi)\hat{e}(t+\xi)d\xi = \sum_{p=1}^{N}\int_0^1\hat{e}(t+\hat{\theta}_p+\kappa h_2)^T(H_{p-1}-H_p)\hat{e}(t+\hat{\theta}_p+\kappa h_2)d\kappa \tag{15e}$$

$$\int_{-\tau_1}^{0}\int_{-\tau_1}^{0}\hat{e}(t+s)^T\left(\frac{\partial}{\partial s}R(s,\xi)+\frac{\partial}{\partial\xi}R(s,\xi)\right)\hat{e}(t+\xi)dsd\xi = h_1\sum_{q=1}^{N}\sum_{p=1}^{N}\int_0^1\hat{e}(t+\theta_p+\beta h_1)^T(R_{p-1,q-1}-R_{p,q})\times\hat{e}(t+\theta_p+\kappa h_1)d\kappa d\beta \tag{15f}$$

$$\int_{-\tau_2}^{0}\int_{-\tau_2}^{0}\hat{e}(t+s)^T\left(\frac{\partial}{\partial s}T(s,\xi)+\frac{\partial}{\partial\xi}T(s,\xi)\right)\hat{e}(t+\xi)dsd\xi = h_2\sum_{q=1}^{N}\sum_{p=1}^{N}\int_0^1\hat{e}(t+\hat{\theta}_p+\beta h_2)^T(T_{p-1,q-1}-T_{p,q})\times\hat{e}(t+\hat{\theta}_p+\kappa h_2)d\kappa d\beta \tag{15g}$$

$$2\hat{e}(t)^T\int_{-\tau_1}^{0}\left(-\dot{Q}(\xi)+R(0,\xi)\right)\hat{e}(t+\xi)d\xi = 2h_1\hat{e}(t)^T\sum_{p=1}^{N}\int_0^1\left(2Q_p^a+R_{0,p}^s+(1-2\kappa)R_{0,p}^a\right)\times\hat{e}(t+\theta_p+\kappa h_1)d\kappa \tag{15h}$$

$$2\hat{e}(t)^T\int_{-\tau_2}^{0}T(0,\xi)\hat{e}(t+\xi)d\xi = 2h_2\hat{e}(t)^T\sum_{p=1}^{N}\int_0^1\left(T_{0,p}^s+(1-2\kappa)T_{0,p}^a\right)\hat{e}(t+\hat{\theta}_p+\kappa h_2)d\kappa. \tag{15i}$$

Now, from (4) and (13)–(15), one has

$$\begin{aligned}\hat{z}_e(t)^T\hat{z}_e(t)-\gamma^2\hat{w}(t)^T\hat{w}(t)+\dot{V}(t) &\le \chi_e^T(t)\Bigg(\Xi_e+D^s\tilde{U}_1D^{s^T}+O^s\tilde{U}_2O^{s^T}\\ &\quad+\frac{1}{3}\left(D^a\tilde{U}_1D^{a^T}+O^a\tilde{U}_2O^{a^T}\right)\Bigg)\chi_e(t)\\ &\quad-\int_0^1\int_0^1\phi_e(\kappa;\alpha)^TR_{ds}\phi_e\ (s;\alpha)]\,d\kappa\,ds\\ &\quad-\int_0^1\int_0^1\hat{\phi}_e(\kappa;\alpha)^TT_{ds}\hat{\phi}_e(s;\alpha)d\kappa\,ds\\ &\quad-\int_0^1\phi_D(\kappa;\alpha)^T\Theta_1\phi_D(\kappa;\alpha)d\kappa\\ &\quad-\int_0^1\phi_O(\kappa;\alpha)^T\Theta_2\phi_O(\kappa;\alpha)d\kappa\end{aligned} \tag{16}$$

with $\chi_e(t)=\mathrm{col}\{\chi(t),\dot{\hat{e}}(t-\tau_2),\int_{t-\tau_3}^{t}\hat{e}(\theta)d\theta,\hat{\psi}_1(\hat{e}(t)),\hat{\psi}_2(\hat{e}(t)),\hat{\psi}_3(\hat{e}(t)),\int_{t-\tau_3}^{t}e^{\alpha(t-\theta)}\hat{\psi}_3(\hat{e}(\theta))\,d\theta,\ \hat{\psi}_2(\hat{e}(t-\tau_1)),\ \hat{w}(t)\}$, $\chi_D(t):=(D^s+(1-2\kappa)D^a)^T\chi_e(t)$, $\chi_O(t):=(O^s+(1-2\kappa)O^a)^T\chi_e(t)$, $\phi_D(\kappa;\alpha):=[\chi_D(t)^T,\phi_e(\kappa;\alpha)^T]^T$, $\phi_o(\kappa;\alpha):=[\chi_o(t)^T,\hat{\phi}_e(\kappa;\alpha)^T]^T$, $\phi_e(\kappa;\alpha)=\mathrm{col}\{\hat{e}(t+\theta_1+\kappa h_1),\hat{e}(t+\theta_2+\kappa h_1),\ldots,\hat{e}(t+\theta_N+\kappa h_1)\}$, $\hat{\phi}_e(\kappa;\alpha)=\mathrm{col}\{\hat{e}(t+\hat{\theta}_1+\kappa h_2),\hat{e}(t+\hat{\theta}_2+\kappa h_2),\ldots,\hat{e}(t+\hat{\theta}_N+\kappa h_2)\}$, and

$$\Theta_1:=\begin{bmatrix}\tilde{U}_1 & -I\\ * & S_d\end{bmatrix}\quad\Theta_2:=\begin{bmatrix}\tilde{U}_2 & -I\\ * & H_d\end{bmatrix}.$$

Ξ_e, $\tilde{\Sigma}_{11}$, and $\tilde{\Sigma}_{15}$ are expressed in the equations shown at the bottom of the next page.

Let $\zeta_i=\mathrm{diag}\{\tilde{U}_i,I\}$ with $\tilde{U}_i=\overline{U}_i^{-1}$. Premultiplying ζ_i and postmultiplying ζ_i^T to the LMIs (6b) and (6c), one obtains

$\Theta_i > 0$, $i = 1, 2$. Then, using the Jensen inequality (Lemma 1 in the Appendix) to the forth and fifth terms in (16), we have

$$\int_0^1 \phi_D(\kappa;\alpha)^T \Theta_1 \phi_D(\kappa;\alpha) d\kappa \geq [\, \chi_e^T(t) \quad \phi_e(\kappa;\alpha)^T \,] \begin{bmatrix} D^s \tilde{U}_1 D^{s^T} & -D^s \\ * & S_d \end{bmatrix} \begin{bmatrix} \chi_e(t) \\ \phi_e(\kappa;\alpha) \end{bmatrix}$$

$$\int_0^1 \phi_O(\kappa;\alpha)^T \Theta_2 \phi_O(\kappa;\alpha) d\kappa \geq [\, \chi_e^T(t) \quad \hat{\phi}_e(\kappa;\alpha)^T \,] \begin{bmatrix} O^s \tilde{U}_2 O^{s^T} & -O^s \\ * & H_d \end{bmatrix} \begin{bmatrix} \chi_e(t) \\ \hat{\phi}_e(\kappa;\alpha) \end{bmatrix}.$$

Using the preceding inequalities in (16), we conclude that

$$\hat{z}_e(t)^T \hat{z}_e(t) - \gamma^2 \hat{w}(t)^T \hat{w}(t) + \dot{V}(t) \leq \tilde{\chi}_e(t)^T \tilde{\Xi}_e \tilde{\chi}_e(t) \quad (17)$$

where $\tilde{\chi}_e(t) = [\chi_e(t)^T, \int_0^1 \phi_e(\kappa;\alpha)^T d\kappa, \int_0^1 \hat{\phi}_e(\kappa;\alpha)^T d\kappa]^T$, and

$$\tilde{\Xi}_e = \begin{bmatrix} \Xi_e + \frac{1}{3}\left(D^a \tilde{U}_1 D^{a^T} + O^a \tilde{U}_2 O^{a^T}\right) & -D^s & -O^s \\ * & -S_d - R_{ds} & 0 \\ * & * & -H_d - T_{ds} \end{bmatrix}. \quad (18)$$

On the other hand, for a prescribed $\gamma > 0$ and under zero initial conditions, J_∞ can be rewritten as

$$J_\infty \leq \int_0^\infty \left[\hat{z}_e(t)^T \hat{z}_e(t) - \gamma^2 \hat{w}(t)^T \hat{w}(t)\right] dt + V(t)|_{t\to\infty} - V(t)|_{t=0}$$
$$= \int_0^\infty \left[\hat{z}_e(t)^T \hat{z}_e(t) - \gamma^2 \hat{w}(t)^T \hat{w}(t) + \dot{V}(t)\right] dt \quad (19)$$

and the condition $\hat{z}_e(t)^T \hat{z}_e(t) - \gamma^2 \hat{w}(t)^T \hat{w}(t) + \dot{V}(t) < 0$ means that the condition $\tilde{\Xi}_e < 0$ satisfies the H_∞ performance measure, and by applying the Schur complement, one gets

$$\begin{bmatrix} \Xi_e & D^s & O^s & D^a & O^a \\ * & -S_d - R_{ds} & 0 & 0 & 0 \\ * & * & -H_d - T_{ds} & 0 & 0 \\ * & * & * & -3\overline{U}_1 & 0 \\ * & * & * & * & -3\overline{U}_2 \end{bmatrix} < 0. \quad (20)$$

Then, we choose $P_3 = \delta P_2$, $\delta \in R$, where δ is a tuning scalar parameter (which may be restrictive). Note that the matrix P_2 is nonsingular due to the fact that the only matrix that can be negative definite in the second block on the diagonal of (20) is $-\delta\,\mathrm{sym}(P_2) + U_1$. Therefore, considering $L_i = P_2^T K_i$, $i = 1, 2, 3$, results in the LMI (6d). Moreover, the condition $J_\infty < 0$ for $w(t) \equiv 0$ implies $\dot{V}(t) < 0$. Then, we have $V(t) < V(0)$. From (7) and (8), one gets

$$V(0) \leq \Delta_1 \|\zeta\|^2 + \Delta_2 \|\dot{\zeta}\|^2. \quad (21)$$

Moreover, from [24, Prop. 5.20], it is clear that the LKF condition $V(t) \geq \varepsilon |e(t)|^2$ is satisfied if the LMI (6a) is satisfied. In this case, we obtain $V(t) \geq e^{2\alpha t} \lambda_{\min}(\Sigma) |e(t)|^2$ with the matrix Σ in (6a). Therefore, we have

$$|e(t)|^2 \leq \frac{\max\{\Delta_1, \Delta_2\}}{\lambda_{\min}(\Sigma)} e^{-2\alpha t} \left[\|\zeta\|^2 + \|\dot{\zeta}\|^2\right].$$

That is

$$|e(t)| \leq \sqrt{\frac{\max\{\Delta_1, \Delta_2\}}{\lambda_{\min}(\Sigma)}} e^{-\alpha t} \left[\|\zeta\| + \|\dot{\zeta}\|\right]$$

which shows that the synchronization error network (4) with (5) is globally exponentially stable and has the exponential decay rate α. This completes the proof. ■

Remark 4: If we are interested in further simplification in the LMIs (6), the arbitrary matrices $\overline{U}_1$ and $\overline{U}_2$ can be eliminated from the inequality $\tilde{\Xi}_e < 0$ in (18), or equivalently from the LMI (6d) in Theorem 1, using [24, Prop. B6] to yield the following matrix inequality:

$$\begin{bmatrix} \Xi_e & D^s & O^s & D^a & O^a \\ * & -S_d - R_{ds} & 0 & 0 & 0 \\ * & * & -H_d - T_{ds} & 0 & 0 \\ * & * & * & -3S_d & 0 \\ * & * & * & * & -3H_d \end{bmatrix} < 0.$$

In this case, the LMIs (6b) and (6c) are eliminated from the conditions in Theorem 1. Therefore, it can be seen that the results in Theorem 1 are less conservative than those in [24].

The results given in Theorem 1 are derived for the MSNNs (1) and (2), where the delays τ_i are available. However, in many situations, the information on the delays is *a priori* unknown. In this case, it is assumed that $\tau_i \in [0, \overline{\tau}_i]$. Then, we have the following corollary (its proof is straightforward and hence omitted).

Corollary 1: Let $h_i = \overline{\tau}_i / N$, $i = 1, 2$, be given for any positive integer N. Under Assumptions 1 and 2, a memoryless coupling in the form $\hat{u}(t) = K_1 \hat{e}(t)$ exists such that the controlled

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$$\Xi_e = \begin{bmatrix} \tilde{\Sigma}_{11} & \begin{bmatrix} -P_2^T K_2 - Q_N + e^{\alpha\tau_1} C_1^T C_2 \\ -P_3^T K_2 \end{bmatrix} & \begin{bmatrix} -\alpha P_2^T W_3 e^{\alpha\tau_2} \\ -\alpha P_3^T W_3 e^{\alpha\tau_2} \end{bmatrix} & \begin{bmatrix} P_2^T W_3 e^{\alpha\tau_2} \\ P_3^T W_3 e^{\alpha\tau_2} \end{bmatrix} & \begin{bmatrix} -P_2^T K_3 \\ -P_3^T K_3 \end{bmatrix} & \tilde{\Sigma}_{15} \\ * & -S_N + e^{2\alpha\tau_1} C_2^T C_2 & 0 & 0 & 0 & 0 \\ * & * & -H_N & 0 & 0 & 0 \\ * & * & * & -U_1 & 0 & 0 \\ * & * & * & * & -U_2 & 0 \\ * & * & * & * & * & \Sigma_{55} \end{bmatrix}$$

$$\tilde{\Sigma}_{11} = \hat{\Sigma}_{11} + \mathrm{diag}\{-F^+ \Lambda_1 F^- - G^+ \Lambda_2 G^- - H^+ \Lambda_3 H^-, 0\}$$

$$\tilde{\Sigma}_{15} = \begin{bmatrix} \begin{bmatrix} P_2^T W_1 + \frac{1}{2}(F^+ + F^-)\Lambda_1 \\ P_3^T W_1 \end{bmatrix} & \begin{bmatrix} \frac{1}{2}(G^+ + G^-)\Lambda_2 \\ 0 \end{bmatrix} & \begin{bmatrix} \frac{1}{2}(H^+ + H^-)\Lambda_3 \\ 0 \end{bmatrix} & \begin{bmatrix} P_2^T W_4 + C_1^T C_3 \\ P_3^T W_4 \end{bmatrix} & \begin{bmatrix} P_2^T W_2 e^{\alpha\tau_1} \\ P_3^T W_2 e^{\alpha\tau_1} \end{bmatrix} & \begin{bmatrix} -P_2^T E \\ -P_3^T E \end{bmatrix} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

slave system (2) exponentially synchronizes with the master system (1) with the H_∞ performance level $\gamma > 0$ and an exponential decay rate $\alpha > 0$ for any $\tau_i \in [0, \overline{\tau}_i]$, if there exist some scalars δ, σ_i, ρ_i, and λ_i $(i = 1, 2, \ldots, N)$, matrices P_2, L_1, Q_i, S_i, H_i, $R_{i,j} = R_{i,j}^T$, and $T_{i,j} = T_{i,j}^T$ $(i, j = 0, 1, \ldots, N)$, and positive-definite matrices P_1, Z_1, Z_2, U_1, U_2, $\overline{U}_1$, and $\overline{U}_2$ satisfying the LMIs (6a)–(6c) and

$$\begin{bmatrix} \overline{\Xi}_e & D^s & O^s & D^a & O^a \\ * & -S_d - R_{ds} & 0 & 0 & 0 \\ * & * & -H_d - T_{ds} & 0 & 0 \\ * & * & * & -3\overline{U}_1 & 0 \\ * & * & * & * & -3\overline{U}_2 \end{bmatrix} < 0$$

where $\overline{\Xi}_e$, $\overline{\Sigma}_{11}$, $\overline{\Sigma}_{15}$, and $\overline{\Sigma}_{55}$ are expressed in the equations shown at the bottom of the page. The decay coefficient can be calculated by

$$M = \sqrt{\frac{\max\{\Delta_1, \Delta_2\}}{\lambda_{\min}(\Sigma)}}$$

where $\Delta_1 = \lambda_{\max}(P_1) + \overline{\tau}_1 \lambda_{\max}(Q_p)_{p=0}^N + \overline{\tau}_1 \lambda_{\max}(G^{+T} Z_1 G^+) + \overline{\tau}_1 \lambda_{\max}(S_p)_{p=0}^N + \overline{\tau}_1^2 \lambda_{\max}(R_{p,p})_{p=0}^N + \overline{\tau}_2 \lambda_{\max}(H_p)_{p=0}^N + \overline{\tau}_2^2 \lambda_{\max}(T_{p,p})_{p=0}^N + (1/2)\overline{\tau}_3^3 (\lambda_{\max}(H^{+T} Z_2 H^+) + \lambda_{\max}(U_2))$, and $\Delta_2 = \overline{\tau}_2 \lambda_{\max}(U_1)$. Moreover, the controller gains are given by $K_1 = (P_2^T)^{-1} L_1$.

Remark 5: In Theorem 1, we provide a new exponential H_∞ synchronization scheme for a class of MSNNs with delayed feedback control, including discrete and distributed delay terms, by using a general DLKF. The results are expressed within the framework of LMIs, which can easily be computed by the interior-pint method. It is also observed that the LMIs (6) are linear in the set of scalars δ, σ_i, ρ_i, and λ_i, matrices P_2, L_1, L_2, L_3, Q_i, S_i, H_i, $R_{i,j} = R_{i,j}^T$, $T_{i,j} = T_{i,j}^T$, P_1, Z_1, Z_2, U_1, U_2, $\overline{U}_1$, and $\overline{U}_2$ $(i, j = 0, 1, \ldots, N)$, and the scalar γ^2 for some given scalars N, α, τ_1, τ_2, and τ_3. Then, the optimal solution to obtain the minimum disturbance attenuation level, i.e., γ_{optimal}, can be found by solving the following convex optimization problem:

Min $\quad \lambda$

subject to the LMIs (6) with $\lambda := \gamma^2$.

IV. Uncertainty Characterization

In this section, we will discuss the uncertainty characterization for the MSNNs (1) and (2) with different neutral, discrete, and distributed delays.

A. Polytopic Uncertainty

The first class of uncertainty frequently encountered in practice is the polytopic uncertainty [24]. In this case, the matrices of the MSNNs (1) and (2) are not exactly known, except that they are within a compact set Ω, which is denoted by

$$\Omega = [\, A \quad W_1 \quad W_2 \quad W_3 \quad W_4 \quad E \,].$$

We assume that

$$\Omega = \sum_{j=1}^{q} s_j \Omega_j \tag{22}$$

for some scalars s_j, satisfying

$$0 \leq s_j \leq 1, \qquad \sum_{j=1}^{q} s_j = 1 \tag{23}$$

where the q vertices of the polytope are described by

$$\Omega_j = \left[\, A^{(j)} \quad W_1^{(j)} \quad W_2^{(j)} \quad W_3^{(j)} \quad W_4^{(j)} \quad E^{(j)} \,\right]. \tag{24}$$

To take into account the polytopic uncertainty in the exponential H_∞ synchronization problem of the MSNNs (1) and (2), we derive the following result from applying the same transformation that was used in deriving Theorem 1.

Theorem 2: Let $h_i = \tau_i / N$, $i = 1, 2$, be given for any positive integer N. Under Assumptions 1 and 2, if the uncertainty set Ω is polytopic with vertices Ω_j, $j = 1, 2, \ldots, q$, then the MSNNs described by (1) and (2) and (22)–(24) are globally exponentially stable with the H_∞ performance level $\gamma > 0$ and an exponential decay rate $\alpha > 0$, if there exist some scalars δ, σ_i, ρ_i, and λ_i $(i = 1, 2, \ldots, N)$, matrices P_2, L_1, L_2, L_3, Q_i, S_i, H_i, $R_{i,j} = R_{i,j}^T$, and $T_{i,j} = T_{i,j}^T$ $(i, j = 0, 1, \ldots, N)$, and

$$\overline{\Xi}_e = \begin{bmatrix} \overline{\Sigma}_{11} & \begin{bmatrix} e^{\alpha\tau_1} C_1^T C_2 - Q_N \\ 0 \end{bmatrix} & \begin{bmatrix} -\alpha P_2^T W_3 e^{\alpha\overline{\tau}_2} \\ -\alpha\delta P_2^T W_3 e^{\alpha\overline{\tau}_2} \end{bmatrix} & \begin{bmatrix} P_2^T W_3 e^{\alpha\overline{\tau}_2} \\ \delta P_2^T W_3 e^{\alpha\overline{\tau}_2} \end{bmatrix} & 0 & \overline{\Sigma}_{15} \\ * & -S_N + e^{2\alpha\overline{\tau}_1} C_2^T C_2 & 0 & 0 & 0 & 0 \\ * & * & -H_N & 0 & 0 & 0 \\ * & * & * & -U_1 & 0 & 0 \\ * & * & * & * & -U_2 & 0 \\ * & * & * & * & * & \overline{\Sigma}_{55} \end{bmatrix}$$

$$\overline{\Sigma}_{11} = \text{sym}\left(\begin{bmatrix} -P_2^T(A - \alpha I) - L_1 & P_1 - P_2^T \\ -\delta P_2^T (A - \alpha I) - \delta L_1 & -\delta P_2^T \end{bmatrix} \right) + \text{diag}\left\{ \text{sym}(Q_0) + S_0 + H_0 + \overline{\tau}_3^2 U_2 - F^+ \Lambda_1 F^- - G^+ \Lambda_2 G^- - H^+ \Lambda_3 H^-, U_1 \right\}$$

$$\overline{\Sigma}_{15} = \begin{bmatrix} \begin{bmatrix} P_2^T W_1 + \frac{1}{2}(F^+ + F^-)\Lambda_1 \\ \delta P_2^T W_1 \end{bmatrix} & \begin{bmatrix} \frac{1}{2}(G^+ + G^-)\Lambda_2 \\ 0 \end{bmatrix} & \begin{bmatrix} \frac{1}{2}(H^+ + H^-)\Lambda_3 \\ 0 \end{bmatrix} & \begin{bmatrix} P_2^T W_4 + C_1^T C_3 \\ \delta P_2^T W_4 \end{bmatrix} & \begin{bmatrix} P_2^T W_2 e^{\alpha\overline{\tau}_1} \\ \delta P_2^T W_2 e^{\alpha\overline{\tau}_1} \end{bmatrix} & \begin{bmatrix} -P_2^T E \\ -\delta P_2^T E \end{bmatrix} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overline{\Sigma}_{55} = \text{diag}\left\{ -\Lambda_1, Z_1 - \Lambda_2, \overline{\tau}_3^2 Z_2 - \Lambda_3, -e^{-2\alpha\overline{\tau}_3} Z_2 + C_3^T C_3, -Z_1, -\gamma^2 I \right\}$$

positive-definite matrices P_1, Z_1, Z_2, U_1, U_2, $\overline{U}_1$, and $\overline{U}_2$ such that the LMIs (6) are satisfied for all

$$\begin{aligned}&[A \quad W_1 \quad W_2 \quad W_3 \quad W_4 \quad E] \\ &\quad = \left[A^{(j)} \quad W_1^{(j)} \quad W_2^{(j)} \quad W_3^{(j)} \quad W_4^{(j)} \quad E^{(j)}\right], \\ &\qquad\qquad j = 1, 2, \ldots, q. \end{aligned} \tag{25}$$

Then, the controller gains in (5) are given by $K_i = (P_2^T)^{-1} L_i \ (i = 1, 2, 3)$.

Proof: It directly follows from the Proof of Theorem 1 and using properties of (22)–(24). ■

B. Norm-Bounded Uncertainty

There are also other uncertainties that cannot reasonably be modeled by a polytopic uncertainty set with a number of vertices. In such a case, it is assumed that the deviation of the system parameters of an uncertain system from their nominal values is norm bounded [24]. This kind of uncertainties often appears in modeled NNs mainly due to the modeling error, external disturbance, and parameter fluctuation during the implementation, and such deviations and perturbations are usually bounded. To reflect such a reality, consider the MSNNs (1) and (2) with

$$A + \Delta A(t), \quad W_i + \Delta W_i(t), \quad E + \Delta E(t) \tag{26}$$

where the time-varying structured uncertainties $\Delta A(t)$, $\Delta W_i(t)$, and $\Delta E(t)$ are said to be admissible if the following form holds:

$$[\Delta A(t) \ \ \Delta W_i(t) \ \ \Delta E(t)] = M_1 \Delta(t) \, [L_a \ \ L_{w_i} \ \ L_e] \tag{27}$$

where L_a, L_{w_i}, L_e are constant matrices with appropriate dimensions, and $\Delta(t)$ is an unknown, real, and possibly time-varying matrix with Lebesgue measurable elements, and its Euclidean norm satisfies

$$\|\Delta(t)\| \leq 1 \qquad \forall t. \tag{28}$$

In this section, we modify Assumption 1 to enable the application of the Lyapunov's method for the stability of the uncertain MSNNs (1) and (2) with (26)–(28).

Assumption 3: Let the difference operator $\nabla x_t = x(t) - (W_3 + \Delta W_3(t)) x(t - \tau_2)$ be delay-independently stable with respect to all delays, and a sufficient condition is that all the eigenvalues of the matrix $W_3 + \Delta W_3(t)$ lie inside the unit circle.

Theorem 3: Let $h_i = \tau_i / N$, $i = 1, 2$, be given for any positive integer N. Under Assumptions 2 and 3, the MSNNs described by (1) and (2) and admissible uncertainties (26)–(28) are globally exponentially stable with the H_∞ performance level $\gamma > 0$ and an exponential decay rate $\alpha > 0$, if there exist some scalars $\mu > 0$, δ, σ_i, ρ_i, and λ_i $(i = 1, 2, \ldots, N)$, matrices P_2, L_1, L_2, L_3, Q_i, S_i, H_i, $R_{i,j} = R_{i,j}^T$, and $T_{i,j} = T_{i,j}^T$ $(i, j = 0, 1, \ldots, N)$, and positive-definite matrices P_1, Z_1, Z_2, U_1, U_2, $\overline{U}_1$, and $\overline{U}_2$ such that the LMIs (6a)–(6c) and the following LMI are feasible:

$$\begin{bmatrix} \Pi & \Gamma_d & \mu\Gamma_e \\ * & -\mu I & 0 \\ * & * & -\mu I \end{bmatrix} < 0 \tag{29}$$

where $\Gamma_e = [-L_a, 0, 0, -\alpha e^{\alpha\tau_2} L_{w_3}, e^{\alpha\tau_2} L_{w_3}, L_{w_1}, 0, 0, L_{w_4}, L_{w_2}, L_e, \underbrace{0, \cdots, 0}_{(4N)\text{ elements}}]$, and $\Gamma_d = [M_1^T P_2, \ \delta M_1^T P_2, \ \underbrace{0, \cdots, 0}_{(4N+9)\text{ elements}}]^T$.

Then, the controller gains in (5) are given by $K_i = (P_2^T)^{-1} L_i \ (i = 1, 2, 3)$.

Proof: If the state-space matrices A, $W_1, \ldots, W_4$, and E in (6d) are replaced with $A + M_1 \Delta(t) L_a$, $W_1 + M_1 \Delta(t) L_{w_1}, \ldots, W_4 + M_1 \Delta(t) L_{w_4}$, and $E + M_1 \Delta(t) L_e$, respectively, then the inequality (6d) is equivalent to the following condition:

$$\Pi + \text{sym}\left(\Gamma_d^T \Delta(t) \, \Gamma_e\right) < 0. \tag{30}$$

By Lemma 2 (in the Appendix), a necessary and sufficient condition for (30) is that there exists a scalar $\mu > 0$ such that

$$\Pi + \mu^{-1} \Gamma_d^T \Gamma_d + \mu \Gamma_e^T \Gamma_e < 0. \tag{31}$$

Then, applying Schur complements, we find that (31) is equivalent to (29). ■

Remark 6: It can easily be seen that the results of this paper are quite different from existing results in [4] in the following perspectives: 1) The delayed NN structure in [4] considers NNs with time-varying discrete delays and, in comparison with our case, does not center on mixed time delays, i.e., the results in [4] cannot directly be applied to the NNs with different neutral, discrete, and distributed delays. 2) In [4], the authors design a control input associated with the state feedback to synchronize the MSNNs such as the elements of the gain matrix are determined by checking a certain Hamiltonian matrix if its eigenvalues lie on the imaginary axis or not. Furthermore, according to [4, Remark 3], it is not simple to find the analytical solutions (if they exist) for the condition of the main theorem in this reference. However, they can numerically be solved in almost all cases by an eigenvalue-solver MATLAB and a trial-and-error procedure, but in our case, the control input depends on the discrete and distributed delays, and the control gain matrices can be calculated by systematically solving some LMIs. Therefore, this algorithm is faster than the proposed algorithm in [4]. 3) In [4], using the inequality bounding technique [25] employed for all cross terms encountered in their analysis conditions may produce conservative results in comparison with the present paper.

Remark 7: The reduced conservatism of Theorems 1–3 benefit from the construction of the new DLKF in (8), introducing some free-weighting matrices to express the relationship among the system matrices, utilizing a general form of the activation functions, and neither the model transformation approach nor any bounding technique is needed to estimate the inner product of the involved crossing terms (see, for instance, [4]). It can easily be seen that the results of this paper are quite different from most existing results in the literature in the following perspective: Theoretically, the exponential synchronization problem of MSNNs with mixed neutral, discrete, and distributed delays is much more complicated, particularly for the case where the delays are different. In this paper, the derived sufficient conditions are convex and neutral-delay-dependent, discrete-delay-dependent, and distributed-delay-dependent, which make the treatment in this paper more general with less conservativism in comparison with most existing results in the literature, which are independent of the neutral or distributed delays (see, for instance, [4], [32], and [52]).

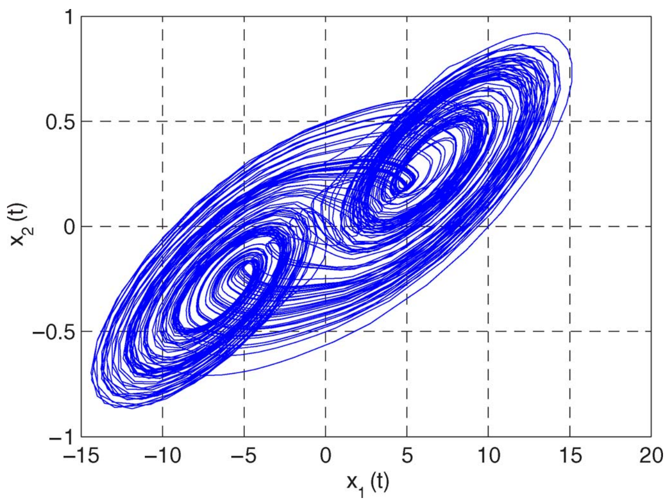

Fig. 1. x_1–x_2 plot.

TABLE I
γ_{optimal} COMPARISON W.R.T. N AND α

	$\alpha=0.5$	$\alpha=1$	$\alpha=1.5$
$N=1$	0.3065	0.3250	0.3345
$N=2$	0.2920	0.3205	0.3285
$N=3$	0.2895	0.3165	0.3215

V. NUMERICAL RESULTS

Let us consider the MSNNs (1) and (2) with the following matrices:

$$A = I_2 \qquad W_1 = \begin{bmatrix} 1+\frac{\pi}{4} & 20 \\ 0.1 & 1+\frac{\pi}{4} \end{bmatrix}$$

$$W_2 = \begin{bmatrix} -1.3\sqrt{2}\frac{\pi}{4} & 0.1 \\ 0.1 & -1.3\sqrt{2}\frac{\pi}{4} \end{bmatrix} \quad W_3 = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$$

$$W_4 = \begin{bmatrix} 2+\frac{\pi}{2} & 40 \\ 0.2 & 2+\frac{\pi}{2} \end{bmatrix} \qquad C_1 = C_2 = C_3 = 1$$

$$E = [1 \quad 1]^T \qquad \tau_1 = 1$$

$$\tau_2 = 0.5 \qquad \tau_3 = 0.2$$

where $0 \le c \le 1$, and $f(x_i(t)) = g(x_i(t)) = h(x_i(t)) = 0.5(|x_i(t)+1| - |x_i(t)-1|)$ are monotonically increasing activation functions, which are monotone increasing and globally Lipschitz continuous.

In the case of $W_3 = W_4 = 0$, the chaotic behavior of NN (1) with the aforementioned parameters has been investigated in [4], [9], and [52]. The chaotic trajectory of the model, i.e., the x_1–x_2 plane, for the delay $\tau_1 = 0.95$ with the initial condition $x(s) = [-0.1, 0.3]^T \ \forall s \in [-1, 0]$ is plotted in Fig. 1.

Assume that $c = 0.1$. It is required to design a driving signal $\hat{u}(t)$ of the form (5) such that the MSNNs (1) and (2) with the aforementioned parameters are exponentially synchronized with the H_∞ performance measure. To this end, in light of Theorem 1, the LMIs (6) are solved using the MATLAB LMI Control Toolbox [58] for different values of the parameter N, i.e., $N \in \{1, 2, 3\}$, and different values of the exponential decay rate α, i.e., $\alpha \in \{0.5, 1, 1.5\}$, and corresponding values of the parameter γ_{optimal} are obtained and shown in Table I. It is easily seen that, for a fixed value of the exponential decay rate α, the parameter γ_{optimal} is decreased as the parameter N is increased, and for a fixed value of the parameter N, the parameter γ_{optimal} is increased as the parameter α is increased. Moreover, the control gains $K_i \ (i = 1, 2, 3)$ for $\alpha = 0.5$ are also calculated and illustrated in Table II. For simulation purposes, an exogenous disturbance input is set as

$$w(t) = \frac{100}{8+5t^2}, \qquad t \ge 0$$

which belongs to $[0, \infty)$ and is imposed on the system. Now, by considering $N = 3$ and initial conditions $\phi(t) = [0.5 \quad -0.7]^T$ and $\varphi(t) = [-0.4 \quad 0.1]^T$, and applying the delayed feedback control (5) with the parameters available in Table II, the phase trajectories of the network and the synchronization errors between the MSNNs are shown in Fig. 2. It shows that the synchronization errors exponentially converge to zero. The simulation results imply that the MSNNs under consideration are globally exponentially synchronized.

TABLE II
CONTROLLER GAINS (WITH $\alpha = 0.5$) W.R.T. N

	$N=1$	$N=2$	$N=3$
K_1	$10^3 \cdot \begin{bmatrix} 1.2784 & 0.3765 \\ 0.5684 & 1.2305 \end{bmatrix}$	$10^3 \cdot \begin{bmatrix} 1.7759 & 0.3182 \\ 0.3237 & 0.2737 \end{bmatrix}$	$10^3 \cdot \begin{bmatrix} 2.5903 & 1.0947 \\ 1.3883 & 2.3299 \end{bmatrix}$
K_2	$\begin{bmatrix} -0.2117 & 0.2374 \\ 0.1541 & -0.2781 \end{bmatrix}$	$\begin{bmatrix} 0.5864 & 0.0560 \\ 0.0676 & 0.2468 \end{bmatrix}$	$\begin{bmatrix} -1.0150 & -0.4911 \\ -0.3406 & -0.8680 \end{bmatrix}$
K_3	$\begin{bmatrix} -0.0965 & -0.1318 \\ -0.1093 & -0.1201 \end{bmatrix}$	$\begin{bmatrix} 0.2450 & 0.3790 \\ 0.4570 & 0.8645 \end{bmatrix}$	$\begin{bmatrix} 0.0864 & 0.0560 \\ 0.0676 & 0.0468 \end{bmatrix}$

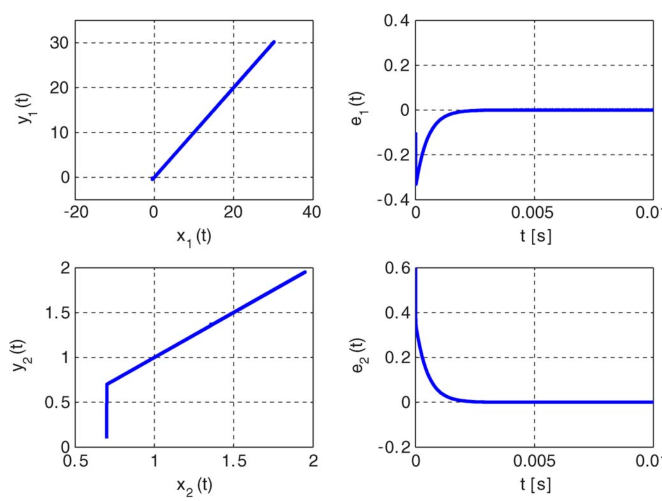

Fig. 2. Phase trajectories of the MSNNs and the synchronization errors.

VI. CONCLUSION

This paper has presented the exponential H_∞ synchronization problem for uncertain MSNNs with mixed time delays, where the mixed delays comprise different neutral, discrete, and distributed time delays. Both the polytopic and the norm-bounded uncertainty cases were separately studied. An appropriate DLKF and some free-weighting matrices were utilized to establish some delay-dependent sufficient conditions for designing delayed state-feedback control as a synchronization law by convex optimization over LMIs under less restrictive conditions. It was shown that the synchronization law guaranteed the exponential H_∞ synchronization of the two coupled MSNNs regardless of their initial states. Detailed comparisons with existing results were made, and numerical simulations were carried out to demonstrate the effectiveness of the established synchronization laws.

APPENDIX

Lemma 1 [25] (Jensen's inequality): Given a positive-definite matrix $P \in \Re^{n\times n}$ and two scalars $b > a \geq 0$ for any vector $x(t) \in \Re^n$, we have

$$\int_{t-b}^{t-a} x^T(\omega)Px(\omega)d\omega \geq \frac{1}{b-a}\left(\int_{t-b}^{t-a} x(\omega)d\omega\right)^T P \left(\int_{t-b}^{t-a} x(\omega)d\omega\right).$$

Lemma 2 [57]: Given matrices $Y = Y^T$, D, E, and F of appropriate dimensions with $F^TF \leq I$, then the matrix inequality $Y + \mathrm{sym}(DFE) < 0$ holds for all F if and only if there exists a scalar $\varepsilon > 0$ such that $Y + \varepsilon DD^T + \varepsilon^{-1}E^TE < 0$.

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