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Hyperbolic discount curves: a reply to Ainslie

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Abstract Ainslie (Theory and Decision, 73, 3–34, 2012) challenges our interpretation of the properties of hyperbolic discount curves in an iterated prisoners' dilemma (IPD) model. In this reply, we discuss the emergence of hyperbolic discount functions in the behavioral economics literature and evaluate their properties. Furthermore, we present a summarized version of our IPD model and evaluate Ainslie's points of contention.

Keywords Hyperbolic · Discount function · Iterated prisoners' dilemma

JEL Classification C72 · C73 · D90

1 Introduction

Ainslie (2012, p. 27) (hereafter simply Ainslie) questions our understanding of the properties of hyperbolic discount curves as modeled in Musau (2009) (hereafter [M]). We are grateful for the opportunity to provide a more structured discussion of our model and evaluate Ainslie's points of contention. Evidently, our conclusions were not clearly stated.

The paper is organized as follows: Sect. 2 provides a discussion of the emergence of hyperbolic discount functions in the behavioral economics literature and evaluates their

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properties. Section 3 highlights the relevance of calculating incentives to cooperate in the iterated prisoners' dilemma (IPD) under non-exponential discounting. Section 4 presents a summarized version of the model and results in [M]. Section 5 responds to Ainslie's points with reference to the previous sections. Eventually, Sect. 6 concludes.

2 Hyperbolic discount functions

Hyperbolic discount functions incorporate two key behavioral features namely:

(i) Extreme impatience for payoffs occurring in the immediate future.
(ii) Declining discount rates over time.

In Sect. 2.4 of [M], several hyperbolic functional forms are surveyed but the most common version is Mazur (1987):

$$D(t) = (1 + kt)^{-1} \tag{1}$$

where $D(t)$ is the discount function, k is the discount rate, and t represents a time delay. Before characterizing the mathematical properties of this function (and hence hyperbolic discounting functions in general), it is appropriate to first motivate their emergence in the behavioral economics and psychology literatures.

2.1 Pre-hyperbolic discounting: The discounted utility model

Samuelson (1937) (hereafter [S]) proposed the discounted utility model (hereafter DU model) for representing intertemporal preferences. A fundamental point in [S] is the observation that representing tradeoffs at different points in time requires a cardinal measure of utility (Assumption I, p. 156). Since the DU model's introduction, it has dominated economic analysis of intertemporal choice (Loewenstein and Prelec 1992, p. 573).

In the DU model, an economic agent is represented as selecting between choices based on a weighted sum of utilities: the weights being represented as discount factors. The main underlying assumption of the DU model is that the discount factor is constant over time (Assumption III in [S], p. 156). As an example, assume that the DU model represents the decline in a kid's utility for reading comic books. If the kid were to evaluate his utility for reading comics three years from now in relation to two years from now, then the percentage decline would equal that of two years from now in relation to a year from now. The same holds for any evaluation that is done for some pair t and $t - 1$ years from now ($t \geq 1$). The DU model's constant discount rate assumption thus leads to an exponential discounting function—commonly associated with financial calculations of present value. Formally, the functional form in [S] is expressed in the following way in discrete time:

$$U(c_1, c_2, \ldots, c_N) = \sum_{\tau=1}^{N} \delta^{\tau-1} u(c_\tau) \tag{2}$$

where $U(\cdot)$ is the time-separable intertemporal utility function, $(c_1, c_2, \ldots, c_N)$ represents a consumption profile: c_τ denotes consumption in period τ, $u(\cdot)$ is the cardinal instantaneous utility function and δ is the discount factor. In terms of the discount factor and the discount rate (denoted k), the following inverse relation holds:

$$\delta = (1 + k)^{-1} \tag{3}$$

In [M], it is argued that a reason for the DU model's dominance of economic analysis of intertemporal choice relates to desirable mathematical properties inherent in exponential discounting. Two properties identified are exponential discounting's stationarity property which leads to dynamically consistent choices and its mathematical tractability (refer to Sect. 2.1.2 for illustrated examples). However, a list of anomalies have been identified in the DU model that diminish its adequacy in modeling the behavior of economic agents.

2.2 DU model anomalies and origins of hyperbolic functions

The literature on behavioral anomalies in the DU model arising from empirical and experimental studies in behavioral economics and psychology is extensive (refer to Fredrick et al. 2002 for a survey). In the following, we describe five main anomalies:

1. Falling discount rates—Relative to exponential discounting, individuals discount the near future at a higher rate and the far future at a lower rate.
2. Preference reversal—For mutually exclusive rewards X at τ and Y at $\tau + s$, an individual at period t indicates preference for X. The same individual at period t indicates preference for Y at $\tau + k + s$ over X at $\tau + k$ (where τ is some point in time, s is a short time delay and k is a long time delay).
3. Magnitude of payoff—For positive payoffs, larger absolute amounts are discounted at a lower rate than smaller amounts.
4. Sign of payoff—Positive payoffs (gains) are discounted at a higher rate than negative payoffs (losses).
5. Framing—Individuals' willingness to accept (WTA) for delaying a real reward from period t to $t + k$ is greater than their willingness to pay (WTP) to speed up its receipt from $t + k$ to t ($k > 0$ days).

The anomalies described above were formally documented long after concerns were raised regarding the empirical validity of the DU model. Two decades after its introduction, economist Robert Strotz was first to claim that the DU model was not a normatively representative model of intertemporal choice (Strotz 1956). He did not specify an alternative model but noted that any non-exponential discount function would lead to time-inconsistent preferences.

A discount function depicting the specific form of inconsistency emerged following findings from animal behavior experiments. Herrnstein (1961) observed that subjects approximately sample two concurrently available streams of rewards in proportion to the mean rates, immediacies, and sizes of the rewards (the "matching law").[1] Following this, Ainslie (1975) proposed that the matching law if applied to individual, discrete choices between smaller sooner (SS) and larger later (LL) rewards would imply the following predictions:

(i) The relationship between decline in rewarding effect and delay is better characterized by a value function that is inversely proportional to delay (a hyperbolic discount function) than a value function that declines by a constant proportion of residual value per unit of delay (an exponential discount function).
(ii) The hyperbolic discount function will lead to preference reversals of the sort noted by Strotz. [2,3]

Section 2.4 reviews properties of hyperbolic discount functions and illustrates how they account for some of the behavioral anomalies of the DU model described in this section. However, an approximation of the hyperbolic discount function in discrete time is the quasi-hyperbolic or "$\beta - \delta$" discount function. The function has gained wide popularity among behavioral economists in recent times and we next review its properties.

2.3 The quasi-hyperbolic discount function

The quasi-hyperbolic specification was proposed by Phelps and Pollak (1968) in a model of intergenerational altruism.[4] It takes the form:

$$U = u(c_0) + \beta\delta u(c_1) + \beta\delta^2 u(c_2) + \beta\delta^3 u(c_3) + \ldots, \quad 0 < \beta < 1, \quad 0 < \delta < 1 \tag{4}$$

where the parameter β reflects "myopia" and all other variables as defined in the DU model in Sect. 2.1 (p. 186).[5] The function can be expressed as the following set of discrete values $\{1, \beta\delta, \beta\delta^2, \beta\delta^3, \ldots\}$ and is such that:

$$D(t) = \begin{cases} 1 & \text{if } t = 0 \\ \beta\delta^t & \text{if } t > 0 \end{cases} \tag{5}$$

[1] In Herrnstein's experiment, pigeons in an operant chamber could peck at one of two response-keys, each of which was on a variable interval (VI) reinforcement schedule. The experiment used concurrent schedules of intermittent reward (VI-VI).

[2] Many writers in behavioral economics get this point wrong. In fact, no paper until Ainslie (1975) pointed out that the "matching relationship" would be a hyperbola if applied to individual, discrete choices, and thus cause preference reversals.

[3] Ainslie's hyperbolic function is such that events τ periods away are discounted with factor $\frac{1}{\tau}$.

[4] The function was later used by Laibson (1997) to model intrapersonal dynamic conflict.

[5] Myopia in common usage is near-sightedness. O'Donoghue and Rabin (1999) define the term as "present-biased" time preference in the context of intertemporal choice.

An analysis of the model in Eq. 4 shows that it is identical to the DU model if $\beta = 1$. To interpret the role of β, the model can be rewritten as follows:

$$U = u(c_o) + \beta[\delta u(c_1) + \delta^2 u(c_2) + \delta^3 u(c_3) + \ldots] \tag{6}$$

β can thus be interpreted as a weight factor applicable to all future periods. For example, if $\beta < 1$ say $\beta \simeq 0.33\ldots$ and $\delta \simeq 1$, we have the following set of values:

$$\{1, \beta\delta, \beta\delta^2, \beta\delta^3, \ldots\} = \{1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots\} \tag{7}$$

implying,

$$U = u(c_o) + \frac{1}{3}[u(c_1) + u(c_2) + u(c_3) \ldots] \tag{8}$$

From Eq. 8, we can derive some properties of a quasi-hyperbolic discount function:

(i) All periods in the future are worth less relative to the present (for example, $\frac{1}{3}$ in Eq. 8).
(ii) All discounting beyond the exponential occurs between the present and the immediate future (resulting from the effect of β).
(iii) Between future periods, there is no additional discounting.

The quasi-hyperbolic discount function is thus able to account for extreme impatience that individuals exhibit for payoffs occurring in the immediate future through the parameter β. However, the fact that it is a hybrid of exponential discounting implies that payoffs occurring at periods $t > 1$ are discounted at a constant rate.

2.4 Properties of hyperbolic discount functions

In order to derive properties of hyperbolic discount functions, we analyze Mazur's functional form in Eq. 1. Two necessary conditions that a function must satisfy to be a discount function are:

Axiom 1 $D(0) = 1$, that is, no discounting of the present.
Axiom 2 $D'(t) < 0$, that is, $D(t)$ must be strictly monotone decreasing.

It is easy to verify that the hyperbolic discount function in Eq. 1 satisfies Axiom 1 and Axiom 2. For $t = 0$, the value of the function is equal to unity.

$$D(0) = \frac{1}{1 + (k \times 0)} = 1 \tag{9}$$

Similarly, the first-order derivative of the function with respect to both of its arguments is negative implying that $D(t)$ is strictly monotone decreasing.

$$\frac{d}{dt}\left(\frac{1}{1+kt}\right) = -\frac{k}{(1+kt)^2} < 0, \quad t > 0, \; k > 0 \tag{10}$$

$$\frac{d}{dk}\left(\frac{1}{1+kt}\right) = -\frac{t}{(1+kt)^2} < 0, \quad t > 0, \; k > 0 \tag{11}$$

Evaluating the limits of the function, we note that $D(t) \in (0, 1]$

$$\lim_{k\to\infty} D(t) = 0 \;\wedge\; \lim_{t\to\infty} D(t) = 0 \tag{12}$$

Therefore, a low value of the hyperbolic discount function can be attributed to a long time delay t, or a high discount rate k, or both. The elasticity of the hyperbolic discount function with respect to a time-delay t (denoted η_h) is approximately equal to -1 for all t.[6] This is given by:

$$\eta_h = \frac{tD'(t)}{D(t)} = \frac{-kt(1+kt)^{-2}}{(1+kt)^{-1}} = -kt(1+kt)^{-1} \tag{13}$$

It should be noted that in Eq. 1, only the normalizing constant in the denominator prevents this value from strictly being equal to -1.[7] In a different hyperbolic specification such as Ainslie (1975) where $D(t) = 1/t$, this elasticity is exactly equal to -1 implying that the hyperbolic discount curve is inversely proportional to delay.

We plot a hyperbolic discount curve and an exponential discount curve in Fig. 1. As can be observed from the plot, the hyperbolic curve initially is steeper relative to the exponential curve but eventually flattens out as the time delay increases.

To get an intuition of this, we can evaluate the first-order derivative of the function with respect to a time-delay at different intervals.[8] Setting $k = 1$, and choosing some low t, say $t = 0.1$, the value of $D'(t)$ in Eq. 10 is equal to -0.8264. Conversely, choosing a high t, say $t = 10$, the value of $D'(t)$ is -0.0083. Thus, the approximate change in the function value as the time delay increases by one unit is significantly smaller for longer delays. The implication is that a hyperbolic discount function implies a monotonically falling discount rate.

Empirically, Fredrick et al. (2002) note that when mathematical functions are explicitly fit to measured data, a hyperbolic functional form is seen to fit the data better than the exponential functional form which implies a constant discount rate (p. 360). As a consequence, the hyperbolic discount function is able to explain dynamically inconsistent behavior observed in experiments involving both human and animal subjects.

[6] The elasticity of $D(t)$ with respect to t represents the ratio of the incremental change of the logarithm of $D(t)$ with respect to the incremental change of the logarithm of t.

[7] The constant in the denominator of Eq. 1 ensures that the value of the function is equal to 1 if either $k = 0$ or $t = 0$. Otherwise, the value of the function is not defined at this point.

[8] The derivative of $D(t)$ with respect to t measures the change in the function as the time delay changes marginally holding the discount rate k constant.

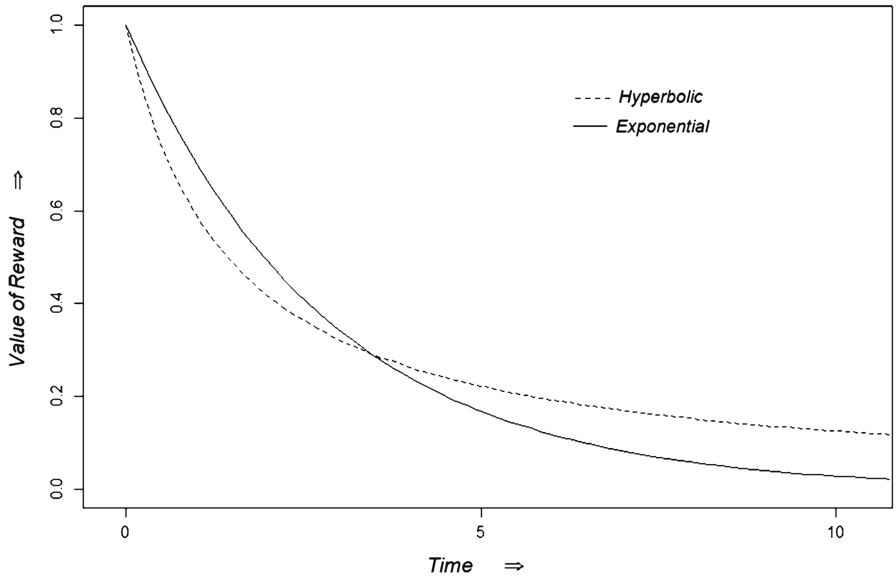

Fig. 1 A comparison of exponential and hyperbolic discounting

Ainslie illustrates this property of the function by comparing an SS reward occurring at some period τ and an LL reward at $\tau + s$, where s is a fixed time lag. He notes that the LL reward is preferred when both τ and $\tau + s$ are distant but as one approaches τ, there is a shift in preference to the SS reward (p. 6).

A restrictive property of hyperbolic discount functions is their mathematical intractability. For example, when evaluating an infinite series of payoffs, the following series does not converge:

$$\frac{1}{1+k} + \frac{1}{1+2k} + \frac{1}{1+3k} + \cdots \tag{14}$$

One therefore has to perform a new computation at every point in time a payoff occurs. This feature of hyperbolic discount functions limits their use in applied work.

3 Discounting and the IPD model

Before presenting our IPD model under different forms of discounting in Sect. 4, we first provide a basis for such an analysis. Many long-run relationships between two parties in various economic contexts can be described as self-enforcing agreements. According to Telser (1980), self-enforcing agreements are characterized by the following features: (i) each party unilaterally decides whether it is in its interest to continue dealing with the other party; (ii) if one party violates the terms of the agreement, then the only recourse is for the other party to terminate the agreement upon discovering the violation; and (iii) no external party intervenes to enforce the agreement, to establish whether violations have occurred, to assess damages, or to impose penalties.

Such relational contracts arise since it is costly or impossible in many instances to rely on third parties, e.g., courts, to enforce agreements and assess damages resulting from violations. Within economics, multi-period principal-agent models illustrate self-enforcing contracts and have been considered in as diverse contexts as labor markets (Harris and Holmstrom 1982; Shapiro and Stiglitz 1984), credit markets (Bulow and Rogoff 1989; Albuquerque and Hopenhayn 2004), and international trade (Thomas and Worrall 1994).

The qualitative features of self-enforcing contacts allows them to be represented in the form of IPDs. The standard approach within economics and social psychology is to equate cooperation in the IPD with trust and trustworthiness of players (refer to Deutsch 1960; La Porta et al. 1997). Single defections in the form of breaches of trust render the relational contract void, for example, the worker losing her job after repeated instances of shirking.

So far, there has not been much interest among economists and other social scientists in calculating the effect of delay discounting on interpersonal IPD incentives since this adds little to IPD contingencies. Therefore, not much work exists beyond early folk-theorem exercises which typically utilize exponential discounting. However, intrapersonal, i.e., intertemporal, IPDs should not even exist under exponential discounting, and discerning their outcome values under non-exponential discounting provides the key to whether preferences will reverse as a function of time.

The issue is central in understanding impulsiveness and self-control, a principal topic in behavioral economics. Ross (2010) categorizes economic models that depict self-control lapses (procrastination) into either of two main categories.[9] The first, labeled *prior-strategy* models, consider an individual as unable to exert influence on her future selves, and thereby, she must set up her future incentives beforehand.[10] The second labeled *dual-motivation* models separate the planner from the doer, usually by pitting one faculty against another, e.g., weak flesh versus a sovereign will.[11]

None of the existing models utilize hyperbolic discounting, and as Ainslie observes, they all stop short of recursive self-prediction (p. 21). Consider the case of a worker who voluntarily contributes to her retirement fund each month: It usually is the case that her short-term and long-term interests are in conflict. The short-term self will want to spend the monthly paycheck, including that part that is set aside for retirement contribution. The long-term self, on the other hand, will not want to jeopardize the retirement fund and will aim at consistently making contributions until the desired target is met. What existing behavioral economics models cannot depict is the intrapersonal simultaneous contest between self-regulation and temptation.

Trigger strategies employed in proofs of the folk theorem (for example, Rubinstein 1979) and also utilized in the model is Sect. 4 fit some of these intrapersonal conflict situations such as the example of the worker saving for retirement, or a recovering alcoholic contemplating a lapse. For each single choice in a series of choices, e.g., whether to make the monthly contribution in the case of the worker, or whether to take

[9] The existing models primarily utilize $\beta - \delta$ discounting.

[10] It is usual practice in economics to model a time-inconsistent agent as a sequence of sub-agents, in effect splitting her up on diachronic dimensions (see Ross 2005).

[11] Refer to Ainslie (2012) for a review of the models.

a single drink in case of the recovering alcoholic, the credibility of future contributions and future sobriety, respectively, is put at stake.

The hypothesis that motivates our model in Sect. 4, referred to as *reward bundling*, is proposed by Ainslie's prediction 4b (p. 17). Specifically, individuals who frame their current choices as predictive of similar future choices "bundle" their expectations of these choices into just such a series (Ainslie 1975, 1992, 2001). The inherent tendency to prefer small short-term rewards combined with imperfect self-prediction results in limited conflict among successive motivational states (or selves[12]) which may be resolved by defining a variant of the IPD among the selves. Defection in the present increases the likelihood of defection in the future not from a motive of retaliation but by making cooperation seem likely to be wasted.[13]

Evidence of reward bundling comes from an experimental study by Hofmeyr et al. (2010) where participants consisting of regular smokers (exemplifying addicted individuals) and a group of non-smokers choose between small, short-term and larger, long-term monetary rewards over a sequence of four decisions with a lag of two weeks between decisions. Decisions are framed as either independent, or part of a series. Hofmeyr et al. observe a significant increase in preference for long-term rewards among the group of smokers when decisions are framed as part of a series.[14] This effect has also been documented by Kirby and Guastello (2001) on undergraduate students using both monetary rewards and slices of pizza.

4 An iterated prisoners' dilemma model

The following is a summarized version of the model in [M] (Sect. 3; p. 29). The interaction between two firms is modeled in the form of an IPD. The firms can choose between two actions at each period: *Cooperate* or *Defect*. If both firms play *Cooperate*, then each obtains a payoff of $C \in \mathbb{R}$. If both firms play *Defect*, then each obtains a payoff of $D \in \mathbb{R}$. If either firm plays *Defect*, then it obtains a payoff of $A \in \mathbb{R}$ in the event that the other firm plays *Cooperate*. The latter obtains a payoff of $Z \in \mathbb{R}$. The following inequality holds with respect to the magnitude of these payoffs: $A > C > D > Z$. The stage IPD game is a triplet: $G = (N, S, \pi)$ where

- N is the set of players: $N = \{1, 2\}$.
- S is the set of pure strategy profiles: $S = \times_{i \in N} S_i$
  - (i) S_i denotes the strategy set for player $i \in N$. In the model description, each player $i \in N$ has two strategies.[15] We denote these strategies s_i^1 and s_i^2: $s_i^1 =$ *Cooperate*, s_i^2= *Defect*. Therefore, $S_i = \{s_i^1, s_i^2\}$.
  - (ii) From (i), it follows that $S = S_1 \times S_2 \Rightarrow S = \{(s_i^1, s_i^1), (s_i^1, s_i^2), (s_i^2, s_i^1), (s_i^2, s_i^2)\}$. The elements in S represent the outcomes of the game. It is convenient

[12] "selves" here representing "oneself in different motivational states".

[13] To draw an analogy with our earlier discussion, the limited conflict described here is also a feature of interpersonal bargaining where it gives rise to self-enforcing agreements.

[14] Notably, the effect in not observed among the group of non smokers.

[15] In the normal form specification, strategies are equivalent to actions.

Fig. 2 The stage IPD game in matrix form

		Firm 2	
		Cooperate	Defect
Firm 1	Cooperate	**C, C**	**Z, A**
	Defect	**A, Z**	**D, D**

from a notational point of view to define these elements in the following way: $s^1 = (s_i^1, s_i^1)$, $s^2 = (s_i^1, s_i^2)$, $s^3 = (s_i^2, s_i^1)$, $s^4 = (s_i^2, s_i^2)$.

– π is the combined payoff function: $\pi : S \to \mathbb{R}^2$
  π is a mapping from S to the cartesian plane. We use the notation $\pi^i(s) \in \mathbb{R}$ to denote the payoff to player $i \in N$ under the outcome s.

Figure 2 exhibits the normal form game in matrix form. A characterizing feature of the stage IPD game is the existence of a Nash-Equilibrium outcome (*Defect, Defect*) that is Pareto-dominated by a non-Nash-Equilibrium outcome (*Cooperate, Cooperate*).[16]

In the following, we analyze the IPD game under different forms of discount functions. It is worth noting that in a finitely repeated version of the game, the unique sub-game perfect equilibrium (SPE) outcome is the Nash-Equilibrium outcome of the stage game where both firms play *Defect* (refer to Appendix 1 for a proof). However, if the game is repeated infinitely or has an unknown end-point, then the outcome (*Cooperate, Cooperate*) can emerge as an SPE under certain conditions.[17] Before proceeding to characterize these conditions as it relates to players' time preference, we limit our analysis to one kind of strategy called trigger strategies. We follow the definition by Shy (1995) adapted to our setting.

Definition $D-1$: Firm $i \in N$ is said to be playing a *trigger strategy* if for every period τ; $\tau = 1, 2, \ldots,$

$$s_\tau^i = \begin{cases} Cooperate & \text{as long as } s_\tau^i = s_\tau^j = \text{Cooperate } \forall t = 1, 2, \tau - 1. \\ Defect & \text{otherwise} \end{cases}$$

The trigger strategy states that firm $i \in N$ plays *Cooperate* as long as both itself and the other firm have not deviated from this strategy. In the case of a single deviation, firm i plays *Defect* forever.

Proposition 1 *Under exponential discounting, the outcome where both firms play their trigger strategy is SPE if:* $\delta \geq \frac{A-C}{A-D}$.

Proof We consider two possible cases under the trigger strategy.

[16] The Prisoners' Dilemma game was originally framed by Merrill Flood and Melvin Dresher working at RAND Corporation in 1950 and later formalized by Albert W. Tucker.

[17] This is an implication of the folk theorem which states that in repeated games, conditional on players' minimax conditions being satisfied, any outcome is a feasible solution concept.

Case 1: "*Mutual Cooperation*" – Both firms choose the strategy $s_i^1 = Cooperate$ for $[t = 1, 2, \ldots]$.
Case 2: "*Unilateral Defection*" – Firm i chooses the strategy $s_i^2 = Defect$ and firm j chooses the strategy $s_i^1 = Cooperate$ at $t = 1$; $(i \neq j)$.

□

With no loss of generality, suppose that firm i = firm 1. Under *Case 1*, firm 1 earns a payoff of C at each period since $\pi^1(s^1) = C$. Over an infinite time horizon, the present value (PV) of this sum of payoffs under exponential discounting is:

$$C + \delta C + \delta^2 C + \cdots = C\left(\frac{1}{1-\delta}\right) \tag{15}$$

Under *Case 2*, firm 1 earns a payoff of A at the first period since $\pi^1(s^3) = A$. After the first period, the "trigger" defined in $D - 1$ is pulled and firm 1 thus earns a payoff of D at each subsequent period since $\pi^1(s^4) = D$. The PV of this sum of payoffs under exponential discounting is:

$$A + \delta D + \delta^2 D + \delta^3 D + \cdots = A + D\left(\frac{\delta}{1-\delta}\right) \tag{16}$$

Taking Eqs. 15 and 16 and solving for δ yields the result in Proposition 1 (refer to Appendix 2 for a step by step derivation).[18]

Proposition 2 *Under quasi-hyperbolic discounting, the outcome where both firms play their trigger strategy is SPE if* $\delta \geq \frac{A-C}{\beta(C-D)+A-C}$.

Proof We employ Phelps and Pollak (1968) specification of a quasi-hyperbolic function to analyze the model.[19] Taking *Case 1* described in the proof of Proposition 1, the PV of the resulting sum of payoffs under quasi discounting is:

$$C + \beta\delta C + \beta\delta^2 C + \beta\delta^3 C + \cdots = C + \beta C\left(\frac{\delta}{1-\delta}\right) \tag{17}$$

Under *Case 2*, the sum of the resulting payoffs under quasi-hyperbolic discounting is:

$$A + \beta\delta D + \beta\delta^2 D + \beta\delta^3 D + \cdots = A + \beta D\left(\frac{\delta}{1-\delta}\right) \tag{18}$$

Taking Eqs. 17 and 18 and solving for δ yields the following result (refer to Appendix 3 for a step by step derivation):[20]

[18] This result is standard since most folk-theorem analysis employ an exponential discount function.

[19] refer to Sect. 2.3 for a summary of the function.

[20] Streich and Levy (2007) obtain the same conditions for the discount factor when comparing a tit-for-tat strategy versus an always-defect strategy in the same game.

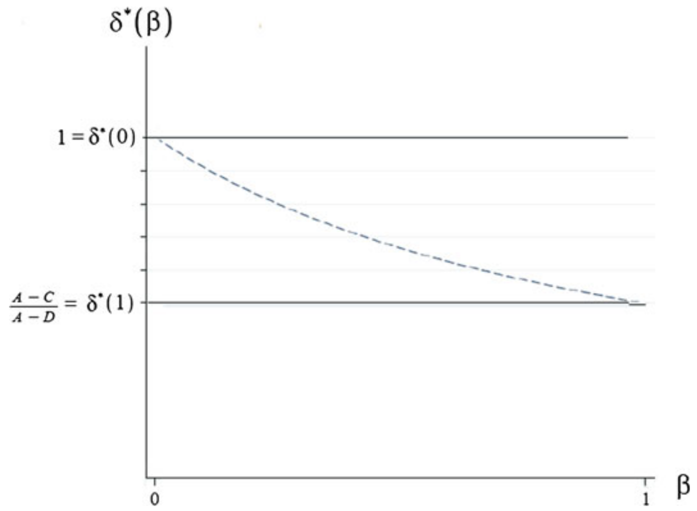

Fig. 3 Domain and range of $\delta^*(\beta)$

$$\delta \geq \frac{A - C}{\beta(C - D) + A - C} \equiv \delta^*(\beta) \tag{19}$$

Taking the derivatives of $\delta^*(\beta)$, we establish that the function is decreasing in the parameter β (refer to Appendix 4 for details):

$$\frac{d}{d\beta}\,\delta^*(\beta) < 0 \quad \wedge \quad \frac{d^2}{d\beta^2}\,\delta^*(\beta) > 0 \tag{20}$$

Figure 3 exhibits the inverse relationship between β and $\delta^*(\beta)$. From Eq. 19, we establish the following limits for the function:

$$\lim_{\beta \to 0} \delta^*(\beta) = 1 \quad \wedge \quad \lim_{\beta \to 1} \delta^*(\beta) = \frac{A - C}{A - D} \quad \text{where} \quad \left(0 < \frac{A - C}{A - D} < 1\right) \tag{21}$$

Therefore, the discount factor under quasi-hyperbolic discounting necessary to sustain the outcome (*Cooperate, Cooperate*) as an SPE equals that of exponential discounting if β is equal to 1. However, if $\beta < 1$, the implication is that the firms require a higher discount factor for the cooperative outcome to emerge.[21]

Proposition 3 *Under hyperbolic discounting, there does not exist a level of δ for which trigger strategies constitute an SPE.*

Ainslie's point of contention relates to Proposition 3. Therefore, we leave the proposition stated and discuss it in the next section.

[21] β here reflects the degree of "present-biased" time preferences (refer to Sect. 2.3).

5 Discussion

The quasi-hyperbolic discount function reviewed in Sect. 2.3 to date still remains the most widely utilized approximation of a hyperbolic discount function in discrete time. However, the function does not account for declining discount rates over time—an key empirical anomaly of the DU model. As Ainslie observes:

> Most behavioral economists have settled on [the *quasi-hyperbolic discount function* in representing intertemporal preferences] more from a desire to *preserve the tractability* of classical economic discount functions than from either parsimony or a *need to fit experience* (p.4; italics added)

This point alludes to our Proposition 3. In illustrating how to represent declining discount rates in discrete time, we run into the problem of specifying too many parameters ([M], p. 36). Consider the following model of discounting:

$$U = u(c_0) + u(c_1)\beta_1\delta + u(c_2)\beta_1\beta_2\delta^2 + \cdots \tag{22}$$

where β_i is a weighting factor ($i = 1, 2, \ldots$), and all other variables as defined in the DU model in Sect. 2.1. The model can thus be expressed as the following infinite set of discrete values:

$$\{1,\ \beta_1\delta,\ \beta_1\beta_2\delta^2,\ \ldots\} \tag{23}$$

The function is arbitrary since we specify no constraints on the β parameter values. However, its advantage is that it has intuitive appeal since it exemplifies the fact that the change in discounting between any periods t and $t+1$ is $\beta_{t+1}\delta$, whereas under exponential and generally quasi-hyperbolic discounting, this change would be a constant δ. By defining $\beta_1\delta = (1+k_h)^{-1}$, $\beta_1\beta_2\delta^2 = (1+2k_h)^{-1}$, and so on, where k_h is the hyperbolic discount rate, one is able to specify a sequence in which the discount rate falls over time in line with Mazur's functional form in Eq. 1. For $\delta^t = (1+k_e)^{-t}$ where k_e is the exponential discount rate:

$$\beta_1 = \frac{1+k_e}{1+k_h},\quad \beta_2 = \frac{(1+k_e)^2}{\beta_1(1+2k_h)} = \frac{(1+k_e)(1+k_h)}{1+2k_h},\ \text{and so on.} \tag{24}$$

In general for $n > 1$:

$$\beta_n = \frac{(1+k_e)^n}{(1+nk_h)\prod_{i=1}^{n-1}\beta_i} = \frac{(1+k_e)(1+(n-1)k_h)}{1+nk_h} \tag{25}$$

The function explicitly points to a tractability issue that would arise if one were to employ it to evaluate a large series of payoffs. Since it has an infinite number of parameters, it cannot be evaluated analytically. To illustrate this, consider *Case 1* in Sect. 4 where both firms choose the strategy s_i^1 = *Cooperate* for [$t = 1, 2, \ldots$]. The

Fig. 4 The stage IPD game with numerical payoffs

		Firm 2	
		Cooperate	Defect
Firm 1	Cooperate	2, 2	0, 3
	Defect	3, 0	1, 1

PV of the payoffs under hyperbolic discounting is thus:

$$C + \beta_1 \delta C + \beta_1 \beta_2 \delta^2 C + \cdots \tag{26}$$

Conversely, under *Case 2*, the PV of the payoffs is:

$$A + \beta_1 \delta D + \beta_1 \beta_2 \delta^2 D + \cdots \tag{27}$$

Taking Eqs. 26 and 27, it is apparent that one is not able to analytically solve for the level of δ for which the PV of payoffs in the former exceeds the PV of payoffs in the latter. Therefore, no analytical solution for δ exists hence our Proposition 3.

In this regard, we admit that we were a bit careless in omitting the term "analytical" in the proposition which led to Ainslie incorrectly interpreting our results. The fact that no analytical solution exists does not mean that no solution exists. One may, for instance, use iterative estimation methods to find solutions in specific contexts. However, note that analytical expressions such as those in Proposition 1 and Proposition 2 are general and hold for any value of payoffs and parameters.[22]

Ainslie's prediction on the incentives to cooperate in the IPD under hyperbolic discounting is qualitative:

> ... although *hyperbolic curves* are steeper than exponential ones at short delays, they are *decreasingly steep with longer delays* and become less steep than exponential curves, leading to *more incentive to cooperate* as more delayed rewards are taken into account (p.27; italics added).

We agree with this prediction. However, if we are to test its validity, then we need to specify a numerical example. Do we wish to do this? We think yes.

5.1 A numerical example

Consider the stage IPD game in Fig. 4:

The outcome (*Cooperate, Cooperate*) is SPE under exponential discounting if:

$$2 + 2\delta + 2\delta^2 + \ldots \geq 3 + \delta + \delta^2 + \delta^3 + \cdots \tag{28}$$

[22] In our case, therefore, one may specify within limits any set of values for A, C, D, Z, and β and obtain a value for δ for which (*Cooperate, Cooperate*) constitutes an SPE.

Table 1 Exponential discounting (break-even discount rate $k = 1 \Leftrightarrow \delta = 0.5$)

Period	Payoffs			Discounted payoffs	
	Case 1	*Case 2*		*Case 1*	*Case 2*
$t = 0$	2	3		2	3
$t = 1$	2	1		1	0.5
$t = 2$	2	1		0.5	0.25
$t = 3$	2	1		0.25	0.125
$t = 4$	2	1		0.125	0.0625
$t = 5$	2	1		0.0625	0.03125
$t = 6$	2	1		0.03125	0.015625
$t = 7$	2	1		0.015625	0.0078125
$t = 8$	2	1		0.0078125	0.00390625
$t = 9$	2	1		0.00390625	0.001953125
$t = 10$	2	1		0.001953125	0.000976563
$t = 11$	2	1		0.000976563	0.000488281
$t = 12$	2	1		0.000488281	0.000244141
$t = 13$	2	1		0.000244141	0.00012207
$t = 14$	2	1		0.00012207	6.10352E-05
$t = 15$	2	1		6.10352E-05	3.05176E-05
			$\sum$	3.999938965	3.999969482

From Proposition 1, it follows that:

$$\delta \geq \frac{3-2}{3-1} \Leftrightarrow \delta \geq \frac{1}{2}. \tag{29}$$

Table 1 shows that the sum of discounted payoffs for both the right hand and left hand member of Eq. 28 converges to 4 given $\delta = 0.5$ (or $k = 1$) from Eq. 29. This result that can be established analytically thanks to properties of the exponential discount function:[23]

$$2\sum_{t=0}^{\infty} 0.5^t = 2\left(\frac{1}{1-0.5}\right) = 4 \tag{30}$$

$$3 + \sum_{t=1}^{\infty} 0.5^t = 3 + \left(\frac{0.5}{1-0.5}\right) = 4 \tag{31}$$

For values of the discount factor greater than 0.5, the PV of payoffs under *Case 1* in Sect. 4 exceeds the PV of payoffs under *Case 2*.

[23] In particular, $\sum_{t=0}^{\infty} \delta^t = \frac{1}{1-\delta}$ and $\sum_{t=1}^{\infty} \delta^t = \frac{\delta}{1-\delta}$.

Table 2 Sum of discounted payoffs: hyperbolic discounting

Discount rate	Time delay			
	$t = 1, 2, \ldots, 10$	$t = 1, 2, \ldots, 20$	$t = 1, 2, \ldots, 100$	$t = 1, 2, \ldots, 1000$
$k = 0.01^a$	20.971 (12.486)	38.298 (21.149)	140.131 (72.065)	480.672 (242.336)
$k = 0.1^a$	15.375 (9.688)	23.320 (13.660)	49.065 (26.533)	93.329 (48.664)
$k = 1^a$	6.040 (5.020)	7.291 (5.645)	10.395 (7.197)	14.973 (9.486)
$k = 2^a$	4.362 (4.181)	5.008 (4.504)	6.579 (5.289)	8.872 (6.436)
$k = 5$	3.064 (3.532)	3.328 (3.664)	3.960 (3.980)	4.879 (4.439)
$k = 10$	2.557 (3.278)	2.690 (3.345)	3.007 (3.503)	3.466 (3.733)
$k = 20$	2.285 (3.143)	2.352 (3.176)	2.511 (3.255)	2.741 (3.370)
$k = 100$	2.058 (3.029)	2.072 (3.036)	2.103 (3.052)	2.149 (3.075)

Mutual Cooperation (Unilateral Defection)
[a] The outcome (*Cooperate, Cooperate*) is SPE

Under quasi-hyperbolic discounting, the outcome (*Cooperate, Cooperate*) is SPE if:

$$2 + 2\beta\delta + 2\beta\delta^2 + \ldots \geq 3 + \beta\delta + \beta\delta^2 + \cdots \tag{32}$$

From Proposition 2, it follows that:

$$\delta \geq \frac{3-2}{\beta(2-1)+3-2} \Leftrightarrow \delta \geq \frac{1}{\beta+1}. \tag{33}$$

Except in the case where β=1 and thus quasi-hyperbolic discounting is equivalent to exponential discounting, the implication is that with “present-biased” time preferences ($0 \leq \beta < 1$), firm $i \in N$ requires a higher δ for (*Cooperate, Cooperate*) to emerge as an SPE. The requirement is that the value of δ lie in the half closed interval (0.5, 1].

Under hyperbolic discounting, to prove that the strategy where both firms play their trigger strategy is SPE, it is sufficient to establish that there exists a discount rate under which the PV of payoffs under *Mutual Cooperation* exceeds the PV of payoffs under *Unilateral Defection*. Employing Mazur’s specification in Eq. 1, we compute the sum of discounted payoffs for delays $t_1 = 1, 2, \ldots, 10$; $t_2 = 1, 2, \ldots, 20$; $t_3 = 1, 2, \ldots, 100$; and $t_4 = 1, 2, \ldots, 1000$; and discount rates $k_1 = 0.01$, $k_2 = 0.1$, $k_3 = 1$, $k_4 = 2$, $k_5 = 5$, $k_6 = 10$, $k_7 = 20$, and $k_8 = 100$. Table 2 summarizes results of the computations.

As is evident, more delayed payoffs are prominent for low values of the hyperbolic discount rate implying that the strategy where both firms play *Cooperate* is preferred. The break-even discount rate under hyperbolic discounting ($k > 2$) is much higher than under exponential discounting ($k = 1$) consistent with Ainslie’s prediction.

6 Concluding remarks

Ainslie criticizes existing models of internal self-control in behavioral economics (hereafter BE) noting that they all attempt to make a single equilibrium preference predictable from a person's prior incentives:

> …"$\beta-\delta$" *delay discount functions* [have] been widely *justified* by the assumption that a person's *intertemporal inconsistency (impulsiveness) can be accounted for* by the arousal of appetite for visceral rewards. Although arousal is clearly a factor in some cases of intertemporal inconsistency, it cannot be blamed for others, and furthermore *does not necessarily imply* ["$\beta-\delta$"] *discounting*. (p.1; italics added)

This criticism is confined to one class of BE models but it generally extends to others. As Frydman and Goldberg observe in their textbook *Imperfect Knowledge Economics*, BE models although putatively embrace "psychological realism" are similar to neoclassical economics models in the sense that they fully pre-specify the causal mechanism that underpins change (2007, p. 12). This approach inevitably leads to an inadequacy on the part of the models in representing observed behavior since not all aspects of individual decision-making can be represented in terms of causal variables.

Herbert Simon, widely considered the founding father of the subdiscipline, echoes the same sentiments in a critique of Ariel Rubinstein's textbook on bounded rationality:

> out of the rich collection of examples expounded in your lectures, I simply do not see how they lead to the kind of economic theory that we should all be seeking: a theory that *describes real-world phenomena* …not all phenomena that we imagine, but *those that actually occur* (Rubinstein 1998 p.190; italics added).

Neoclassical economics models have been criticized on the basis of their failure to conform to reality. The criticisms directed towards BE models should be taken seriously if such models are to distinguish themselves from their predecessors.

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Appendix 1: finitely iterated prisoners' dilemma

Consider the IPD model described in Sect. 4 and suppose that the game is repeated T times in periods $1, 2, \ldots, T$ where $T \in \mathbb{Z}$ s.t. $1 \leq T \geq \infty$. If this is common knowledge, we prove that the game has a unique SPE in which each firm plays *Defect* at all periods. We split the proof into 2 parts.

[*Part 1:* First, we determine the Nash equilibrium of the stage game using the best response function of firm $i \in N$:

Definition $D-2$: In a 2 player game, the best response function of player i is the function $R^i(s_j)$ that for every given strategy s_j of player j assigns a strategy $s_i = R^i(s_j)$ that maximizes player i's payoff $\pi^i(s_i, s_j)$ (Shy 1995, p. 21).

From the model description in Sect. 4, the best response function of firm $i \in N$ is:

$$R^i(s_j) = \begin{cases} Defect \ \ if \ s_j \ = \ Cooperate \\ Defect \ \ if \ s_j \ = \ Defect \end{cases} \tag{34}$$

Definition $D - 3$: An outcome $\hat{s} = (\hat{s}^1, \hat{s}^2, \ldots, \hat{s}^N)$ (where $\hat{s}^i \in S_i$ for every $i = 1, 2, \ldots, N$) is said to be a Nash equilibrium (NE) if for every player i, $\pi^i(\hat{s}^i, \hat{s}^{\neg i}) \geq \pi^i(s^i, \hat{s}^{\neg i})$ for every $s^i \in S_i$ (Shy 1995, p.18).

For the outcome $s^1 = (s_i^1, s_i^1)$; $\pi^1(s_i^1, s_i^1) = C < \pi^1(s_i^2, s_i^1) = A$, contradicting $D - 3 \Rightarrow s^1$ is not NE.

For the outcome $s^2 = (s_i^1, s_i^2)$; $\pi^1(s_i^1, s_i^2) = Z < \pi^1(s_i^2, s_i^2) = D$, contradicting $D - 3 \Rightarrow s^2$ is not NE.

For the outcome $s^3 = (s_i^2, s_i^1)$; $\pi^2(s_i^2, s_i^1) = Z < \pi^2(s_i^2, s_i^2) = D$, contradicting $D - 3 \Rightarrow s^3$ is not NE.

For the outcome $s^4 = (s_i^2, s_i^2)$; $\pi^1(s_i^2, s_i^2) = D > \pi^1(s_i^1, s_i^2) = Z$ & $\pi^2(s_i^2, s_i^2) = D > \pi^2(s_i^2, s_i^1) = Z \Rightarrow s^4$ is NE.

The outcome $s^4 = $ (*Defect, Defect*) thus constitutes an equilibrium in dominant strategies (EDS) and a unique NE for the stage IPD game.

[*Part 2:* Having established that (*Defect, Defect*) is an NE of the stage game, we suppose that both firms have played the IPD game in $T - 1$ periods and they are ready to play for one last time in period T. At this point, the game is identical to the stage game and firm $i \in N$ plays its dominant strategy *Defect* (refer to the best response function of firm i in Eq. 34). Therefore, the outcome of the game is the NE of the stage game (*Defect, Defect*). Now consider the game in period $T - 1$. Both firms know that following this period, they will have one game to play and the outcome of the game involves both playing *Defect*. Again, at this period, both firms will play their dominant strategy resulting in the outcome (*Defect, Defect*). Using backward induction, we note that at each period $T - 2, T - 3, \ldots, 1$, the outcome where both firms play *Defect* will result hence SPE. □

Appendix 2: cooperation under exponential discounting

Consider *Case 1* and *Case 2* defined in Sect. 4. We prove that the outcome (*Cooperate, Cooperate*) is SPE under exponential discounting if $\delta \geq \frac{A-C}{A-D}$.

- The sum of discounted payoffs under *Case 1* is given by:

$$C + \delta C + \delta^2 C + \ldots \tag{35}$$

To find the sum of the series in Eq. 35, we exploit a property of the exponential discount function.

Claim: The following sum, $1 + \delta + \delta^2 + \ldots$, converges to $\frac{1}{1-\delta}$ if $\delta < 1$.

Proof Define the partial sums of the series as follows: $s_1 = 1$, $s_2 = 1 + \delta$, $s_3 = 1 + \delta + \delta^2$,..., $s_n = 1 + \delta + \cdots + \delta^{n-1}$ where s_i represents the ith partial sum

$(i = 1, 2, \ldots n)$. Multiply s_n by δ and obtain $\delta s_n = \delta + \delta^2 + \cdots + \delta^n$. Subtract δs_n from s_n and obtain: $s_n - \delta s_n = 1 - \delta^n$. Solve for s_n: $s_n = \frac{1-\delta^n}{1-\delta}$, $(\delta \neq 1)$. Finally taking the value for s_n, note that if $|\delta| < 1$ then $\delta^n \to 0$ as $n \to \infty$ and $s_n \to \frac{1}{1-\delta}$. □

- From this property, we establish that the sum in Eq. 35 is $C\left(\frac{1}{1-\delta}\right)$
- The sum of discounted payoffs under *Case 2* is given by:

$$A + \delta D + \delta^2 D + \cdots = A + D\left(\frac{\delta}{1-\delta}\right) \tag{36}$$

- For the outcome (*Cooperate, Cooperate*) to be an SPE, we require that:

$$C\left(\frac{1}{1-\delta}\right) \geq A + D\left(\frac{\delta}{1-\delta}\right) \tag{37}$$

$$\Leftrightarrow \frac{C}{1-\delta} \geq A + \frac{\delta D}{1-\delta} \Leftrightarrow \frac{C - \delta D}{1-\delta} \geq A \Leftrightarrow C - \delta D \geq A - \delta A$$

$$\Leftrightarrow \delta(A - D) \geq A - C \Leftrightarrow \delta \geq \frac{A-C}{A-D} \qquad \square$$

Appendix 3: cooperation under quasi-hyperbolic discounting

Consider *Case 1* and *Case 2* defined in Sect. 4. We prove that the outcome (*Cooperate, Cooperate*) is SPE under quasi-hyperbolic discounting if $\delta \geq \frac{A-C}{\beta(C-D)+A-C}$.

- The sum of discounted payoffs under *Case 1* is given by:

$$C + \beta\delta C + \beta\delta^2 C + \cdots \tag{38}$$

Similarly, we exploit the convergence property of exponential discounting to determine this sum. The sum in Eq. 38 is thus:

$$\beta C\left(\frac{\delta}{1-\delta}\right) + C$$

- The sum of discounted payoffs under *Case 2* is given by:

$$A + \beta\delta D + \beta\delta^2 D + \cdots \tag{39}$$

The sum in Eq. 39 is:

$$\beta D\left(\frac{\delta}{1-\delta}\right) + A$$

– For the outcome (*Cooperate, Cooperate*) to be an SPE, we require that:

$$\beta C\left(\frac{\delta}{1-\delta}\right)+C \geq \beta D\left(\frac{\delta}{1-\delta}\right)+A \tag{40}$$

$$\Leftrightarrow \frac{\delta(\beta C-\beta D)}{1-\delta} \geq A-C \Leftrightarrow \delta(\beta C-\beta D) \geq (A-C)(1-\delta)$$

$$\Leftrightarrow \delta(\beta C-\beta D+A-C) \geq A-C \Leftrightarrow \delta \geq \frac{A-C}{\beta(C-D)+A-C} \qquad \square$$

Appendix 4: analysis of the break-even quasi-hyperbolic discount factor

We show that the following relation in Eq. 20 holds:

$$\frac{d}{d\beta}\,\delta^*(\beta)<0$$

Define $(A-C)$ as α and $(C-D)$ as γ in Eq. 19 and re-write $\delta^*(\beta)$ as follows:

$$\delta^*(\beta)=\frac{A-C}{\beta(C-D)+A-C}=\frac{\alpha}{\beta\gamma+\alpha}$$

Differentiating $\delta^*(\beta)$ with respect to β, we obtain:

$$\frac{d}{d\beta}\left(\frac{\alpha}{\beta\gamma+\alpha}\right)=-\frac{\alpha\gamma}{(\beta\gamma+\alpha)^2}$$

From the model description in Sect. 4, we have that $A > C > D$ implying $\alpha > 0$ and $\gamma > 0$:

$$\alpha=\underbrace{(A-C)}_{+} \quad \gamma=\underbrace{(C-D)}_{+} \quad \Rightarrow (A-C)(C-D) > 0$$

Therefore, we establish the result in Eq. 20:

$$-\frac{\alpha\gamma}{(\beta\gamma+\alpha)^2} < 0 \Leftrightarrow -\frac{(A-C)(C-D)}{(\beta(C-D)+(A-C))^2} < 0$$

Similarly, we show that the following relation holds for the second order derivative:

$$\frac{d^2}{d\beta^2}\,\delta^*(\beta)>0$$

$$\frac{d^2}{d\beta^2}\left(\frac{A-C}{\beta(C-D)+A-C}\right)=\frac{d}{d\beta}\left(\frac{-\alpha\gamma}{(\beta\gamma+\alpha)^2}\right)=-\alpha\gamma\,\frac{d}{d\beta}\left(\frac{1}{(\beta\gamma+\alpha)^2}\right)$$

$$=-\alpha\gamma\cdot\frac{-2\gamma(\beta\gamma+\alpha)}{(\beta\gamma+\alpha)^4}=2\alpha\gamma^2\,\frac{1}{(\beta\gamma+\alpha)^3}>0$$

$$\Leftrightarrow\ \frac{2(A-C)(C-D)^2}{(\beta(C-D)+(A-C))^3}>0$$

□

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